H2 Mathematics Textbook

CHOO YAN MIN

This version: 16 November 2021.

Latest version here.

Revision in progress.[1]

Latest changes: Currently revising Part V (Calculus).

[1] This textbook was first completed in Jul 2016. For 1.5 years afterwards, only minor changes were made. But starting Feb 2018, I'll be completely rewriting this textbook. In particular, I'll be working to **(a)** slow down the pace of this textbook; and **(b)** explain things more clearly and simply. Less importantly, I'll also be removing the old (9740) material that is no longer on the current (9758) syllabus and making trivial formatting changes (to make the book beautifuller).

As with everything I do, please let me know if you spot any errors or have any feedback. Thank you.

☺ Errors? Feedback? Email me! ☺

With your help, I plan to keep improving this textbook.

Please refer to the latest version and also mention the relevant page number(s) (bottom left).

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The first thing to understand is that mathematics is an art.

— Paul Lockhart (2009).

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. *... Beauty is the first test: there is no permanent place in the world for ugly mathematics.*

— G.H. Hardy (1940).

Le savant n'étudie pas la nature parce que cela est utile; il l'étudie parce qu'il y prend plaisir et il y prend plaisir parce qu'elle est belle. Si la nature n'était pas belle, elle ne vaudrait pas la peine d'être connue, la vie ne vaudrait pas la peine d'être vécue.

The scientist does not study nature because it is useful to do so. He studies it because he takes pleasure in it, and he takes pleasure in it because it is beautiful. If nature were not beautiful it would not be worth knowing, and life would not be worth living.

— Henri Poincaré (1908, 1914t).

whoever does not love and admire mathematics for its own internal splendours, knows nothing whatever about it.

— Michael Polanyi (1959).

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About this Book

This textbook is for H2 Maths students in Singapore (hence the occasional Singlish and TLAs[2]). It exactly follows the **latest 9758 syllabus**.[3] (Of course, I hope that anyone else in the world will also find this useful!)

- **FREE!** This book is free. But if you paid any money for it, I certainly hope your money is going to me! This book is free because

1. It is a shameless advertising vehicle for my awesome tutoring services.
2. The marginal cost of reproducing this book is zero and I am a benevolent maximiser of social welfare. If you don't understand what that last sentence means, you should read my economics textbooks.[4] (Quick translation: *I'm a very nice guy.* ☺)

- **HELP ME IMPROVE THIS BOOK!** Feel free to email me if

1. There are any errors in this book. Please let me know even if it's something as trivial as a spelling mistake, an extra space, a grammatical error, or an incorrect/broken link.
2. You have absolutely any suggestions for improvement.
3. Any part of this book is less than crystal clear.

If at any point in this textbook, you've read the same passage a few times, tried to reason it through, and still find things confusing, then it is **a failure on my part**—I have failed to explain things clearly and simply. Please let me know and I will try to rewrite it so that it's clearer and simpler. (There is also the possibility that I simply made some mistake or typo! So please let me know if there's anything confusing!)

I deeply value any feedback, because I'd like to keep improving this textbook for the benefit of everyone! I am very grateful to all the kind folks who've already written in, allowing me to rid this book of more than a few embarrassing errors.

- **LyX rocks!**[5] A big thank you to all who've contributed to the LyX project.

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[2]Three Letter Abbreviations. In the US for example, abbreviations are viewed by some as dumbing down. In contrast, in Singapore, the ability to use as many abbreviations as possible and even create one's own abbreviations is nearly a mark of intelligence. There shall therefore be very many abbreviations in this textbook.

[3]In 2017, the current 9758 syllabus was examined for the first time and the previous 9740 syllabus for the last time.

[4]I'm working on these. You can find half-completed versions on my website.

[5]This book was written using LyX. LaTeX is the typesetting program used by most economists and scientists. But LaTeX can be annoying to use. LyX is a user-friendly, GUI, for-dummies version of LaTeX. With LyX, you can actually clearly see on screen the equations you're typing, as you're typing them.

It is quite pointless to try working out one's maths in LaTeX because you can't clearly tell what you're writing. In contrast, it is perfectly feasible and indeed easy to work out one's maths in LyX. For example, if you have countless lines of tedious algebra to do, you can do it in LyX and copy-paste/document every step of the way. Otherwise you'd probably be doing it on pen and paper which is messy and which you'll probably misplace.

LyX has boosted my productivity by countless hours over the years and you should use LyX too!

- ***Maths*, *math*, and *matzz***

This book uses British English. And so the word *mathematics* shall be abbreviated as *maths* (and not *math*). By the way, Singaporeans used to pronounce *maths* as *matzz* (similar to how they pronounce *clothes* as *klotes*). But at some point between 2005 and 2015, perhaps after watching too many American TV and movies, Singaporeans decided they'd switch to the American *math*. Perhaps in 50 years, we'll all be Amos Yees trying to speak annoying pseudo American English. But in the meantime, I'll continue to say *matzz*. This is my way of promoting and preserving Singlish (and also sticking it to the ghost of LKY).

- **Hyperlinks**

This book has clickable hyperlinks that bring you to some other location in this book, a web-page, or email link.

After clicking on any such hyperlink, clicking on "Back" or "Previous View" available on many PDF readers (on desktop Adobe Acrobat, "Alt-Left Arrow" works for me) should go back to where you were previously. Try it!

Tips for the Student

- **Read maths slowly**

Reading maths is not like reading *Harry Potter*. Most of *Harry Potter* is fluff. You can skip 100 pages of *Harry Potter* without missing a beat.[6]

In contrast, maths has little fluff. Skip or misapprehend one line and it will cost you.

So, go slowly. Dwell upon and carefully consider **every** sentence in this textbook. Make sure you **completely** understand what each statement says and why it is true. Reading maths is very different from reading most other subject matter.

If you don't quite understand some material, you might be tempted to move forward anyway. Don't. In maths, later material usually builds on earlier material. So, if you simply move forward, then yea sure, this may save you some time and frustration in the short run, but it will almost always cost you far more in the long run.

Better then to stop right there. Keep working on it until you "get" it. Help is all around—ask a friend or a teacher. Feel free to even email me! (I'm always interested to know what the common points of confusion are and how I can better clear them up.)

- **Do *every* example and exercise**

Carefully **work** through **every** example and exercise. Merely moving your eyeballs is not the same as working. Working means having pencil and paper by your side and **going through each example/exercise word-by-word, line-by-line**.

For example, we might write, "$x^2 - y^2 = 0$. So, $(x-y)(x+y) = 0$. Thus, $x = y$ or $x = -y$." If it's not obvious why the first sentence implies the second and the third, stop right there and work on it until you understand why. And again, if you can't figure it out yourself, don't be shy about asking someone for help. Don't just let your eyeballs fly over these sentences and pretend that your brain "gets" it. Such self-deceit will only cost you in the long run.

The Chinese believe in "eating bitterness" and work for work's sake. That is not my view. The exercises in this textbook are not to make you suffer or somehow strengthen your moral fibre. Instead, as with learning to ride a bike or swim, practice makes perfect. The best way to learn and master any material is by practising, doing, and occasionally failing. You may struggle initially and get a few bruises, but eventually, you'll get good.

To repeat, I strongly advise that you do **every** exercise in this textbook. They'll help you learn. They'll also serve as a check that you've actually "got it". And if you haven't, then well, as mentioned, either keep working until you get it or go out and get help. The seemingly easy way out of pretending you've got it and flipping to the next page is actually the hard way out, because it will only cost you more grief in the long run.

- **Construct *your own* examples**

> *A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.*
>
> — Paul Halmos (1983).

---

[6]Spoiler: Voldemort dies, Harry Potter lives happily ever after.

the surest kind of knowledge is what you construct yourself.

— Judea Pearl (2018).

- **Confused? Good!**

誨女知之乎！知之為知之，不知為不知，是知也。

shall I teach you what wisdom means? To know what you know and know what you do not know—this then is wisdom.

— *Analects of Confucius* (1998t).

he fancies he knows something, although he knows nothing, whereas I, as I do not know any thing, so I do not fancy I do. In this trifling particular, then, I appear to be wiser than him, because I do not fancy I know what I do not know.

— Plato's Socrates (4th century BC, 1854t).

he discovered that, when he imagined his education was completed, it had in fact not commenced; and that, although he had been at a public school and a university, he in fact knew nothing. To be conscious that you are ignorant is a great step to knowledge.

— Benjamin Disraeli (1845).

erste Vorbedingung des Lernens: das Wissen des Nichtwissens.

The first prerequisite for learning anything is … the knowledge that we do not know.

— Gottlob Frege (1884, 1950t).

There are always some students who, when probed, say they are not at all confused and have no questions to ask. In my experience, these are usually precisely the worst students. These students have such poor understanding of the material that **they do not know what they do not know**.

And so, when you find yourself confused, don't panic or despair. Your confusion is actually good news! To be confused is to **know that you do not know** and thus to have taken your first step towards wisdom. You now know what you need to work on and what questions to ask your friends and teachers.

Of course, merely knowing what you do not know is not enough. You need to actually act on this as well. Be proactive: your education and your life are in your own hands!

- **Made a mistake? Good!**

> *Making mistakes is the key to making progress. ...*
>
> *Mistakes are not just opportunities for learning; they are, in an important sense, the* only *opportunity for learning ...*
>
> *The chief trick to making good mistakes is not to hide them—especially not from yourself. Instead of turning away in denial when you make a mistake, you should become a connoisseur of your own mistakes, turning them over in your mind as if they were works of art, which in a way they are. The fundamental reaction to any mistake ought to be this: "Well, I won't do* that *again!"*
>
> — Daniel Dennett (2013).

- **If you can't explain something simply, you don't understand it well enough**[7]

> [Richard] *Feynman was once asked by a Caltech faculty member to explain why spin 1/2 particles obey Fermi-Dirac statistics. He gauged his audience perfectly and said, "I'll prepare a freshman lecture on it." But a few days later he returned and said, "You know, I couldn't do it. I couldn't reduce it to the freshman level. That means we really don't understand it."*
>
> — David L. Goodstein (1989).

> *if I can't explain something I'm doing to a group of bright undergraduates, I don't really understand it myself*
>
> — Daniel Dennett (2013).

> *There is a view in some philosophical circles that anything that can be understood by people who have not studied philosophy is not profound enough to be worth saying. To the contrary, I suspect that whatever cannot be said clearly is probably not being thought clearly either.*
>
> — Peter Singer (2016).

The above saying is useful for the teacher. But it also yields the learner the following useful corollary (in mathematics, a **corollary** is a statement that follows readily from another):

> To test whether you understand something well, try explaining it simply.

This learning technique may be dubbed **learning by teaching**.[8]

---

[7]This or some similar saying is often misattributed to Einstein. But as Einstein himself once said, "73% of Einstein quotes are misattributed."

[8]In Latin, *Docendo discimus—by teaching, we learn.*

> *I have learned much from my teachers and even more from my friends, but from my students I have learned more than from all of them.*
>
> — Rabbi Hanina (*Talmud*).

> *When I write equations, I have a very clear idea of who my readers are. Not so when I write for the general public—an entirely new adventure for me. Strange, but this new experience has been one of the most rewarding educational trips of my life. The need to shape ideas in your language, to guess your background, your questions, and your reactions, did more to sharpen my understanding of causality than all the equations I have written prior to writing this book.*
>
> — Judea Pearl (2017).

One way to implement this technique[9] is for you and your friends to get together explain concepts to each other aloud. For this technique to work, you and your friends must be demanding of and challenge each other. Do not rest until all are satisfied that the concept has been clearly, simply, and correctly explained.

- **Lessons from the science of learning**

Unfortunately, the science of learning is still very much in its infancy. But according to this short review of the literature—"How We Learn: What Works, What Doesn't" (2013), the two most effective study/learning techniques are

1. **Self-testing**; and
2. **Distributed** or **spaced practice**.[10]

The next three are

3. **Elaborative interrogation**;
4. **Self-explanation**; and
5. **Interleaved practice**.[11]

Two commonly used but *ineffective* techniques are **(a) highlighting**; and **(b) rereading**.

Again, the science of learning is very much in its infancy, so you should take with a large dose of salt any advice (including mine about learning by teaching). Nonetheless, there's probably no harm trying out different techniques and seeing what works for you.

[9] Another way is to hand over the classroom to students. But this is probably too adventurous in Singapore, where the closest thing is probably the officially sanctioned and of course graded project work presentation.

[10] Say a student has a 20-minute session of PE thrice a week for three weeks. Plan A: An hour of aerobics during Week 1, an hour of basketball during Week 2, and an hour of cricket during Week 3. Plan B (distributed or space practice): Each week, do 20 minutes of each sport.

[11] Say a student has five addition problems and five multiplication problems. Plan A: Do them in this order—$A_1A_2A_3A_4A_5M_1M_2M_3M_4M_5$. Plan B (interleaved practice): Do $A_1M_1A_2M_2A_3M_3A_4M_4A_5M_5$.

Miscellaneous Tips for the Student

- **During the A-Level exam, you get a "List MF26: List of Formulae and Statistical Tables for Mathematics and Further Mathematics" (PDF)**

So there's no need to memorise all the formulae that are already on this list. (Note though that your JC may or may not give you List MF26 during your JC common tests and exams.)

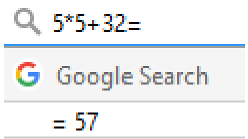
- **Remember your O-Level Maths & 'A' Maths?**

You've probably forgotten some (or most?) of it. But unfortunately, you are still assumed to know *all* of O-Level Maths (2019 syllabus) *and* "some" (OK, more like a lot) of Additional Maths (2019 syllabus). See your A-Level syllabus (pp. 14–15) for what you need to know from 'A' Maths. (To take H2 Maths, most JCs require that you at least passed 'A' Maths.)[12]

Littered around this textbook are occasional "O-Level Reviews" (e.g. Ch. 5). These reviews will usually be very quick and hopefully you'll have no difficulty with them. But if you do, go back and review your O-Level Maths and 'A' Maths!

- **Web-based calculators**

Google is probably the quickest for simple calculations. Type anything into your browser's Google search bar and the answer will instantly show up:



Wolfram Alpha is somewhat more advanced (but also slower). Enter $\sin x$ for example and you'll get graphs, the derivative, the indefinite integral, the Maclaurin series, and a bunch of other stuff you neither know nor care about. In this textbook, you'll sometimes see (usually at the end of an example or an exercise answer) a clickable Wolfram Alpha logo that will bring you to the relevant computation on Wolfram Alpha.

Symbolab is a much less powerful alternative to Wolfram Alpha. However, it's perfectly good for simple algebra and somewhat quicker, so you may sometimes prefer using it.

The **Derivative Calculator** and the **Integral Calculator** are probably unbeatable for the specific purposes of differentiation and integration. Both give step-by-step solutions for anything you want to differentiate or integrate. As with Wolfram Alpha, you'll sometimes see clickable logos and that'll bring you to the relevant computations. (Note that unfortunately, after clicking on these logos, you'll also have to either click "Go!" or hit Enter.)[13]

I also made this Collection of Spreadsheets. These are for doing tedious and repetitive calculations you'll often encounter in H2 Maths (with vectors, complex numbers, etc.).[14]

[12] Some *kiasu* JCs, like HCI, even require that you got at least a B3 for both Maths & Additional Maths.
[13] The site author informed me that not having a direct link was deliberate and defensive.
[14] As with anything I do, I welcome any feedback on these spreadsheets. (Perhaps in the future I will make a more attractive version.)

- **Other online resources**

There are way too many websites that try to cover primary, secondary, and lower-level undergraduate maths. Unfortunately, some of them are awful and get things wrong.

Three websites I like (though are probably a bit advanced for JC students) are

1. The Stack Exchange (SE) family of Q&A websites.[15]

Just for maths alone, there are *three* SE sites![16]

(a) At **MATHEMATICS**, you can ask questions and often get them answered fairly promptly. Note though that this site is mostly frequented by fairly advanced users of maths (including many mathematicians), so they can be pretty impatient and quick to downvote questions they perceive to be "stupid". Nonetheless, if you make an effort to write down a carefully crafted question and show also that you've made some effort to look for an answer (either on your own or online), they can be very helpful.

(b) Mathematics Educators beta focuses on pedagogy (teaching and learning). Unfortunately, it's not as active as **MATHEMATICS**. Nonetheless, many of the discussions there are filled with insights—insights that I've tried to incorporate into this textbook.

(c) math***overflow*** is for research-level mathematics and is thus way beyond anything that's of use to us.

In this textbook, you'll occasionally see and logos. Click/touch them and you'll be sent to related SE discussions.

2. ProofWiki gives succinct and rigorous definitions and proofs. Unfortunately it is very incomplete.
3. Mathworld.Wolfram is also great, but at times excessively encyclopaedic, at the cost of clarity and brevity.

And of course, you can find countless free maths textbooks online (some less legal than others). One wonderful and (mostly) legal resource is the Internet Archive, which has many old books—including very many that were scanned by Google but which are no longer available on Google Books.

Two totally illegal[17] resources are Library Genesis for books and Sci-Hub for articles.[18] And of course, an old reliable is BitTorrent. (I have, of course, never used any of these illegal resources, but I hear they are great.)

[15]The flagship SE site is stack**overflow** where you can ask any programming question and (often) see it answered amazingly quickly. For computing in general, there are also *many* other SE sites.

SE sites are like Yahoo! Answers or Quora, but less stupid. The worst SE site is probably Politics beta, but even there, the average question or answer is probably better than that on Yahoo! Answers or Quora.

[16]There are also many other wonderful SE sites that you should explore. Unfortunately, despite my magnificent contributions, Economics beta is not exactly thriving. It seems that economists, unlike programmers or mathematicians, have learnt all too well that contributing to the public good is folly.

[17]Well, depending on which jurisdiction you live in. Of course, in Singapore, unless told otherwise, you should assume that everything is illegal.

[18]Note though that these sites are constantly playing whac-a-mole with the fascist authorities and so the URLs often change. If the links given here aren't correct, the first thing you should do is to let me know so I can correct these broken links. Then simply google to find the current working URLs. For Sci-Hub, the WIKIPEDIA page usually lists the latest up-and-running URLs.

Use of Graphing Calculators

You are required to know how to **use a graphing calculator**.[19]

This textbook will give only a few examples involving graphing calculators. There is no better way of learning to use it than to play around with it yourself. By the time you sit down for your A-Level exams, you should have had plenty of practice with it.

You can also use any of the seven calculators in the following list:[20]

LIST OF APPROVED GRAPHING CALCULATORS[1]

The following graphing calculator models are approved for use ONLY in subjects examined at H1, H2 and H3 Levels of the A-Level curriculum.
Note: All graphing calculators must be reset prior to any examination.

S/N	Calculator Brand	Calculator Model	Operating System	Approved Period[1]
1	CASIO	FX-9860GIIs	OS 2.04 OS 2.09	2014 – 2022
2	TEXAS INSTRUMENTS	TI-84 Plus	OS 2.43 OS 2.53MP OS 2.55MP	2006 – 2024
3		TI-84 Plus Silver Edition		2006 – 2024
4		TI-84 Plus Pocket SE		2012 – 2021
5		TI-84 Plus C Silver Edition	OS 4.0 OS 4.2	2015 – 2023
6		TI-84 Plus CE	OS 5.0.1 OS 5.1.5 OS 5.2.0 OS 5.2.2 OS 5.3.0 OS 5.3.1 OS 5.4.0	2017 – 2024

This textbook will stick with the TI-84 PLUS Silver Edition, OS 2.55MP[21] which I'll simply call the **TI84**. (My understanding is that most students use a TI calculator and that the five approved TI calculators are pretty similar.)

I'll always start each example with the calculator freshly reset.[22]

[19]Pretty bizarre that in this age of the smartphone, they want you to learn how to use these clunky and now-useless devices from the '80s and '90s. It is the equivalent of learning to program a VCR. (The TI-81 was designed in the 1980s and first sold in 1990. The TI-84 PLUS was first sold in 2004 and represents only a modest improvement over the original TI-81.)
IMHO it'd be much better to teach you to some simple programming or Excel (or whatever spreadsheet program). "B-b-but ... how would such learning be tested in an exam format?" Ay, there's the rub. In the Singapore education system, anything that cannot be "examified" is not worth learning.
Of course, there are some folks over in Texas who don't mind. Nor do those lucky few MOE teachers and administrators who get to go on all-expenses-paid "business" trips to Texas to "learn" more about the calculators.

[20]This list was last updated by SEAB on 2019-10-31.

[21]You can download the operating system through the TI website.

[22]I've never actually bought or owned a graphing calculator. All my screenshots here of the TI84 are actually from the emulator Wabbitemu. (Yup, you can download Atari or DOS emulators to play decades-old games; and you can likewise download an emulator for this decades-old piece of junk.)

Exam Tips for *Towkays*

Use your graphing calculator as much as possible. You are *always* allowed to use your graphing calculator.

Some exam questions will explicitly instruct you not to use your calculator, but this just means that your written answer should not include any hint that you used your calculator. (Nonetheless, you can still cheat and use your calculator for guidance and to check your answer.)

Instructions to not use your calculator include:

- **"Do not use a calculator in answering this question."**
- "Without using your calculator ..."
- "Use a non-calculator method ..."
- "Find the exact value of ..."
- "Express your answer in terms of $\sqrt{3}$ or π."

In *Spider-Man: Homecoming* (2017), Spider-Man uses a TI calculator to bust out of the "most secure facility on the Eastern Seaboard"—the Damage Control Deep Storage Vault (also known as the Department of Damage Control Vault).[23]

[23] This scene takes place at about 57:30–58:00 of the movie (YouTube clip).

They removed the brand name and also the model name, so I can't quite tell what model it is. One website claims it's a TI-86, while another claims it's an exact match to the TI-83 PLUS.

Zooming in, it looks like Spider-Man is doing some sort of combinatorics on his notepad. I can't tell what exactly's on the calculator screen. Many of the buttons on the calculator seem to be permanently depressed, which is weird.

Slowing down, we see that he first punches in $\boxed{5}$ $\boxed{8}$ $\boxed{(}$, then a moment later $\boxed{5}$ $\boxed{7}$ (or maybe $\boxed{2}$ $\boxed{6}$)—so it's probably all just rubbish that he's punching in. The latter keystrokes do get him out of the place though. (That's all for my brilliant movie analysis of the week.)

Apps You Can Use on Your TI Calculator

On 2020-04-03, I received this information from SEAB:

> The following applications for Texas Instruments graphing calculators are approved for use in the GCE A-Level national examinations:
>
> 1. Finance
> 2. Conics
> 3. Inequalz
> 4. PlySmlt2
> 5. Transfrm
>
> Please ensure that your graphing calculator is installed with the approved operating system. The Singapore Reset (pressing "On" key while holding the "2" and "8" keys) will reset the calculator without deleting these applications.

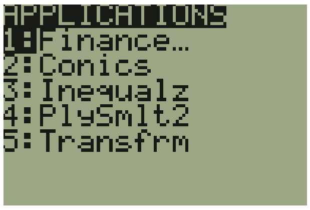
On my (emulated) TI-84 Plus Silver Edition, the Finance app was already pre-installed. So I simply had to google, download, install the other four apps.[24]

After doing so, here's what my "APPLICATIONS" screen looks like after executing these two steps:

1. Press ON to turn on your calculator.
2. Press APPS to bring up the "APPLICATIONS" screen.

Step 1.

Step 2.



(Screenshots are of the end of each Step.)

[24] Webpages to download these four apps for the TI-83 PLUS or TI-84 PLUS: Conics, Inequalz (seems like we're supposed to use the "International" version), PlySmlt2, and Transfrm (links retrieved on 2020-04-04)—ZIP file containing these four apps and the accompanying guidebooks (all written in 2001 or 2002!). (Unfortunately, the guidebook they have for PlySmlt2 seems to be for an old version of the same app.) Note that if you have the TI-84 PLUS CE, you may have to look for different webpages from which to download the appropriate apps.

Preface/Rant

When you're very structured almost like a religion … Uniforms, uniforms, uniforms … everybody is the same. Look at structured societies like Singapore where bad behaviour isn't tolerated. You are extremely punished. Where are the creative people? Where are the great artists? Where are the great musicians? Where are the great singers? Where are the great writers? Where are the athletes? All the creative elements seem to disappear.

— Steve Wozniak (2011).[25]

That ghost, authoritarianism, sees education as a way to instill in all students the same knowledge and skills deemed valuable by the authority.

— Zhao Yong (2014).

The most dangerous man, to any government, is the man who is able to think things out for himself.

— H.L. Mencken (1919).

Divide students into two extremes:

- Type 1 students are happy to learn absolutely nothing, so long as they get an A.
- Type 2 would rather learn a lot, even if this means getting a C.

The good Singaporean is taught that pragmatism is the highest virtue (obedience is second). She is thus also trained to be a Type 1 student (and indeed a Type 1 human being).

If you're a Type 1 student, then this textbook may not be the best use of your time,[26] though you may still find the exercises and Ten Year Series (TYS) questions useful. (But do read Why Even Type 1 Pragmatists Should Read This Textbook below.)

Of course, any careful student of this textbook will be rewarded with an A. But getting an A is not the goal of this textbook. Instead, the goal is to **impart genuine understanding**.

Contrast this with the goal of the Singapore education system: **Create a docile labour force that generates GDP growth.**

At first glance, these two goals do not seem to be in conflict. After all, we'd expect a student who genuinely understands her H2 Maths to also contribute to GDP growth.

[25] 2011 BBC interview. Also quoted in Zhao (2012, p. 103). Unfortunately, the audio at the given BBC link is broken. (Over the years, I've reported this broken audio to the BBC at least thrice, but it's never been fixed. I have not been able to find this audio anywhere else.)

[26] The efficient Type 1 student may be interested in these resources: **(a)** The *H2 Mathematics CheatSheet*, all the formulae you'll ever need on two sides of an A4 sheet of paper (this was written in 2016 and I hope to update it "soon"); **(b)** The *H1 Mathematics Textbook*, which is written more simply and covers a subset of the H2 syllabus; **(c)** The *H2 Maths Exercise Book* (coming "soon"), which teaches you how to mindlessly apply formulae and give the "correct" answer to every exam question; **(d)** My **totally awesome tuition classes**!

The conflict only arises with the keyword *docile*. An education system that imparts genuine understanding tends also to encourage independent thinking and discourage docility.

On the one hand, to maximise GDP growth, *Gahmen* wants "creative" innovators. On the other, it doesn't want too much of a challenge to the status quo (especially politically). Its goal is thus to turn docile test-taking drones into docile creative innovators.

Unfortunately, docility and creative innovation are not compatible. A populace trained to avoid the slightest transgression is not one that is capable of producing anything new. The result is lip service to buzzwords like "creativity" and half-hearted education reform. Once a decade or so, some technocrat comes up with an inane four-letter campaign (FLC) like TSLN 1997 and TLLM 2005 that brings us precisely nowhere.[27]

To this ambivalence and pussy-footing, add **(a)** the deep-rooted East Asian love of exams and rote-learning; and **(b)** the elitist British educational system we inherited.[28] Altogether, despite superficial appearances to the contrary, we've had very little change over the years. Administrators, teachers, and students alike remain completely fixated on exams.[29]

Singapore produces world champion test-takers and gold medallists at the various International Olympiads. But as currently constituted, the Singapore education system **will never produce a Fields Medallist or a Nobel Laureate**. And as Steve Wozniak suggests, Singapore **will never produce a world-beating innovator like an Apple or a Google**. The reason is that unlike taking tests (be it your J1 Promos or your IMO), such endeavours require more than mere monkey-see-monkey-do mimicry.[30]

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[27] In 1997, *Thinking Schools, Learning Nation* (or, as was joked, *Sinking Schools, Burning Nation*). In 2006, *Teach Less, Learn More*. These campaigns may now be found in the ash heap of history (alongside such gems as *Goal 2010*). The current FLC is probably ESGS (*Every School a Good School*).

[28] From a 2013-04-12 *Straits Times* interview with Tharman:

ST: DPM, why do you think we are the way we are?

Tharman: Well, we inherited the British system, which is quite academically biased and in Britain, of course, quite an elitist system. We also inherited a Chinese education culture, which is also quite academically oriented, a strong emphasis on values and character education but quite academically oriented and quite test-oriented. And I think the combination of a British and East Asian educational ethos has created a particular form of meritocracy which achieved a lot in 40 years. But as we go forward and we think about the type of inclusive society we want, it's not just about wages, which we are working on, it's about how you view yourself and others at the workplace, wherever you live, how we view fellow Singaporeans, do you view them as equals, do you do things together. That has to start from young and it has to continue through life.

[29] To be fair, I should highly commend the recent *Learn for Life* changes. These changes were announced in late 2018 and will be implemented in 2019–21. They include the removal of *all* weighted assessments for P1 and P2; and mid-year exams for P3, P5, S1, and S3. Also, in changes that had already been announced earlier, from 2021, PSLE T-scores will be replaced by wider scoring bands. In my opinion, all of these changes reduce the obsession with exams and should ipso facto be commended.

Unfortunately, to my knowledge, *Gahmen* has yet to announce any such changes for JC students, Singapore university admissions, and Singapore universities. In 2019, the Singapore Management University (SMU)—which has always tried to brand itself as hip, relaxed, and American—even went out of its way to condemn and criticise a Canadian SMU professor for giving As to all 169 of his students. (see 2019-05-25 *Straits Times* story reproduced at *The Star* and archived here).

[30] Here are two common excuses for why Singapore has produced no Nobel Laureates: Singapore **(a)** has a small population; and **(b)** was until fairly recently very poor. But consider Denmark (population 5.8M), Finland (5.5M), and Norway (5.3M), whose populations are similar to or even smaller than Singapore's (5.6M) and that were producing Nobel Laureates when they were *far* poorer than Singapore is today (whether in absolute terms or relative to, say, the US). We could also point to tiny Saint Lucia (180,000) with her two Nobel Laureates. I am thus accepting bets for this proposition: "By 2050, no born-and-bred Singaporean will have won a Fields medal or a Nobel Prize (Peace excluded)." (We can work out what

How and Why JC Maths Fails Our Students

I have a study in mind: Gather all the students who got As for their A-Level maths exams $x = 5, 10, 15, 20, \ldots$ years *after* the exam. Get them to do that *exact same* A-Level exam they took x years ago. Also, ask if they remember anything from their JC maths education or if they believe it had any value whatsoever. I suspect that most will score close to zero, remember absolutely nothing, and consider their JC maths education to have been completely worthless. If these suspicions are correct, then JC maths education has no value, except as a **selection device**.

(But of course, selection devices are of paramount importance in elitist, social-Darwinist Singapore. Grades help differentiate the President's Scholar from the "mere" PSC scholar, the lowly McDonald's employee from the *dalit* cleaner, and those who should reproduce from those who shouldn't.[31] Grades even decide whether a sex offender can avoid prison.[32])

How is it that the vast majority of even those who got As on their A-Level maths exams x years ago can remember nearly nothing whatsoever? What explains this colossal failure of JC maths education? The explanation is simple:

In Singapore, testing isn't everything; it's the only thing.

Testing does have a place in the educator's toolkit. But it shouldn't be the *only* tool, as is the case in Singapore (and the rest of Confucian East Asia).

Moreover, as currently constructed, Singapore's system of testing does not test for genuine understanding. Instead, students are tested on whether they've mastered the East Asian "skills" of mimicry, following instructions, and reproducing recipes, formulae, and algorithms. In other words, students are tested on whether they're well-trained, obedient monkeys capable of performing tricks they've practised *over and over and over* again.

In *A Mathematician's Lament*, Paul Lockhart describes maths education in the US as "stupid and boring", "formulaic", and "mindless" "pseudo-mathematics".[33] The same may be said of maths education in Singapore.

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exactly "born-and-bred" means.)

[31] One will recall the Graduate Mothers' Priority Scheme and Small Family Incentive Scheme. While those two Orwellian/Nazi schemes have since been scrapped, the Social Development Unit (SDU) lives on, though now rebranded the Social Development Network (SDN). The SDU was established "to encourage social interaction and marriage among graduate singles" at around the same time (1984) as the aforementioned schemes .

[32] In 2018, NUS student Terence Siow Kai Yuan molested a woman *thrice*. He first molested her twice on an MRT train. When she alighted at Serangoon station, he too alighted and followed her in order to molest her a third time. Siow also admitted to having committed similar acts since at least 2016 and admitted to being "unable to recall the number of times he had committed such acts". On 2019-09-25, noting that Siow's "academic results show he has the 'potential to excel in life'", District Judge Jasvender Kaur sentenced him to only 21 months probation and 150 hours community service.

Also in 2018, another NUS student Nicholas Lim filmed a female student Monica Baey in the shower without her consent. The police gave him a conditional warning, because a "prosecution, with a possible jail sentence, will likely ruin his entire future, with a permanent criminal record." This incident came to light in early 2019 only because the victim Baey relentlessly called for attention. Importantly, Baey had the aid of social media—in 2005 say, this incident would not have seen the light of day, much less shame NUS and the police into giving *any* response.

[33] By the way, Lockhart explains why this is so and what maths really is far more eloquently and clearly than I ever could. I strongly recommend that every student and instructor of maths read *A Mathematician's Lament*. There are two versions—a 2002 25-page PDF that circulated online and a 2009 book version.

As currently taught and tested for, JC maths is a mere and mindless collection of isolated recipes. Little or no understanding is required. Where do these mysterious recipes come from? Why do they "work"? No matter. Instead, all that's required is for the student to *mug*[34] and reproduce these recipes during exams.

Imagine a maths education system in which students were simply required to *mug* a million-digit number. Those who correctly write out the first 10 000 digits get an A; those who manage only 5 000 get a B; etc. Those who manage 100 000 digits get government scholarships, with the n President's Scholarships going to the top n *muggers*.

This imaginary system sounds absurd, but isn't really all that different from the existing system (or how the Chinese imperial examinations worked for many centuries). Indeed, it'd probably be an improvement. If we replaced the existing maths "education" system with the simple goal of *mugging* a million-digit number, then

- Students would understand no less of maths than they do now.
- Students would find maths no more "stupid and boring" than they do now.
- We'd have a selection device that's no worse than the present one. (Under the present system, students are selected and sorted based on their ability to *mug* and regurgitate isolated recipes that are no less random or arbitrary than a million-digit number.)
- Resources currently dedicated to JC maths education would be freed for other uses.

"[M]emory works far better when you learn networks of facts rather than facts in isolation," writes the British mathematician Timothy Gowers (2012) in a critique of A-Level maths. I agree. When we learn a network of facts and seek to *understand* why something works, we are far more likely to remember what we've learnt and make use of what we've learnt.

Moreover, further learning also becomes easier. It becomes easier to acquire and assimilate even more facts, knowledge, and wisdom.

In contrast, when we *mug* isolated facts and recipes that we do not understand, we aren't likely to remember them mere months after our A-Level exam. When in a few years, we actually have to make use of the material that we were supposed to have learnt in JC, we find ourselves having to learn everything from scratch. What then was the point? We may as well have spent those two years *mugging* a random million-digit number.

[34]To *mug* is to study hard and, especially, to engage in rote learning or memorisation. I consider *mug* to be Singlish and so italicise it.

It seems though that the phrase *mug up* may have originated in Britain. To my knowledge, this phrase isn't in current usage in Britain. I have however come across South Asians using *mug up* in this sense. (In efficient Singlish though, the preposition *up* is simply dropped.)

Work Work Work Work Work Work

Pre-tertiary maths education in Singapore is no more "stupid and boring" than in the US (or elsewhere). The difference is that by the time the typical student in the US (or elsewhere) completes high school, she will only have squandered a very small portion of her life on such "mindless" and "formulaic" "pseudo-maths".

The same cannot be said for the typical Singaporean student. By the time she turns 18, she will have—just for the single subject of maths alone—clocked many thousands of hours attending school and tuition classes; doing homework, practice exam questions, assessment books, and TYS; taking common tests, promos, prelims, mid-year exams, and end-of-year exams; *ad infinitum*, *ad nauseam*.

The Confucian East Asian countries[35] perform splendidly on international tests. For example, in the 2018 Programme for International Student Assessment (PISA) Maths test, they swept the top seven spots (see p. li).[36]

"What," the Western educator enquires, "is the magic here?"[37] But there is no magic. To me, the explanation for why East Asian students do so well on these tests is obvious:

They're forced to work their butts off.

While the American teen is "wasting" her time on typical, "useless" teenager-ly pursuits, the Singaporean teen is seated obediently in front of his desk, doing yet another soul-crushing TYS question. And once in a while, children as young as ten commit suicide due to poor exam results.[38] Kids the world over commit suicide for a variety of reasons, but only in East Asia do they regularly do so because of exams and schoolwork.

In South Korea, legislators have even passed (ill-enforced) laws barring *hagwons* (private cram schools) from operating past 10 p.m. To the American teen, it is mind-blowing that **(a)** anyone would be in school past 10 p.m.; or that **(b)** this practice would grow so common that legislators saw fit to take action. But to the East Asian, this isn't at all strange.

To me, the fact that East Asian students bust their butts is *obviously* the single most important explanation for why they do so well on international tests. Yet strangely, in the countless papers and books that I've come across seeking to explain why some countries do better than others, this explanation is rarely ever considered.[39]

[35] China, Hong Kong, Japan, Korea, Macao, Singapore, Taiwan, and Viet Nam.

[36] Viet Nam's results have been omitted from the headline charts in the 2018 edition. I'm not sure why exactly and haven't looked into this matter. But we have for example this somewhat-cryptic statement: "the statistical uniqueness of Viet Nam's response data implies that performance in Viet Nam cannot be validly reported on the same PISA scale as performance in other countries" ("PISA 2018 Results, Volume I, What Students Know and Can Do", 2019, p. 190, PDF).

[37] In the US, "Singapore Math" has acquired something of a mythical status. As with weight loss, Americans are constantly on the lookout for some magic, painless solution to their mediocre education systems.

[38] In 2001, ten-year-old Lysher Loh jumped to her death from her fifth floor apartment. She "had been disappointed with her mid-year examination results and had found the workload heavy." She had also "told her maid Lorna Flores two weeks before her death she did not want to be reincarnated as a human being because she never wanted to have to do homework again." In 2016, an 11-year-old "killed himself over his exam results by jumping from his bedroom window in the 17th-storey flat". More suicides here.

[39] This explanation is neglected in part because the relevant statistics are difficult and expensive to collect. But a bigger reason for such neglect is probably that writers on education (be it in the West or in East Asia) are simply not quite aware of how much harder students in East Asia are made to work.

A closely related explanation is that East Asians are *trained* from young to take every test seriously. In contrast, US kids couldn't care less about some inconsequential PISA test.[40]

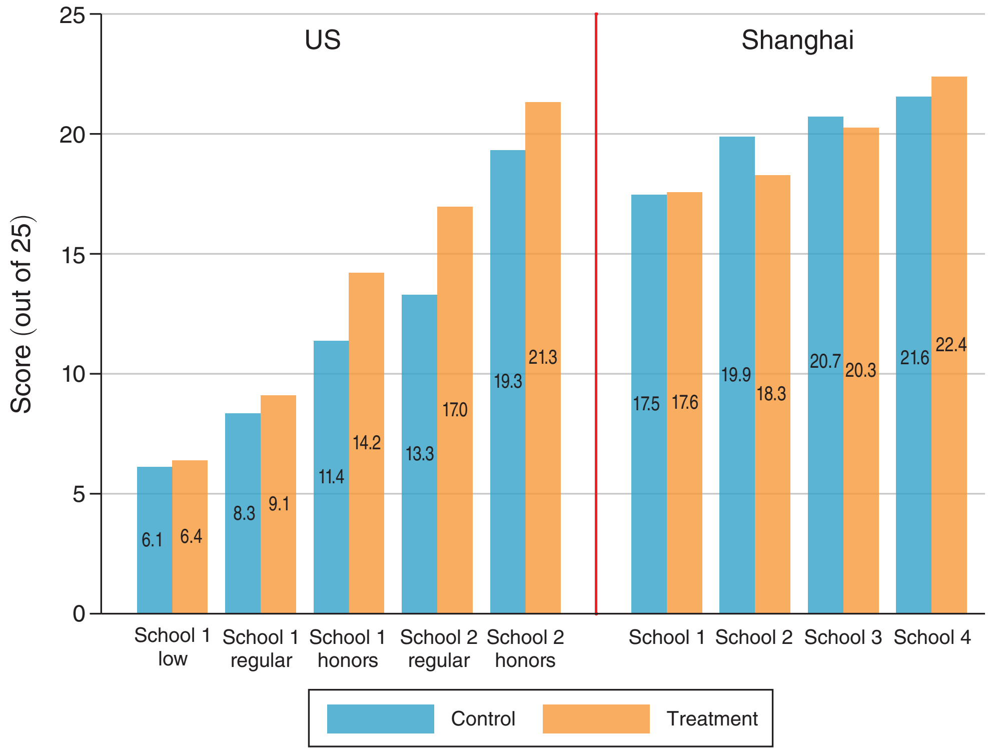
And yes, the PISA test is truly inconsequential. Each individual student's results are never revealed (not to the student herself, her teachers, her parents, her school, or anyone else). Students are simply forced to take a two-hour test[41] whose results they'll never know and which will have absolutely no consequence on their lives.

Under such circumstances, I suspect the *truly* intelligent and educated kid would simply click through the test as quickly as possible. In contrast, the dull-witted, obedient, and well-trained student would actually take the test seriously.

This explanation has received some academic attention. For example, in "Measuring Success in Education: The Role of Effort on the Test Itself" (2019), experimenters went to two high schools in the US[42] and four in Shanghai.[43] Students at these schools[44] were made to do a 25-minute, 25-question PISA-like maths test.

At each school, students were divided into a control group and a treatment group. Students in the control group got nothing. Each student in the treatment group was given US$25 or ¥90 upfront, with US$1 or ¥3.60 later taken away for each wrong answer.[45]

In the US, the treatment groups did significantly better than the control groups. In contrast, in Shanghai, the treatment groups did no better:



[40] East Asians care about exams as much as Americans care about sports, and vice versa.

[41] The standard PISA test is two hours. More time may be spent on other questionnaires and assessments.

[42] "[A] high-performing private boarding school and a large public school with both low- and average-performing students" (p. 295).

[43] "[O]ne below-average-performing school, one school with performance that is just above average, and two schools with performance that is well above average" (p. 295).

[44] "All students present on the day of testing took part in the experiment" (p. 295).

[45] Using *The Economist's* Big Mac Index, the authors reckoned that ¥90 was the purchasing-power equivalent of US$25 (see their p. 295 and n. 11).

This suggests that robotic Shanghai students have been trained and conditioned to *always* try their hardest, whether or not there's any financial incentive. In contrast, US students may not be trying particularly hard when the stakes are low (or zero as is the case with PISA), but *will* try a little harder when there's some financial incentive.

Note also that

> *Importantly, students learn about the incentive just before taking the test, so any impact on performance can only operate through increased effort on the test itself rather than through, for example, better preparation or more studying.*

Hence, any remaining performance gap between US and Shanghai students could very well be eliminated if US students were incentivised (through carrot and stick) to prepare and work half as hard as Shanghai students.[46]

In two separate studies, the authors arrive at similar conclusions:

- "a country can improve its ranking by up to 15 places by encouraging its own students to take the exam seriously";[47]
- "our measures of student effort explain between 32 and 38 percent of the variation in test scores across countries".[48]

Altogether then, there is very little that Western educators can learn from East Asia. The only lesson is this: *If you want your students to do well on PISA-like tests, then*

- *Train them from young to take every test seriously;* and
- *Force them to work their butts off.*

This may mean destroyed childhoods and adolescences, plus the occasional academic suicide. But that, surely, is a small price to pay for topping the PISA charts.

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[46] Two more factors that may explain East Asians' stellar performance (more pure speculation on my part):

1. In the US, it would take a very unusual teacher or student to devote even a second preparing for PISA. In contrast, at many East Asian schools, many teachers and students will likely have spent at least some time preparing for it.

2. In the peculiar case of China, there will likely also be a "patriotism effect": Students, teachers, and administrators may be conditioned and exhorted to think, "It is our patriotic duty to do our best in order to show the rest of the world that China is great and glorious! China is no longer the Sick Man of Asia!" (Cue Chinese heroes defeating Somali pirates to the tune of *March of the Volunteers*.) Chinese teachers and students will therefore try harder than even other East Asians.

[47] "Taking PISA Seriously: How Accurate Are Low-Stakes Exams?" (2021).

[48] "When Students Don't Care: Reexamining International Differences in Achievement and Student Effort" (2019).

Genuine Understanding

Again, the goal of this textbook is to **impart genuine understanding**. I suspect the sincere pursuit of this goal will do more to promote GDP growth than any of *Gahmen's* current educational policies.

But quite aside from any such instrumental value, I believe that a genuine understanding of maths (and indeed any other material) is **intrinsically valuable**. (GDP growth is lovely, but despite what *Gahmen* would have you believe, it is not all that makes life worthwhile.) And heck, learning can even be that three-letter F word banished from the Singapore education system—and apparently also from playgrounds and HDB void decks:

These were actual signs posted in Singapore. They went viral and were removed in June 2013 (*New Paper* story) and March 2016 (*Straits Times* story), respectively.

Now, what do I mean by "imparting genuine understanding"?

Personal anecdote: As a JC student, I remember being deeply mystified by why the scalar product had such a simple algebraic definition and yet could at the same time also tell us about the cosine of the angle between the two vectors. I never figured it out. But this didn't matter, because this was simply "yet another formula" that we learnt for the sole purpose of answering exam questions.[49]

I remember being confused about the difference between the sample mean, the mean of the sample mean, the variance of the sample mean, and the sample variance. But this confusion didn't matter, because once again, all we needed to do to get an A was to mindlessly apply formulae and algorithms. Monkey see, monkey do.

[49] I remember complaining about this to a classmate, who responded along these lines, "But that's how we've always been taught maths *what*. It's just a bunch of formulae." Of course, my classmate was right. Today, the intellectually curious student can easily find the answer on the internet. But at that time (2001–02), the internet was not quite as developed and so one could not easily find answers online.

The non-East Asian reader may be wondering why I didn't just ask a teacher if I wanted to know. The reason is that in East Asia, the student is prohibited from asking questions. Now, this prohibition is not absolute (indeed, happily, it has been gradually softening at least in Singapore). Nonetheless, this prohibition sufficed to deter me from asking too many questions. See e.g. Chuah (2010) for more about how this cultural phenomenon works.

This textbook is thus partly in response to my unhappy and unsatisfactory experience as a cog in the Singapore educational system. In other words, this is the textbook I wish I had had when I was a JC student.

In the Singapore JC approach, teachers work through one example after another, demonstrating to students *that* something "works". The student is then expected to faithfully replicate such "workings" on the exams. If the student acquires any understanding of what she's doing, then that's a nice but not terribly important bonus—what's far more important is that the student produces the "correct" answers on the exams.

Such perverse priorities are reversed in this textbook. Our foremost goal will always be to explain and *understand* why something is true. To then also be able to ace the exams is a natural but unimportant side effect.

Almost all results are proven. I try to supply the intuition for each result in the simplest possible terms. Many proofs are relegated to the appendices, but where a proof is especially simple and beautiful, I encourage the student to savour it by leaving it in the main text (often as a guided exercise). In the rare instances where proofs are entirely omitted from this book—usually because they are too advanced—I make sure to clearly state so, lest the student be confused over where a result comes from or whether it's supposed to be obvious.

This textbook follows the Singapore A-Level syllabus.[50] And so, a good deal of mindless formulae is unavoidable. Even so, I try in this textbook to give the student a tiny glimpse of what maths really is—"*the art of explanation*".[51] I try to plant a thoughtcrime in the student's mind: Maths is not merely another pain to be endured, but can at times be a joy. And so for example, this textbook explains

- A bit of intuition behind differentiation, integration, and the Fundamental Theorems of Calculus. (To get an A, no understanding of these is necessary. Instead, one need merely know how to "do" differentiation and integration problems.)
- A bit of intuition behind why the Maclaurin series "works". (To get an A, it suffices to mindlessly write out various Maclaurin series without having the slightest clue what they are or where they come from.)
- Why the Central Limit Theorem (CLT) is so amazing. (To get an A, one need merely treat the CLT as yet another mysterious mathematical "trick" whose sole purpose is to solve exam questions. It isn't necessary to appreciate why it is so amazing, where it might possibly come, or what relevance it has to everyday life.)
- Why it is terribly wrong to believe that "*a high correlation coefficient means a good model*". (Yet this is exactly your A-Level examiners seem to believe—see Ch. 132.9.)

[50] A quixotic, longer-term goal of mine is to entirely change that too. As Lockhart says, "[T]hrow the stupid curriculum and textbooks out the window!"
[51] Lockhart (2002, 2009).

Why Even Type 1 Pragmatists Should Read This Textbook

Two reasons:

1. The A-Level exams now include more curveball or out-of-syllabus questions.

Previously, the A-Level exam questions were always perfectly predictable. If you had no problem doing past-year exam questions, then you'd have no problem getting an A.

But starting in 2017 (coinciding with the new and supposedly reduced 9758 syllabus), curveball questions now carry a weight of perhaps 10–20%. For example, in 2017, out of absolutely nowhere, students were suddenly asked to use something called D'Alembert's ratio test and to explain whether a series converges (see Exercise 575).

I can find no official, publicly available statement announcing this change (much less explaining it). I have heard only that JC maths teachers were informed by MOE of this change ahead of time. My guess is that this is the MOE's highly creative method of creating creative students.

In my humble opinion, this change is cow manure. Challenging students with curveball questions is a perfectly good idea. What isn't a good idea is to pose such questions on the high-stakes A-Level exam, thereby exerting additional pressure on the already pressure-cooked Singapore student.[52]

But I will confess that selfishly, I welcome this change because it increases the value of this textbook. The student who carefully studies this textbook will be rewarded with a true and deep understanding of all the H2 Maths material and hence be fully prepared to bat away any curveball.

Take for example N2017/I/6. This unfamiliar problem will likely have come as a shock to the Singapore monkey drilled a thousand times over to "do" computational problems involving 3D geometry, without ever understanding what he was doing. In contrast, any student who bothered to read Part III of this textbook even once will always have understood what she'd been "doing" and will thus have breezed through this problem.

[52]Of course, the ancient East Asian education system has conditioned students, teachers, and administrators to believe that anything that isn't on the exam is worthless. And so, it is difficult to persuade them to learn or teach any material that isn't also on the exams.

Difficult yes, impossible no. Overcoming this ancient mindset will require years of hard work, cultural change, and courage. Simply adding curveball questions to the A-Level exams certainly does not help and indeed merely serves to reinforce that mindset: "Oh you have to learn this too, not because it's interesting or intrinsically valuable, but because it might show up on your exam."

2. If you're intending to do more maths in the future (this includes not just maths, but also physics, economics, engineering, and many other subjects), then this textbook will actually save you time in the long run.

Merely doing well in A-Level H2 Maths may give you the false illusion that you've actually learnt or understood the material. Down the road, this may cost you more time.

Another personal anecdote: When I began my undergraduate studies, I was still the typical *kiasu* Singaporean monkey trained to believe that life was a competitive, Social Darwinist race. And so I skipped a whole bunch of lower level maths classes (Calc. I, Calc. II, Statistics, and Linear Algebra), *thinking I had already covered all the material back in JC*.

On paper, I may indeed have covered all this material. But in practice, all I'd learnt in JC was monkey-see-monkey-do. I'd learnt enough to do well on the exams, but not enough to actually *understand* or use any of the material.

It was only many years later, with the benefit of hindsight, that I began to see how much of a mistake I had made. Skipping those classes saved me time and put me "ahead of the race" in the short run. But in the long run, this actually cost me dearly. I would actually have saved more time by not skipping those seemingly elementary lower level classes![53]

And it wasn't just that I had to spend time relearning everything from scratch. I also had to spend time *unlearning* and repairing the damage caused by my JC maths education.

This textbook thus offers the sort of A-Level maths education I wished I had received.[54] You'll be spending two years on H2 Maths anyway. And so, instead of wasting these two years learning mindless recipes which **(a)** you'll forget a few months after your A-Level exam; and **(b)** will do you more harm in the long-run, why not spend these two years actually *learning* and *understanding* material that you'll appreciate, remember, and can actually make use of at any point in your life?

And of course, in my completely humble and unbiased opinion, the best way to learn and understand H2 Maths is by studying this textbook.

I conclude this Preface/Rant by expressing my hope that even if you the instructor or student do not use this textbook as your primary instructional or learning material, you will still find it perfectly useful as an authoritative and reliable reference.

P.S. This textbook is far from perfect. To steal a certain neighbourhood school's motto, *the best is yet to be*. I hope to keep improving this textbook, but I can only do so with your help. So **if you have any feedback or spot any errors, please feel free to email me**. (As you can tell, I am pretty merciless about criticising others. So please don't be shy about pointing out the many foolish mistakes that are surely still lurking in this textbook.)

[53] And of course, I would've saved even more time by skipping the Singapore education system, but *alas*, that wasn't an option.

[54] Here I intend absolutely no knock or diss on my JC Maths teachers. They and indeed most of my Singapore teachers were generally pretty good (especially when compared to the US). They did the best they could—within the stultifying confines of the Singapore educational system. My critique here applies to that **system**. As the hip-hop cliché goes, I don't hate the player(s); I hate the game.

PISA 2018 Maths Results

B-S-J-Z	591	**NZ**	494	**Brunei**	430
Singapore	569	**Portugal**	492	**Kazakhstan**	423
Macao	558	**Australia**	491	**Moldova**	421
Hong Kong	551	**Russia**	488	**Baku**	420
Taiwan	531	**Italy**	487	**Thailand**	419
Japan	527	**Slovakia**	486	**Uruguay**	418
Korea	526	**Spain**	483	**Chile**	417
Estonia	523	**Luxembourg**	483	**Qatar**	414
Netherlands	519	**Hungary**	481	**Mexico**	409
Poland	516	**Lithuania**	481	**Bosnia & Herz.**	406
Switzerland	515	**US**	478	**Costa Rica**	402
Canada	512	**Belarus**	472	**Jordan**	400
Denmark	509	**Malta**	472	**Peru**	400
Slovenia	509	**Croatia**	464	**Georgia**	398
Belgium	508	**Israel**	463	**N. Macedonia**	394
Finland	507	**Turkey**	454	**Lebanon**	393
Sweden	502	**Ukraine**	453	**Colombia**	391
UK	502	**Greece**	451	**Brazil**	384
Norway	501	**Cyprus**	451	**Argentina**	379
Ireland	500	**Serbia**	448	**Indonesia**	379
Germany	500	**Malaysia**	440	**Saudi Arabia**	373
Czechia	499	**Albania**	437	**Morocco**	368
Austria	499	**Bulgaria**	436	**Kosovo**	366
Latvia	496	**UAE**	435	**Panama**	353
France	495	**Romania**	430	**Philippines**	353
Iceland	495	**Montenegro**	430	**Dominican R.**	325

Notes: "B-S-J-Z" = "Beijing-Shanghai-Jiangsu-Zhejiang"; Baku is in Azerbaijan; I've abbreviated or otherwise slightly changed some country names.[55]

Source: "PISA 2018 Results, Volume I, What Students Know and Can Do" (2019, pp. 17–18, PDF).

[55] And yes, I consider Macao, Hong Kong, and Taiwan countries that are separate from China. I hope the PRC *gahmen* and people will forgive me for this egregious sin (please don't kidnap me and force me to make a TV confession).

Excerpts from "The Ugly Models" by Martha Nussbaum (2010)

What do educators in Singapore and China do? By their own internal accounts, they do a great deal of rote learning and "teaching to the test." Even if our sole goal was to produce students who would contribute maximally to national economic growth—the primary, avowed goal of education in Singapore and China—we should reject their strategies, just as they themselves have rejected them. In recent years, both nations have conducted major educational reforms, concluding that a successful economy requires nourishing analytical abilities, active problem-solving, and the imagination required for innovation. In other words, neither country has adopted a broader conception of education's goal, but both have realized that even that narrow goal of economic enrichment is not well served by a system focused on rote learning. ...

Singapore ... reformed its education policy in 2003 and 2004, allegedly moving away from rote learning toward a more "child-centered" approach in which children are understood as "proactive agents." Rejecting "repetitious exercises and worksheets," the reformed curriculum conceives of teachers as "co-learners with their students, instead of providers of solutions." It emphasizes both analytical ability and "aesthetics and creative expression, environmental awareness ... and self and social awareness." ... Singapore and China are trying to move toward open-ended progressive education that cultivates student creativity ...

Observers ... conclude that the reforms have not really been implemented. Teacher pay is still linked to test scores, and thus the incentive structure to effectuate real change is lacking. In general, it's a lot easier to move toward rote learning than to move away from it, since teaching of the sort Dewey and Tagore recommended requires resourcefulness and perception, and it is always easier to follow a formula.

Moreover, the reforms are cabined by these authoritarian nations' fear of true critical freedom. In Singapore, nobody even attempts to use the new techniques when teaching about politics and contemporary problems. "Citizenship education" typically takes the form of analyzing a problem, proposing several possible solutions, and then demonstrating how the one chosen by government is the right one for Singapore. In universities, some instructors attempt a more genuinely open approach, but the government has a way of suing professors for libel if they criticize the government in class, and even a small number of high-profile cases chills debate. One professor of communications (who has since left Singapore) reported on a recent attempt to lead a discussion of the libel suits in her class: "I can feel the fear in the room. ... You can cut it with a knife." Nor are foreign visitors immune: NYU's film school[56] *has been encouraged to set up a Singapore branch, but informed that films made in the program may not be shown outside the campus. ...*

Singapore and China are terrible models of education for any nation that aspires to remain a pluralistic democracy. They have not succeeded on their own business-oriented terms, and they have energetically suppressed imagination and analysis when it comes to the future of the nation and the tough choices that lie before it.

[56] In 2003, Singapore launched its "Global Schoolhouse" initiative to "bring in good foreign institutions as partners and attract 'large numbers of international students'" (2021). This initiative failed, taking down with it NYU Tisch Asia (RIP 2012), Johns Hopkins Singapore (2006), NYU@NUS (2013), Chicago Booth Asia eMBA (2013), MIT-SUTD (2017), and Yale-NUS College (2021).

Perhaps ambitious (and greedy) administrators at US universities will now finally heed the words of long-time Harvard president Derek Bok (2013): "I had my own run-in with Lee Kuan Yew some years ago ... Nothing in that experience would tempt me to try to establish a Harvard College in Singapore."

Part 0.
A Few Basics

Revision in progress (November 2021).

And hence messy at the moment.
Appy polly loggies for any inconvenience caused.

The glory of [maths] is its complete irrelevance *to our lives. That's why it's so fun!*

— Paul Lockhart (2002, 2009).

I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.

— G.H. Hardy (1940).

1. Just To Be Clear

In this textbook, we'll stick to these standard conventions:

- **Greater than** means "*strictly* greater than" (>). So I won't bother saying "strictly", unless it's something I want to emphasise.
- Similarly, **less than** means "*strictly* less than" (<).
- If I want to say **greater than or equal to** (≥) or **smaller than or equal to** (≤), I'll say exactly that.
- **Positive** means "greater than zero" (>0) and **non-positive** means "less than or equal to zero" (≤ 0).
- Similarly, **negative** means "less than zero" (<0) and **non-negative** means "greater than or equal to zero" (≥ 0).
- Zero is **neither positive nor negative**. Instead, it is **both non-negative and non-positive**.[57]

Names of some punctuation marks:

Left		Right		A pair of	
Parenthesis	(	**Parenthesis**	)	**Parentheses**	()
Bracket	[	**Bracket**	]	**Brackets**	[]
Brace	{	**Brace**	}	**Braces**	{}

Remark 1. Some writers refer to (), [], and {} as **round**, **square**, and **curly brackets**—we'll avoid these terms. Instead, as stated above, we'll strictly refer to (), [], and {} as **parentheses**, **brackets**, and **braces**.[58]

- The symbol ∵ and ∴ stand for the words **because** and **therefore**.
- The punctuation mark " means **ditto** or "the same as above/before":

Example 1. Tokyo is the capital of Japan.
Beijing " China.
Lima " Peru.

- The multiplication symbol · is (sometimes) preferred to × because there is (sometimes) the slight risk of confusing × with the letter x.

[57] But in France, *positif* and *négatif* mean ≥ 0 and ≤ 0, so that 0 is both *positif and négatif* (Wiktionary).

[58] There is actually another pair of brackets ⟨⟩ called **angle brackets**. If we're using angle brackets, then we'll want to be careful to distinguish them from [] by referring to the latter as **square brackets**. Happily, we won't be using angle brackets at all in this textbook. And so, we'll simply call [] **brackets**.

2. PSLE Review: Division

Example 2. Consider $9 \div 4$: $$\frac{9}{4} = 2\frac{1}{4}.$$

We call

- 9 the **dividend**
- 4 the **divisor**
- $q = 2$ the **quotient**—q is the largest integer such that $4q \leq 9$
- $r = 1$ the **remainder**—r is defined so that $9 = 4q + r$ or equivalently,

$$r = 9 - 4q = 9 - 4 \times 2 = 1.$$

Example 3. Consider $17 \div 3$: $$\frac{17}{3} = 5\frac{2}{3}.$$

We call

- 17 the **dividend**
- 3 the **divisor**
- $q = 5$ the **quotient**—q is the largest integer such that $3q \leq 17$
- $r = 2$ the **remainder**—r is defined so that $17 = 3q + r$ or equivalently,

$$r = 17 - 3q = 17 - 3 \times 5 = 2.$$

The above two examples used an algorithm[59] called **Euclidean division**:

Definition 1. (Euclidean division) Given positive integers x and d,[60] let q be the largest integer that satisfies $dq \leq x$. Then let $r = x - dq \geq 0$.

In the equation $x = dq + r$, we call x the *dividend*, d the *divisor*, q the *quotient*, and r the *remainder*.

And if $r = 0$, then we say that x divided by d *leaves no remainder* and that d is a *factor* of (or *divides*) x.

It is possible to prove that thus defined, **the quotient and remainder are unique.**[61]

Example 4. Find $95 \div 4$ using Euclidean division:

1. Find the largest integer q such that $4q \leq 95$: $q = 23$.
2. Now compute the remainder: $r = 95 - 4q = 95 - 4 \times 23 = 3$.
3. Conclude: $\dfrac{95}{4} = 23\dfrac{3}{4}$.

[59] *Algorithm* is just a fancy synonym for *method*.

[60] For simplicity, this definition deals only with the case where $x, d > 0$. For a more general definition that covers also the cases where x and/or d may be negative, see Definition 262 (Appendices).

[61] See Theorem 50 (Appendices).

Example 5. Find $18 \div 6$ using Euclidean division:

1. Find the largest integer q such that $6q \leq 18$: $q = 3$.
2. Now compute the remainder: $r = 18 - 6q = 18 - 6 \times 3 = 0$.
3. Conclude: $\dfrac{18}{6} = 3$.

Since $r = 0$, we say that $18 \div 6$ **leaves no remainder** and that 6 **is a factor of** or **divides** 18.

2.1. Long Division

Remember **long division** from primary school? It's simply Euclidean division repeated at each decimal place:

Example 6. Find $95 \div 4$ using long division. Seven steps:

1. First, draw two lines, one horizontal and one vertical.[62] Then write the dividend 95 "inside" and the divisor 4 on the "left".

Long division *is* repeated Euclidean division—specifically, we do Euclidean division at each decimal place (ones, tens, hundreds, etc., but starting from the biggest place):

2. Starting at the tens place, we ask, "What is the largest integer such that $40q_1 \leq 95$?" Answer: $q_1 = 2$. So, we write 2 above the horizontal line, at the tens place.
3. Compute $40q_1 = 40 \times 2 = 80$ and write "80" below our dividend 95. Also, draw a horizontal line below "80".
4. Compute the remainder $95 - 80 = 15$ and write "15" below the line just drawn.

Step 1.

$$\begin{array}{r|r} \hline 4 & 95 \end{array}$$

Step 2.

$$\begin{array}{r|r} & 2 \\ \cline{2-2} 4 & 95 \end{array}$$

Step 3.

$$\begin{array}{r|r} & 2 \\ \cline{2-2} 4 & 95 \\ & 80 \\ \cline{2-2} \end{array}$$

Step 4.

$$\begin{array}{r|r} & 2 \\ \cline{2-2} 4 & 95 \\ & 80 \\ \cline{2-2} & 15 \end{array}$$

Step 6.

$$\begin{array}{r|r} & 23 \\ \cline{2-2} 4 & 95 \\ & 80 \\ \cline{2-2} & 15 \\ & 12 \\ \cline{2-2} \end{array}$$

Step 7.

$$\begin{array}{r|r} & 23 \\ \cline{2-2} 4 & 95 \\ & 80 \\ \cline{2-2} & 15 \\ & 12 \\ \cline{2-2} & 3 \end{array}$$

5. Now move to the ones place. Ask, "What is the largest integer such that $4q_0 \leq 15$?" Answer: $q_0 = 3$. So, write 3 above the top horizontal line, at the ones place.
6. Compute $4q_0 = 4 \times 3 = 12$ and write "12" below our remainder "15". Also, draw a horizontal line below "12".
7. Compute the remainder $15 - 12 = 3$ and write "3" below the line just drawn.

We're done! The quotient is 23 and the remainder is 3:

$$\frac{95}{4} = 23\frac{3}{4}.$$

We could actually keep going, repeating the same procedure for subsequent decimal places: the tenths, hundredths, etc. If we did this, we'd find that

$$\frac{95}{4} = 23.75.$$

[62]The vertical line is actually usually a curve. But it was too much trouble/work getting this typesetting to work and look nice. So, I've simply settled for a vertical line.

Example 7. Find $87 \div 7$ using long division:

$$
\begin{array}{rr}
 & 12 \\
\cline{2-2}
7 & \multicolumn{1}{|r}{87} \\
 & 70 \\
\cline{2-2}
 & 17 \\
 & 14 \\
\cline{2-2}
 & 3
\end{array}
$$

Thus,
$$\frac{87}{7} = 12\frac{3}{7}.$$

Example 8. Find $912 \div 17$ using long division:

$$
\begin{array}{rr}
 & 53 \\
\cline{2-2}
17 & \multicolumn{1}{|r}{912} \\
 & 850 \\
\cline{2-2}
 & 62 \\
 & 51 \\
\cline{2-2}
 & 11
\end{array}
$$

Thus,
$$\frac{912}{17} = 53\frac{11}{17}.$$

Exercise 1. Find $8\,057 \div 39$ using long division. Identify the dividend, divisor, quotient, and remainder. (Answer on p. 1747.)

2.2. The Cardinal Sin of Dividing By Zero

Dividing by zero is a common mistake. It's easy to avoid this mistake when the divisor is obviously a big fat zero. It's harder when the divisor is an unknown constant or variable that may or may not be zero.

Example 9. Solve $x(x-1) = (2x-2)(x-1)$. (That is, find the values of x for which the equation is true. We call such values of x the *solutions* or *roots* to the equation.)

Here's a wrong answer: "Divide both sides by $x-1$ to get $x = 2x-2$. So $x = 2$." ✗

Here's a correct answer that considers two possible cases:

Case 1. If $x-1=0$, then the equation is true. So, $x=1$ is a possible solution.

Case 2. If $x-1\neq 0$, then we can divide both sides by $x-1$ to get $x = 2x-2$. So, $x=2$ is another possible solution.

Conclusion. The two possible solutions are $x=1$ and $x=2$.

Moral of the story. Dividing by zero may cause us to lose perfectly valid solutions. So, always make sure the divisor is non-zero. If you're not sure whether it equals zero, then break up your analysis into two cases, as was done in the above example:

Case 1. The quantity equals zero (and see what happens in this case).

Case 2. The quantity is non-zero (in which case go ahead and divide).

Exercise 2. What's wrong with the following seven-step "proof" that $1 = 0$?

1. Let x and y be positive numbers such that $x = y$.
2. Square both sides: $x^2 = y^2$.
3. Rearrange: $x^2 - y^2 = 0$.
4. Factorise: $(x-y)(x+y) = 0$.
5. Divide both sides by $x-y$ to get $x+y=0$.
6. Since $x=y$, plug $y=x$ into the above equation to get $2x=0$.
7. Divide both sides by $2x$ to get $1=0$. (Answer on p. 1747.)

Exercise 3. Identify any error(s) below.[63] (Answer on p. 1747.)

Example 13.

$\dfrac{42x}{x-2} = \dfrac{35x}{x-3}$: divide both numerators by x, and you will have $\dfrac{42}{x-2} = \dfrac{35}{x-3}$; therefore $42 = \dfrac{35x-70}{x-3}$; therefore $42x - 126 = 35x - 70$; therefore $42x - 35x - 126 = -70$, that is, $7x - 126 = -70$; therefore $7x = 126 - 70$, that is, $7x = 56$; and $x = 8$.

[63] This was taken from an 18th-century textbook (*The Elements of Algebra*, 1740, p. 103), written by one Nicholas Saunderson (1682–1739). Saunderson was the fourth Lucasian Professor of Mathematics (at the University of Cambridge)—this is a fancy title that's been held by Isaac Newton (1642–1727) and Stephen Hawking (1942–2018).

2.3. Is One Divided by Zero Infinity?

By the way, let's take this opportunity to clear up a related and popular misconception. You've probably heard someone saying that

$$\frac{1}{0} = \infty. \qquad ✗$$

This is wrong. Instead,
$$\frac{1}{0} \neq \infty. \qquad ✓$$

Any number divided by zero is **undefined**.[64]

Undefined is the mathematician's way of saying

You haven't told me what you're talking about.
So what you're saying is meaningless.

And so, the following five expressions are all undefined:

$$\frac{1}{0}, \quad \frac{50}{0}, \quad , \frac{-17.1}{0}, \quad \frac{\infty}{0}, \quad \frac{-\infty}{0}.$$

[64]In the context of limits, the expression $0 \div 0$ is sometimes also called an **indeterminate form**, but this isn't something H2 Maths students need care about.

3. Logic

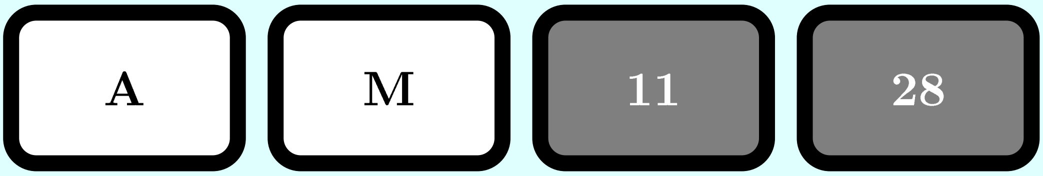
Big surprise—you've secretly been using logic your whole life.

Logic isn't explicitly on your H2 Maths syllabus.[65] But spending an hour or two on logic pays huge dividends—you'll learn to reason better, both in maths and in everyday life.

This chapter is thus a brief and gentle introduction to logic. Here we merely present some of the most basic and useful results from logic. (If you truly can't be bothered, please at least check out the one-page summary of this chapter on p. 41.)

First, try this appetiser.[66]

Example 10. (Wason Four-Card Puzzle) In a special deck of cards, each card has a letter on one side and a number on the other. You are shown these four cards:



Shanmugam the Liar now comes along and makes this claim:

"*If a card has a vowel on one side, then it has an even number on the other side.*"

You suspect that Shanmugam is wrong. To prove him wrong, which of the above four cards should you turn over? (The goal is to turn over as few cards as possible.)[67]

The above puzzle baffles most who are untrained in logical thinking—most turn over A (correct) and 28 (wrong). Right now, that probably includes you. But by p. 26 of this textbook, you'll have had some training in logic and thus be able to solve this puzzle easily.

By the way, psychologists[68] have found that people find it easier to solve an entirely equivalent but less abstract version of the above puzzle:

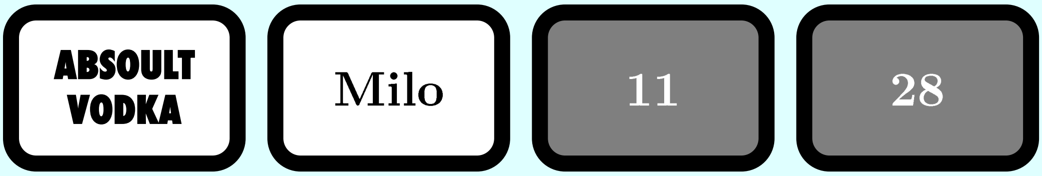
[65]If it were up to me, the H2 Maths syllabus would devote at least a little time to logic. Instead, that time is spent on learning to mindlessly compute such objects as the volume of the revolution of a curve around the y-axis. Which is a doggie trick that (1) students will forget two weeks after the final A-Level exam; and (2) is completely useless unless you're planning to be an engineer or physicist, in which case it is still completely useless since down the road, you'll be learning it again (and probably more properly).

[66]This is a slightly reworded version of Wason (1966, pp. 145–146).

[67]Answer: A and 11. We'll explain this on p. 26.

[68]Griggs and Cox (1982).

Example 11. (Wason Four-Card Puzzle, Alcohol Version) In a special deck of cards, each card has a person's age on one side and the drink she's having on the other. You are shown these four cards:



Shanmugam the Liar now comes along and makes this claim:

"*If a person is having alcohol, then she must be at least 18.*"

You suspect that Shanmugam is wrong. To prove him wrong, which of the above four cards should you turn over? (The goal is to turn over as few cards as possible.)[69]

(Absolut Vodka is alcoholic; Milo isn't.)

[69]Answer: Absolut Vodka and 11.

Example 12. Let A, B, C, and D be these **statements**:

A:	"Germany is in Europe."	✓	(True)
B:	"Germany is in Asia."	✗	(False)
C:	"$1 + 1 = 2$."	✓	(True)
D:	"$1 + 1 = 3$."	✗	(False)

Statements A and C are **true**, while B and D are **false**.

(We'll use ✓ and ✗ as shorthand for true and false.)

Example 13. Let M, N, and O be these statements:

M: "$x > 0$."
N: "$x > 1$."
O: "x is a positive number."

Note that the truth values of M, N, and O depend on the value of x. That is, whether each statement is true or false depends on the value of x.

So, without further information, each of these three statements can neither be said to be true nor said to be false. Instead, we say that each is **indeterminate**.

But if we're told that

- $x = 5$, then all three statements are true.
- $x = 0.5$, then statements M and O are true, while N is false.
- $x = -1$, then all three statements are false.

Remark 2. In this textbook, we'll leave the terms **statement**, **true**, **false**, and **indeterminate** undefined.

3.2. The Conjunction AND and the Disjunction OR

Example 14. As before, let A, B, C, and D be these statements:

A:	"Germany is in Europe."	✓
B:	"Germany is in Asia."	✗
C:	"$1+1=2$."	✓
D:	"$1+1=3$."	✗

Using the logical connective **AND** (called the **conjunction**), we can form these statements (which we *also* call **conjunctions**):

A AND B:	"Germany is in Europe AND Germany is in Asia."	✗
A AND C:	"Germany is in Europe AND $1+1=2$."	✓
B AND D:	"Germany is in Asia AND $1+1=3$."	✗

- The conjunction A AND B is false because B is false.
- The conjunction A AND C is true because both A and C are true.
- The conjunction B AND D is false because B is false. (Indeed, D is also false.)

Using the logical connective **OR** (called the **disjunction**) we can form these statements (which we also call **disjunctions**):

A OR B:	"Germany is in Europe OR Germany is in Asia."	✗
A OR C:	"Germany is in Europe OR $1+1=2$."	✓
B OR D:	"Germany is in Asia OR $1+1=3$."	✗

- The disjunction A OR B is true because A is true.
- The disjunction A OR C is true because A is true. (Indeed, C is also true.)
- The disjunction B OR D is false because both B and D are false.

Definition 2. Let P and Q be statements. The *conjunction of* P *and* Q is the statement denoted P AND Q and defined as

- True if **both** P and Q are true; and
- False if **at least one** of P or Q is false.

The *disjunction of* P *and* Q is the statement denoted P OR Q and defined as

- True if **at least one** of P or Q is true; and
- False if **both** P and Q are false.

Remark 3. Given the statements P and Q, the conjunction P AND Q (or disjunction P OR Q) is itself a statement and may be called a *compound* statement.

Exercise 4. Continue with the last example. Explain if each statement is true or false.

(a) B AND C. **(b)** A AND D. **(c)** C AND D. **(d)** B OR C.
(e) A OR D. **(f)** C OR D. (Answer on p. 1748.)

3.3. The Negation NOT

Informally, a statement's **negation** is the "opposite" or contradictory statement. Formally,

Definition 3. Let P be a statement. The *negation of* P is the statement denoted NOT−P and defined as

true if P is false and false if P is true.

Example 15. As before, let A, B, C, and D be these statements:

A:	"Germany is in Europe."	✓
B:	"Germany is in Asia."	✗
C:	"$1+1=2$."	✓
D:	"$1+1=3$."	✗

The **negations** of statements A, B, C, and D are simply these statements:

NOT−A:	"Germany is not in Europe."	✗
NOT−B:	"Germany is not in Asia."	✓
NOT−C:	"$1+1\neq 2$."	✗
NOT−D:	"$1+1\neq 3$."	✓

- Since A and C are true, their negations NOT−A and NOT−C must be false.
- Since B and D are false, their negations NOT−B and NOT−D must be true.

The negation of the negation simply brings us back to the original statement:

NOT−NOT−A:	"Germany is in Europe."	✓
NOT−NOT−B:	"Germany is in Asia."	✗
NOT−NOT−C:	"$1+1=2$."	✓
NOT−NOT−D:	"$1+1=3$."	✗

Exercise 5. Let E: "It's raining", F: "The grass is wet", G: "I'm sleeping", and H: "My eyes are shut".

Write down NOT−E, NOT−F, NOT−G, and NOT−H. (Answer on p. 1748.)

Remark 4. We've just learnt about the three basic **logical connectives**: **AND**, **OR**, and **NOT**. Using these three connectives, we can build ever more statements.

Remark 5. Instead of **AND-**, **OR-**, and **NOT-**, others may use the notation $\wedge$, $\vee$, and $\neg$ (or $\sim$). That is, instead of P AND Q, P OR Q, and NOT−P, they may write $P \wedge Q$, $P \vee Q$, and $\neg P$ (or $\sim P$). (This textbook won't use such notation.)

3.4. Equivalence $\iff$

Example 16. Let x be a real number and M, N, and O be these statements:

$$M: \text{ "}x > 0.\text{"}$$
$$N: \text{ "}x > 1.\text{"}$$
$$O: \text{ "}x \text{ is a positive number."}$$

Observe that whatever x is, M and O always have the same truth value—i.e. whenever M is true, O is also true; and whenever M is false, O is also false. So, we say that M and O are **equivalent** and write

$$M \iff O.$$

We will also say that "*P is equivalent to Q*" or "*P if and only if Q*".[70]

In contrast, if $x = 0.5$, then M is true while N is false. Since M and N do not always have the same truth value, we say that M and N are **not** equivalent and write

$$M \not\Longleftrightarrow N.$$

We will also say that "P is **not** equivalent to Q".

So, if we can find even one single instance where the statements P and Q have different truth values, then $P \not\Longleftrightarrow Q$. If we can't, then $P \iff Q$.

Definition 4. Let P and Q be statements. Suppose P and Q always have the same truth value (i.e. whenever P is true, Q is also true; and whenever P is false, Q is also false). Then we say that P and Q are *equivalent* and write

$$P \iff Q.$$

We will also say that "*P is equivalent to Q*" or "*P if and only if Q*".

Example 17. Let x be a real number and α and β be these statements:

$$\alpha: \text{ "}x = 3\text{";} \qquad \text{and} \qquad \beta: \text{ "}x + 2 = 5\text{".}$$

If α is true, then β is true. And if α is false, then β is also false. Since α and β always have the same truth value, we say that α and β are **equivalent** and write $\alpha \iff \beta$.

[70] *If and only if* is sometimes shortened to *iff*.

Example 18. Let A, B, C, and D be these statements:

A: "Germany is in Europe." ✓
B: "Germany is in Asia." ✗
C: "$1 + 1 = 2$." ✓
D: "$1 + 1 = 3$." ✗

Both statements A and C are always true. So, $A \iff C$.

Both statements B and D are always false. So, $B \iff D$.

Also, $A \not\iff B$, $A \not\iff D$, $C \not\iff B$, and $C \not\iff D$.

Exercise 6. Explain if each pair of statements is equivalent. (Answer on p. 1748.)

(a) N: "$x > 1$"; and O: "x is a positive number".
(b) α: "$x = 3$"; and γ: "$x^2 = 9$".
(c) ☺: "$2 \times 3 = 6$"; and ☹: "$10 - 3 = 7$".
(d) ☼: "$1 + 1 = 2$"; and ●: "$x = 5$".
(e) ✡: "$1 + 1 = 3$"; and ●: "$x = 5$".

3.5. De Morgan's Laws: Negating the Conjunction and Disjunction

The **negation** of P AND Q is denoted NOT– (P AND Q).

Remark 6. Note the use of **parentheses**.

In arithmetic, the statement $2+3\times 7=23$ differs from the statement $(2+3)\times 7=35$.

Likewise, in logic, the statement NOT– (P AND Q) differs from NOT–P AND Q.

In both arithmetic and logic, we sometimes add parentheses to be clear about the order of operations. Indeed, just to be extra clear, we sometimes add parentheses even when they aren't strictly necessary.[71]

Fact 1. *Suppose P and Q are statements. Then*

$$\text{NOT–}(P \text{ AND } Q) \qquad \Longleftrightarrow \qquad \text{NOT–}P \text{ OR NOT–}Q.$$

Proof. See p. 1546 (Appendices). □

Example 19. Let A: "Germany is in Europe"; B: "Germany is in Asia"; C: "$1+1=2$", and D: "$1+1=3$".

Consider these three statements:

1. A AND B: "Germany is in Europe AND Germany is in Asia." ✗
2. C AND D: "$1+1=2$ AND $1+1=3$." ✗
3. A AND C: "Germany is in Europe AND $1+1=2$." ✓

Now consider these three statements' **negations**:

1. NOT– (A AND B) is true. There are two ways to see this:
   - Since A AND B is false, its negation NOT– (A AND B) must be true.
   - By Fact 1, NOT– (A AND B) is equivalent to (NOT–A OR NOT–B): "Germany is not in Europe OR Germany is not in Asia". Which is true because NOT–B: "Germany is not in Asia" is true.
2. NOT– (C AND D) is true. Two ways to see this:
   - Since C AND D is false, its negation NOT– (C AND D) must be true.
   - By Fact 1, NOT– (C AND D) is equivalent to (NOT–C OR NOT–D): "$1+1\neq 2$ OR $1+1\neq 3$". Which is true because NOT–D: "$1+1\neq 3$" is true.
3. NOT– (A AND C) is false. Two ways to see this:
   - Since A AND C is true, its negation NOT– (A AND C) must be false.
   - By Fact 1, NOT– (A AND C) is equivalent to (NOT–A OR NOT–C): "Germany is not in Europe OR $1+1\neq 2$". Which is false because both NOT–A: "Germany is not in Europe" and NOT–C: "$1+1\neq 2$" are false.

[71] In arithmetic, we have the PEMDAS order of operations convention: Parentheses, Exponentiation, Multiplication, Division, Addition, Subtraction. It turns out that in logic, the (usual) convention is Parentheses, NOT, AND, OR, $\Longrightarrow$, $\Longleftrightarrow$. (But this isn't something A-Level Maths students need know, so where there is any risk of doubt, I'll use parentheses.)

Exercise 7. Continue with the last example. Is the negation of each statement below true? (Answer on p. 1748.)

(a) B AND C. **(b)** A AND D. **(c)** B AND D.

The **negation** of P OR Q is denoted NOT–(P OR Q).

Fact 2. *Suppose P and Q are statements. Then*

$$\text{NOT–}(P \text{ OR } Q) \quad \iff \quad \text{NOT–}P \text{ AND NOT–}Q.$$

Proof. See p. 1547 (Appendices). □

Remark 7. Together, Facts **1** and **2** are called **De Morgan's Laws**. W

Example 20. As before, let A: "Germany is in Europe"; B: "Germany is in Asia"; C: "$1+1=2$", and D: "$1+1=3$".

Consider these three statements:

1. A OR B: "Germany is in Europe OR Germany is in Asia." ✓
2. C OR D: "$1+1=2$ OR $1+1=3$." ✓
3. A OR C: "Germany is in Europe OR $1+1=2$." ✓

Now consider these three statements' **negations**:

1. NOT–(A OR B) is false. There are two ways to see this:
   - Since A OR B is true, its negation NOT–(A OR B) must be false.
   - By Fact 2, NOT–(A OR B) is equivalent to (NOT–A AND NOT–B): "Germany is not in Europe AND Germany is not in Asia". Which is false because NOT–A: "Germany is not in Europe" is false.
2. NOT–(C OR D) is false. Two ways to see this:
   - Since C OR D is true, its negation NOT–(C OR D) must be false.
   - By Fact 2, NOT–(C OR D) is equivalent to (NOT–C AND NOT–D): "$1+1\neq 2$ AND $1+1\neq 3$". Which is false because NOT–C: "$1+1\neq 2$" is false.
3. NOT–(A OR C) is false. Two ways to see this:
   - Since A OR C is true, its negation NOT–(A OR C) must be false.
   - By Fact 2, NOT–(A OR C) is equivalent to (NOT–A AND NOT–C): "Germany is not in Europe AND $1+1\neq 2$". Which is false because NOT–A: "Germany is not in Europe" is false. Indeed, NOT–C: "$1+1\neq 2$" is also false.

Exercise 8. Continue with the last example. Is the negation of each statement below true? (Answer on p. 1749.)

(a) B OR C. **(b)** A OR D. **(c)** B OR D.

3.6. The Implication $P \implies Q$

The statement "$P \implies Q$" is read aloud as "If P, then Q" or "P implies Q".

Example 21. Let E: "It's raining" and F: "The grass is wet".
Then the **implication** $E \implies F$ is this statement:

"If it's raining, then the grass is wet."

Or equivalently, "That it's raining implies that the grass is wet."

Which is true.[72]

Example 22. Let G: "I'm sleeping" and H: "My eyes are shut".
Then the implication $G \implies H$ is this statement:

"If I'm sleeping, then my eyes are shut."

Or equivalently, "That I'm sleeping implies that my eyes are shut."

Which is true.[73]

Example 23. Let M: "$x > 0$" and N: "$x > 1$".
Then the implication $M \implies N$ is this statement:

"If $x > 0$, then $x > 1$."

Or equivalently, "That $x > 0$ implies that $x > 1$."

Which is false (counterexample: $x = 0.5$).

Example 24. Let α: "$x = 3$." and γ: "$x^2 = 9$."
Then the implication $\gamma \implies \alpha$ is this statement:

"If $x^2 = 9$, then $x = 3$."

Or equivalently, "That $x^2 = 9$ implies that $x = 3$."

Which is false (counterexample: $x = -3$).

This all seems simple enough. However, you may find the formal definition of $P \implies Q$ a little strange and unintuitive:

[72] Provided the grass is outdoors.
[73] For most healthy human beings.

Definition 5. Given statements P and Q, the *implication* $P \implies Q$ is defined as

$$\text{NOT}-P \text{ OR } Q.$$

That is, $P \implies Q$ is the statement defined as

- True if P is false OR Q is true; and
- False if P is true AND Q is false.

Given the implication $P \implies Q$, we call P the *hypothesis*, *premise*, or *antecedent*; and Q the *conclusion* or *consequent*.

What's most confusing about the above definition is this: From a false hypothesis, *any* conclusion may be drawn!

That is, If P is false, then $P \implies Q$ is always true!

Example 25. Consider these three statements:

1. "If goldfish can walk on land, then pigs can fly."
2. "If there are 17 washing machines on Mars, then I am a billionaire."
3. "If $x > 0$ and $x < 0$, then $\sqrt{2} = 77$."

Pigs cannot fly, I am not a billionaire, and $\sqrt{2} \neq 77$. It may thus seem that all three statements must be false.

But strangely enough, all three are true! To see why, use the above Definition to rewrite the three statements as

1. "Goldfish cannot walk on land OR pigs can fly." Which is true, because "Goldfish cannot walk on land" is true.
2. "There aren't 17 washing machines on Mars OR I am a billionaire." Which is true, because "There aren't 17 washing machines on Mars" is true.
3. "NOT-$(x > 0$ AND $x < 0)$ OR $\sqrt{2} = 77$." Which is true, because "NOT-$(x > 0$ AND $x < 0)$" is true—it is impossible that x is both more than AND less than 0.

The examples here illustrate that from a false hypothesis, *any* conclusion may be drawn!

Exercise 9. Is each statement true? (Answer on p. 1749.)

(a) "If Tin Pei Ling (TPL) is a genius, then the Nazis won World War II (WW2)."

(b) "If TPL is a genius, then the Allies won WW2."

(c) "If π is rational, then I am the king of the world."

(d) "If π is rational, then Lee Hsien Loong is Lee Kuan Yew's son."

3.7. Other Ways to Express $P \implies Q$ (Optional)

Eskimos supposedly have 50 different words for *snow*, presumably because snow is so important and ubiquitous in their lives.[74]

Similarly, because the implication $P \implies Q$ is so important and ubiquitous in both mathematics/logic and everyday life, we have many equivalent ways to express it:

Example 26. Let E: "It is raining" and F: "The grass is wet". Then **all** of the following statements are exactly equivalent:

Maths/Logic	**Everyday English**
$E \implies F$	That it's raining implies that the grass is wet.
$E \implies F$	It's raining only if the grass is wet.[75]
If E, then F.	If it's raining, then the grass is wet.
If E, F.	If it's raining, the grass is wet.
F if E.	The grass is wet if it's raining.
F when E.	The grass is wet when (or whenever) it's raining.
F follows from E.	That the grass is wet follows from the fact that it's raining.
E is sufficient for F.	That it's raining is sufficient for the grass to be wet.
F is necessary for E.	It is necessary that the grass is wet, for it to be raining.

But don't worry. The above is just FYI.

In the main text of this textbook, we'll avoid using the terms "only if", "sufficient", and "necessary". Instead, we'll stick to using only these three (equivalent) statements:

"$P \implies Q$", "P implies Q", and "If P, then Q".

Exercise 10. Let G: "I'm sleeping" and H: "My eyes are shut". Construct the exact same table as we just did, but for $G \implies H$. (Answer on p. 1749)

[74] See this *Washington Post* story: "There really are 50 Eskimo words for 'snow'".

[75] It's far from obvious, but *implies* is logically equivalent to *only if*. And thus, the symbol $\implies$ can be read aloud not only as *implies*, but also as *only if*.

Example 27. Let I: "x is German" and J: "x is European".

Consider the implication $I \implies J$:

"If x is German, then x is European."

Which of the following correctly negates $I \implies J$? That is, which of the following statements is NOT– $(I \implies J)$?

(a) "If x is German, then x is **not** European."

(b) "If x is **not** German, then x is European."

(c) "x is German and **not** European."

(d) "x is European and **not** German."

This is tricky and you should take as long as you need to think about it, before reading the answer/explanation on the next page. The point of this exercise is to demonstrate to yourself that it isn't obvious what the negation of an implication is. (Or if it's obvious, it'll demonstrate that you're pretty smart.)

(As I've repeatedly stressed, do not do the intellectually lazy thing of skipping ahead. Give it at least three minutes of honest effort before going to the next page.)

Or don't.

Whatevs.

If you dun care, I oso dun care.

(*Example continues on the next page ...*)

It will be easy to negate any implication with this result:

Fact 3. *Suppose P and Q are statements. Then*

$$\text{NOT}-(P \implies Q) \qquad \iff \qquad (P \text{ AND NOT}-Q).$$

Proof. See Exercise 11. $\square$

Exercise 11. Prove the above Fact.[76] (Answer on p. 1749.)

(... *Example continued from the previous page.*)

Continue to let I: "x is German" and J: "x is European".

The implication $I \implies J$ is, "If x is German, then x is European."

So, its negation, NOT$-(I \implies J)$, is by Fact 3 equivalent to

$$I \text{ AND NOT-}J\text{: "}x \text{ is German AND } x \text{ is not European".}$$

Exercise 12. Let K: "x is donzer" and L: "x is kiki". Consider this implication:

$$K \implies L\text{: "If } x \text{ is donzer, then } x \text{ is kiki."}$$

Which statement below negates the above implication? That is, which is NOT$-(K \implies L)$?

(a) "If x is kiki, then x is **not** donzer."

(b) "x is kiki and **not** donzer."

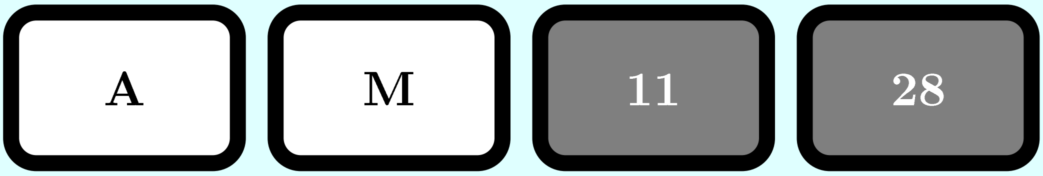
(c) "If x is donzer, then x is **not** kiki."

(d) "x is donzer and **not** kiki." (Answer on p. 1749.)

[76]Hint: Look at the definition of $P \implies Q$ (Definition 5). What is its negation?

We can now solve the Wason Four-Card Puzzle (reproduced):

Example 10. (Wason Four-Card Puzzle) In a special deck of cards, each card has a letter on one side and a number on the other. You are shown these four cards:



Shanmugam the Liar now comes along and makes this claim:

> "*If a card has a vowel on one side, then it has an even number on the other side.*"

You suspect that Shanmugam is wrong. To prove him wrong, which of the above four cards should you turn over? (The goal is to turn over as few cards as possible.)

Answer: Turn over A and 11.

Explanation I. By Fact 3, the negation of $P \implies Q$ is P AND NOT-Q. So, the negation of Shanmugam's claim is

> "A card has a vowel on one side AND an odd number on the other side."

Hence, turn over

- Any vowel to see if it has an odd number on the other side (this would prove Shanmugam wrong); and
- Any odd number to see if it has vowel on the other side (").

In case you weren't convinced, here's **Explanation II**, which doesn't directly use Fact 3. We can call this the brute-force case-by-case method:

1. An odd number behind A would prove Shanmugam wrong. So, we *should* turn over A.
2. An odd number behind M would not prove Shanmugam wrong. Nor would an even number. So, we *needn't* turn over M.
3. A vowel behind 11 would prove Shanmugam wrong. So, we *should* turn over 11.
4. A vowel behind 28 would not prove Shanmugam wrong. Nor would a consonant. So, we *needn't* turn over 28.

3.9. The Converse $Q \implies P$

Informally, **converse** = "flip". Formally,

Definition 6. Given the implication $P \implies Q$, its *converse* is the statement $Q \implies P$.

Example 28. Let E: "It's raining" and F: "The grass is wet". The implication $E \implies F$ is

"If it's raining, then the grass is wet."

The converse of the implication $E \implies F$ is the implication $F \implies E$:

"If the grass is wet, then it's raining."

Or equivalently, "That the grass is wet implies that it's raining."

Note that $F \implies E$ is false. One way to show a statement is false is to provide a counterexample. Here we want to provide a counterexample where F is true (the grass is wet) but E is false (it isn't raining). We can easily think of such counterexamples:

1. The rain just stopped.
2. Someone is watering the grass.
3. A dog is peeing on the grass.
4. You are peeing on the grass.

In the above example, an implication $E \implies F$ is true but its converse $F \implies E$ is false.

We now give an example where an implication is true *and* its converse is also true:

Example 29. Let A: "Germany is in Europe" and ; B: "Germany is in Asia"; C: "$1+1=2$", and D: "$1+1=3$".

The implication $A \implies C$: "If Germany is in Europe, then $1+1=2$" is true. Its converse $C \implies A$: "If $1+1=2$, then Germany is in Europe" is also true.

The implication $B \implies D$: "If Germany is in Asia, then $1+1=3$" is true. (Why?)[77] Its converse $D \implies B$: "If $1+1=3$, then Germany is in Asia" is also true. (Why?)[78]

The above two examples show that a true implication could have a converse that's either true or false. This is stated as Fact 4(a) below.

Now, what about a false implication? Perhaps surprisingly, a false implication's converse *must* be true. This is stated as Fact 4(b) below. Example:

[77] The hypothesis B is false and so the implication $B \implies D$ is automatically true.
[78] The hypothesis D is false and so the implication $D \implies B$ is automatically true.

Example 30. Let A: "Germany is in Europe" and ; B: "Germany is in Asia"; C: "$1+1=2$", and D: "$1+1=3$".

The implication $A \implies D$: "If Germany is in Europe, then $1+1=3$" is false. But its converse $D \implies A$: "If $1+1=3$, then Germany is in Europe" is true. (Why?)[79]

The implication $C \implies B$: "If $1+1=2$, then Germany is in Asia" is false. But its converse $B \implies C$: "If Germany is in Asia, then $1+1=3$" is true. (Why?)[80]

Fact 4. *Let P and Q be statements.*

(a) *If $P \implies Q$ is true, then $Q \implies P$ can be true or false.*

(b) *If $P \implies Q$ is false, then $Q \implies P$ must be true.*

Proof. By definition, $P \implies Q$ is equivalent to NOT$-P$ OR Q; similarly, $Q \implies P$ is equivalent to NOT$-Q$ OR P.

(a) Suppose $P \implies Q$ is true. Then either NOT$-P$ or Q is true—equivalently, P is false or Q is true. Let's look at the four possible cases:

1. P is false and Q is true, in which case $Q \implies P$ is false;
2. P is false and Q is false, in which case $Q \implies P$ is true;
3. Q is true and P is false, in which case $Q \implies P$ is false;
4. Q is true and P is true, in which case $Q \implies P$ is true.

So indeed, $Q \implies P$ can be true or false.

(b) Suppose $P \implies Q$ is false. Then both NOT$-P$ and Q are false—equivalently, P is true and Q is false. So, $Q \implies P$ is true. □

[79]The hypothesis D is false and so the implication $D \implies A$ is automatically true.
[80]The hypothesis B is false and so the implication $B \implies C$ is automatically true.

Exercise 13. Let G: "I'm sleeping"; H: "My eyes are shut"; M: "$x > 0$"; N: "$x > 1$"; α: "$x = 3$"; and γ: "$x^2 = 9$". For each statement below, explain whether it's true, write down its converse, and explain whether this converse is true. (Answer on p. 1750.)

(a) $G \implies H$. **(b)** $M \implies N$. **(c)** $\gamma \implies \alpha$.

Exercise 14. In Exercise 9, we showed that each statement below is true. Now, write down its converse and explain whether this converse is true. (Answer on p. 1750.)

(a) "If Tin Pei Ling (TPL) is a genius, then the Nazis won World War II (WW2)."

(b) "If TPL is a genius, then the Allies won WW2."

(c) "If π is rational, then I am the king of the world."

(d) "If π is rational, then Lee Hsien Loong is Lee Kuan Yew's son."

Exercise 15. Let A: "Germany is in Europe", B: "Germany is in Asia", C: "$1 + 1 = 2$", and D: "$1 + 1 = 3$". For each statement below, explain whether it's true, write down its converse, and explain whether this converse is true. (Answer on p. 1750.)

(a) $A \implies B$. **(b)** $A \implies C$. **(c)** $A \implies D$. **(d)** $C \implies D$.

Exercise 16. Fill in each blank below with **(i)** *must be true*; **(ii)** *must be false*; or **(iii)** *could be true or false*. (Answer on p. 1751.)

(a) If P is true, then $P \implies Q$ _______ and $Q \implies P$ _______.

(b) If P is false, then $P \implies Q$ _______ and $Q \implies P$ _______.

(c) If $P \implies Q$ is true, then $Q \implies P$ _______.

(d) If $P \implies Q$ is false, then $Q \implies P$ _______.

3.10. Affirming the Consequent (or The Fallacy of the Converse)

Affirming the consequent or **the fallacy of the converse** is a common error that people make in everyday life:

Example 31. 1. "If it's raining, then the grass is wet."

2. We observe, "The grass is wet."

3. We incorrectly conclude, "Therefore, it's raining."[81]

That the grass is wet does not imply that it's raining. The grass may be wet because (a) it just stopped raining; (b) someone is watering it; (c) a dog is peeing on it; (d) you are peeing on it; or a billion other reasons.

Formally, the fallacy takes this form:

1. "$P \implies Q$."
2. "Q."
3. "Therefore, P."

Example 32. 1. "If John has been doing chemotherapy, then John is bald."

2. We observe, "John is bald."

3. We incorrectly conclude, "Therefore, John is undergoing chemotherapy."

That John is bald does not imply he's has been doing chemotherapy. John may be bald because (a) he's suffering from male-pattern baldness; (b) he likes to keep his scalp clean-shaven; or a billion other reasons.

Example 33. 1. "If Mary drinks a lot of Coke, then Mary is fat."

2. We observe, "Mary is fat."

3. We incorrectly conclude, "Therefore, Mary drinks a lot of Coke."

That Mary is fat does not imply she drinks a lot of Coke. Mary may be fat because (a) she drinks a lot of Pepsi; (b) eats a lot of junk food; (c) suffers from low metabolism; or a billion other reasons.

When spelt out so explicitly, **affirming the consequent** or **the fallacy of the converse** seems rather silly. But unfortunately, people make this error all the time. Hopefully you'll now be able to avoid it.

Exercise 17. Is the following chain of reasoning valid?

1. "If Warren Buffett owns Google, then Warren Buffett is rich."
2. "Warren Buffett is rich."
3. "Therefore, Warren Buffett owns Google." (Answer on p. 1751.)

[81] By the way, such a chain of reasoning (whether valid or invalid) is traditionally called a **syllogism**. A syllogism has two or more statements called *premises*, followed by a conclusion.

3.11. The Contrapositive NOT–$Q \implies$ NOT–P

Informally, **contrapositive** = "Flip and negate both". Formally,

Definition 7. Let P and Q be statements. Given the implication $P \implies Q$, its *contrapositive* is the statement NOT–$Q \implies$ NOT–P.

Example 34. Let I: "x is German"; and J: "x is European".

Consider the implication $I \implies J$ and its contrapositive NOT–$J \implies$ NOT–I:

$$I \implies J\text{: "If } x \text{ is German, then } x \text{ is European."}$$
$$\text{NOT–}J \implies \text{NOT–}I\text{: "If } x \text{ is not European, then } x \text{ is not German."}$$

Both $I \implies J$ and its contrapositive NOT–$J \implies$ NOT–I are true.

Next, consider the converse $J \implies I$ and *its* contrapositive NOT–$I \implies$ NOT–J:

$$J \implies I\text{: "If } x \text{ is European, then } x \text{ is German."}$$
$$\text{NOT–}I \implies \text{NOT–}J\text{: "If } x \text{ is not German, then } x \text{ is not European."}$$

Both $J \implies I$ and its contrapositive NOT–$I \implies$ NOT–J are false.

Example 35. Let E: "It's raining" and F: "The grass is wet".

Consider the implication $E \implies F$ and its contrapositive NOT–$F \implies$ NOT–E:

$$E \implies F\text{: "If it's raining, then the grass is wet."}$$
$$\text{NOT–}F \implies \text{NOT–}E\text{: "If the grass is not wet, then it's not raining."}$$

Both $E \implies F$ and its contrapositive NOT–$F \implies$ NOT–E are true.

Next, consider the converse $F \implies E$ and *its* contrapositive NOT–$E \implies$ NOT–F:

$$F \implies E\text{: "If the grass is wet, then it's raining."}$$
$$\text{NOT–}E \implies \text{NOT–}F\text{: "If it's not raining, then the grass is not wet."}$$

Both $E \implies F$ and its contrapositive NOT–$F \implies$ NOT–E are false.

As the above examples suggest, **every implication is equivalent to its contrapositive**:

Fact 5. *Suppose P and Q are statements. Then*

$$(P \implies Q) \qquad \iff \qquad (\text{NOT–}Q \implies \text{NOT–}P).$$

Proof. By Definition 5, $P \implies Q$ is NOT–P OR Q. And NOT–$Q \implies$ NOT–P is Q OR NOT–P. Hence, $P \implies Q$ and NOT–$Q \implies$ NOT–P are equivalent.[82] □

[82]This proof implicitly assumes that the logical connective OR is commutative.

Fact 5 is especially useful on those occasions when it's hard to prove an implication but easy to prove its contrapositive:[83]

Example 36. It's not obvious how we can prove this implication:

$$\text{If } x^4 - x^3 + x^2 \neq 1, \text{ then } x \neq 1.$$

But proving its contrapositive is easy:

$$\text{If } x = 1, \text{ then } x^4 - x^3 + x^2 = 1.$$

Proof. Simply plug in $x = 1$ to verify that $x^4 - x^3 + x^2 = 1$. □

Example 37. It's not obvious how we can prove this implication:

$$\text{If } x^2 \text{ is even, then } x \text{ is even.}$$

But proving its contrapositive is easy:

$$\text{If } x \text{ is odd, then } x^2 \text{ is odd.}$$

Proof. Let $x = 2k + 1$ where k is some integer. Then $x^2 = 4k^2 + 4k + 1$, which is odd. □

But don't worry. In A-Level Maths, you won't be required to write any proofs; the above is just FYI and to illustrate why the contrapositive is useful.

Exercise 18. The statement "If x is German, then x is European" is true. Which statement below is its contrapositive? For each statement, explain whether it *must* be true, *must* be false, or could be either true or false. (Answer on p. 1751.)

(a) "If x is European, then x is German."

(b) "If x is not German, then x is not European."

(c) "If x is not German, then x is European."

(d) "If x is not European, then x is not German."

(e) "If x is not European, then x is German."

[83] These examples are from ✿.

3.12. $(P \implies Q \text{ AND } Q \implies P) \iff (P \iff Q)$

Example 38. As before, let

- I: "x is German."
- J: "x is European."
- M: "$x > 0$."
- N: "$x > 1$."
- O: "x is a positive number."

To show that two statements are equivalent, we can use Definition 4. So for example,

- $M \iff O$, because both statements always have the same truth value.
- $I \not\Longleftrightarrow J$ (counterexample: if x is Emmanuel Macron, then I is false while J is true).
- $M \not\Longleftrightarrow N$ (counterexample: if $x = 0.5$, then M is true while N is false).

Alternatively, we can use this fact:

Fact 6. *Suppose P and Q are statements. Then*

$$(P \implies Q \text{ AND } Q \implies P) \qquad \iff \qquad (P \iff Q).$$

Proof. See p. 1547 (Appendices). □

So, to show that $P \iff Q$, we can show that both $P \implies Q$ and $Q \implies P$ are true. And to show that $P \not\Longleftrightarrow Q$, we can show that either $P \implies Q$ or $Q \implies P$ is false.

Example 39. The implication $M \implies O$ ("if $x > 0$, then x is a positive number") is true.

Also, the implication $O \implies M$ ("if x is a positive number, then $x > 0$") is true.

So, by Fact 6, $M \iff O$.

Example 40. The implication $J \implies I$ ("if x is European, then x is German") is false.

So, by Fact 6, $I \not\Longleftrightarrow J$.

Example 41. The implication $N \implies M$ ("if $x > 1$, then $x > 0$") is false.

So, by Fact 6, $M \not\Longleftrightarrow N$.

Exercise 19. Continue with the last example: Is $N \iff O$ true? (Answer on p. 1751.)

Exercise 20. Are any two of the three statements X, Y, and Z equivalent? (Answer on p. 1751.)

X: "John is a Singapore citizen."

Y: "John has a National Registration Identity Card (NRIC)."

Z: "John has a pink NRIC."

If you aren't Singaporean, use these X, Y, and Z instead:

X: "John is an animal."

Y: "John is human."

Z: "John is a living thing."

3.13. The Four Categorical Statements and Their Negations

In mathematics and logic, the word **some** means **at least one**.

Here are the **four categorical statements**:[84]

1. **All Yes**[85] "All S are P" (or "Every S is P").
2. **All No** "No S is P" (or "Every S is NOT–P").
3. **Some Yes** "Some S is P."
4. **Some No** "Some S is NOT–P."

In each, we call S the **subject** and P the **predicate**.

Example 42. We give six examples of each categorical statement:

	All Yes ("All S are P")
1.	"All Koreans are Asian." (Or, "Every Korean is Asian.")
2.	"All Germans are European." (Or, "Every German is European.")
3.	"All animals are dogs." (Or, "Every animal is a dog.")
4.	"All mammals are bats." (Or, "Every mammal is a bat.")
5.	"All Koreans eat dogs." (Or, "Every Korean eats dogs.")
6.	"All Germans eat bats." (Or, "Every German eats bats.")

	All No ("No S is P")
7.	"No Korean is Asian." (Or, "Every Korean is not Asian.")
8.	"No German is European." (Or, "Every German is not European.")
9.	"No animal is a dog." (Or, "Every animal is not a dog.")
10.	"No mammal is a bat." (Or, "Every mammal is not a bat.")
11.	"No Korean eats dogs." (Or, "Every Korean does not eat dogs.")
12.	"No German eats bats." (Or, "Every German does not eat bats.")

The six subjects used are "Korean", "German", "animal", "mammal", "Korean", and "German".

The six predicates used are "Asian", "European", "a dog", "a bat", "eats dogs" (or "a dog-eater"), and "eats bats" (or "a bat-eater").

(*Example continues on the next page ...*)

[84]See Definition 263 (Appendices) for the formal definitions.

(... *Example continued from the previous page.*)

	Some Yes ("Some S is P")
13.	"Some Korean is Asian." (Or, "At least one Korean is Asian.")
14.	"Some German is European." (Or, "At least one German is European.")
15.	"Some animal is a dog." (Or, "At least one animal is a dog.")
16.	"Some mammal is a bat." (Or, "At least one mammal is a bat.")
17.	"Some Korean eats dogs." (Or, "At least one Korean eats dogs.")
18.	"Some German eats bats." (Or, "At least one German eats bats.")

	Some No ("Some S is NOT–P**")**
19.	"Some Korean is not Asian." (Or, "At least one Korean is not Asian.")
20.	"Some German is not European." (Or, "At least one German is not European.")
21.	"Some animals is not a dog." (Or, "At least one animal is not a dog.")
22.	"Some mammal is not a bat." (Or, "At least one mammal is not a bat.")
23.	"Some Korean does not eat dogs." (Or, "At least one Korean does not eat dogs.")
24.	"Some German does not eat bats." (Or, "At least one German does not eat bats.")

Exercise 21. For each given pair of subject S and predicate P, write down the corresponding All Yes, All No, Some Yes, and Some No statements. (Answer on p. 1752.)

	S	P
(a)	Donzer	Kiki
(b)	Donzer	Cancer
(c)	Bachelor	Married
(d)	Bachelor	Smoke

Exercise 22. Explain whether each statement is true. (Answer on p. 1752.)

(a) Corresponding All Yes and All No statements are always negations of each other.

(b) Corresponding Some Yes and Some No statements are always negations of each other.

The last exercise showed that All Yes isn't the negation of All No. Similarly, Some Yes isn't the negation of Some No. So, what then is the correct negation of each categorical statement?

Example 43. Consider the All Yes statement, "All Koreans eat dogs."

To show it's false, we need merely show that there is at least one Korean who doesn't eat dogs. So, its negation is

"Some (i.e. at least one) Korean does not eat dogs."

Which is a Some No statement.

Example 44. Consider the All No statement, "No Korean eats dogs," (or, "All Koreans do not eat dogs.")

To prove it's false, we need merely show that there is at least one Korean who eats dogs. So, its negation is

"Some (i.e. at least one) Korean eats dogs".

Which is a Some Yes statement.

In general,

Fact 7. (a) *The negation of an All Yes statement is the corresponding Some No statement.*

(b) *The negation of an All No statement is the corresponding Some Yes statement.*

Proof. See p. 1549 (Appendices). □

Example 45. The same examples as before, with their negations:

	All Yes ("All S are P")	**Negation: Some No ("Some S is NOT–P")**
1.	"All Koreans are Asian."	"Some Korean is not Asian."
2.	"All Germans are European."	"Some German is not European."
3.	"All animals are dogs."	"Some animal is not a dog."
4.	"All mammals are bats."	"Some mammal is not a bat."
5.	"All Koreans eat dogs."	"Some Korean does not eat dogs."
6.	"All Germans eat bats."	"Some German does not eat bats."

	All No ("No S is P")	**Negation: Some Yes ("Some S is P")**
7.	"No Korean is Asian."	"Some Korean is Asian."
8.	"No German is European."	"Some German is European."
9.	"No animal is a dog."	"Some animal is a dog."
10.	"No mammal is a bat."	"Some mammal is a bat."
11.	"No Korean eats dogs."	"Some Korean eats dogs."
12.	"No German eats bats."	"Some German eats bats."

Exercise 23. For each statements below, identify whether it's an All Yes, All No, Some Yes, or Some No statement. Also, write down its negation. (Answer on p. 1752.)

	Statement
(a)	"All donzers are kiki."
(b)	"No donzer is kiki."
(c)	"Some donzer is kiki."
(d)	"Some donzer is not kiki."
(e)	"All bachelors are married."
(f)	"No bachelor is married."
(g)	"Some bachelor is married."
(h)	"Some bachelor is not married."
(i)	"All donzers cause cancer."
(j)	"No donzer causes cancer."
(k)	"Some donzer causes cancer."
(l)	"Some donzer does not cause cancer."
(m)	"All bachelors smoke."
(n)	"No bachelor smokes."
(o)	"Some bachelor smokes."
(p)	"Some bachelor does not smoke."

Exercise 24. While trying to excuse the less-than-perfect play of a basketball player, a commentator remarks, (Answer on p. 1753.)

"Everybody is not LeBron James."

(a) Rewrite this statement into the form of a categorical statement. Identify the type of categorical statement, the subject, and the predicate.

(b) Write down this statement's negation.

(c) Is the commentator's statement is true? If false, what should he have said instead?

Exercise 25. Suppose this statement is true: "Every breadfruit is either donzer or kiki." Based solely on this given statement, explain whether each statement below must be true, must be false, or could be true or false. (Answer on p. 1753.)

(a) "Every breadfruit is both donzer and kiki."

(b) "Every breadfruit is donzer."

(c) "Every breadfruit is kiki."

(d) "Some breadfruit is donzer."

(e) "Some breadfruit is kiki."

(f) "No breadfruit is donzer."

(g) "No breadfruit is kiki."

(h) "Every non-breadfruit is neither donzer nor kiki."

(i) "Every non-breadfruit is not donzer."

(j) "Every non-breadfruit is not kiki."

(k) "Some non-breadfruit is not donzer."

(l) "Some non-breadfruit is not kiki."

(m) "No non-breadfruit is donzer."

(n) "No non-breadfruit is kiki."

Exercise 26. Negate each statement. (Answer on p. 1753.)

(a) "Every prime number is odd."

(b) "Every prime number greater than 2 is odd."

(c) "At least one rational number is greater than π."

(d) "No number is both negative and imaginary."

(e) "Every number is both positive and irrational."

3.14. Chapter Summary

- The **conjunction** P AND Q is true if *both* P and Q are true.
- The **disjunction** P OR Q is true if *either* P or Q is true.
- The **negation** NOT–P is true if P is false and false if P is true.
- P and Q are **equivalent** (written $P \iff Q$) if both always have the same truth value.
- **De Morgan's Laws**:
  - NOT–(P AND Q) $\iff$ (NOT–P OR NOT–Q);
  - NOT–(P OR Q) $\iff$ (NOT–P AND NOT–Q).
- The **implication** $P \implies Q$ is
  - *Defined* as NOT–P OR Q;
  - Equivalent to its **contrapositive** NOT–$Q \implies$ NOT–P;
  - **Negated** by P AND NOT-Q;
  - **Not** generally equivalent to its **converse** $Q \implies P$;
    * If $P \implies Q$ is true, then $Q \implies P$ could be true or false.
    * If $P \implies Q$ is false, then $Q \implies P$ must be true.
- The **four categorical statements** and their negations:

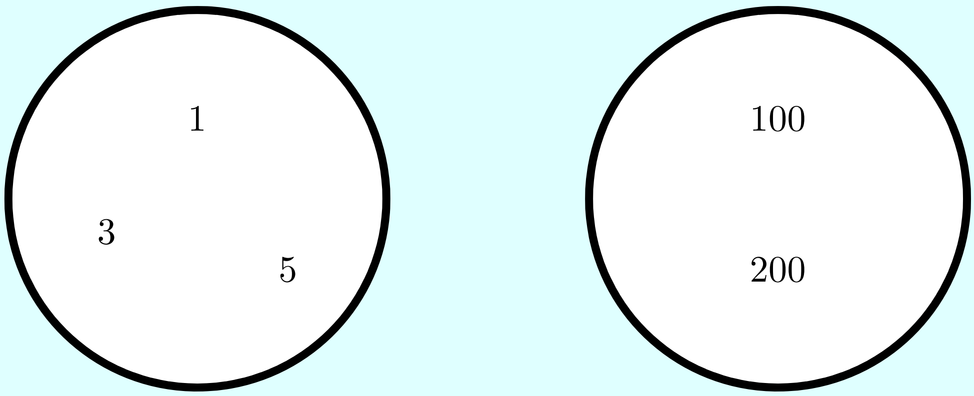
Statement	Negation
All Yes: "All S are P."	**Some No**: "Some S is NOT–P."
All No: "No S is P."	**Some Yes**: "Some S is P."

We call S the **subject** and P the **predicate**.

4. Sets

The **set** is the basic building block of mathematics. Informally, a set is a "container" or "box" whose contents we call its **elements** (or **members**).

Example 46. Let $A = \{1, 3, 5\}$ and $B = \{100, 200\}$.



The set A The set B

The set A contains three elements—namely, the numbers 1, 3, and 5. Informally, it is a "box" containing the numbers 1, 3, and 5.

The set B contains three elements—namely, the numbers 100 and 200. Informally, it is a "box" containing the numbers 100 and 200.

Note that when we talk about a set, we refer to *both* the box *and* the things inside it.

Mathematical punctuation:

- **Braces** $\{\}$—are used to denote the "container".
- A **comma** means "and" and is used to separate the elements within a set.

Exercise 27. Write down C, the set of the first 7 positive integers. (Answer on p. 1754.)

Exercise 28. Write down D, the set of even prime numbers. (Answer on p. 1754.)

4.1. The Elements of a Set Can Be Pretty Much *Anything*

Note that for the A-Levels and hence also in this textbook, the elements of a set will almost always be numbers. However, in general, they needn't be numbers; they can be pretty much *any* **objects** whatsoever:[86]

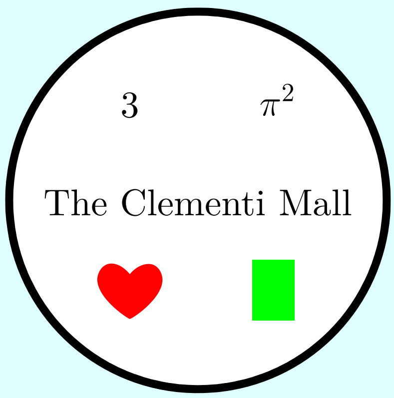
Example 47. Let V be the set of the four largest cities in the US. Then

$$V = \{\text{New York City, Los Angeles, Chicago, Houston}\}.$$

Example 48. Let L be the set of suits in the game of bridge. Then

$$L = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}.$$

Example 49. Let $E = \{3, \pi^2, \text{ The Clementi Mall, Love, the colour green}\}$.



The set E

The set E contains exactly five elements: two numbers (3 and π^2); a shopping centre (The Clementi Mall); an abstract concept called love (denoted in the figure above by a red heart); and even the colour green (denoted by a green rectangle).

Example 50. Let S be the set of Singapore citizens. Then S contains about 3.5M elements,[87] three of which are Lee Hsien Loong, Ho Ching, and Chee Soon Juan.

Example 51. Let U be the set of United Nations (UN) member states. Then U contains exactly 193 elements,[88] three of which are Afghanistan, Singapore, and Zimbabwe.

Exercise 29. Write down X, the set of Singapore Prime Ministers (both past and present). (Answer on p. 1754.)

Remark 8. We'll leave the terms **set**, **element**, and **object** undefined.

---

[86] Actually, there *are* some restrictions on what can go into a set, but these technicalities are beyond the scope of the A-Levels.

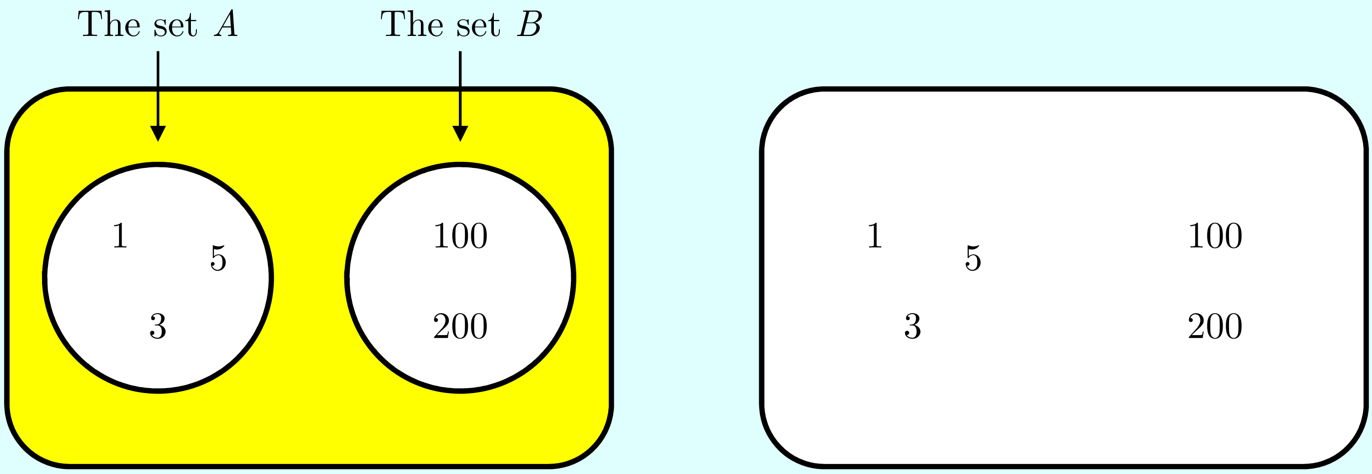
[87] According to SingStat, the number of Singapore citizens in 2017 was about 3,439,200.

[88] This UN webpage states that the most recent and 193rd state to join the UN was South Sudan in 2011.

A set can even contain other sets:

Example 52. Let F be the set that contains the sets $A = \{1, 3, 5\}$ and $B = \{100, 200\}$.

That is, let
$$F = \{A, B\} = \{\{1, 3, 5\}, \{100, 200\}\}.$$



The set F The set G

Now consider instead this set:

$$G = \{1, 3, 5, 100, 200\}.$$

The sets F and G look very similar. So, is G the same set as F?

Nope. The set F contains exactly two elements, namely the sets A and B.

In contrast, G contains exactly five elements, namely the numbers 1, 3, 5, 100, and 200. And so, F and G are not the same.

You can think of F as a box that itself contains two boxes—namely, A and B, each of which contain some numbers. In contrast, G is a box that contains no boxes; instead, it simply contains five numbers.

The following exercises continue with the above example:

Exercise 30. Let H be the set whose elements are F and G. (Answer on p. 1754.)

(a) How many elements does H have?

(b) Write down H without using the letters A, B, F, or G.

Exercise 31. Let I be the set whose elements are A, B, and G. (Answer on p. 1754.)

(a) How many elements does I have?

(b) Write down I without using the letters A, B, F, or G.

(c) Compare the sets H (from the previous Exercise) and I. Are they the same?

4.2. In $\in$ and Not In $\notin$

Mathematical punctuation:

- $\in$ means "is in" (or "is an element of");
- $\notin$ means "is not in" (or "is not an element of").

Example 53. Let $J = \{1, 2, 3, 4, 5, 6, 7\}$. Then

$$1 \in J, \quad 2 \in J, \quad 3 \in J, \quad 4 \in J, \quad 5 \in J, \quad 6 \in J, \quad 7 \in J.$$

You can read these statements aloud as "1 is in J" (or "1 is an element of J"), "2 is in J" (or "2 is an element of J"), "3 is in J" (or "3 is an element of J"), etc.

We can also write $1, 2, 3 \in J$ (read aloud as "1, 2, and 3 are in J" or "1, 2, and 3 are elements of J").

Also, $8 \notin J$, $9 \notin J$, $10 \notin J$, etc. (read aloud as "8 is not in J" or "8 is not an element of J", "9 is not in J" or "9 is not an element of J", "10 is not in J" or "10 is not an element of J", etc.).

We can also write $8, 9, 10 \notin J$ (read aloud as "8, 9, and 10 are not in J" or "8, 9, and 10 are not elements of J").

Example 54. Let $K = \{\text{Cow, Chicken}\}$. Then

- Cow $\in K$ ("Cow is in K" or "Cow is an element of K").
- Chicken $\in K$ ("Chicken is in K" or "Chicken is an element of K").
- Cow, Chicken $\in K$ ("Cow and Chicken are in K" or "Cow and Chicken are elements of K").
- Cat $\notin K$ ("Cat is not in K" or "Cat is not an element of K").

Exercise 32. Fill in the blanks with either $\in$ or $\notin$. (Answer on p. 1754)

(a) Los Angeles _____ The set of the four largest cities in the US.

(b) Tharman _____ The set of Singapore Prime Ministers (past and present).

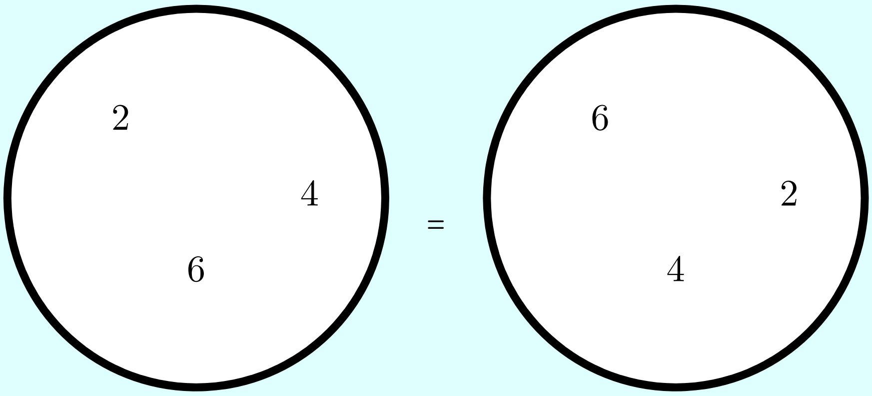
4.3. The Order of the Elements Doesn't Matter

Definition 8. Two sets A and B are *equal* if every element that is in A is also in B and every element that is in B is also in A.

One implication of the above definition[89] is that **the order in which we write out the elements of a set does not matter**:

Example 55. Let $A = \{2, 4, 6\}$ and $B = \{6, 2, 4\}$.

Then $A = B$ because every element that is in A is also in B and every element that is in B is also in A.

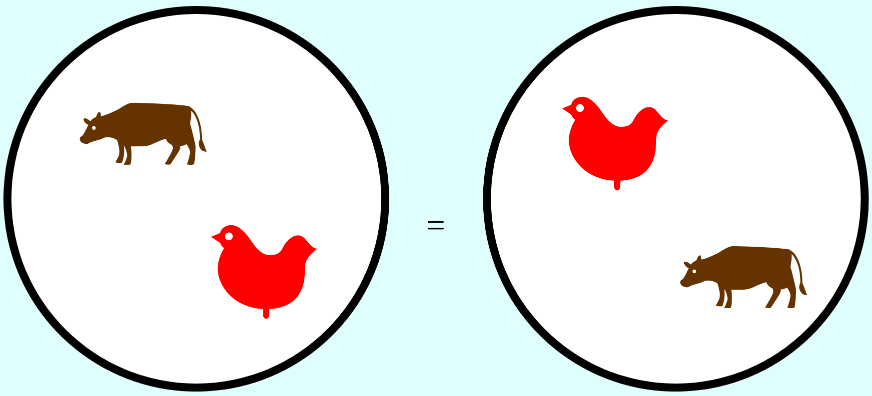


The set A The set B

In fact, $\{2, 4, 6\} = \{2, 6, 4\} = \{4, 2, 6\} = \{4, 6, 2\} = \{6, 2, 4\} = \{6, 4, 2\}$.

Example 56. Let $C = \{\text{Cow, Chicken}\}$ and $D = \{\text{Chicken, Cow}\}$.

Then $C = D$ because every element that is in C is also in D and every element that is in D is also in C.



The set C The set D

Exercise 33. Are the two given sets equal? (Answer on p. 1754.)

(a) $\{1, 2, 3\}$ and $\{3, 2, 1\}$. **(b)** $\{\{1\}, 2, 3\}$ and $\{\{3\}, 2, 1\}$.

[89]Actually, in set theory, this is not a definition, but an axiom (known as the Axiom of Extensionality). But here for simplicity, I'll just call it a definition.

4.4. $\mathrm{n}(S)$ Is the Number of Elements in the Set S

Let S be a set. Then the number of elements in S[90] is denoted by

$$\mathrm{n}(s).$$

Example 57. $\mathrm{n}(\{2,4,6\}) = 3$.

Example 58. $\mathrm{n}(\{\text{Cow, Chicken}\}) = 2$.

Exercise 34. Let X be the set of Singapore Prime Ministers (past and present). Then what is $\mathrm{n}(X)$? (Answer on p. 1754.)

Remark 9. Note that most writers denote the number of elements in the set S by $|S|$.[91]

But for some reason, your A-Level syllabus (p. 16) instead uses the notation $\mathrm{n}(S)$, so that's what we'll have to use too.

4.5. The Ellipsis "..." Means Continue in the Obvious Fashion

More mathematical punctuation—the *ellipsis* "..." means "continue in the obvious fashion".

Example 59. L is the set of all odd positive integers smaller than 100. So in set notation, we can write $L = \{1,3,5,7,9,11,\ldots,99\}$.

Example 60. M is the set of all negative integers greater than -100. So in set notation, we can write $M = \{-99,-98,-97,\ldots,-2,-1\}$.

What is "obvious" to you may not be obvious to your reader. So only use the ellipsis when you're confident it will be obvious to your reader! And as I did with the sets above, never be shy to write a few more of the set's elements (doing so costs you nothing except maybe a few more seconds and some ink).

Exercise 35. In the above examples, what are $\mathrm{n}(L)$ and $\mathrm{n}(M)$? (Answer on p. 1754.)

Exercise 36. Let N be the set of even integers greater than 100 but smaller than 1,000. Write down N in set notation. (Answer on p. 1754.)

[90] Or the *cardinality of* S.

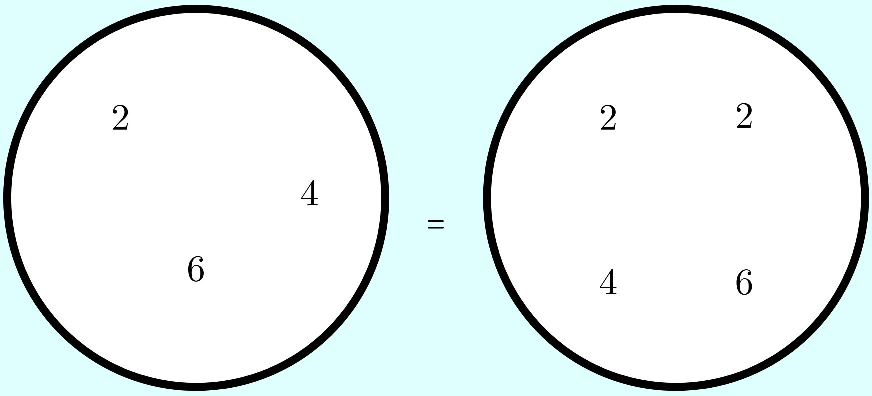
[91] Some also use card A. See ISO 80000-2:2009, Item No. 2-5.5.

4.6. Repeated Elements Don't Count

Another implication of Definition 8 is that **repeated elements don't count** (they're simply ignored):

Example 61. Let $A = \{2, 4, 6\}$ and $B = \{2, 2, 4, 6\}$.

Then $A = B$ because every element that is in A is also in B and every element that is in B is also in A.

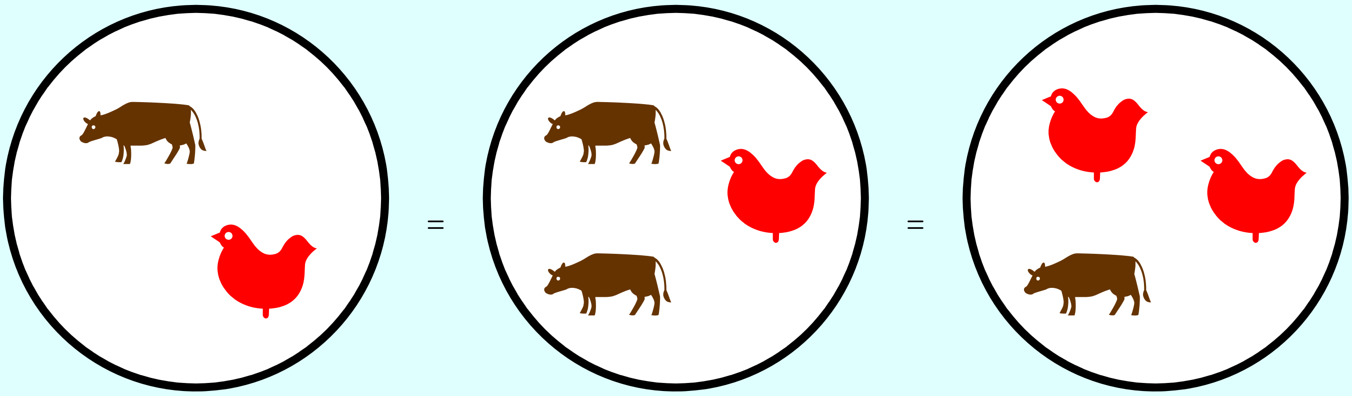


The set A The set B

In fact, $\{2, 4, 6\} = \{2, 2, 4, 6\} = \{4, 2, 6, 2, 2, 6, 4, 2\}$.

Moreover, $\mathrm{n}(\{2, 4, 6\}) = \mathrm{n}(\{2, 2, 4, 6\}) = \mathrm{n}(\{4, 2, 6, 2, 2, 6, 4, 2\}) = 3$.

Example 62. {Cow, Chicken} = {Cow, Cow, Chicken} = {Chicken, Cow, Chicken}.



Moreover,

$$\mathrm{n}(\{\text{Cow, Chicken}\}) = \mathrm{n}(\{\text{Cow, Cow, Chicken}\}) = \mathrm{n}(\{\text{Chicken, Cow, Chicken}\}) = 2.$$

Exercise 37. Let W = {Apple, Apple, Apple, Banana, Banana, Apple}. Find $\mathrm{n}(W)$. Also, rewrite W more simply. (Answer on p. 1754.)

Exercise 38. Let C be the set of even prime numbers. Find $\mathrm{n}(C)$. (Answer on p. 1754.)

4.7. $\mathbb{R}$ Is the Set of Real Numbers

> *Like most students entering college, mathematicians of the midnineteenth century thought they understood real numbers. In fact, the real number line turned out to be much subtler and more complicated than they imagined.*
>
> — David M. Bressoud (2008).

> *Few mathematical structures have undergone as many revisions or have been presented in as many guises as the real numbers. Every generation re-examines the reals in the light of its values and mathematical objectives.*
>
> — Faltin, Metropolis, Ross, and Rota (1975).

So far, we've encountered only finite sets, i.e. sets with finitely many elements.

In this and the next two subchapters, we introduce several infinite sets, i.e. sets with *infinitely* many elements.

First, we have the **set of real numbers** (or simply **reals**):

Definition 9. The *set of real numbers* is denoted $\mathbb{R}$.

Example 63. 16, -1.87, $\pi \approx 3.14159$, and $\sqrt{2} \approx 1.41421$ are all real numbers.

Now, by the way, what exactly is a real number? This sounds like a "dumb" question, but is actually a profound one that was satisfactorily resolved only from the late 19th century. Indeed, this question is a little beyond the scope of the A Levels.

And so for the A Levels, we'll simply pretend—as we did in secondary school—that "everyone knows" what real numbers are (even though, as the quotes above suggest, they actually don't). We shall not attempt to define or construct the real numbers.

4.8. $\mathbb{Z}$ Is the Set of Integers

> Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.
>
> *God made the integers, the rest is the work of man.*
>
> — Leopold Kronecker (1886).[92]

Next, $\mathbb{Z}$ is the set of **integers**:

Definition 10. The *set of integers*, denoted $\mathbb{Z}$, is

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Z is for *Zahl*, German for *number*.

Example 64. 16 is an integer, while -1.87, π, and $\sqrt{2}$ are non-integers.

Note that with an infinite set, we cannot explicitly list out all its elements. And so, when writing out an infinite set, we'll sometimes find it helpful to use the ellipsis.

When writing out $\mathbb{Z}$ above, we used two ellipses. The first ellipsis says we continue "leftwards" in the "obvious" fashion, with -4, -5, -6, etc. The second says we continue "rightwards" in the "obvious" fashion, with 4, 5, 6, etc.

Exercise 39. H is the set of all prime numbers. With the aid of an ellipsis, write down H in set notation. (Answer on p. 1754.)

Definition 11. An *even integer* is any element of the set $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$.
An *odd integer* is any element of the set $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$.

It's not on your syllabus, but a **natural number** is simply any positive integer:[93]

Definition 12. The *set of natural numbers*, denoted $\mathbb{N}$, is

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

[92] According to Heinrich Weber (in his 1893 obituary for Kronecker), this quoted remark was made by Kronecker at an 1886 lecture to the Berliner Naturforscher-Versammlung.

[93] There's actually some debate as to whether $\mathbb{N}$ should include 0.

4.9. $\mathbb{Q}$ Is the Set of Rational Numbers

Definition 13. A *rational number* (or simply *rational*) is any real number that can be expressed as the ratio of two integers.

Any other real number is called an *irrational number* (or simply *irrational*).

Definition 14. The *set of rational numbers* is denoted $\mathbb{Q}$.

Q is for *quotient*.[94]

Example 65. $16 \in \mathbb{Q}$ because we can express 16 as the ratio of two integers (e.g. 16/1).

$-1.87 \in \mathbb{Q}$ because we can express -1.87 as the ratio of two integers (e.g. $-187/100$).

Example 66. $\sqrt{2}, \pi \notin \mathbb{Q}$. In words: "$\sqrt{2}$ and π are not elements of the set of rational numbers." Or more simply: "$\sqrt{2}$ and π are irrational."

(Note though that this is far from obvious. It takes a little work to prove that $\sqrt{2}$ is irrational and even more work to prove that π is irrational.)

As you probably already know, a number is rational if and only if its digits eventually recur:[95]

Example 67. $1/3 = 0.33333\cdots = 0.\overline{3}$ is rational and sure enough, it has the recurring digit 3. We will use the overbar to denote recurring digit(s).[96]

$1/7 = 0.142857142857142857\cdots = 0.\overline{142857}$ is rational and sure enough, it has the recurring digits 142857.

Similarly, $16 = 16.000\cdots = 16.\overline{0}$ and $1.87 = 1.87000\cdots = 1.87$ are rational and have the recurring digit 0. (Of course, when the recurring digit is 0, we usually don't bother writing it.)

Example 68. $\sqrt{2} \approx 1.4142135623\ldots$ and $\pi \approx 3.1415926535\ldots$ are irrational. And sure enough, their digits never recur.

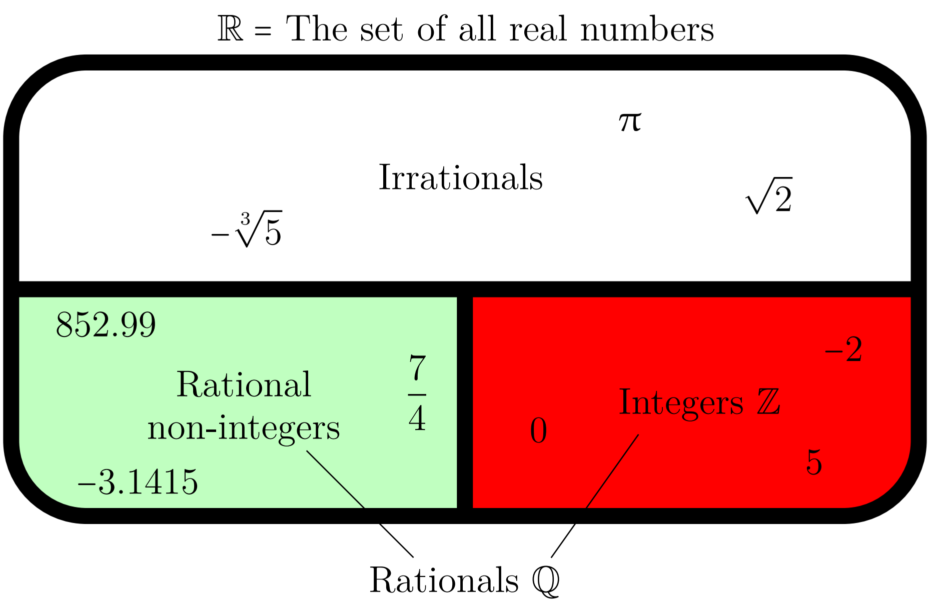
(But again, this is far from obvious and takes some work to prove.)

[94] Also *quoziente* in Italian, *Quotient* in German, and *quotient* in French.

[95] See Proposition 25 (Appendices).

[96] Some writers also use an overdot instead: $1/3 = 0.33333\cdots = 0.\dot{3}$. But we won't.

4.10. A Venn Diagram of the Real Numbers



All reals are either rational or irrational (and never both).

All rationals are either integers or non-integers (and never both).

Five remarks:

4. $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Q}$ are infinite sets (i.e. they contain infinitely many elements).

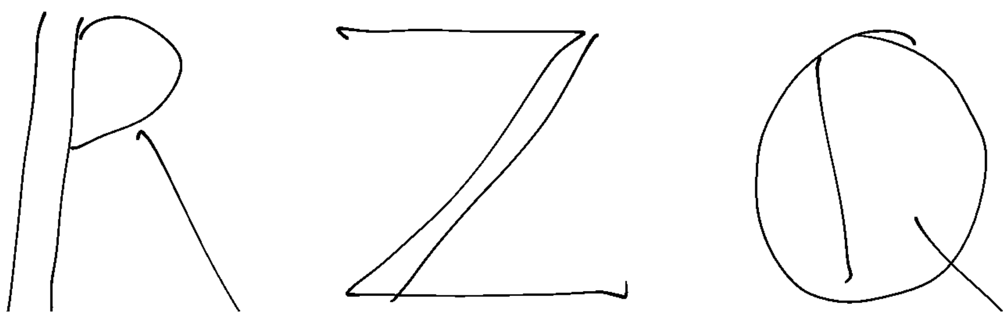
   We may thus write $\mathrm{n}(\mathbb{R}) = \mathrm{n}(\mathbb{Z}) = \mathrm{n}(\mathbb{Q}) = \infty$.

5. **Infinity** (∞) and **negative infinity** ($-\infty$) are **not** numbers.[97]

As with real numbers, we won't try to formally define what either ∞ or $-\infty$ is.[98] We'll just repeat and emphasise the key point here:

Infinity is not a number.

3. The areas in the above Venn diagram may incorrectly suggest that there are "as many" rationals as irrationals. But this is false—it turns out (and this is beyond A-Levels) there's a precise sense in which there are "more" irrationals than rationals.[99]
4. This textbook follows your A-Level syllabus and exams in using **blackboard bold font** ($\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$). Note though that some other writers use **bold font** (**R**, **Z**, and **Q**).
5. We can handwrite $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Q}$ as an R, Z, and Q with an extra vertical line on the left, an extra diagonal line "inside", an extra vertical line "inside":



[97]Actually, some writers in some contexts find it convenient to call ∞ and $-\infty$ numbers—for example, they may call ∞ and $-\infty$ **extended real numbers**. But this textbook will keep it simple and insist that neither ∞ nor $-\infty$ is a number.

[98]Indeed, there's no need to define either ∞ or $-\infty$ at all. There isn't even any need to think of either as some definite object or "thing". Instead, either is usually simply used as shorthand for expressing some long-winded statement. For example, we can think of the statement "$\mathrm{n}(\mathbb{Q}) = \infty$" as equivalent to "the number of elements in $\mathbb{Q}$ is unbounded from above".

[99]And perhaps surprisingly, there are "as many" integers as rational non-integers.

4.11. More Notation: $^+$, $^-$, and $_0$

To create a new set that contains only the positive elements of the old set, append a superscript plus sign ($^+$) to the name of a set:

1. $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ is the set of all positive integers.
2. $\mathbb{Q}^+$ is the set of all positive rational numbers.
3. $\mathbb{R}^+$ is the set of all positive real numbers.

To create a new set that contains only the negative elements of the old set, append a superscript minus sign ($^-$) to the name of a set:

1. $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$ is the set of all negative integers.
2. $\mathbb{Q}^-$ is the set of all negative rational numbers.
3. $\mathbb{R}^-$ is the set of all negative real numbers.

To add the number 0 to a set, append a subscript zero ($_0$) to its name:

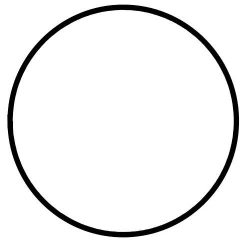
1. $\mathbb{Z}_0^+ = \{0, 1, 2, 3, \ldots\}$ is the set of all non-negative integers.
2. $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ is the set of all non-positive integers.
3. $\mathbb{Q}_0^+$ is the set of all non-negative rational numbers.
4. $\mathbb{Q}_0^-$ is the set of all non-positive rational numbers.
5. $\mathbb{R}_0^+$ is the set of all non-negative real numbers.
6. $\mathbb{R}_0^-$ is the set of all non-positive real numbers.

Exercise 40. Which of the sets introduced above are finite? (Answer on p. 1754.)

Remark 10. The three pieces of notation introduced on this page ($^+$, $^-$, and $_0$) aren't terribly important or widely used. I give them a quick mention only because they're listed on p. 16 of your A-Level syllabus.

4.12. The Empty Set $\varnothing$

A set can contain any number of elements. Indeed, it can even contain **zero** elements:



The empty set $\{\}$ or $\varnothing$.

> **Definition 15.** The *empty set* $\{\}$ is the set that contains zero elements (and is often also denoted $\varnothing$).

Informally, the empty set $\{\} = \varnothing$ is the "container" with nothing inside. Hence the name.

> **Example 69.** In 2016, the set of all Singapore Ministers who are younger than 30 is $\{\}$ or $\varnothing$. This means there is no Singapore Minister who is younger than 30.

> **Example 70.** The set of all even prime numbers greater than 2 is $\{\}$ or $\varnothing$. This means there is no even prime number that is greater than 2.

> **Example 71.** The set of numbers that are greater than 4 and smaller than 4 is $\{\}$ or $\varnothing$. This means there is no number that is simultaneously greater than 4 and smaller than 4.

Example 72. The set $\{\varnothing\}$ is not the same as the set $\varnothing$.

$\{\varnothing\}$ is a set containing a single element, namely the empty set.

Informally, $\{\varnothing\}$ is a box containing an empty box—it is **not** empty.

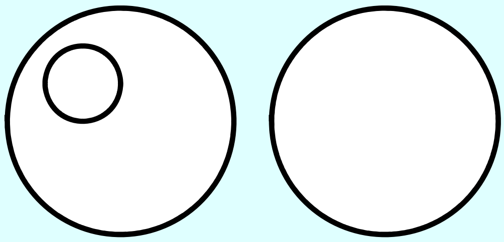
In contrast, $\varnothing$ is the empty set.

Informally, it is simply an empty box.

We can also rewrite the two sets as

$$\{\varnothing\} = \{\{\}\} \qquad \text{and} \qquad \varnothing = \{\}.$$

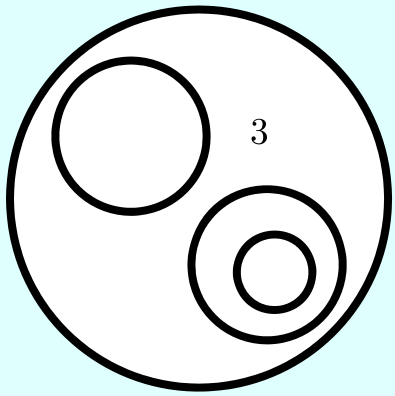
Now it is perhaps clearer that $\{\{\}\} \neq \{\}$.



$\{\varnothing\} = \{\{\}\}$ $\qquad$ $\varnothing = \{\}$

Example 73. The set $\{\varnothing, 3, \{\varnothing\}\}$ is the set containing *exactly* three elements, namely the empty set, the number 3, and a set containing the empty set.

Take care to note that this set does **not** contain four elements.



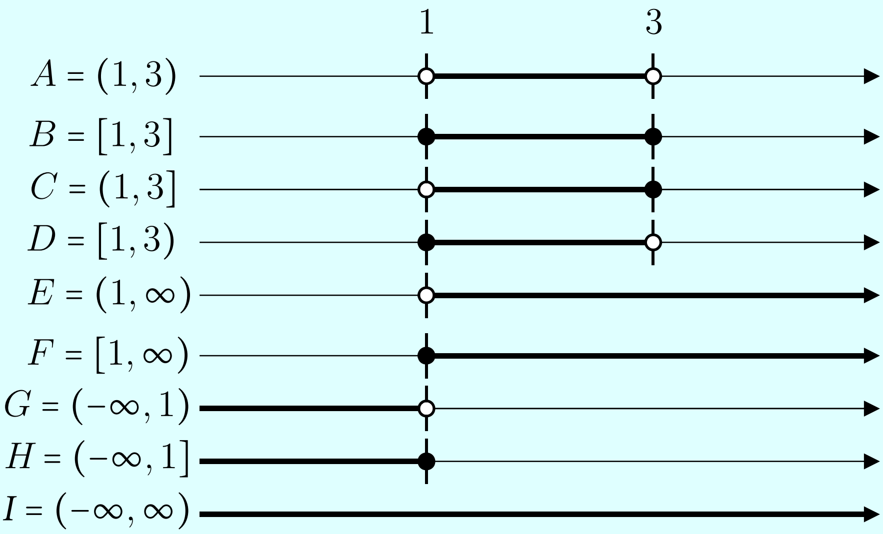
$\{\varnothing, 3, \{\varnothing\}\}$

> **Exercise 41.** Tricky: Let $S = \{\{\{\}\}, \varnothing, \{\varnothing\}, \{\}\}$. What is $\mathrm{n}(S)$? (Answer on p. 1754.)

4.13. Intervals

Using **left parenthesis** (, **right parenthesis**), **left bracket** [, and **right bracket**], we can write **intervals of real numbers**:

Example 74. Here are nine **intervals**:



(a) The interval $A = (1,3)$ is the set of real numbers that are > 1 and < 3.
(b) The interval $B = [1,3]$ " ≥ 1 and ≤ 3.
(c) The interval $C = (1,3]$ " > 1 and ≤ 3.
(d) The interval $D = [1,3)$ " ≥ 1 and < 3.
(e) The interval $E = (1,\infty)$ " > 1.
(f) The interval $F = [1,\infty)$ " ≥ 1.
(g) The interval $G = (-\infty,1)$ " < 1.
(h) The interval $H = (-\infty,1]$ " ≤ 1.
(i) The interval $I = (-\infty,\infty)$ denotes the set of *all* real numbers. That is, $(-\infty,\infty) = \mathbb{R}$.

Given below are the **left** and **right endpoints** of the above nine intervals:

	A	B	C	D	E	F	G	H	I
Left endpoint	1	1	1	1	1	1	$-\infty$	$-\infty$	$-\infty$
Right endpoint	3	3	3	3	∞	∞	1	1	∞

Definition 16. Let a and b be real numbers with $b \geq a$. An *interval* is any one of these nine sets:

(a) The interval (a, b) is the set of real numbers that are $> a$ and $< b$.
(b) The interval $[a, b]$ " $\geq a$ and $\leq b$.
(c) The interval $(a, b]$ " $> a$ and $\leq b$.
(d) The interval $[a, b)$ " $\geq a$ and $< b$.
(e) The interval (a, ∞) " $> a$.
(f) The interval $[a, \infty)$ " $\geq a$.
(g) The interval $(-\infty, a)$ " $< a$.
(h) The interval $(-\infty, a]$ " $\leq a$.
(i) The interval $(-\infty, \infty)$ is the set of *all* real numbers. That is, $(-\infty, \infty) = \mathbb{R}$.

Given below are the *left* and *right endpoints* of the above nine intervals:

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**
Left endpoint	a	a	a	a	a	a	$-\infty$	$-\infty$	$-\infty$
Right endpoint	b	b	b	b	∞	∞	a	a	∞

By the way, we call

- (a, b) an **open interval**.
- $[a, b]$ a **closed interval**.
- $(a, b]$ a **half-closed interval**, a **half-open interval**, a **left-half-open interval**, or a **right-half-closed interval**.
- $[a, b)$ a **half-closed interval**, a **half-open interval**, a **left-half-closed interval**, or a **right-half-open interval**.

Exercise 42. Let $A = [1, 1]$, $B = (1, 1)$, $C = (1, 1]$, $D = [1, 1)$, and $E = (1, 1.01)$. Find the number of elements in each of these five sets. Express each set in another way. It turns out that each of the sets A, B, C, and D is called a *degenerate interval*, while E is called a *non-degenerate interval*. Can you write down some possible definitions of the two terms *degenerate interval* and *non-degenerate interval*? (Answer on p. 1755.)

Exercise 43. Express $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}_0^+$, $\mathbb{R}^-$, and $\mathbb{R}_0^-$ in interval notation. (Answer on p. 1755.)

Remark 11. Some writers (usually French-speakers) use **reverse bracket notation**: They write $]a, b[$, $]a, b]$, or $[a, b[$ instead of (a, b), $(a, b]$, or $[a, b)$. We will **not** do so in this textbook.[100]

[100] One good argument in favour of reverse bracket notation is that it avoids confusing the open interval (a, b) with the **ordered pair** (a, b) (we'll learn more about ordered pairs in Ch. 7). However, by and large, the reverse bracket notation remains uncommon, except in continental Europe and especially France (where it was introduced by the Bourbaki group).

4.14. Subset Of $\subseteq$

Definition 17. Let A and B be sets. If every element of A is an element of B, then we call A a *subset of* B and write

$$A \subseteq B.$$

Otherwise, we say that A is *not a subset of* B and write $A \nsubseteq B$.

Example 75. Let $M = \{1,2\}$, $N = \{1,2,3\}$, and $O = \{1,2,4,5\}$, and $P = \{3,2,1\}$. Then

M is a subset of N, O, and P	$M \subseteq N$, $M \subseteq O$, and $M \subseteq P$
N is a subset of P, but not of M or O	$N \subseteq P$, but $N \nsubseteq M$ and $N \nsubseteq O$
O is not a subset of M, N, or P	$O \nsubseteq M$, $O \nsubseteq N$, and $O \nsubseteq P$
P is a subset of N, but not of M or O	$P \subseteq N$, but $P \nsubseteq M$ and $P \nsubseteq O$

Observe that $N \subseteq P$ and $P \subseteq N$. Indeed, the sets $N = P$. We have the following fact:

Fact 8. *Two sets are equal* $\iff$ *They are subsets of each other.*

Proof. Two sets A and B are equal

$\iff$ Every element in A is in B and every element in B is in A (Definition 8)

$\iff$ A is a subset of B and B is a subset of A (Definition 17). □

Exercise 44. Consider $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. (Answer on p. 1755.)

(a) Is $\mathbb{Z}$ a subset of $\mathbb{Q}$ or $\mathbb{R}$?

(b) Is $\mathbb{Q}$ a subset of $\mathbb{Z}$ or $\mathbb{R}$?

(c) Is $\mathbb{R}$ a subset of $\mathbb{Z}$ or $\mathbb{Q}$?

Exercise 45. Explain whether this statement is true: (Answer on p. 1755.)

> "The set of current Singapore Prime Minister(s) is a subset of the set of current Singapore Minister(s)."

Exercise 46. Let A and B be sets. Explain whether each of the following statements is generally true. (If false, give a counterexample.) (Answer on p. 1755.)

(a) $A \subseteq B \implies A = B$.

(b) $B \subseteq A \implies A = B$.

(c) $A = B \implies A \subseteq B$.

(d) $A = B \implies B \subseteq A$.

(e) $A = B \iff A \subseteq B$.

(f) $A = B \iff B \subseteq A$.

4.15. Proper Subset Of $\subset$

Definition 18. Let A and B be sets. If $A \subseteq B$ but $A \neq B$, then we call A a *proper subset of B* and write

$$A \subset B.$$

Otherwise, we say that A is *not* a proper subset of B and write $A \not\subset B$.

Example 76. Let $M = \{1, 2\}$, $N = \{1, 2, 3\}$, and $O = \{1, 2, 4, 5\}$, and $P = \{3, 2, 1\}$. Then

M is a proper subset of N, O, and P	$M \subset N$, $M \subset O$, and $M \subset P$
N is not a proper subset of M, O, or P	$N \not\subset M$, $N \not\subset O$, $N \not\subset P$
O is not a proper subset of M, N, or P	$O \not\subset M$, $O \not\subset N$, $O \not\subset P$
P is not a proper subset of M, N, or O	$P \not\subset M$, $P \not\subset N$, $P \not\subset O$

Observe that $N = P$ so that by Definition 18, $N \not\subset P$ and $P \not\subset N$.

Remark 12. The A-Level syllabus (p. 16) uses the symbol $\subseteq$ to mean "subset of" and $\subset$ to mean "proper subset of". So this textbook will do the same.

However, confusingly enough, some writers use the symbol $\subset$ to mean "subset of" and $\subsetneq$ to mean "proper subset of". We will **not** follow such practice in this textbook. This is just so you know, in case you get confused when reading other mathematical texts!

Exercise 47. Let S be the set of all squares and R be the set of all rectangles. Is $S \subset R$? (Answer on p. 1756.)

Exercise 48. Does $A \subseteq B$ imply that $A \subset B$? (Answer on p. 1756.)

Exercise 49. Does $A \subset B$ imply that $A \subseteq B$? (Answer on p. 1756.)

Exercise 50. Explain whether this statement is true: "If A is a subset of B, then A is either a proper subset of or is equal to B." (Answer on p. 1756.)

Remark 13. If A is a (proper) subset of B, then we are very much tempted to say that A is "(strictly) smaller than" B—or equivalently, that B is "(strictly) larger than" A. Informally and intuitively, this is perhaps permissible. However, we shall

avoid speaking of one set as being "(strictly) smaller or larger than" (or "the same size as") another.

The reason is the "(proper) subset of" relation is merely one way by which we can compare the "sizes" of sets. There are at least two other ways, called **cardinality** and **measure**.

Example: Suppose $A = [0, 1]$ and $B = [0, 2]$. Then A is a proper subset of B, so that in this sense, we may consider A to be "strictly smaller than" B. However, it turns out that

- The cardinality of A is the same as that of B. So, in the sense of **cardinality**, A is "the same size as" B.
- The measure of A is strictly smaller than that of B. So, in the sense of **measure**, A is "strictly smaller than" B.

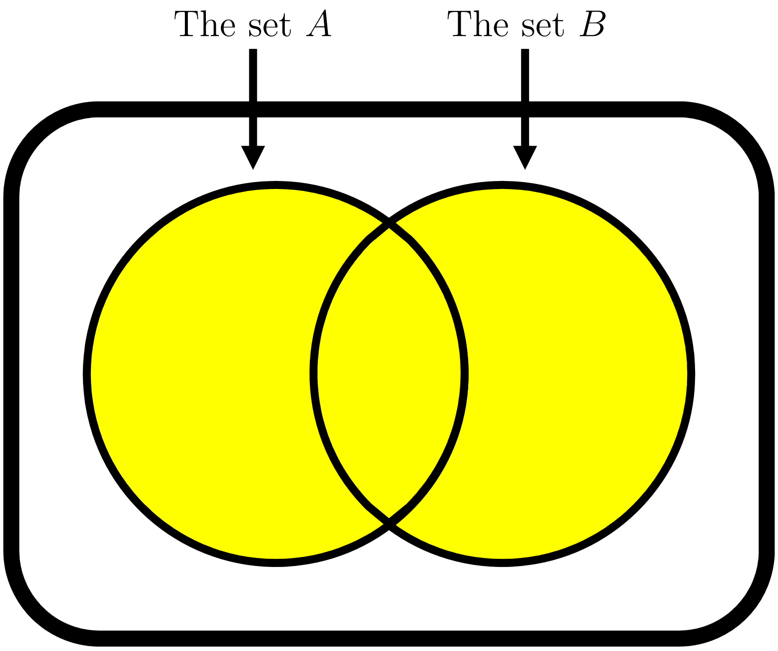
Happily, for A-Level Maths, you needn't know anything about the concepts of **cardinality** and **measure**. This remark is merely to explain to you why we avoid speaking of one set being "(strictly) smaller or larger than" (or "the same size as") another.

Example 77. Again, let $M = \{1, 2\}$ and $N = \{1, 2, 3\}$. So, $M \subset N$.

While tempting, we shall avoid saying that M is (strictly) smaller than N.

4.16. Union $\cup$

Definition 19. Let A and B be sets. The *union* of A and B, denoted $A \cup B$, is the set of elements that are in A or B.



$A \cup B$ is the yellow region.

Tip: "U" for Union.

Example 78. Let $T = \{1, 2\}$, $U = \{3, 4\}$, and $V = \{1, 2, 3\}$. Then

$$T \cup U = \{1, 2, 3, 4\},\ T \cup V = \{1, 2, 3\},\ U \cup V = \{1, 2, 3, 4\},\ T \cup U \cup V = \{1, 2, 3, 4\}.$$

Exercise 51. Rewrite each set more simply: (Answer on p. 1756.)

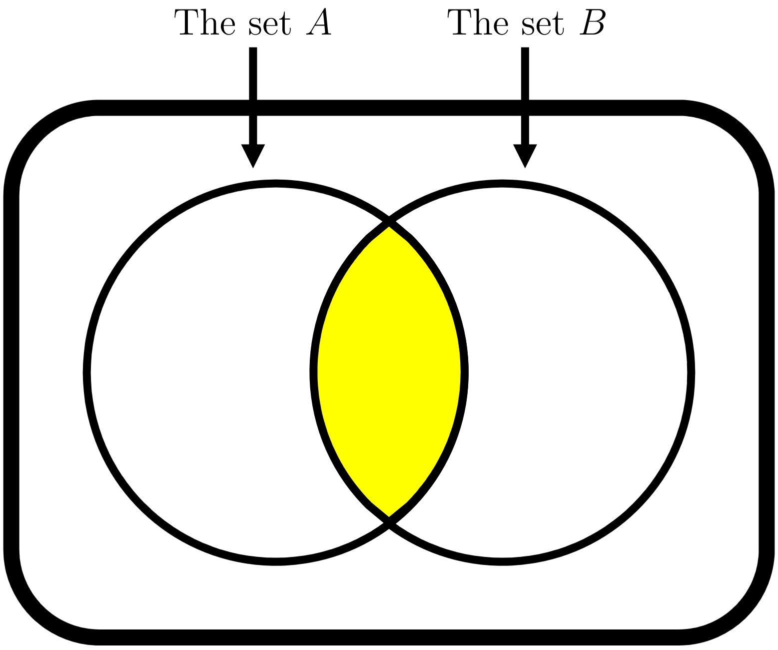
(a) $[1, 2] \cup [2, 3]$ **(b)** $(-\infty, -3) \cup [-16, 7)$ **(c)** $\{0\} \cup \mathbb{Z}^+$

Exercise 52. What is the union of each pair of sets? (Answer on p. 1756.)

(a) The set of squares S and the set of rectangles R.

(b) The set of rationals and the set of irrationals.

4.17. Intersection $\cap$



$A \cap B$ is the yellow region.

Definition 20. Let A and B be sets. The *intersection of A and B*, denoted $A \cap B$, is the set of elements that are in A and B. Two sets *intersect* if their intersection contains at least one element.

Equivalently, two sets A and B *intersect* if their intersection is non-empty, i.e.

$$A \cap B \neq \varnothing.$$

Definition 21. Two sets are *mutually exclusive* or *disjoint* if they do not intersect (i.e. their intersection is empty).

Example 79. Let $T = \{1,2\}$, $U = \{3,4\}$, and $V = \{1,2,3\}$. Then

$$T \cap U = \varnothing, \quad T \cap V = \{1,2\}, \quad U \cap V = \{3\}, \quad T \cap U \cap V = \varnothing.$$

The intersection of T and U is empty. Hence, T and U are mutually exclusive or disjoint.

The intersection of the three sets T, U, *and* V is also empty. However, the three sets T, U, *and* V are **not** mutually exclusive or disjoint:

Definition 22. Three or more sets are *mutually exclusive* or *disjoint* if every two of them are mutually exclusive or disjoint.

Example 80. Let $T = \{1, 2\}$, $U = \{3, 4\}$, and $W = \{5, 6\}$. Then

$$T \cap U \cap W \stackrel{1}{=} \varnothing.$$

Note though that $\stackrel{1}{=}$ (the above equation) does not suffice for declaring that T, U, and W are mutually exclusive.

Instead and as per Definition 22, T, U, and W are mutually exclusive or disjoint because *every two of them* are mutually exclusive or disjoint:

$$T \cap U = \varnothing, \quad T \cap W = \varnothing, \quad U \cap W = \varnothing.$$

Exercise 53. Rewrite each set more simply: (Answer on p. 1756.)

(a) $(4, 7] \cap (6, 9)$ **(b)** $[1, 2] \cap [5, 6]$ **(c)** $(-\infty, -3) \cap [-16, 7)$

Exercise 54. What is the intersection of each pair of sets?(Answer on p. 1756.)

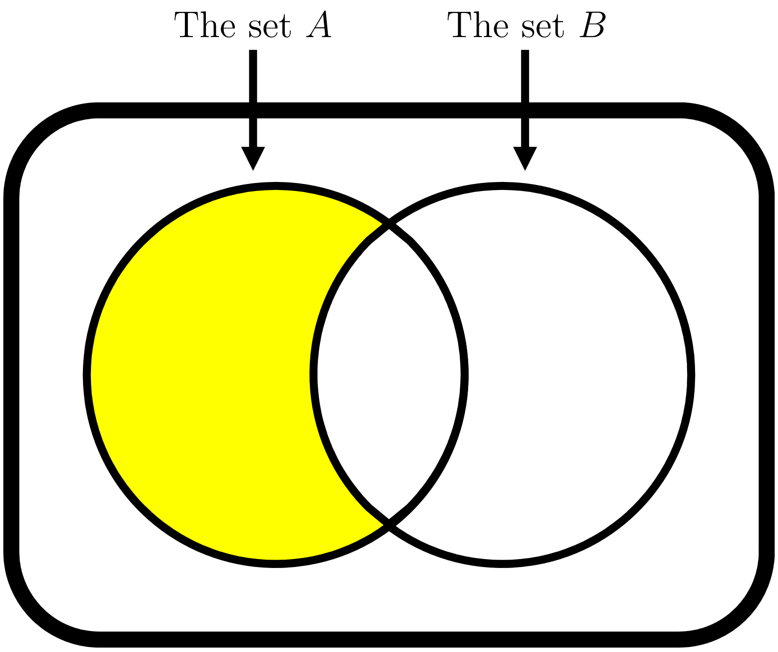
(a) The set of squares S and the set of rectangles R.

(b) The set of rationals and the set of irrationals.

4.18. Set Minus $\smallsetminus$

The **set minus** (sometimes also called **set difference**) sign $\smallsetminus$ is very convenient. Sadly, it does **not** appear in the A-Level syllabus. Nonetheless, it's worth a quick mention and I'll sometimes use it in this textbook.

Definition 23. Suppose A and B are sets. Then A *set minus* B, denoted $A \smallsetminus B$, is the set of elements that are in A and not in B



$A \smallsetminus B$ is the yellow region.

Example 81. Let $T = \{1, 2\}$, $U = \{3, 4\}$, and $V = \{1, 2, 3\}$. Then

$$T \smallsetminus U = T, \qquad T \smallsetminus V = \varnothing, \qquad U \smallsetminus V = \{4\}.$$

Exercise 55. In the above example, what are $V \smallsetminus T$ and $V \smallsetminus U$? (Answer on p. 1756.)

Examples of why the set minus notation is sometimes very convenient and helps us avoid ugly monstrosities:

Example 82. Consider the set of all real numbers except 1.

Without the set minus sign, we'd write this as $(-\infty, 1) \cup (1, \infty)$.

With the set minus sign, we can more simply write $\mathbb{R} \smallsetminus \{1\}$.

Example 83. Consider the set of all real numbers that aren't integers.

Without the set minus sign, we'd write this as $\cdots \cup (-3, -2) \cup (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3) \ldots$.

With the set minus sign, we can more simply write $\mathbb{R} \smallsetminus \mathbb{Z}$.

4.19. The Universal Set $\mathscr{E}$

The **universal set**, denoted $\mathscr{E}$ (that's a squiggly E), is the set of "all" elements.[101] Note though that what we mean by "*all*" elements depends on the context:

Example 84. In the context of a roll of a six-sided die, the universal set might be the set of all possible outcomes:

$$\mathscr{E} = \{1, 2, 3, 4, 5, 6\}.$$

Example 85. In the context of a spin of a European-style roulette wheel, the universal set might be is the set of all possible outcomes:

$$\mathscr{E} = \{0, 1, 2, 3, \ldots, 36\}.$$

Note that in American-style roulette, there is a 38th possible outcome—double zero 00. And so in the American context, the universal set might instead be:

$$\mathscr{E} = \{00, 0, 1, 2, 3, \ldots, 36\}.$$

(It looks like Marina Bay Sands does American-style roulette.)

Example 86. In the context of a game of chess, the universal set might be the set of all possible outcomes for White:

$$\mathscr{E} = \{\text{Win, Lose, Draw}\}.$$

Remark 14. I give the universal set notation $\mathscr{E}$ a quick mention here only because it appears on your A-Level syllabus (p. 16). We will rarely (if ever) make use of this piece of notation in this textbook.

Exercise 56. For each context, write down the universal set and its number of elements.

(a) 4D.

(b) Sex (as on a Singapore NRIC).

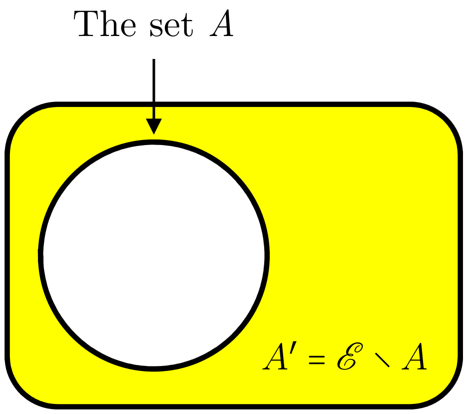
(c) Singapore's official language(s).

(d) Singapore's national language(s). (Answer on p. 1756.)

[101] This is not a precise definition. Indeed, there can be no single precise definition of the universal set since what this is depends on the context.
Note also that the "set of all objects" leads to a contradiction (see e.g. Russell's paradox) and thus does not exist.

4.20. The Set Complement A'

Definition 24. Let A be a set. The *set complement* of A, denoted A', is the set of all elements that are not in A.



Equivalently: $A' = \mathscr{E} \setminus A$. That is, A' is the universal set $\mathscr{E}$ minus those elements in A.

In the figure (right), A' is the yellow region.

To repeat, what the universal set $\mathscr{E}$ is (i.e. what we mean by "all" elements) depends on the context:

Example 87. Let $A = \{2, 3\}$. If the relevant context is the roll of a die, then

$$\mathscr{E} = \{1, 2, 3, 4, 5, 6\} \qquad \text{and} \qquad A' = \{1, 2, 3, 4, 5, 6\} \setminus A = \{1, 4, 5, 6\}.$$

But if instead the relevant context is a European-style roulette wheel spin, then

$$\mathscr{E} = \{0, 1, 2, \ldots, 36\} \qquad \text{and} \qquad A' = \{0, 1, 2, \ldots, 36\} \setminus A = \{0, 1, 4, 5, 6, \ldots, 36\}.$$

And if instead the relevant context is the positive integers smaller than 10, then

$$\mathscr{E} = \{1, 2, \ldots, 9\} \qquad \text{and} \qquad A' = \{1, 2, \ldots, 9\} \setminus A = \{1, 4, 5, 6, 7, 8, 9\}.$$

Example 88. Let $B = \{2, 4, 6, \ldots\}$. If the relevant context is the positive integers, then

$$\mathscr{E} = \mathbb{Z}^+ \qquad \text{and} \qquad B' = \mathbb{Z}^+ \setminus \{2, 4, 6, 8, \ldots\} = \{1, 3, 5, 7, \ldots\}.$$

That is, B' is simply the set of positive odd numbers.

But if instead the relevant context is the set of *all* integers, then

$$\mathscr{E} = \mathbb{Z} \qquad \text{and} \qquad B' = \mathbb{Z} \setminus \{2, 4, 6, 8, \ldots\} = \mathbb{Z}_0^- \cup \{1, 3, 5, 7, \ldots\}.$$

That is, B' is the set of positive odd numbers and all non-positive integers.

Example 89. Let $C = \mathbb{R}^+ = (0, \infty)$. If the relevant context is *all* real numbers, then

$$\mathscr{E} = \mathbb{R} \qquad \text{and} \qquad C' = \mathbb{R} \setminus C = \mathbb{R}_0^-.$$

That is, C' is simply the set of non-positive reals.

But if instead the relevant context is the set of reals greater than or equal to -1, then

$$\mathscr{E} = [-1, \infty) \qquad \text{and} \qquad C' = [-1\infty) \setminus C = [-1, 0].$$

That is, C' is the set of reals between -1 and 0 (inclusive).

Remark 15. Just so you know, instead of A', some writers write A^c or $\overline{A}$.

Exercise 57. For each given context and set S, write down S'. (Answer on p. 1756.)

(a) Context: 4D; $S = \{3000, 3001, 3002, \ldots, 3999\}$.

(b) Context: Temperature (in degrees Celsius); $S = (30, \infty)$.

4.21. De Morgan's Laws

It turns out there's a deep connection between logic and set theory. In particular,

The intersection $\cap$ corresponds to the logical connective AND (the conjunction).
The union $\cup$ corresponds to the logical connective OR (the disjunction).

Earlier, for logic, we had **De Morgan's Laws** (Facts 1 and 2)

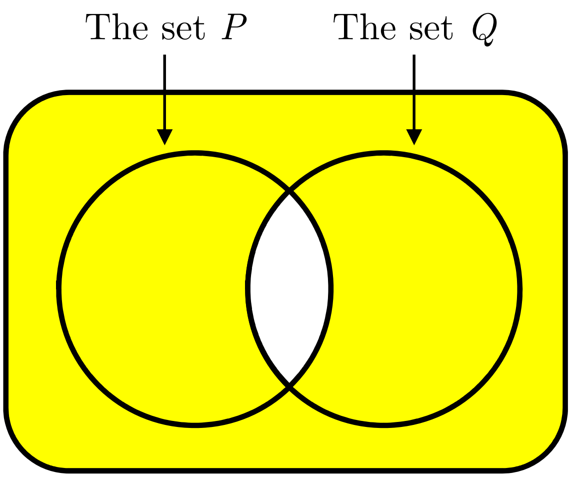
(a) The negation of the conjunction P AND Q is the disjunction NOT−P OR NOT−Q.

(b) The negation of the disjunction P OR Q is the conjunction NOT−P AND NOT−Q.

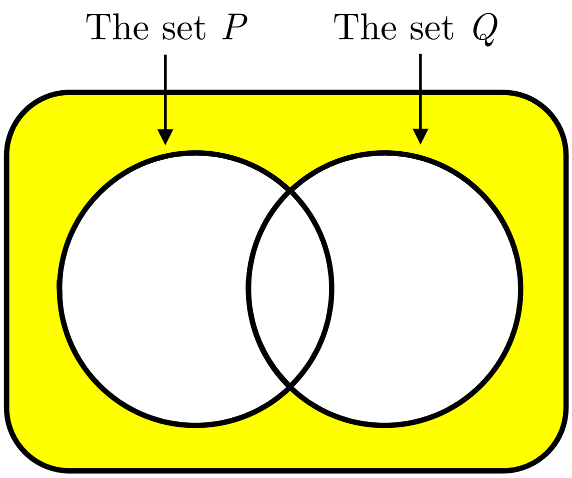
Correspondingly, for set theory, we also have these **De Morgan's Laws**:

(a) The complement of the intersection $P \cap Q$ is the union $P' \cup Q'$.

(b) The complement of the union $P \cup Q$ is the intersection $P' \cap Q'$.



$(P \cap Q)' = P' \cup Q'$ is in yellow.



$(P \cup Q)' = P' \cap Q'$ is in yellow.

For future reference, let's jot this down as a formal result:

Fact 9. *Suppose P and Q are sets. Then*

(a) $(P \cap Q)' = P' \cup Q'$ *and* **(b)** $(P \cup Q)' = P' \cap Q'$.

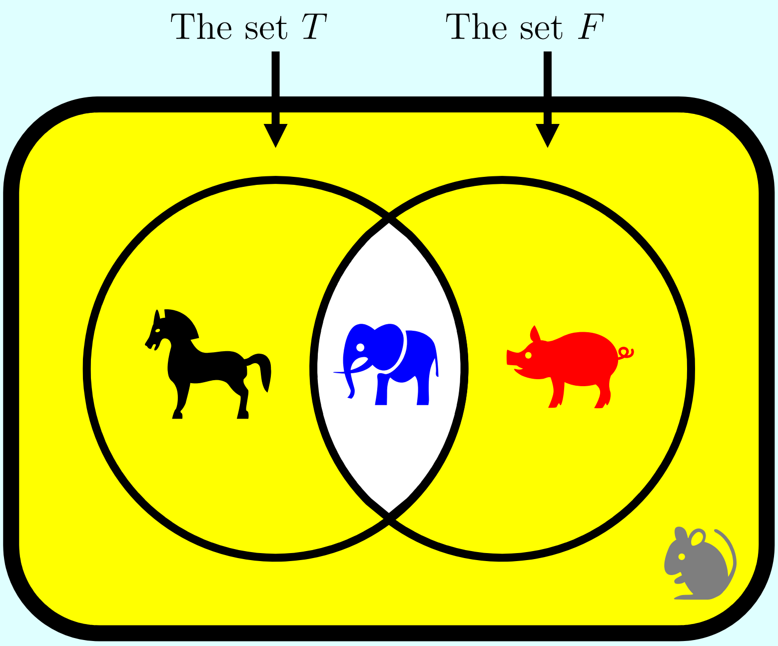
Proof. See p. 1553 (Appendices). □

Example 90. Suppose that every animal is **(i)** either tall or short; and **(ii)** either fat or lean. So for example, the **horse** is tall and lean, the **elephant** is tall and fat, the **pig** is short and fat, and the **mouse** is short and lean.

Let T be the set of tall animals and F be the set of fat animals.

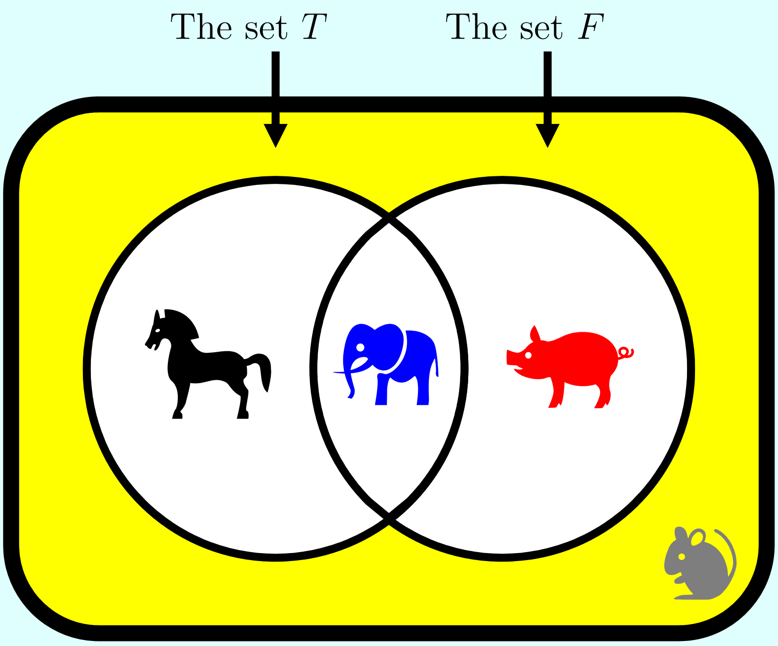
Consider $T \cap F$, the set of animals that are tall AND fat. Its complement is the set of animals that are short OR lean (yellow region below):

$$(T \cap F)' = T' \cup F'.$$



Consider $T \cup F$, the set of animals that are tall OR fat. Its complement is the set of animals that are short AND lean (yellow region below):

$$(T \cup F)' = T' \cap F'.$$



4.22. Set-Builder Notation (or Set Comprehension)

Previously, we simply wrote out a set using the method of **set enumeration**. That is, we simply **enumerated** (i.e. listed out) all their elements:

Example 91. The set of Singapore PMs (both past and present) is

$$S = \{\text{Lee Kuan Yew, Goh Chok Tong, Lee Hsien Loong}\}.$$

Where a set has too many elements to list out, we can use the ellipsis "…". But this too counts as the method of set enumeration:

Example 92. The set of integers greater than 100 is

$$T = \{101, 102, 103, 104, \ldots\}.$$

We now introduce a second method of writing out a set, called **set-builder notation** or **set comprehension**:

Example 93. The set of Singapore PMs (both past and present) is

$$S = \{x : x \text{ has ever been the Singapore PM}\}.$$

In set-builder notation, the mathematical punctuation mark *colon* ":" means **such that**. Following the colon is the property or criterion that x must satisfy in order to be an element of the set. Hence, S is

"the set of all objects x **such that** x has ever been the Singapore PM".

Note that the letter or symbol x is simply a **dummy** or **placeholder variable**. We could have replaced x with any other letter or indeed any other symbol and the set would have been unchanged. For example, we could've replaced x with the letter y:

$$S = \{y : y \text{ has ever been the Singapore PM}\}.$$

Or even a smiley face ☺:

$$S = \{☺ : ☺ \text{ has ever been the Singapore PM}\}.$$

Remark 16. A **dummy variable** is also called a **placeholder**, **dead**, or **bound variable**.

Example 94. Let $T = \{x : x \in \mathbb{Z}, x > 100\}$. (Recall that the comma "," means AND.) Then T is

the set of all objects x such that x is an integer AND $x > 100$.

Observe that this time, following the colon are *two* properties or criteria that x must satisfy in order to be an element of the set.

We could also have written T as "the set of all integers x such that $x > 100$":

$$T = \{x \in \mathbb{Z} : x > 100\}.$$

Remark 17. In maths (as in natural language), there are often many ways of saying the same thing. Here we've written the set of Singapore PMs (past and present) in four different ways and the set of integers greater than 100 in three different ways.

So, if there are many ways to say the same thing, then how should we say it? Well, in maths (as in natural language), you should always strive to express yourself as clearly and simply as possible. This will require using your judgment, experience, and wisdom.

Example 95. Let S be the set of ASEAN members.

Using **set enumeration**, we'd write

$$S = \{\text{Brunei, Cambodia, Indonesia, Laos, Malaysia, Myanmar,}$$
$$\text{the Philippines, Singapore, Thailand, Vietnam}\}.$$

Using **set-builder notation** or **set comprehension**, we'd write

$$S = \{x : x \text{ is a member of ASEAN}\}.$$

This is "the set of all objects x such that x is a member of ASEAN".

Example 96. Let Familee be the set of individuals who have ever been members of Singapore's Royal or First Family. Consider

$$A = \{x : x \text{ has ever been the Singapore PM, } x \notin \text{Familee}\}.$$

In words, A is "the set of all objects x such that x has ever been the Singapore PM AND x has never been a member of the Familee". And so,

$$A = \{\text{Goh Chok Tong}\}.$$

Example 97. Let $B = \{x : x \text{ is a member of ASEAN}, x \text{ has fewer than } 5,000 \text{ islands}\}$.

In words, B is "the set of all objects x such that x is a member of ASEAN AND x has fewer than $5,000$ islands". So, using set enumeration, we'd write

$$B = \{\text{Brunei, Cambodia, Laos, Malaysia, Myanmar, Singapore, Thailand, Vietnam}\}.$$

Exercise 58. Rewrite the sets S and T using set enumeration. (Answer on p. 1756.)

$$S = \{x : x \text{ is a child of Lee Kuan Yew}\},$$
$$T = \{x : x \text{ is a child of Lee Kuan Yew, } x \text{ has never been the Singapore PM}\}.$$

Exercise 59. What is the following set? (Answer on p. 1756.)

$$\{x : x \text{ has ever been the Singapore PM}\} \cap \text{Familee}.$$

We've used set-builder notation to describe sets of prime ministers or countries. But more commonly, we'll use it to describe sets of numbers:

Example 98. Let $A = \left\{x : x^2 - 1 = 0\right\}$.

In words, A is "the set of all objects x such that $x^2 - 1 = 0$".

Hence, $$A = \{-1, 1\}.$$

Example 99. Let $B = \left\{x : x^2 - 1 = 0, x > 0\right\}$.

In words, B is "the set of all objects x such that $x^2 - 1 = 0$ and $x > 0$".

Hence, $$B = \{1\}.$$

We could also have written, $$B = \left\{x \in \mathbb{R}^+ : x^2 - 1 = 0\right\}.$$

This says that B is "the set of all positive real numbers x such that $x^2 - 1 = 0$".

Example 100. Consider $C = \{x \in \mathbb{Z} : 3.5 < x < 5.5\}$.

In words, C is "the set of all integers x such that $3.5 < x < 5.5$.

Hence, $$C = \{4, 5\}.$$

Example 101. The set of positive reals may be written

$$\mathbb{R}^+ = (0, \infty) = \{x \in \mathbb{R} : x > 0\}.$$

In words, this is "the set of all reals x such that x is greater than 0".

Similarly, the set of non-negative reals may be written as

$$\mathbb{R}_0^+ = [0, \infty) = \{x \in \mathbb{R} : x \geq 0\}.$$

In words, this is "the set of all reals x such that x is greater than or equal to 0".

Similarly, we may write

$$\begin{aligned} \mathbb{Q}^+ &= \{x \in \mathbb{Q} : x > 0\}, \qquad & \mathbb{Z}^+ &= \{x \in \mathbb{Z} : x > 0\}, \\ \mathbb{Q}_0^+ &= \{x \in \mathbb{Q} : x \geq 0\}, \qquad & \mathbb{Z}_0^+ &= \{x \in \mathbb{Z} : x \geq 0\}. \end{aligned}$$

Exercise 60. Write out, in words, what $\mathbb{Q}^+$, $\mathbb{Q}_0^+$, $\mathbb{Z}^+$, and $\mathbb{Z}_0^+$ are. (Answer on p. 1756.)

Example 102. Let S be the set of positive even numbers. Then we may write

$$S = \{2, 4, 6, 8, \dots\} = \{x : x = 2k, k \in \mathbb{Z}^+\}.$$

In words, S is "the set of all objects x such that x equals $2k$ AND k is a positive integer". Notice we've introduced a *second* **dummy variable** k to help us describe the set.

Again, both symbols x and k could've been replaced by any other symbols and we'd still have the same set. For example,

$$\begin{aligned} S = \{2, 4, 6, 8, \dots\} &= \{p : p = 2q, q \in \mathbb{Z}^+\} \\ &= \{\star : \star = 2\blacklozenge, \blacklozenge \in \mathbb{Z}^+\} \\ &= \{\smiley : \smiley = 2\blacksquare, \blacksquare \in \mathbb{Z}^+\}. \end{aligned}$$

(Of course, letters like x, k, p, and q are customary and hence preferable to weird symbols like shapes and faces.)

Here's another way to write down S:

$$S = \{2, 4, 6, 8, \dots\} = \{x : x/2 \in \mathbb{Z}^+\}.$$

In words, S is "the set of all elements x such that x divided by 2 is a positive integer".

And here's another:

$$S = \{2, 4, 6, 8, \dots\} = \{2k : k \in \mathbb{Z}^+\}.$$

In words, S is "the set of all elements $2k$ such that k is a positive integer".

Exercise 61. Let $a \le b$. Rewrite each set in set-builder notation. (Answer on p. 1757.)

(a)	$\mathbb{R}^-$	**(b)**	$\mathbb{Q}^-$	**(c)**	$\mathbb{Z}^-$	**(d)**	$\mathbb{R}_0^-$
(e)	$\mathbb{Q}_0^-$	**(f)**	$\mathbb{Z}_0^-$	**(g)**	(a, b)	**(h)**	$[a, b]$
(i)	$(a, b]$	**(j)**	$[a, b)$	**(k)**	$(-\infty, -3) \cup (5, \infty)$		

(l) $(-\infty, 1] \cup (2, 3) \cup (3, \infty)$ **(m)** $(-\infty, 3) \cap (0, 7)$

(n) The set of negative even numbers

(o) The set of positive odd numbers

(p) The set of negative odd numbers.

(q) $\{\pi, 4\pi, 7\pi, 10\pi, \dots\}$

(r) $\{-2\pi, \pi, 4\pi, 7\pi, 10\pi, \dots\}$.

Remark 18. Following the A-Level syllabus (p. 16), in set-builder notation, we use the colon ":" to mean **such that**. Note though that some writers use the pipe "|" instead.

4.23. Chapter Summary

- Informally, a **set** is a container or box whose objects we call its **elements**.

  Whenever we talk about a set, we refer to **both** the container and the objects inside.
- $\in$ means **in** and $\notin$ means **not in**.
- The ellipsis "…" means **continue in the obvious fashion**.
- The **order of the elements doesn't matter** and **repeated elements don't count**:

$$\{1,2,3\} = \{3,3,2,1,3,2,1,1,1,1,2,3\}.$$

- $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Q}$ are the sets of **reals**, **integers**, and **rationals**.[102]
- $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_0^+$, and $\mathbb{R}_0^-$ are the sets of positive reals, negative reals, non-negative reals, and non-positive reals. Similarly, we have

$$\mathbb{Q}^+, \mathbb{Q}^-, \mathbb{Q}_0^+, \text{ and } \mathbb{Q}_0^-; \qquad \text{and} \qquad \mathbb{Z}^+, \mathbb{Z}^-, \mathbb{Z}_0^+, \text{ and } \mathbb{Z}_0^-.$$

- The **empty set** $\{\} = \varnothing$ is the set with zero elements.
- **Interval notation**. Let $a < b$. Then

  1. (a,b) is the set of reals between a (excluded) and b (excluded).
  2. $[a,b]$ is the set of reals between a (included) and b (included).
  3. $(a,b]$ is the set of reals between a (excluded) and b (included).
  4. $[a,b)$ is the set of reals between a (included) and b (included).

- A is a **subset** of B—denoted $A \subseteq B$—if every element in A is also in B.
- A is a **proper subset** of B—denoted $A \subset B$—if $A \subseteq B$ and $A \neq B$.
- $A \cup B$, the **union** of A and B, is the set of elements that are in A OR B.
- $A \cap B$, the **intersection** of A and B, is the set of elements that are in A AND B.
- $A \setminus B$ (or $A - B$), A **set minus** B, is the set of elements that are in A AND **not** in B.
- The **universal set** $\mathscr{E}$ is the set of "all" elements (where "all" depends on the context).
- The **set complement** A' is the set of elements that are not in A.
- **De Morgan's Laws**: $(P \cap Q)' = P' \cup Q'$ and $(P \cup Q)' = P' \cap Q'$.
- **Set-builder notation** (or **set comprehension**). The set of objects satisfying the property P is denoted $\{x : P(x)\}$.

Exercise 62. Write each set more simply. (Answer on p. 1757.)

(a) $\mathbb{R} \setminus \mathbb{R}^+$.

(b) $\mathbb{R} \setminus (\mathbb{Q} \cup \mathbb{Z})$.

(c) $[1,6] \setminus ((3,5) \cap (1,4))$.

(d) $\{1,5,9,13,\dots\} \cap \{2,4,6,8,\dots\}$.

(e) $\{2,5,8,11,\dots\} \cap \{2,4,6,8,\dots\}$.

(f) $(0,5] \cap ([1,8] \cap [5,9))'$.

[102] Not on your syllabus: $\mathbb{N}$ is the set of **natural numbers**.

5. O-Level Review

5.1. Some Mathematical Vocabulary

Example 103. Consider the **equation** $1 + 4 = 2 + 3$.

$$\overbrace{\underset{\substack{\downarrow\\ \text{Term}}}{1} \quad + \quad \underset{\substack{\downarrow\\ \text{Term}}}{4}}^{\text{LHS expression}} \quad = \quad \overbrace{\underset{\substack{\downarrow\\ \text{Term}}}{2} \quad + \quad \underset{\substack{\downarrow\\ \text{Term}}}{3}}^{\text{RHS expression}}$$

The **expression** on the equation's **left-hand-side (LHS)** is $1 + 4$ and contains the two **terms** 1 and 4.

The expression on the **right-hand-side (RHS)** is $2 + 3$ and contains the two terms 2 and 3.

Informally, the difference between an **equation** and an **expression** is this:

An equation contains an equals sign; an expression does not.

Example 104. Consider the equation $y = 5x + 6$.

$$\overset{\substack{\text{Variable}\\ \uparrow}}{y} \quad = \quad \underset{\text{Coefficient} \swarrow \quad \searrow \text{Variable}}{5x} \quad + \quad \overset{\substack{\text{Constant term}\\ \uparrow}}{6}$$

This equation contains two **variables**: x and y.

The **coefficient** on x is 5. The coefficient on y is 1.

The **constant term** or more simply **constant** is 6. (The constant term is simply any real number[103] that does not involve any variables.)

Example 105. Consider the **inequality** $8x^2 + 4x + 3 > y - 2$.

$$\underset{\substack{\downarrow\\ \text{Coefficient}}}{8x^2} \quad + \quad \underset{\swarrow \quad \searrow \text{Variable}}{4x} \quad + \quad \overset{\text{Constant term}}{3 \nearrow} \quad > \quad \underset{\swarrow}{y} \quad - \quad \overset{\nwarrow}{2}$$

This inequality contains two variables: x and y.

The coefficients on x^2 and x are 8 and 4. The coefficient on y is 1.

The constant terms on the LHS is 3. The constant terms on the RHS -2.

[103]When we study complex numbers in Part IV, constants will also include complex numbers.

5.2. The Absolute Value or Modulus Function

Definition 25. The *absolute value* (or *modulus*) *function*, denoted $|\cdot|$, is defined for all $x \in \mathbb{R}$ by

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

Example 106. $|5| = 5$, $|-5| = 5$, $|0| = 0$.

Each of the following three equations is **not** generally true.

$$\left|\frac{a}{b}\right| = \frac{|a|}{b}. \text{ ✗} \qquad \left|\frac{a}{b}\right| = \frac{a}{|b|}. \text{ ✗} \qquad \left|\frac{a}{b}\right| = \frac{a}{b}. \text{ ✗}$$

Example 107. $\left|\frac{6}{-2}\right| \neq \frac{|6|}{-2}$ because $\left|\frac{6}{-2}\right| = |3| = 3$ but $\frac{|6|}{-2} = \frac{6}{-2} = -3$.

Example 108. $\left|\frac{-6}{2}\right| \neq \frac{-6}{|2|}$ because $\left|\frac{-6}{2}\right| = |3| = 3$ but $\frac{-6}{|2|} = \frac{-6}{2} = -3$.

Example 109. $\left|\frac{-6}{2}\right| \neq \frac{-6}{2}$ because $\left|\frac{-6}{2}\right| = |3| = 3$ but $\frac{-6}{2} = -3$.

What's true is this:

Fact 10. *If $a, b \in \mathbb{R}$ with $b \neq 0$, then* $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

Proof. If $a = 0$, then $\left|\frac{a}{b}\right| = |0| = 0 = \frac{0}{|b|} = \frac{|0|}{|b|} = \frac{|a|}{|b|}$.

Now suppose $a \neq 0$.

Case 1. a and b have the same signs.

Then either $|a|/|b| = a/b$ or $|a|/|b| = (-a)/(-b) = a/b$. Moreover, $a/b > 0$ so that $|a/b| = a/b$.

Case 2. a and b have opposite signs.

Then either $|a|/|b| = (-a)/b$ or $|a|/|b| = a/(-b)$. Moreover, $a/b < 0$ so that $|a/b| = -a/b$. □

If the above proof doesn't convince you, hopefully the following examples do:

Example 110. $\left|\frac{6}{2}\right| = \frac{|6|}{|2|}$ because $\left|\frac{6}{2}\right| = |3| = 3$ and $\frac{|6|}{|2|} = \frac{6}{2} = 3$.

Example 111. $\left|\frac{-6}{2}\right| = \frac{|-6|}{|2|}$ because $\left|\frac{-6}{2}\right| = |-3| = 3$ and $\frac{|-6|}{|2|} = \frac{6}{2} = 3$.

Example 112. $\left|\frac{6}{-2}\right| = \frac{|6|}{|-2|}$ because $\left|\frac{6}{-2}\right| = |-3| = 3$ and $\frac{|6|}{|-2|} = \frac{6}{2} = 3$.

Example 113. $\left|\frac{-6}{-2}\right| = \frac{|-6|}{|-2|}$ because $\left|\frac{-6}{-2}\right| = |3| = 3$ and $\frac{|-6|}{|-2|} = \frac{6}{2} = 3$.

We also have this similar result:

Fact 11. *If $a, b \in \mathbb{R}$, then $|ab| = |a|\,|b|$.*

Proof. If $a, b \geq 0$, then $|ab| = ab = |a|\,|b|$. ✓ If $a, b < 0$, then $|ab| = ab = |a|\,|b|$. ✓ If $a \geq 0$ and $b < 0$, then $|ab| = -ab = |a|\,|b|$. ✓ If $a < 0$ and $b \geq 0$, then $|ab| = -ab = |a|\,|b|$. ✓ □

Example 114. $|6 \times 2| = |6|\,|2|$ because $|6 \times 2| = |12| = 12$ and $|6|\,|2| = 6 \times 2 = 12$.

Example 115. $|-6 \times 2| = |-6|\,|2|$ because $|-6 \times 2| = |-12| = 12$ and $|-6|\,|2| = 6 \times 2 = 12$.

Example 116. $|6 \times (-2)| = |6|\,|-2|$ because $|6 \times (-2)| = |-12| = 12$ and $|6|\,|-2| = 6 \times 2 = 12$.

Example 117. $|(-6)(-2)| = |-6|\,|-2|$ because $|(-6)(-2)| = |12| = 12$ and $|-6|\,|-2| = 6 \times 2 = 12$.

5.3. The Factorial $n!$

Definition 26. Let $n \in \mathbb{Z}_0^+$. Then *n-factorial*, denoted $n!$, is the number defined by

$$n! = \begin{cases} 1, & \text{for } n = 0, \\ 1 \times 2 \times \cdots \times n, & \text{for } n > 0. \end{cases}$$

Or equivalently,[104]

$$n! = \begin{cases} 1, & \text{for } n = 0, \\ (n-1)! \times n, & \text{for } n > 0. \end{cases}$$

And so,

$$\begin{aligned} 0! &= 1, \\ 1! &= 0! \times 1 = 1, \\ 2! &= 1! \times 2 = 1 \times 2, \\ 3! &= 2! \times 3 = 1 \times 2 \times 3, \\ 4! &= 3! \times 4 = 1 \times 2 \times 3 \times 4, \\ 5! &= 4! \times 5 = 1 \times 2 \times 3 \times 4 \times 5, \\ 6! &= 5! \times 6 = 1 \times 2 \times 3 \times \cdots \times 6, \\ &\vdots \end{aligned}$$

You may be wondering, "Why is 0! defined as 1?" It turns out this is the definition that causes us the least overall inconvenience.[105]

[104]This latter equivalent definition is an example of a **recursive definition**.
[105]For example, with the Maclaurin series (Ch. 102) and combinatorics (Part VI).

5.4. Exponents

Definition 27. Suppose $b \neq 0$ and $x \in \mathbb{Z}$. Then b *to the power of* x, denoted b^x, is the number defined by

$$b^x = \begin{cases} \underbrace{b \cdot b \cdot \cdots \cdot b}_{x \text{ times}}, & \text{for } x > 0, \\ 1, & \text{for } x = 0, \\ \dfrac{1}{b^{|x|}}, & \text{for } x < 0. \end{cases}$$

Given the number b^x, we call b its *base* and x its *exponent*.

Example 118. $2^1 = 2$, $2^2 = 2 \cdot 2 = 4$, $2^3 = 2 \cdot 2 \cdot 2 = 8$, $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$,
$2^{20} = \underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{20 \text{ times}} = 1\,048\,576$.

Example 119. $2^{-1} = \dfrac{1}{2} = 0.5$, $2^{-2} = \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{4} = 0.25$, $2^{-3} = \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{8} = 0.125$,
$2^{-4} = \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{16} = 0.0625$,

$$2^{-20} = \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \cdots \cdot \frac{1}{2}}_{20 \text{ times}} = \frac{1}{1\,048\,576} = 0.000\,000\,953\,674\,316\,406\,25.$$

Example 120. $\left(\dfrac{1}{2}\right)^1 = \dfrac{1}{2}$, $\left(\dfrac{1}{2}\right)^2 = \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{4}$, $\left(\dfrac{1}{2}\right)^3 = \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{8}$, $\left(\dfrac{1}{2}\right)^4 = \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{16}$,
$\left(\dfrac{1}{2}\right)^{20} = \underbrace{\dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \cdots \cdot \dfrac{1}{2}}_{20 \text{ times}} = \dfrac{1}{1\,048\,576}$.

Example 121. $\left(\dfrac{1}{2}\right)^{-1} = 2$, $\left(\dfrac{1}{2}\right)^{-2} = 2 \cdot 2 = 4$, $\left(\dfrac{1}{2}\right)^{-3} = 2 \cdot 2 \cdot 2 = 8$, $\left(\dfrac{1}{2}\right)^{-4} = 2 \cdot 2 \cdot 2 \cdot 2 = 16$,
$\left(\dfrac{1}{2}\right)^{-20} = \underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{20 \text{ times}} = 1\,048\,576$.

Example 122. $2^0 = 1$, $\left(\dfrac{1}{2}\right)^0 = 1$.

Note well that Definition 27 covers **only** the case where the base b is **non-zero** and the exponent x is an **integer**.

We now cover also the case where the base b is 0 and the exponent x is *any* real number:

Definition 28. Suppose $x \in \mathbb{R}$. Then 0 *to the power of* x, denoted 0^x, is the number defined by

$$0^x = \begin{cases} 0, & \text{for } x > 0, \\ 1, & \text{for } x = 0, \\ \text{Undefined}, & \text{for } x < 0. \end{cases}$$

Example 123. $0^3 = 0$, $0^\pi = 0$, $0^{100} = 0$.

0^{-3}, $0^{-\pi}$, and 0^{-100} are undefined.

$0^0 = 1$. There's actually nothing "obvious" or "natural" about defining $0^0 = 1$. Similar to $0! = 1$, this is merely the definition that will cause us the least inconvenience.[106]

Next, we'll define **roots**. But to do so, we'll need to make use of this result:

Theorem 1. *Suppose $b > 0$ and x is a non-zero integer. Then there exists a unique positive real number a such that*

$$a^x = b.$$

Proof. Omitted.[107] □

Example 124. Let $b = 18.73$ and $x = 39$. Then Theorem 1 says that there exists $a > 0$ such that

$$a^{39} = 18.73.$$

Definition 29. Let $b > 0$ and x be a non-zero integer. By Theorem 1, there exists a unique positive real number a such that

$$a^x = b.$$

We call a the *xth root of* b and write $a = \sqrt[x]{b}$.

We'll also say a is b *to the power of* $1/x$ and write $a = b^{1/x}$.

Remark 19. To be clear, $\sqrt[x]{b}$ and $b^{1/x}$ are simply two different ways to write the exact same thing.

[106]One convenience this definition affords is this: For all $b \in \mathbb{R}$, we simply have $b^0 = 1$.
Some writers argue that we should simply leave 0^0 undefined.
But I think $0^0 = 1$ is the definition that will cause us the least inconvenience in A-Level Maths.

[107]See e.g. Rudin (*Principles of Mathematical Analysis*, 1976, p. 10, Theorem 1.21). Note that our Theorem 1 also allows for the possibility that the exponent x is a negative integer.

Example 125. By Definition 29,

$$4^{\frac{1}{2}} = \sqrt[2]{4} = \sqrt{4} = 2, \qquad \left(\frac{1}{4}\right)^{\frac{1}{2}} = 0.25^{\frac{1}{2}} = \sqrt[2]{\frac{1}{4}} = \sqrt[2]{0.25} = \sqrt{0.25} = 0.5 = \frac{1}{2},$$

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2, \qquad \left(\frac{1}{8}\right)^{\frac{1}{3}} = 0.125^{\frac{1}{3}} = \sqrt[3]{\frac{1}{8}} = \sqrt[3]{0.125} = 0.5 = \frac{1}{2},$$

$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2, \qquad \left(\frac{1}{16}\right)^{\frac{1}{4}} = 0.0625^{\frac{1}{4}} = \sqrt[3]{\frac{1}{8}} = \sqrt[4]{0.0625} = 0.5 = \frac{1}{2}.$$

Also, $$1\,048\,576^{\frac{1}{20}} = \sqrt[20]{1\,048\,576} = 2.$$

And, $$\left(\frac{1}{1\,048\,576}\right)^{\frac{1}{20}} = 0.000\,000\,953\,674\,316\,406\,25^{\frac{1}{20}} = \sqrt[20]{\frac{1}{1\,048\,576}}$$
$$= \sqrt[20]{0.000\,000\,953\,674\,316\,406\,25} = 0.5 = \frac{1}{2}.$$

Definition 29 requires that $b > 0$. If $b < 0$, then the definition doesn't apply:

Example 126. There is no positive number (or indeed any real number) a such that

$$a^2 = -1.$$

So, for now, we'll leave $-1^{1/2} = \sqrt{-1}$ undefined.

(Spoiler: Later on in Part IV (Complex Numbers), we will define $\mathrm{i} = -1^{1/2} = \sqrt{-1}$.)

Definition 29 also requires that $x \neq 0$. If $x = 0$, then $\sqrt[0]{b}$ or $b^{\frac{1}{0}}$ is simply **undefined**.

For some special cases, we have these customary notation and terminology:

- If $x = 1$, we won't write $a = \sqrt[1]{b}$. Instead, we'll simply write $a = b$.
- If $x = 2$, we won't write $a = \sqrt[2]{b}$. Instead, we'll simply write $a = \sqrt{b}$. Moreover, we'll usually call it the **square root** of b.
- Similarly, we'll usually call $a = \sqrt[3]{b}$ the **cube root** of b.

We next extend our definition of b^x to the case where $b > 0$ and x is *any* rational number:

Definition 30. Let $b > 0$ and $x \in \mathbb{Q}$. Suppose $x = m/n$ for some integers m and $n \neq 0$. Then *b to the power of x* is denoted b^x and is the number defined as

$$b^x = \left(b^{\frac{1}{n}}\right)^m.$$

So, $$b^x = b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{b}\right)^m.$$

Example 127. By Definition 30,

$$\begin{aligned}
2^{1.5} = 2^{\frac{3}{2}} &= \left(\sqrt{2}\right)^3 &&= (1.414\dots)^3 &&= 2.828\dots,\\
2^{13.83} = 2^{\frac{1\,383}{100}} &= \left(\sqrt[100]{2}\right)^{1383} &&= (1.007\dots)^{1\,383} &&= 14\,562.798\dots,\\
2^{2.\overline{3}} = 2^{\frac{7}{3}} &= \left(\sqrt[3]{2}\right)^7 &&= (1.259\dots)^7 &&= 5.039\dots.
\end{aligned}$$

Proposition 1. (Laws of Exponents) *Suppose $a, b > 0$ and $x, y \in \mathbb{R}$. Then*

(a) $b^x b^y = b^{x+y}$. **(b)** $b^{-x} = \dfrac{1}{b^x}$. **(c)** $\dfrac{b^x}{b^y} = b^{x-y}$. **(d)** $(b^x)^y = b^{xy}$. **(e)** $(ab)^x = a^x b^x$.

Remark 20. Note well the requirement in Proposition 1 that the base b be positive. Otherwise we can "prove" that $-1 = 1$:

$$-1 = -1^1 = (-1)^{2\times\frac{1}{2}} \stackrel{\star}{=} \left[(-1)^2\right]^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1.$$

At $\stackrel{\star}{=}$, we applied Proposition 1(d). But this is an error because the base -1 is not positive, so that Proposition 1 cannot be applied.

Our partial proof below covers only the case where $x, y \in \mathbb{Z}^+$ (exponents are positive integers). (This partial proof can easily be extended to cover also the case where x and/or y are non-positive integers.)

For a more complete proof covering also the case where x and y are *any* real numbers, see p. 1726 (Appendices).

Proof (partial). Suppose $x, y \in \mathbb{Z}^+$. Below, $\stackrel{\star}{=}$ will denote the use of Definition 27.

(a) $b^x b^y \stackrel{\star}{=} \underbrace{b \cdot b \cdot \dots \cdot b}_{x \text{ times}} \times \underbrace{b \cdot b \cdot \dots \cdot b}_{y \text{ times}} \stackrel{\star}{=} \underbrace{b \cdot b \cdot \dots \cdot b}_{x+y \text{ times}} = b^{x+y}$.

(b) Since $x \geq 0$, we have $|-x| = x$ and hence $b^{-x} \stackrel{\star}{=} \dfrac{1}{b^{|-x|}} = \dfrac{1}{b^x}$.

(c) If $x \geq y$ so that $x - y \geq 0$, then

$$b^{x-y} \stackrel{\star}{=} \underbrace{b \cdot b \cdot \dots \cdot b}_{x-y \text{ times}} = \underbrace{b \cdot b \cdot \dots \cdot b}_{x-y \text{ times}} \times \frac{\overbrace{b \cdot b \cdot \dots \cdot b}^{y \text{ times}}}{\underbrace{b \cdot b \cdot \dots \cdot b}_{y \text{ times}}} \stackrel{\star}{=} \frac{\overbrace{b \cdot b \cdot \dots \cdot b}^{x \text{ times}}}{\underbrace{b \cdot b \cdot \dots \cdot b}_{y \text{ times}}} = \frac{b^x}{b^y}.$$

If instead $x < y$ so that $x - y < 0$ and $|x - y| = y - x$, then

$$b^{x-y} \stackrel{\star}{=} \frac{1}{b^{|x-y|}} = \frac{1}{\underbrace{b \cdot b \cdot \dots \cdot b}_{|x-y|=y-x \text{ times}}} = \frac{1}{\underbrace{b \cdot b \cdot \dots \cdot b}_{y-x \text{ times}}} \times \frac{\overbrace{b \cdot b \cdot \dots \cdot b}^{x \text{ times}}}{\underbrace{b \cdot b \cdot \dots \cdot b}_{x \text{ times}}} = \frac{\overbrace{b \cdot b \cdot \dots \cdot b}^{x \text{ times}}}{\underbrace{b \cdot b \cdot \dots \cdot b}_{y \text{ times}}} = \frac{b^x}{b^y}.$$

(d) $(b^x)^y \stackrel{\star}{=} (\underbrace{b \cdot b \cdot \dots \cdot b}_{x \textbf{ times}})^y \stackrel{\star}{=} \overbrace{(\underbrace{b \cdot b \cdot \dots \cdot b}_{x \textbf{ times}}) \times \dots \times (\underbrace{b \cdot b \cdot \dots \cdot b}_{x \textbf{ times}})}^{y \textbf{ times}} = \underbrace{b \cdot b \cdot \dots \cdot b}_{xy \textbf{ times}} \stackrel{\star}{=} b^{xy}$.

(e) $(ab)^x \stackrel{\star}{=} \underbrace{(ab)(ab)\dots(ab)}_{x \text{ times}} = \underbrace{a \cdot a \cdot \dots \cdot a}_{x \text{ times}} \times \underbrace{b \cdot b \cdot \dots \cdot b}_{x \text{ times}} \stackrel{\star}{=} a^x b^x$. □

Exercise 63. Let $x \in \mathbb{R}$. Simplify each expression. (Answer on p. 1758.)

(a) $\dfrac{5^{4x} \cdot 25^{1-x}}{5^{2x+1} + 3 \cdot 25^x + 17 \cdot 5^{2x}}$. **(b)** $\sqrt{2}\dfrac{8^{x+2} - 34 \cdot 2^{3x}}{\sqrt{8}^{2x+1}}$.

Exercise 64. Let $b > 0$ and $x, y \in \mathbb{Z}^+$. Explain whether each equation is generally true.

(a) $b^{(x^y)} = b^{xy}$. **(b)** $(b^x)^y = b^{xy}$. (Answer on p. 1758.)

Remark 21. We have defined b^x to cover several cases.

However, we still have not defined b^x in these two cases:

1. The base b is positive and the exponent x is irrational.
2. The base b is negative and the exponent x is a non-integer.

Unfortunately, defining these is somewhat beyond the scope of H2 Maths.

Yet at the same time, in H2 Maths, we will often deal with the above two cases of b^x, which is a little discomforting.

One comfort is that with Case 1, Proposition 1 (the Laws of Exponents) still holds.

With Case 2, those Laws of Exponents don't always hold and we should be a little more careful. Nonetheless, for H2 Maths, there is probably little danger in simply assuming that these Laws always hold.

5.5. The Square Root Refers to the Positive Square Root

By Definition 29,

The square root refers to the positive square root.

Example 128. It is true that the equation $b^2 = 25$ has two solutions, namely,

$$b = 5 \qquad \text{and} \qquad b = -5.$$

However, each of the following four statements is **false**:

$$25^{\frac{1}{2}} = \pm 5. \text{ ✗} \qquad 25^{\frac{1}{2}} = -5. \text{ ✗} \qquad \sqrt{25} = \pm 5. \text{ ✗} \qquad \sqrt{25} = -5. \text{ ✗}$$

By Definition 29, $25^{1/2}$ or $\sqrt{25}$ is a *positive* number. So,

$$25^{1/2} = \sqrt{25} = 5. \qquad \checkmark$$

To talk about the *negative* square root -5, (simply) stick a minus sign in front:

$$-25^{1/2} = -\sqrt{25} = -5 \qquad \checkmark$$

In general, given *any* $b > 0$, ***the* square root** of b is the positive number denoted

$$\sqrt{b} > 0.$$

The *negative* square root of b is the negative number has a minus sign in front:

$$-\sqrt{b} < 0.$$

This seems like a trivial and "obvious" point. You may find it strange that I have devoted this subchapter to this point. But overlooking this point is a common source of error:

Exercise 65. True or false: "$\sqrt{x^2} = x$ for all $x \in \mathbb{R}$." (Answer on p. 1758.)

Exercise 66. True or false: "For all $x \in \mathbb{R} \setminus \{0\}$, $\dfrac{\sqrt{x^2}}{x} \overset{1}{=} \dfrac{\sqrt{x^2}}{\sqrt{x^2}} \overset{2}{=} 1$." (Answer on p. 1758.)

Exercise 67. True or false: "For all $x \in \mathbb{R} \setminus \{0\}$, $\dfrac{x}{\sqrt{x^2}} \overset{1}{=} \dfrac{\sqrt{x^2}}{\sqrt{x^2}} \overset{2}{=} 1$." (Answer on p. 1758.)

Remark 22. Because of the "ambiguity" just discussed, some writers refer to $\sqrt{25}$ as the **principal** square root of the number 25. More generally, given any non-negative number x, they call the positive number $\sqrt{x}$ the **principal** square root of x.

We shall however have no use for the term **principal** in this textbook. The reason is that there is in fact no ambiguity at all—we have in this textbook clearly defined the positive number $\sqrt{25}$ to be *the* square root of 25. There is thus no need to refer to $\sqrt{25}$ as the *principal* square root of 25.

Remark 23. Definition 29 is to some extent an **arbitrary convention**, similar to whether we drive on the left or right side. Singapore drives on the left side of the road (because Singapore was part of the British Empire). But Singapore could just as easily drive on the right side of the road (which is what nearly all of the world except the former British Empire does). Except it doesn't and it would be quite unwise to try to go against the accepted convention.

Here similarly, Definition 29 says that *the* square root (of a positive number b) is itself a positive number. But mathematicians could just as easily have chosen the opposite convention. Except they didn't and it would be quite unwise to try to go against the accepted convention.

It's not really important which convention a community chooses. What's important is that everyone in that community knows what the convention is and agrees to use it, hence reducing confusion and accidents (whether in maths or on the road).

In Exercise 65, we gave this false statement:

$$\text{For all } x \in \mathbb{R}, \sqrt{x^2} = x. \qquad ✗$$

The correct statement should instead be this:

Fact 12. *For all* $x \in \mathbb{R}$, $\sqrt{x^2} = |x|$.

Proof. If $x \geq 0$, then $\sqrt{x^2} = x = |x|$. If $x < 0$, then $\sqrt{x^2} = -x = |x|$. $\square$

Here's another result we'll find useful: Given any $x \neq 0$, the expression

$$\frac{|x|}{x} \qquad \text{or} \qquad \frac{x}{|x|}$$

gives us its **sign**. That is,

Fact 13. $\dfrac{x}{|x|} = \dfrac{|x|}{x} = \begin{cases} 1, & \textit{for } x > 0, \\ -1, & \textit{for } x < 0. \end{cases}$

Proof. If $x > 0$, then $x/|x| = x/x = 1$ and $|x|/x = 1/(|x|/x) = 1/1 = 1$.
If $x < 0$, then $x/|x| = x/(-x) = -1$ and $|x|/x = 1/(|x|/x) = 1/(-1) = -1$. $\square$

Example 129. In Exercise 66, we gave this false statement:

$$\text{For all } x \in \mathbb{R} \setminus \{0\}, \frac{\sqrt{x^2}}{x} = \frac{\sqrt{x^2}}{\sqrt{x^2}} = 1. \qquad ✗$$

Using the above two facts, the correct statement should instead be this:

$$\text{For all } x \in \mathbb{R} \setminus \{0\}, \frac{\sqrt{x^2}}{x} = \frac{|x|}{x} = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases} \qquad ✓$$

Example 130. Similarly, in Exercise 67, we gave this false statement:

$$\text{For all } x \in \mathbb{R} \setminus \{0\}, \frac{x}{\sqrt{x^2}} = \frac{\sqrt{x^2}}{\sqrt{x^2}} = 1. \qquad ✗$$

Using the above two facts, the correct statement should instead be this:

$$\text{For all } x \in \mathbb{R} \setminus \{0\}, \frac{x}{\sqrt{x^2}} = \frac{x}{|x|} = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases} \qquad ✓$$

5.6. Rationalising the Denominator with a Surd

The term **surd** usually refers to a square root $\sqrt{\cdot}$.

Example 131. $\dfrac{1}{\sqrt{2}} \overset{\text{TOT}}{=} \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.

At $\overset{\text{TOT}}{=}$, we used what I'll call the **Times One Trick (TOT)**. It is of course always legitimate to multiply any quantity by 1, though one might wonder why we'd ever want to do that. The above is one example of when the TOT comes in handy.

Definition 31. The *conjugate* of $a+b$ is $a-b$. We call $a+b$ and $a-b$ a *conjugate pair*.

Given a denominator with a surd, we can often rationalise it by using the TOT, the conjugate, and the identity $(a+b)(a-b) = a^2 - b^2$:

Example 132. $\dfrac{1}{1+\sqrt{2}} \overset{\text{TOT}}{=} \dfrac{1}{1+\sqrt{2}} \times \dfrac{1-\sqrt{2}}{1-\sqrt{2}} = \dfrac{1-\sqrt{2}}{1^2-\left(\sqrt{2}\right)^2} = \dfrac{1-\sqrt{2}}{1-2} = \dfrac{1-\sqrt{2}}{-1} = \sqrt{2}-1.$

We take this opportunity to talk about the $\pm$ and $\mp$ notation:

Example 133. We have $\dfrac{1}{2+\sqrt{3}} \overset{\text{TOT}}{=} \dfrac{1}{2+\sqrt{3}} \times \dfrac{2-\sqrt{3}}{2-\sqrt{3}} = \dfrac{2-\sqrt{3}}{2^2-3} = 2-\sqrt{3}.$ $\star$

And, $\dfrac{1}{2-\sqrt{3}} \overset{\text{TOT}}{=} \dfrac{1}{2-\sqrt{3}} \times \dfrac{2+\sqrt{3}}{2+\sqrt{3}} = \dfrac{2+\sqrt{3}}{2^2-3} = 2+\sqrt{3}.$ ☽

Using **plus-minus** $\pm$ and **minus-plus** $\mp$ notation, we can combine the two statements $\star$ and ☽ into this single statement:

$$\frac{1}{2\pm\sqrt{3}} = \frac{1}{2\pm\sqrt{3}}\frac{2\mp\sqrt{3}}{2\mp\sqrt{3}} = \frac{2\mp\sqrt{3}}{2^2-3} = 2\mp\sqrt{3}.$$

The minus-plus notation $\mp$ indicates the "opposite" of $\pm$. So here,

- If $\pm$ is $+$, then $\mp$ is $-$ and we have $\star$. And
- If $\pm$ is $-$, then $\mp$ is $+$ and we have ☽.

Remark 24. We **only** ever use $\mp$ **after** we've already used $\pm$ and have a need to indicate that the signs go the opposite way. So for example, we'd never write

$$x^2 = 17 \quad \Longrightarrow \quad x = \mp\sqrt{17},$$

because we can simply write $\quad x^2 = 17 \quad \Longrightarrow \quad x = \pm\sqrt{17}.$

Following this last line, if we subsequently have a need to discuss $-x$, then that's when we might use $\mp$:

$$-x = \mp\sqrt{17}.$$

Exercise 68. Let $x, y \in \mathbb{R}$, with $y \neq 0$. Prove the following. (Answer on p. **1758**.)

$$\frac{1}{\frac{x}{y} \pm \sqrt{\frac{x^2}{y^2}+1}} = -\frac{x}{y} \pm \sqrt{\frac{x^2}{y^2}+1}.$$

Exercise 69. Let $a, b, c \in \mathbb{R}$ with $a \neq 0$ and $b^2 - 4ac > 0$. Prove the following.[108]

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2c}{-b \mp \sqrt{b^2-4ac}}.$$

(Answer on p. 1760.)

[108]The LHS is the quadratic formula in familiar form. The RHS is also the quadratic formula, but in an alternative and less familiar form.

5.7. Logarithms

Informally, logarithms are simply the inverse of exponents. A bit more formally,

Definition 32. Let $x \in \mathbb{R}$ and $b, n > 0$ with $b \neq 1$. If $b^x = n$, then we call x the *base-b logarithm of n* and write

$$x = \log_b n.$$

Example 134. $2^3 = 8 \iff 3 = \log_2 8$.

Example 135. $3^4 = 81 \iff 4 = \log_3 81$.

Example 136. $4^5 = 1\,024 \iff 5 = \log_4 1\,024$.

You should find these **Laws of Logarithms** familiar:

Proposition 2. (Laws of Logarithms) *Let $x \in \mathbb{R}$ and $a, b, c > 0$ with $b \neq 1$. Then*

(a) $\log_b 1 = 0$

(b) $\log_b b = 1$

(c) $\log_b b^x = x$

(d) $b^{\log_b a} = a$

(e) $c \log_b a = \log_b a^c$

(f) $\log_b \frac{1}{a} = -\log_b a$ **(Logarithm of Reciprocal)**

(g) $\log_b (ac) = \log_b a + \log_b c$ **(Sum of Logarithms)**

(h) $\log_b \frac{a}{c} = \log_b a - \log_b c$ **(Difference of Logarithms)**

(i) $\log_b a = \frac{\log_c a}{\log_c b}$, *for $c \neq 1$* **(Change of Base)**

(j) $\log_{a^b} c = \frac{1}{b} \log_a c$, *for $a \neq 1$*

Proof. In this proof, $\stackrel{\star}{\Longrightarrow}$ denotes the use of Definition 32.

(a) $$b^0 = 1 \stackrel{\star}{\iff} \log_b 1 = 0.$$

(b) $$b^1 = b \stackrel{\star}{\iff} \log_b b = 1.$$

(c) $$b^x = b^x \stackrel{\star}{\iff} \log_b b^x = x.$$

(d) $$\text{Let } y = \log_b a \overset{\star}{\iff} b^y = a \iff b^{\log_b a} = a.$$

(e) First use Proposition 1(d):

$$b^{c\log_b a} = \left(b^{\log_b a}\right)^c \overset{(d)}{=} a^c \overset{\star}{\iff} c\log_b a = \log_b a^c.$$

(f) $$\log_b \frac{1}{a} = \log_b a^{-1} \overset{\text{(e)}}{=} -\log_b a.$$

(g) First use Proposition 1(a):

$$b^{\log_b a+\log_b c} = b^{\log_b a}b^{\log_b c} \overset{(d)}{=} ac \overset{\star}{\iff} \log_b(ac) = \log_b a + \log_b c.$$

(h) $$\log_b \frac{a}{c} = \log_b\left(a\cdot\frac{1}{c}\right) \overset{\text{(g)}}{=} \log_b a + \log_b \frac{1}{c} \overset{\text{(f)}}{=} \log_b a - \log_b \frac{1}{c}.$$

(i) Let $y = \log_b a$ or $b^y = a$. Let $z = \log_c a$ or $c^z = a$.

Hence, $b^y = c^z$ or $z = \log_c b^y = y\log_c b$ or $\log_c a = \log_b a \log_c b$ or $\log_b a = \dfrac{\log_c a}{\log_c b}$.

(k) $$\log_{a^b} c \overset{\text{(i)}}{=} \frac{\log_a c}{\log_a a^b} \overset{(c)}{=} \frac{\log_a c}{b} = \frac{1}{b}\log_a c.$$ □

Exercise 70. Show that each expression equals 2. (Answer on p. 1760.)

(a) $\log_2 32 + \log_3 \dfrac{1}{27}$ **(b)** $\log_3 45 - \log_9 25$ **(c)** $\log_{16} 768 - \log_2 \sqrt[4]{3}$

Remark 25. Some (including your TI84) write $\log x$ to mean **the base-10 logarithm** of x. Still others write $\log x$ to mean the **natural logarithm** of x. (Note that at this point, we still haven't discussed what the **natural logarithm function** is. We will do so only in Ch. 28.)

We have thus a rather confused situation with different writers using different notation.

In this textbook, we'll stick strictly to this notation on your H2 Maths syllabus (p. 18):

$\log_a x$	logarithm to the base a of x
$\ln x$	natural logarithm of x
$\lg x$	logarithm of x to base 10

We will only ever write $\log_b x$ and even then fairly rarely. We will **never** write $\log x$.

5.8. Polynomials

Definition 33. Let $c_0, c_1, \ldots, c_n$ be constants, with $c_n \neq 0$. The expression

$$c_n x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0$$

is called an *nth-degree polynomial (in one variable x)*. We also call

- Each $c_i x^i$ the *ith-degree term* (or more simply the *ith term*);
- Each c_i the *ith coefficient on x^i* (or the *ith-degree coefficient*, or the *ith coefficient*);
- The 0th coefficient c_0 the *constant term* (or, more simply, the *constant*).

A *(nth-degree) polynomial equation (in one variable x)* is any equation

$$c_n x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0 = 0,$$

or any equation that can be rewritten in the above form.

As usual, in the above definition, x is merely a **dummy variable** that can be replaced with any other symbol, like y, z, ☺, or ★.

Example 137. The expression $7x - 3$ is a 1**st-degree** or **linear polynomial**.

The equation $7x - 3 = 0$ is a 1st-degree **polynomial** (or **linear**) **equation**.

$$\overbrace{\underbrace{7}_{\text{1st-degree coefficient}}\ x}^{\text{1st-degree term}}\ \overbrace{\underbrace{-3}_{\text{0th-degree coefficient}}}^{\text{0th-degree term}}$$

	Term	**Coefficients**
0th-degree	-3	-3
1st-degree	$7x$	7

The 1**st-degree term** is $7x$.

The 1**st-degree coefficient** is 7.

The 0**th-degree term**, the **constant term**, the **constant**, *and* the 0**th-degree coefficient** is -3.

Example 138. The expression $3x^2 + 4x - 5$ is a **2nd-degree** or **quadratic polynomial**. The equation $3x^2 + 4x - 5 = 0$ is a 2nd-degree **polynomial** (or **quadratic**) **equation**.

$$\overbrace{\underbrace{3}_{\text{2nd-degree coefficient}}\ x^2}^{\text{2nd-degree term}} + \overbrace{\underbrace{4}_{\text{1st-degree coefficient}}\ x}^{\text{1st-degree term}}\ \overbrace{\underbrace{-5}_{\text{0th-degree coefficient}}}^{\text{0th-degree term}}$$

	Term	**Coefficient**
0th-degree	-5	-5
1st-degree	$4x$	4
2nd-degree	$3x^2$	3

Example 139. The expression $-5x^3 + 2x + 9$ is a **3rd-degree** or **cubic polynomial**. The equation $-5x^3 + 2x + 9 = 0$ is a 3rd-degree **polynomial** (or **cubic**) **equation**.

When a particular coefficient is 0, we usually don't bother writing out that term (as is the case here with the 2nd-degree term).

$$\text{3rd-degree coefficient} \leftarrow \overbrace{-5x^3}^{\text{3rd-degree term}} + \overbrace{\underbrace{0}_{\text{2nd-degree coefficient}}x^2}^{\text{2nd-degree term}} + \overbrace{\underbrace{2}_{\text{1st-degree coefficient}}x}^{\text{1st-degree term}} + \underbrace{9}_{\text{0th-degree term \& coefficient}}$$

	Term	**Coefficient**
0th-degree	9	9
1st-degree	$2x$	2
2nd-degree	$0x^2$	0
3rd-degree	$-5x^3$	-5

You get the idea. We also have **4th-, 5th-, 6th-, ... degree** (or **quartic, quintic, sextic, ...**) **polynomials** and **equations**.

Example 140. Technically, the expression 7 could be regarded as a **0th-degree polynomial**, because $7 = 7x^0$. Indeed, in certain contexts, it may be convenient to refer to the expression 7 as such.

But more commonly, we'll simply call the expression 7 a **constant**.

This textbook will look mostly only at polynomials in **one variable**. So unless otherwise stated, when we say polynomial, we'll **always** mean a polynomial in one variable.[109]

[109]But in case you were wondering, an example of **a polynomial in two variables** is $Ax + Bxy + Cy$. In general, the degree of each term in a polynomial is the sum of the exponents on the variables. And the polynomial's degree is simply the highest such degree. So here, the term Ax has degree 1, Bxy has degree 2, and Cy has degree 1. So this polynomial is of degree 2.

Another example of a polynomial in two variables is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F$. Despite looking more complicated than the previous example, this polynomial also has degree 2 because the greatest sum of exponents on any term is again 2.

And by the way, as Ch. 142.20 (Appendices) discusses, the conic section is, in general, described by this 2nd-degree polynomial equation in two variables:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

6. O-Level Review: Inequalities

Ch. 41 (Solving Inequalities) will go through inequalities in greater depth. This chapter simply reviews some basics you should find at least vaguely familiar:

6.1. The Sign of the Product of Two Real Numbers

Given $a, b \in \mathbb{R}$, what is the sign of ab?

1. If $a, b > 0$, then $ab > 0$.

Example 141. If $a = 2 > 0$ and $b = 3 > 0$, then $ab = 6 > 0$.

2. If $a, b < 0$, then $ab > 0$.

Example 142. If $a = -2 < 0$ and $b = -3 < 0$, then $ab = 6 > 0$.

3. If a and b have opposite signs (i.e., $a > 0$ AND $b < 0$ or $a < 0$ AND $b > 0$), then $ab < 0$.

Example 143. If $a = 2 > 0$ and $b = -3 < 0$, then $ab = -6 < 0$.

Example 144. If $a = -2 < 0$ and $b = 3 > 0$, then $ab = -6 < 0$.

4. And of course, if either $a = 0$ or $b = 0$, then $ab = 0$.

Example 145. If $a = 0$ and $b = 3 > 0$, then $ab = 0$.

Example 146. If $a = -2 < 0$ and $b = 0$, then $ab = 0$.

Exercise 71. Given some $x \in \mathbb{R}$, what are the sign of $7x$ and $-7x$? (Answer on p. 1760.)

More generally,

Fact 14. *Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Suppose $P = a_1 a_2 \ldots a_n$. Then*

(a) $P = 0 \iff$ *At least one of a_1, a_2, ..., and a_n is zero.*

(b) $P > 0 \iff$ *An even number of a_1, a_2, ..., and a_n are negative (and the rest positive).*

(c) $P < 0 \iff$ *An odd number of a_1, a_2, ..., and a_n are negative (and the rest positive).*

Proof. See p. 1612 (Appendices). □

Example 147. XXX

Exercise 72. XXX (Answer on p. 94.)

A72.

6.2. Multiplying an Inequality by an Unknown Number

Given the equation $a = b$, we can multiply by *any* number x to get

$$ax = bx. \qquad \checkmark$$

That is, an equality is preserved when we multiply it by any number.

In contrast, an inequality is not always preserved when we multiply it by a number:

Example 148. Consider the inequality $2 > 1$.

(a) It is **preserved** if we multiply it by a **positive** number like 3:

$$2 \times 3 > 1 \times 3 \qquad \text{or} \qquad 6 > 3.$$

(b) It is **reversed** (or **flipped**) if we multiply it by a **negative** number like -3:

$$2 \times (-3) < 1 \times (-3) \qquad \text{or} \qquad -6 < -3.$$

(c) It becomes an **equality** if we multiply it by **zero**:

$$2 \times 0 = 1 \times 0 \qquad \text{or} \qquad 0 = 0.$$

The above seems obvious. But a common mistake is to multiply an inequality by some unknown number x and expect it to be preserved:

Example 149. Let $x \in \mathbb{R}$. Beng reasons, "We know that $8 > 5$. Therefore $8x > 5x$."

Beng's reasoning is wrong:

(a) If $x > 0$, then yea, Beng happens to be correct and $8x > 5x$. ✓

(b) If $x < 0$, then $8x < 5x$ and he's wrong. ✗

(c) If $x = 0$, then $8x = 5x\ (= 0)$ and he's again wrong. ✗

In general,

Fact 15. *Let $a, b, x \in \mathbb{R}$. Suppose $a > b$.*

(a) *If $x > 0$, then $ax > bx$.*

(b) *If $x < 0$, then $ax < bx$.*

(c) *If $x = 0$, then $ax = bx = 0$.*

Analogous results also hold if we replace $a > b$ with $a \geq b$.

Proof. Since $a > b$, we have $a - b > 0$.

(a) If $x > 0$, then by Fact **??**(b), $(a - b)\,x > 0$ or $ax - bx > 0$ or $ax > bx$.

(b) If $x < 0$, then by Fact **??**(c), $(a - b)\,x < 0$ or $ax - bx < 0$ or $ax < bx$.

(c) If $x = 0$, then by Fact **??**(a), $ax = 0$ and $bx = 0$.

The proof of the last sentence is similar and omitted. □

So, in general, we may have to break our analysis into two (or three) cases, depending on whether x is positive, negative, or zero:

Example 150. Suppose $a, b, x \in \mathbb{R}$ with $x \neq 0$ and

$$\frac{a}{x} \overset{1}{>} \frac{b}{x}.$$

What can we say about a and b?

Beng reasons, "Multiply $\overset{1}{>}$ by x to conclude that $a > b$." ✗

As usual, Beng is wrong. We must break our analysis down into two cases:

(a) If $x > 0$, then yea, he happens to be correct—we can indeed multiply $\overset{1}{>}$ by x to conclude that $a > b$.

(b) But if $x < 0$, then multiplying $\overset{1}{>}$ by x reverses the inequality, so that $a < b$.

Correct reasoning:

$$\text{If } x > 0, \text{ then } a > b. \qquad \text{But if } x < 0, \text{ then } a < b.$$

By the way, why did we specify that $x \neq 0$?[110]

Exercise 73. Solve each inequality. (Answers on p. 1760.)

(a) $\dfrac{x-1}{-4} > 0$. **(b)** $\dfrac{-1}{-4} > 0$. **(c)** $\dfrac{1}{-4} > 0$. **(d)** $\dfrac{2x+1}{3x+2} > 0$. **(e)** $\dfrac{-3x-18}{9x-14} > 0$.

[110]If $x = 0$, then a/x and b/x would involve the cardinal sin of dividing by zero.

6.3. The Sign of the Reciprocal of A Real Number

Given $a \in \mathbb{R}$, what is the sign of $\frac{1}{a}$?

1. If $a > 0$, then $\frac{1}{a} > 0$.

Example 151. If $a = 2 > 0$, then $\frac{1}{a} = \frac{1}{2} > 0$.

2. If $a < 0$, then $\frac{1}{a} < 0$.

Example 152. If $a = -2 < 0$, then $\frac{1}{a} = \frac{1}{-2} = -\frac{1}{2} < 0$.

And of course,

3. If $a = 0$, then $\frac{1}{a}$ is undefined.

The following result summarises the above observations:

Fact 16. *If $a \neq 0$, then a has the same sign as $1/a$.*

Proof. Since $a \cdot \frac{1}{a} = 1 > 0$, by Fact **??**, a and $\frac{1}{a}$ must have the same sign. □

Equivalently, $a > 0 \iff \frac{1}{a} > 0$ and $a < 0 \iff \frac{1}{a} < 0$.

6.4. Does Taking Reciprocals Always Preserve an Inequality?

No:

Fact 17. *Suppose $a > b$ (with $a, b \neq 0$).*

(a) *If a and b have the same sign, then* $\dfrac{1}{a} < \dfrac{1}{b}$. **(Reciprocation reverses order)**

(b) *If a and b have opposite signs, then* $\dfrac{1}{a} > \dfrac{1}{b}$. **(Reciprocation preserves order)**

Proof. **(a)** Since $a > b$, $b - a \overset{1}{<} 0$.

Since a and b have the same sign, $ab > 0$ and so by Fact 16, $\dfrac{1}{ab} > 0$.

By $\overset{1}{<}$ and Fact 15(a), $\dfrac{b-a}{ab} < 0$. Rearranging, $\dfrac{b}{ab} < \dfrac{a}{ab}$. Equivalently, $\dfrac{1}{a} < \dfrac{1}{b}$.

(b) Since $a > b$ and a and b have opposite signs, it must be that $a > 0$ and $b < 0$.[111] So, by Fact 16, $1/a > 0$, $1/b < 0$. Hence, $1/a > 1/b$. □

Example 153. XXX

Example 154. XXX

Exercise 74. XXX (Answer on p. 98.)

A74.

[111] It cannot be that $a < 0$ and $b > 0$.

6.5. Does Taking Square Roots Always Preserve an Inequality?

Fact 18. *Let $a, b \geq 0$. If $a > b$, then $\sqrt{a} > \sqrt{b}$.*

Proof. First, $\sqrt{a}, \sqrt{b} \geq 0$. So, $\left(\sqrt{a}\right)^2 = a$ and $\sqrt{b} = b$.

Hence, if $\sqrt{a} \leq \sqrt{b}$, then $a = \left(\sqrt{a}\right)^2 \leq \left(\sqrt{b}\right)^2 = b$.

Thus, by the contrapositive of the last implication, we have the claim. □

Next, if $a < 0$ or $b < 0$, then $\sqrt{a}$ or $\sqrt{b}$ is simply undefined, so that the statement $\sqrt{a} > \sqrt{b}$ is meaningless.[112]

Altogether, the answer to the above question is, "Yes, taking square roots always preserves an inequality."

6.6. Does Squaring Always Preserve an Inequality?

No:

Example 155. $-2 > -3$ but $(-2)^2 < (-3)^2$

Example 156. $5 > -8$ but $5^2 < (-8)^2$

Example 157. $5 > -5$ but $5^2 = (-5)^2$

To properly answer the titular question, we first need some preliminaries:

Fact 19. *Suppose $a \in \mathbb{R}$. Then*

(a) $a \neq 0 \iff a^2 > 0$; *and*

(b) $a = 0 \iff a^2 = 0$.

Example 158. $2 \neq 0 \iff 2^2 > 0$

Example 159. $-2 \neq 0 \iff (-2)^2 > 0$

Proof. **(a)** By Fact 14(b).

(b) ($\Longrightarrow$) If $a = 0$, then $a^2 = 0 \times 0 = 0$.

($\Longleftarrow$) If $a \neq 0$, then by **(a)**, $a^2 > 0$ and in particular $a^2 \neq 0$. □

[112] Even allowing for complex numbers, if either c or d is not real, then the statement $c > d$ still remains meaningless (Ch. 79.3).

Fact 20. *For every $a \in \mathbb{R}$, we have $a^2 = |a|^2$.*

Proof. If $a \geq 0$, then $|a| = a$ and so $|a|^2 = a^2$.
If $a < 0$, then $|a| = -a$ and so $|a|^2 = (-a)^2 = a^2$. □

The titular question is somewhat tricky and requires distinguishing between several cases:

Fact 21. *Let $a, b \in \mathbb{R}$. Suppose $a \overset{1}{>} b$.*

(a) *If $a \leq 0$, then $a^2 < b^2$.*
(b) *If $a > 0$ and $b \geq 0$, then $a^2 > b^2$.*
(c) *Suppose $a > 0$ and $b < 0$.*

(i) *If $a > |b|$, then $a^2 > b^2$.*
(ii) *If $a < |b|$, then $a^2 < b^2$.*
(iii)
(iv) *If $a = |b|$, then $a^2 = b^2$.*

Proof. Below, $\star$ denotes the use of Fact 20.

(a) Suppose $a \leq 0$. Then by $\overset{1}{>}$, $b < 0$.

Also, $|a| = -a \overset{1}{<} -b = |b|$

So, $a^2 \overset{\star}{=} |a|^2 < |b|^2 \overset{\star}{=} b^2$.

(b) Suppose $a > 0$ and $b \geq 0$. Then $a = |a| > |b| = b$. So, $a^2 \overset{\star}{=} |a|^2 > |b|^2 \overset{\star}{=} b^2$.

(c) Suppose $a > 0$ and $b < 0$.

(c)(i) Suppose $a > |b|$. Then $a^2 > |b|^2 \overset{\star}{=} b^2$.

(c)(ii) Suppose $a < |b|$. Then $a^2 < |b|^2 \overset{\star}{=} b^2$.

(c)(iii) Suppose $a = |b|$. Then $a^2 = |b|^2 \overset{\star}{=} b^2$. □

6.7. Does Exponentiation Always Preserve an Inequality?

No:[113]

Example 160. $5 > 3$ but $5^{-2} < 3^{-2}$ $(1/25 < 1/9)$

Example 161. $5 > 3$ but $5^0 = 3^0$ $(1 = 1)$

Example 162. $-2 > -7$ but $(-2)^2 < (-7)^2$ $(4 < 49)$

Example 163. $3 > -5$ but $3^2 < (-5)^2$ $(9 < 25)$

So, "if $a > b$, then $a^x > b^x$" is **not** always true. Nonetheless, it is true if $a, b, x > 0$:

Fact 22. *Let $x > 0$ and $b \geq 0$. If $a > b$, then $a^x > b^x$.*

Example 164. $5 > 3$ and $5^2 > 3^2$ $(25 > 9)$

Example 165. $5 > 3$ and $5^\pi \overset{1}{>} 3^\pi$.

Again, in the main text, we haven't actually defined b^x in the case where x is irrational. Nonetheless, $\overset{1}{>}$ is true, with $156.99 \approx 5^\pi > 3^\pi \approx 31.544$.

Example 166. $5 > 0$ and $5^4 > 0^4$ $(625 > 0)$

To prove Fact 22, we'll make use of these "obvious" results:

Fact 23. *Let $x > 0$.*

(a) *If $b > 0$, then $b^x > 0$.*

(b) *If $b > 1$, then $b^x > 1$.*

Proof. See p. 1726 (Appendices). □

We can now prove Fact 22:

Proof. Since $a, x > 0$, by Fact 22(a), $a^x > 0$.

If $b = 0$, then $b^x = 0^x = 0$, so that $a^x > b^x$.

If $b > 0$, then $b^x > 0$, so that $\dfrac{a^x}{b^x} \overset{2}{=} \left(\dfrac{a}{b}\right)^x \overset{3}{>} 1$ ($\overset{2}{=}$ and $\overset{3}{>}$ use Proposition 1 and Fact 22). □

[113] By the way, the previous three subchapters simply looked at special cases of exponentiation: Given b^x,

- Taking reciprocals is equivalent to setting $x = -1$;
- Taking square roots is equivalent to setting $x = 1/2$;
- Squaring is equivalent to setting $x = 2$.

Part I.
Functions and Graphs

Revision in progress (November 2021).

And hence messy at the moment.
Appy polly loggies for any inconvenience caused.

[A] large part of mathematics which became useful developed with absolutely no desire to be useful, and in a situation where nobody could possibly know in what area it would become useful; and there were no general indications that it ever would be so. By and large it is uniformly true in mathematics that there is a time lapse between a mathematical discovery and the moment when it is useful; and that this lapse of time can be anything from thirty to a hundred years, in some cases even more ...

This is true for all of science. Successes were largely due to forgetting completely about what one ultimately wanted, or whether one wanted anything ultimately; in refusing to investigate things which profit, and in relying solely on guidance by criteria of intellectual elegance; it was by following this rule that one actually got ahead in the long run, much better than any strictly utilitarian course would have permitted.

— John von Neumann (1954).

7. Graphs

7.1. Ordered Pairs

Recall that with sets, the order of the elements doesn't matter:

Example 167. {Cow, Chicken} = {Chicken, Cow}.

Example 168. $\{-5, 4\} = \{4, -5\}$.

We now introduce a new mathematical object called an **ordered pair** (a, b). Like the sets {Cow, Chicken} and $\{-5, 4\}$, you can think of an ordered pair as a container with two objects.

But unlike sets, with ordered pairs, the **order matters** (hence the name).[114]

Definition 34. Given the ordered pair (a, b), we call a its *first* or *x-coordinate* and b its *second* or *y-coordinate.*

Two ordered pairs are equal if and only if both their x- and y-coordinates are equal.

Fact 24. *Suppose a, b, x, and y are objects; and (a, b) and (x, y) are ordered pairs. Then*

$$(a, b) = (x, y) \qquad \iff \qquad a = x \text{ AND } b = y.$$

Proof. See p. 1554 in the Appendices. □

Example 169. Consider these two ordered pairs: (Cow, Chicken) and (Chicken, Cow).

The ordered pair (Cow, Chicken) has x-coordinate Cow and y-coordinate Chicken.

The ordered pair (Chicken, Cow) has x-coordinate Chicken and y-coordinate Cow.

Since these two ordered pairs have different x- and y-coordinates, they are not equal:

$$(\text{Cow, Chicken}) \neq (\text{Chicken, Cow}).$$

This is in contrast to what we saw above with sets, where:

$$\{\text{Cow, Chicken}\} = \{\text{Chicken, Cow}\}.$$

To distinguish an ordered pair from a set with two elements, we use **parentheses** (instead of braces). Be very clear that the ordered pair (a, b) is a completely different mathematical object from the set $\{a, b\}$:

$$(\text{Cow, Chicken}) \neq \{\text{Cow, Chicken}\};$$

and, $$(\text{Chicken, Cow}) \neq \{\text{Chicken, Cow}\}.$$

[114] For the formal definition of an ordered pair, see p. 1554 in the Appendices.

Example 170. Consider these two ordered pairs:

$$(\text{-}5,4) \qquad \text{and} \qquad (4,\text{-}5).$$

The ordered pair $(-5,4)$ has x-coordinate -5 and y-coordinate 4.

In contrast, the ordered pair $(4,-5)$ has x-coordinate 4 and y-coordinate -5.

Since these two ordered pairs have different x- and y-coordinates, they are not equal:

$$(-5,4) \neq (4,-5)$$

This is in contrast to what we saw above with sets, where $\{-5,4\} = \{4,-5\}$.

Again, be very clear that the ordered pair (a,b) and the set $\{a,b\}$ are two completely different mathematical objects:

$$(-5,4) \neq \{-5,4\} \qquad \text{and} \qquad (4,-5) \neq \{4,-5\}.$$

It is possible that in an ordered pair (a,b), we have $a = b$:

Example 171. Here are four ordered pairs that are distinct (or not equal):

(Cow, Chicken), (Chicken, Cow), (Cow, Cow), and (Chicken, Chicken).

The ordered pair (Cow, Cow) has x-coordinate Cow and y-coordinate Cow.

The ordered pair (Chicken, Chicken) has x-coordinate Chicken and y-coordinate Chicken.

Example 172. Here are four distinct ordered pairs:

$$(1,\text{-}5), \quad (\text{-}5,1), \quad (1,1), \quad \text{and} \quad (\text{-}5,\text{-}5).$$

The ordered pair $(1,1)$ has x-coordinate 1 and y-coordinate 1.

The ordered pair $(-5,-5)$ has x-coordinate -5 and y-coordinate -5.

Remark 26. Confusingly, $(-5, 4)$ can denote two entirely different things:

- In Ch. 4.13, we learnt that $(-5, 4)$ is an **interval**—in particular, it denotes the set of real numbers between -5 and 4.
- Here we learn that $(-5, 4)$ denotes the ordered pair with the x-coordinate -5 and the y-coordinate 4.

This is an unfortunate and confusing situation. But don't worry.

In the *Oxford English Dictionary*, the word *run* has 645 different meanings.[115] But with a little experience, one rarely has trouble telling from the context which of these is meant when someone says uses that word.

Likewise, you'll rarely have trouble telling from the context whether by $(-5, 4)$, the writer means a set of real numbers or an ordered pair.

[115]The *OED* editor Peter Gilliver spent nine months working on the word *run* (*New York Times*, 2011). Previously, *set* had the most different meanings, at 430 (*Guinness Book of World Records*).

7.2. The Cartesian Plane

Rather than ordered pairs of cows and chickens, we'll usually be concerned with **ordered pairs of real numbers**:

Definition 35. A *point* is any ordered pair of real numbers.

Example 173. $A = (-5, 4)$, $B = (1, 1)$, and $C = (2, -3)$ are points
$D = (\text{Cow}, \text{ Chicken})$ and $E = (3, \text{ Chicken})$ are not.

Definition 36. The *cartesian plane* is the set of all points:

$$\{(x, y) : x, y \in \mathbb{R}\}.$$

Fun Fact

The **cartesian plane**[116] is named after René Descartes (1596–1650), who's also the same dude who came up with "*Cogito ergo sum*" ("I think, therefore I am"). W

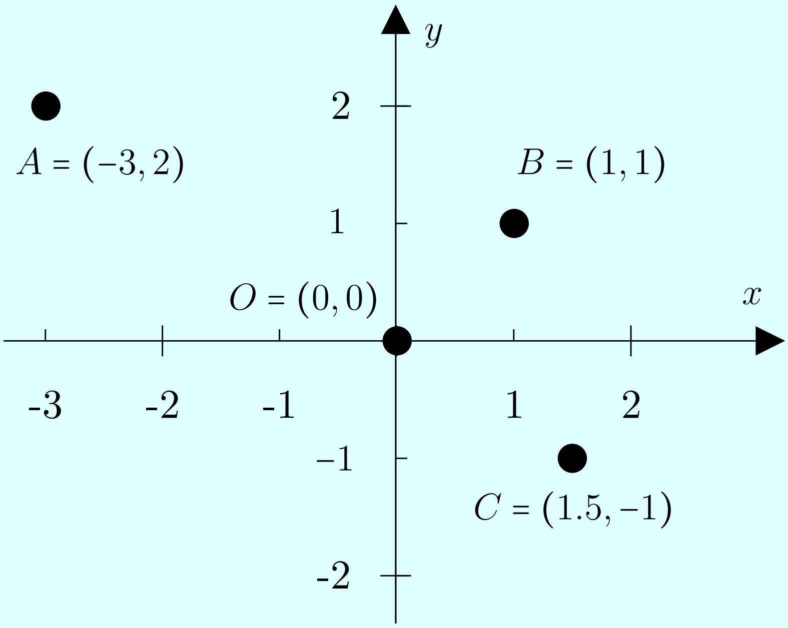
Definition 37. The *origin* is the point $O = (0, 0)$.

[116]There's some disagreement over whether to capitalise *cartesian* here—see e.g. mathoverflow.
Indeed, in your H2 Maths syllabus, it used to be capitalised on p. 19 but not on pp. 7–8! (My guess is that while pp. 1–15 were written by the local Singapore authorities, pp. 16–20 were simply copy-pasted from some standard Cambridge notation template.)
[*2020 update: They've now corrected this in the latest 2021 syllabus. Now* cartesian *on p. 19 is also not capitalised.*]
My personal preference is to capitalise *cartesian*, but it seems that the A-Level exams do not do so. I shall therefore follow the sacred A-Level exams by **not** capitalising *cartesian*.

Example 174. The cartesian plane below is centred on the origin $O = (0,0)$ (x-coordinate 0 and y-coordinate 0) and stretches out to $\pm\infty$ in both the x- and y-directions.

Three points are depicted:

- $A = (-3,2)$has x-coordinate -3 and y-coordinate 2;
- $B = (1,1)$ has x-coordinate 1 and y-coordinate 1; and
- $C = (1.5,-1)$ has x-coordinate 1.5 and y-coordinate -1.



When depicting the cartesian plane, it is customary (and helpful) to draw the x-axis (or horizontal axis) and y-axis (or vertical axis).

The point at which these two axes intersect is the origin $O = (0,0)$.

Remark 27. For now, we'll be concerned only with the cartesian plane, which is a two-dimensional space.[117]

And so, for now, whenever we say **point**, it should be clear that we're talking about an ordered pair of real numbers.

Note though that more generally, it is also to talk points in a one-dimensional space and a three-dimensional space:

- A *point (in one-dimensional space)* is simply any real number.
- As we'll learn in Part III (Vectors), a *point (in three-dimensional space)* is any ordered **triple** of real numbers.

By the way and just so you know, a point (in any-dimensional space) is itself a zero-dimensional object.

[117]Formally, we may define a two-dimensional space to be any subset of the cartesian plane $\mathbb{R}^2$. Similarly, a one-dimensional space is any subset of the **real number line** $\mathbb{R}$ and a three-dimensional space is any subset of $\mathbb{R}^3$. See Definition 271 (Appendices).

7.3. A Graph is *Any* Set of Points

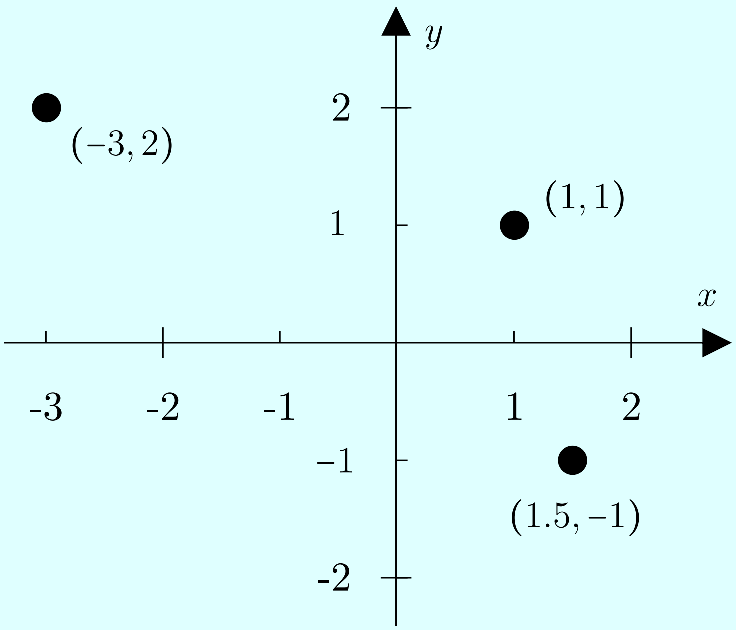
You're probably used to thinking of a **graph** (or a **curve**) as a "drawing". But formally,

Definition 38. A *graph* (or *curve*) is any set of points.

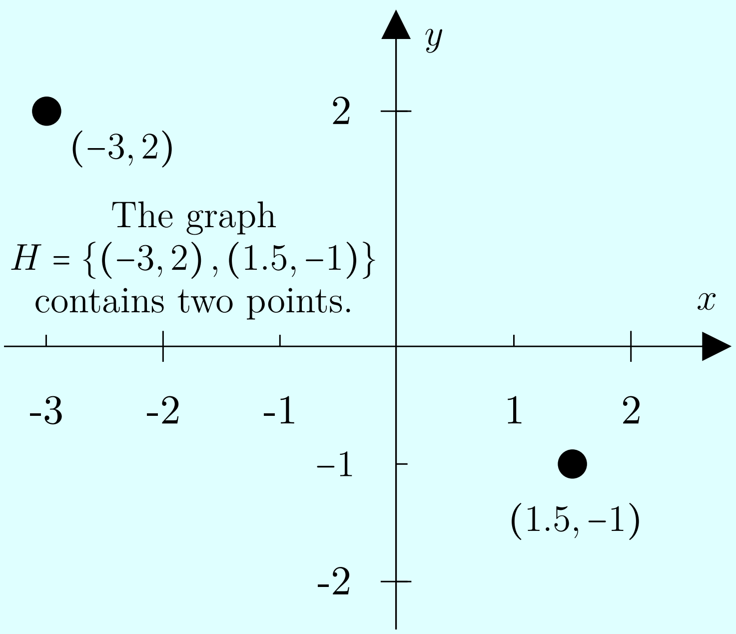
So equivalently, a *graph* is any subset of the cartesian plane.

Example 175. The set $G = \{(-3,2), (1,1), (1.5,-1)\}$ contains three points. And so by definition, G is also a graph.

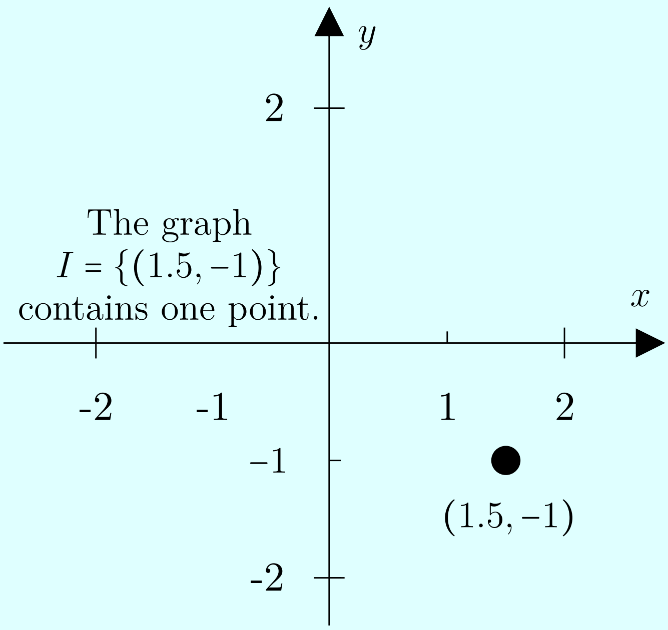
We've defined *graph* as a noun (it is a set of points). But at the slight risk of confusion, we'll also use *graph* as the **verb** meaning **to draw a graph**. So, we can either say, "The graph G is drawn below," or, "G is graphed below".



Example 176. The set $H = \{(-3,2), (1.5,-1)\}$ contains two points. And so by definition, H is also a graph.



Example 177. The set $I = \{(1.5, -11)\}$ contains one point. And so by definition, I is also a graph.



If a set contains at least one element that isn't a point, then it isn't a graph:

Example 178. Consider $J = \{(-5, 4), \text{ Love}\}$.

The set J contains two elements—the point $(-5, 4)$ and the abstract concept called Love. Since J contains at least one element that isn't a point (i.e. an ordered pair of real numbers), J is not a graph.

Example 179. Consider $K = \{(-5, 4), (1, 1), 1\}$.

The set K contains three elements—the points $(-5, 4)$ and $(1, 1)$, and the number 1. Since K contains at least one element that isn't a point (i.e. an ordered pair of real numbers), K is not a graph.[118]

Example 180. The set $\mathbb{R}$ contains at least one element that isn't a point (for example, 5). Indeed, $\mathbb{R}$ contains no points at all and infinitely many elements that are not points. Thus, $\mathbb{R}$ is not a graph.

(The same is true of $\mathbb{Q}$ and $\mathbb{Z}$.)

You may be used to thinking of a graph as a "drawing". But you should now think of a graph as being simply **a set of points**. A "drawing" of a graph is not the graph itself, but merely a visual aid.[119]

[118] As explained in Remark 27, a real number can also be regarded as a point in one-dimensional space. However, as stated earlier, for now, whenever we say *point*, we mean a point in two-dimensional space. That is, a point is strictly an ordered pair of real numbers. And so, following this usage, the number 1 here is **not** a point.

[119] Albeit a tremendously helpful one. Indeed, **analytic** or **cartesian geometry** was one of the major milestones in the history of mathematics. The idea of combining algebra and geometry is today "obvious" even to the secondary school student, but wasn't always so to mathematicians.

Remark 28. Your A-Level exams seem to use the terms **graph** and **curve** interchangeably (i.e. as entirely equivalent synonyms), so that's what this textbook will do too.

7.4. The Graph of An Equation

In the previous subchapter, we looked at graphs that were simply sets of random, isolated points.

But going forward, we'll be looking mostly at **graphs of equations** (and shortly, also of functions):

Definition 39. The *graph of an equation* is the set of points for which the equation is true.

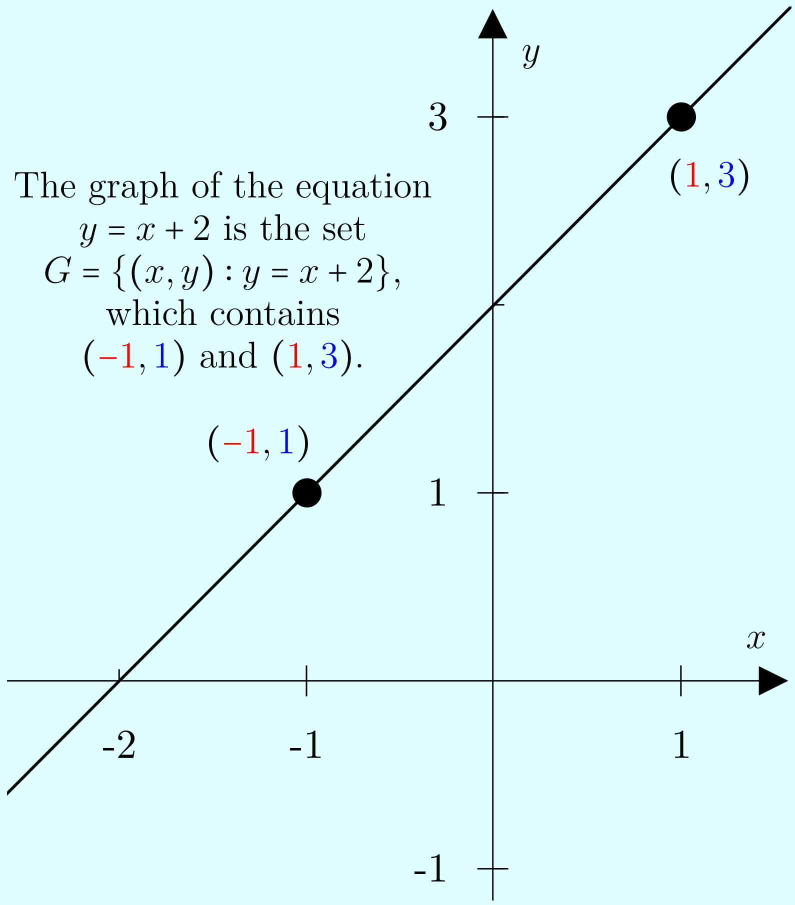
Example 181. Let G be the graph of the equation $y = x + 2$. Then

$$G = \{(x, y) : y = x + 2\}.$$

In words, G is the set of points (x, y) for which the equation $y = x + 2$ is true. And so, G contains

$$(-1, 1), \text{ because } 1 = -1 + 2; \qquad \text{and} \qquad (1, 3), \text{ because } 3 = 1 + 2.$$

We can say, "Below we've graphed the equation $y = x + 2$" or more simply, "Below is graphed $y = x + 2$," or, "Below is graphed G".



We'll also say that the equation $y = x + 2$ *describes* a **line**—specifically, the line with **gradient** 1 and which passes through the point $(0, 2)$.

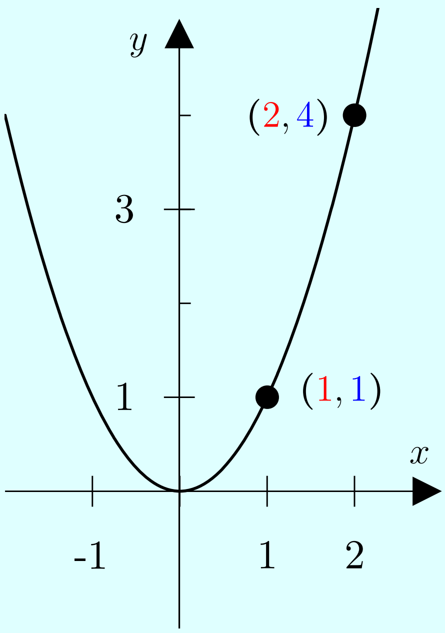
(We'll learn more about lines and gradients in Ch. 8.)

Example 182. Let H be the graph of the equation $y = x^2$. Then

$$H = \left\{(x, y) : y = x^2\right\}.$$

In words, H is the set of points (x, y) for which the equation $y = x^2$ is true. And so, H contains

$$(1, 1), \text{ because } 1 = 1^2; \qquad \text{and} \qquad (2, 4), \text{ because } 4 = 2^2.$$



We'll also say that the equation $y = x^2$ *describes* a parabola.

7.5. Graphing with the TI84

Our first examples of using the TI84 (see “Use of Graphing Calculators”, p. xxxvii):

Example 183. Graph the equation $y = x^2$.

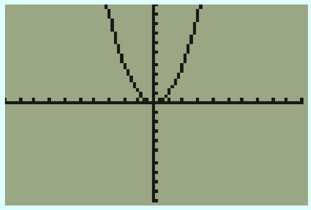
1. Press ON to turn on your calculator.
2. Press Y= to bring up the Y= editor.
3. Press X,T,θ,n to enter “X”; then x^2 to enter the squared “^{2}” symbol.
4. Now press GRAPH and the calculator will graph $y = x^2$.

Step 1.

Step 2.

Step 3.

Step 4.



(Screenshots are of the end of each Step.)

Example 184. XXX

Exercise 75. Graph these equations. (Answer on p. 1762.)

(a) $y = e^x$. **(b)** $y = 3x + 2$. **(c)** $y = 2x^2 + 1$.

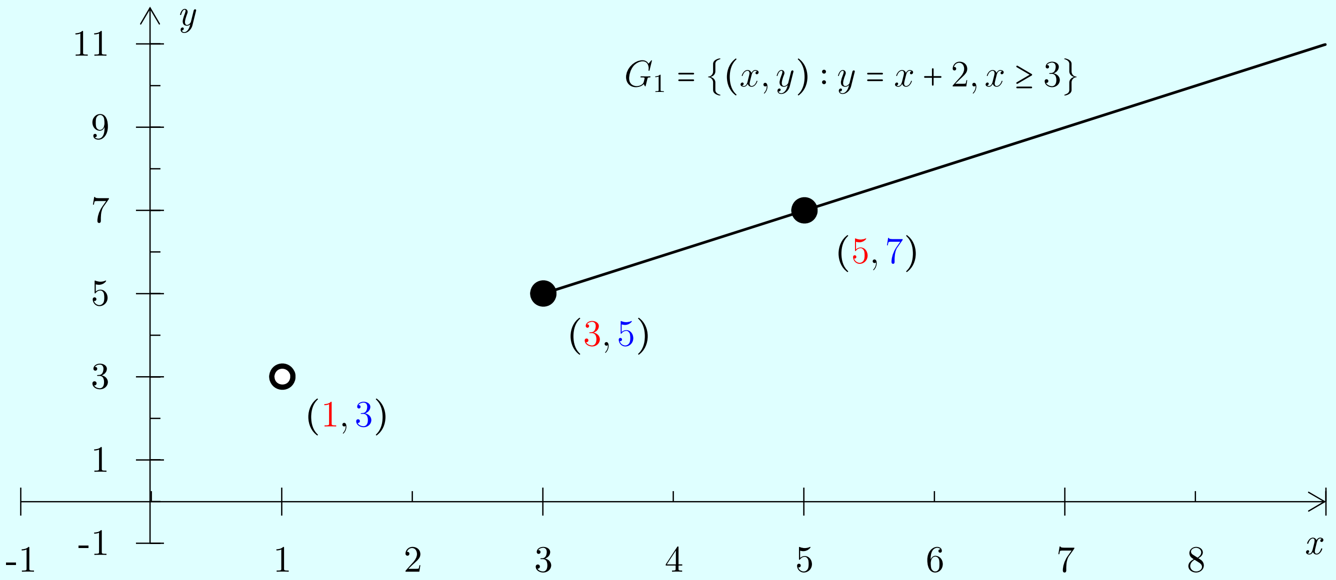
Example 185. Let G_1 be the graph of the equation $y = x + 2$ with the constraint $x \geq 3$. Equivalently, let G_1 be the graph of $y = x + 2$, $x \geq 3$. Then

$$G_1 = \{(x, y) : y = x + 2,\ x \geq 3\}.$$

In words, G_1 is the set of points (x, y) for which "$y = x + 2$ AND $x \geq 3$" is true. And so, G_1 contains

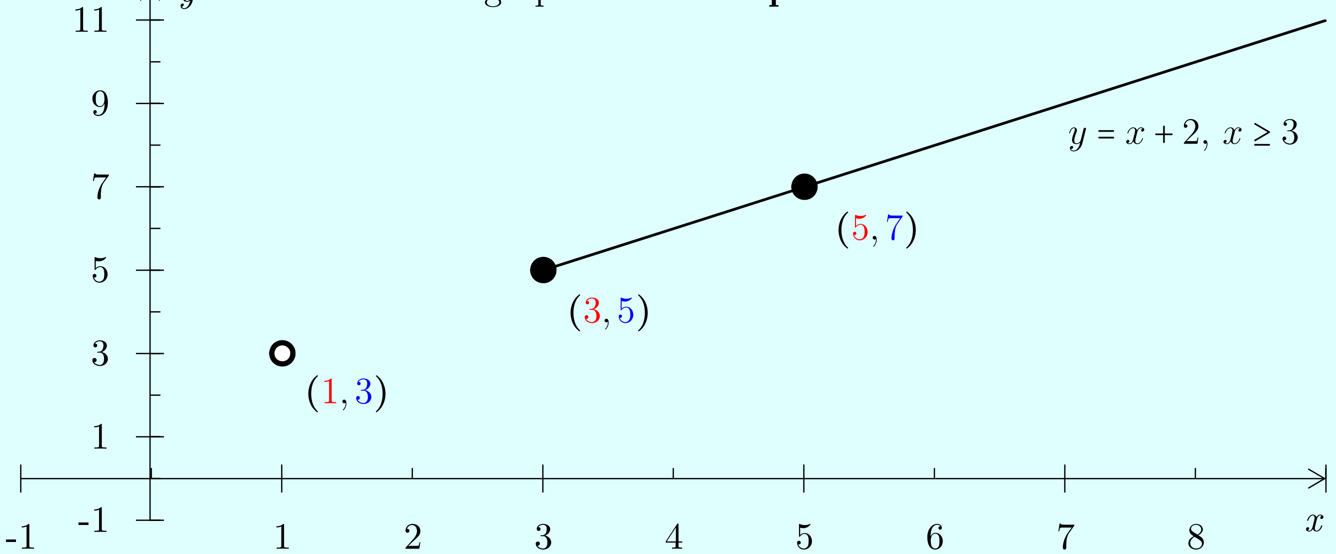
- $(5, 7)$, because $7 = 5 + 2$ AND $5 \geq 3$.
- $(3, 5)$, because $5 = 3 + 2$ AND $3 \geq 3$.

In contrast, G_1 does **not** contain $(1, 3)$, because $1 \ngeq 3$.



Above we labelled our graph as $G_1 = \{(x, y) : y = x + 2,\ x \geq 3\}$. But going forward, we'll be a little lazy/sloppy and simply label it as $y = x + 2$, $x \geq 3$ (as done below), with the understanding that this is the graph that satisfies the equation and constraint contained in that label. Nonetheless, you should always remember:

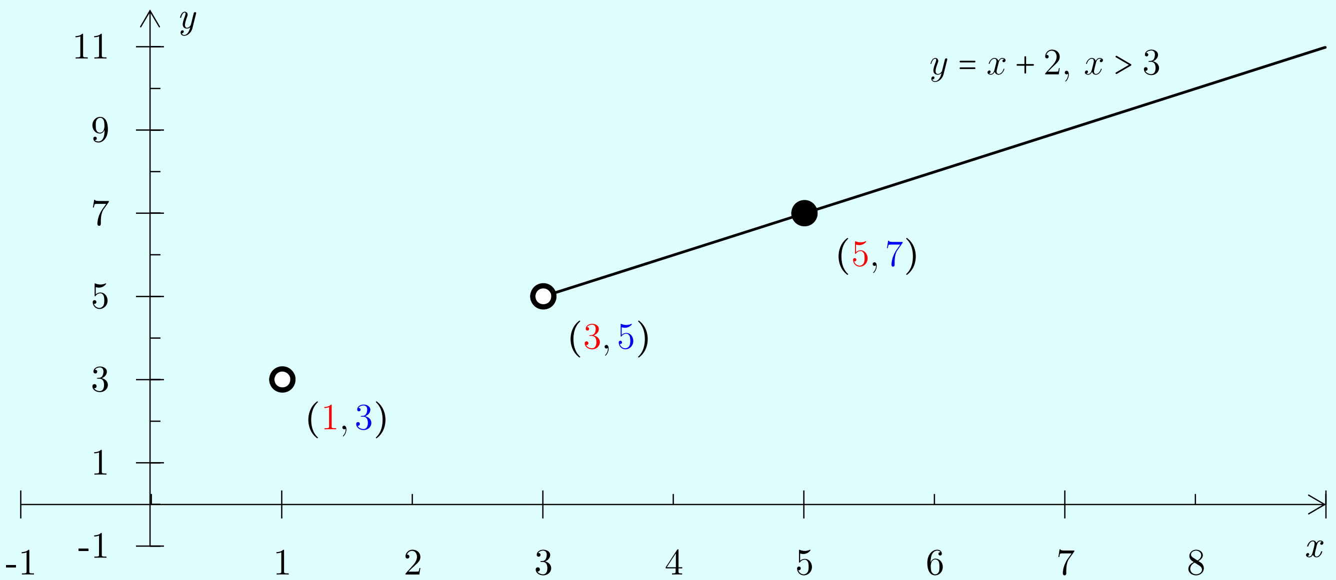
A graph is a **set of points**.



Example 186. Let G_2 be the graph of $y = x + 2$, $x > 3$. Then

$$G_2 = \{(x, y) : y = x + 2,\ x > 3\}.$$

The set G_2 is exactly the same as G_1 but with one difference—the constraint (inequality) is now strict, so that this time, G_2 does **not** contain $(3, 5)$.



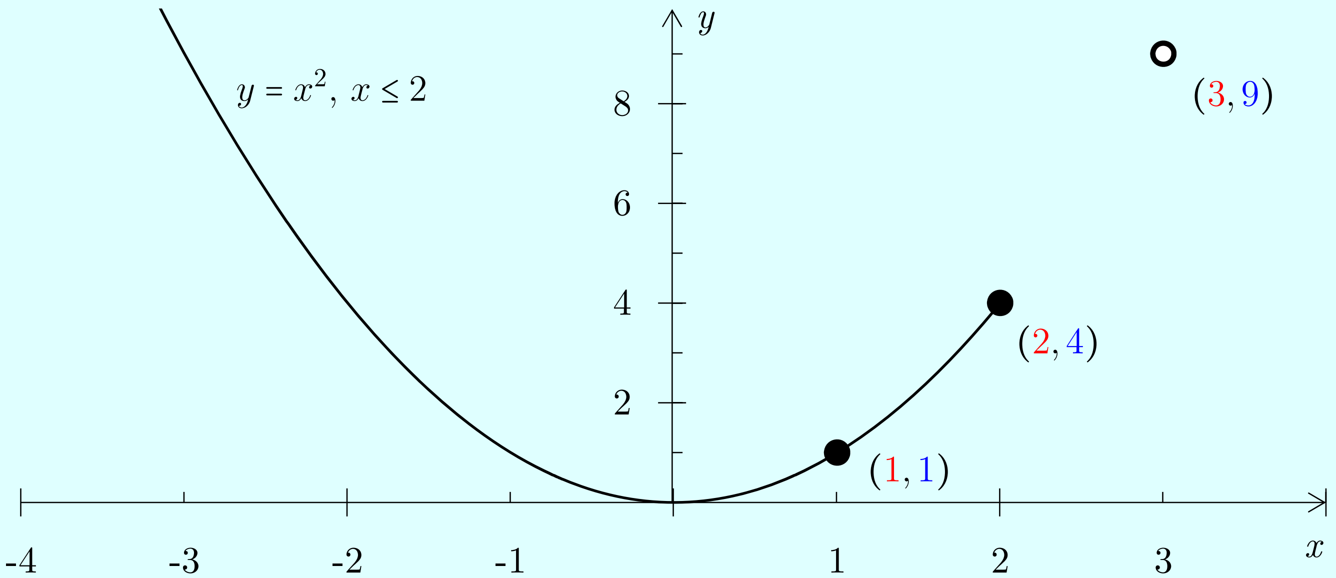
Example 187. Let H_1 be the graph of $y = x^2$, $x \leq 2$. Then

$$H_1 = \left\{(x, y) : y = x^2,\ x \leq 2\right\}.$$

In words, H_1 is the set of points (x, y) for which "$y = x^2$ AND $x \leq 2$" is true. And so, H_1 contains

- $(1, 1)$, because $1^2 = 1$ AND $1 \leq 2$.
- $(2, 4)$, because $2^2 = 4$ AND $2 \leq 2$.

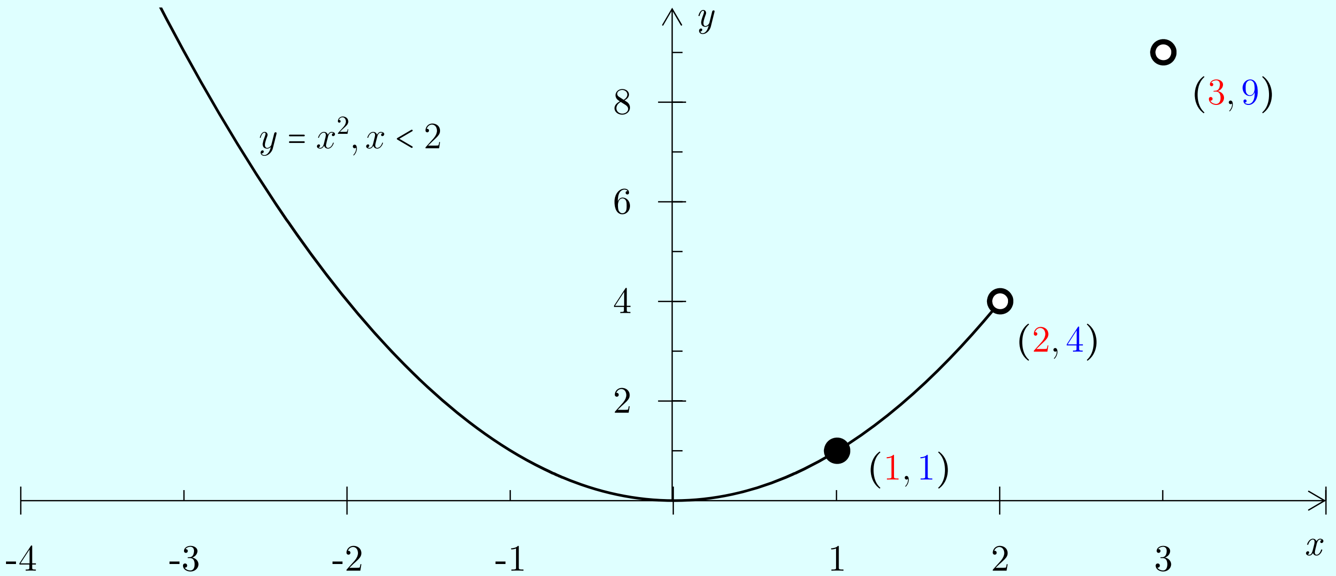
In contrast, H_1 does **not** contain $(3, 9)$, because $3 \nleq 2$.



Example 188. Let H_2 be the graph of $y = x^2$, $x < 2$. Then

$$H_2 = \left\{(x, y) : y = x^2, x < 2\right\}.$$

The set H_2 is exactly the same as H_1 but with one difference—the constraint (inequality) is now strict, so that this time, H_2 does **not** contain $(2, 4)$.

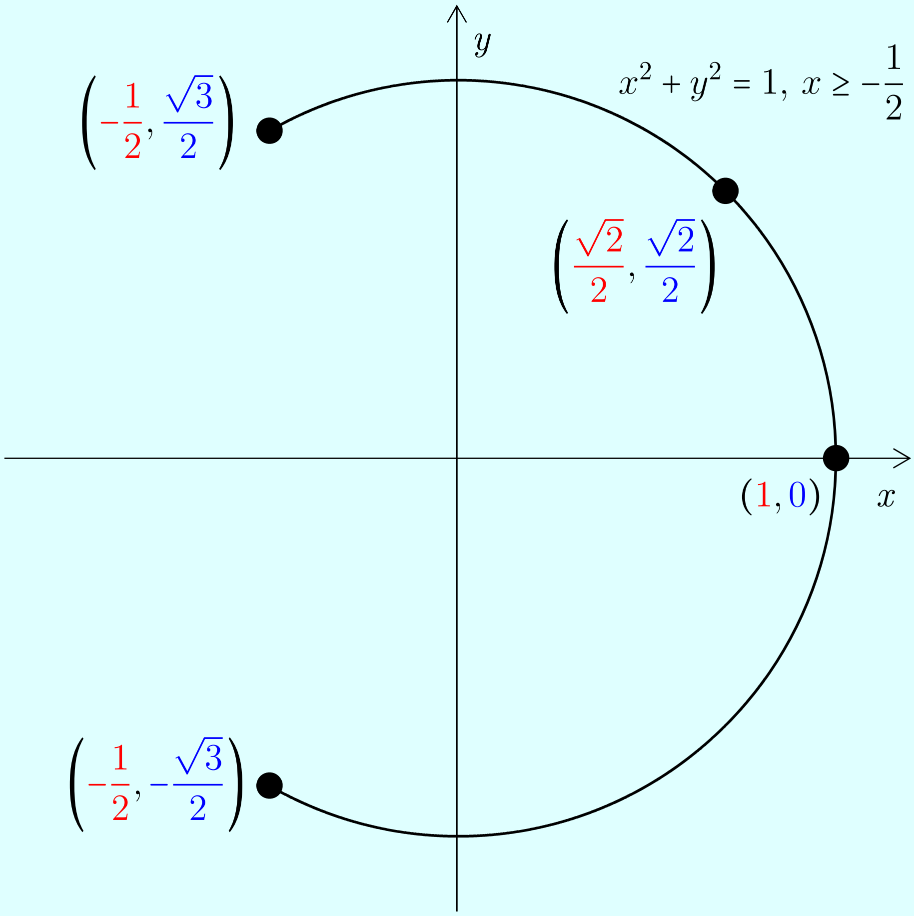


Example 189. Let I_1 be the graph of $x^2 + y^2 = 1$, $x \geq -\frac{1}{2}$. Then

$$I_1 = \left\{(x,y) : x^2 + y^2 = 1,\ x \geq -\frac{1}{2}\right\}.$$

In words, I_1 is the set of points (x,y) for which "$x^2 + y^2 = 1$ AND $x \geq -\frac{1}{2}$" is true. And so, I_1 contains

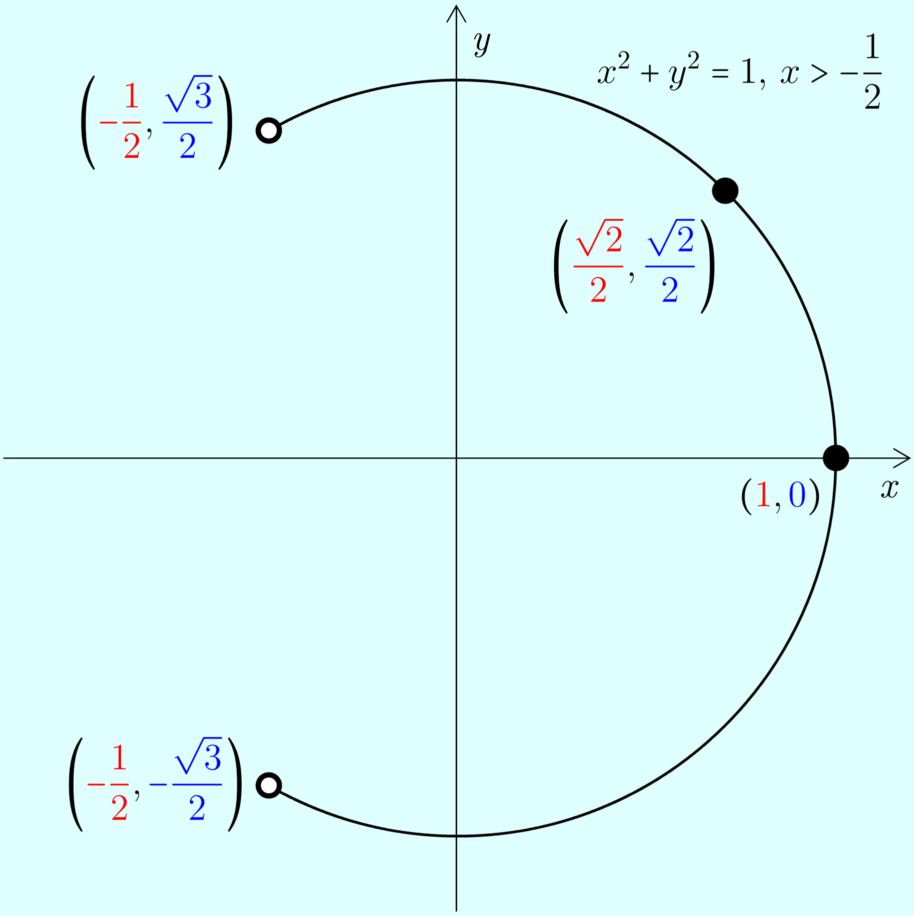
- $(1,0)$, because $1^2 + 0^2 = 1$ AND $1 \geq -0.5$.
- $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, because $\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$ AND $\frac{\sqrt{2}}{2} \geq -\frac{1}{2}$.
- $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, because $\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$ AND $-\frac{1}{2} \geq -\frac{1}{2}$.
- $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, because $\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = 1$ AND $-\frac{1}{2} \geq -\frac{1}{2}$.



Example 190. Let I_2 be the graph of $x^2 + y^2 = 1$, $x > -\dfrac{1}{2}$. Then

$$I_2 = \left\{ (x, y) : x^2 + y^2 = 1,\ x > -\frac{1}{2} \right\}.$$

The set I_2 is exactly the same as I_1 but with one difference—the constraint (inequality) is now strict, so that this time, I_2 does **not** contain $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$ or $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2} \right)$.

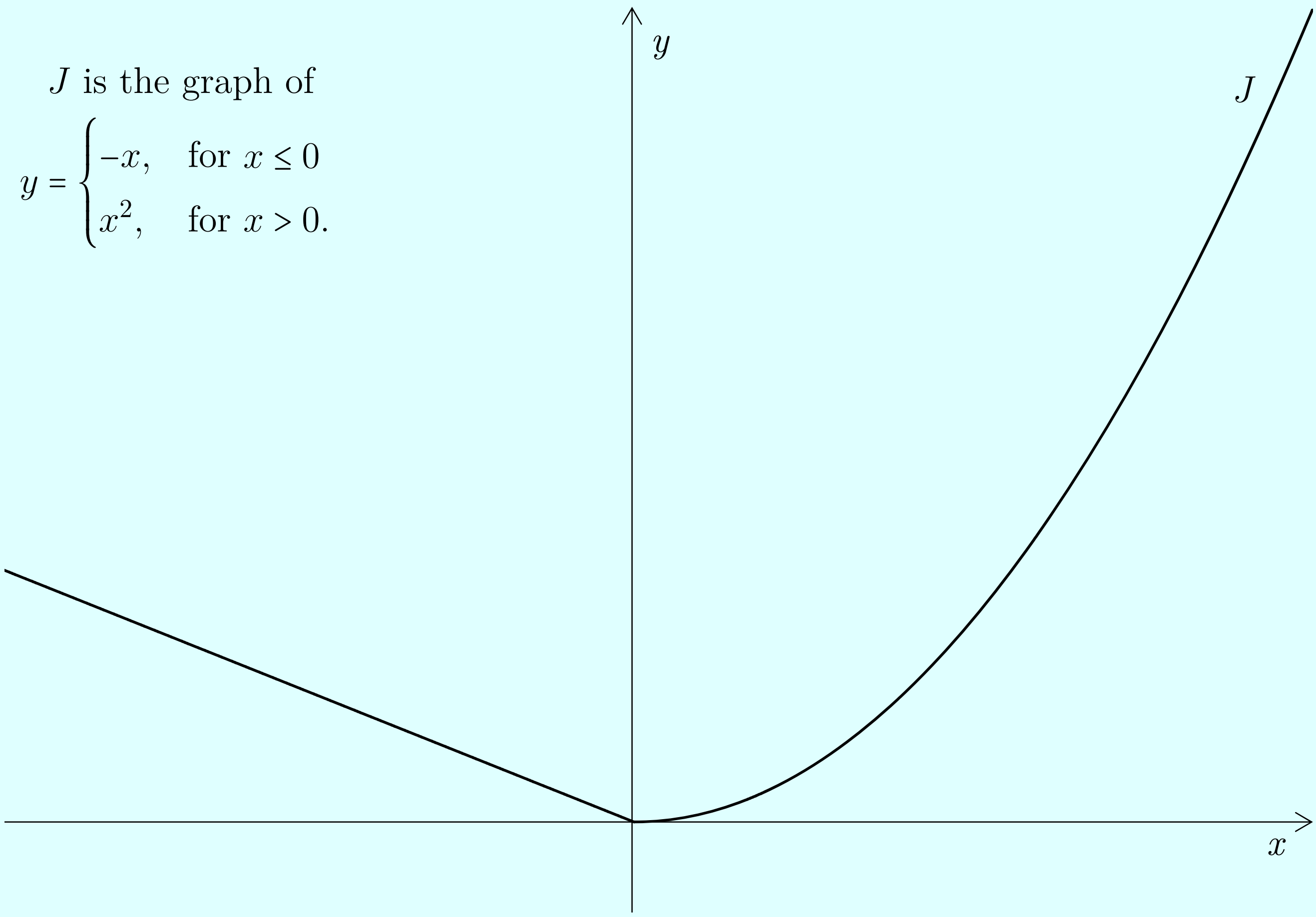


More generally, a graph can also be of multiple equations with multiple constraints:

Example 191. Let J be the graph of

$$y = \begin{cases} -x, & \text{for } x \le 0, \\ x^2, & \text{for } x > 0. \end{cases}$$

In words, J is the set of points for which either "$y = -x$ AND $x \le 0$" or "$y = x^2$ AND $x > 0$" is true.



We can actually write down J using set-builder notation and the union operator:

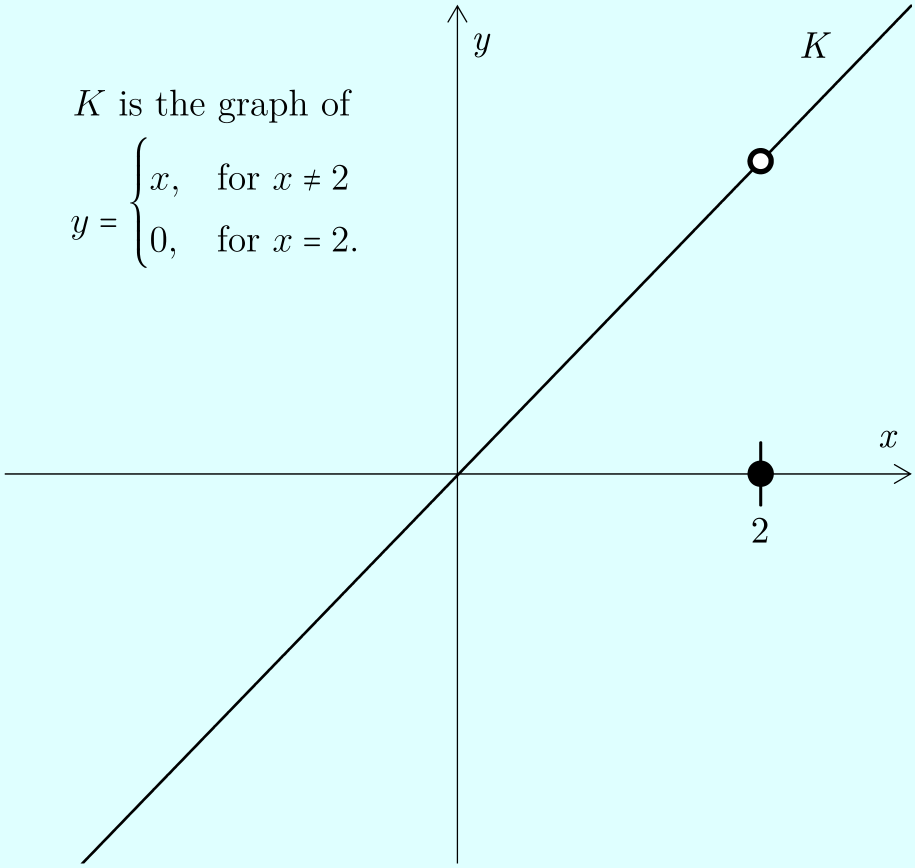
$$J = \{(x, y) : y = -x, x \le 0\} \cup \{(x, y) : y = x^2, x > 0\}.$$

But this is cumbersome and difficult to read. And so, we'll usually simply specify J as was done above.

Example 192. Let K be the graph of

$$y = \begin{cases} x, & \text{for } x \neq 2, \\ 0, & \text{for } x = 2. \end{cases}$$

In words, K is the set of points for which either "$y = x$ AND $x \neq 2$" or "$y = 0$ AND $x = 2$" is true.



Exercise 76. Graph each of the following. (Answers on p. 1763.)

(a) $y = e^x$, $-1 \le x < 2$.

(b) $y = 3x + 2$, $-1 \le x < 2$.

(c) $y = 2x^2 + 1$, $-1 \le x < 2$.

Exercise 77. Graph each of the following. (Answer on p. 1764.)

(a) $y = \begin{cases} x + 1, & \text{for } x \le 0, \\ x - 1, & \text{for } x > 0. \end{cases}$

(b) $y = \begin{cases} x + 1, & \text{for } x < 0, \\ x - 1, & \text{for } x \ge 0. \end{cases}$

7.7. Intercepts and Roots

In secondary school, you learnt about x-**intercepts**, y-**intercepts**, and **solutions** (or **roots**). We now give their formal definitions:

Definition 40. Let G be a graph.

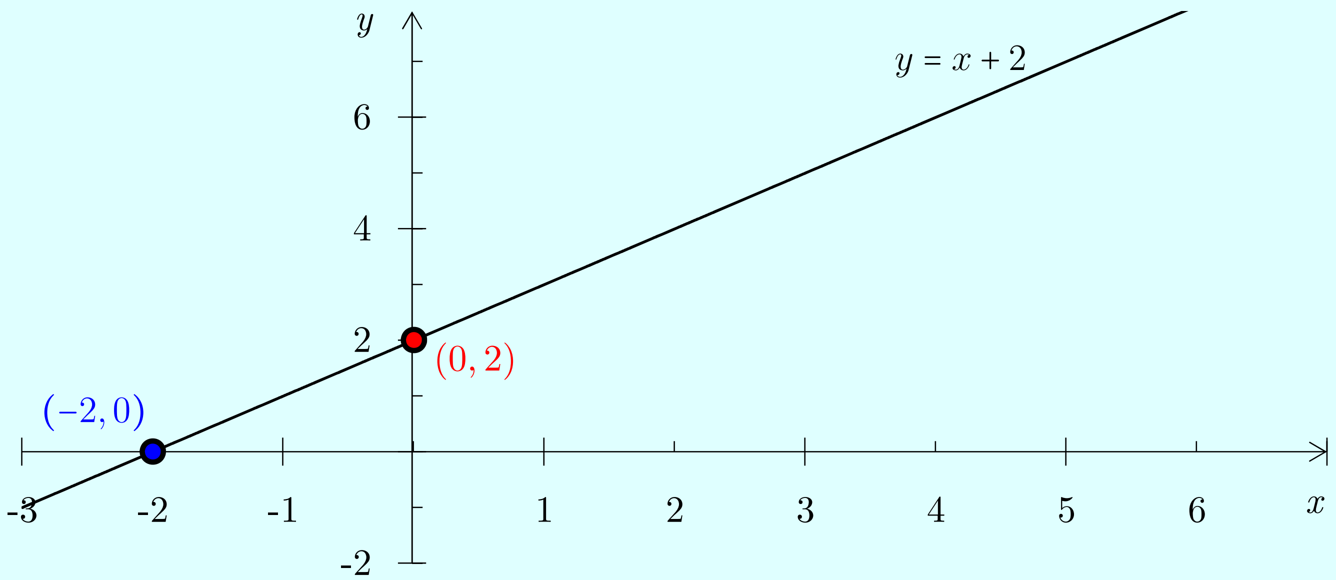
- If the point $(0, b)$ is in G, then we call $(0, b)$ a *vertical* or *y-intercept* of G.
- If the point $(a, 0)$ is in G, then we call $(a, 0)$ a *horizontal* or *x-intercept* of G.

  If, moreover, G is the graph of an equation, then we also call a a *solution* or *root* of that equation.

Example 193. The graph of the equation $y = x + 2$ has horizontal or x-intercept $(-2, 0)$ and vertical or y-intercept $(0, 2)$.

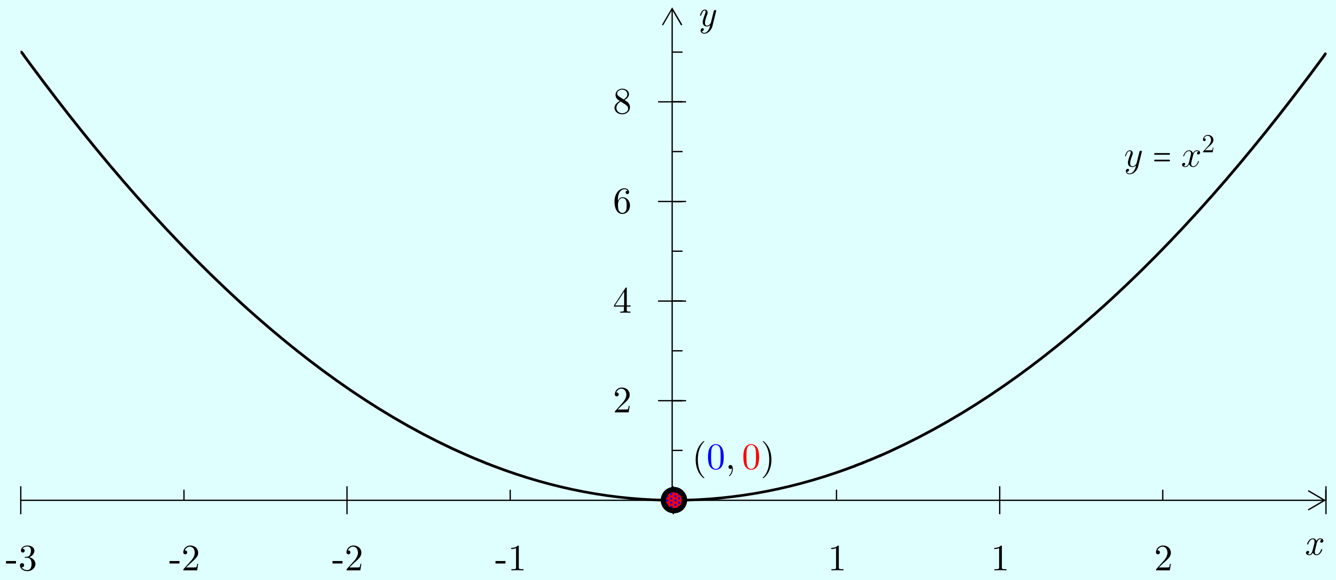
Or more simply, "The equation $y = x + 2$ has horizontal or x-intercept $(-2, 0)$ and vertical or y-intercept $(0, 2)$."

We also say that -2 is a **solution** (or **root**) of the equation $y = x + 2$.

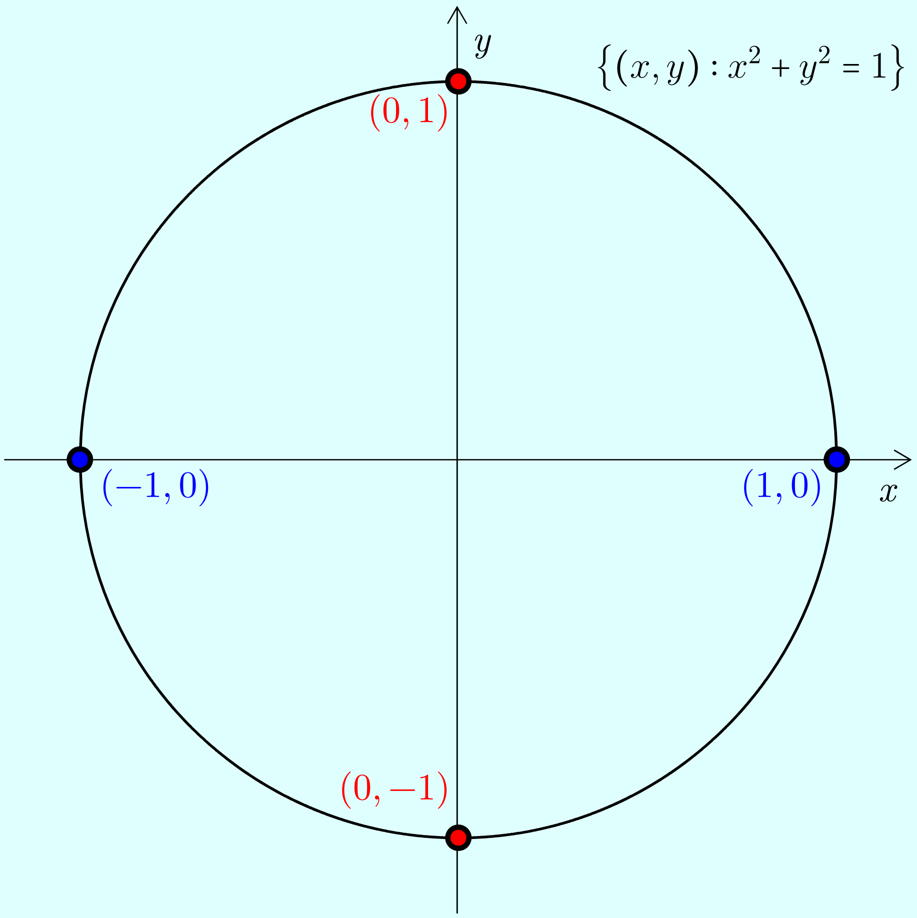


Remark 29. Just so you know, some writers (including your TI84) also call a *solution* (or *root*) a **zero**. So in the above example, they'd say that the **zero** of the equation $y = x + 2$ is -2. However, we will avoid using the term **zero** in this textbook because it does not appear on your A-Level exams or syllabus.

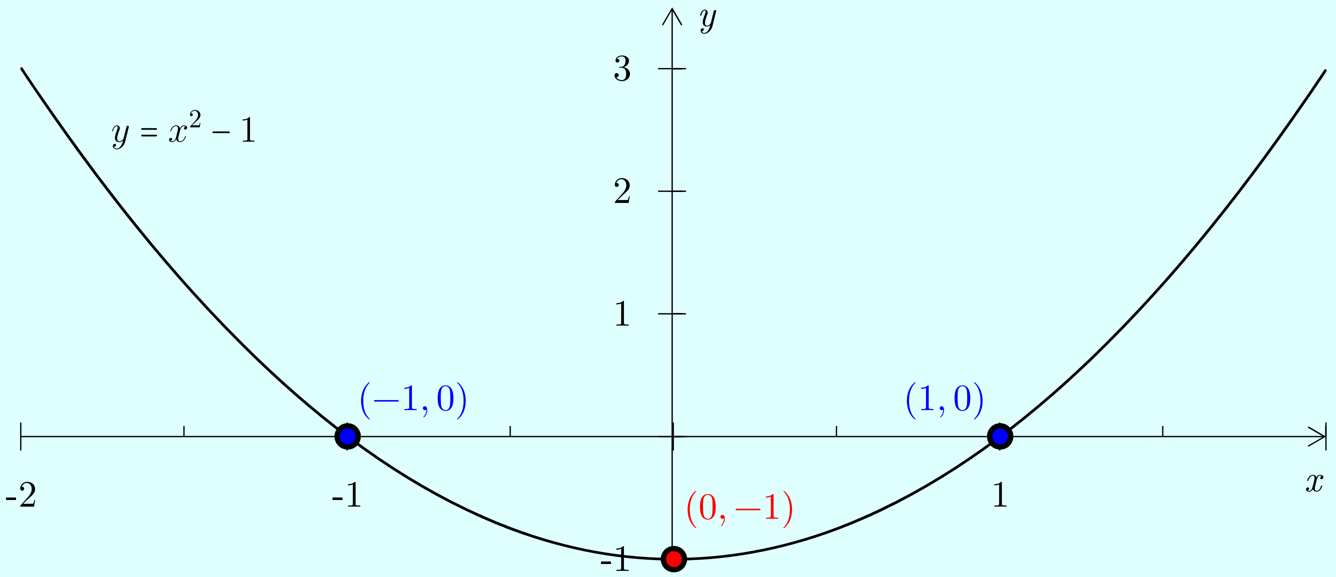
Example 194. Consider the equation $y = x^2$. The point $(0, 0)$ is *both* its (only) x-intercept and its (only) y-intercept. Also, the equation has solution (or root) 0.



Example 195. The equation $x^2 + y^2 = 1$ has two x-intercepts $(-1, 0)$ and $(1, 0)$, two y-intercepts $(0, -1)$ and $(0, 1)$, and two solutions (or roots) -1 and 1.



Example 196. The equation $y = x^2 - 1$ has two x-intercepts $(-1, 0)$ and $(1, 0)$, one y-intercept -1, and two solutions (or roots) -1 and 1.



Exercise 78. For each equation, write down any x-intercept(s), y-intercept(s), and solutions (or roots). (Answer on p. 1764.)

(a) $y = 2$.

(b) $y = x^2 - 4$.

(c) $y = x^2 + 2x + 1$.

(d) $y = x^2 + 2x + 2$.

8. Lines

You already know what a **line** is, informally and intuitively. Here's a formal definition:[120]

Definition 41. A *line* is the graph of any equation

$$ax + by + c = 0,$$

where $a, b, c \in \mathbb{R}$ and at least one of a or b is non-zero.

You may find this definition a little puzzling—didn't we always simply write lines as

$$y = dx + e?$$

It turns out that if $b \neq 0$, then $ax + by + c = 0$ and $y = dx + e$ are equivalent, as we now prove:

$$ax + by + c = 0 \qquad \iff \qquad by = -ax - c \qquad \iff \qquad y = \underbrace{-\frac{a}{b}}_{d} x \underbrace{-\frac{c}{b}}_{e}.$$

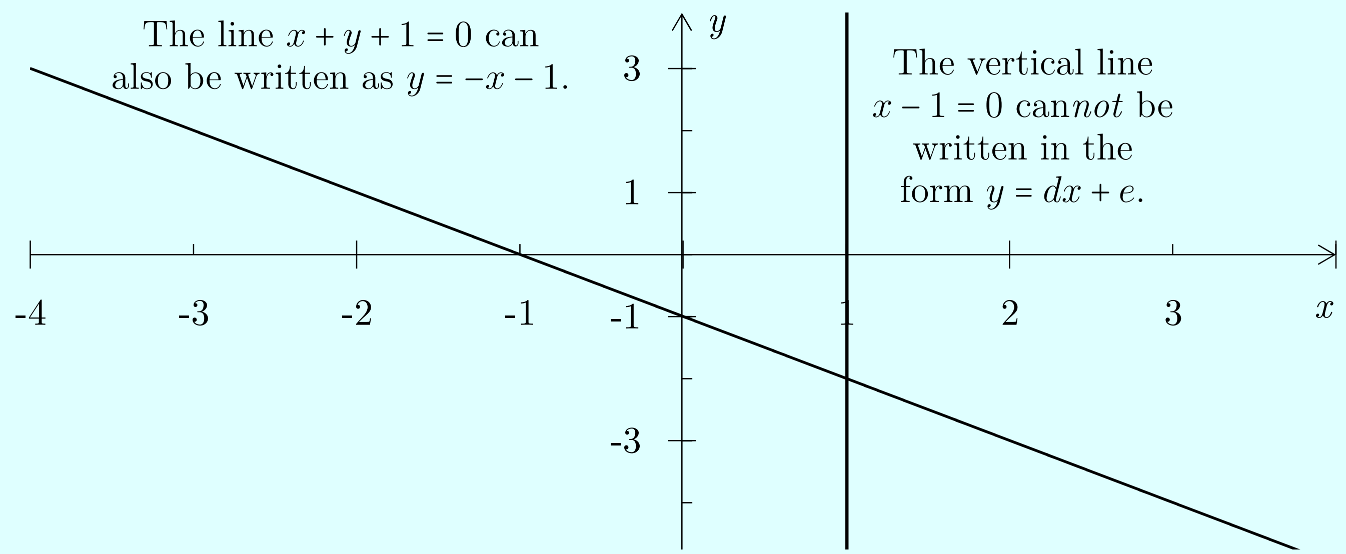
Writing a line as $y = dx + e$ has two advantages—it immediately tells us that its

gradient[121] is d and y**-intercept** is $(0, e)$.

But writing a line as $y = dx + e$ also has one big disadvantage—it can't describe the case where $b = 0$, i.e. **vertical lines**:

Example 197. The line $x + y + 1 = 0$ can also be written as $y = -x - 1$.

In contrast, the vertical line $x - 1 = 0$ can*not* be written in the form $y = dx + e$.



Here's what we'll do in this textbook: If we know *for sure* that a line isn't vertical, then we'll write it in the form $y = dx + e$, because of the two advantages. Otherwise, we'll write it as $ax + by + c = 0$.

[120] Definition 41 covers only lines in 2D space. In Part IV (Vectors), we'll learn a more general definition (Definition 143) that covers also lines in higher-dimensional spaces and supersedes Definition 41.

[121] We'll formally define the **gradient** of a line in Ch. 8.3.

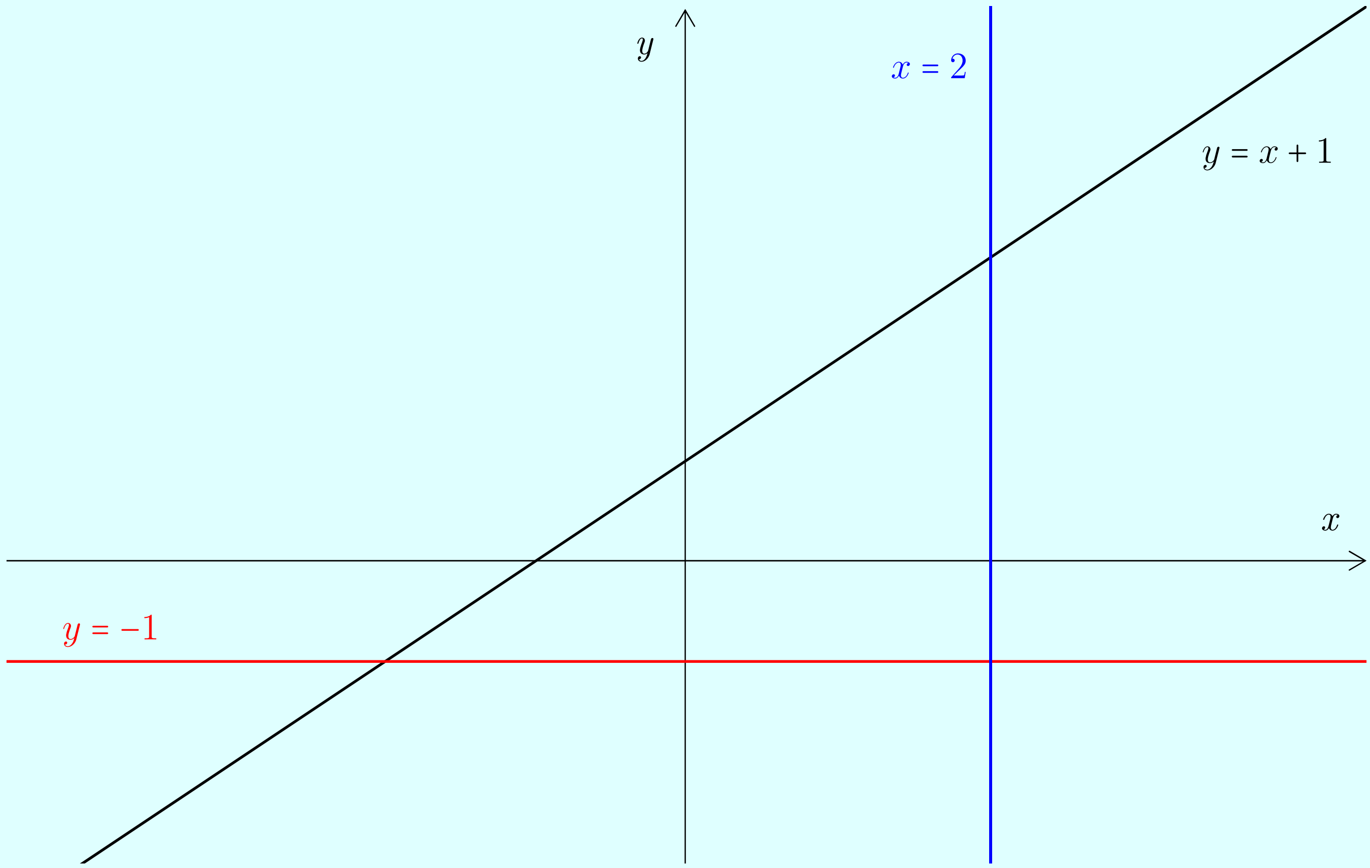
Definition 42. A line is

(a) *Horizontal* if any two points (in that line) have the same y-coordinate;

(b) *Vertical* if any two points have the same x-coordinate; and

(c) *Oblique* (or *slanted*) if it is neither horizontal nor vertical.

Example 198. The line $y = -1$ is horizontal, the line $x = 2$ is vertical, and the line $y = x + 1$ is oblique.



Fact 25. *Suppose $ax + by + c = 0$ describes a line. Then*

(a) *The line is horizontal* $\iff$ $a = 0$.

(b) *The line is vertical* $\iff$ $b = 0$.

Proof. See p. 1557 in the Appendices. □

Example 199. The line $y = -1$ is horizontal because the coefficient on x is zero.

The line $x = 2$ is vertical because the coefficient on y is zero.

The line $y = x + 1$ is oblique because the coefficients on x and y are both non-zero.

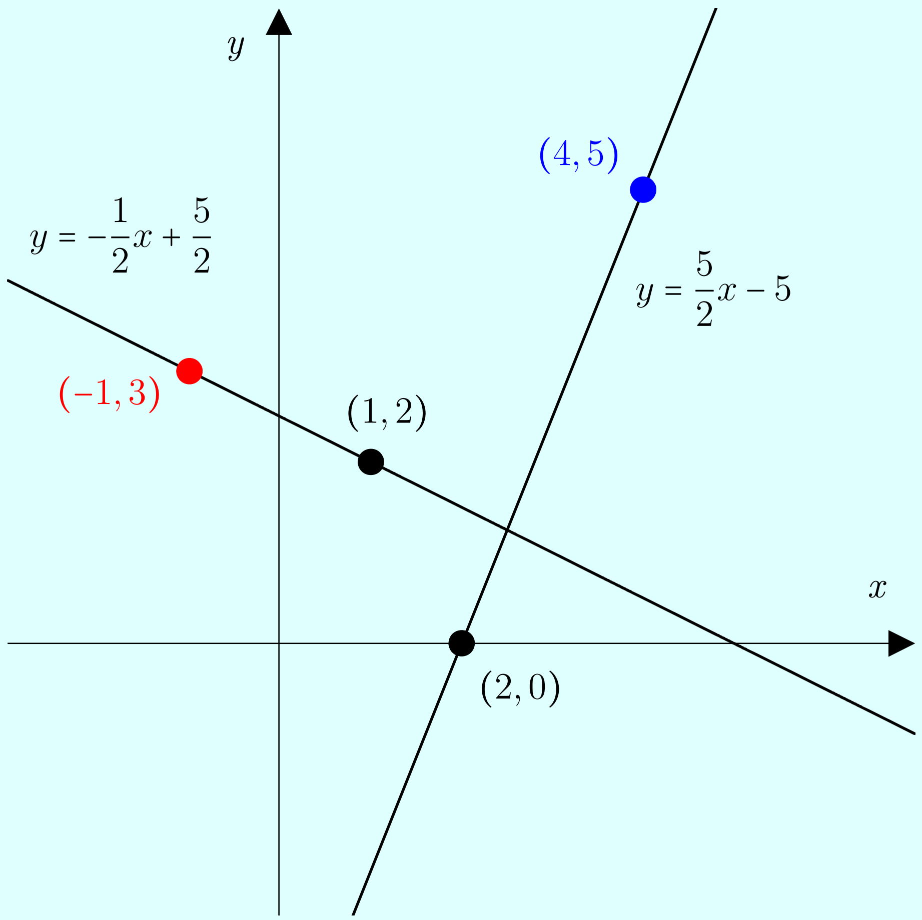
Fact 26. *The unique line that contains both of the distinct points* (x_1, y_1) *and* (x_2, y_2) *is*

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1).$$

Proof. See p. 1558 in the Appendices. □

Example 200. The line containing the points $(1, 2)$ and $(-1, 3)$ is

$$(-1 - 1)(y - 2) = (3 - 2)(x - 1) \qquad \text{or} \qquad -2y = x - 5 \qquad \text{or} \qquad y = -\frac{1}{2}x + \frac{5}{2}.$$



The line containing the points $(2, 0)$ and $(4, 5)$ is

$$(4 - 2)(y - 0) = (5 - 0)(x - 2) \qquad \text{or} \qquad 2y = 5x - 10 \qquad \text{or} \qquad y = \frac{5}{2}x - 5.$$

If $x_1 \neq x_2$ and $y_1 \neq y_2$, then we can rewrite the equation in Fact 26 as

$$\frac{y - y_1}{y_2 - y_1} \stackrel{\star}{=} \frac{x - x_1}{x_2 - x_1}.$$

This latter equation is perhaps more familiar and easier to remember. But of course, it's illegal if $x_1 = x_2$ or $y_1 = y_2$, which is why we sometimes prefer the equation in Fact 26.

We reuse the last example:

Example 201. The line containing the points $(1,2)$ and $(-1,3)$ is

$$\frac{y-2}{3-2} \overset{\star}{=} \frac{x-1}{-1-1} \qquad \text{or} \qquad y-2=-\frac{1}{2}(x-1) \qquad \text{or} \qquad y=-\frac{1}{2}x+\frac{5}{2}.$$

The line containing the points $(2,0)$ and $(4,5)$ is

$$\frac{y-0}{5-0} \overset{\star}{=} \frac{x-2}{4-2} \qquad \text{or} \qquad y=\frac{5}{2}(x-2)=\frac{5}{2}x-5.$$

Exercise 79. Find the line that contains each pair of points.

(a) $(4,5)$ and $(7,9)$ **(b)** $(1,2)$ and $(-1,-3)$ (Answer on p. 1765.)

8.3. The Gradient of a Line

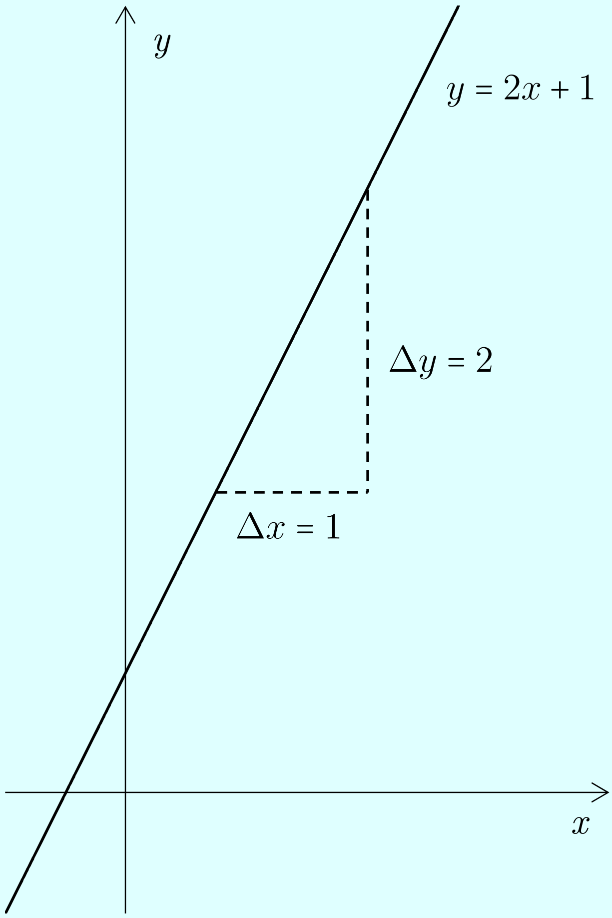
Informally, the **gradient** (or **slope**) of a line is

- "Rise over Run";
- $\Delta y / \Delta x$ or "Change in y over change in x" (Δ is the upper-case Greek letter *delta*);
- The answer to this question:

  "If we move 1 unit to the right while remaining along the line, by how many units will we move up?"

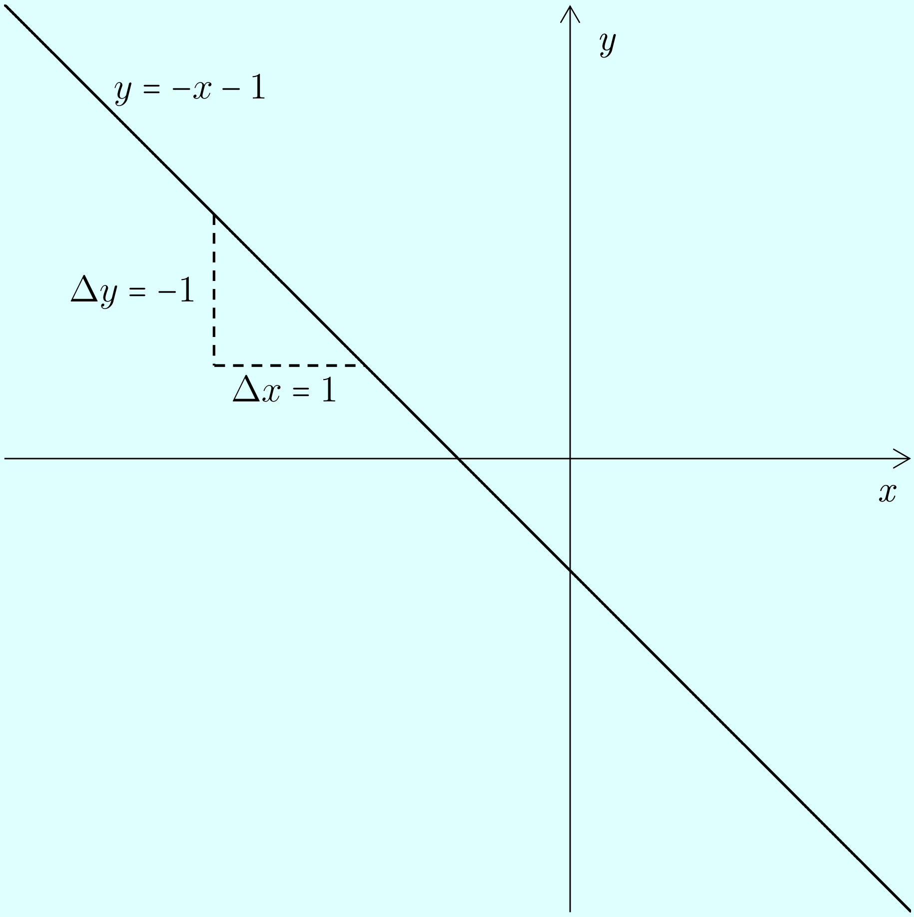
Example 202. The gradient of the line $y = 2x + 1$ is

$$\frac{\Delta y}{\Delta x} = \frac{2}{1} = 2.$$



Example 203. The gradient of the line $y = -x - 1$ is

$$\frac{\Delta y}{\Delta x} = \frac{-1}{1} = -1.$$



Remark 30. **Slope** is a perfectly good synonym for **gradient**. However, your A-Level syllabus and exams do not use the word *slope*. And so we'll stick to using only the word *gradient*.

In general, to find a line's **gradient**, pick two distinct points on the line (x_1, y_1) and (x_2, y_2), then compute

$$\text{Gradient} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Figure to be
inserted here.

One caveat is that if the line is vertical, then $x_1 = x_2$ so that our last expression has denominator 0 and is thus undefined. And so, if a line is vertical, we'll simply say that its gradient is undefined.

Let's jot the above down as a formal definition:

Definition 43. Suppose a non-vertical line contains the distinct points (x_1, y_1) and (x_2, y_2). Then its *gradient* is the number defined as

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

If a line is vertical, then its gradient is simply left undefined.

Fact 27. *Suppose* $ax + by + c = 0$ *is a non-vertical line. Then its gradient is* $-a/b$.

Proof. Since the line $ax + by + c = 0$ is non-vertical, $b \neq 0$ (Fact 25).

Case 1. If $a = 0$, then the line contains the points $\left(0, -\frac{c}{b}\right)$ and $\left(1, -\frac{c}{b}\right)$, so that by Definition 43, its gradient is

$$\frac{-\frac{c}{b} - \left(-\frac{c}{b}\right)}{1 - 0} = 0 = -\frac{a}{b}. \qquad \checkmark$$

Case 2. If $a \neq 0$, then the line contains the points $\left(0, -\frac{c}{b}\right)$ and $\left(1, -\frac{a+c}{b}\right)$, so that by Definition 43, its gradient is

$$\frac{-\frac{a+c}{b} - \left(-\frac{c}{b}\right)}{1 - 0} = -\frac{a}{b}. \qquad \square$$

Example 204. XXX

Example 205. XXX

Exercise 80. XXX (Answer on p. 131.)

A80.

Fact 28. *The line that contains the point* (x_1, y_1) *and has gradient* m *is*

$$y - y_1 = m(x - x_1).$$

Proof. Let the line be $ax + by + c \stackrel{1}{=} 0$.

The line's gradient is $m \stackrel{2}{=} -a/b$. Plugging $\stackrel{2}{=}$ into $\stackrel{1}{=}$ yields $y + \frac{c}{b} \stackrel{3}{=} mx$.[122]

Also, the line contains (x_1, y_1) — plugging this into $\stackrel{3}{=}$ yields $y_1 + \frac{c}{b} = mx_1$ or $\frac{c}{b} \stackrel{4}{=} mx_1 - y_1$.

Plug $\stackrel{4}{=}$ into $\stackrel{3}{=}$ to get $y + mx_1 - y_1 = mx$ or $y - y_1 = m(x - x_1)$. □

Example 206. The line that contains the point $(-1, 2)$ and has gradient 3 is[123]

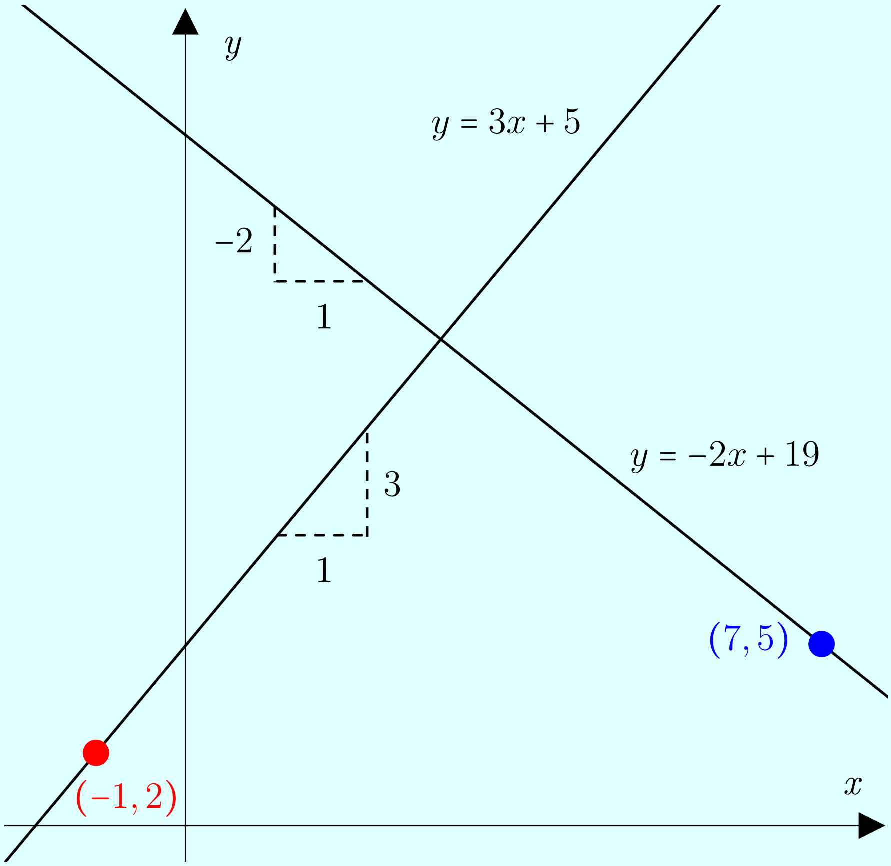
$$y - 2 = 3[x - (-1)],$$

or, $$y = 3x + 5.$$

The line that contains the point $(7, 5)$ and has gradient -2 is

$$y - 5 = -2(x - 7),$$

or, $$y = -2x + 19.$$



Exercise 81. Find the line with the given point and gradient. (Answer on p. 1765.)

(a) Point $(4, 5)$ and gradient 3 **(b)** Point $(1, 2)$ and gradient -2

[122]Since the line's gradient is defined, $b \neq 0$.

[123]It looks like Wolfram Alpha understands *slope* but not *gradient*.

8.5. Parallel and Perpendicular Lines

Definition 44 (*informal*)**.** Two lines are *parallel* if their gradients are equal.

If two lines l and m are parallel, we'll write $l \parallel m$; if not, we'll write $l \nparallel m$.

Definition 45 (*informal*)**.** Two lines are *perpendicular* if

(a) Their gradients are negative reciprocals of each other; or

(b) One line is vertical while the other is horizontal.

If two lines l and m are perpendicular, then we will also write $l \perp m$.

If two lines l and m are perpendicular, we'll write $l \perp m$; if they aren't, we'll write $l \not\perp m$.

In Part IV (Vectors), we'll give Definition 150, which is a more general definition of when two lines are parallel or perpendicular and which will supersede the above two definitions.

Example 207. XXX

Example 208. XXX

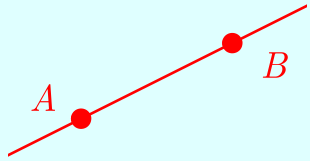
Exercise 82. XXX (Answer on p. 133.)

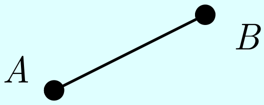
A82.

8.6. Lines vs Line Segments vs Rays

Example 209. Let A and B be points.

The **line** AB contains both A and B. It has infinite length, extending "forever" in both directions.

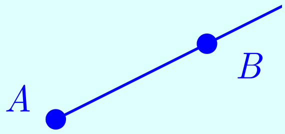


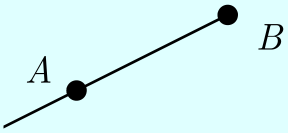


In contrast, the **line segment** AB has finite length—it contains A, B, and every point between, but **no other point**.

With lines and line segments, the order in which we write A and B doesn't matter. The line AB is the same as the line BA. And the line segment AB is the same as the line segment BA. But with **rays**, the order in which we write A and B *does* matter.

The **ray** AB is the "half-infinite line" that starts at A, passes through B, and also contains every point "beyond" B.





In contrast, the **ray** BA is the "half-infinite line" that starts at B, passes through A, and also contains every point "beyond" A.

The rays AB and BA are distinct. Both have infinite length.

It makes no sense to speak of the length of a line or a ray (because these have infinite length). We can however speak of a line segment's (finite) length.[124]

Remark 31. Confusingly, some other writers use *ray* to mean a (finite) line segment. But this textbook will strictly reserve the word *ray* to mean a "half-infinite line".

Remark 32. According to ISO 80000-2: 2009,[125] $\overline{AB}$ may be used to denote the line segment from point A to point B. This notation is convenient and allows us to distinguish between the line AB and the line segment $\overline{AB}$.

However, your A-Level syllabus and exams (see e.g. N2007/I/6) do not seem to use this notation.[126] And so, this textbook shall not do so either.

For us, the only way to tell whether "AB" refers to a line or a line segment is to see if we say "the line AB" or "the line segment AB". Thus, we must always be **absolutely clear** if we're talking about a line or a line segment.

[124]For the formal definitions of a line, a line segment, and a ray, see Definition 273 in the Appendices.

[125]See Item No. 2-8.4. The 40-something-page PDF costs a mind-blowing 158 Swiss Francs or about S$214 at the ISO store. As always, you may or may not be able to find free versions of this document elsewhere on the interwebz.

[126]Indeed, they don't seem to be very careful about distinguishing between lines and line segments.

9. Distance

9.1. The Distance between Two Points on the Real Number Line

On the real number line, the distance between two points (or real numbers) is simply the magnitude of their difference. Formally,

Definition 46. Let A and B be two real numbers. The *distance between* A *and* B, denoted $|AB|$, is the number defined by

$$|AB| = |B - A|.$$

Example 210. The distance between $A = 3$ and $B = 5$ is

$$|AB| = |B - A| = |5 - 3| = |2| = 2.$$

Example 211. The distance between $C = 7$ and $D = -2$ is

$$|CD| = |D - C| = |-2 - 7| = |-9| = 9.$$

Of course, the distance between A and B is the same as the distance between B and A:

$$|AB| = |B - A| = |A - B| = |BA|.$$

Exercise 83. Let $A = 5$, $B = -7$, and $C = 1$. What are $|AB|$, $|AC|$, and $|BC|$?(Answer on p. 1766.)

9.2. The Distance between Two Points on the Cartesian Plane

Definition 47. In a right triangle, we call the side facing the right angle the triangle's *hypotenuse* and the other two sides its *legs*.

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two points on the cartesian plane. How should we define the distance between A and B?

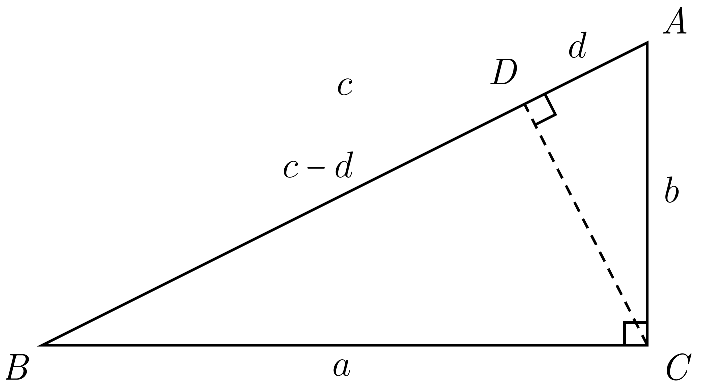
To motivate this definition, let's review Pythagoras' Theorem:

Theorem 2. (Pythagoras' Theorem) *Suppose a right triangle has legs of lengths a and b, and hypotenuse of length c. Then*

$$a^2 + b^2 = c^2.$$

Proof. Construct the triangle ABC, where the vertices A, B, and C are opposite the sides of lengths a, b, and c, respectively.

Let D be the point on AB that is the base of the perpendicular from the point C:

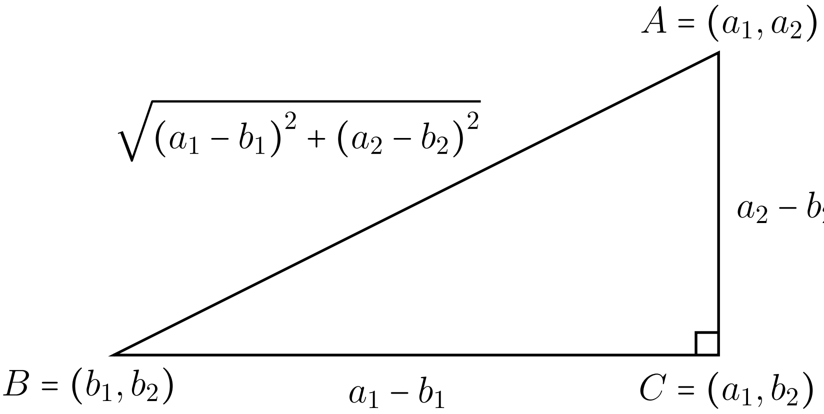


The triangles ABC and CBD are similar. Hence, $\dfrac{a}{c} = \dfrac{c-d}{a}$ or $a^2 \stackrel{1}{=} c(c-d)$.

The triangles ABC and ACD are similar. Hence, $\dfrac{b}{c} = \dfrac{d}{b}$ or $b^2 \stackrel{2}{=} cd$.

Now, $\stackrel{1}{=} + \stackrel{2}{=}$ yields $a^2 + b^2 = c(c-d) + cd = c(c-d+d) = c^2$. $\square$

So, if we draw $A = (a_1, a_2)$ and $B = (b_1, b_2)$ as two vertices of a right triangle, then by Pythagoras' Theorem, the length of the hypotenuse AB should simply be $\sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$:



The foregoing discussion suggests this formal definition of **distance**:

Definition 48. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be points in the cartesian plane. The *distance between A and B*, denoted $|AB|$, is the number defined by

$$|AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

Also, we define the *length* of the line segment AB to be $|AB|$.

Example 212. The distance between $A = (3, 6)$ and $B = (5, -2)$ is

$$|AB| = \sqrt{(3-5)^2 + [6-(-2)]^2} = \sqrt{2^2 + 8^2} = \sqrt{68}.$$

Example 213. The distance between $C = (1, 7)$ and $D = (0, -2)$ is

$$|CD| = \sqrt{(1-0)^2 + [7-(-2)]^2} = \sqrt{1^2 + 9^2} = \sqrt{82}.$$

Of course, the distance between A and B is the same as the distance between B and A:

$$|AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2} = |BA|.$$

(We say that the distance or length operator $|\cdot|$ is **commutative**, just as addition and multiplication are: $a + b = b + a$ and $ab = ba$.)

Remark 33. Later in Part IV (Vectors), we'll again use the Pythagoras' Theorem to motivate a similar definition for the distance between two points in *three*-dimensional space.

Exercise 84. Let $A = (5, -1)$, $B = (-7, 0)$, and $C = (1, 2)$. What are $|AB|$, $|AC|$, and $|BC|$? (Answer on p. 1766.)

9.3. Closeness (or Nearness)

Definition 49. Let A, B, and C be points.

If $|AB| < |AC|$, then we say that B is *closer* (or *nearer*) *to A than C*.

If $|AB| = |AC|$, then we say that B is *as close* (or *near*) *to A as C*.

If $|AB| > |AC|$, then we say that B is *further from A than C*.

Let S be a set of points. We say that $D \in S$ is the point in S that's *closest to A* if $|AD| \leq |AE|$ for every point $E \in S$.

Example 214. Let $A = (3,6)$, $B = (5,-2)$, and $C = (1,7)$. We have

$$|AB| = \sqrt{(3-5)^2 + [6-(-2)]^2} = \sqrt{2^2+8^2} = \sqrt{68},$$
$$|AC| = \sqrt{(3-1)^2 + (6-7)^2} = \sqrt{2^2+1^2} = \sqrt{5},$$
$$|BC| = \sqrt{(5-1)^2 + (-2-7)^2} = \sqrt{4^2+9^2} = \sqrt{117}.$$

Since $|AC| < |AB|$, C is closer to A than B. Equivalently, B is further from A than C.

Since $|AB| = |BA| < |BC|$, A is closer to B than C. Equivalently, C is further from B than A.

Example 215. The point on the line $y = 0$ closest to the point $(1,2)$ is the point $(1,0)$.

Exercise 85. Let $A = (5,-1)$, $B = (-7,0)$, and $C = (1,2)$. Fill in the blanks:

(a) A is ________ B than C.

(b) C is ________ A than B. (Answer on p. 1766.)

Exercise 86. Find the point on the line $x = 3$ that is closest to the point $(5,6)$. (Answer on p. 1766.)

10. Circles

The **unit circle centred on the origin** (the word *unit* means the circle has radius 1) is graphed below.

This is the set of points (x, y) whose distance from the origin $(0, 0)$ is 1, i.e.

$$\sqrt{(x-0)^2 + (y-0)^2} = 1.$$

Or equivalently and more elegantly,

$$x^2 + y^2 = 1.$$

The above discussion motivates this definition:

Definition 50. The *unit circle centred on the origin* is the graph of $x^2 + y^2 = 1$.

That is, the **unit circle centred on the origin** *is* this set:

$$G = \left\{(x, y) : x^2 + y^2 = 1\right\}.$$

Figure to be
inserted here.

Example 216. The point $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle because it satisfies the equation $x^2 + y^2 = 1$:

$$\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = 1.$$

Example 217. The point $\left(0.1, -\sqrt{0.99}\right)$ is on the unit circle because it satisfies the equation $x^2 + y^2 = 1$:

$$0.1^2 + \left(-\sqrt{0.99}\right)^2 = 0.01 + 0.99 = 1.$$

Next, the **circle of radius r centred on (a, b)** is graphed below.

This is the set of points (x, y) whose distance from the point (a, b) is r, i.e.

$$\sqrt{(x-a)^2 + (y-b)^2} = r.$$

Or equivalently,

$$(x-a)^2 + (y-b)^2 = r^2.$$

The above discussion motivates this definition:

Definition 51. Let (a, b) be a point in the cartesian plane and $r \geq 0$. The *circle of radius r centred on* (a, b) is the graph of

$$(x-a)^2 + (y-b)^2 = r^2.$$

The circle's *radius* is the number r, its *centre* is the point (a, b), and its *diameter* that is the number $2r$.

Equivalently, the **circle of radius r centred on (a, b)** *is* this set:

$$G = \left\{(x, y) : (x-a)^2 + (y-b)^2 = r^2\right\}.$$

Figure to be
inserted here.

Example 218. XXX

Example 219. XXX

Example 220. XXX

Exercise 87. **(a)** What is the graph of each of these three equations?

(i) $(x-5)^2+(y-1)^2=16$ **(ii)** $(x+3)^2+(y+2)^2=2$ **(iii)** $(x+1)^2+y^2=81$

(b) Write down the equation that corresponds to each of these three circles:

(i) The circle of radius 5 centred on $(-1,2)$.
(ii) The circle of radius 1 centred on $(3,-2)$.
(iii) The circle of radius $\sqrt{3}$ centred on $(0,0)$.

For each of the six circles given above in **(a)** and **(b)**,

(c) Does it contain the point $(8,2)$?

(d) What are its centre, radius, and diameter? (Answer on p. 1766.)

Example 221. Graph the equation $x^2 + y^2 = 1$.

Unfortunately, you cannot directly input this equation, because the piece of junk that is the TI84 requires that you enter equations with y on the left hand side. And so here, we'll have to tell the TI84 to graph **two** separate equations: $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$.

1. Press ON to turn on your calculator.
2. Press Y= to bring up the Y= editor.

Most buttons on the TI84 have *three* different functions. Simply pressing a button executes the function that's printed *on* the button itself. Pressing the 2ND button and then a button executes the function that's printed in blue *above* the button. And pressing the ALPHA and then a button executes the function that's printed in green *above* the button.

3. Press the 2ND button and then the x^2 button to execute the $\sqrt{\ }$ function and enter "$\sqrt{\vdots}$". Next press 1 − X,T,θ,n x^2. Altogether you've entered $\sqrt{1 - x^2}$.

Warning: Confusingly, the TI84 has two different minus buttons: − and (-). The − is used as the minus operation in the middle of an expression, as was just done. In contrast, the (-) is used to denote negative numbers at the beginning of an expression, as we'll do in Step 5 (below). Unfortunately, you really have to key in the correct button, or else you'll get a frustrating error message.

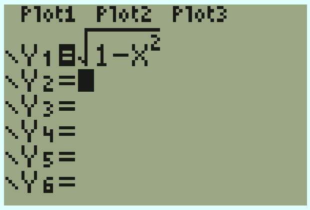
4. Now press ENTER and the blinking cursor will move down, to the right of "Y_2 =".

Step 1.

Step 2.

Step 3.

Step 4.



(*Example continues on the next page ...*)

(*Example continues on the next page ...*)

We'll now enter the second equation.

5. Press the (-) button. Now repeat what we did in step 3 above: Press the blue 2ND button and then $\sqrt{\ }$ (which corresponds to the x^2 button) to enter "$\sqrt{\ }$". Next press 1 − X,T,θ,n x^2 to enter "$1 - X^2$". Altogether you will have entered $-\sqrt{1-x^2}$.
6. Now press GRAPH and the calculator will graph both $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$.

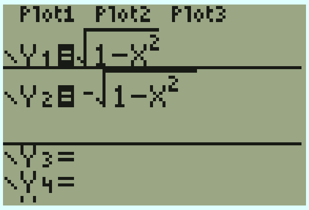
Notice the graphs are very small. To zoom in,

7. Press the ZOOM button to bring up a menu of ZOOM options.
8. Press 2 to select the Zoom In option. Nothing seems to happen. But now press ENTER and the TI84 will zoom in a little for you.

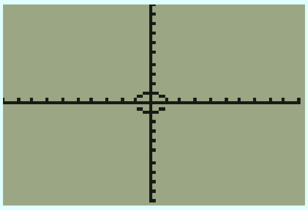
We expected to see a perfect circle—instead, we get an ellipse. Hm, what's going on? The reason is that by default, the x- and y- axes are scaled differently. To set them to the same scale,

9. Press the ZOOM button again to bring up the ZOOM menu of options. Press 5 to select the ZSquare option. The TI84 will adjust the x- and y- axes so that they have the same scale and thus give us a perfect circle.

Step 5.

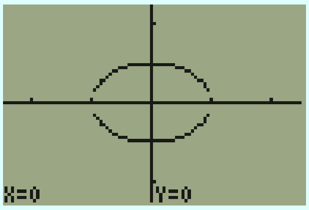


Step 6.

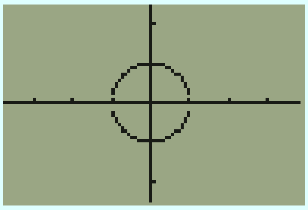


Step 7.

Step 8.



Step 9.



P.S. An alternative to Step 5 is to enter "$-Y_1$" instead of "$-\sqrt{1-X^2}$". To do so, replace Step 5 with these instructions: First press (-) to enter the minus sign, as was done in Step 5. Next press VARS to bring up the VARS menu. Then press ⟩ to go to the Y-VARS menu. Now press ENTER to select "1: Function...". Press ENTER again to select "1: Y_1". Altogether, we will have entered "$Y_2 = -Y_1$". Now go to Step 6.

Exercise 88. Graph $(x-3)^2 + (y+2)^2$ on your TI84. (Answer on p. 143.)

A88. TBD.

10.2. The Number π (optional)

Theorem 3. *The ratio of a circle's area to the square of its radius is constant.*

Proof. Omitted. See e.g. Euclid's *Elements* (c. 300 BC, XII.2—Joyce, 1997).[127] □

Definition 52. The *circumference* of a circle is its length.

Remark 34. In writing down the above definition, we're actually cheating a little: So far we've only defined what the length of a line segment is; we have **not** defined the "length" of any curve (e.g. a circle).

It turns out that the question of the "length" of a curve is a little harder than one might think. In particular, it is beyond the scope of H2 Maths (and is dealt with in H2 Further Maths, but only cursorily and as a mindless formula that students are to mug and apply).

Nonetheless, just to screw students over, the question of length abruptly appeared as a curveball question in the 2018 A-Level H2 Maths exam (see Exercise 673).

Theorem 4. *The area of a circle with radius r and circumference C is $Cr/2$.*

Proof. Omitted. This result was first proven by Archimedes (c. 250 BC, *Measurement of a Circle*, Proposition 1—see e.g. Heath, 1897). □

Corollary 1. *The ratio of a circle's circumference to its radius is constant.*

Proof. Suppose two circles have circumferences C_1 and C_2 and radii r_1 and r_2.

Then by Theorem 4, their areas are $C_1r_1/2$ and $C_2r_2/2$.

And by Theorem 3,

$$\frac{C_1r_1/2}{r_1^2} = \frac{C_2r_2/2}{r_2^2} \qquad \text{or} \qquad \frac{C_1}{r_1} = \frac{C_2}{r_2}.$$

We've just shown that the ratio of a circle's circumference to its radius is constant. □

Given Corollary 1, we can define the number π:

Definition 53. The real number π is the ratio of a circle's circumference to its diameter.

Remark 35. The above is the definition of π that should be familiar to you from primary school and will be good enough for us.

Note though that more advanced mathematical texts will usually adopt some other definition of π—for example, after defining the sine function sin, they might define π to be the smallest positive number such that $\sin \pi = 0$.

Fact 29. *A circle with radius r has circumference $2\pi r$ and area πr^2.*

[127] But this result was probably proven before Euclid.

Proof. Let C be the circle's circumference.

By Definition 53, $\pi = C/(2r)$. Rearranging, $C = 2\pi r$.

By Theorem 4, the circle's area is $Cr/2 = (2\pi r)\,r/2 = \pi r^2$. □

Fun Fact

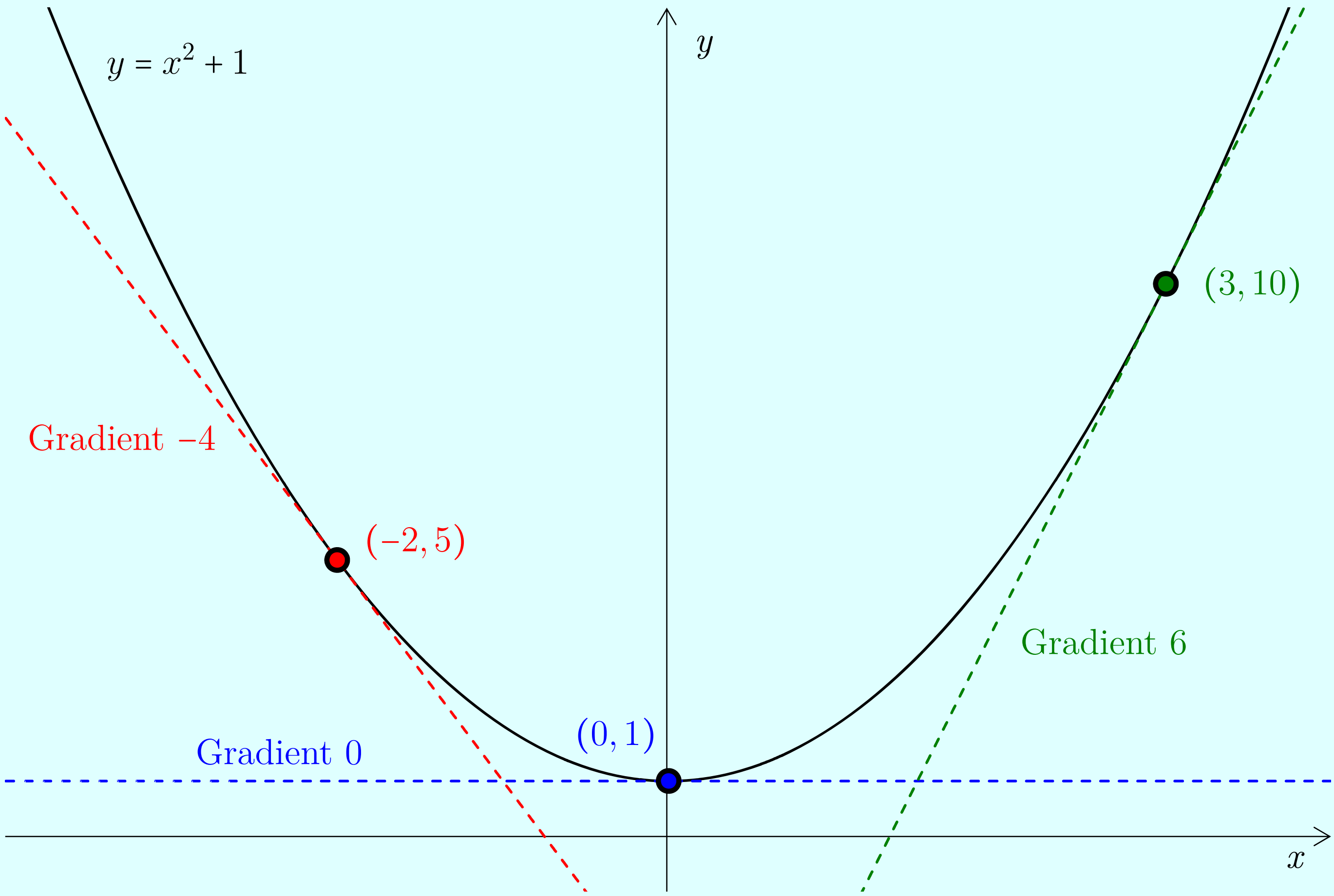
There's a movement to replace $\pi = 3.14\ldots$ with $\tau = 6.28\ldots$ as the "standard" constant, where $\tau = 6.28\ldots$ is the ratio of a circle's circumference to its *radius*. See "The Tau Manifesto" or *Wall Street Journal* (2020-03-13).

11. Tangent Lines, Gradients, and Stationary Points

11.1. Tangent Lines and the Gradient of a Graph at a Point

Example 222. Graphed below is the equation $y = x^2 + 1$.

The red line is tangent to the graph at the point $(-2, 5)$. It can be shown that this line's gradient is -4. So, the graph's gradient at this point is -4.



The blue line is tangent to the graph at the point $(0, 1)$. It can be shown that this line's gradient is 0. So, the graph's gradient at this point is 0.

The green line is tangent to the graph at the point $(3, 10)$. It can be shown that this line's gradient is 6. So, the graph's gradient at this point is 6.

Here's the informal definition of **tangent lines** you probably learnt in secondary school:

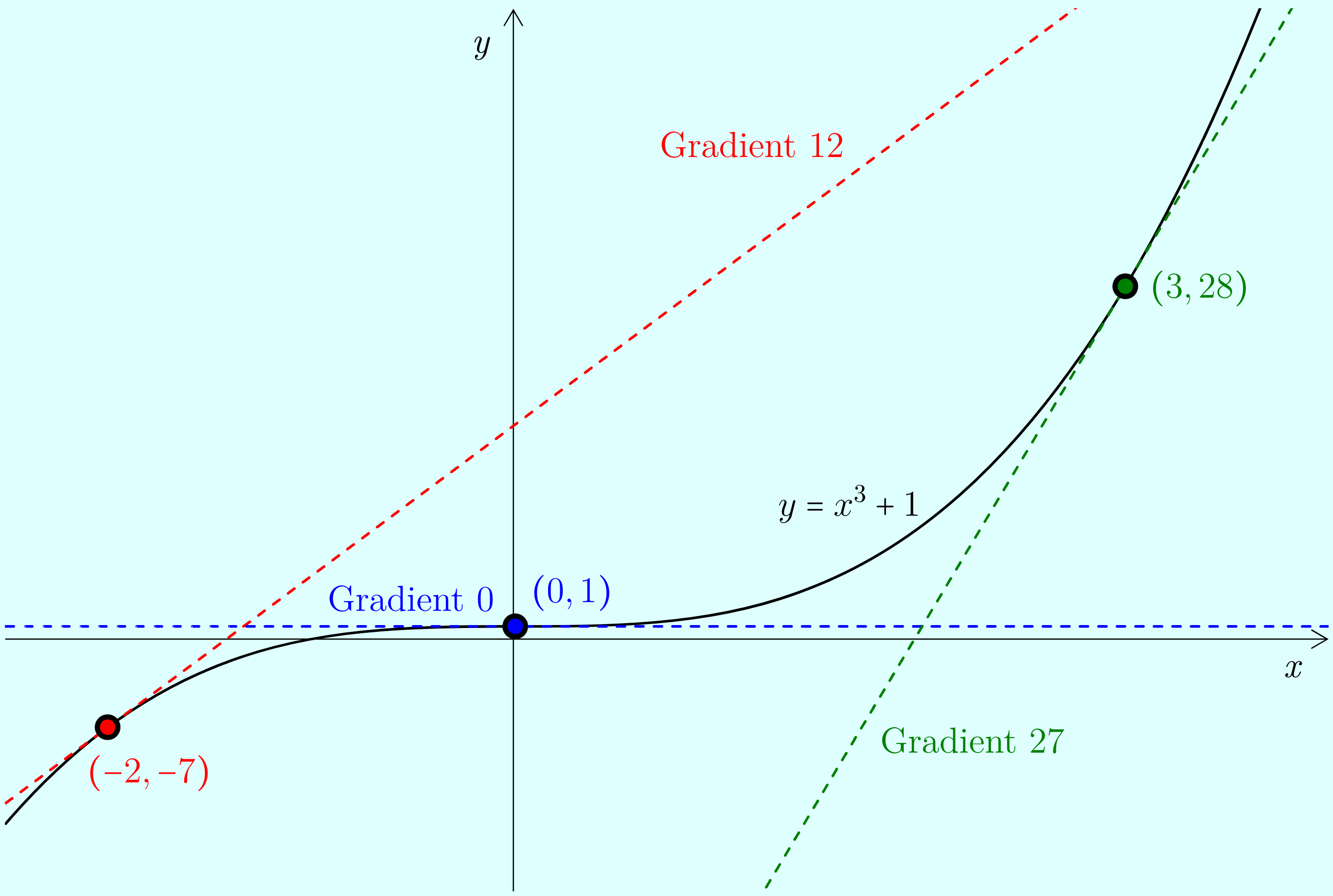
Definition 54 (*informal*)**.** A *tangent line (to a graph at a point)* is a line that "just" touches that graph at that point.

Definition 55 (*informal*)**.** The *gradient (of a graph at a point)* is the gradient of the tangent line (to that graph at that point).

Tangent comes from the Latin for *touching*. Later on in Part V (Calculus), we'll define tangent lines more formally and precisely (see Definition XXX). But for now, these definitions will be good enough.

Example 223. Consider the graph of $y = x^3 + 1$.

The red line is tangent to the graph at the point $(-2, -7)$. It can be shown that this line's gradient is 12. So, the graph's gradient at this point is 12.



The blue line is tangent to the graph at the point $(0, 1)$. It can be shown that this line's gradient is 0. So, the graph's gradient at this point is 0.

The green line is tangent to the graph at the point $(3, 28)$. It can be shown that this line's gradient is 27. So, the graph's gradient at this point is 27.

Example 224. XXX

Example 225. XXX

Example 226. XXX

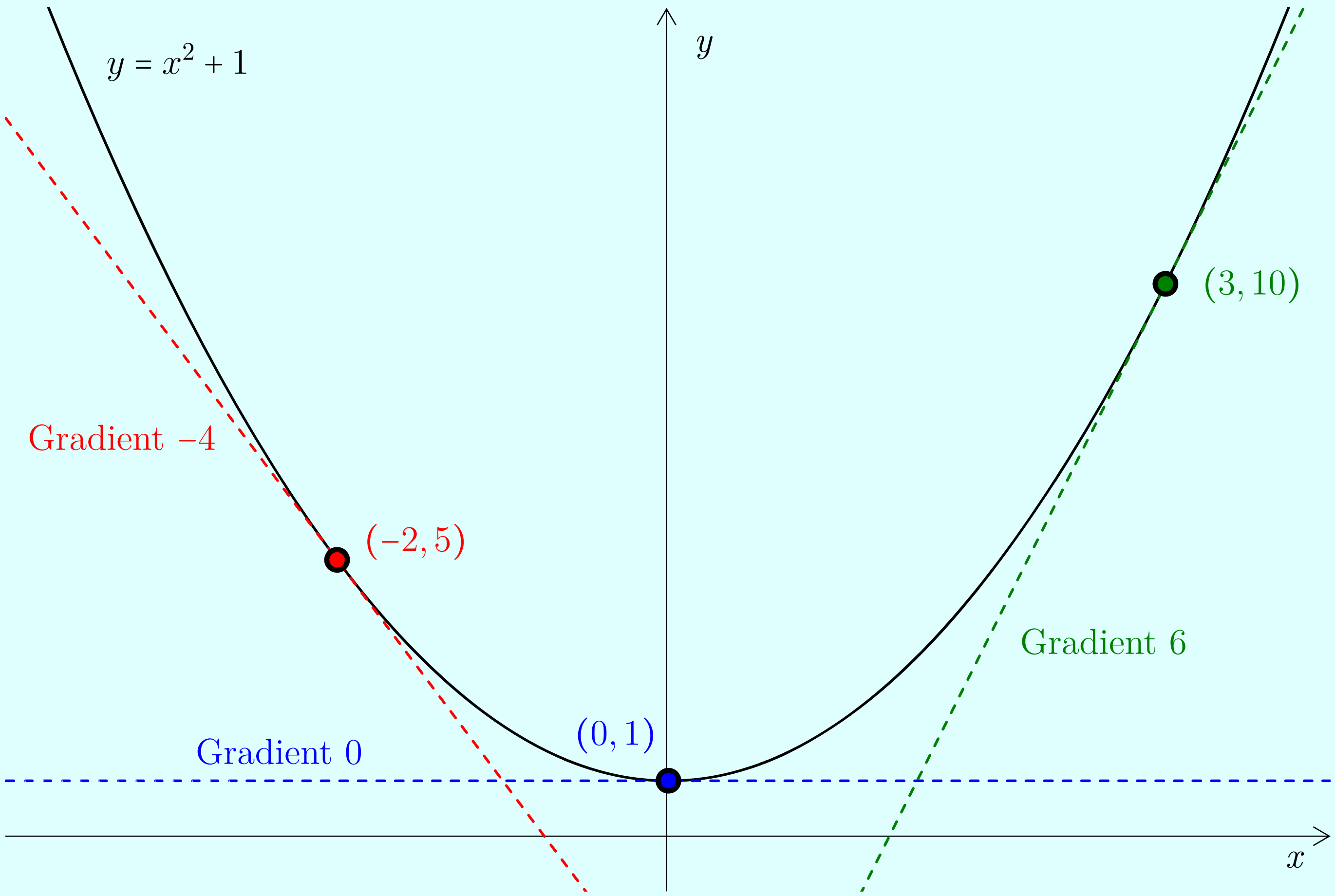
11.2. Stationary Points

Definition 56 (*informal*)**.** A point of a graph is called a *stationary point* if the gradient of the graph at that point is zero.[128]

Equivalently, a point of a graph is called a *stationary point* if the tangent line (to that graph at that point) is horizontal.

Example 227. Graphed below is the equation $y = x^2 + 1$.

The point $(0,1)$ is a stationary point of the graph because the gradient of the graph at that point is zero (or equivalently, the tangent line to the graph at that point is horizontal).



The points $(-2,5)$ and $(3,10)$ are not stationary points of the graph.

Example 228. XXX

Example 229. XXX

Example 230. XXX

12. Maximum, Minimum, and Turning Points

In secondary school, you learnt about **maximum points** and **minimum points**.

We'll now relearn these, but in a little more depth.

12.1. Maximum Points

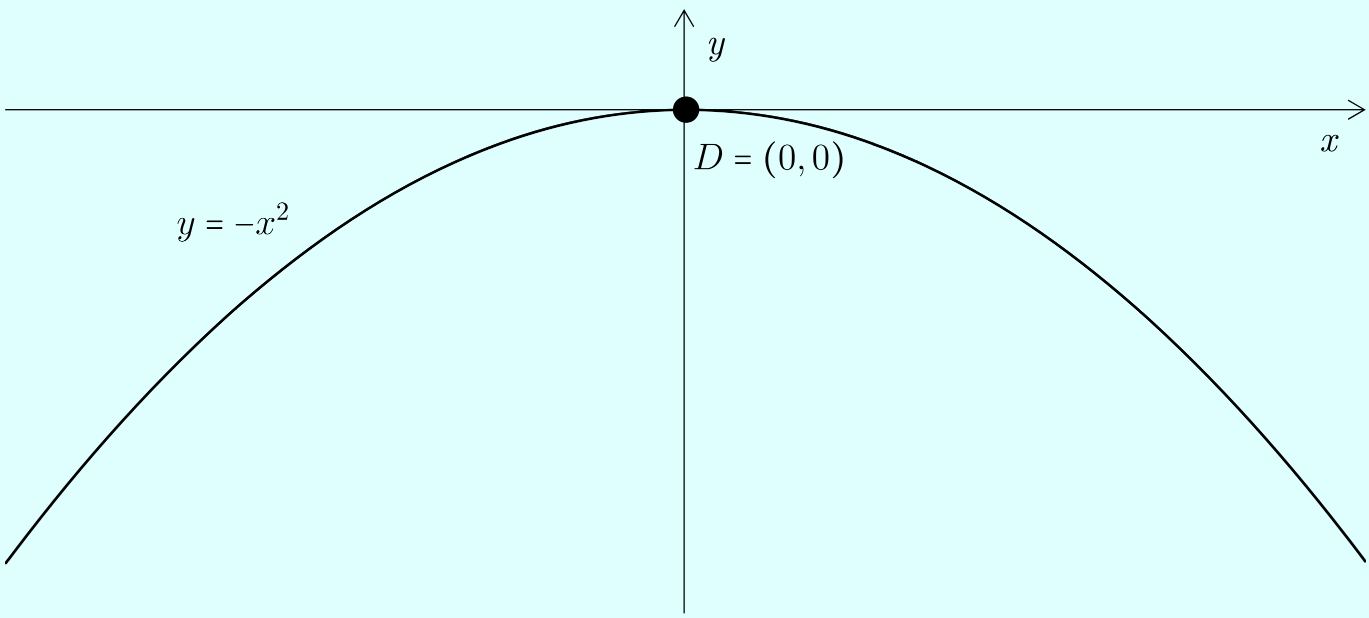
In particular, for maximum points, we'll learn to make these two distinctions:

- **Global** vs **local** maximum points;
- **Strict** vs **non-strict** maximum points.

Informally, a **global maximum (point)** of a graph is a point that's at least as high as any other point (that's also in that graph).[129]

And a ***strict* global maximum (point)** of a graph is a point that's *strictly* higher than *every* other point (that's also in that graph).

Example 231. Consider $y = -x^2$. The point $D = (0,0)$ is a **global maximum** (of the graph of $y = -x^2$), because it is at least as high as any other point (in that graph).



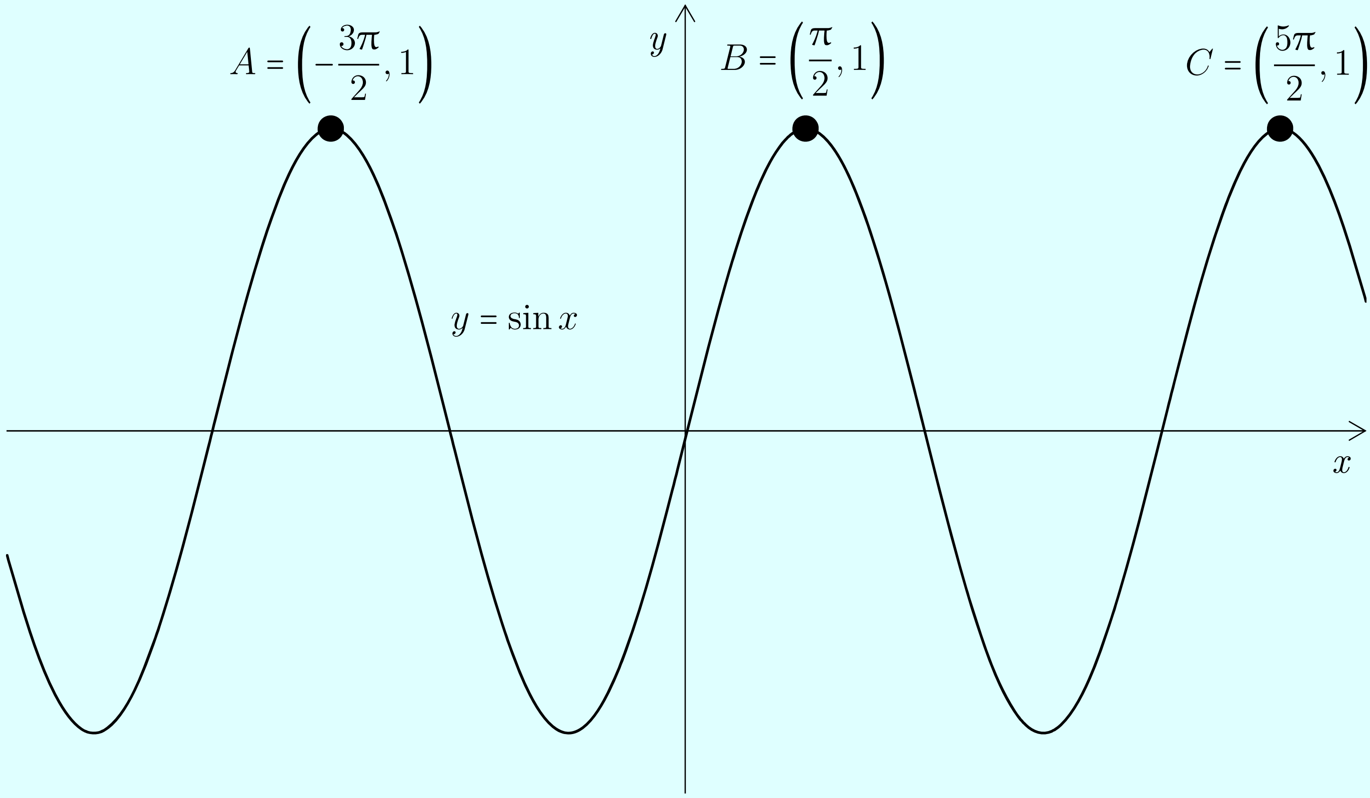
Indeed, it is also a **strict global maximum**, because it is *strictly* higher than every other point.

[129]Here, we'll merely give the informal definitions of the eight types of extrema introduced. For their formal definitions, see Definition 277 (Appendices).

Example 232. Consider $y = \sin x$ and these three points:

$$A = \left(-\frac{3\pi}{2}, 1\right), \qquad B = \left(\frac{\pi}{2}, 1\right), \qquad \text{and} \qquad C = \left(\frac{5\pi}{2}, 1\right).$$

Each of A, B, and C is a global maximum (of the graph of $y = \sin x$), because each is at least as high as any other point (in that graph).



Indeed, $y = \sin x$ has infinitely many global maxima: For *every* integer k, the following point is a global maximum:

$$\left(\left(2k + \frac{1}{2}\right)\pi, 1\right).$$

Note though that $y = \sin x$ has no *strict* global maximum, because no point is *strictly* higher than every other point.

Obviously, if a point is strictly higher than any other point, then it must also be at least as high as any other point. And so,

Fact 30. *Every strict global maximum is also a global maximum.*

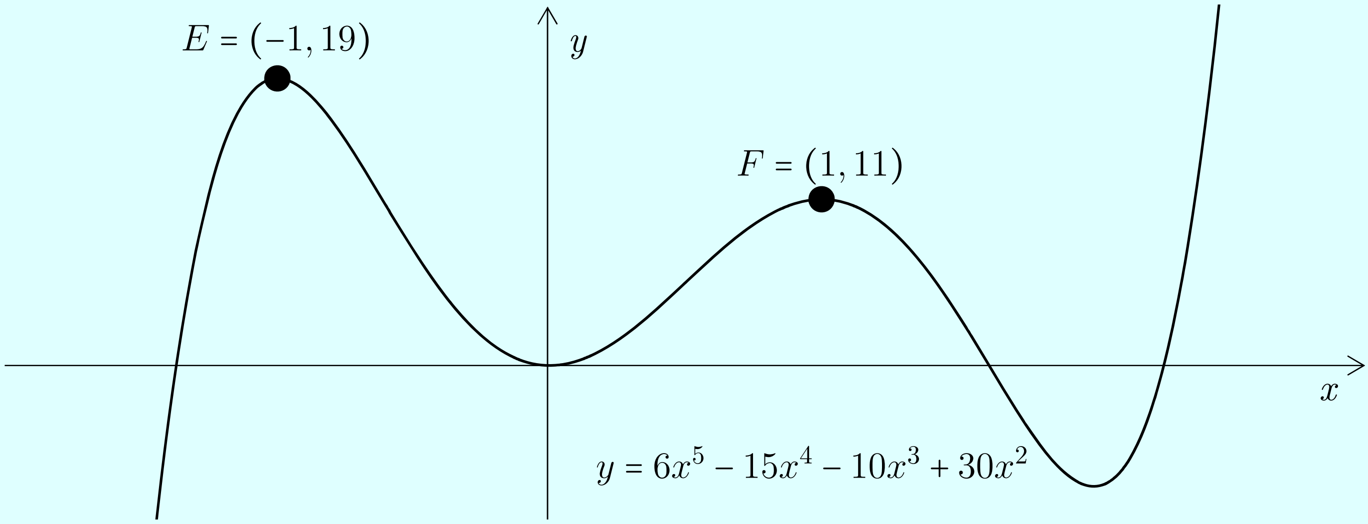
However, the converse is not true. That is, a global maximum need not be strict. (In the above example, each of A, B, and C is a global maximum, but none is a *strict* global maximum. Indeed, the graph of $y = \sin x$ has no strict global maximum.)

We next introduce the concept of a ***local* maximum**:

Informally, a **local maximum (point)** of a graph is a point that's at least as high as any "nearby" point (that's also in that graph).

Example 233. Consider the graph of $y = 6x^5 - 15x^4 - 10x^3 + 30x^2$.

Neither $E = (-1, 19)$ nor $F = (1, 11)$ is a global maximum, because neither is at least as high as any other point.



However, each of E and F is a local maximum, because each is at least as high as any "nearby" point.

Informally, a ***strict* local maximum (point)** of a graph is a point that's *strictly* higher than any "nearby" point (that's also in that graph).

Example 234. In the last example, the points E and F are also *strict* local maxima, because each is strictly higher than any "nearby" point.

Again and obviously, if a point is strictly higher than any "nearby" point, then it must also be at least as high as any "nearby" point. And so in general,

Fact 31. *Every strict local maximum is also a local maximum.*

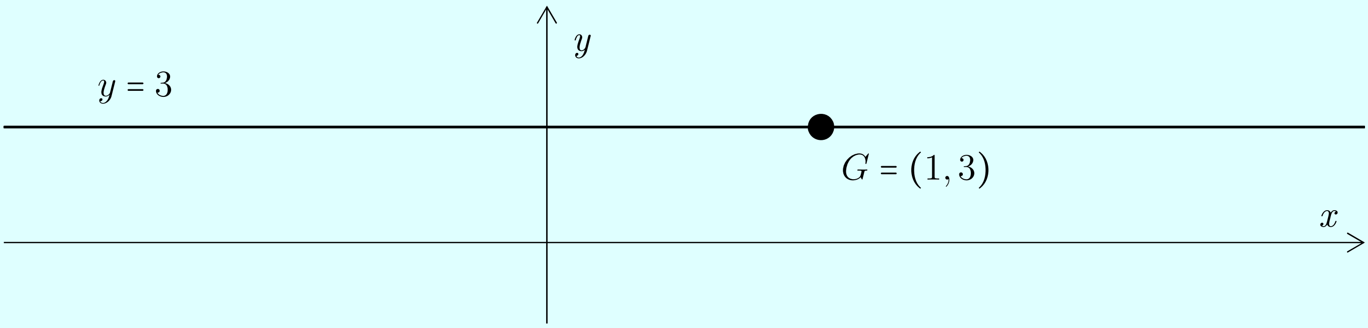
Again, the converse is not true. That is, a local maximum need not be strict:

Example 235. Consider $y = 3$. The point $G = (1, 3)$ is a local maximum, because it's at least as high as any "nearby" point. However, it is **not** a strict local maximum, because it is not strictly higher than any "nearby" point.

Actually, *every* point in $y = 3$ is a local maximum (though not a strict local maximum)!

Indeed, *every* point in $y = 3$ is a *global* maximum (though not a strict global maximum)!

(The last two sentences may seem puzzling. If so, take a moment to convince yourself that they're true.)



12.2. Minimum Points

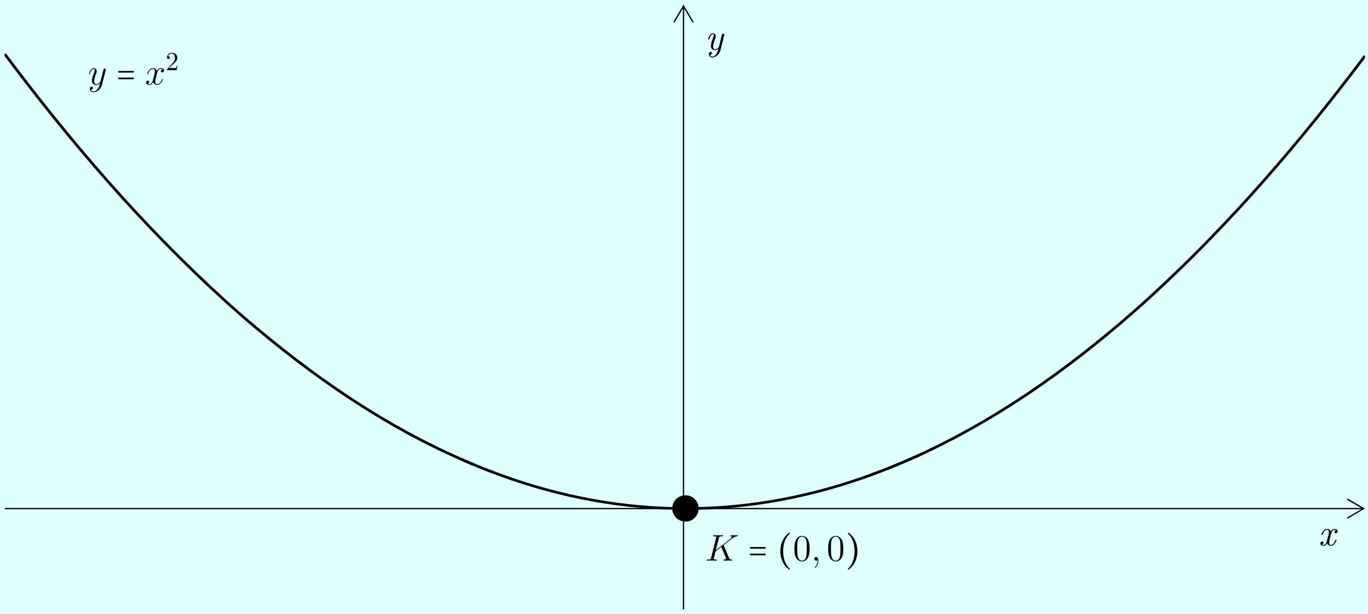
Similarly, for minimum points, we can make the same two distinctions:

- **Global** vs **local** minimum points;
- **Strict** vs **non-strict** minimum points.

Informally, a **global minimum (point)** of a graph is a point that's at least as low as any other point (that's also in that graph).

And a **strict global minimum (point)** of a graph is a point that's (strictly) lower than any other point (that's also in that graph).

Example 236. Consider $y = x^2$. The point $K = (0, 0)$ is a **global minimum** (of the graph of $y = x^2$), because it is at least as low as any other point (in that graph).

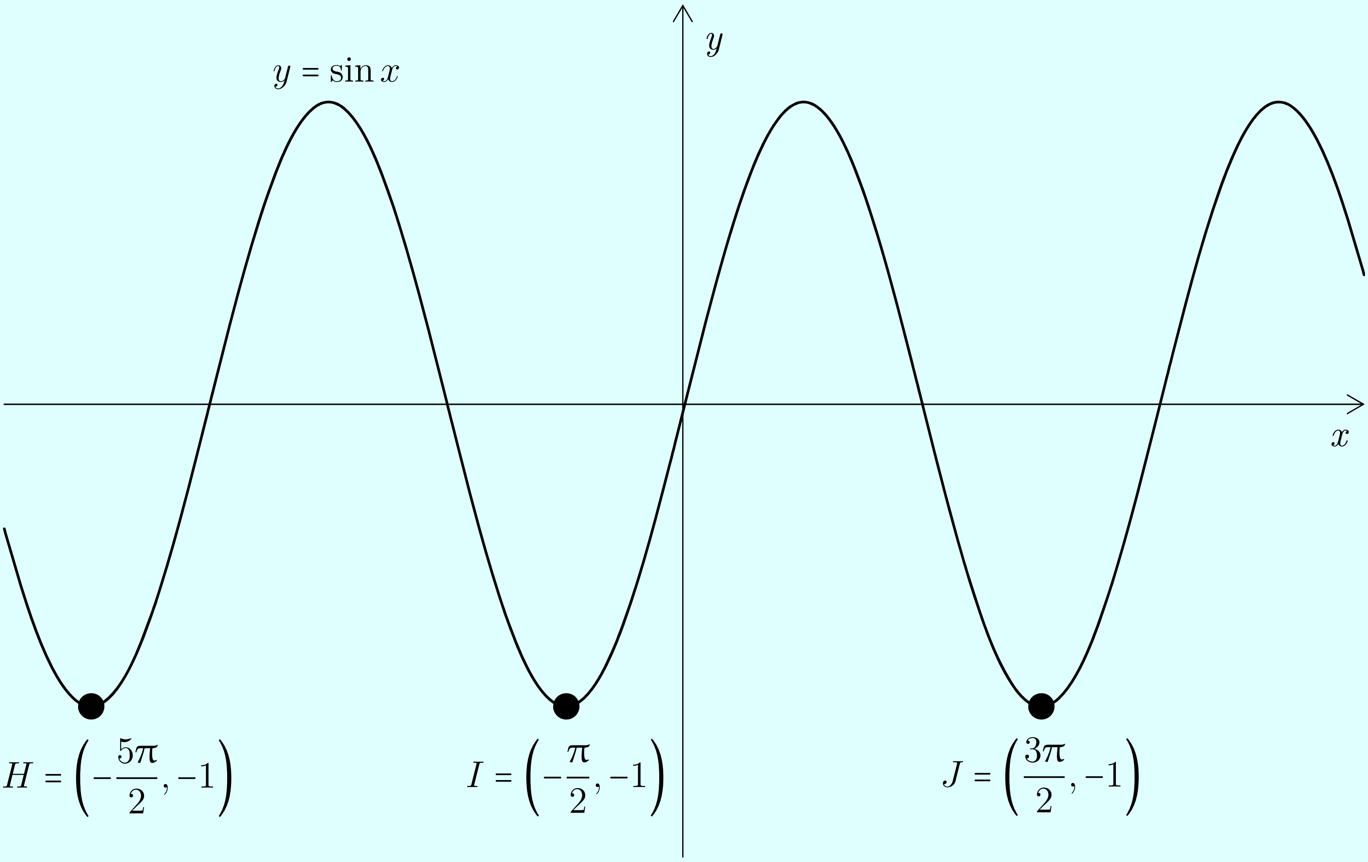


Indeed, it is also a **strict global minimum**, because it is (strictly) lower than every other point.

Example 237. Consider $y = \sin x$ and these three points:

$$H = \left(-\frac{5\pi}{2}, -1\right), \qquad I = \left(-\frac{\pi}{2}, -1\right), \quad \text{and} \quad J = \left(\frac{3\pi}{2}, -1\right).$$

Each of H, I, and J is a global minimum, because each is at least as low as every other point (in the graph of $y = \sin x$).



Indeed, $y = \sin x$ has infinitely many global minima: For *every* integer k, the following point is a global maximum:

$$\left(\left(2k - \frac{1}{2}\right)\pi, 1\right).$$

Note though that $y = \sin x$ has no *strict* global minimum, because no point is *strictly* lower than every other point.

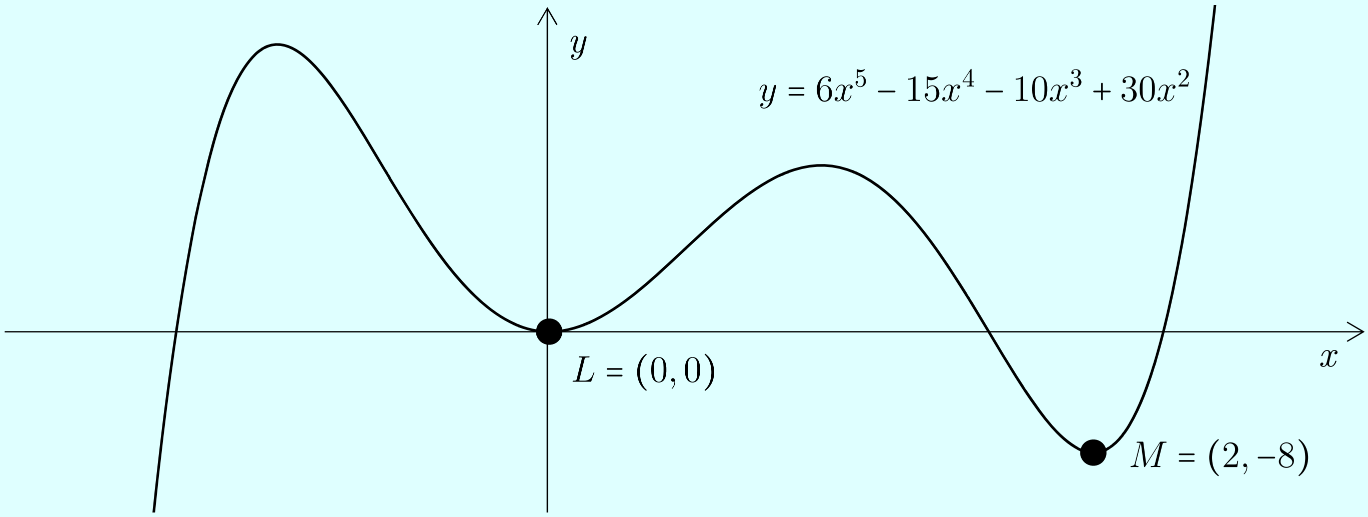
Obviously, if a point is strictly lower than any other point, then it must also be at least as low as any other point. And so,

Fact 32. *Every strict global minimum is also a global minimum.*

However, the converse is not true. That is, a global maximum need not be strict. (In the above example, each of H, I, and J is a global minimum, but none is a *strict* global minimum. Indeed, the graph of $y = \sin x$ has no strict global minimum.)

Informally, a **local minimum (point)** of a graph is a point that's at least as low as any "nearby" point (that's also in that graph).

Example 238. In the graph of $y = 6x^5 - 15x^4 - 10x^3 + 30x^2$, neither $L = (0, 0)$ nor $M = (2, -8)$ is a global minimum, because neither is at least as low as every other point (in that graph). However, each is a local minimum, because each is at least as low as any "*nearby*" point.



Informally, a ***strict* local minimum (point)** of a graph is a point that's *strictly* lower than any "nearby" point (that's also in that graph).

Example 239. In the last example, the points L and M are also *strict* local minima, because each is *strictly* lower than any "nearby" point.

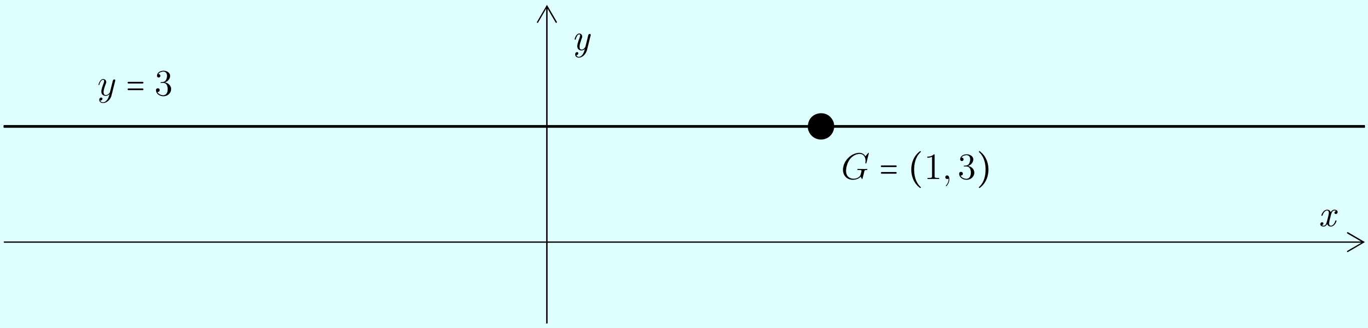
Again and obviously, if a point is strictly lower than any "nearby" point, then it must also be at least as low as any "nearby" point. And so,

Fact 33. *Every strict local minimum is also a local minimum.*

However, the converse is not true. That is, a local minimum need not be strict:

Example 240. Consider $y = 3$. The point $G = (1, 3)$ is a local minimum, because it's at least as low as any "nearby" point. However, it is **not** a strict local minimum, because it is not strictly lower than any "nearby" point.

Actually, *every* point in $y = 3$ is a local minimum (though not a strict local minimum)! (This statement may initially seem puzzling, but you should take a moment to convince yourself that it is true.)



Indeed, *every* point in $y = 3$ is a *global* minimum (though not a strict global minimum)! (Ditto.)

12.3. Extrema

It will be convenient to have a word for all maximum and minimum points.

Definition 57. An *extremum* (plural: *extrema*) is any maximum or minimum point. A *strict extremum* is any strict maximum or minimum point.

So, so far in this chapter, we've learnt about eight types of extrema, now summarised:

(a) A **global maximum** is at least as high as any other point.
(b) The **strict global maximum** is higher than any other point.
(c) A **local maximum** is at least as high as any "nearby" point.
(d) A **strict local maximum** is higher than any "nearby" point.
(e) A **global minimum** is at least as low as any other point.
(f) The **strict global minimum** is lower than any other point.
(g) A **local minimum** is at least as low as any "nearby" point.
(h) A **strict local minimum** is lower than any "nearby" point.

Exercise 89. Explain whether each statement is true. (Answer on p. 1769.)

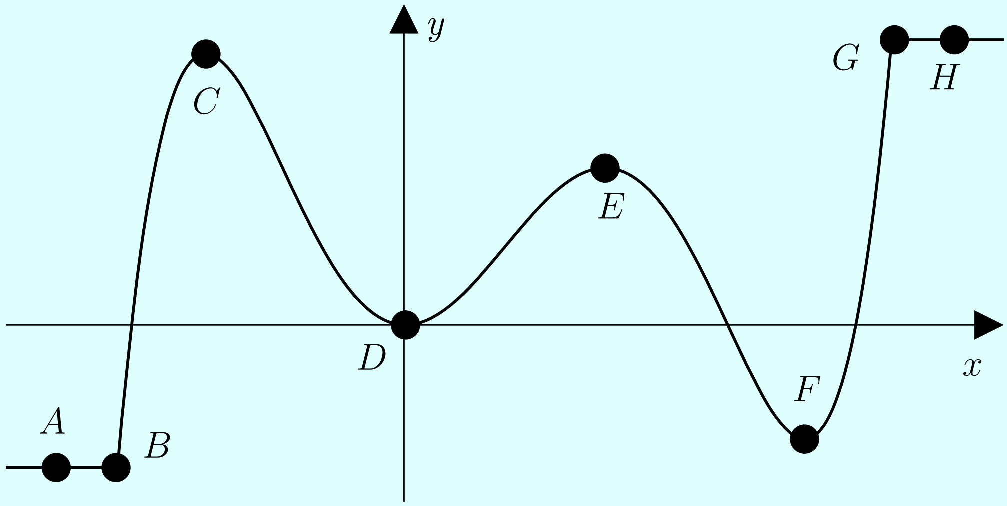
(a) A global maximum must also be a strict local maximum.
(b) A global maximum must also be a local maximum.
(c) A strict global maximum must also be a global maximum, strict local maximum, and local maximum.
(d) A global maximum cannot be a local minimum.

12.4. Turning Points

Informally, a turning point is any point at which a graph "turns". Formally,

Definition 58. A *turning point* is a point that's both a stationary point and a strict local maximum or minimum.

Example 241. The table below says that $A = (-8, -81)$ in the given graph is a local maximum, a global minimum, *and* a local minimum, but is not any of the other five types of extrema and is not a turning point. (Verify that the entire table is correct.)



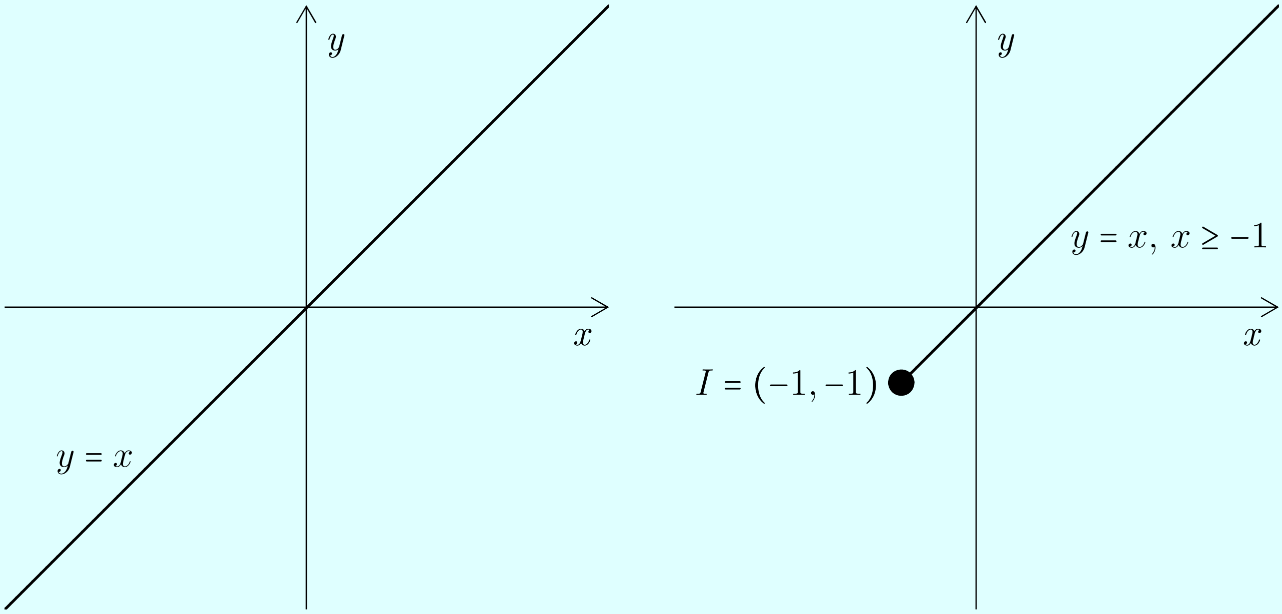
	A	B	C	D	E	F	G	H
Global maximum							✓	✓
Strict global maximum								
Local maximum	✓		✓		✓		✓	✓
Strict local maximum			✓		✓			
Global minimum	✓	✓						
Strict global minimum								
Local minimum	✓	✓		✓		✓		✓
Strict local minimum				✓		✓		
Turning point			✓	✓	✓	✓		

Besides the eight points A–H, does this graph have any other extrema?[130]

[130]Yes. In fact, there are *infinitely* many other extrema:

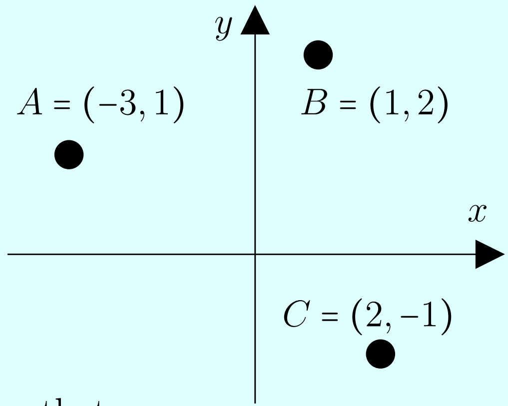
For *every* $p < -3$, the point $(p, -81)$ is—like A—a global minimum, local maximum, *and* local minimum. And for *every* $q > 5$, the point $(q, 125)$ is—like H—a global maximum, local maximum, *and* local minimum.

Example 242. The graph of $y = x$ (left) has no extrema.



In contrast, the graph of $y = x$ with the constraint $x \geq -1$ (right) has one extremum, namely $I = (-1, -1)$, which is a global minimum, strict global minimum, local minimum, and strict local minimum. Observe though that I is not a turning point.

Example 243. Consider the graph $G = \{A, B, C\}$.



Informally, the points A, B, and C aren't "close" to each other. (More formally, they are **isolated points**.)[131]

And so, interestingly, each of A, B, and C is a *strict local maximum* of G. This is because each of A, B, and C has no "nearby" point. It is thus trivially or *vacuously* true that each of A, B, and C is strictly higher than any "nearby" point.

By the same token, it is likewise trivially or vacuously true that each of A, B, and C is a *strict local minimum* of G.

Note though that there's only one *strict* global maximum or global maximum, namely A.

Likewise, there is only one *strict* global minimum or global minimum, namely C.

Observe also that none of A, B, or C is a turning point.

Exercise 90. Identify each graph's extrema and turning points. (Answer on p. 1767.)

(a) $y = x^2 + 1$. **(b)** $y = x^2 + 1, \quad -1 \leq x \leq 1$.

(c) $y = \cos x$. **(d)** $y = \cos x, \quad -1 \leq x \leq 1$.

[131] Informally, a point of a set is **isolated** if no other point in the same set is "nearby". For the formal definition, see Definition 276 (Appendices).

13. Solutions and Solution Sets

To **solve an equation** is to find *all* its **solutions** (or **roots**). Equivalently, to solve an equation is to find that equation's **solution set**.

Example 244. Solve the equation $x - 1 = 0$ ($x \in \mathbb{R}$).

Here are three perfectly good (and equivalent) answers to the above problem:

- "$x = 1$."
- "$x \in \{1\}$."
- "The equation's **solution set** is $\{1\}$."

Example 245. Solve the equation $x + 5 = 8$ ($x \in \mathbb{R}$).

Here are three perfectly good (and equivalent) answers to the above problem:

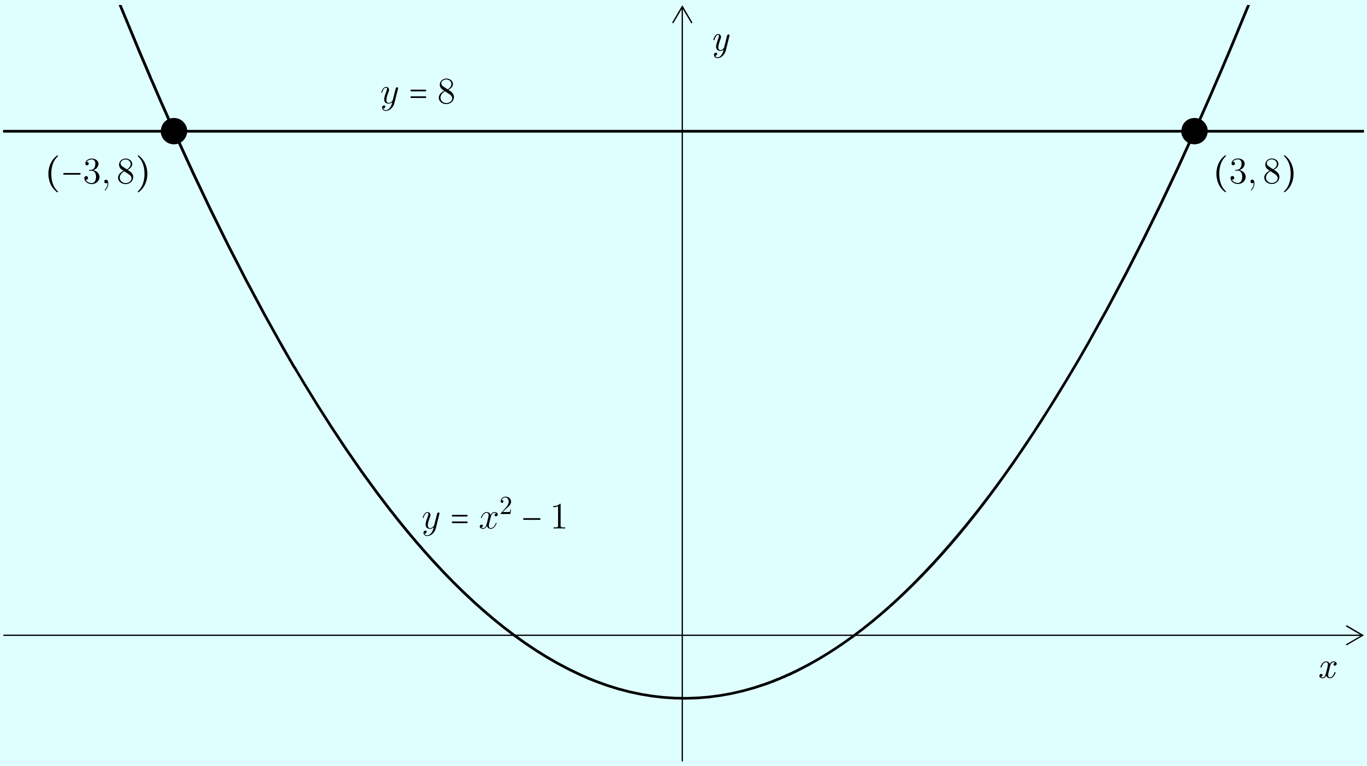
- "$x = 3$."
- "$x \in \{3\}$."
- "The equation's **solution set** is $\{3\}$."

The above two examples were easy enough to solve. In the next chapter, we'll review the solution to the quadratic equation $ax^2 + bx + c = 0$. For now, here's a quick example:

Example 246. Consider the equation $x^2 - 1 = 8$ ($x \in \mathbb{R}$).

Let's see how graphs can help us solve this equation.

Graph $y = x^2 - 1$ and $y = 8$. We find that these two graphs intersect at $x = -3$ and $x = 3$.

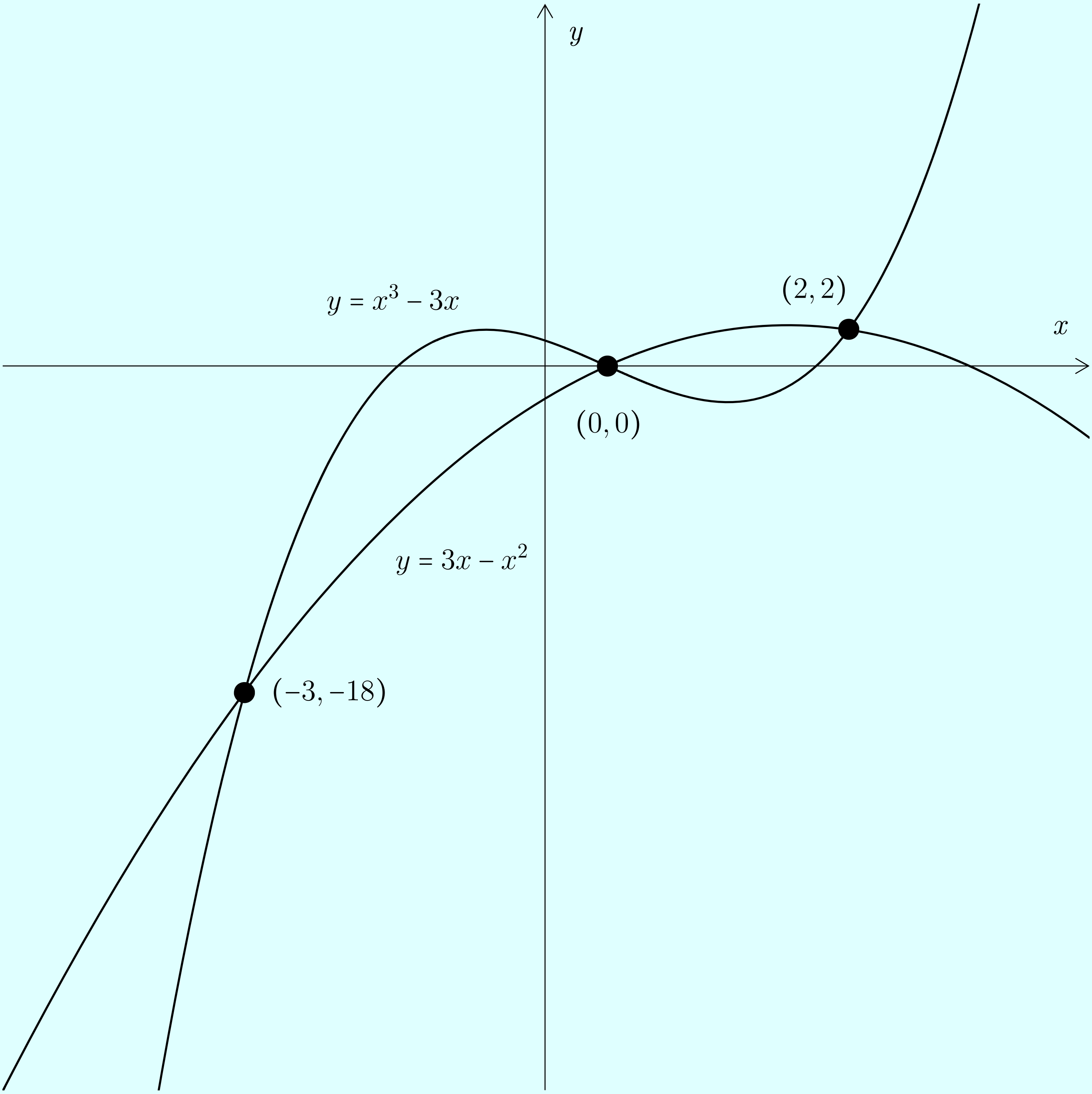


Thus, the equation $x^2 - 1 = 8$ ($x \in \mathbb{R}$) has *two* solutions: -3 and 3. Equivalently, its solution set is $\{-3, 3\}$ or $\{\pm 3\}$.

Example of a **cubic equation**:

Example 247. Consider the equation $x^3 - 3x = 3x - x^2$ $(x \in \mathbb{R})$.

Graph $y = x^3 - 3x$ and $y = 3x - x^2$. We find that these two graphs intersect at $x = -3$, $x = 0$, and $x = 2$.



Thus, the equation $x^3 - 3x = 3x - x^2$ $(x \in \mathbb{R})$ has *three* solutions: -3, 0, and 2. Equivalently, its solution set is $\{-3, 0, 2\}$.

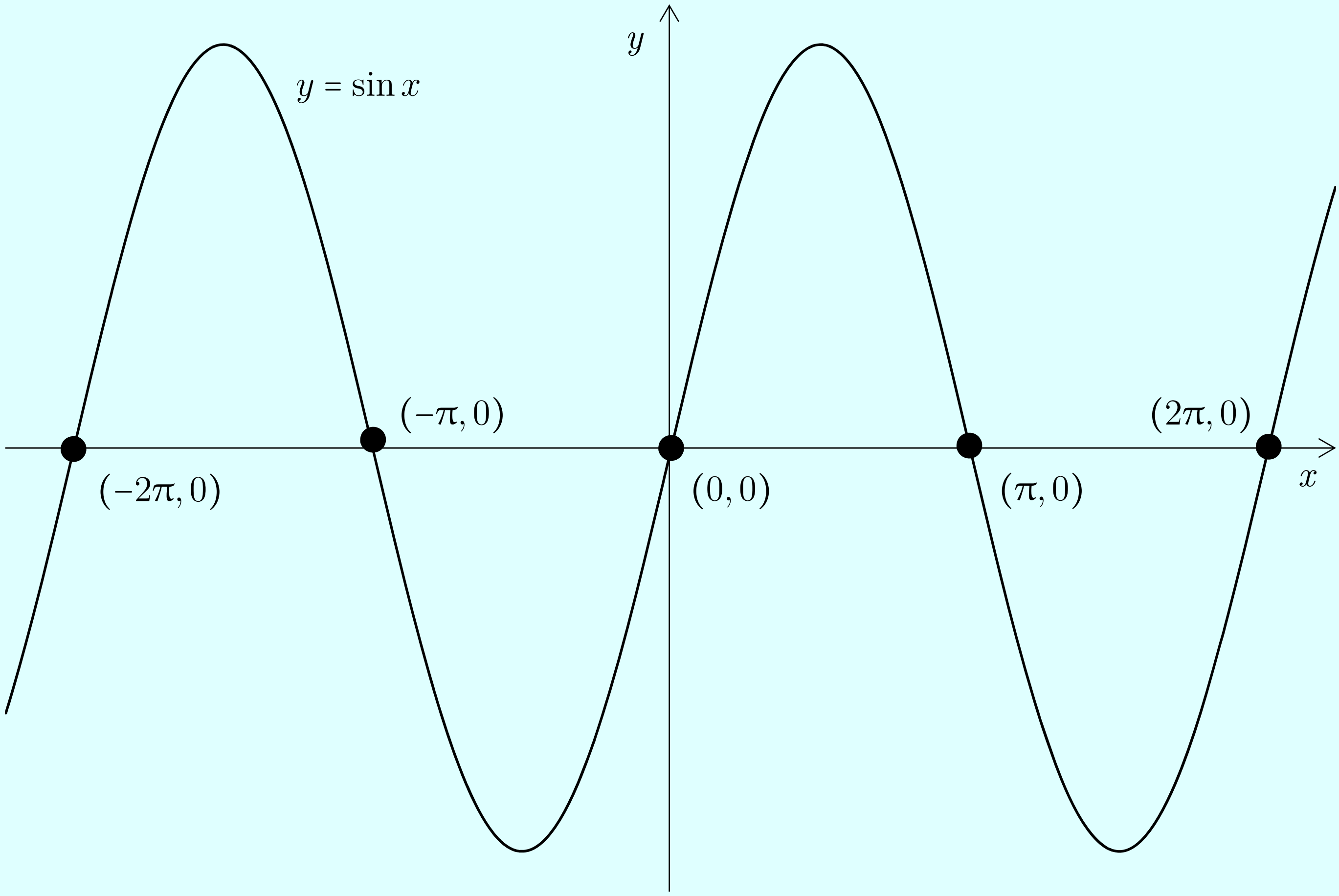
By the way, we won't be learning to solve the general cubic equation. But we will be learning to solve specific instances of the cubic equation that are not too difficult to factorise (Ch. 38).

Also, we *are* required to know how to use a graphing calculator to find the solutions to this and indeed just about any equation. We'll learn how to do so later.

Like the above examples, most equations we'll encounter will have only finitely many solutions. But this isn't always the case:

Example 248. Consider the equation $\sin x = 0$ $(x \in \mathbb{R})$.

It has infinitely many solutions—for every $k \in \mathbb{Z}$, $k\pi$ is a solution. The equation's solution set is $\{k\pi : k \in \mathbb{Z}\}$.



We now look at **inequalities**. To **solve an inequality** is to find *all* its **solutions** (or **roots**). Equivalently, to solve an inequality is to find its **solution set**.

Inequalities usually have infinitely many solutions:

Example 249. Solve $x - 1 \overset{1}{>} 0$ ($x \in \mathbb{R}$).

Three perfectly good (and equivalent) answers to the above problem:

- "$x > 1$."
- "$x \in (1, \infty)$."
- "The solution set of $\overset{1}{>}$ is $(1, \infty)$."

Example 250. Solve $x + 5 \overset{2}{\geq} 8$ ($x \in \mathbb{R}$).

Three perfectly good (and equivalent) answers to the above problem:

- "$x \geq 3$."
- "$x \in [3, \infty)$."
- "The solution set of $\overset{2}{\geq}$ is $[3, \infty)$."

The above two inequalities were easy enough to solve. In Ch. 41.3, we'll learn the general solution to the quadratic inequality $ax^2 + bx + c > 0$. For now, here's a quick example:

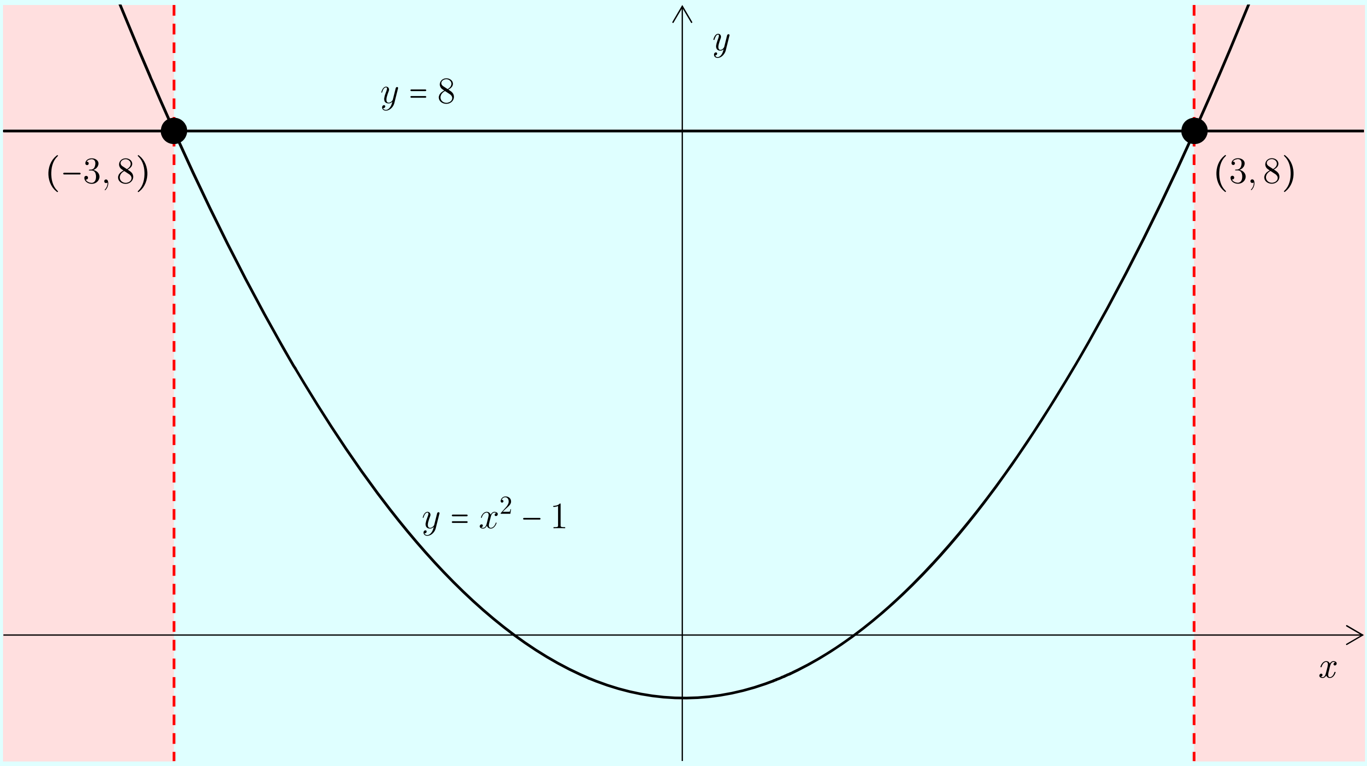
Example 251. Solve $x^2 - 1 > 8$ ($x \in \mathbb{R}$).

Again, let's see how graphs can help us solve this inequality. Graph $y = x^2 - 1$ and $y = 8$. We find that the graph of $y = x^2 - 1$ is above that of $y = 8$ in the light-red regions indicated below (**excluding** the dotted red lines).

Thus, three perfectly good (and equivalent) answers to the above problem are

- "$x < -3$, $x > 3$."
- "$x \in (-\infty, -3) \cup (3, \infty) = \mathbb{R} \setminus [-3, 3]$."
- "The solution set is $(-\infty, -3) \cup (3, \infty) = \mathbb{R} \setminus [-3, 3]$."

Also, we may say in words that every real number *except* those between -3 and 3 (inclusive) is a solution.



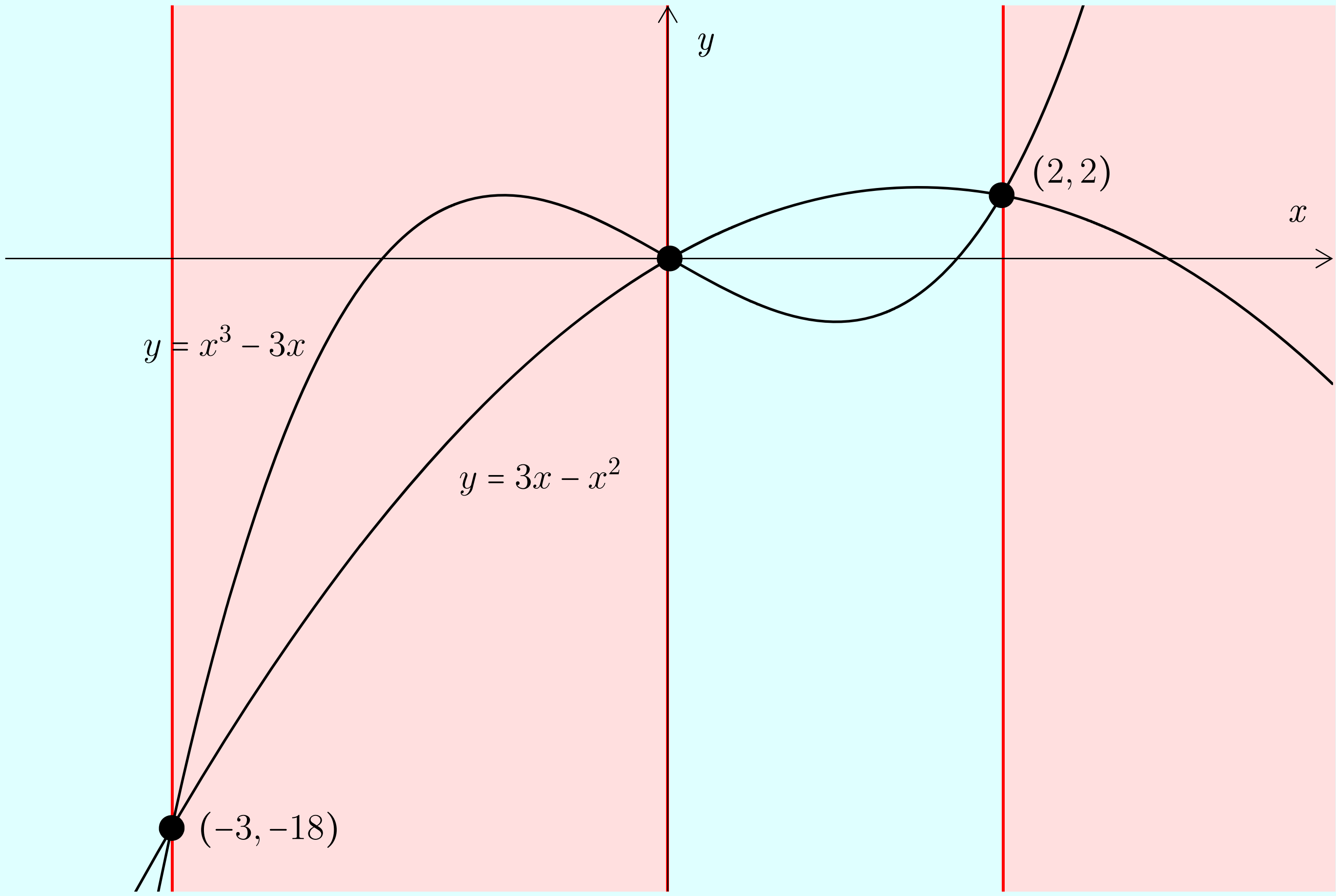
Example 252. Solve $x^3 - 3x \geq 3x - x^2$.

Graph $y = x^3 - 3x$ and $y = 3x - x^2$. We find that the graph of $y = x^3 - 3x$ is above that of $y = 3x - x^2$ in the light-red regions (**including** the solid red lines) indicated below.

Thus, three perfectly good (and equivalent) answers to the above problem are

- "$-3 \leq x \leq 0$, $x \geq 2$."
- "$x \in [-3, 0] \cup [2, \infty)$."
- "The solution set is $[-3, 0] \cup [2, \infty)$."

Also, we may say in words that every real number between -3 and 0 (inclusive) and every real number greater than or equal to 2 is a solution.



Here are the formal definitions of a **solution** and a **solution set**:

Definition 59. Given an equation (or inequality) in one variable, any number that satisfies the equation (or inequality) is called a *solution* (or *root*) of that equation (or inequality). And the set of all such solutions is called the *solution set* of that equation (or inequality).

Remark 36. The A-Level exams don't seem to ever mention the concept of a **solution set**. Nonetheless, it is quite convenient and this textbook will use it.

For now, we'll be dealing only with real numbers. And so for now, we'll be looking only at real solutions and real solution sets.

But in Part III (Complex Numbers), we'll learn also that solutions can include non-real or

complex numbers. Here's a quick preview:

Example 253. The equation $x^2+1=0$ ($x \in \mathbb{R}$) has **no solution**. (We say that its solution set is $\varnothing$, the empty set.) But this is only because we've specified that $x \in \mathbb{R}$.

In contrast, the equation $x^2+1=0$ ($x \in \mathbb{C}$) has *two* solutions: $-\mathrm{i}$ and i. (We say that its solution set is $\{-\mathrm{i}, \mathrm{i}\}$.)

14. O-Level Review: The Quadratic Equation $y = ax^2 + bx + c$

In this chapter, we'll review the **quadratic equation**:

$$0 = ax^2 + bx + c \qquad (a \neq 0).$$

(If $a = 0$, then the equation is not quadratic but linear: $0 = bx + c$.)

Let's find the **solutions** (or **roots**) of this quadratic equation:

Divide both sides by $a \neq 0$: $$0 = x^2 + \frac{b}{a}x + \frac{c}{a}.$$

Add and subtract $\dfrac{b^2}{4a^2}$: $$0 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}. \qquad ☺$$

By the way, ☺ is an example of the **Plus Zero Trick**:[132] It is always perfectly legitimate to add zero to *any* expression. So, given any quantity q (in this case $q = b^2/4a^2$), it is also always legitimate to add $+q - q = 0$ to any expression.

Complete the square: $$0 = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}.$$

Rearrange: $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}.$$

Take the square root: $$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

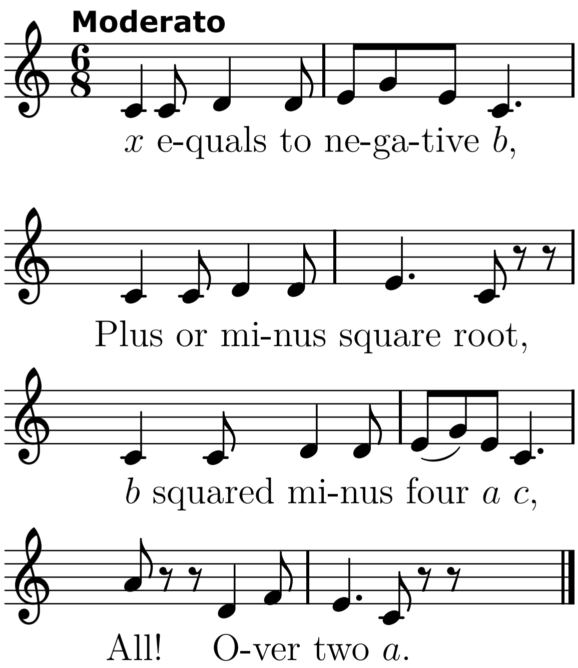
Rearrange to get the **quadratic formula** (i.e. the two solutions or roots of the quadratic equation):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quadratic formula is **not** on the List of Formulae (MF26) and so sadly, you'll have to memorise it. Unfortunately, I've never come across a good mnemonic that works (for me). Look for one that works for you. (Lemme know if you think you've found a good one!)

[132] Tao (2016) calls this the "middle-man trick".

One mnemonic is to sing the quadratic formula to the tune of Pop Goes the Weasel.[133] Arranged by a musical genius so that each syllable matches each note:



Definition 60. The *discriminant* of the quadratic equation $0 = ax^2 + bx + c$ is $b^2 - 4ac$.

Remark 37. Some writers denote the discriminant by the symbol D or Δ.[134] (Your H2 Maths syllabus and exams don't seem to do so, so neither shall we.)

On the next page, Fact 34 summarises the key features of the quadratic equation and also reviews some of the concepts we've gone through in previous chapters. Six examples follow.

[133] I checked out about ten versions on YouTube and unfortunately I found all of them to be very annoying and cannot recommend them. Maybe I'll make one—don't worry, I won't be the one singing.

[134] The symbol Δ is the upper-case Greek letter *delta*.

By the way, the equation $y = ax^2 + bx + c$ is *also* a quadratic equation, but in two variables.

Fact 34. *Consider the graph of the quadratic equation $y = ax^2 + bx + c$.*

(a) *The only y-intercept is $(0, c)$.*

(b) *There are two, one, or zero x-intercepts, depending on the sign of $b^2 - 4ac$:*

- **(i)** *If $b^2 - 4ac > 0$, then there are two x-intercepts:*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

And $$ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right).$$

- **(ii)** *If $b^2 - 4ac = 0$, then there is one x-intercept $x = -b/2a$ (where the graph just touches the x-axis).*

And $$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2.$$

- **(iii)** *If $b^2 - 4ac < 0$, then there are no x-intercepts. There is also no way to factorise the quadratic polynomial $ax^2 + bx + c$ (unless we use complex numbers).*

(c) *It is symmetric in the vertical line $x = -b/2a$.*

(d) *The only turning point is $\left(-b/2a, -b^2/4a + c\right)$, which is a strict global* **(i)** *minimum if $a > 0$; or* **(ii)** *maximum if $a < 0$.*

Proof. See p. 1568 in the Appendices. □

Corollary 2 (informal)**.** *Let G be the graph of the quadratic equation $y = ax^2 + bx + c$.*

(a) *If $a > 0$, then G is $\cup$-shaped.*

(b) *If $a < 0$, then G is $\cap$-shaped.*

"*Proof.*" "Clearly", the graph is either $\cup$-shaped or $\cap$-shaped.[135]

(a) By Fact 34(d)(i), if $a > 0$, then the graph has a strict global minimum. So, the graph must be $\cup$-shaped.

(b) By Fact 34(d)(ii), if $a < 0$, then the graph has a strict global maximum. So, the graph must be $\cap$-shaped. □

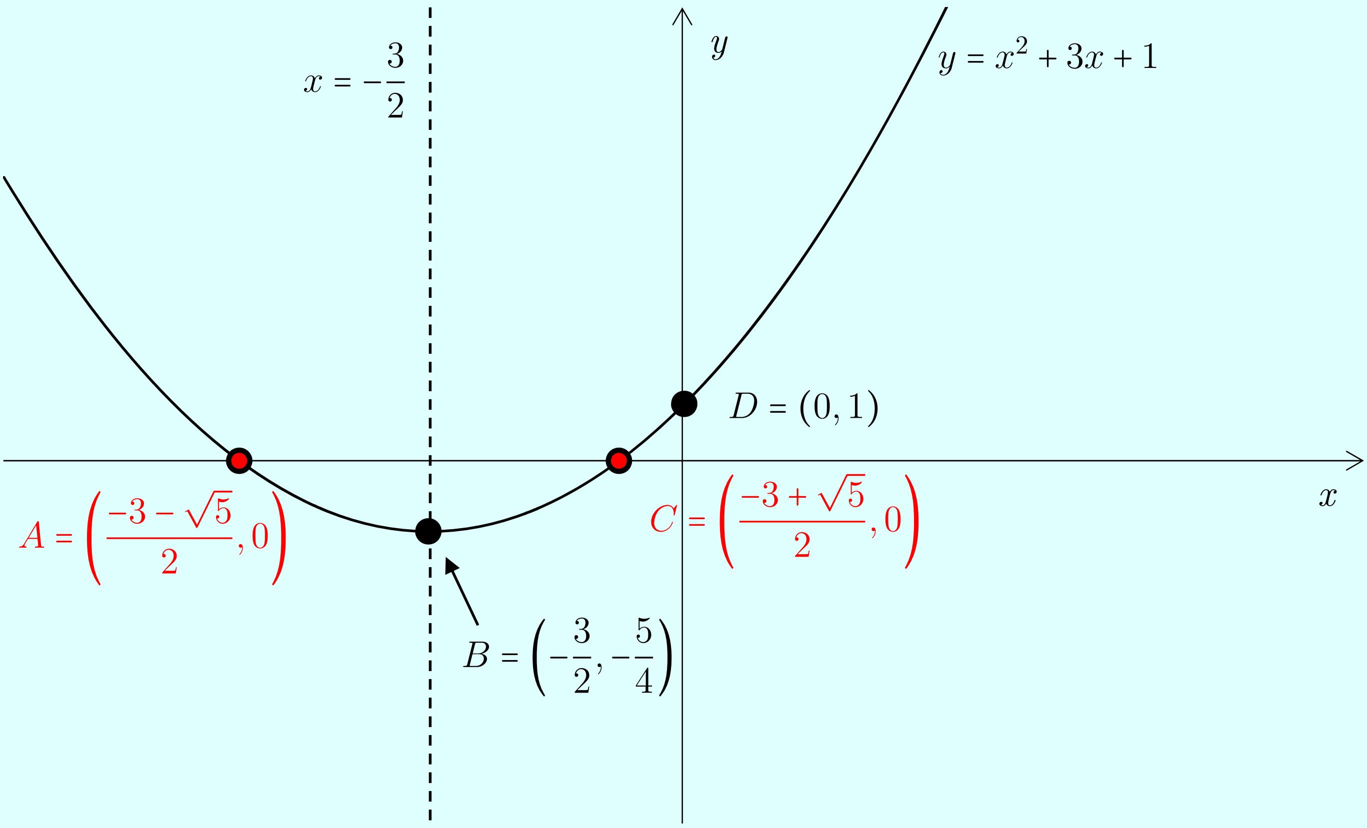
Remark 38. In mathematics, a **corollary** is a statement that follows readily from another result. (Here for example, Corollary 2 follows readily from Fact 34.)

[135] It is this bit here that makes the result informal. We have to more precisely define the terms "$\cup$-shaped" and "$\cap$-shaped". One simple possibility is to replace these two terms with the terms *convex* and *concave*. However, we'll only be introducing these latter two terms in Part V (Calculus). So, for now, we'll just stick with the informal "$\cup$-shaped" and "$\cap$-shaped".

We can distinguish between **six cases** of the quadratic equation, depending on whether of $a \gtrless 0$ and whether $b^2 - 4ac \lesseqqgtr 0$.

Here are six examples to illustrate the six cases:

Example 254. Consider the quadratic equation $y = x^2 + 3x + 1$.



1. The only y-intercept is $D = (0, 1)$.
2. The equation $0 = x^2 + 3x + 1$ has two real solutions or roots:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 1}}{2 \times 1} = \frac{-3 \pm \sqrt{5}}{2}.$$

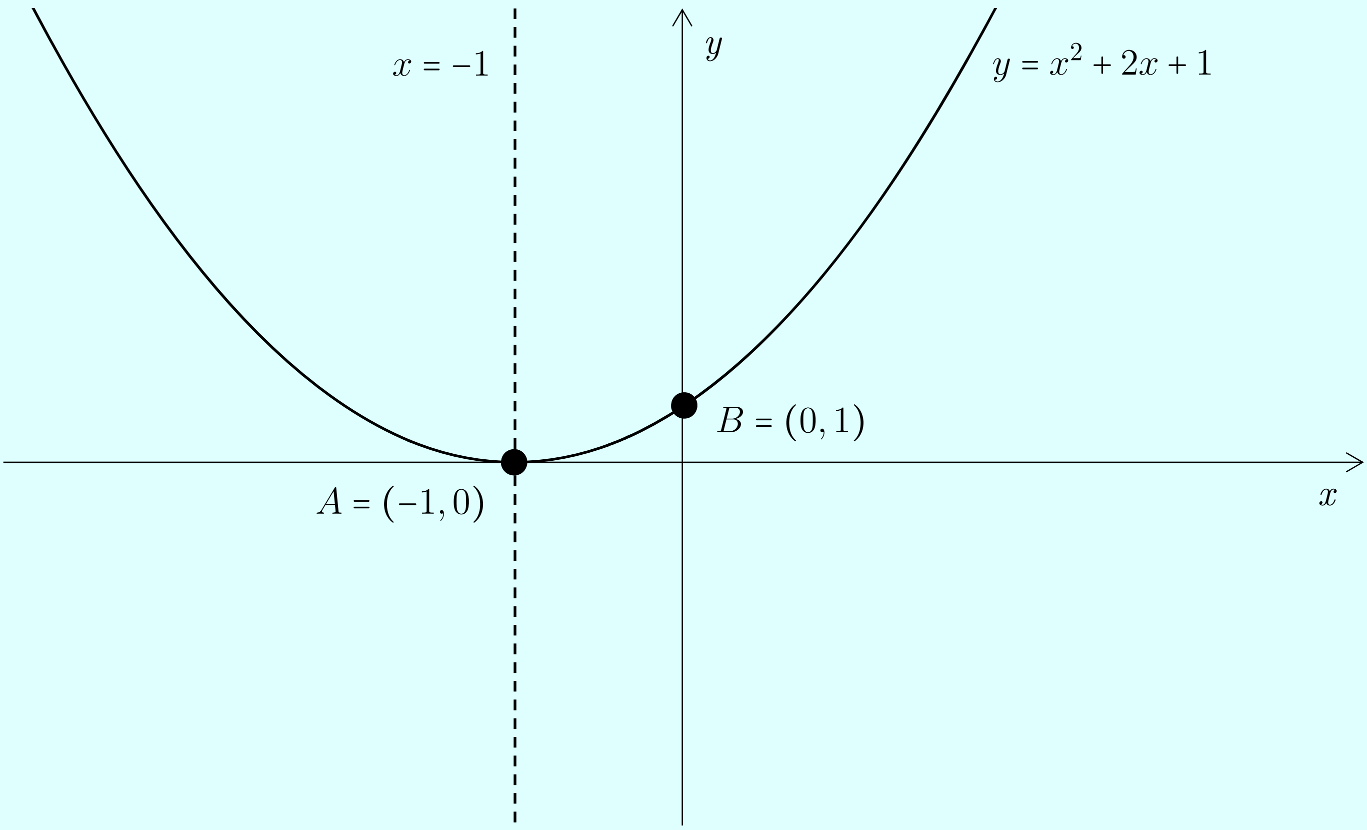
   So, the two x-intercepts are $A = \left(\frac{-3-\sqrt{5}}{2}, 0\right)$ and $C = \left(\frac{-3+\sqrt{5}}{2}, 0\right)$.

   And we may write $\quad x^2 + 3x + 1 = \left(x - \frac{-3-\sqrt{5}}{2}\right)\left(x - \frac{-3+\sqrt{5}}{2}\right)$.

3. The graph is symmetric in the line $x = -\frac{b}{2a} = -\frac{3}{2 \times 1} = -\frac{3}{2}$. (We'll learn more about symmetry in Ch. 16.)

4. The only turning point is $B = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right) = \left(-\frac{3}{2}, 1 - \frac{3^2}{4 \times 1}\right) = \left(-\frac{3}{2}, -\frac{5}{4}\right)$.
   Since $a = 1 > 0$, B is the strict global minimum and the graph is ∪-shaped.

Example 255. Consider the quadratic equation $y = x^2 + 2x + 1$.



1. There is one y-intercept: $B = (0,1)$.
2. The equation $y = x^2 + 2x + 1$ has one real root:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 1}}{2 \times 1} = \frac{-2 \pm \sqrt{0}}{2} = -1.$$

And so, there is one x-intercept, where the graph just touches the x-axis:

$$A = (-1,0).$$

We can factorise the quadratic polynomial:

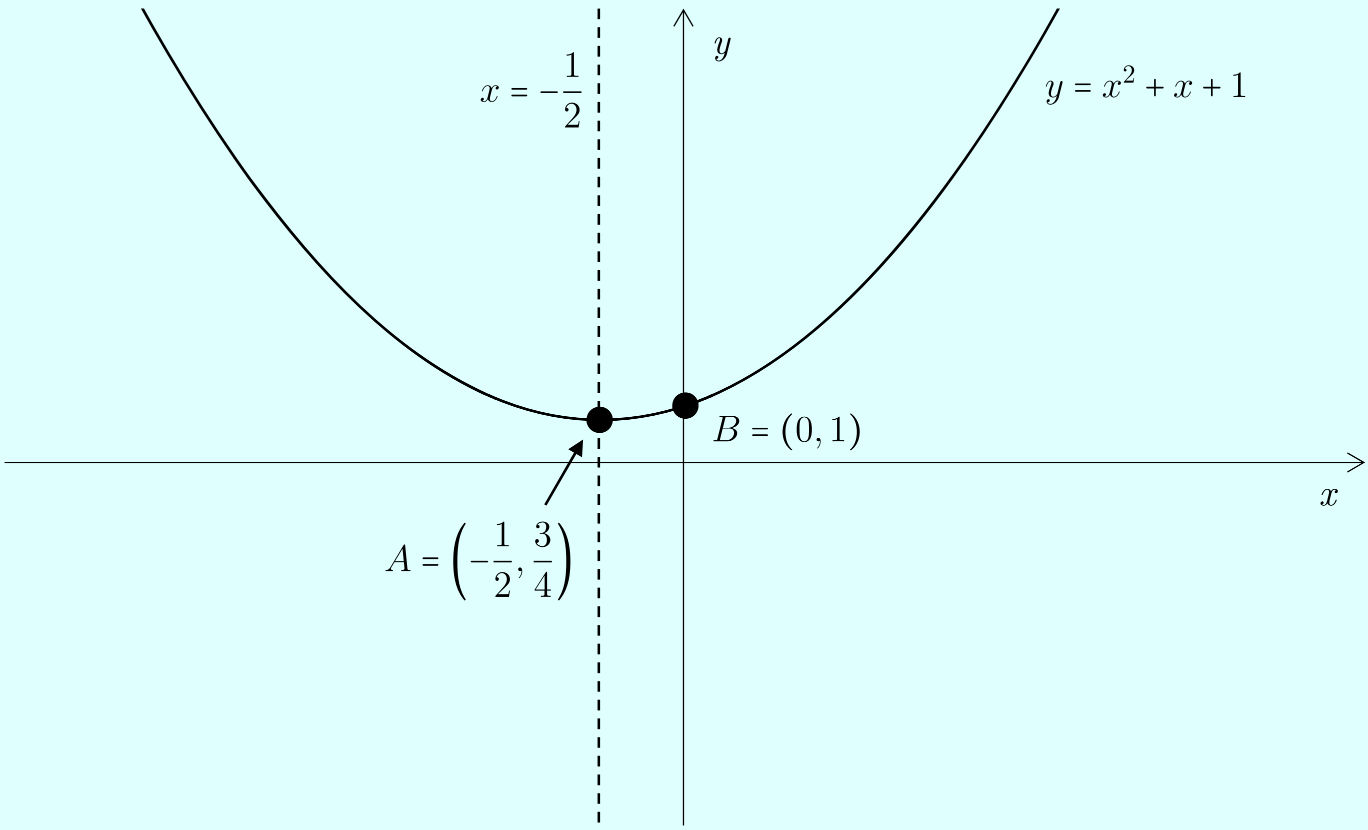
$$x^2 + 2x + 1 = [x - (-1)]^2 = (x+1)^2.$$

3. There is one (vertical) line of symmetry: $x = -\dfrac{b}{2a} = -\dfrac{2}{2 \times 1} = -1$.
4. In general, if a quadratic equation has only one real root, then the turning point is also the x-intercept:

$$A = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right) = \left(-1, 1 - \frac{2^2}{4 \times 1}\right) = (-1,0).$$

5. Since the coefficient 1 on x^2 is positive, the graph is ∪-shaped, with the turning point being the strict global minimum.

Example 256. Consider the quadratic equation $y = x^2 + x + 1$.



1. There is one y-intercept: $B = (0, 1)$.
2. The equation $y = x^2 + x + 1$ has **no** real roots:

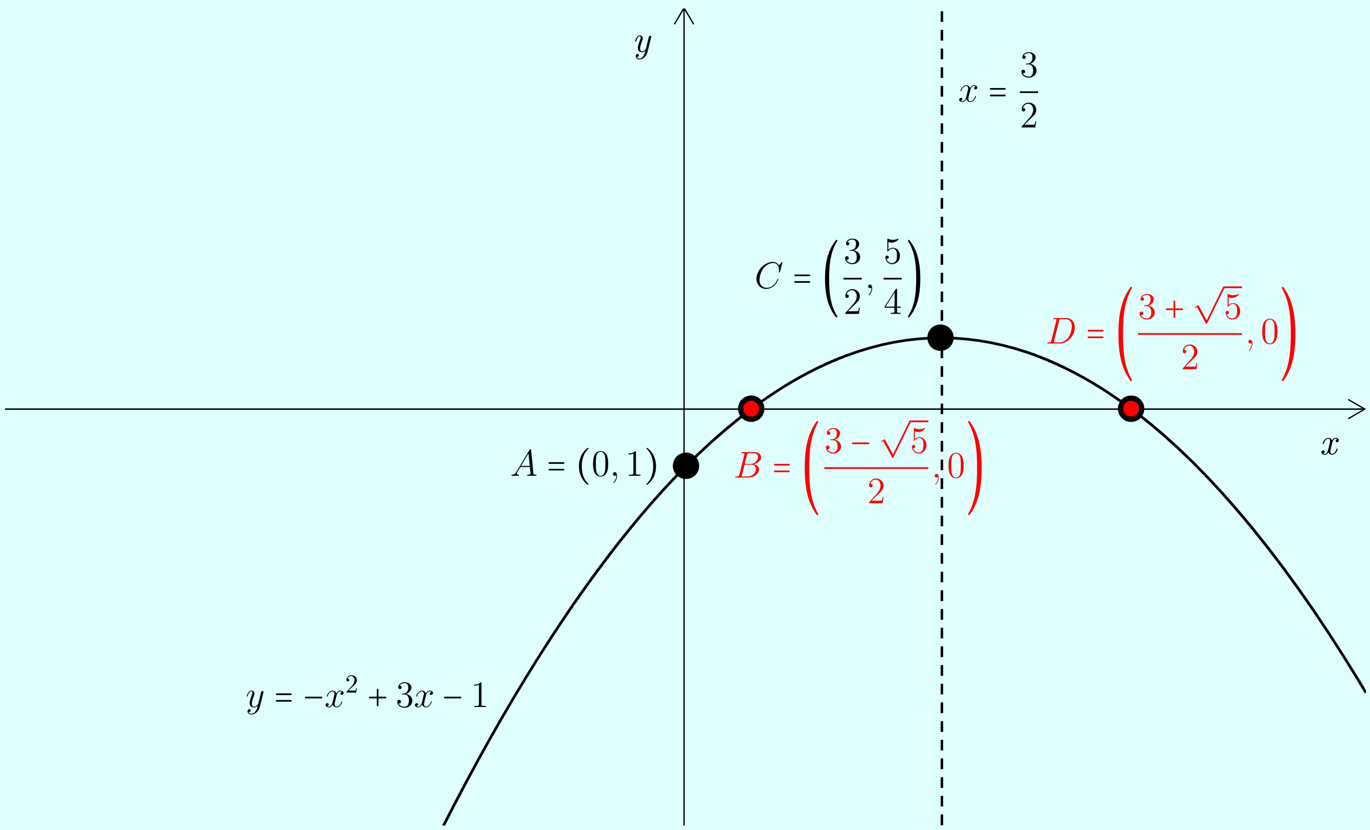
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2 \times 1} = \frac{-1 \pm \sqrt{-3}}{2}.$$

   And so, there are no x-intercepts. We can*not* factorise the quadratic polynomial (unless we use complex numbers).
3. There is one (vertical) line of symmetry: $x = -\frac{b}{2a} = -\frac{1}{2 \times 1} = -\frac{1}{2}$.
4. The one turning point is

$$A = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right) = \left(-\frac{1}{2}, 1 - \frac{1^2}{4 \times 1}\right) = \left(-\frac{1}{2}, \frac{3}{4}\right).$$

5. Since the coefficient 1 on x^2 is positive, the graph is ∪-shaped, with the turning point being the strict global minimum.

Example 257. Consider the quadratic equation $y = -x^2 + 3x - 1$.



1. There is one y-intercept: $A = (0, 1)$.
2. The equation $y = -x^2 + 3x - 1$ has two real roots:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \times (-1) \times (-1)}}{2 \times (-1)} = \frac{-3 \pm \sqrt{5}}{-2} = \frac{3 \mp \sqrt{5}}{2}.$$

   And so, there are two x-intercepts:

$$B = \left(\frac{3-\sqrt{5}}{2}, 0\right) \text{ and } D = \left(\frac{3+\sqrt{5}}{2}, 0\right).$$

   We can factorise the quadratic polynomial:

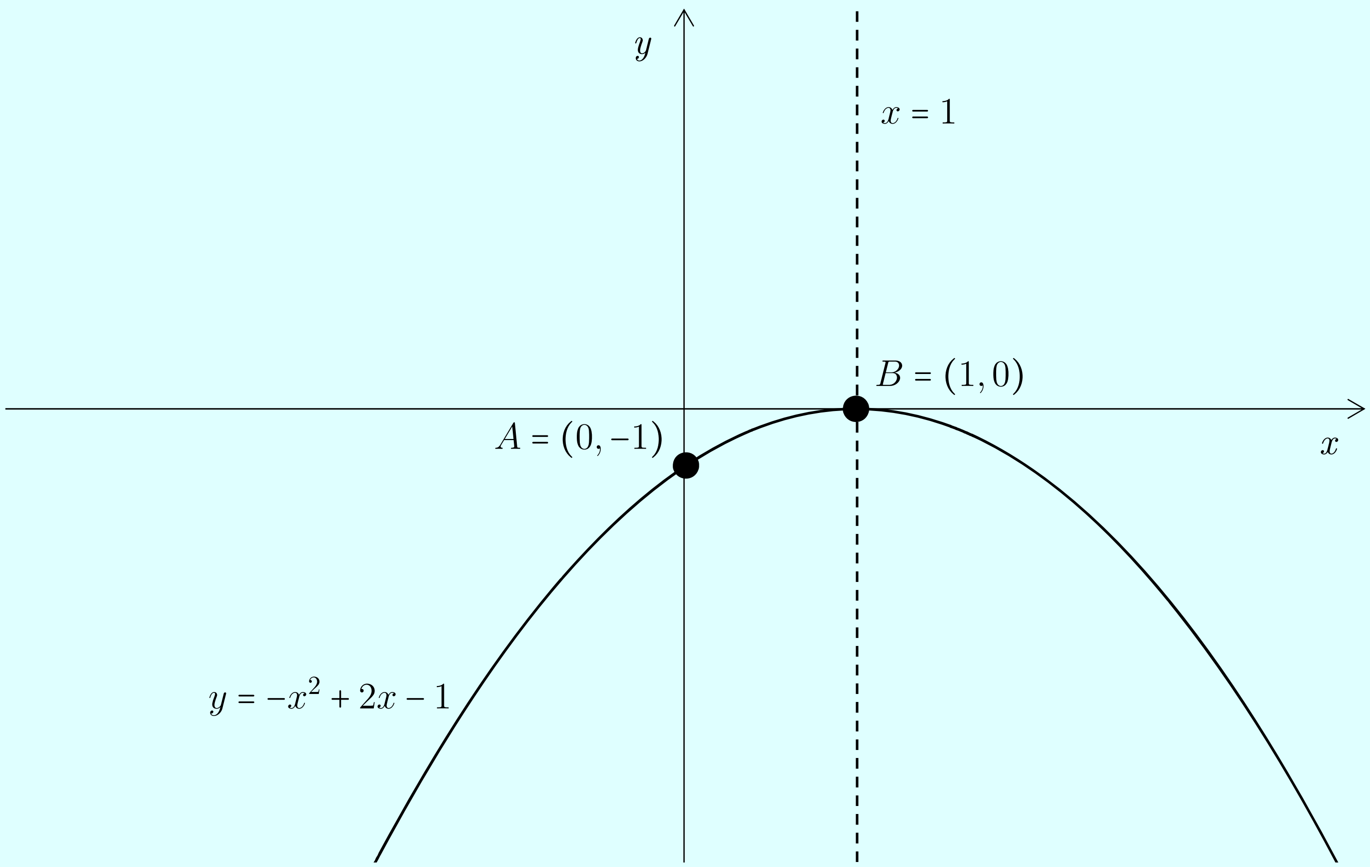
$$-x^2 + 3x - 1 = -\left(x - \frac{3-\sqrt{5}}{2}\right)\left(x - \frac{3+\sqrt{5}}{2}\right).$$

3. There is one (vertical) line of symmetry: $x = -\frac{b}{2a} = -\frac{3}{2 \times (-1)} = \frac{3}{2}$.
4. The one turning point is

$$C = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right) = \left(\frac{3}{2}, -1 - \frac{3^2}{4 \times (-1)}\right) = \left(\frac{3}{2}, \frac{5}{4}\right).$$

5. Since the coefficient -1 on x^2 is negative, the graph is $\cap$-shaped, with the turning point being the strict global maximum.

Example 258. Consider the quadratic equation $y = -x^2 + 2x - 1$.



1. There is one y-intercept: $A = (0, -1)$.
2. The equation $y = -x^2 + 2x - 1$ has one real root:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \times (-1) \times (-1)}}{2 \times (-1)} = \frac{-2 \pm \sqrt{0}}{-2} = 1.$$

   And so, there is one x-intercept, which is also where it just touches the x-axis:

$$B = (1, 0).$$

   We can factorise the quadratic polynomial:

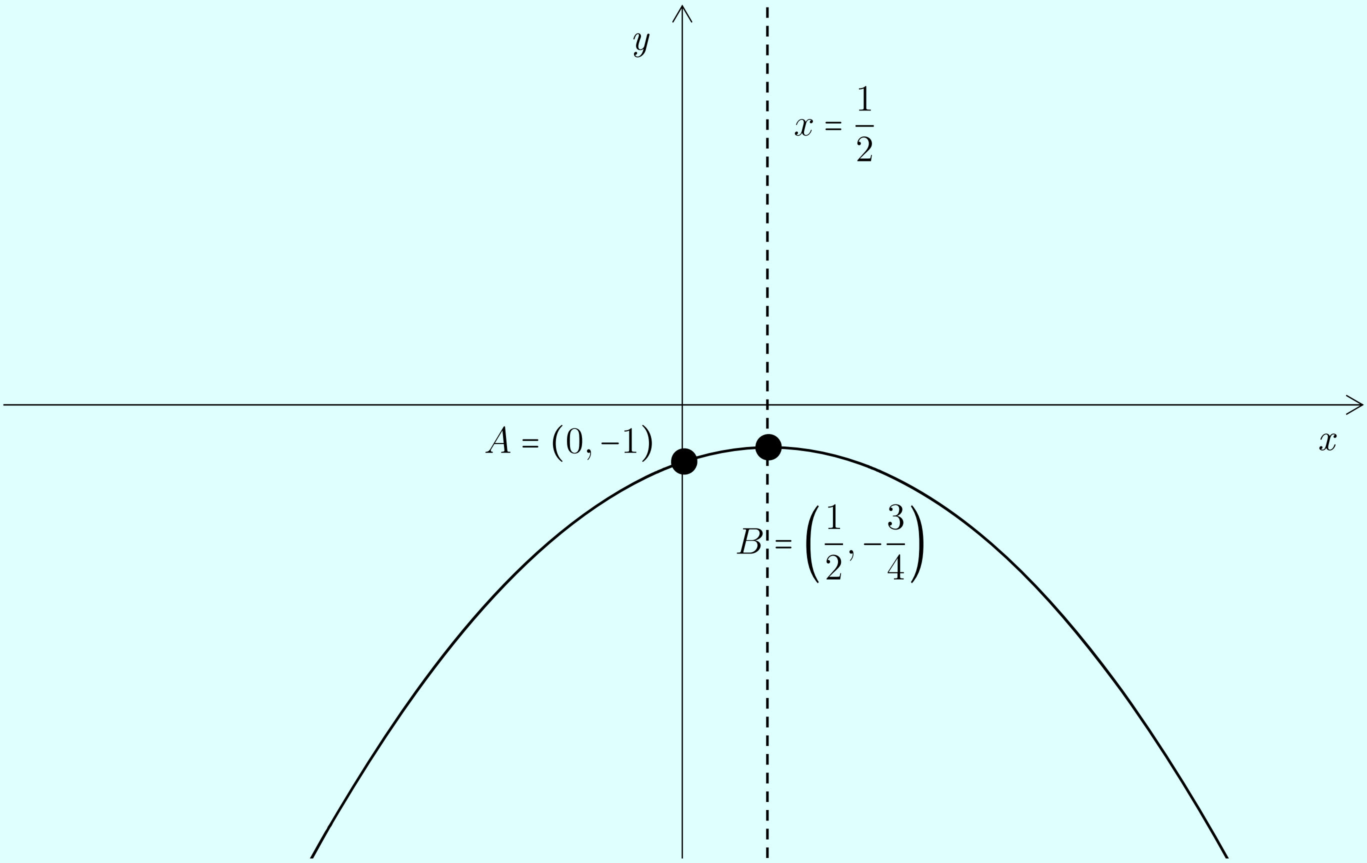
$$-x^2 + 2x - 1 = -(x-1)^2.$$

3. There is one (vertical) line of symmetry: $x = -\dfrac{b}{2a} = -\dfrac{2}{2 \times (-1)} = 1$.
4. In general, if a quadratic equation has only one real root, then the turning point is also the x-intercept:

$$B = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right) = \left(1, 1 - \frac{2^2}{4 \times (-1)}\right) = (1, 0).$$

5. Since the coefficient -1 on x^2 is negative, the graph is ∩-shaped, with the turning point being the strict global maximum.

Example 259. Consider the quadratic equation $y = -x^2 + x - 1$.



1. There is one y-intercept: $A = (0, -1)$.
2. The equation $y = -x^2 + x - 1$ has **no** real roots:

$$x = \frac{-b \pm \sqrt{2^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4 \times (-1) \times (-1)}}{2 \times (-1)} = \frac{-1 \pm \sqrt{-3}}{-2}.$$

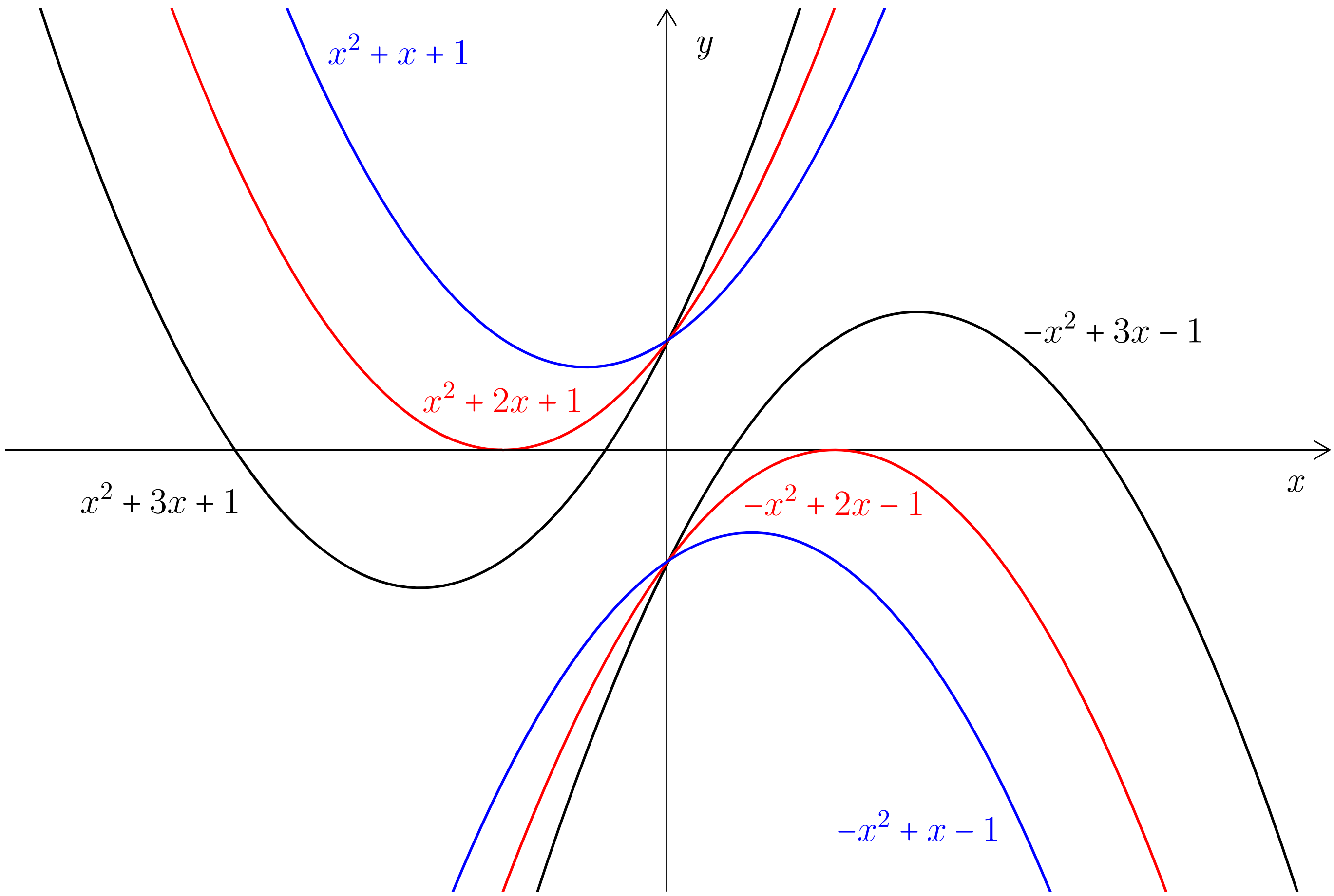
   And so, there are no x-intercepts. We can*not* factorise the quadratic polynomial (unless we use complex numbers).

3. There is one (vertical) line of symmetry: $x = -\frac{b}{2a} = -\frac{1}{2 \times (-1)} = \frac{1}{2}$.
4. The one turning point is

$$B = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right) = \left(\frac{1}{2}, -1 - \frac{1^2}{4 \times (-1)}\right) = \left(\frac{1}{2}, -\frac{3}{4}\right).$$

5. Since the coefficient -1 on x^2 is negative, the graph is $\cap$-shaped, with the turning point being the strict global maximum.

The last six examples in one figure:



Exercise 91. Sketch each graph, identifying any intercepts, lines of symmetry, turning points, and extrema.

(a) $y = 2x^2 + x + 1$.

(b) $y = -2x^2 + x + 1$.

(c) $y = x^2 + 4x + 4$. (Answers on p. 1771.)

15. The Distance Between a Point and a Line

Definition 61. The *distance between a point A and a graph G* is the distance between A and B, where B is the point on G that's closest to A.

The above definition is general in that G can be *any* graph (i.e. *any* set of points). But to keep things simple, we'll look only at cases where G is a line.

Remark 39. In the trivial case where A is in G, the point on G that's closest to A is A itself. And so, the distance between A and G is of course simply zero.

We can find B—the point on l that's closest to A—using what we've learnt about quadratic equations:

Example 260. Let $A = (5, 1)$ be a point and the line l be (the graph of) $y = 2x$.

Pick[136] any arbitrary point $P \overset{1}{=} (p, 2p)$ on l, where $p \in \mathbb{R}$.

By Definition 48, the distance between A and P is

$$\sqrt{(p-5)^2 + (2p-1)^2} \overset{2}{=} \sqrt{5p^2 - 14p + 26}.$$

This last surd expression is minimised when the quadratic expression inside is minimised.[137] By Fact 34(d), this occurs when

$$p = -\frac{\text{“}b\text{”}}{2\text{“}a\text{”}} = -\frac{-14}{2(5)} = 1.4.$$

Now, plug $p = 1.4$ into $\overset{1}{=}$ to get $B = (1.4, 2.8)$, the point on l that's closest to A.

By Definition 61, the distance between A and l is simply the distance between A and B, which we get by plugging $p = 1.4$ into the surd expression for distance in $\overset{2}{=}$:

$$\left.\sqrt{5p^2 - 14p + 26}\right|_{p=1.4} = \sqrt{5(1.4)^2 - 14 \times 1.4 + 26} = \sqrt{16.2}.$$

[136] Clarification in case you're not sure what's going on here: Here we first set the x-coordinate of P to be any arbitrary real number p. Then, since P is on l, its y-coordinate must be $2p$.

[137] Actually, to justify this assertion, we also need to show that $\sqrt{\cdot}$ is an increasing function. But we'll gloss over this for now as we haven't yet introduced the concepts of a function and when a function is increasing.

Example 261. Let $A = (-4,0)$ be a point and the line l be $y = -x/3 + 2$.

Pick any arbitrary point $P \stackrel{1}{=} (p, -p/3 + 2)$ on l.

By Definition 48, the distance between A and P is

$$\sqrt{(p+4)^2 + (-p/3+2)^2} \stackrel{2}{=} \sqrt{10p^2/9 + 20p/3 + 20}.$$

Again, the above expression is minimised when

$$p = -\frac{\text{“}b\text{”}}{2\text{“}a\text{”}} = -\frac{20/3}{2\,(10/9)} = -3.$$

Now, plug $p = -3$ into $\stackrel{1}{=}$ to get $B = (-3,3)$, the point on l that's closest to A.

Also, plug $p = -3$ into $\stackrel{2}{=}$ to get the distance between A and lL

$$\left.\sqrt{10p^2/9 + 20p/3 + 20}\right|_{p=-3} = \sqrt{10\,(-3)^2/9 + 20\,(-3)/3 + 20} = \sqrt{10}.$$

We can use the “quadratic equations" idea illustrated in the above examples to find the unique point B on the line l that's closest to A:

Proposition 3. *The unique point on the line $ax + by + c = 0$ that is closest to the point (p,q) is*

$$B = \left(p - a\frac{ap+bq+c}{a^2+b^2}, q - b\frac{ap+bq+c}{a^2+b^2}\right).$$

Proof. The proof is not difficult conceptually, but does involve tedious and messy algebra. So, we'll relegate it to the Appendices—see p. 142.6. □

Knowing the point B on the line l that's closest to A, we can easily find the distance between A and l (this is simply the distance between A and B):

Corollary 3. *The distance between a point (p,q) and a line $ax + by + c = 0$ is*

$$\frac{|ap+bq+c|}{\sqrt{a^2+b^2}}.$$

Proof. By Proposition 3, the point on the line that is closest to (p,q) is

$$\left(p - a\frac{ap+bq+c}{a^2+b^2}, q - b\frac{ap+bq+c}{a^2+b^2}\right).$$

So, the distance between this point and (p,q) is

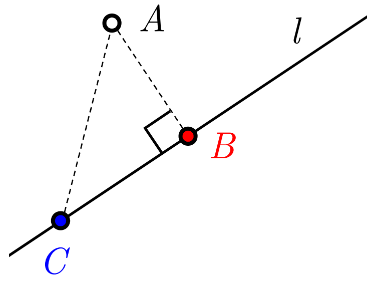
$$\sqrt{\left[p-\left(p-a\frac{ap+bq+c}{a^2+b^2}\right)\right]^2+\left[q-\left(q-b\frac{ap+bq+c}{a^2+b^2}\right)\right]^2}$$

$$=\sqrt{\left(a\frac{ap+bq+c}{a^2+b^2}\right)^2+\left(b\frac{ap+bq+c}{a^2+b^2}\right)^2}=\left|\frac{ap+bq+c}{a^2+b^2}\right|\sqrt{a^2+b^2}=\frac{|ap+bq+c|}{\sqrt{a^2+b^2}}. \qquad \square$$

Remark 40. Do **not** try mugging the above two results. What's more important is to understand the methods used and be able to find the closest point B in any specific case, as illustrated in the examples and exercises here.

15.1. Foot of the Perpendicular (from a Point to a Line)

It turns out that if B is the point on the line l that's closest to the point A, then l is perpendicular to the line AB.



Moreover, the converse is also true: If l is perpendicular to AB (where B is some point on l), then B is the point on l that's closest to A.

The next result simply combines these last two assertions:

Fact 35. *Let A be a point that is not on the line l. Suppose B is a point on l. Then*

$$B \text{ is the point on } l \text{ that is closest to } A \qquad \Longleftrightarrow \qquad l \perp AB.$$

Proof. Here we'll prove only $\Longleftarrow$. (For a proof of $\Longrightarrow$, see p. 1564 Appendices.)

($\Longleftarrow$) Suppose $l \perp AB$. Let $C \neq B$ be any other point on l. Then ABC forms a right triangle[138] with hypotenuse AC.

By Pythagoras' Theorem, $|AC| = \sqrt{|AB|^2 + |BC|^2} > |AB|$. So, B is closer to A than C.

Since C was arbitrarily chosen, we conclude that B is closer to A than any other point on l. □

It will be convenient to give B a special name in the case where $l \perp AB$:

Definition 62. Let A be a point that isn't on the line l. The *foot of the perpendicular from A to l* is the (unique) point B on l such that $AB \perp l$.

Altogether, given a point A not on the line l,

1. By Proposition 3, there exists a unique point B on the line l that's closest to A;
2. Moreover, by Fact 35($\Longrightarrow$), B is the foot of the perpendicular from A to l.
3. Now, B is also the unique foot of the perpendicular. This is because if there were some other point $C \neq B$ that were also the foot of the perpendicular from A to l, then by Fact 35($\Longleftarrow$), C would also be a point on l that's closest to A, contradicting Statement 1.

In summary,

Corollary 4. *Suppose A is a point not on the line l. Then there exists a point B that is both* **(a)** *the unique point on l that's closest to A; and* **(b)** *the unique point on l such that $l \perp AB$.*

[138] This proof may be considered slightly informal as it makes use of geometry and triangles. In Ch. 62 (Part IV, Vectors), we'll prove this result again using the language of vectors.

Corollary 5. *Let l be the line $ax + by + c = 0$. Suppose A is a point not on l. Then the point B that is both* **(a)** *the unique point on l that's closest to A; and* **(b)** *the unique point on l such that $l \perp AB$ is*

$$B = \left(p - a\frac{ap + bq + c}{a^2 + b^2}, q - b\frac{ap + bq + c}{a^2 + b^2} \right).$$

16. Reflection and Symmetry

16.1. The Reflection of a Point in a Point

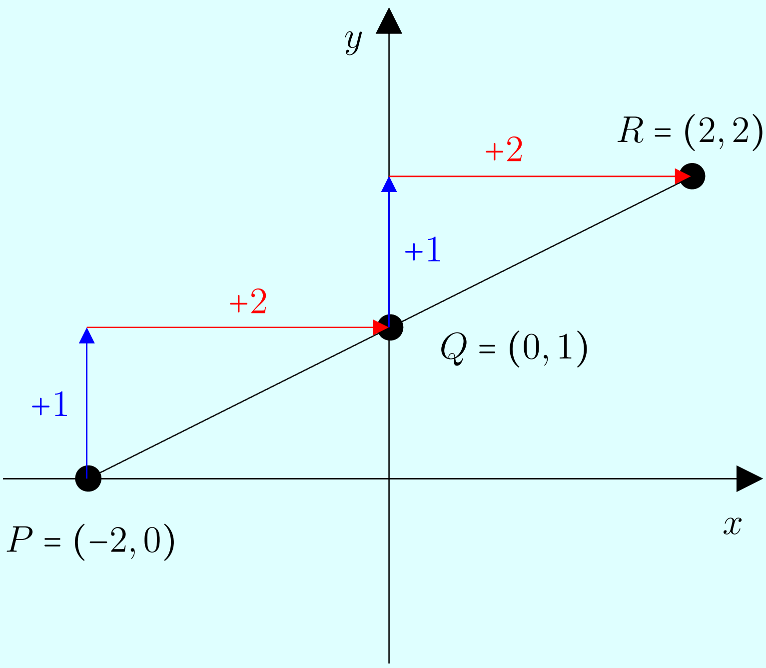
Example 262. Let $P = (-2, 0)$ and $Q = (0, 1)$ be points.

Suppose we **reflect** P in Q to produce the point R. Then what would R be?

Well, Common Sense suggests that the reflection R should satisfy these two properties:

(a) $|PQ| = |QR|$ **(b)** R is on the line PQ.

(Of course, we should also have $R \neq P$.)



How can properties **(a)** and **(b)** be satisfied? Common Sense suggests that

- Since $Q_x - P_x = +2$, we must also have $R_x - Q_x = +2$.
- Since $Q_y - P_y = +1$, we must also have $R_y - Q_y = +1$.

Hence, $P = (-2, 0)$ in $Q = (0, 1)$ is $R = (2, 2)$.

Intuitively, it seems "obvious" that there should be exactly one point R that satisfies both properties **(a)** and **(b)**. Happily, it turns out that this intuition is correct.[139]

And so, we are allowed to write down this formal definition of a **reflection**:

[139]See Proposition 26 (Appendices).

Definition 63. Let P and Q be distinct points. The *reflection of P in Q* is the unique point $R \neq P$ that satisfies these two properties:

(a) $|PQ| = |QR|$. **(b)** R is on the line PQ.

By Common Sense, if $P = (a,b)$ and $Q = (c,d)$, then the reflection of P in Q should be the point

$$R = \left(c + {\color{red}(c-a)}, d + {\color{blue}(d-b)}\right) = (2c-a, 2d-b).$$

For future reference, let's jot this down as a formal result:

Fact 36. *If $P = (a,b)$ and $Q = (c,d)$ are distinct points, then the reflection of P in Q is the point*

$$R = (2c-a, 2d-b).$$

Proof. See Exercise 93. □

Rather than *mug* Fact 36, it's probably easier to find reflection points by directly using our Common-Sense Reasoning:

Example 263. The reflection of $(1,2)$ in $(5,8)$ is $(9,14)$.

$$(1,2) \xrightarrow[+6]{+4} (5,8) \xrightarrow[+6]{+4} (9,14)$$

Exercise 92. What is the reflection of $(8,5)$ in $(-2,4)$? (Answer on p. 1770.)

Exercise 93. Prove Fact 36—to do so, simply verify that the point $R = (2c-a, 2d-b)$ satisfies properties **(a)** and **(b)**. (Answer on p. 1770.)

16.2. The Reflection of a Point in a Line

Definition 64. The *reflection* of a point P in a graph is the reflection of P in the point on the graph that's closest to P.

The above definition is general in that G can be *any* graph (i.e. *any* set of points). But to keep things simple, we'll look only at cases where G is a line.

Example 264. Find the distance between the point $P = (-3, 2)$ and the line l given by $y = 2x + 3$.

Let Q be the point on l that's closest to P.

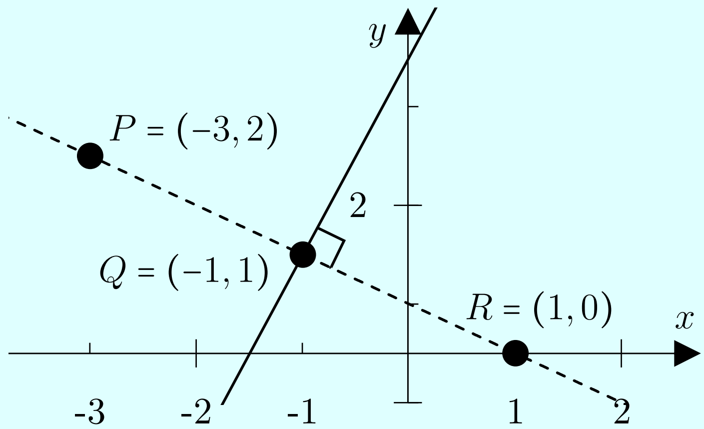
Let's use what we learnt in Ch. 15 to find Q:

Let $S \overset{1}{=} (s, 2s + 3)$ be an arbitrary point on l. Then the distance between P and S is

$$\sqrt{(s+3)^2 + (2s+3-2)^2} = \sqrt{5s^2 + 10s + 10},$$

which is minimised at $s = -10/(2 \times 5) = -1$.

Plug this into $\overset{1}{=}$ to get $Q = (-1, 1)$. (As we learnt in Ch. 15, Q is also called the foot of the perpendicular from P to l and $l \perp PQ$.)



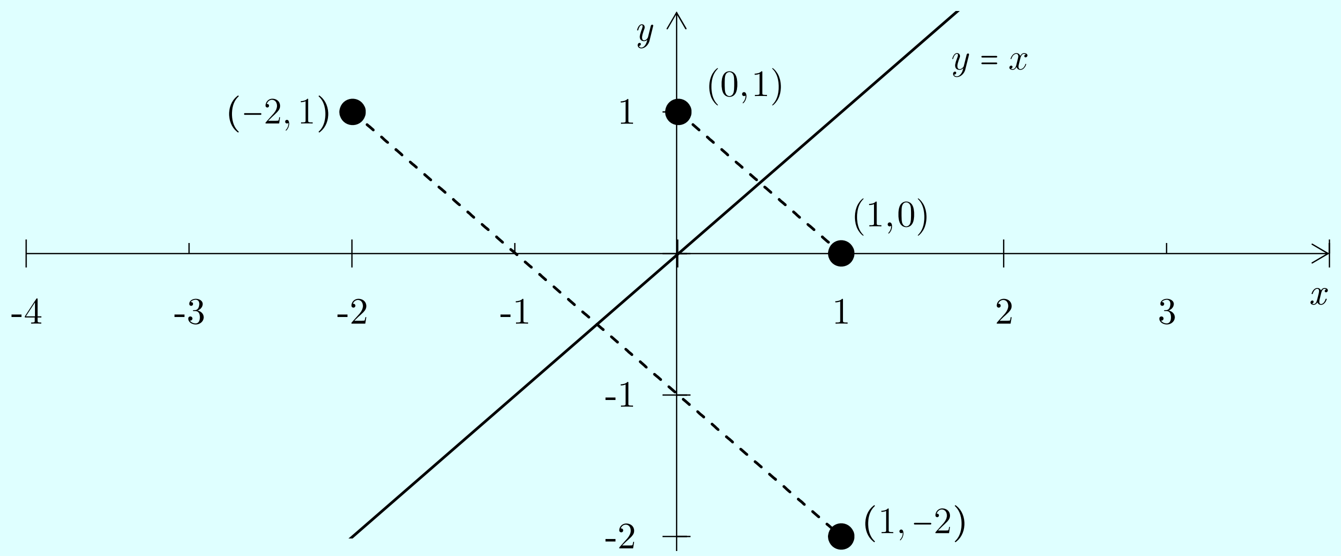
And now, by Fact 36, the reflection of P in Q is $R = (1, 0)$.

By Definition 64, R is the reflection of P in l.

Some simple but useful results:

Fact 37. *The reflection of the point (p, q) in the line $y = x$ is the point (q, p).*

Example 265. The reflection of $(-2, 1)$ in $y = x$ is $(1, -2)$. The reflection of $(0, 1)$ in $y = x$ is $(1, 0)$.



We'll give two proofs of the above result. First, a formal proof using the usual "quadratic equations" idea:

Proof. The distance between (p, q) and an arbitrary point (a, a) on $y = x$ is

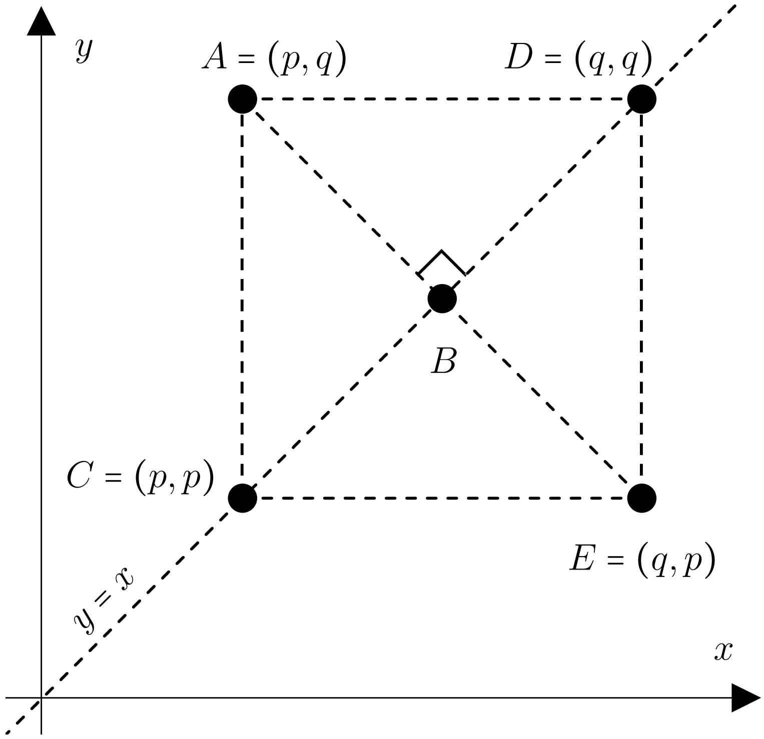
$$\sqrt{(p-a)^2 + (q-a)^2} = \sqrt{2a^2 - 2(p+q)a + p^2 + q^2}.$$

This is minimised at $a = 2(p+q)/2 \cdot 2 = (p+q)/2$. So, the point on $y = x$ closest to (p, q) is $D = ((p+q)/2, (p+q)/2)$. Hence, the reflection of (p, q) in D and thus also in the line $y = x$ is (q, p). □

Second, an informal proof-by-picture:

Proof. Let $A = (p, q)$ and B be the point on $y = x$ that's closest to A.

Mark also the points $C = (p, p)$, $D = (q, q)$, and $E = (q, p)$. Clearly, $ACED$ is a square.



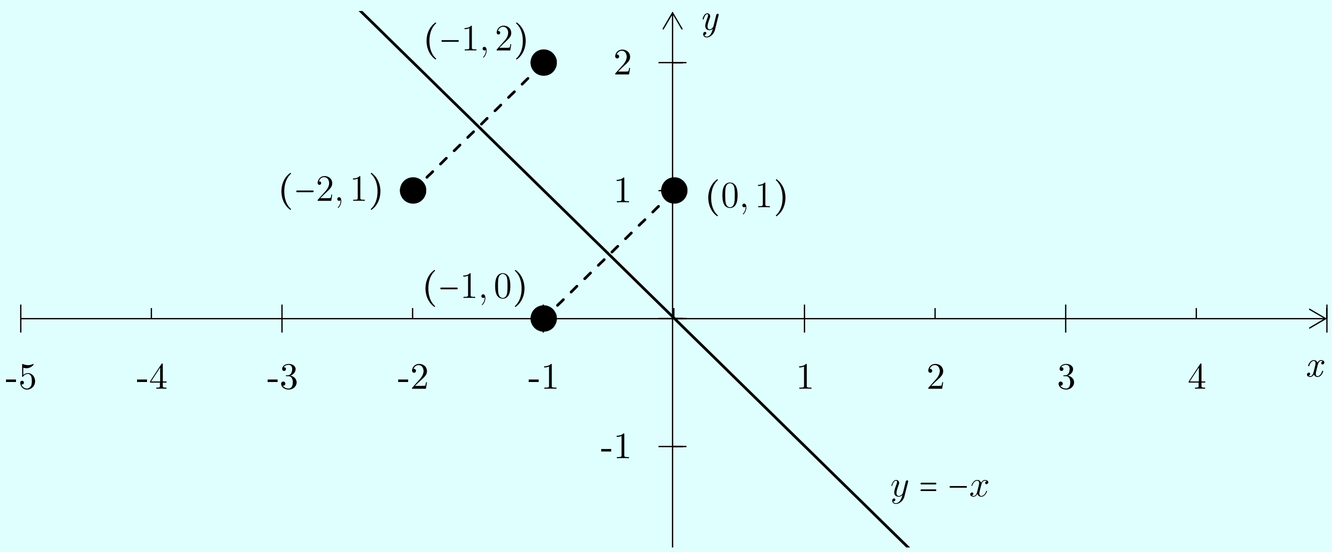
The lines AE and $y = x$ are the square's diagonals, which are perpendicular to each other. (This is a result from geometry we haven't formally stated and proven in this textbook.)

Moreover, $|AB| = |BE|$.

Altogether, by Definition 63, E is the reflection of A in B. Hence, by Definition 64, E is also the reflection of A in l. $\square$

Fact 38. *The reflection of the point (p,q) in the line $y = -x$ is the point $(-q,-p)$.*

Example 266. The reflection of $(-2,1)$ in $y = -x$ is $(-1,2)$. The reflection of $(0,1)$ in $y = x$ is $(-1,0)$.



Again, we'll give two proofs of the above result. First, a formal proof using the usual "quadratic equations" idea:

Proof. The distance between (p,q) and an arbitrary point $(a,-a)$ on $y = -x$ is

$$\sqrt{(p-a)^2+(q+a)^2}=\sqrt{2a^2+2(q-p)a+p^2+q^2}$$

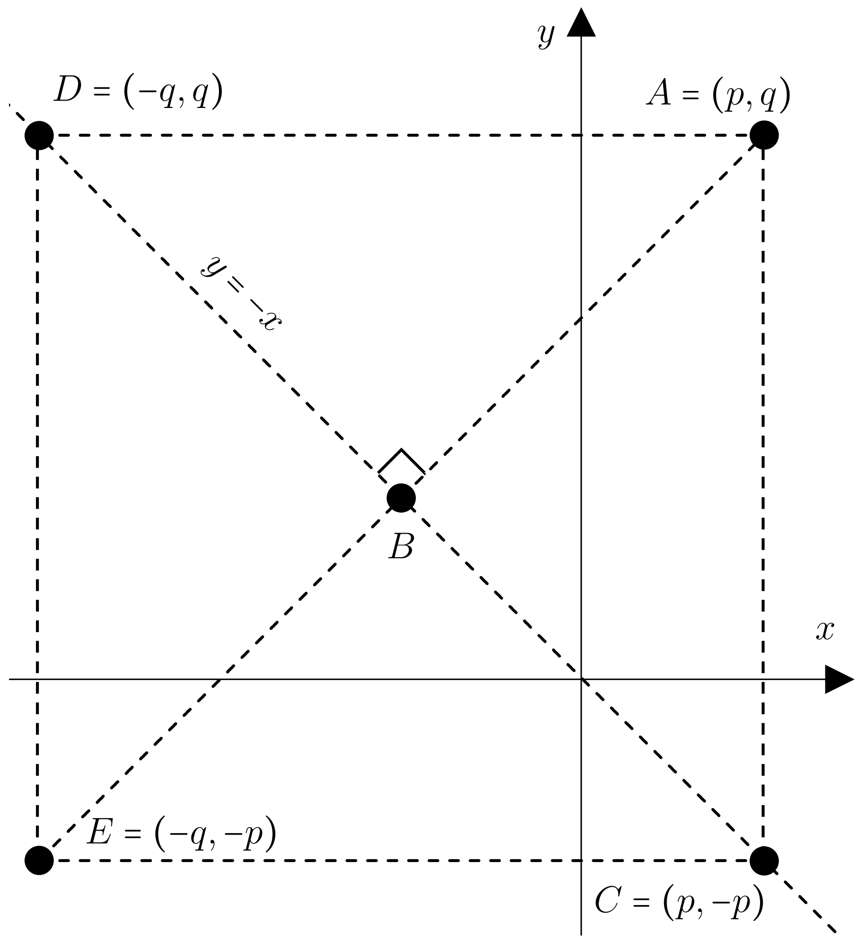
This is minimised at $a=2(q-p)/(2\cdot 2)=(q-p)/2$. So, the point on $y=-x$ closest to (p,q) is $D=((q-p)/2,(p-q)/2)$. Hence, the reflection of (p,q) in D and thus also in the line $y=-x$ is $(-q,-p)$. □

Second, an informal proof-by-picture:

Proof. Informal proof-by-picture:

Let $A=(p,q)$ and B be the point on $y=-x$ that's closest to A.

Mark also the points $C=(p,-p)$, $D=(-q,q)$, and $E=(-q,-p)$. Clearly, $ACED$ is a square (four sides have equal length and four angles are right).



The lines AE and $y=-x$ are the square's diagonals, which are perpendicular to each other. (This is a result from geometry we haven't formally stated and proven in this textbook.)

Moreover, $|AB|=|BE|$.

Altogether, by Definition 63, E is the reflection of A in B. Hence, by Definition 64, E is also the reflection of A in l. □

Fact 39. *The reflection of the point (p,q) in the vertical line $x=d$ is the point $(2d-p,q)$.*

Example 267. XXX

Again, we'll give two proofs of the above result. First, a formal proof using the usual "quadratic equations" idea:

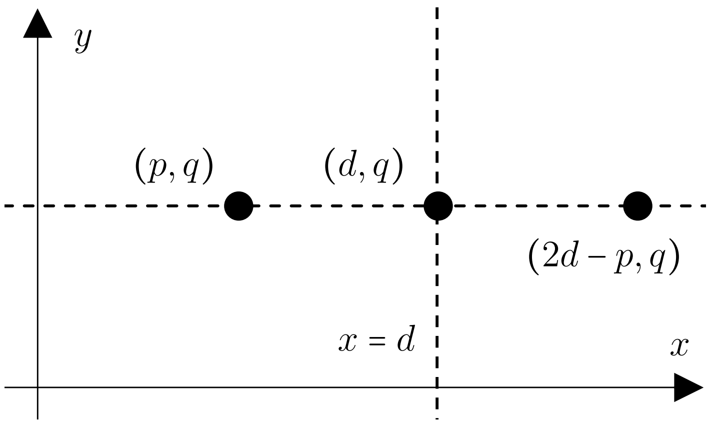
Proof. The distance between (p,q) and an arbitrary point (d,y) on $x=d$ is

$$\sqrt{(p-d)^2+(q-y)^2} = \sqrt{y^2-2qy+p^2+q^2+d^2-2pd}.$$

This is minimised at $y = -(-2q)/(2\cdot 1) = q$. So, the point on $x=d$ closest to (p,q) is $A=(d,q)$. Hence, the reflection of (p,q) in A and thus also in the line $x=d$ is $(2d-p,q)$. □

Second, an informal proof-by-picture:

Proof. Informal proof-by-picture:[140]



"Clearly", the point on $x=d$ that's closest to (p,q) is (d,q).

And by Fact 36, the reflection of (p,q) in (d,q) is $(2d-p,q)$. Hence, by Definition 64, the reflection of (p,q) in $x=d$ is $(2d-p,q)$. □

Fact 40. *The reflection of the point (p,q) in the horizontal line $y=d$ is the point $(p,2d-q)$.*

Example 268. XXX

Again, we'll give two proofs of the above result. First, a formal proof using the usual "quadratic equations" idea:

Proof. The distance between (p,q) and an arbitrary point (x,d) on $y=d$ is

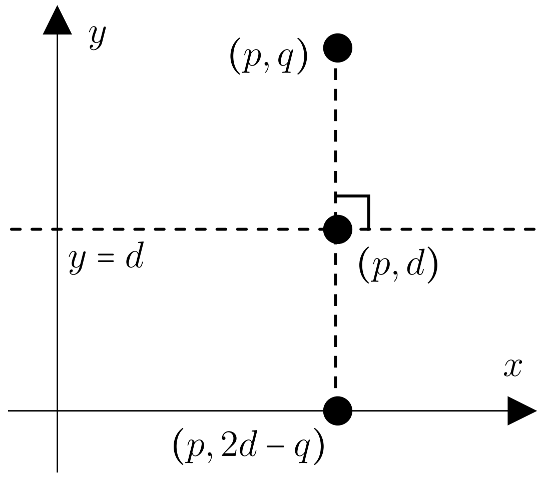
$$\sqrt{(p-x)^2+(q-d)^2} = \sqrt{x^2-2px+p^2+q^2+d^2-qd}.$$

This is minimised at $x = -(-2p)/(2\cdot 1) = p$. So, the point on $y=d$ closest to (p,q) is $A=(p,d)$. Hence, the reflection of (p,q) in A and thus also in the line $y=d$ is $(p,2d-q)$. □

Second, an informal proof-by-picture:

Proof. Informal proof-by-picture:

[140]For a formal proof, see p. **??** (Appendices).



"Clearly", the point on $y=d$ that's closest to (p,q) is (p,d).

And by Fact 36, the reflection of (p,q) in (p,d) is $(p,2d-q)$. Hence, by Definition 64, the reflection of (p,q) in $y=d$ is $(p,2d-q)$. □

Most generally,

Fact 41. *The reflection of the point* (p,q) *in the line* $ax+by+c=0$ *is the point*

$$\left(p-2a\frac{ap+bq+c}{a^2+b^2},q-2b\frac{ap+bq+c}{a^2+b^2}\right).$$

Proof. By Proposition 3, the point on the line $ax+by+c=0$ that is closest to (p,q) is

$$\left(p-a\frac{ap+bq+c}{a^2+b^2},q-b\frac{ap+bq+c}{a^2+b^2}\right).$$

So, the reflection of (p,q) in $ax+by+c=0$ is the reflection of (p,q) in the above point. And by Fact 36, this reflection point is

$$\left(2\left(p-a\frac{ap+bq+c}{a^2+b^2}\right)-p,2\left(q-b\frac{ap+bq+c}{a^2+b^2}\right)-q\right)=\left(p-2a\frac{ap+bq+c}{a^2+b^2},q-2b\frac{ap+bq+c}{a^2+b^2}\right).$$ □

As with some similar results in Ch. 15, you should definitely **not** bother *mugging* Fact 41. Instead, seek to understand the methods by which it was derived.

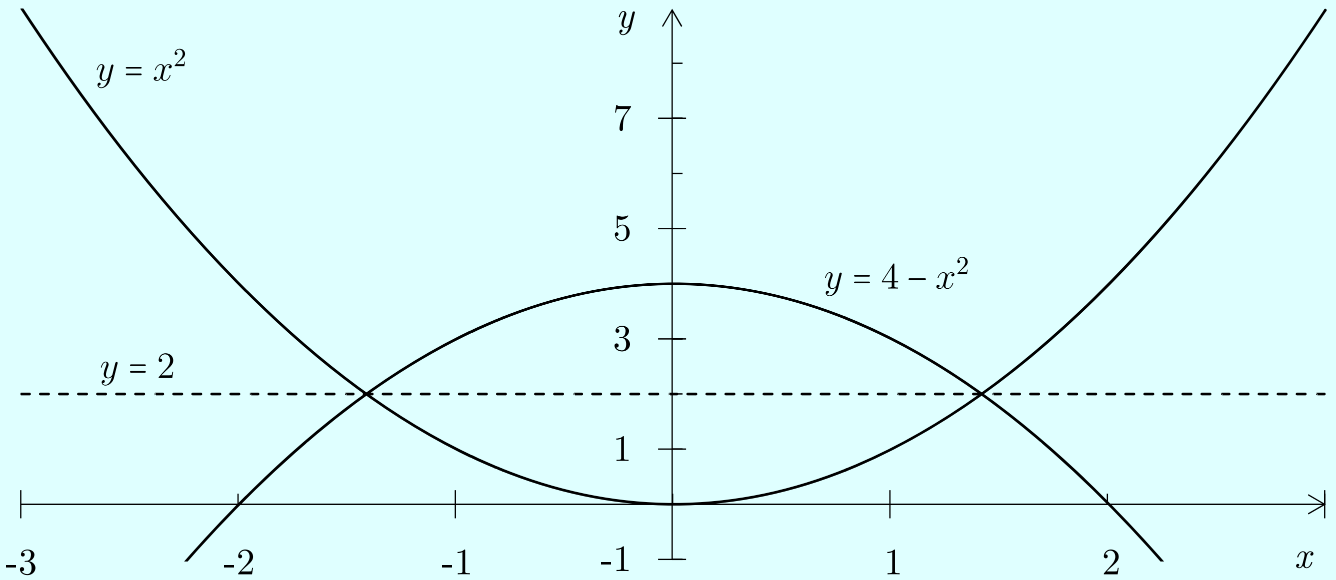
Exercise 94. Find the reflections of $(3,2)$ in $y=x$ and $y=-x$. (Answer on p. 1770.)

16.3. Lines of Symmetry

Definition 65. Let G and L be graphs. The *reflection of G in L* is the set of points obtained by reflecting every point in G in L.

Again, the above definition is general in that L can be *any* graph. But again, to keep things simple, we'll look only at cases where L is a line:

Example 269. By informal observation,[141] the reflection of $y = x^2$ in the line $y = 2$ is $y = 4 - x^2$.



Definition 66. Suppose the reflection of the graph G in the line l is G itself. Then we say that G is *symmetric in l*, or that that l is a *line* (or *axis*) *of symmetry* for G.

Example 270. By informal observation,[142] the graph of $y = x^2$ is symmetric in the line $x = 0$ (also the y-axis). Equivalently, $x = 0$ is a line (or axis) of symmetry for $y = x^2$.

Figure to be
inserted here.

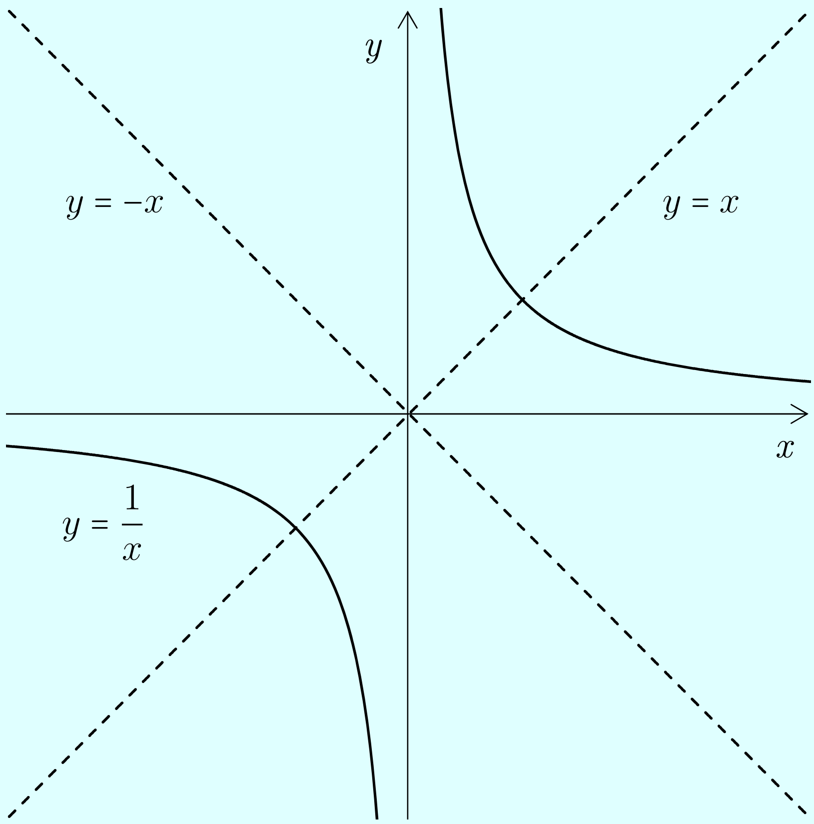
[141] Here's a formal proof: By Fact 40, the closest point on $y = 2$ to any point (a, a^2) in $y = x^2$ is $(a, 2)$. The reflection of (a, a^2) in $(a, 2)$ is the point $(a, 4 - a^2)$. So, the reflection of $y = x^2$ in the line $y = 2$ is $y = 4 - x^2$.

[142] Formal proof: By Fact 39, the closest point on $x = 0$ to any point (a, a^2) in $y = x^2$ is $(0, a^2)$. The reflection of (a, a^2) in $(0, a^2)$ is the point $(-a, a^2)$, which is also a point in the graph of $y = x^2$. We've just shown that the reflection of any point in the graph of $y = x^2$ in the line $x = 0$ is itself also in the graph of $y = x^2$. So, the graph of $y = x^2$ is its own reflection in the line $x = 0$.

Example 271. Let G be the graph of $y = 1/x$. We show that G is symmetric in the lines **(a)** $y = x$; and **(b)** $y = -x$:

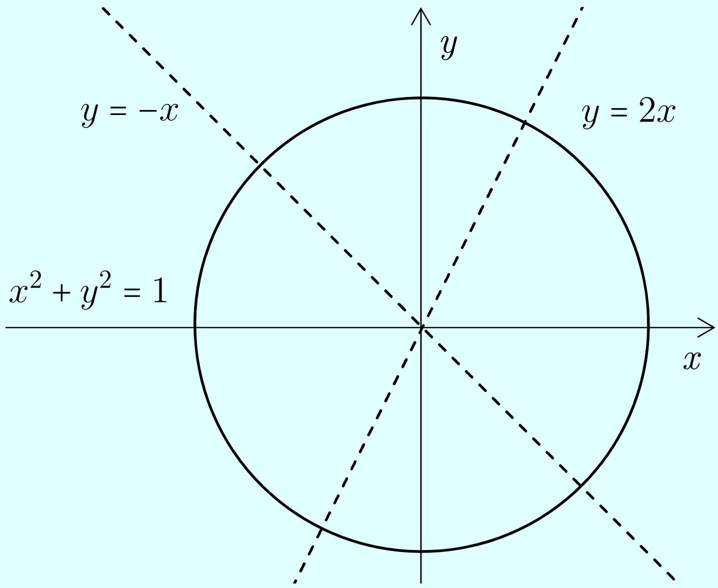
Pick any point $(a, 1/a)$ in G:

(a) By Fact 37, the reflection of any point $(a, 1/a)$ in G in the line $y = x$ is $(1/a, a)$, which is also in G. We've just shown that the reflection of any point in G in the line $y = x$ is also in G. So, the reflection of G in $y = x$ is G. Hence, G is symmetric in $y = x$.



(b) By Fact 38, the reflection of any point $(a, 1/a)$ in G in the line $y = -x$ is $(-1/a, -a)$, which is in G. We've just shown that the reflection of any point in G in the line $y = -x$ is itself also in G. So, the reflection of G in $y = -x$ is G. Hence, G is symmetric in $y = -x$.

Example 272. By informal observation,[143] the graph of $x^2 + y^2 = 1$ is symmetric in *every* line through the origin.



For example, it is symmetric in the lines $y = 2x$ and $y = -x$.

Exercise 95. Find any line(s) of symmetry for $y = x^2 + 2x + 2$. (Answer on p. 1770.)

[143]The formal proof is not conceptually difficult, but does involve tedious and messy algebra. So, I've relegated it to the Appendices—see Fact 252.

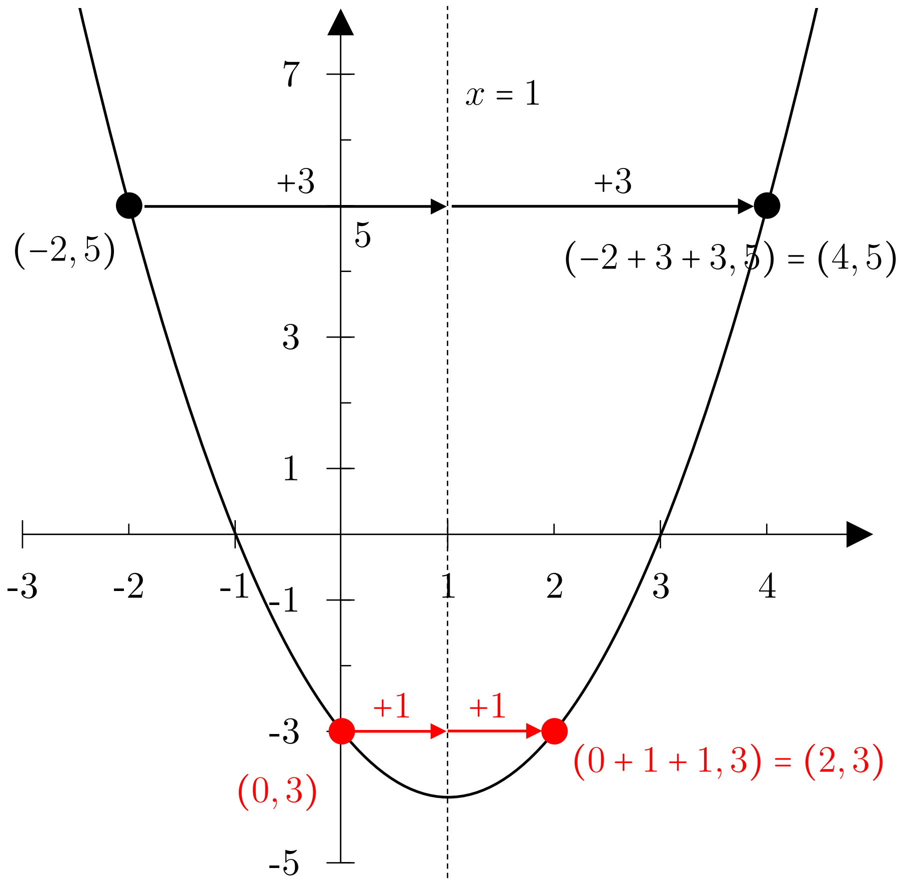
16.4. Symmetry in Vertical and Horizontal Lines

Fact 42. *Suppose G is a graph and $a \in \mathbb{R}$. Then*

$$\begin{array}{rcl} \textit{G is symmetric in} & & \textit{For every } (b,c) \in G, \\ \textit{the vertical line } x = a & \iff & (2a-b,c) \in G. \end{array}$$

Proof. Immediate from Fact 39: The reflection of the point (b,c) in the vertical line $x = a$ is the point $(2a-b,c)$. □

Illustration of above result:

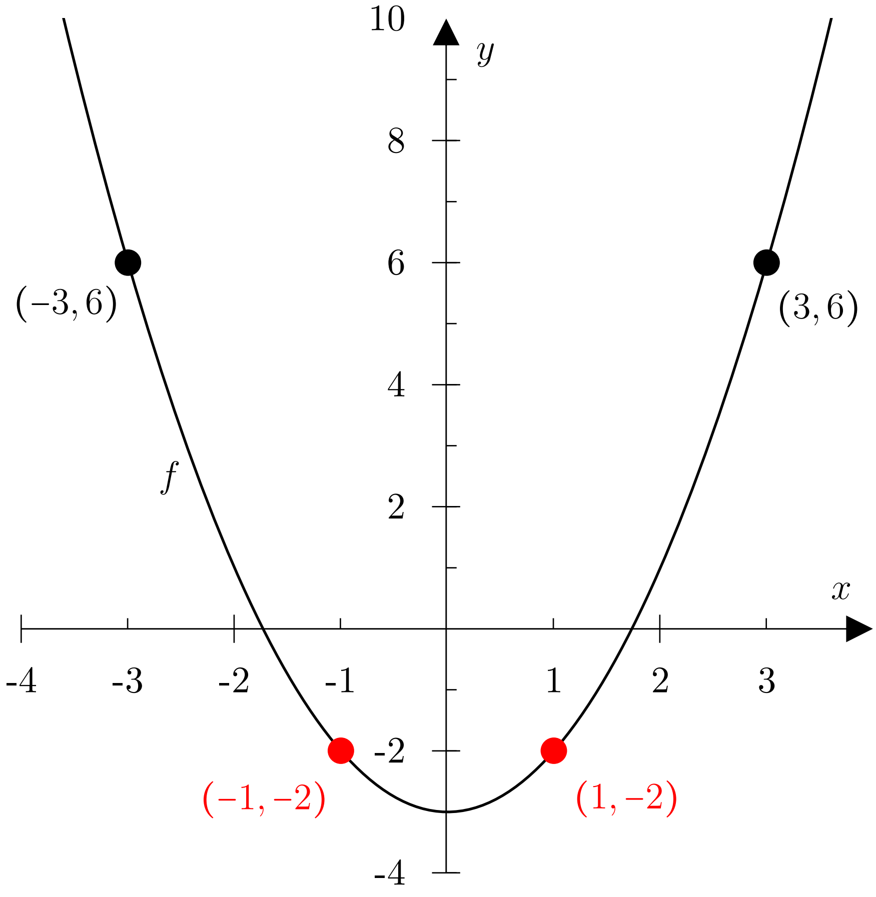


Plug $a = 0$ into the above result to get the next result:

Corollary 6. *Suppose G is a graph. Then*

$$\begin{array}{rcl} \textit{G is symmetric in the} & & \textit{For every } (b,c) \in G, \\ \textit{vertical line } x = 0 \ (\textit{y-axis}) & \iff & (-b,c) \in G. \end{array}$$

Illustration of above result:



Fact 43. *Suppose G is a graph and $a \in \mathbb{R}$. Then*

G is symmetric in the horizontal line $y = a$	$\iff$	*For every $(b,c) \in G$, $(b, 2a - c) \in G$.*

Proof. Immediate from Fact 40: The reflection of the point (b,c) in the horizontal line $y = a$ is the point $(b, 2a - c)$. $\square$

Illustration of above result:

Figure to be
inserted here.

17. Functions

Undoubtedly the most important concept in all of mathematics is that of a function—in almost every branch of modern mathematics functions turn out to be the central objects of investigation.

— Michael Spivak (1994).

In secondary school, you probably learnt to describe **functions** like this:

"Let $f(x) = 2x$ be a function."

Strictly speaking,[144] the above description of functions is incorrect. In particular, it suffers from these two big problems:

1. It fails to make any mention of the **domain** and the **codomain**.

   Whenever we specify a function, we *must* also specify the domain and codomain.[145]

2. It incorrectly suggests that a function must always be some sort of a "formula".

   But it needn't be. A function simply maps or assigns every element in the domain to (exactly) one element in the codomain. There need be nothing logical or formulaic about how this mapping or assignment is done. We will illustrate this important point with *many* examples below.

Here is one correct way to describe functions:[146]

Definition 67 (*informal*)**.** A *function* consists of these three objects:

1. A set called the *domain*;
2. A set called the *codomain*; and
3. A *mapping rule* that maps (or assigns) **every** element in the domain to **(exactly) one** element in the codomain.

Two very simple examples of functions:

[144] Pedagogical note: Many bits of H2 Maths are equivalent to first- and even second-year university courses in many countries. (Well, at least on paper. See my Preface/Rant.) My view is that while the above incorrect description of a function may have been suitable at earlier stages of a student's maths education, it is no longer so at this stage. Definition 67 (below) is not at all difficult (especially when compared with a lot of the junk that's already in H2 Maths). At the cost of very little additional pain and time, the student gains a far better understanding of what functions are and how they work, and thus saves herself more grief in the long run.

[145] Things become *so* much simpler if we make it clear from the outset and indeed insist that the very definition of a function includes the specification of a domain and codomain. Instead we have the present rigmarole where we ask students to explain which values "to exclude" from a function's domain, as if we were doing some ad hoc repair to make the function "work".

[146] Definition 67 will serve us very well. Note though that it is a little informal. For the formal definition, see Definition 280 (Appendices).

Example 273. Let f be the function with

1. **Domain:** The set $\{1, 5, 3\}$.
2. **Codomain:** The set $\{100, 200\}$.
3. **Mapping rule:** $f(1) = 100$, $f(5) = 100$, $f(3) = 200$.

The function f is well-defined, because *every* element in the domain maps to or "hits"[147] *(exactly) one* element in the codomain:

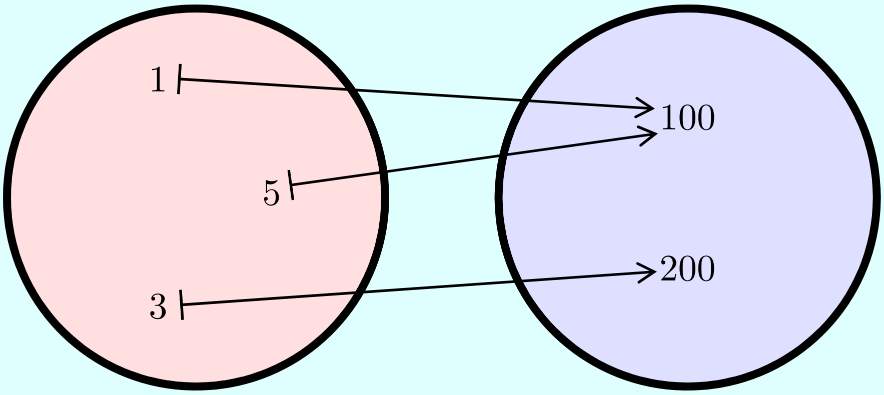
$$1 \text{ "hits" } 100, \qquad 5 \text{ "hits" } 100, \qquad 3 \text{ "hits" } 200.$$

There is no apparent logic or formula behind how the mapping rule works.

Why should 1 "hit" 100, 5 "hit" 100, and 3 "hit" 200?

No matter. To qualify as a well-defined function, all that matters is that *every* element in the domain is mapped to or "hits" *(exactly) one* element in the codomain. Which is true of f. So, f is well-defined.

The function f



Domain f Codomain f

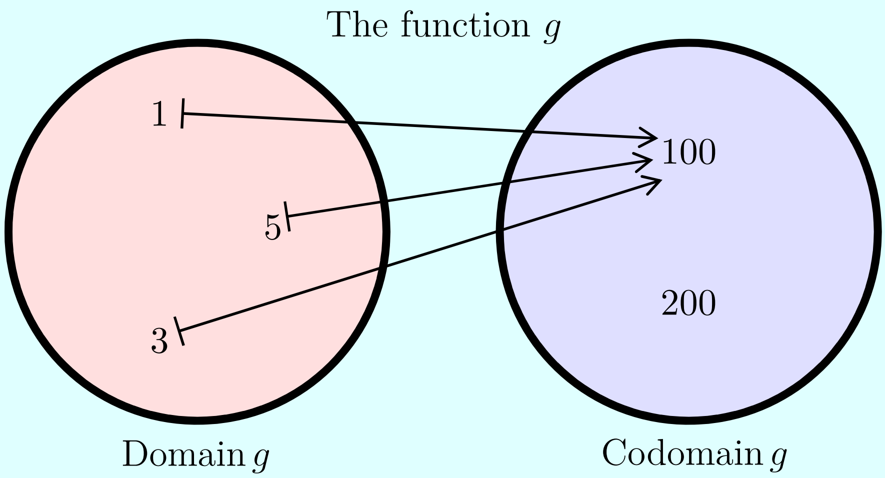
[147]The term "hits" is informal (hence the scare quotes). The formal term is *maps to*. But "hits" is probably clearer, more evocative, and easier to understand.

Example 274. Let g be the function with

1. **Domain:** The set $\{1, 5, 3\}$.
2. **Codomain:** The set $\{100, 200\}$.
3. **Mapping rule:** $g(1) = 100$, $g(5) = 100$, $g(3) = 100$.

Observe that the element 200 in the codomain is not "hit". Nonetheless, g is again a well-defined function, because it satisfies the requirement that *every* element in the domain maps to or "hits" *(exactly) one element* in the codomain:

1 "hits" 100,
5 "hits" 100,
3 "hits" 100.



For a function to be well-defined, there is **no** requirement that every element in the codomain be "hit".

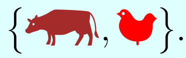
And so here, even though 200 in the codomain isn't "hit", g is a perfectly well-defined function.

The concept of a function is of great generality. To qualify as a function, all we require is that **every** element in the domain maps to or “hit” **(exactly) one** element in the codomain. In each of our above examples, both the domain and codomain were sets of real numbers. But in general, they could be *any* sets whatsoever.

We’ll use these six symbols in this chapter and again in Ch. 24:

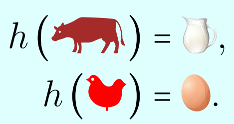
Cow	Chicken	Dog	Produces eggs	Guards the home	Produces milk

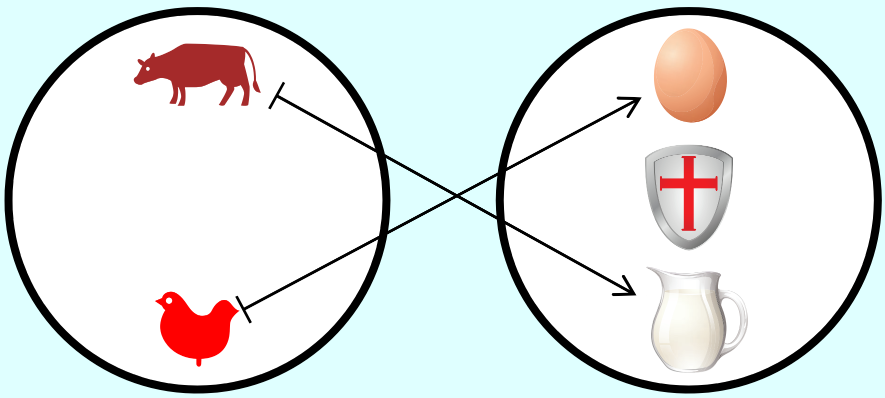
Example 275. Let h be the function with

1. **Domain:** The set .
2. **Codomain:** The set .
3. **Mapping rule:** “Match the animal to its role.”

Here the mapping rule is informally stated in words.

Nonetheless, it is clear enough. If we wanted to, we could write it out formally, like this:

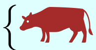
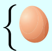
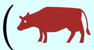
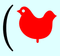


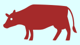


The function h is well-defined, because it satisfies the requirement that every element in the domain “hits” (exactly) one element in the codomain.

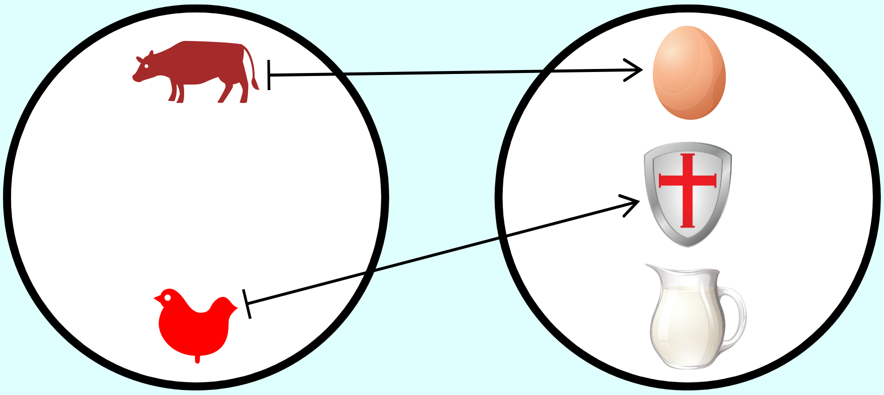
In the above example, the mapping rule “makes sense”—we simply map each animal to its role. In the next example, the mapping rule “makes no sense”. Nonetheless and all the same, our function is perfectly well-defined:

Example 276. Let i be the function with

1. **Domain:** The set $\{$, $\}$.
2. **Codomain:** The set $\{$, , $\}$.
3. **Mapping rule:** $i\big($$\big) =$ , $i\big($$\big) =$ .

This time, the mapping rule "makes no sense"—it maps  to  and  to . Nonetheless, this is a well-defined function, because it maps every element in the domain to (exactly) one element in the codomain.

The function i



Domain i Codomain i

Another pair of examples:

Example 277. Let j be the function with

1. **Domain:** The set of UN member states.
2. **Codomain:** The set of ISO three-letter country codes.
3. **Mapping rule:** "Match the state to its code."

The domain, codomain, and mapping rule are all informally stated in words. Nonetheless, they are clear enough.

The domain contains 193 elements, namely Afghanistan, Albania, Algeria, ..., Zambia, and Zimbabwe (list). The codomain contains 249 elements, namely ABW, AFG, AGO, AIA, ALA, ..., ZMB, and ZWE (list).[148]

And the mapping rule maps every element in the domain to (exactly) one element in the codomain. For example,

$$\begin{aligned} j\left(\text{The United States of America}\right) &= \text{USA}, \\ j\left(\text{China}\right) &= \text{CHN}, \\ j\left(\text{Singapore}\right) &= \text{SGP}. \end{aligned}$$

Altogether then, j is a well-defined function, because it satisfies the requirement that every element in the domain "hits" exactly one element in the codomain.

Again, a very similar example, but this time the mapping rule "makes no sense":

Example 278. Let k be the function with

1. **Domain:** The set of UN member states.
2. **Codomain:** The set of ISO three-letter country codes.
3. **Mapping rule:** For every x, $k\left(x\right) = \text{SGP}$.

The function k simply maps every UN member state to the three-letter code SGP. This is strange and seems to make no sense. Nonetheless, k is a well-defined function, because it satisfies the requirement that every element in the domain "hits" exactly one element in the codomain.

So! Specifying a function is jolly simple. Simply write down the

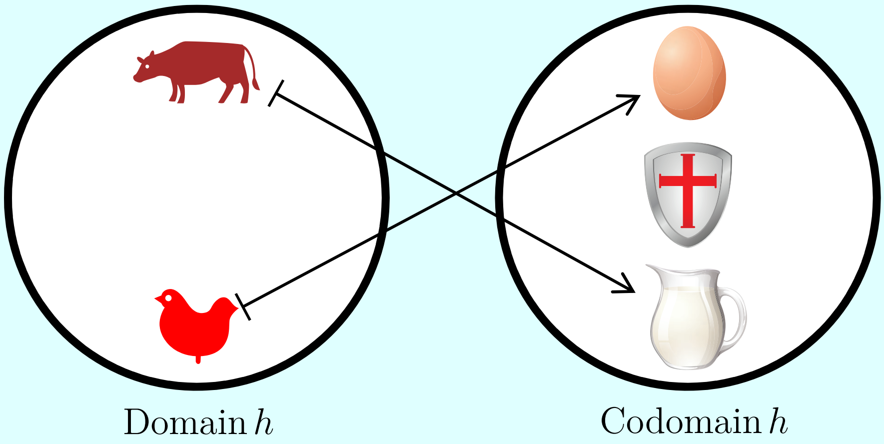
1. **Domain** (could be any set);
2. **Codomain** (could be any set);
3. **Mapping rule** (maps **every** element in the domain to **exactly one** element in the codomain).

[148] There are more ISO three-letter country codes than UN members because not every "country" is a UN member. For example, **Greenland** is assigned the ISO code GRL, but is a territory of Denmark and is not a UN member state. The **Holy See** (or **Vatican City**) is a fully sovereign state and is assigned the ISO code VAT, but is not a UN member state. **Taiwan** is, for all intents and purposes, a fully sovereign state and is assigned the ISO code TWN, but is not a UN member state.

Now, a function consists of the above three pieces. Hence—and here's a somewhat subtle point—two functions are identical (i.e. equal) if and only if they have the same domain, codomain, *and* mapping rule.

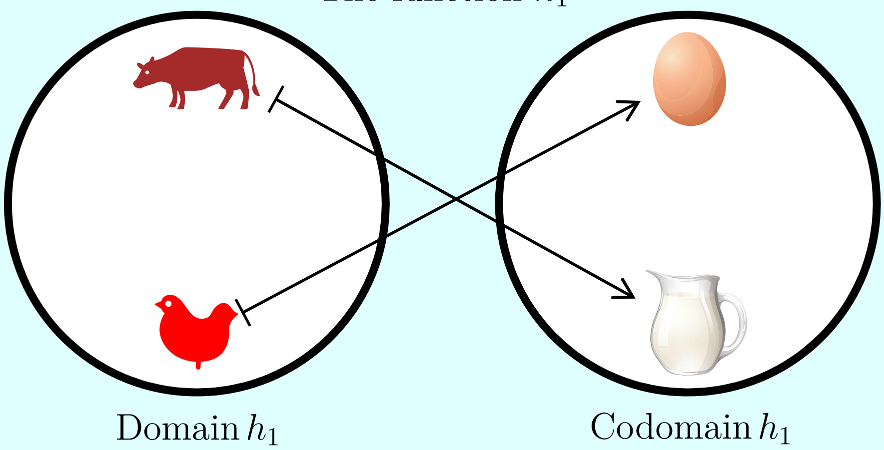
Example 279. Recall from above the function h:

The function h



Now consider the very similar-looking function h_1:

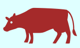
The function h_1



The functions h and h_1 look very similar. Indeed, they have the same domain and mapping rule.

However, their codomains are different. And so h and h_1 are distinct:[149]

$$h \neq h_1.$$

One might think, "*Aiyah*, the codomain not very important *wat*. Both h and h_1 map  to  and  to . So just call them the same function *lah*!" But this is wrong.

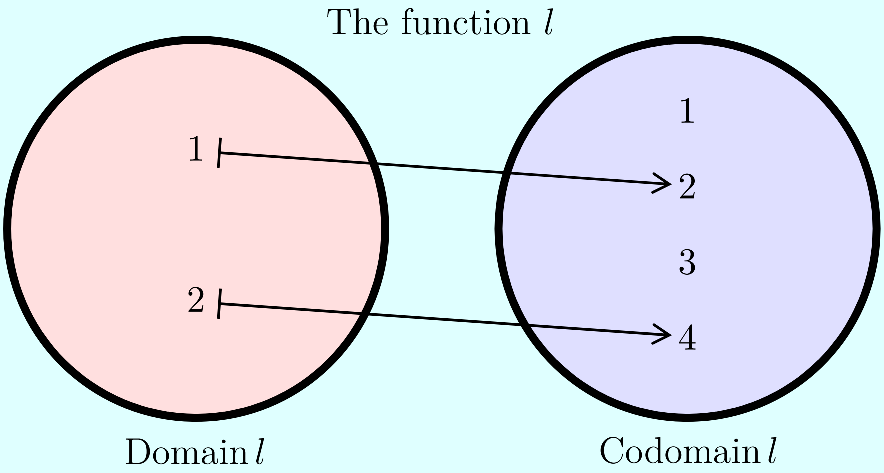
Again, to reiterate, stress, and emphasise, a function consists of three pieces: the domain, the codomain, and the mapping rule. So,

$$\text{Two functions are identical} \iff \text{Same domain, codomain, and mapping rule.}$$

[149] *Distinct* is a synonym for *not equal*.

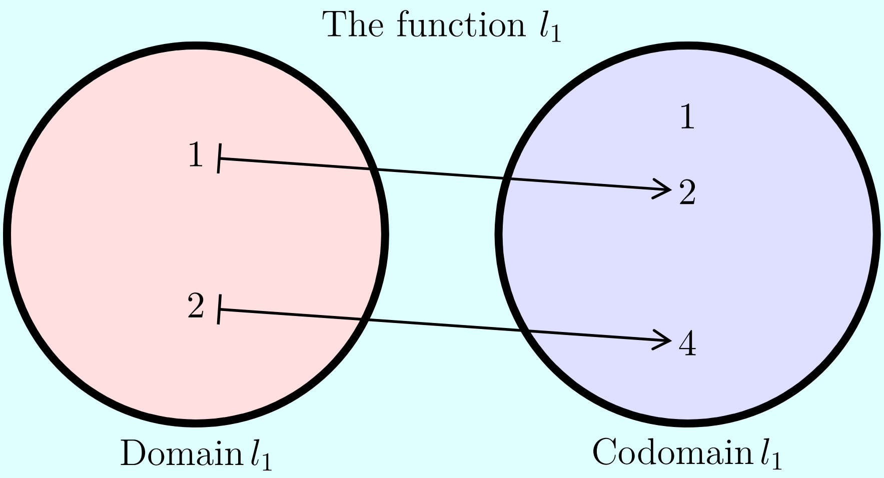
Example 280. Let l be the function with

1. **Domain:** The set $\{1, 2\}$.
2. **Codomain:** The set $\{1, 2, 3, 4\}$.
3. **Mapping rule:** For every x, $l(x) = 2x$.



Now consider the very similar-looking function l_1, which has

1. **Domain:** The set $\{1, 2\}$.
2. **Codomain:** The set $\{1, 2, 4\}$.
3. **Mapping rule:** For every x, $l_1(x) = 2x$.



Again, the functions l and l_1 look very similar—they have the same domain and mapping rule. However, their codomains are different. And so, l and l_1 are distinct:

$$l \neq l_1.$$

Exercise 96. *Fill in the blanks.* (Answer on p. 1772.)

A function consists of ______ objects: namely, the ______, the ______, and the ______.

Exercise 97. *Fill in the blanks with* "can be any set"*;* "must be $\mathbb{R}$"*; or* "must be a subset of $\mathbb{R}$"*.* (Answer on p. 1772.)

In general, the domain _________; and the codomain ________.

Exercise 98. *Fill in the blanks with "at least one element"; "every element"; or "exactly one element".* (Answer on p. 1772.)

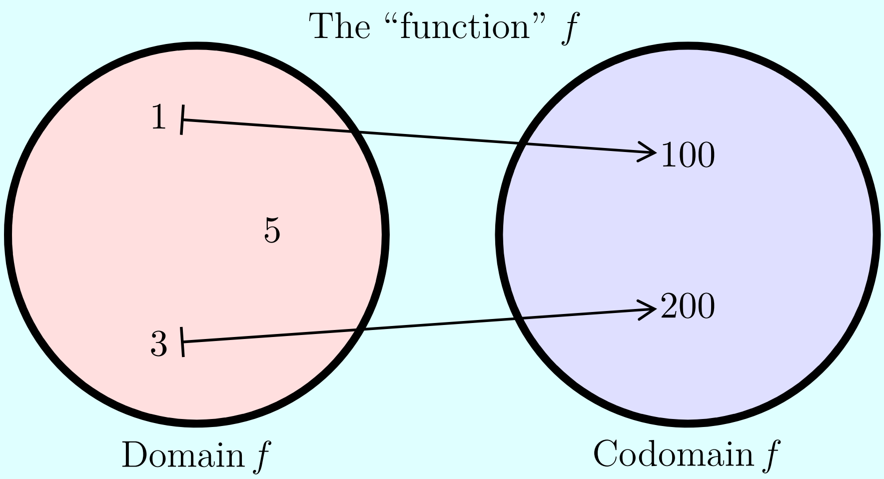
A function maps ________ in its domain to ________ in its codomain.

17.1. What Functions *Aren't*

To better understand what functions are, we'll now look at what functions *aren't*. That is, we'll now look at examples of **non-functions**:

Example 281. It is alleged that f is the function with

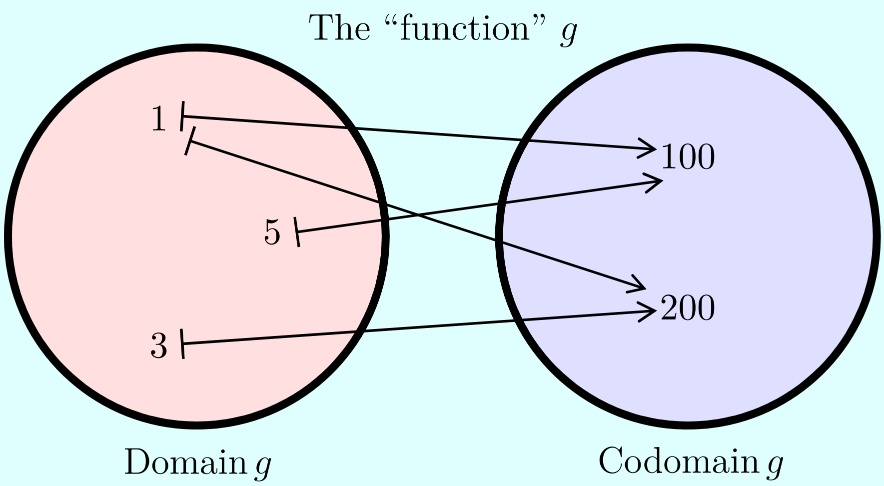
1. **Domain:** The set $\{1, 5, 3\}$.
2. **Codomain:** The set $\{100, 200\}$.
3. **Mapping rule:** $f(1) = 100$, $f(3) = 200$.



Unfortunately, f is **not** a function, because f fails to map **every** element in the domain to (exactly) one element in the codomain—in particular, f fails to map 5 to any element in the codomain.

Example 282. It is alleged that g is the function with

1. **Domain:** The set $\{1, 5, 3\}$.
2. **Codomain:** The set $\{100, 200\}$.
3. **Mapping rule:** $g(1) = 100$, $g(1) = 200$, $g(5) = 100$, $g(3) = 200$.



Unfortunately, g is **not** a function, because g fails to map every element in the domain to **(exactly) one** element in the codomain—in particular, g maps 1 to *two* elements, namely 100 and 200.

Example 283. It is alleged that h is the function with

1. **Domain:** The set of ISO three-letter country codes.
2. **Codomain:** The set of UN member states.
3. **Mapping rule:** "Match the code to the corresponding state."

Unfortunately, h is **not** a function, because h fails to map **every** element in the domain to (exactly) one element in the codomain.

For example, h fails to map VAT in the domain to any element in the codomain. This is because VAT corresponds to the Holy See (or the Vatican City), which is **not** a UN member state and thus not an element of the codomain.

Example 284. It is alleged that i is the function with

1. **Domain:** The set of real numbers $\mathbb{R}$.
2. **Codomain:** The set of real numbers $\mathbb{R}$.
3. **Mapping rule:** For all x, $i(x) = \sqrt{x}$.

Unfortunately, i is **not** a function, because i fails to map **every** element in the domain to (exactly) one element in the codomain.

For example, i fails to map -1 in the domain to any element in the codomain. This is because $\sqrt{-1}$ is not a real number.

Exercise 99. Which of these four alleged functions are well-defined? (Answer on p. 1772.)

(a) The function a has domain {🐄, 🐔, 🐕}, codomain {🥚, 🛡️, 🥛}, and mapping rule "match the animal to its role".

(b) The function b has domain {🐄, 🐔, 🐕}, codomain {🥚, 🥛}, and mapping rule "match the animal to its role".

(c) The function c has domain the set of UN member states, codomain the set of cities, and mapping rule "match the state to its most splendid city".

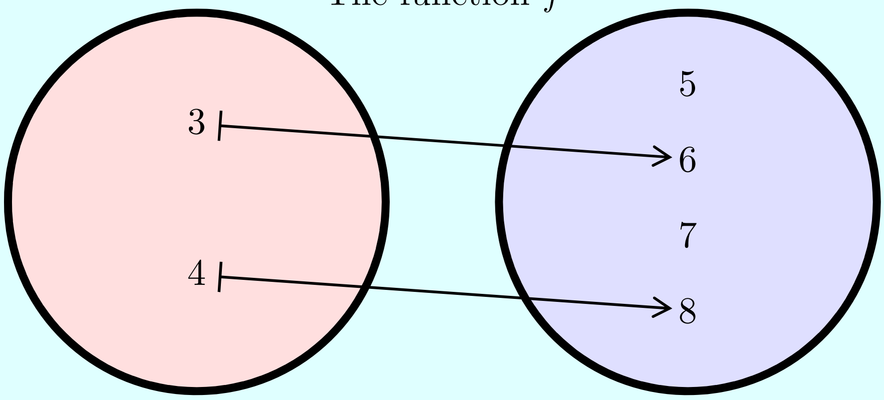
(d) The function d has domain the set of UN member states, codomain the set of cities, and mapping rule "match the state to a city with over 10M people".

17.2. Notation for Functions

As shorthand, we'll denote a function f's domain and codomain by Domain f and Codomain f.

Example 285. Let f be the function with

1. **Domain:** The set $\{3, 4\}$.
2. **Codomain:** The set $\{5, 6, 7, 8\}$.
3. **Mapping rule:** "Double it."



We may write

$$\text{Domain } f = \{3, 4\},$$
$$\text{Codomain } f = \{5, 6, 7, 8\}.$$

Remark 41. The notation Domain f and Codomain f are not standard.

For example, in place of Domain f, others may instead write Domain (f), Dom f, Dom (f), D, D_f, $D(f)$, $\mathcal{D}$, $\mathcal{D}_f$, or $\mathcal{D}(f)$—among countless other variations.

Your A-Level maths syllabus and exams do not seem to have any shorthand notation at all for "the domain of the function f" and "the codomain of the function f". Nevertheless, this textbook will use Domain f and Codomain f because they are convenient.

We now discuss how the **mapping rule** is written.

To write down a **mapping rule**, there is usually **more than one way**. In the above example, the mapping rule was written informally as "Double it". But we could also have written it more formally as

$$\text{"For every } x \in \text{Domain } f \text{, we have } f(x) = 2x\text{."}$$

Alternatively, we could've written the mapping rule *explicitly*, stating what each and every element in the domain is to be mapped to:

$$\text{"}f(3) = 6, \qquad f(4) = 8\text{."}$$

The mathematical punctuation mark $\mapsto$ means **maps to**. And so, here's yet another way to write the mapping rule:

"$f: \color{red}{3} \mapsto \color{blue}{6}$, $\quad f: \color{red}{4} \mapsto \color{blue}{8}$."

So, altogether, in the above example alone, we've given four different but entirely equivalent ways to write out the mapping rule.

You can choose to write the mapping rule however you like. What's important is that you make clear *how* the mapping rule maps **each** element in the domain to **(exactly) one** element in the codomain. If you haven't made it sufficiently clear, then you have failed to communicate to others what your function is and your function is not well-defined.

Here are eight ways to say aloud "$f(4) = 8$" or "$f: 4 \mapsto 8$":

"f of 4 is 8."	"f of 4 equals 8."	"f of 4 is equal to 8."
"f maps 4 to 8."	"The value of f at 4 is 8."	"f evaluated at 4 is 8."
"The value of f when applied to 4 is 8."		"The image of 4 under f is 8."

If $x \notin \text{Domain}\, f$, then $f(x)$ is simply **undefined**. So in the last example, $0 \notin \text{Domain}\, f$; and thus, $f(0)$ is simply undefined.

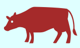
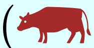
Example 286. Let g be the function with

1. **Domain:** The set of positive integers $\mathbb{Z}^+$.
2. **Codomain:** The set of real numbers $\mathbb{R}$.
3. **Mapping rule:** "Double it."

Then we have, for example,

$$g(1) = 2, \qquad g(2) = 4, \quad \text{and} \quad g(3) = 6.$$

Or equivalently, $\quad g: 1 \mapsto 2,\ g: 2 \mapsto 4$, and $g: 3 \mapsto 6$.

Since $-1, 1.5, \pi, 0 \notin \text{Domain}\, g$, $g(-1)$, $g(1.5)$, $g(\pi)$, and $g(0)$ are undefined.
Likewise, since ,  $\notin \text{Domain}\, g$, $g($$)$ and $g($$)$ are undefined.

The mapping rule, "Double it," was somewhat informal. We could've written it more formally as

$$\text{For every } x \in \mathbb{Z}^+, \text{ we have } g(x) = 2x.$$

With the aid of an ellipsis, we could also have written it down explicitly:[150]

$$g(1) = 2,\ g(2) = 4,\ g(3) = 6,\ g(4) = 8,\ \ldots$$

Or equivalently, $\quad g: 1 \mapsto 2,\ g: 2 \mapsto 4,\ g: 3 \mapsto 6,\ g: 4 \mapsto 8,\ \ldots$

So far, we've written down functions with the aid of tables and/or figures. But going forward, we'll want to **write down functions more concisely** and without the aid of

[150]But in general, this may not be possible.

tables or figures.

Here are nine entirely equivalent and formal ways to fully write down the function g from the last example:

1. Let g be the function that maps every element x in the domain $\mathbb{Z}^+$ to the element $2x$ in the codomain $\mathbb{R}$.
2. Let g be the function that has domain $\mathbb{Z}^+$, codomain $\mathbb{R}$, and maps every $x \in \mathbb{Z}$ to $2x \in \mathbb{R}$.
3. Let $g : \mathbb{Z}^+ \to \mathbb{R}$ be the function defined by $g(x) = 2x$.
4. Let $g : \mathbb{Z}^+ \to \mathbb{R}$ be the function defined by $g : x \mapsto 2x$.
5. Let $g : \mathbb{Z}^+ \to \mathbb{R}$ be defined by $g(x) = 2x$.
6. Let $g : \mathbb{Z}^+ \to \mathbb{R}$ be defined by $g : x \mapsto 2x$.
7. Define $g : \mathbb{Z}^+ \to \mathbb{R}$ by $g(x) = 2x$.
8. Define $g : \mathbb{Z}^+ \to \mathbb{R}$ by $g : x \mapsto 2x$.
9. Define $g : \mathbb{Z}^+ \to \mathbb{R}$ by $x \mapsto 2x$.

In Statements 3–9, the **domain** comes after the colon ":", the **codomain** after the right arrow "→", and the **mapping rule** at the end of the statement.

The general version of Statement 9 is

$$\text{Define } f : \overset{\substack{\text{Domain } f \\ \nwarrow}}{A} \to \overset{\substack{\text{Codomain } f \\ \nearrow}}{B} \text{ by } \underbrace{x \mapsto f(x)}_{\text{Mapping rule}}.$$

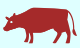
In words, the function f is defined to have domain A, codomain B, and mapping rule $x \mapsto f(x)$.

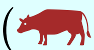
Example 287. Define $h : \mathbb{R}^+ \to \mathbb{R}$ by $h(x) = 2x$.

The domain of h is $\mathbb{R}^+$ (the set of positive real numbers). The codomain is $\mathbb{R}$ (the set of real numbers). Expressed informally in words, the mapping rule is "Double it".

So for example, $\quad h(1) = 2,\ h(2.3) = 4.6, \text{ and } h(3) = 6.$

Or equivalently, $\quad h : 1 \mapsto 2,\ h : 2.3 \mapsto 4.6, \text{ and } h : 3 \mapsto 6.$

Observe that $-1, 0,$ ,  $\notin \mathbb{R}^+ = \text{Domain}\, h$. Thus, the following are all undefined:

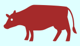
$h(-1)$, $h(0)$, $h($$)$, and $h($$)$.

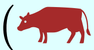
Example 288. Define $i : \mathbb{Z} \to \mathbb{R}$ by $i(x) = x^2$.

The domain of i is $\mathbb{Z}$ (the set of integers). The codomain is $\mathbb{R}$ (the set of real numbers). Expressed informally in words, the mapping rule is "Square it".

So for example, $\quad i(1) = 1$, $i(7) = 49$, and $i(-6) = 36$.

Or equivalently, $\quad i : 1 \mapsto 1$, $i : 7 \mapsto 49$, and $i : -6 \mapsto 36$.

Observe that $1.5, \pi,$ ,  $\notin \mathbb{Z} = \text{Domain}\, i$. Thus, the following are all undefined:

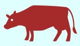
$h(1.5)$, $i(\pi)$, $i($$)$, and $i($$)$.

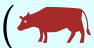
Example 289. Define $j : \mathbb{R}^+ \to \mathbb{R}$ by $j(x) = x^2$.

The domain of j is $\mathbb{R}^+$ (the set of positive real numbers). The codomain is $\mathbb{R}$ (the set of real numbers). Expressed informally in words, the mapping rule is "Square it".

So for example, $\quad j(1) = 1$, $j(2.2) = 4.84$, and $j(6) = 36$.

Or equivalently, $\quad j : 1 \mapsto 1$, $j : 2.2 \mapsto 4.84$, and $j : 6 \mapsto 36$.

Observe that $-1, -3.2,$ ,  $\notin \mathbb{Z} = \text{Domain}\, j$. Thus, the following are all undefined:

$j(-1)$, $j(-3.2)$, $j($$)$, and $j($$)$.

Note that as usual, the symbol x here is merely a **dummy variable** that can be replaced by any other symbol, such as y, z, ☺, or $\bigstar$. So, we could also have written this example's first sentence as either

"Define $j : \mathbb{R}^+ \to \mathbb{R}$ by $j(y) = y^2$" $\qquad$ or $\qquad$ "Define $j : \mathbb{R}^+ \to \mathbb{R}$ by $j(☺) = ☺^2$".

(Consider the last four examples. Is $g = h$? Is $i = j$?)[151]

In the above examples, we actually cheated a little. Or rather, we took it for granted that the specified mapping rule applied to *all* elements in the domain. If we wanted to be extra careful/pedantic, then here's what we should "really" have written:

Example 290. Define $g : \mathbb{Z} \to \mathbb{R}$ by $g(x) = 2x$ *for all* $x \in \mathbb{Z}$.

Example 291. Define $h : \mathbb{R}^+ \to \mathbb{R}$ by $h(x) = 2x$ *for all* $x \in \mathbb{R}^+$.

Example 292. Define $i : \mathbb{Z} \to \mathbb{R}$ by $i(x) = x^2$ *for all* $x \in \mathbb{Z}$.

Example 293. Define $j : \mathbb{R}^+ \to \mathbb{R}$ by $j(x) = x^2$ *for all* $x \in \mathbb{R}^+$.

This is because we can have a **piecewise function**, where there are different mapping

[151] The functions g and h have different domains and so $g \neq h$. Likewise, the functions i and j have different domains and so $i \neq j$.

rules for different "sections", "pieces", or intervals of the domain:

Example 294. Define $k : \mathbb{Z} \to \mathbb{R}$ by

$$k(x) = \begin{cases} 2x, & \text{for } x \leq 5, \\ x + 1, & \text{for } x > 5. \end{cases}$$

In words, the function k doubles integers that are less than or equal to 5, but adds one to those that are greater than 5.

So for example, $k(-10) = -20$, $k(0) = 0$, and $k(4) = 8$.

But, $k(7) = 8$, $k(15) = 16$, and $k(100) = 101$.

Example 295. Define $l : \mathbb{R}^+ \to \mathbb{R}$ by

$$l(x) = \begin{cases} x^2, & \text{for } x \leq 5, \\ x^3, & \text{for } x > 5. \end{cases}$$

In words, the function l squares positive real numbers that are less than or equal to 5, but cubes those that are greater than 5.

So for example, $l(1) = 1$, $l(1.1) = 1.21$, and $l(4) = 16$.

But, $l(5.3) = 148.877$, $l(7) = 343$, and $l(9) = 729$.

Exercise 100. Evaluate each function at 1. (Answer on p. 1772.)

$$\begin{aligned}\text{Define}\quad a:\mathbb{R}\to\mathbb{R} \quad &\text{by}\quad a(x)=x+1;\\ b:[-1,1]\to\mathbb{R} \quad &\text{by}\quad b(x)=17x;\\ c:\mathbb{Z}^+\to\mathbb{R} \quad &\text{by}\quad c(x)=3^x;\\ d:\mathbb{Z}^-\to\mathbb{R} \quad &\text{by}\quad d(x)=3^x;\\ e:\mathbb{R}\to\mathbb{R} \quad &\text{by}\quad e(x)=17.\end{aligned}$$

Exercise 101. Four functions are given below. (Answer on p. 1772.)

(a) Verify that each is well-defined.

(b) Which (if any) of them are equivalent?

Function	Domain	Codomain	Mapping rule
a	$\{1,2\}$	$\{1,2,3,4\}$	"Double it"
b	$\{1,2,3\}$	$\{1,2,3,4,5,6\}$	"Double it"
c	$\{x\in\mathbb{Z}^+ : x<4\}$	$\{1,2,3,4,5,6\}$	"Double it"
d	$[0,4)\cap\mathbb{Z}$	$(-3,6]\cap\mathbb{Z}_0^+$	"Double it"

Exercise 102. Define $f:\mathbb{R}^+\to\mathbb{R}$ by "round off to the nearest integer (and round **up** half-integers)". (Answer on p. 1772.)

(a) What are $f(3)$, $f(\pi)$, $f(3.5)$, $f(3.88)$, and $f(0)$?

Would the function still be well-defined if we changed , respectively,

(b) Its domain to $\mathbb{R}$ and codomain $\mathbb{Z}$?

(c) Its domain to $\mathbb{Z}$ and codomain $\mathbb{R}$?

Exercise 103. Below, the functions f, g, and h are described in words. Write down a formal and concise statement that defines each. (Answer on p. 1772.)

(a) The function f has domain the set of positive integers, codomain the set of integers, and mapping rule, "Triple it".

(b) The function g has domain the set of negative real numbers, codomain the set of non-zero real numbers, and mapping rule, "Square it".

(c) The function h has domain the set of even integers, codomain the set of positive even integers, and mapping rule, "Square it".

17.3. Warning: f and $f(x)$ Refer to Different Things

If you cannot say what you mean, Your Majesty, you will never mean what you say and a gentleman should always mean what he says.

— Peter O'Toole as Reginald Johnson, in *The Last Emperor* (1987).

A common mistake is to believe that $f(x)$ denotes a function. But **this is wrong.**

f and $f(x)$ refer to two different things.

f denotes a function. $f(x)$ denotes the value of f at x.

In the next two examples, S is the set of human beings.

Example 296. Let $h : S \to \mathbb{R}$ be the height function. That is, h gives each human being's height (rounded to the nearest centimetre).

So for example, h (Joseph Schooling) = 184.

h (Joseph Schooling) is **not** a function.

Instead, h (Joseph Schooling) is the **number** 184.

It would be silly to say that h (Joseph Schooling) is a function, because it isn't—h (Joseph Schooling) *is* the number 184.

And in general, for any human being x, $h(x)$ is her height (rounded to the nearest cm). Again, $h(x)$ is **not** a function—it's x's height (rounded to the nearest cm), which is a **number**.

Example 297. Let $w : S \to \mathbb{R}$ be the weight function. That is, w gives each human being's weight (rounded to the nearest kilogram).

So for example, w (Joseph Schooling) = 74.

w (Joseph Schooling) is **not** a function.

Instead, w (Joseph Schooling) is the **number** 74.

It would be silly to say that w (Joseph Schooling) is a function, because it isn't—w (Joseph Schooling) *is* the number 74.

And in general, for any human being x, $w(x)$ is her weight (rounded to the nearest kg). Again, $w(x)$ is **not** a function—it's x's weight (rounded to the nearest kg), which is a **number**.

This may seem like an excessively pedantic distinction. But maths *is* precise and pedantic. In maths, we are gentlemen (and ladies) who say what we mean and mean what we say. There is no room for ambiguity or alternative interpretations.

17.4. Nice Functions

Example 298. Consider the function $f : [1,5] \to \mathbb{R}$ defined by $f(x) = x^2$.

The domain of f contains only real numbers. Hence, f is a **function of a real variable**.

The codomain of f contains only real numbers. Hence, f is a **real-valued function**.

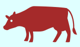
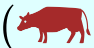
Both the domain and codomain of f contain only real numbers. Hence, f is a **nice function**.

Definition 68. We call a function whose

1. Domain contains only real numbers a *function of a real variable*;
2. Codomain contains only real numbers a *real-valued function* (or more simply, *real function*);
3. Domain and codomain both contain only real numbers a *nice function*.

Remark 42. The terms **function of a real variable** and **real-valued function** (and **real function**) are standard and widely used.

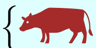
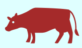
However, the term **nice function** is completely non-standard and used only in this textbook. We use it nonetheless, just so we don't have to keep saying "real-valued function of a real variable".

Example 299. Define the function $g : \{$, $\} \to \mathbb{Z}$ by $g($$) = 0$ and $g($$) = 1$.

Domain g contains objects that aren't real numbers. So, g is **not** a function of a real variable.

Codomain g contains only real numbers. So, g **is** a real-valued function.

Either Domain g or Codomain g contains objects that aren't real numbers. So, g is **not** a nice function.

Example 300. Define the function $h : (-2,5) \to \{$, $\}$ by $h(x) =$ .

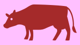
Domain h contains only real numbers. So, h is a function of a real variable.

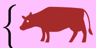
Codomain h contains objects that aren't real numbers. So, h is **not** a real-valued function.

Either Domain h or Codomain h contains objects that aren't real numbers. So, h is **not** a nice function.

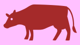
Happily, most functions we'll encounter in H2 Maths are nice functions. That is, we'll often encounter functions like f, but not functions like g or h.

Exercise 104. Explain whether each function is a **(i)** function of a real variable; **(ii)** real-valued function; and **(iii)** nice function. (Answer on p. 1772.)

(a) $i : \{$$,$$, 2\} \to [1,3]$ defined by $i(x) = 3$.

(b) $j : [1,3] \to \{$$,$$, 2\}$ defined by $j(x) = 2$.

(c) $k : \{2\} \to [1,3]$ defined by $k(x) = 1$.

(d) $l : \{2\} \to [1,3] \cup \{$$\}$ defined by $l(x) = 1$.

Exercise 105. What's wrong with the following statement? (Answer on p. 1773.)

"Let $f(x)$ be a function."

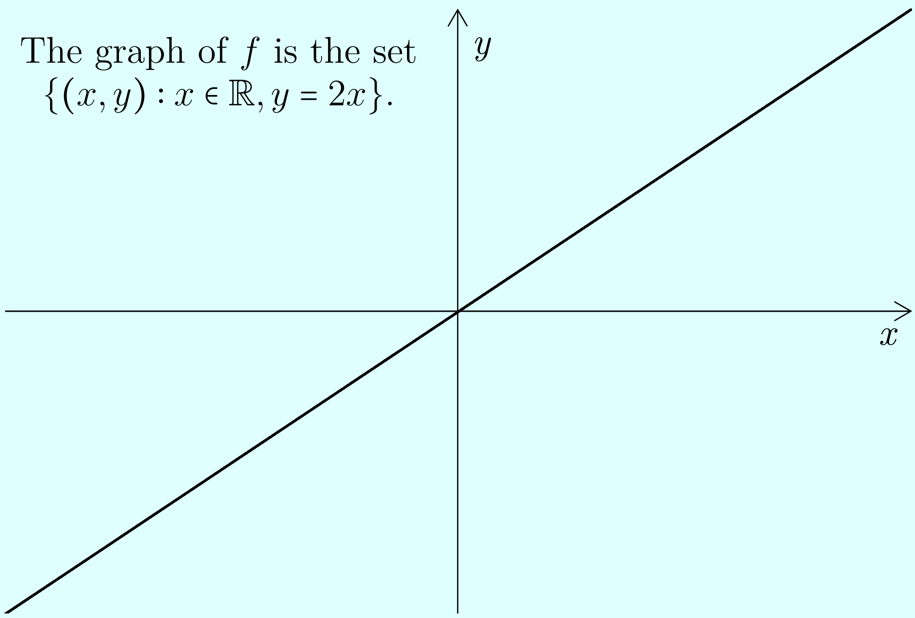
17.5. Graphs of Functions

Given a function f, its **graph** is simply the graph of the equation $y = f(x)$ with the constraint $x \in \text{Domain}\, f$. A little more formally,

Definition 69. Given a function f, its *graph* is this set of points:

$$\{(x, y) : x \in \text{Domain}\, f, y = f(x)\}.$$

Example 301. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$.



By the way, strictly speaking, we should say

"The point $(3, 6)$ is in the graph of f."

That is, we should explicitly state **both** the x- and y-coordinates of any point we are talking about.

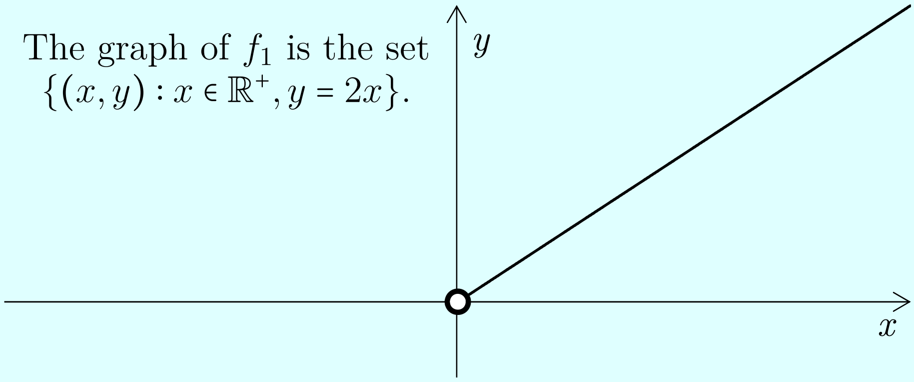
However, since the x-coordinate is sufficient for identifying a point on the graph of a function (why?),[152] we will sometimes be lazy and say things like

"The point $x = 3$ is on the graph of f."

Or even, "The point 3 is on the graph of f."

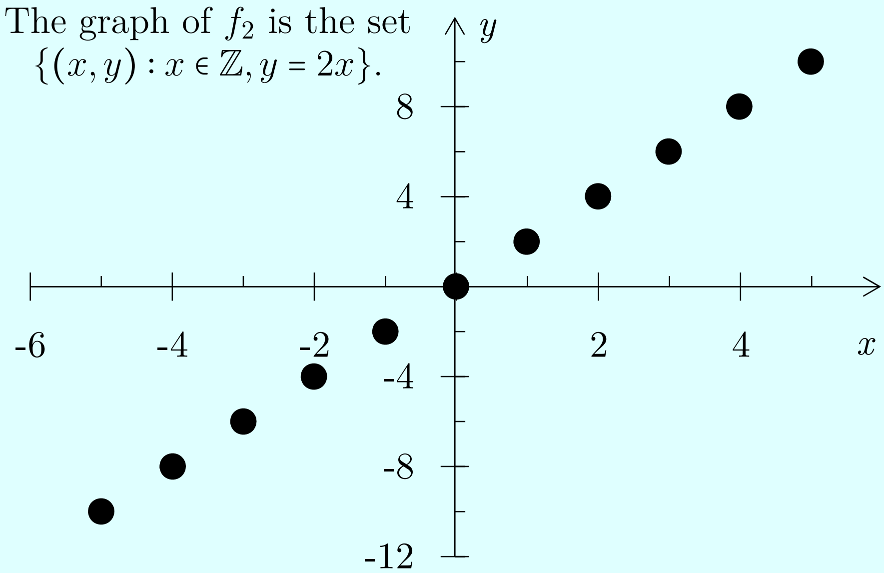
[152] A function maps each x-coordinate to exactly one y-coordinate. So, given only an x-coordinate, we immediately also know what the corresponding y-coordinate is.

Example 302. Define $f_1 : \mathbb{R}^+ \to \mathbb{R}$ by $f_1(x) = 2x$.



Note that the graph of f_1 does **not** contain the point $(0, 0)$, because $0 \notin \mathbb{R}^+$.

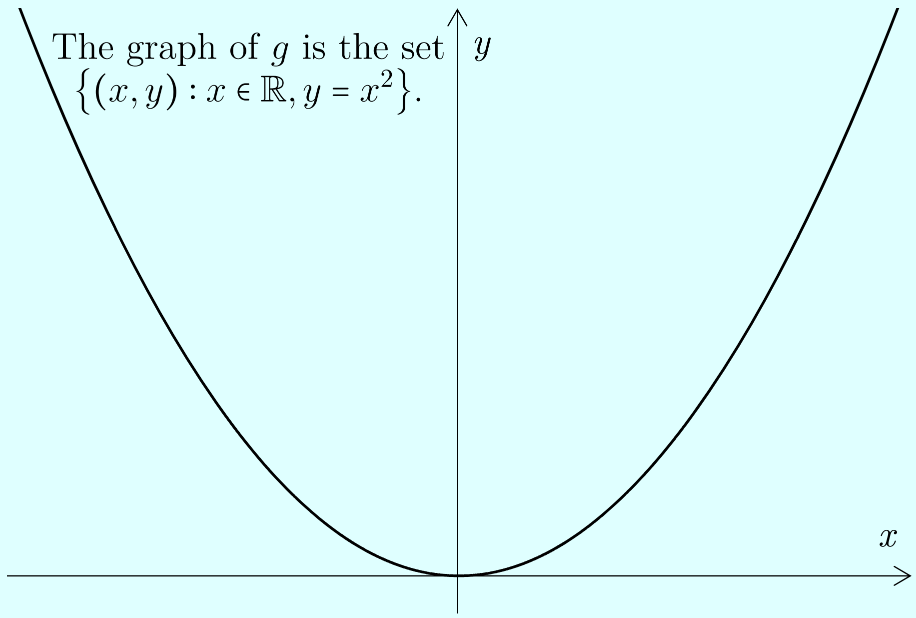
Example 303. Define $f_2 : \mathbb{Z} \to \mathbb{R}$ by $f_2(x) = 2x$.



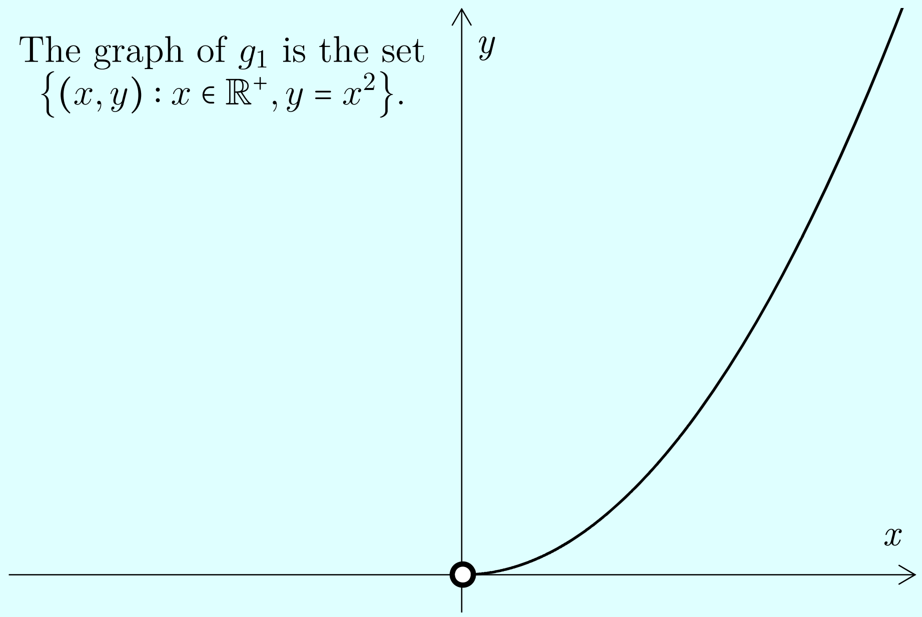
The graph of f_2 is simply this set of isolated points:

$$\{\ldots, (-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6), \ldots\}$$

Example 304. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2$.

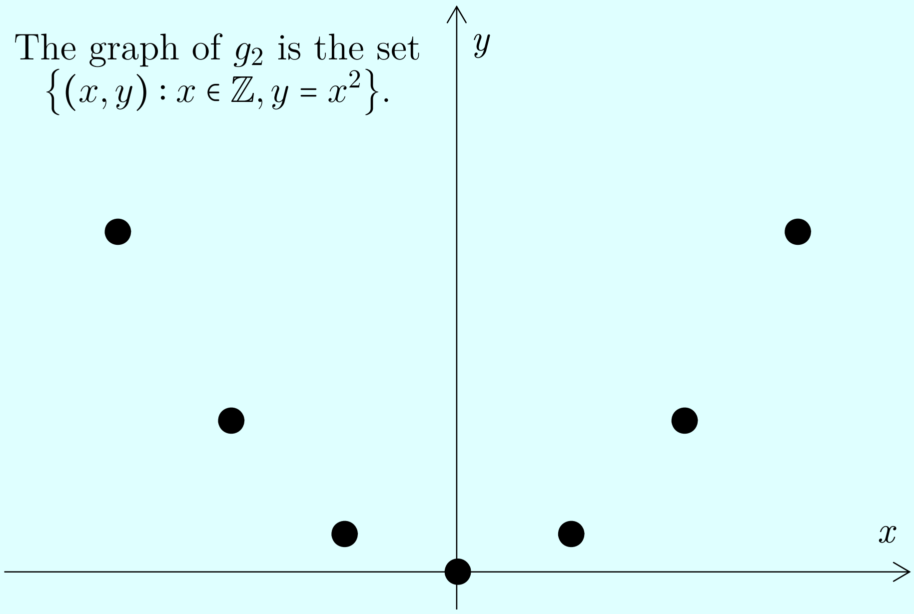


Example 305. Define $g_1 : \mathbb{R}^+ \to \mathbb{R}$ by $g_1(x) = x^2$.



Note that again, the graph of g_1 does **not** contain the point $(0,0)$, because $0 \notin \mathbb{R}^+$.

Example 306. Define $g_2 : \mathbb{Z} \to \mathbb{R}$ by $g_2(x) = x^2$.



The graph of g_2 is simply this set of isolated points:

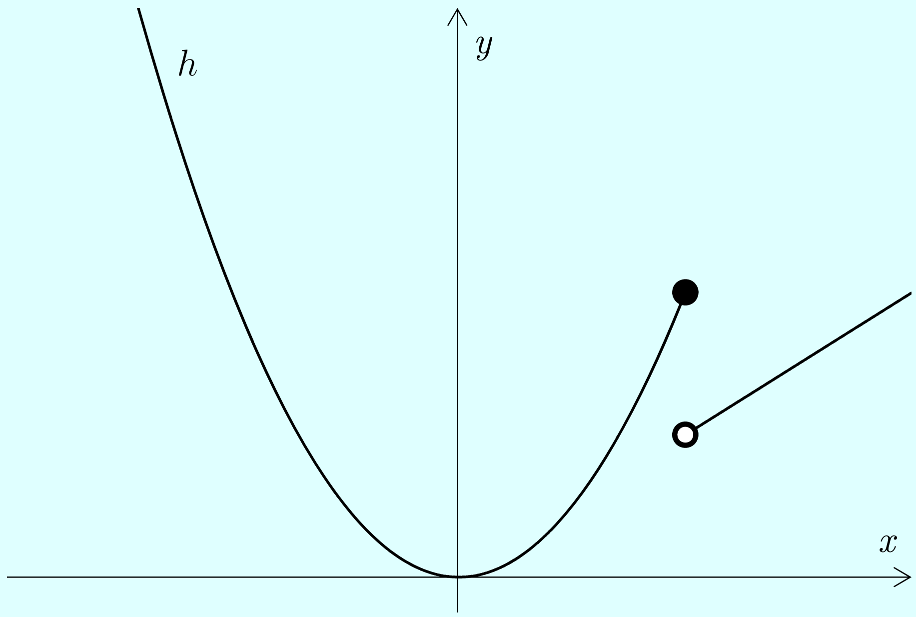
$$\{\ldots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \ldots\}$$

17.6. Piecewise Functions

Informally, a **piecewise function** is a function that has different mapping rules for different intervals (or "pieces"):

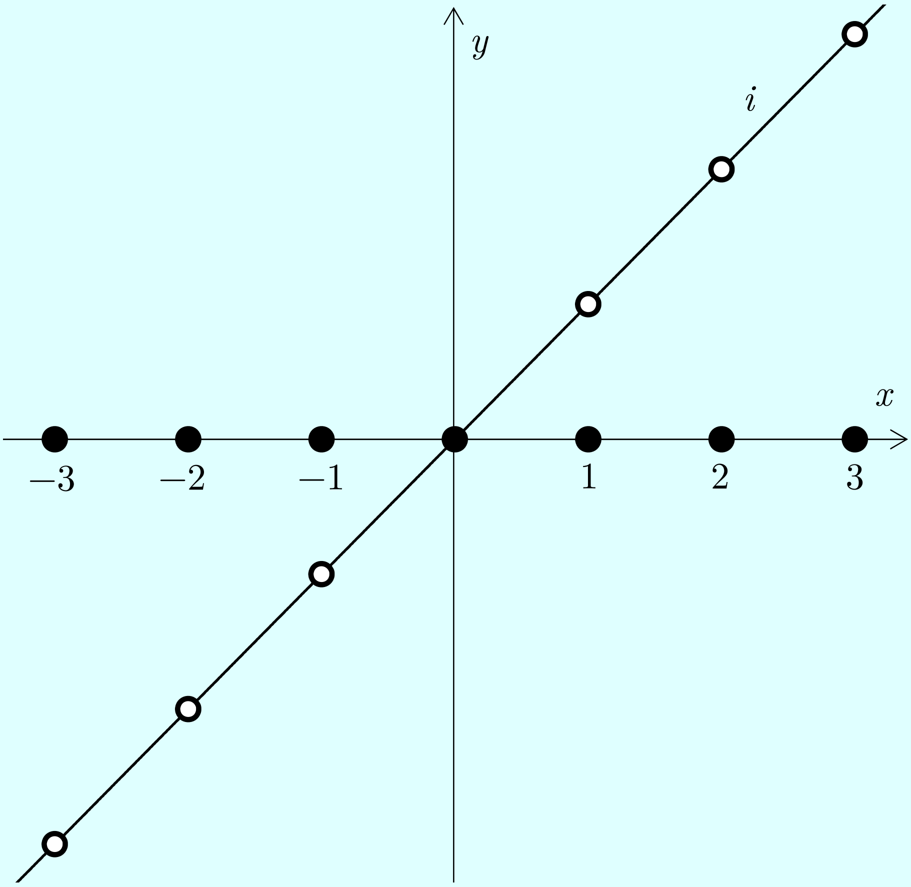
Example 307. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} x^2, & \text{for } x \leq 2, \\ x, & \text{for } x > 2. \end{cases}$$



Example 308. Define $i : \mathbb{R} \to \mathbb{R}$ by

$$i(x) = \begin{cases} x, & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$



An important piecewise function is the **absolute value function** (definition reproduced from Ch. 5.2):

Definition 25. The *absolute value* (or *modulus*) *function*, denoted $|\cdot|$, is defined for all $x \in \mathbb{R}$ by

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

Figure to be
inserted here.

Another important piecewise function is the **sign function**:

Definition 70. The *sign function* $\operatorname{sgn}: \mathbb{R} \to \mathbb{R}$ is defined by

$$\operatorname{sgn} x = \begin{cases} -1, & \text{for } x < 0, \\ 0, & \text{for } x = 0, \\ 1, & \text{for } x > 0. \end{cases}$$

Figure to be inserted here.

Example 309. $\operatorname{sgn} 7 = 1$, $\operatorname{sgn} -5 = -1$, $\operatorname{sgn} 0 = 0$, $\operatorname{sgn} 100 = 1$, $\operatorname{sgn} -531 = -1$.

The absolute value function is closely related to the sign function:

Example 310. $\dfrac{7}{|7|} = \dfrac{7}{7} = 1 = \operatorname{sgn} 7$ and $\dfrac{-5}{|-5|} = \dfrac{-5}{5} = -1 = \operatorname{sgn} -7$.

In Ch. 5.5, we gave this result:

Fact 13. $\dfrac{x}{|x|} = \dfrac{|x|}{x} = \begin{cases} 1, & \textit{for } x > 0, \\ -1, & \textit{for } x < 0. \end{cases}$

Given the sign function, the above result may be restated thus:

Fact 44. $\dfrac{x}{|x|} = \dfrac{|x|}{x} = \operatorname{sgn} x = \begin{cases} 1, & \textit{for } x > 0, \\ -1, & \textit{for } x < 0. \end{cases}$

And so, Examples 129 and 130 from Ch. 5.5 may be restated thus:

Example 311. In Exercise 66 (Ch. 5.5), we gave this false statement:

$$\text{For all } x \in \mathbb{R} \setminus \{0\}, \ \frac{\sqrt{x^2}}{x} = \frac{\sqrt{x^2}}{\sqrt{x^2}} = 1. \qquad ✗$$

The correct statement should instead be this:

$$\text{For all } x \in \mathbb{R} \setminus \{0\}, \ \frac{\sqrt{x^2}}{x} = \frac{|x|}{x} = \operatorname{sgn} x = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases} \qquad ✓$$

Example 312. Similarly, in Exercise 67 (Ch. 5.5), we gave this false statement:

$$\text{For all } x \in \mathbb{R} \setminus \{0\}, \ \frac{x}{\sqrt{x^2}} = \frac{\sqrt{x^2}}{\sqrt{x^2}} = 1. \qquad ✗$$

The correct statement should instead be this:

$$\text{For all } x \in \mathbb{R} \setminus \{0\}, \ \frac{x}{\sqrt{x^2}} = \frac{x}{|x|} = \operatorname{sgn} x = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases} \qquad ✓$$

Exercise 106. Let $A = \{\text{Lion}, \text{Eagle}\}$ and $B = \{\text{Fat}, \text{Tall}\}$. Can we construct a well-defined function using A as the domain and B as the codomain? (Answer on p. 1773.)

Exercise 107. What do we call a function whose ...

(a) Domain contains only real numbers?

(b) Codomain contains only real numbers?

(c) Domain and codomain contain only real numbers? (Answer on p. 1773.)

Exercise 108. Below are 17 alleged functions named a through q, with a given domain, codomain, and mapping rule (the last being given informally in words).

(i) Explain whether each alleged function is actually well-defined.

And if it is,

(ii) Write down a statement that formally and concisely defines it. (Answer on p. 1773.)

Function	**Domain**	**Codomain**	**Mapping rule**
a	$\{5, 6, 7\}$	$\mathbb{Z}$	"Double it"
b	$\{5, 6, 7\}$	$\mathbb{Z}^+$	"Double it"
c	$\{5, 6, 7\}$	$\mathbb{Z}^-$	"Double it"
d	$\{5.4, 6, 7\}$	$\mathbb{Z}$	"Double it"
e	$\{5.5, 6, 7\}$	$\mathbb{Z}$	"Double it"
f	$\{3\}$	$\{3, 4\}$	"Any larger number"
g	$\{3, 3.1\}$	$\{3, 4\}$	"Any larger number"
h	$\{0, 3\}$	$\{3, 4\}$	"Any larger number"
i	$\{3, 4\}$	$\{3, 4\}$	"Any larger number"
j	$\{2, 4\}$	$\{3, 4\}$	"Any *smaller* number"
k	$\{1\}$	$\{1\}$	"Stay the same"
l	$\{1\}$	$\{1, 2\}$	"Stay the same"
m	$\{1, 2\}$	$\{1\}$	"Stay the same"
n	$\mathbb{R}$	$\mathbb{R}$	"Take the square root"
o	$\mathbb{R}$	$\mathbb{R}$	"Take the reciprocal"
p	$\mathbb{R}$	$[0, 1]$	"Add one"
q	$[0, 1]$	$\mathbb{R}$	"Add one"

Exercise 109. Continuing with the above exercise, how can we change the domains of n and o so that n and o become well-defined?[153] (Answer on p. 1774.)

[153]The sharp student may have noticed that one trivial answer here is to simply change the domain to the empty set $\varnothing$. Then it is trivially or *vacuously* true that every element in the domain is mapped to exactly one element in the codomain. If you've noticed and are bothered by this, please change the question to, "What are the '*largest*' subsets of $\mathbb{R}$ to which the domains of n and o can be changed, so

17.7. The Vertical Line Test

Example 313. Let G be the graph of the equation $x^2 + y^2 = 1$. Recall that G is simply the unit circle centred on the origin:

Figure to be
inserted here.

Could G be the graph of a function? That is, does there exist some (nice) function f whose graph is G?

Answer: No.

The reason is that for example, the points $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $\left(-\frac{1}{2}, \frac{1}{4}\right)$ are both in G. So, if G were the graph of f, then we'd have $f\left(\frac{1}{2}\right) = \frac{1}{4}$ *and* $f\left(-\frac{1}{2}\right) = \frac{1}{4}$. That is, f maps the element $\frac{1}{2}$ to *two* elements in the codomain. But this contradicts our requirement that a function maps each element in its domain to *exactly one* element in its codomain.

Hence, G cannot be the graph of any function.

Example 314. XXX

More generally, we have the **Vertical Line Test (VLT)**:

Fact 45. (Vertical Line Test) *Suppose G is a graph (i.e. any set of points in the cartesian plane). Then G is the graph of a nice function $\iff$ No vertical line intersects G more than once.*

Proof. See p. 1572 (Appendices). □

The above result says merely that a graph that fails the VLT can't be the graph of <u>one</u> function. It could however be the (union of the) graphs of two or more functions:

that they become well defined?"

Example 315. Again, let G be the graph of $x^2 + y^2 = 1$. Although G cannot be the graph of one function, it could be the (union of the) graphs of two functions.

Figure to be
inserted here.

Define $f : [-1, 1] \to \mathbb{R}$ by $f(x) = \sqrt{1 - x^2}$ and $h : (-1, 1) \to \mathbb{R}$ by $h(x) = -\sqrt{1 - x^2}$.

Let F and H be the graphs of f and h, respectively.

Observe that $G = F \cup H$. So, we have successfully constructed *two* functions, whose graphs' union form G.

We could of course further split G into the graphs of even more functions.

Figure to be
inserted here.

For example, we could define $f_1 : [-1, 0] \to \mathbb{R}$ by $f_1(x) = \sqrt{1 - x^2}$ and $f_2 : (0, 1] \to \mathbb{R}$ by $f_2(x) = \sqrt{1 - x^2}$. Let F_1 and F_2 be the graphs of f_1 and f_2. Then $G = F_1 \cup F_2 \cup H$.

Example 316. XXX

17.8. Range

Informally, the **range** (of a function) is the set of elements in the codomain that are "hit". Formally,

Definition 71. The *range* of a function f, denoted Range f, is the set defined by

$$\text{Range } f = \{f(x) : x \in \text{Domain } f\}.$$

Example 317. Define $f : [1,2] \to \mathbb{R}$ by $f(x) = x + 3$. Then

- Codomain $f = \mathbb{R}$ (the set of reals).
- Range $f = [4,5]$ (the set of reals between 4 and 5, inclusive).

Figure to be
inserted here.

So, Range $f \subset$ Codomain f and, in particular, Range $f \neq$ Codomain f.[154]

Example 318. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x+3$. Then Codomain $g = \mathbb{R}$ (the set of reals). Every element in the codomain $\mathbb{R}$ is "hit". And so, the range is the same as the codomain: Range $g = \mathbb{R}$.

Figure to be
inserted here.

So, Range g = Codomain g (and Range $g \subseteq$ Codomain g).

The range will **always** be a subset of the codomain. Equivalently, it will always be either a proper subset of the codomain (Example 317) or equal to the codomain (Example 318).

[154] Informally and intuitively, we are tempted to say that the range of f is "strictly smaller than" its codomain. However, as explained in Remark 13, we shall avoid such language.

Example 318 shows that the range can sometimes be identical to the codomain. But as Example 317 shows, this is not generally true—in general, not every element in the codomain will be "hit". Because this is such a common point of confusion, let me repeat:

♡ The range is not the same thing as the codomain. ♡

Remark 43. Unfortunately and very confusingly, some writers (especially in older books) use the term **range** to mean what we call the **codomain** in this textbook. The standard modern usage of the two terms is as given in this textbook and is also what I believe your A-Level examiners have in mind.[155]

Example 319. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = e^x$. Then

$$\text{Codomain}\, h = \mathbb{R} \qquad \text{and} \qquad \text{Range}\, h = \mathbb{R}^+.$$

Figure to be inserted here.

So, $\text{Range}\, h \subset \text{Codomain}\, h$ and, in particular, $\text{Range}\, h \neq \text{Codomain}\, h$.

Example 320. Define $i : \mathbb{R}^+ \to \mathbb{R}$ by $i(x) = \ln x$. Then

$$\text{Codomain}\, i = \mathbb{R} \qquad \text{and} \qquad \text{Range}\, i = \mathbb{R}.$$

Figure to be inserted here.

So, $\text{Range}\, i = \text{Codomain}\, i$ (and $\text{Range}\, i \subseteq \text{Codomain}\, i$).

Remark 44. A synonym for **range** is **image**. But your A-Level syllabus and exams don't use **image**—and so, neither shall we.[156]

[155] Of course, I can't be sure because your A-Level syllabus and exams are never clear about what they mean by the term *range* and never use the term *codomain*.

[156] Some recommend avoiding the term **range** (has different meanings for different writers) and using only **image** (has a single meaning for almost all writers).

Example 321. Define $j : \{1, 2\} \to \mathbb{R}_0^+$ by $j(x) = x + 3$. Then

$$\text{Codomain } j = \mathbb{R}_0^+ \qquad \text{and} \qquad \text{Range } j = \{4, 5\}.$$

Figure to be inserted here.

So, Range $j \subset$ Codomain j and, in particular, Range $j \neq$ Codomain j.

Example 322. Define $k : \{1, 2\} \to \{4, 5\}$ by $k(x) = x + 3$. Then

$$\text{Codomain } j = \{4, 5\} \qquad \text{and} \qquad \text{Range } j = \{4, 5\}.$$

Figure to be inserted here.

So, Range k = Codomain k (and Range $k \subseteq$ Codomain k).

Is the function k the same as the function j in the previous example?[157]

Definition 72. A function whose range and codomain are equal is called *onto*.

[157] No. Although k and j have the same domain and mapping rule (indeed, they even have the same range), their codomains differ. Hence, $k \neq j$.

Example 323. In the above examples, g, i, and k were **onto** because for each, the range was the same as the codomain:

$$\begin{array}{llll} g: \quad \mathbb{R} \to \mathbb{R} & \text{defined by } g(x) = x+3, & \text{Range } g = \mathbb{R}; \\ i: \quad \mathbb{R}^+ \to \mathbb{R} & \quad \text{``} \quad i(x) = \ln x, & \text{Range } i = \mathbb{R}; \\ k: \{1,2\} \to \{4,5\} & \quad \text{``} \quad k(x) = x+3, & \text{Range } k = \{4,5\}. \end{array}$$

In contrast, f, h, and j were **not onto** because for each, the range was a proper subset of the codomain:

$$\begin{array}{llll} f: [1,2] \to \mathbb{R} & \text{defined by } f(x) = x+3, & \text{Range } f = [4,5]; \\ h: \quad \mathbb{R} \to \mathbb{R} & \quad \text{``} \quad h(x) = \mathrm{e}^x, & \text{Range } h = \mathbb{R}^+; \\ j: \{1,2\} \to \mathbb{R}_0^+ & \quad \text{``} \quad j(x) = x+3, & \text{Range } j = [4,5]. \end{array}$$

Remark 45. An exact synonym for **onto** is **surjective**. (And a **surjection** is a function that's onto or surjective.)

Exercise 110. Find the range of each function. Which (if any) of these functions is onto? (Answer on p. 1774.)

(a) Define $a : \mathbb{R}_0^+ \to \mathbb{R}$ by $a(x) = \sqrt{x}$.

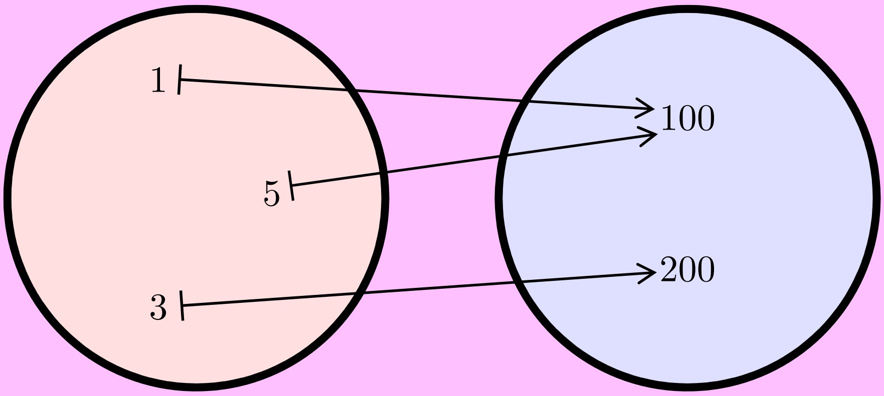
(b) Define $b : \mathbb{Z} \to \mathbb{R}$ by $b(x) = x^2$.

(c) Define $c : \mathbb{Z} \to \mathbb{Z}$ by $c(x) = x^2$.

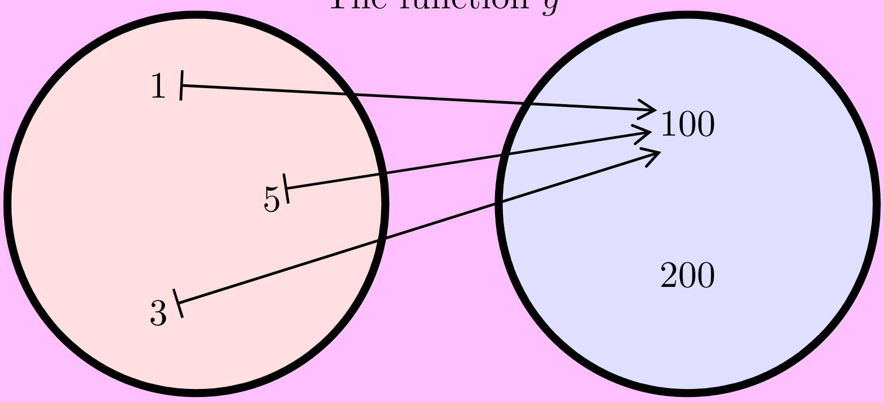
(d) Define $d : \mathbb{Z} \to \mathbb{Z}$ by $d(x) = x + 1$.

(e) Define $e : \mathbb{Z} \to \mathbb{R}$ by $e(x) = x + 1$.

(f) The function f



(g) The function g



Exercise 111. Let f be a function. Of the following six statements, which *must* be true?

(a) Range $f \subseteq$ Domain f.

(b) Range $f \subseteq$ Codomain f.

(c) Range $f \subset$ Domain f.

(d) Range $f \subset$ Codomain f.

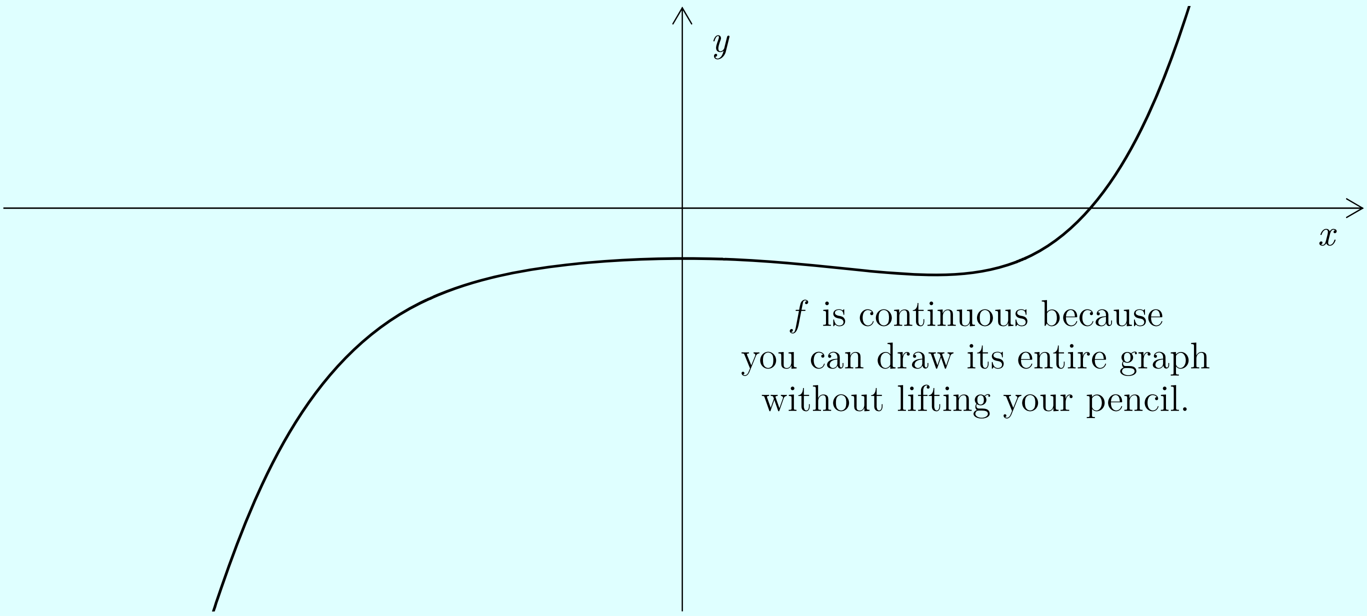
(e) Range $f =$ Domain f.

(f) Range $f =$ Codomain f. (Answer on p. 1775.)

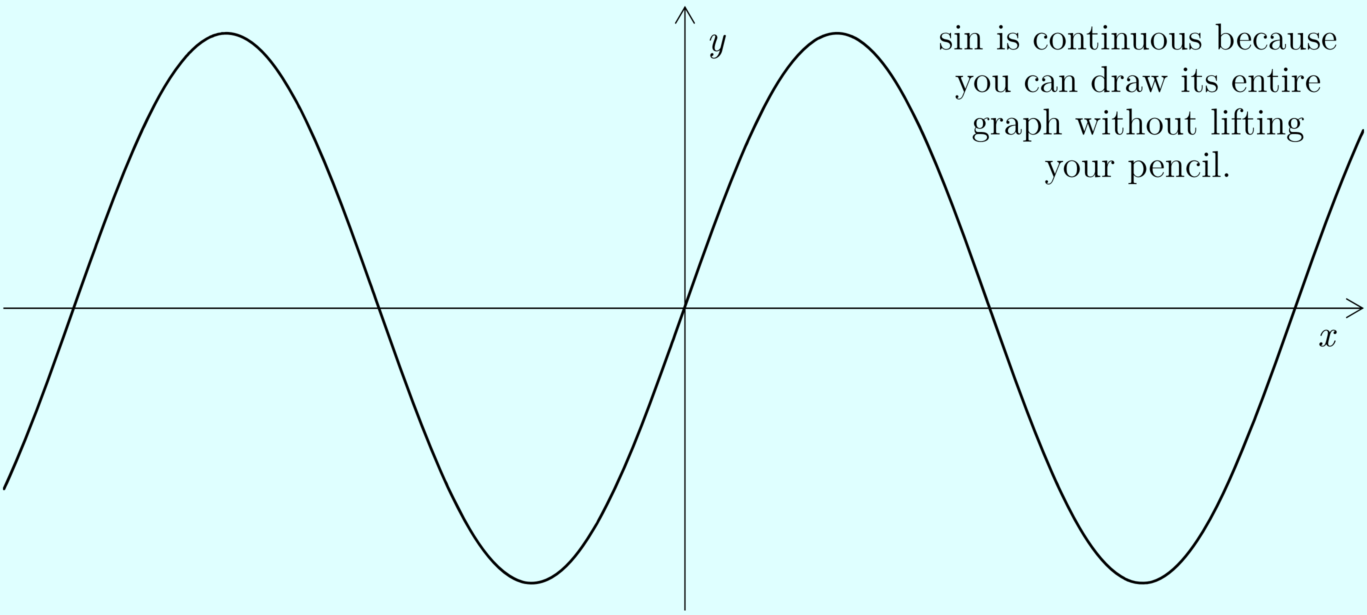
18. An Introduction to Continuity

Informally, if you can draw a function's graph without lifting your pencil, then that function is **continuous**.[158] Most functions we'll encounter in A-Level maths will be continuous:

Example 324. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^5 - x^2 - 1$ is continuous, because you can draw its entire graph without lifting your pencil.

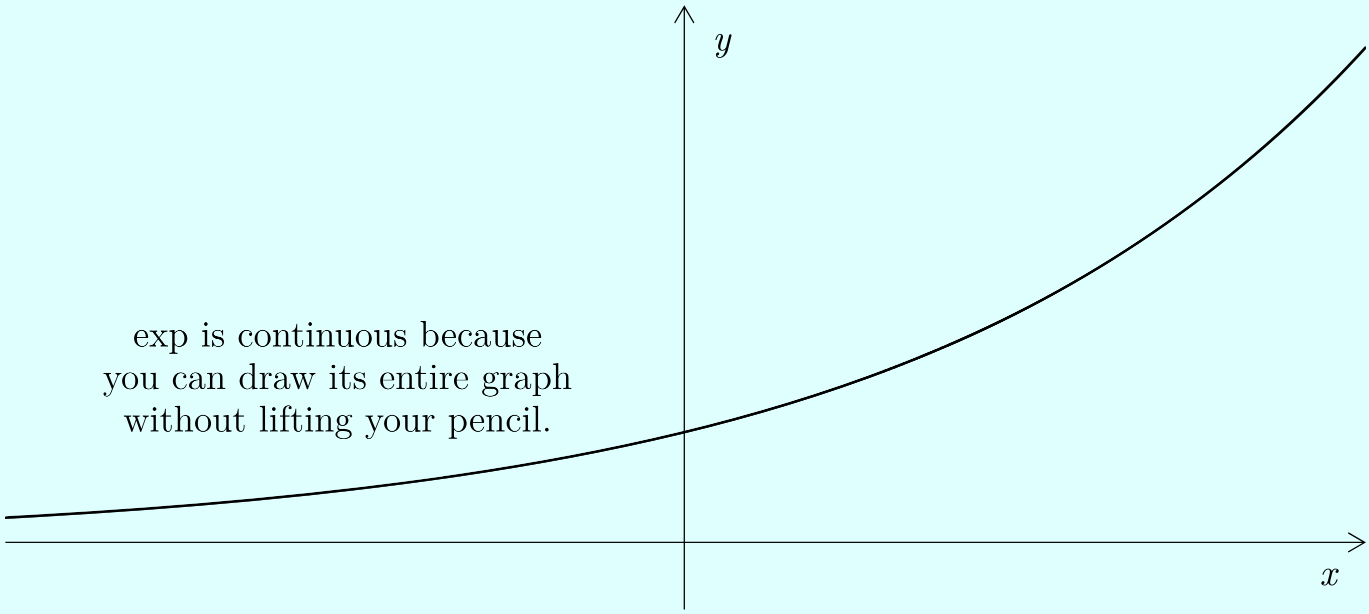


Example 325. The sine function sin is continuous.

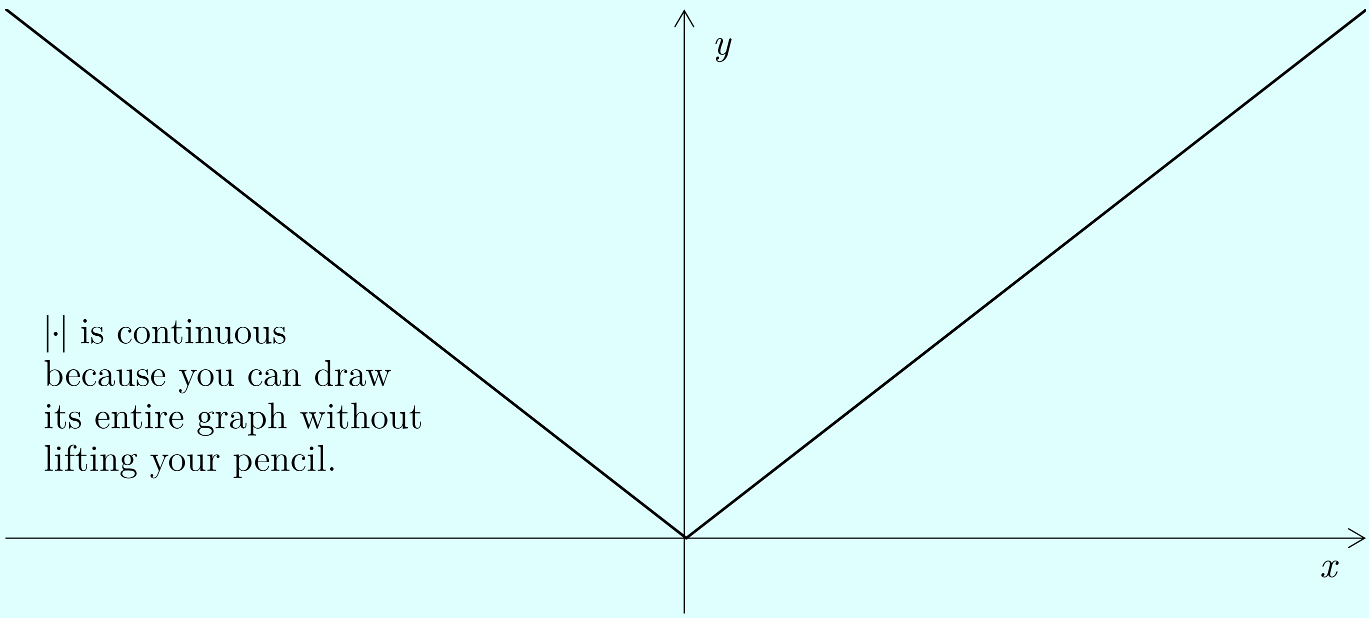


[158] The converse though is false—that is, a function can be continuous even if its graph cannot be drawn without lifting your pencil.

Example 326. The exponential function exp is continuous.



Example 327. The absolute value function $|\cdot|$ is continuous.

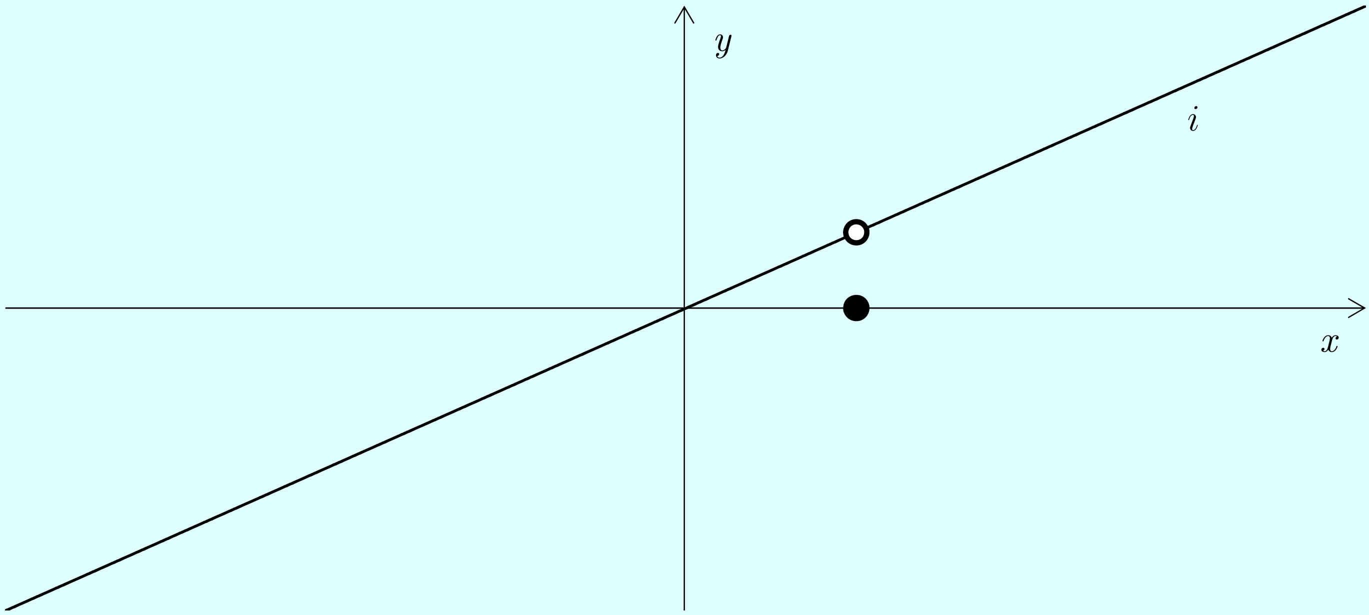


As stated, most functions we'll encounter are continuous. However, we'll occasionally encounter functions that aren't continuous everywhere. For example, functions with "holes" in them are not continuous:

Example 328. Define $i : \mathbb{R} \to \mathbb{R}$ by

$$i(x) = \begin{cases} x & \text{for } x \neq 1, \\ 0 & \text{for } x = 1. \end{cases}$$

The function i is **not** continuous. In Part V (Calculus), we'll learn why precisely this is so. For now, an intuitive understanding will suffice.



Note though that i *is* continuous on $(-\infty, 1)$ because you can draw this portion of the graph without lifting your pencil.

Similarly, i is continuous on $(1, \infty)$ because you can draw this portion of the graph without lifting your pencil.

And so, we can actually say that i is continuous on $(-\infty, 1) \cup (1, \infty)$.

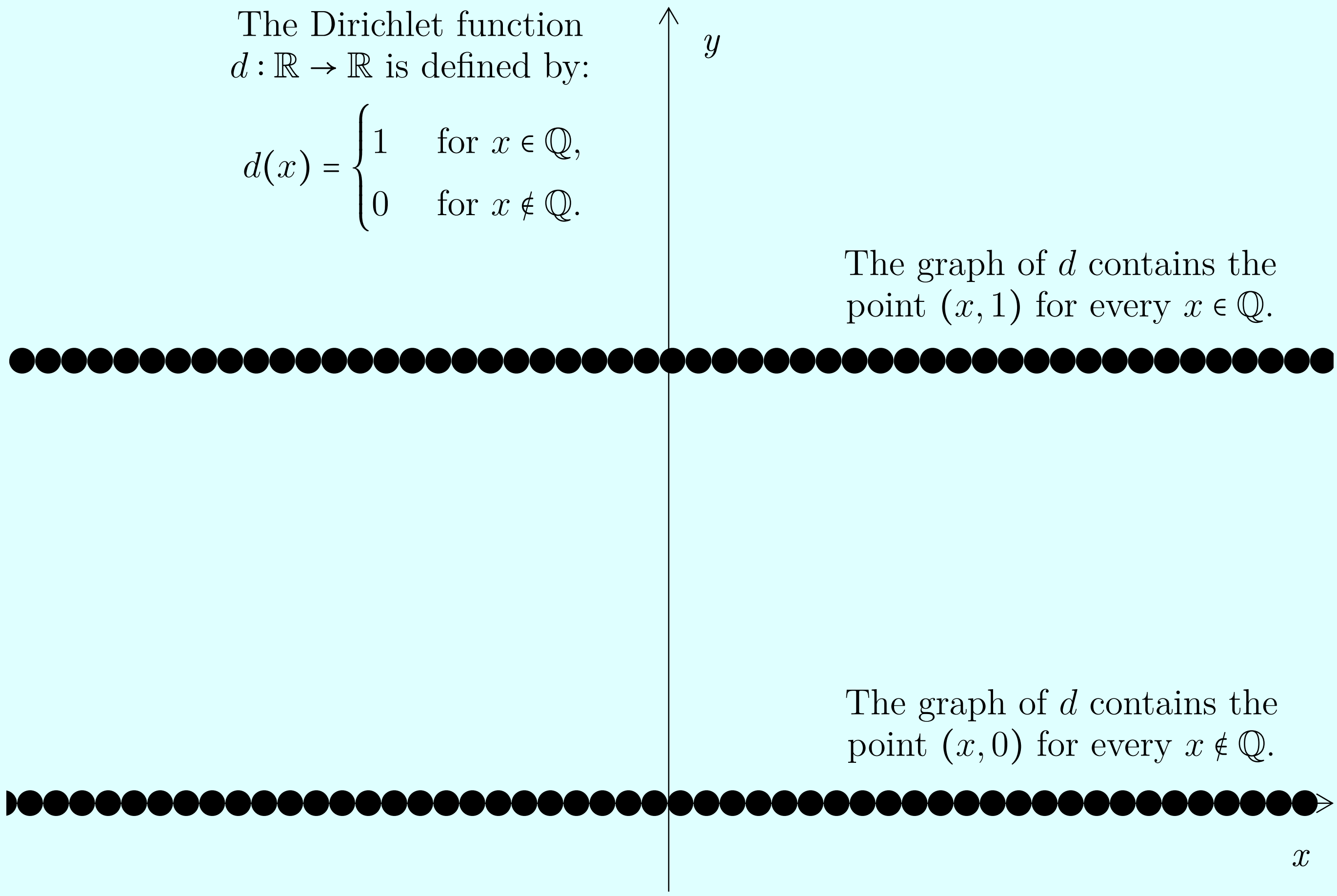
This isn't something you need to know, but just to illustrate, here's a somewhat exotic function whose domain is $\mathbb{R}$ but which is nowhere-continuous.

Example 329. The **Dirichlet function**[159] $d : \mathbb{R} \to \mathbb{R}$ is defined by

$$d(x) = \begin{cases} 1, & \text{for } x \in \mathbb{Q}, \\ 0, & \text{for } x \notin \mathbb{Q}. \end{cases}$$

Then d is **not** continuous everywhere, because to draw its entire graph, you must lift your pencil.

In fact, it's worse than that—d is **nowhere-continuous**! Informally, this means you can't draw more than one point of the graph without lifting your pencil.[160]



[159] Or the **characteristic function of the rationals**. Named after the German mathematician Peter Gustav Lejeune Dirichlet (1805–59). W

[160] We'll revisit this example in Part V (Calculus).

18.1. A Subtle Point about Continuity

We'll formally define continuity only in Part V (Calculus).

It turns out that given a function f, we say that f is

- Either continuous or not (and not both) at each point $x \in \text{Domain } f$ (Definition 198);
- Continuous (and called a continuous function) if it is continuous at every point $x \in \text{Domain } f$ (Definition 199);
- Neither continuous nor discontinuous at every point $x \notin \text{Domain } f$

The last bullet point is somewhat subtle and surprising. Here are three examples to illustrate it:

Example 330. Define $f : [1,2] \cup [3,4] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{for } x \in [1,2], \\ 2, & \text{for } x \in [3,4]. \end{cases}$$

Figure to be inserted here.

It turns out that f is continuous at every $x \in \text{Domain } f = [1,2] \cup [3,4]$.

And so, perhaps surprisingly, f is a continuous function.

Example 331. Define $g : (0,1) \cup (1,2) \cup (2,3) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1, & \text{for } x \in (0,1), \\ 2 & \text{for } x \in (1,2), \\ 3, & \text{for } x \in (2,3). \end{cases}$$

Figure to be inserted here.

It turns out that g is continuous at every $x \in \text{Domain } f = (0,1) \cup (1,2) \cup (2,3)$.

And so, perhaps surprisingly, g is a continuous function.

Example 332. Here's the graph of the tangent function (to be reviewed in Ch. 33):

Figure to be
inserted here.

Again, it turns out that tan is continuous at every $x \in \text{Domain}\tan = \mathbb{R} \setminus \{\text{Odd integer multiples of } \pi/2\}$.

And so, perhaps surprisingly, tan is a continuous function.

The last three examples also show that the (very) informal "you can draw a function's graph without lifting your pencil" definition of continuity is actually wrong![161]

We could try to modify this informal statement to, "For each connected subset of the function's domain, you can draw the graph without lifting your pencil." This is "more" correct but still somewhat imprecise and informal ("lifting your pencil" isn't exactly a precise or formal statement).

Again, the formal and correct definitions of continuity will be given in Part V (Calculus)—Definitions 198 and 199. For now it suffices to be aware of the above subtle point.

[161] More precisely, satisfying this condition is sufficient but not necessary for continuity.

19. When a Function Is Increasing or Decreasing

Informally, we know what it means for a function to be **increasing**, **decreasing**, **strictly increasing**, and/or **strictly decreasing**:

Example 333. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

Figure to be
inserted here.

The function f is **decreasing**—and indeed, **strictly decreasing**—on $\mathbb{R}_0^- = (-\infty, 0]$.

Also, it is **increasing**—and indeed, **strictly increasing**—on $\mathbb{R}_0^+ = [0, \infty)$.

Formal definitions:

Definition 73. Let f be a nice function with domain D. Suppose $S \subseteq D$. Then f is

(a) *Increasing on S* if for any $a, b \in S$ with $a < b$, we have $f(a) \le f(b)$.
(b) *Strictly increasing* " $f(a) < f(b)$.
(c) *Decreasing* " $f(a) \ge f(b)$.
(d) *Strictly decreasing* " $f(a) > f(b)$.

If f is increasing on D, then f is an *increasing function.*

If f is strictly increasing on D, then f is a *strictly increasing function.*

If f is decreasing on D, then f is a *decreasing function.*

If f is strictly decreasing on D, then f is a *strictly decreasing function.*

Example 334. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$.

Figure to be inserted here.

The function g is increasing on $\mathbb{R}$, because if we pick any $a, b \in \mathbb{R}$ with $a < b$, then by informal observation,[162] we have

$$a^3 \leq b^3 \qquad \text{or} \qquad g(a) \leq g(b).$$

Since g is increasing on its domain $\mathbb{R}$, it is an increasing function.

Indeed, by informal observation, we also have

$$a^3 < b^3 \qquad \text{or} \qquad g(a) < g(b),$$

so that g is also *strictly* increasing on its domain $\mathbb{R}$ and is a strictly increasing function.

Fact 46. *Let f be a nice function and $S \subseteq D = \text{Domain } f$.*

(a) *If f is strictly increasing on S, then it is also increasing on S.*

(b) *If f is strictly decreasing on S, then it is also decreasing on S.*

Proof. **(a)** Suppose f is strictly increasing on S. Then by Definition 73(a), $f(a) < f(b)$ for all $a, b \in S$ with $a < b$. So, it must also be true that $f(a) \leq f(b)$ for all $a, b \in S$ with $a < b$. Hence, by Definition 73(b), f is increasing on S.

(b) Similar, omitted. □

Fact 46 says that "strictly increasing implies increasing" and "strictly decreasing implies decreasing". However, the converses of these two statements are false, as the next example shows:

[162]Formal proof that g is strictly increasing (and hence also increasing):
Suppose $x_1 < x_2$. Observe that $x_1^3 - x_2^3 = (x_1 - x_2)\left(x_1^2 + x_1x_2 + x_2^2\right)$. We shall prove that $x_1^3 - x_2^3 < 0$.
Next, $x_1^2 + x_1x_2 + x_2^2 = \dfrac{1}{2}\left(2x_1^2 + 2x_1x_2 + 2x_2^2\right) = \dfrac{1}{2}\left(x_1^2 + x_2^2 + x_1^2 + 2x_1x_2 + x_2^2\right) = \dfrac{1}{2}\left[x_1^2 + x_2^2 + (x_1 + x_2)^2\right]$.
Each of the three terms in brackets is non-negative. Moreover, at least one of x_1 or x_2 is non-zero, so that at least one of x_1^2 or x_2^2 is (strictly) positive. Hence, $x_1^2 + x_1x_2 + x_2^2 > 0$.

Example 335. Consider the function $i : \mathbb{R} \to \mathbb{R}$ defined by $i(x) = 1$.

Figure to be
inserted here.

The function i is increasing on $\mathbb{R}$, because if we pick any $a, b \in \mathbb{R}$ with $a < b$, we have

$$1 \leq 1 \qquad \text{or} \qquad i(a) \leq i(b).$$

However, i is not *strictly* increasing on $\mathbb{R}$, because if we pick any $a, b \in \mathbb{R}$ with $a < b$, we have

$$1 \not< 1 \qquad \text{or} \qquad i(a) \not< i(b).$$

Similarly, i is decreasing on $\mathbb{R}$, because if we pick any $a, b \in \mathbb{R}$ with $a < b$, we have

$$1 \geq 1 \qquad \text{or} \qquad i(a) \geq i(b).$$

However, i is not *strictly* decreasing on $\mathbb{R}$, because if we pick any $a, b \in \mathbb{R}$ with $a < b$, we have

$$1 \not> 1 \qquad \text{or} \qquad i(a) \not> i(b).$$

Altogether, i is both an increasing function and a decreasing function. However, i is neither a strictly increasing function nor a strictly decreasing function.

Exercise 112. Consider the function $j : \mathbb{R} \to \mathbb{R}$ defined by $j(x) = x^4$.

Figure to be
inserted here.

Explain why j is **(a)** strictly decreasing on $(-\infty, 0]$; **(b)** strictly increasing on $[0, \infty)$.

Hence, conclude (*fill in the blanks*): **(c)** By ________, j is also decreasing on ________ and increasing on ________. (Answer on p. 1776.)

Example 336. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = 7x + 5$ and $g(x) = x^3$. Let $k = 2$.

Then we can also define these six functions:

1. $(f+g) : \mathbb{R} \to \mathbb{R}$ by $(f+g)(x) = f(x) + g(x) = 7x + 5 + x^3$;
2. $(f-g) : \mathbb{R} \to \mathbb{R}$ by $(f-g)(x) = f(x) - g(x) = 7x + 5 - x^3$;
3. $(f \cdot g) : \mathbb{R} \to \mathbb{R}$ by $(f \cdot g)(x) = f(x)\,g(x) = (7x+5)\,x^3$;
4. $(kf) : \mathbb{R} \to \mathbb{R}$ by $(kf)(x) = kf(x) = 2(7x+5) = 14x + 10$;
5. $\left(\dfrac{f}{g}\right) : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{7x+5}{x^3}$;
6. $(f+k) : \mathbb{R} \to \mathbb{R}$ by $(f+k)(x) = f(x) + k = 7x + 5 + 2 = 7x + 7$.

For lack of any standard names, I'll call the above six functions the **sum**, **difference**, **product**, **constant multiple**, **quotient**, and **translated functions**, respectively.

Evaluating each of these six functions at 1, we have

1. $(f+g)(1) = 7 \cdot 1 + 5 + 1^3 = 13$;
2. $(f-g)(1) = 7 \cdot 1 + 5 - 1^3 = 11$;
3. $(f \cdot g)(1) = (7 \cdot 1 + 5) \times 1^3 = 12$;
4. $(kf)(1) = 14 \cdot 1 + 10 = 24$;
5. $\left(\dfrac{f}{g}\right)(1) = \dfrac{7 \cdot 1 + 5}{1^3} = 12$;
6. $(f+k)(1) = 7 \cdot 1 + 7 = 14$.

By the way, as usual, parentheses can be helpful for making things clear. Here for example, they make clear that $f+g$ is a single function—it'd be confusing and unclear if we wrote $f + g(1)$ instead of $(f+g)(1)$.

Note the **domain** of each of the above six functions:

For kf and $f+k$, the domain is simply Domain f.

For each of $f+g$, $f-g$, and $f \cdot g$, the domain is Domain $f \cap$ Domain g (i.e. the set of numbers that are in *both* the domains of f and g).

For f/g, it's a little trickier figuring out what the domain is. Observe that $g(0) = 0$. And so, in order for f/g to be well-defined, we must **restrict the domain** by removing the element 0. Otherwise, $(f/g)(0)$ would be undefined and f/g would fail to be a well-defined function. Thus, the domain of f/g is

$$\text{Domain}\, f \cap \text{Domain}\, g \setminus \{x : g(x) = 0\} = \mathbb{R} \setminus \{0\}.$$

In words, the domain of f/g is the set of numbers x that are in *both* the domains of f and g, *except* those for which $g(x)$ equals zero.

Example 337. Define $h, i : [-1, \infty) \to \mathbb{R}$ by $h(x) = x + 1$ and $i(x) = \sqrt{x+1}$. Let $l = 5$. Then we can also define these six functions:

1. $(h+i) : [-1,\infty) \to \mathbb{R}$ by $(h+i)(x) = h(x) + i(x) = x + 1 + \sqrt{x+1}$;

2. $(h-i) : [-1,\infty) \to \mathbb{R}$ by $(h-i)(x) = h(x) - i(x) = x + 1 - \sqrt{x+1}$;

3. $(h \cdot i) : [-1,\infty) \to \mathbb{R}$ by $(h \cdot i)(x) = h(x)\, i(x) = (x+1)\sqrt{x+1}$;

4. $(lh) : [-1,\infty) \to \mathbb{R}$ by $(lh)(x) = lh(x) = 5(x+1) = 5x + 5$;

5. $\left(\frac{h}{i}\right) : (-1,\infty) \to \mathbb{R}$ by $\left(\frac{h}{i}\right)(x) = \frac{h(x)}{i(x)} = \frac{x+1}{\sqrt{x+1}} = \sqrt{x+1}$;

6. $(h+l) : [-1,\infty) \to \mathbb{R}$ by $(h+l)(x) = h(x) + l = x + 1 + 5 = x + 6$.

Evaluating each of these six functions at 1, we have

1. $(h+i)(1) = 1 + 1 + \sqrt{1+1} = 2 + \sqrt{2}$;

2. $(h-i)(1) = 1 + 1 - \sqrt{1+1} = 2 - \sqrt{2}$;

3. $(h \cdot i)(1) = (1+1)\sqrt{1+1} = 2\sqrt{2}$;

4. $(lh)(1) = 5(1+1) = 10$;

5. $\left(\frac{h}{i}\right)(1) = \sqrt{1+1} = \sqrt{2}$;

6. $(h+l)(1) = 1 + 1 + 5 = 7$.

Again, note the **domain** of each of the above six functions:

For each of lh and $h + l$, the domain is simply Domain h.

For each of $h + i$, $h - i$, and $h \cdot i$, the domain is Domain $h \cap$ Domain i (the set of numbers that are in both the domains of h and i).

But for h/i, it's again a little trickier figuring out the domain. Observe that $i(-1) = 0$. And so, in order for h/i to be well-defined, we must **restrict the domain** by removing the element 0. Otherwise, $(h/i)(0)$ would be undefined and h/i would fail to be a well-defined function. Thus, the domain of h/i is

$$\text{Domain}\, h \cap \text{Domain}\, i \setminus \{x : i(x) = 0\} = [-1, \infty) \setminus \{-1\} = (-1, \infty).$$

In words, the domain of h/i is the set of numbers x that are in *both* the domains of h and i, *except* those for which $i(x)$ equals zero.[163]

[163]Here the sharp student may wonder, "But isn't $\sqrt{x+1}$ perfectly well-defined for all $x \in [-1, \infty)$ and, in particular, even for $x = -1$? So couldn't the domain of h/i instead be $[-1, \infty)$?"
Great point! But the thing is, we really want to think of $(h/i)(x)$ as being equal to $h(x)$ divided by $i(x)$ (without doing any simplification beforehand and without thinking of h/i as being simply the "formula" $\sqrt{x+1}$). So, if $i(x)$ is undefined, then we want $(h/i)(x)$ to also be undefined.

Formal definitions of the six functions:

Definition 74. Let $k \in \mathbb{R}$ and $f : A \to B$ and $g : C \to D$ be nice functions. Define[164]

1. The *sum function* $(f+g) : A \cap C \to \mathbb{R}$ by $(f+g)(x) = f(x) + g(x)$;
2. The *difference function* $(f-g) : A \cap C \to \mathbb{R}$ by $(f-g)(x) = f(x) - g(x)$;
3. The *product function* $(f \cdot g) : A \cap C \to \mathbb{R}$ by $(f \cdot g)(x) = f(x) \cdot g(x)$;
4. The *constant multiple function* $(kf) : A \to \mathbb{R}$ by $(kf)(x) = kf(x)$;
5. The *quotient function* $\left(\frac{f}{g}\right) : A \cap C \setminus \{x : g(x) = 0\} \to \mathbb{R}$ by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$;
6. The *translated function* $(f+k) : A \to \mathbb{R}$ by $(f+k)(x) = f(x) + k$.

Remark 46. Again, there are no standard names for the six functions just defined. They are often just left nameless and are sometimes collectively called "arithmetic combinations of functions" (hence the name of this chapter).

The names given above are simply those that I thought make a little sense, though some (especially *translated function*) are a bit awkward.

Remark 47. As we'll learn in the next chapter, fg refers to a function that's entirely different from $f \cdot g$. So take great care to write $f \cdot g$ if that's what you mean.[165]

Exercise 113. Let $k = 2$ and $l = 5$. Define (Answer on p. 1777)

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \text{ by } f(x) = 7x + 5, \\ g &: \mathbb{R} \to \mathbb{R} \text{ by } g(x) = x^3, \\ h &: [-1, \infty) \to \mathbb{R} \text{ by } h(x) = x + 1, \\ i &: [-1, \infty) \to \mathbb{R} \text{ by } i(x) = \sqrt{x+1}. \end{aligned}$$

Evaluate the following.

(a)	$(f+g)(2)$	**(d)**	$(kg)(1)$	**(g)**	$(h+i)(2)$	**(j)**	$(li)(1)$
(b)	$(g-f)(1)$	**(e)**	$(g/f)(1)$	**(h)**	$(i-h)(1)$	**(k)**	$(i/h)(1)$
(c)	$(g \cdot f)(2)$	**(f)**	$(f+k)(1)$	**(i)**	$(i \cdot h)(2)$	**(l)**	$(h+l)(1)$

(k) Write down the domain of each of the functions $f+h$, $f-h$, $f \cdot h$, and f/h. Then define each function.

[164] I've simply set the codomain of each of the six functions as $\mathbb{R}$.

One alternative would be to let the codomains of the six functions be, respectively, $\{y = f(x) + g(x) : x \in A \cap C\}$, $\{y = f(x) - g(x) : x \in A \cap C\}$, $\{y = f(x) g(x) : x \in A \cap C\}$, $\{y = kf(x) : x \in A\}$, $\{y = f(x)/g(x) : x \in A \cap C \setminus \{x : g(x) = 0\}\}$, and $\{y = f(x) + k : x \in A\}$. In which case, each of the six functions would be onto.

[165] Unfortunately and very confusingly, a minority of writers *do* use fg to mean $f \cdot g$. In the A-Level syllabus and exams and in this textbook, we will follow majority practice by insisting that $fg \neq f \cdot g$.

21. Domain Restriction

We can take a given function and create new ones through **domain restrictions**:

Example 338. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=2x$.

Then the **restriction of f to** $[1,2]$ is the function $f\big|_{[1,2]}:[1,2]\to\mathbb{R}$ defined by $f\big|_{[1,2]}(x)=f(x)$.

Figure to be
inserted here.

Note well that the functions f and $f\big|_{[1,2]}$ are distinct: $f\neq f\big|_{[1,2]}$.

Example 339. Define $g:\mathbb{R}_0^+\to\mathbb{R}$ by $g(x)=\sqrt{x}$.

Then the **restriction of g to** $[2,3]$ is the function $g\big|_{[2,3]}:[2,3]\to\mathbb{R}$ defined by $g\big|_{[2,3]}(x)=g(x)$.

Figure to be
inserted here.

Note well that the functions g and $g\big|_{[2,3]}$ are distinct: $g\neq g\big|_{[2,3]}$.

Definition 75. Let $f:A\to B$ be a function and $C\subseteq A$. The *restriction of f to C* is the function $f\big|_C:C\to B$ defined by $f\big|_C(x)=f(x)$.

Remark 48. The notation $f\big|_C$ is not in your A-Level maths syllabus or exams. Nonetheless, it is sufficiently convenient and commonly used that we'll use it in this textbook.

Exercise 114. For each given function, what are its restrictions to $[0,1]$ and $\mathbb{R}^+$? Draw the graphs of the given function and also the two restrictions. (Answer on p. 245.)

(a) xxx

(b) xxx

A114.

22. Composite Functions

Let f and g be nice functions. Earlier we learnt to construct the product function $f \cdot g$ (read aloud as "f times g").

We now learn to construct a new function fg (read aloud simply as "f g"), called the **composite function**.

The functions fg and $f \cdot g$ look similar but are completely different.

Example 340. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$ and $g(x) = x + 1$. Then the **composite function** $fg : \mathbb{R} \to \mathbb{R}$ is defined by

$$(fg)(x) = f(g(x)) = f(x+1) = 2(x+1) = 2x + 2.$$

(To get the composite function fg, plug g into f.)

In contrast, the product function $f \cdot g : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f \cdot g)(x) = f(x) g(x) = (2x)(x+1) = 2x^2 + 2x.$$

(To get the product function $f \cdot g$, multiply f and g.)

The functions fg and $f \cdot g$ are different. For example, the values of each at 0 are

$$(fg)(0) = 2 \times 0 + 2 = 2 \qquad \text{and} \qquad (f \cdot g)(0) = 2 \times 0^2 + 2 \times 0 = 0.$$

Note that $f \cdot g = g \cdot f$, but $fg \neq gf$. We say that the product (of two functions) is **commutative**, but the composition (of two functions) is **not** (generally) so.

Here for example, the composite function $gf : \mathbb{R} \to \mathbb{R}$ is defined by

$$(gf)(x) = g(f(x)) = g(2x) = 2x + 1.$$

(To get the composite function gf, plug f into g.)

The value of gf at 0 is $(gf)(0) = 2 \times 0 + 1 = 1$, which is different from $(fg)(0) = 2$.

Note also that for the composite function fg, we plug g into f. So for example, to compute $fg(7)$, we first compute $g(7) = 7 + 1 = 8$, then compute $f(g(7)) = f(8) = 2 \cdot 8 = 16$. (A common mistake is to do the opposite because one goes instinctively from left to right.)

Conversely, for the composite function gf, we plug f into g. So for example, to compute $gf(7)$, we first compute $f(7) = 2 \cdot 7 = 14$, then compute $g(f(7)) = g(14) = 14 + 1 = 15$. (Ditto.)

Example 341. Define $h, i : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^2 - 1$ and $i(x) = x/2$.

Then the composite function $hi : \mathbb{R} \to \mathbb{R}$ is defined by

$$(hi)(x) = h(i(x)) = h(x/2) = x^2/4 - 1.$$

In contrast, the product function $h \cdot i : \mathbb{R} \to \mathbb{R}$ is defined by

$$(h \cdot i)(x) = h(x)\, i(x) = \left(x^2 - 1\right) x/2 = x^3/2 - x/2.$$

The functions hi and $h \cdot i$ are different. For example, the values of each at 0 are

$$(hi)(0) = 0^2/4 - 1 = -1 \qquad \text{and} \qquad (h \cdot i)(0) = 0^3/2 - 0/2 = 0.$$

Again, note that $h \cdot i = i \cdot h$, but $hi \neq ih$. The composite function $ih : \mathbb{R} \to \mathbb{R}$ is defined by

$$(ih)(x) = i(h(x)) = i\left(x^2 - 1\right) = x^2/2 - 1/2.$$

The value of ih at 0 is $(ih)(0) = \dfrac{0^2}{2} - \dfrac{1}{2} = -\dfrac{1}{2}$, which is different from $(hi)(0) = -1$.

Formal definition of a composite function:

Definition 76. Let f and g be functions with $\text{Range}\, g \subseteq \text{Domain}\, f$. Then the *composite function* fg is the function with

$$\begin{aligned} \text{Domain:} &\quad \text{Domain}\, g; \\ \text{Codomain:} &\quad \text{Codomain}\, f; \text{ and} \\ \text{Mapping rule:} &\quad (fg)(x) = f(g(x)). \end{aligned}$$

So, a composite function's domain is the same as the innermost (or rightmost) function.

And its codomain is the same as the outermost (or leftmost) function.

Remark 49. The composite function fg may also be written $f \circ g$ (read aloud as "f circle g"). This alternative piece of notation is useful for making clear that we do **not** mean $f \cdot g$.

The condition $\text{Range}\, g \subseteq \text{Domain}\, f$ is important.[166] It ensures that for every $x \in \text{Domain}\, g$, we have $g(x) \in \text{Domain}\, f$ and hence that $f(g(x))$ is well-defined.

If this condition fails, then we simply say that the composite function fg **does not exist** or **is undefined**:

[166] Just so you know, many writers take a different approach: They define the composite function $f \circ g$ if and only if $\text{Codomain}\, g = \text{Domain}\, f$.

In contrast, in Definition 76, we use the weaker requirement $\text{Range}\, g \subseteq \text{Domain}\, f$. (Many other writers also do the same as us—see e.g. MathWorld).

Example 342. Define $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \ln x$ and $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x + 1$.

Observe $\text{Range}\, g = \mathbb{R}$ is not a subset of $\text{Domain}\, f = \mathbb{R}^+$. And so for example,

$$(fg)(-5) = f(g(-5)) = f(-5+1) = f(-4)$$

would be undefined. Hence, the composite function fg simply **does not exist**.

The rest of this example is explicitly excluded from your syllabus.[167] *Nonetheless, spending a minute or two reading it will earn you a better understanding of composite functions. (Plus, this is the sort of thing that your A-Level examiners could easily include as a curveball question.)*

Consider the restriction of g to $(-1, \infty)$, i.e. the function $g|_{(-1,\infty)}$.

Observe that $\text{Range}\, g|_{(-1,\infty)} = \mathbb{R}^+$ is a subset of $\text{Domain}\, f = \mathbb{R}^+$.

So, we have the composite function $fg|_{(-1,\infty)} : (-1, \infty) \to \mathbb{R}$ defined by

$$\left(fg|_{(-1,\infty)}\right)(x) = f\left(g|_{(-1,\infty)}(x)\right) = f(x+1) = \ln(x+1).$$

Example 343. Define $i : \mathbb{R}_0^+ \to \mathbb{R}$ by $i(x) = \sqrt{x}$ and $j : \mathbb{R} \to \mathbb{R}$ by $j(x) = x - 3$.

Observe $\text{Range}\, j = \mathbb{R}$ is not a subset of $\text{Domain}\, i = \mathbb{R}_0^+$. And so for example,

$$(ij)(-5) = i(j(-5)) = i(-5-3) = i(-8)$$

would be undefined. Hence, the composite function ij simply **does not exist**.

Again, the rest of this example is explicitly excluded from your syllabus.

Consider the restriction of j to $[3, \infty)$, i.e. the function $j|_{[3,\infty)}$.

Observe that $\text{Range}\, j|_{[3,\infty)} = \mathbb{R}_0^+$ is a subset of $\text{Domain}\, i = \mathbb{R}_0^+$.

So, we have the composite function $ij|_{[3,\infty)} : [3, \infty) \to \mathbb{R}$ defined by

$$\left(ij|_{[3,\infty)}\right)(x) = i\left(j|_{[3,\infty)}(x)\right) = j(x-3) = \sqrt{x-3}.$$

[167] Your syllabus (p. 5) states, "Exclude ... restriction of domain to obtain a composite function."

We can use a single function to build a composite function.

Example 344. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=2x$.

Observe that Range $f=\mathbb{R}$ is a subset of Domain $f=\mathbb{R}$. Hence, the composite function $ff:\mathbb{R}\to\mathbb{R}$ exists and is defined by

$$(ff)(x)=f(f(x))=f(2x)=4x.$$

So for example, $\qquad (ff)(1)=4 \qquad \text{and} \qquad (ff)(3)=12.$

The composite function ff is usually simply written as f^2. So, the above line can also be written as

$$f^2(1)=4 \qquad \text{and} \qquad f^2(3)=12.$$

Next, observe that Range $f^2=\mathbb{R}$ is also a subset of Domain $f=\mathbb{R}$. Hence, the composite function $ff^2=f^3:\mathbb{R}\to\mathbb{R}$ exists and is defined by

$$f^3(x)=ff^2(x)=f\left(f^2(x)\right)=f(4x)=8x.$$

So for example, $\qquad f^3(1)=8 \qquad \text{and} \qquad f^3(3)=24.$

We also have f^4, f^5, etc.:

$$f^4:\mathbb{R}\to\mathbb{R} \text{ is defined by } f^4(x)=16x,$$

$$f^5:\mathbb{R}\to\mathbb{R} \text{ is defined by } f^5(x)=32x,$$

$$\vdots$$

Following common practice (and also your syllabus and exams), we'll use f^2 to mean the composite function ff, f^3 to mean ff^2, f^4 to mean ff^3, etc. For future reference, let's jot this down formally:

Definition 77. Let f be a function. The symbol f^2 denotes the composite function $f\circ f$. For any integer $n\geq 3$, the symbol f^n denotes the composite function $f\circ f^{n-1}$.

Remark 50. Later on in Part V (Calculus), we'll use f', f'', f''', $f^{(4)}$, ..., $f^{(n)}$, ... to denote the "first derivative of", "second derivative of", "third derivative of", "fourth derivative of", ..., "nth derivative of", ... Do not confuse the composite function f^n with the nth derivative $f^{(n)}$.

Example 345. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 1 - x/2$.

Observe that Range $g = \mathbb{R}$ is a subset of Domain $g = \mathbb{R}$. Hence, the composite function $g^2 : \mathbb{R} \to \mathbb{R}$ exists and is defined by

$$g^2(x) = g(g(x)) = g\left(1 - \frac{x}{2}\right) = 1 - \frac{1 - x/2}{2} = \frac{1}{2} + \frac{x}{4}.$$

So for example, $$g^2(1) = \frac{3}{4} \text{ and } g^2(3) = \frac{5}{4}.$$

Next, observe that Range $g^2 = \mathbb{R}$ is also a subset of Domain $g = \mathbb{R}$. Hence, the composite function $g^3 : \mathbb{R} \to \mathbb{R}$ also exists and is defined by

$$g^3(x) = gg^2(x) = g\left(g^2(x)\right) = g\left(\frac{1}{2} + \frac{x}{4}\right) = 1 - \frac{1/2 + x/4}{2} = \frac{3}{4} - \frac{x}{8}.$$

So for example, $$g^3(1) = \frac{5}{8} \text{ and } g^3(3) = \frac{3}{8}.$$

Exercise 115 continues with this example.

Exercise 115. Continue with the above example. (Answer on p. 1778.)

(a) Let $n = 4$.

(i) Explain why the composite function g^n exists.

(ii) Write down the function g^n.

(iii) Evaluate $g^n(1)$ and $g^n(3)$.

(b) Repeat **(a)**, but now for $n = 5$.

(c) Repeat **(a)**, but now for $n = 6$.

The remainder of this question is a little harder, but is also the sort of curveball that the A-Level examiners might throw at you:

So far, we have

$$g(x) = \frac{1}{1} + \left(-\frac{1}{2}\right)x, \qquad g^2(x) = \frac{1}{2} + \left(\frac{1}{4}\right)x, \qquad g^3(x) = \frac{3}{4} + \left(-\frac{1}{8}\right)x,$$

$$g^4(x) = \frac{?}{?} + (?)\,x, \qquad g^5(x) = \frac{?}{?} + (?)\,x, \qquad g^6(x) = \frac{?}{?} + (?)\,x,$$

where the question marks in the second line can be filled in by your answers to parts **(a)**–**(c)**.

As the above suggests, it turns out that for each positive integer n, we have

$$g^n(x) = \frac{a_n}{b_n} + (c_n)\,x.$$

You are given that a general expression for a_n is

$$a_n = \frac{2^n - (-1)^n}{3}.$$

(d) Write down a general expression for b_n and c_n. Hence, write down a general expression for $g^n(x)$.

(e) Explain why as n approaches ∞, each $g^n(x)$ approaches $2/3$.

Exercise 116. For each of **(a)**–**(d)**, determine if the composite functions fg and gf exist, then compute $fg(1)$, $fg(2)$, $gf(1)$, and $gf(2)$ (if these exist). (Answer on p. 1779.)

(a) Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = \mathrm{e}^x$ and $g(x) = x^2 + 1$.

(b) Define $f, g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = 1/x$ and $g(x) = 1/(2x)$.

(c) Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = 1/x$ and $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2 + 1$.

(d) Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = 1/x$ and $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2 - 1$.

Exercise 117. For each of **(a)**–**(d)**, determine if the composite function f^2 exists. If it does, write it down and compute $f^2(1)$ and $f^2(2)$.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \mathrm{e}^x$.

(b) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x + 2$.

(c) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x^2 + 1$.

(d) Define $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \ln x$. (Answer on p. 1780.)

23. One-to-One Functions

Recall that a function is **onto** (or **surjective**) if every element in its codomain is "hit" at least once:

Example 346. XXX

Example 347. XXX

We now introduce a closely related concept: A function is **one-to-one** (or **injective**) if each element in its codomain is "hit" at most once. More precisely,

Definition 78. A function f is *one-to-one* if for all $y \in \text{Codomain}\, f$, there is at most one $x \in \text{Domain}\, f$ such that $f(x) = y$.

Equivalently and graphically,[168]

Fact 47. (Horizontal Line Test) *A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.*

Proof. See p. 1574 (Appendices). □

Example 348. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 2x$.

Observe that $f(-4) = 8$ and $f(2) = 8$—the element 8 in Codomain f is "hit" twice.[169] So, by Definition 78, f is **not** one-to-one.

Figure to be inserted here.

Equivalently, the horizontal line $y = 8$ intersects the graph of f twice.[170] So, by Fact 47, f is **not** one-to-one.

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[168] In the book *Mathematics Education in Singapore* (2019), an unnamed JC maths teacher is quoted as claiming that,

> *to 'proof'* [sic] *that a function is 1-1, one uses the horizontal line test which is incorrect.*

The above quote is false. The horizontal line test is perfectly correct.

The only possible objection is that the horizontal line test is informal as it relies on geometry. But even this objection can be dismissed because the horizontal line test can be stated and proven analytically (as is done here and in the Appendices).

[169] Indeed, every element in Codomain f that's greater than -1 is "hit" twice. (In contrast, -1 is "hit" once and every other element is not "hit" at all.)

[170] Indeed, for every $c > -1$, the horizontal line $y = c$ intersects the graph of f twice.

Example 349. Consider $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x + 1$.

Observe that every element in Codomain g is "hit" at most once.[171] So, by Definition 78, g is one-to-one.

Figure to be inserted here.

Observe also that no horizontal line intersects the graph of g more than once.[172] So, by Fact 47, g is one-to-one.

Example 350. Consider the absolute value function $|\cdot|$.

Observe that $|-2| = 2$ and $|2| = 2$—the element 2 in Codomain $|\cdot|$ is "hit" twice.[173] So, by Definition 78, $|\cdot|$ is **not** one-to-one.

Figure to be inserted here.

Equivalently, the horizontal line $y = 2$ intersects the graph of $|\cdot|$ twice.[174] So, by Fact 47, $|\cdot|$ is **not** one-to-one.

Remark 51. The name **one-to-one** is apt because at most one element in the domain "hits" each element in the codomain.

In contrast, if a function isn't one-to-one, then it is **many-to-one**—"many" (i.e. more than one) elements in the domain "hit" at least one element in the codomain.

Is it possible that a function is one-to-many?[175]

Example 351. XXX

---

[171]Indeed, every element in Codomain g is "hit" *exactly* once.

[172]Indeed, for every $c \in \mathbb{R}$, the horizontal line $y = c$ intersects the graph of g *exactly* once.

[173]Indeed, every element in Codomain $|\cdot|$ that's greater than 0 is "hit" twice. (In contrast, 0 is "hit" once and every other element is not "hit" at all.)

[174]Indeed, for every $c > 0$, the horizontal line $y = c$ intersects the graph of $|\cdot|$ twice.

[175]Nope. A one-to-many function would be one where an element in the domain "hits" "many" (i.e. more than one) elements in the codomain. But this violates our definition of a function—a function maps each element in the domain to *exactly* one element in the codomain.

Example 352. XXX

Remark 52. Just so you know, an exact synonym for **one-to-one** is **injective**. And a one-to-one (or injective) function may be called an **injection**. In this textbook, we will not use the terms *injective* or *injection*.

Exercise 118. XXX (Answer on p. 255.)

A118.

23.1. Two More Characterisations of One-to-One

The following result gives two more **characterisations**[176] of one-to-one functions:

Fact 48. *The function f is one-to-one if and only if for all $x_1, x_2 \in \operatorname{Domain} f$,*

(a) *If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.*

(b) *If $f(x_1) = f(x_2)$, then $x_1 = x_2$.*

Proof. Let P, Q, and R be, respectively, these statements:

- The function f is one-to-one.
- For all $x_1, x_2 \in \operatorname{Domain} f$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
- For all $x_1, x_2 \in \operatorname{Domain} f$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(a) The claim is "$P \iff Q$" is equivalent to "NOT$-P \iff$ NOT$-Q$", which we'll now prove:

	NOT$-P$
$\iff$	"There exists some $y \in \operatorname{Codomain} f$ such that there are more than one $x \in \operatorname{Domain} f$ (call them x_1 and x_2) with $f(x_1) = y$ and $f(x_2) = y$"
$\iff$	NOT$-Q$

(b) In **(a)**, we showed that $P \iff Q$. But by the contrapositive,[177] Q is equivalent to R. Hence, we also have $P \iff R$. □

Example 353. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 2x$.

Observe that $-4 \neq 2$, but $f(-4) = f(2)$. So, by Fact 48(a), f is **not** one-to-one.

Figure to be
inserted here.

Equivalently, $f(-4) = f(2)$, but $-4 \neq 2$. So, by Fact 48(b), f is **not** one-to-one.

[176] A **characterisation** (of a property) is a condition that's equivalent (to that property).

Here, the property is "one-to-one" and the (characterising) condition is "distinct elements are mapped to distinct elements".

Earlier the horizontal line test gave a *graphical* characterisation of one-to-one: Again, the property was "one-to-one", but the (characterising) condition there was "no horizontal line intersects the graph more than once".

[177] If you don't know what the contrapositive is, read Ch. 3.11.

Example 354. Consider the absolute value function $|\cdot|$.

Observe that $-2 \neq 2$, but $f(-2) = f(2)$. So, by Fact 48(a), $|\cdot|$ is **not** one-to-one.

Figure to be
inserted here.

Equivalently, $f(-2) = f(2)$, but $-2 \neq 2$. So, by Fact 48(b), $|\cdot|$ is **not** one-to-one.

Example 355. Consider $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x + 1$.

To prove that g is one-to-one, we can use either Fact 48(a) or (b):

1. Suppose $x_1, x_2 \in \text{Domain}\, g = \mathbb{R}$ with $x_1 \neq x_2$. Then $2x_1 \neq 2x_2$ or $2x_1 + 1 \neq 2x_2 + 1$ or $g(x_1) \neq g(x_2)$. So, by Fact 48(a), g is one-to-one.
2. Suppose $g(x_1) = g(x_2)$. Then $2x_1 + 1 = 2x_2 + 1$ or $2x_1 = 2x_2$ or $x_1 = x_2$. So, by Fact 48(b), g is one-to-one.

Figure to be
inserted here.

Example 356. Consider $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^3$.

To prove that h is one-to-one, we can use either Fact 48(a) or (b):

1. Suppose $x_1, x_2 \in \text{Domain}\, h = \mathbb{R}$ with $x_1 \neq x_2$. Then $x_1^3 \neq x_2^3$ or $h(x_1) \neq h(x_2)$. So, by Fact 48(a), h is one-to-one.
2. Suppose $h(x_1) = h(x_2)$. Then $x_1^3 = x_2^3$ or $x_1 = x_2$. So, by Fact 48(b), h is one-to-one.

Figure to be
inserted here.

Example 357. XXX

Example 358. XXX

Exercise 119. XXX (Answer on p. 258.)

A119.

23.2. A Strictly Increasing/Decreasing Function Is One-to-One

Fact 49. *If a function is strictly increasing (or strictly decreasing), then it is also one-to-one.*

Proof. Let f be a function.

Suppose f is strictly increasing (or strictly decreasing).

Pick any distinct $a, b \in \text{Domain } f$ with $a < b$.

Then $f(a) < f(b)$ (or $f(a) > f(b)$)—and, in particular, $f(a) \neq f(b)$.

Hence, by Fact 48(a), f is one-to-one. □

Example 359. XXX

Example 360. XXX

Unfortunately, the converse of Fact 49 is false:

Example 361. XXX

Happily, Fact 49 has this partial converse. This converse is only *partial* because it adds two assumptions: The function is **(a)** continuous; and **(b)** defined on an interval (i.e. its domain is an interval).

Proposition 4. *Suppose a nice function is continuous and its domain is an interval. If this function is one-to-one, then it is also either strictly increasing or strictly decreasing.*

Proof. See p. 1575 (Appendices). □

Combining Fact 49 and Proposition 4, we have

Corollary 7. *Suppose a nice function f is continuous and its domain is an interval. Then*

$$f \textit{ is one-to-one} \quad \Longleftrightarrow \quad f \textit{ is strictly increasing or strictly decreasing.}$$

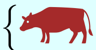
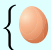
Example 362. XXX

Example 363. XXX

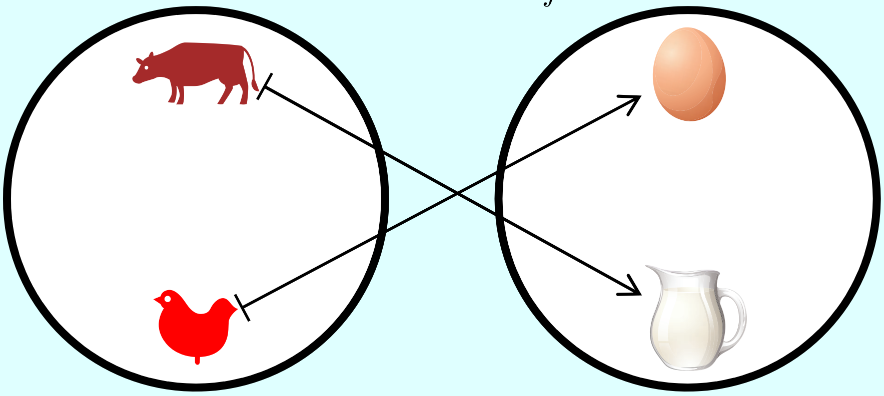
Exercise 120. XXX (Answer on p. 259.)

A120.

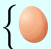
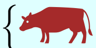
24. Inverse Functions

Example 364. Consider the function $f : \{$, $\} \to \{$, $\}$ defined by "match the animal to its role":

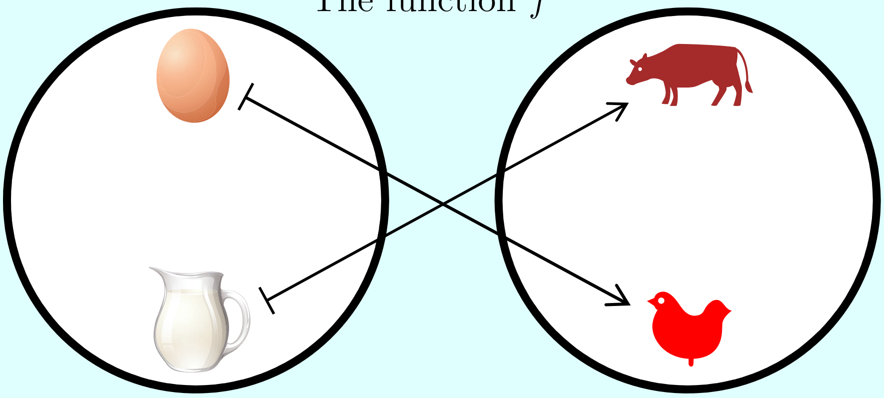
The function f



Domain f Codomain f

Its **inverse function** (or more simply **inverse**) is the function $f^{-1} : \{$, $\} \to \{$, $\}$, defined by "match the role to the corresponding animal":

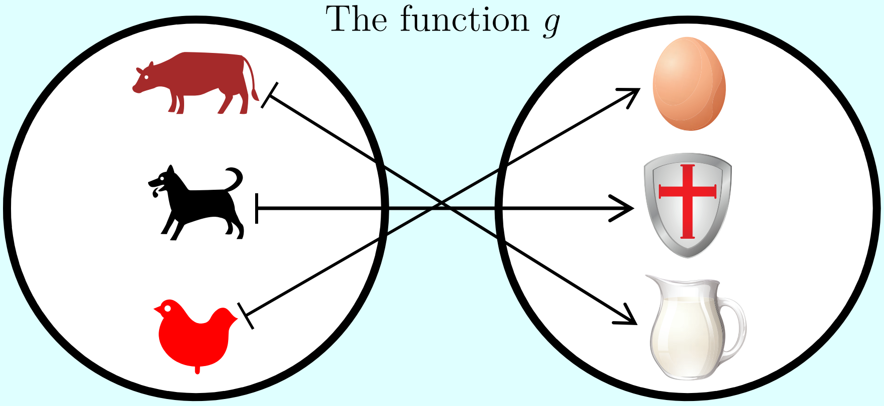
The function f^{-1}



Domain f^{-1} Codomain f^{-1}

Example 365. Consider the function g : {  } $\to$ {  } defined by "match the animal to its role":

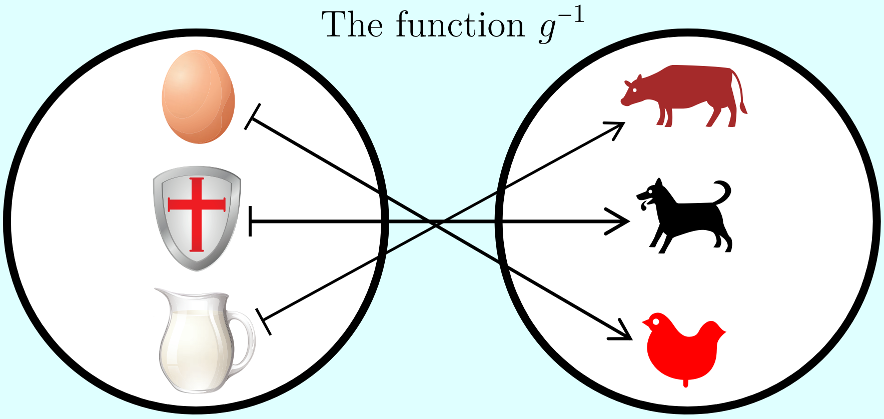
The function g



Domain g Codomain g

Its inverse is the function g^{-1} : {  } $\to$ {  }, defined by "match the role to the corresponding animal":

The function g^{-1}



Domain g^{-1} Codomain g^{-1}

Example 366. Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x + 2$:

Figure to be
inserted here.

Its inverse is the function $h^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $h^{-1}(y) = y - 2$:

Figure to be
inserted here.

Remark 53. In the above example, we use the symbols x and y as our dummy variables for the functions h and h^{-1}, respectively. As usual, these being dummy variables, the symbols x and y can be replaced by any other variable. For example, all of the following would also have been perfectly correct:

1. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(☺) = ☺ + 2$ and $h^{-1} : \mathbb{R} \to \mathbb{R}$ by $h^{-1}(\square) = \square - 2$.
2. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(\star) = \star + 2$ and $h^{-1} : \mathbb{R} \to \mathbb{R}$ by $h^{-1}(☾) = ☾ - 2$.
3. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x + 2$ and $h^{-1} : \mathbb{R} \to \mathbb{R}$ by $h^{-1}(x) = x - 2$.
4. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(y) = y + 2$ and $h^{-1} : \mathbb{R} \to \mathbb{R}$ by $h^{-1}(y) = y - 2$.

In #3, we use the same symbol x twice as the dummy variable for both h and h^{-1}. There is nothing wrong with this. (Indeed, we've always been doing this—e.g., "Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x$ and $g(x) = 5x$.")

Ditto for #4, where we use the same symbol y twice as the dummy variable for both h and h^{-1}.

Example 367. Consider the function $i : [0,1] \to [0,2]$ defined by $i(x) = 2x$:

Figure to be
inserted here.

Its inverse is the function $i^{-1} : [0,2] \to [0,1]$ defined by $i^{-1}(y) = y/2$:

Figure to be
inserted here.

We'll formally define the inverse function only on p. 266. But for now, try this exercise:

Exercise 121. XXX (Answer on p. 263.)

A121.

24.1. Only a One-to-One Function Can Have an Inverse

If a function isn't one-to-one, then it does not have an inverse. Let's see why:

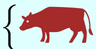
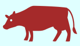
Example 368. Consider the function f : {, , } $\to$ {Yum, Yuck} defined by f () = Yum, f () = Yuck, and f () = Yum:

Figure to be
inserted here.

The element Yum in the codomain is "hit" twice. So, f is not one-to-one.

Let's see what goes wrong if we try to find f^{-1}, the inverse of f:

Figure to be
inserted here.

The "function" f^{-1} is not well-defined because an element in the domain, Yum, is mapped to *two* distinct elements in the codomain.

We may simply say, "f has no inverse," or "the inverse of f does not exist," or "the inverse of f is undefined".

Example 369. Define $g : \{0, 1, 2\} \to \{3, 4\}$ by $g(0) = 4$, $g(1) = 4$, and $g(2) = 3$.

Figure to be
inserted here.

The element 4 in the codomain is "hit" twice. So, g is not one-to-one.

Let's see what goes wrong when we try to find g^{-1}, the inverse of g:

Figure to be
inserted here.

The "function" g^{-1} is not well-defined because an element in the domain, 4, is mapped to two distinct elements in the codomain.

We may simply say, "g has no inverse," or "the inverse of g does not exist," or "the inverse of g is undefined".

We're now ready to give our formal definition of the inverse:

24.2. Formal Definition of the Inverse

Definition 79. Given a one-to-one function f, its *inverse function* (or *inverse*), denoted f^{-1}, is the function with[178]

$$\begin{aligned} \text{Domain:} \quad & \text{Range } f. \\ \text{Codomain:} \quad & \text{Domain } f. \\ \text{Mapping rule:} \quad & y = f(x) \implies f^{-1}(y) = x. \end{aligned}$$

Example 370. XXX

Example 371. XXX

Remark 54. To have an inverse, a function must be one-to-one (every element in its codomain is "hit" at most once), but need not be onto (every element in its codomain is "hit" at least once).[179]

Nonetheless, just to keep things simple, we'll assume that you'll only ever be asked to find the inverse of a function that is also onto.[180] This ensures that the inverse's domain is simply the function's codomain. (Why?)[181]

Again, if a function is not one-to-one, then we need not bother finding its non-existent inverse:

Example 372. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ is not one-to-one.

So, f does not have an inverse. (Or, "the inverse of f does not exist or is undefined.")

Example 373. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is not one-to-one.

So, g does not have an inverse. (Or, "the inverse of g does not exist or is undefined.")

Exercise 122. XXX (Answer on p. 266.)

A122.

[178] Actually, Definition 79 is not quite the standard definition of an inverse function. It turns out under the standard definition, to have an inverse, a function must also be onto. We discuss this in Ch. 142.15 (Appendices).

But this isn't something you need worry about. For A-Level Maths, Definition 79 will be perfectly good and will be used everywhere in this textbook *except* Ch. 142.15 (Appendices).

[179] But see n. 178.

[180] In your A-Level exams, no distinction is ever made between the codomain and the range—the implicit assumption seems to be that they are the same ("usually" or "whenever we need them to be"), so that any function is onto ("usually" or "whenever we need them to be").

[181] Let f be a one-to-one function. By Definition 79, Domain f = Range f. If f is also onto, then Range f = Codomain f, so that Domain f = Codomain f.

24.3. A Method for Showing a Function Is One-to-One *and* Finding Its Inverse

Let $f : A \to B$ be a function with range C. Suppose we know f is one-to-one and its inverse is $f^{-1} : C \to A$ with this mapping rule:

$$y = f(x) \qquad \Longrightarrow \qquad f^{-1}(y) = x.$$

Now, suppose instead we do *not* know whether f is one-to-one. But, we're able to find some function $g : C \to A$ with this mapping rule:

$$y = f(x) \qquad \Longrightarrow \qquad g(y) = x.$$

The function g looks very much like it must be the inverse of f. And so, we ask two questions:

1. Is f even one-to-one?
2. Is g the inverse of f?

Happily, the answer to both of the above questions is affirmative:

Fact 50. *Suppose $f : A \to B$ is a function with range C. Then f is one-to-one if and only if there exists a function $g : C \to A$ where for every $y \in C$, we have*

$$y = f(x) \qquad \Longrightarrow \qquad g(y) = x.$$

(Moreover, if g exists, then by Definition 79, it is the inverse of f.)

Proof. See p. 1576 (Appendices). □

As the following examples show, Fact 50 provides a method for **(a)** showing a function is one-to-one; and **(b)** finding its inverse at the same time. Actually, this method will allow us to skip the work that previously went into **(a)** showing a function is one-to-one.

Example 374. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$.

Let $y \in \text{Range}\, f = \mathbb{R}$.

Since $y \in \text{Range}\, f$, there must exist some $x \in \text{Domain}\, f = \mathbb{R}$ such that $y = f(x)$ or $y = x + 1$.

Do the algebra to get $x = y - 1$.

So, let's construct the function $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(y) = y - 1$.

Observe that for each $y \in \text{Range}\, f = \mathbb{R}$, we do indeed have

$$y = f(x) = x + 1 \qquad \Longrightarrow \qquad f^{-1}(y) = x = y - 1.$$

Hence, by Fact 50, f is one-to-one and its inverse is f^{-1}.

Example 375. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$.

Let $y \in \operatorname{Range} g = \mathbb{R}$.

Since $y \in \operatorname{Range} g$, there must exist some $x \in \operatorname{Domain} g = \mathbb{R}$ such that $y = g(x)$ or $y = 2x$.

Do the algebra to get $x = y/2$.

So, let's construct the function $g^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $g^{-1}(y) = y/2$.

Observe that for each $y \in \operatorname{Range} g = \mathbb{R}$, we do indeed have

$$y = g(x) = 2x \qquad \Longrightarrow \qquad g^{-1}(y) = x = y/2.$$

Hence, by Fact 50, g is one-to-one and its inverse is g^{-1}.

In the above two examples, we were very careful in spelling out each step of the argument. Going forward, we'll be a little lazier and distill the argument into this Three-Step Method:

Three-Step Method to Show a Function Is One-to-One and Find Its Inverse

Consider the function $f : A \to B$ defined by $x \mapsto f(x)$ and with range C.

1. Write $y = f(x)$.
2. Do the algebra to get $x = f^{-1}(y)$.
3. Conclude: f is one-to-one and its inverse is the function $f^{-1} : C \to A$ defined by $y \mapsto f^{-1}(y)$.

Let's redo the above two examples:

Example 376. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$.

1. Write $y = f(x) = x + 1$.
2. Do the algebra: $x = y - 1$.
3. Conclude: f is one-to-one and its inverse is the function $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(y) = y - 1$.

Example 377. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$.

1. Write $y = g(x) = 2x$.
2. Do the algebra: $x = y/2$.
3. Conclude: g is one-to-one and its inverse is the function $g^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $g^{-1}(y) = y/2$.

More examples:

Example 378. Define $h : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ by $h(x) = 1/x$.

1. Write $y = h(x) = 1/x$.
2. Do the algebra: $x = 1/y$.
3. Conclude: h is one-to-one and its inverse is the function $h^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by $h^{-1}(y) = 1/y$.

By the way, observe that $h = h^{-1}$. So, h is an example of a **self-inverse function** (more on this in Ch. 24.7).

Example 379. Define $i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by $i(x) = x^2$.

1. Write $y = i(x) = x^2$.
2. Do the algebra: $x = \pm\sqrt{y}$.

   Since Domain $i = \mathbb{R}_0^+$, we have $x \geq 0$. So, we discard the negative value and are left with $x = \sqrt{y}$.
3. Conclude: i is one-to-one and its inverse is the function $i^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $g^{-1}(y) = \sqrt{y}$.

Example 380. Define $j : \mathbb{R}_0^- \to \mathbb{R}_0^+$ by $j(x) = x^2$.

1. Write $y = j(x) = x^2$.
2. Do the algebra: $x = \pm\sqrt{y}$.

   Since Domain $j = \mathbb{R}_0^-$, we have $x \leq 0$. So, we discard the positive value and are left with $x = -\sqrt{y}$.
3. Conclude: j is one-to-one and its inverse is the function $j^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^-$ defined by $j^{-1}(y) = -\sqrt{y}$.

Example 381. Define $k : \mathbb{R}^+ \to \mathbb{R}^+$ by $k(x) = 1/x^2$.

1. Write $y = k(x) = 1/x^2$.
2. Do the algebra: $x = \pm 1/\sqrt{y}$.

   Since Domain $k = \mathbb{R}^+$, we have $x > 0$. So, we discard the negative value and are left with $x = 1/\sqrt{y}$.
3. Conclude: k is one-to-one and its inverse is $k^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $k^{-1}(y) = 1/\sqrt{y}$.

Example 382. Define $l : \mathbb{R}^- \to \mathbb{R}^+$ by $l(x) = 1/x^2$.

1. Write $y = l(x) = 1/x^2$.
2. Do the algebra: $x = \pm 1/\sqrt{y}$.

   Since Domain $l = \mathbb{R}^-$, we have $x < 0$. So, we discard the positive value and are left with $x = -1/\sqrt{y}$.
3. Conclude: l is one-to-one and its inverse is $l^{-1} : \mathbb{R}^+ \to \mathbb{R}^-$ defined by $l^{-1}(y) = -1/\sqrt{y}$.

Exercise 123. Show that each function is one-to-one and find its inverse.

(a) $a: \mathbb{R} \to \mathbb{R}$ defined by $a(x) = 5x$.
(b) $b: \mathbb{R} \to \mathbb{R}$ defined by $b(x) = x^3$.
(c) $c: \mathbb{R}_0^+ \to [1, \infty)$ defined by $c(x) = x^2 + 1$.
(d) $d: \mathbb{R}_0^- \to [1, \infty)$ defined by $d(x) = x^2 + 1$.
(e) $e: \mathbb{R}_0^+ \to (0, 1]$ defined by $e(x) = 1/\left(x^2 + 1\right)$.
(f) $f: \mathbb{R}_0^- \to (0, 1]$ defined by $f(x) = 1/\left(x^2 + 1\right)$. (Answer on p. 1782.)

Exercise 124. Is each function one-to-one? If it is, find its inverse.

(a) $a: \mathbb{R} \to \mathbb{R}$ defined by $a(x) = x^2 - 1$.
(b) $b: \mathbb{R} \to [-1, \infty)$ defined by $b(x) = x^2 - 1$.
(c) $c: \mathbb{R}_0^+ \to [-1, \infty)$ defined by $c(x) = x^2 - 1$.
(d) $d: \mathbb{R}_0^- \to [-1, \infty)$ defined by $d(x) = x^2 - 1$. (Answer on p. 1782.)

24.4. Inverse Cancellation Laws

Fact 51. *Suppose the function f has inverse f^{-1}. Then*

$$y = f(x) \qquad \Longleftrightarrow \qquad f^{-1}(y) = x.$$

Proof. ($\Longrightarrow$) By Definition 79 (of inverses).
($\Longleftarrow$) See p. 1576 (Appendices). □

Proposition 5. (Inverse Cancellation Laws) *If the function f has inverse f^{-1}, then*

(a) $f^{-1}(f(x)) = x$ *for all* $x \in \text{Domain } f$; *and*

(b) $f\left(f^{-1}(y)\right) = y$ *for all* $y \in \text{Domain } f$.

Proof. **(a)** Let $x \in \text{Domain } f$, $y = f(x)$. By Fact 51 ($\Longrightarrow$), $f^{-1}(y) = x$ or $f^{-1}(f(x)) = x$.
(b) Let $y \in \text{Domain } f$, $x = f^{-1}(y)$. By Fact 51 ($\Longleftarrow$), $f(x) = y$ or $f\left(f^{-1}(y)\right) = y$. □

Example 383. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$.
The inverse of f is the function $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(y) = y - 1$.
And so, by the Cancellation Laws for Inverses (Proposition 5), we have

$$f^{-1}(f(x)) = x, \quad \text{for all } x \in \text{Domain } f = \mathbb{R},$$

$$f\left(f^{-1}(x)\right) = x, \quad \text{for all } x \in \text{Domain } f = \mathbb{R}.$$

For example, $f^{-1}(f(3)) = f^{-1}(4) = 3$ and $f\left(f^{-1}(1)\right) = f(0) = 1$.

Exercise 125. XXX (Answer on p. 271.)

A125.

24.5. The Graphs of f and f^{-1} Are Reflections in the Line $y = x$

We reproduce Fact 51 from Ch. 24.4:

Fact 51. *Suppose the function f has inverse f^{-1}. Then*

$$y = f(x) \qquad \Longleftrightarrow \qquad f^{-1}(y) = x.$$

Fact 51 yields this corollary:

Corollary 8. *Suppose the function f has inverse f^{-1}. Then*

$$(x, y) \text{ is in the graph of } f \qquad \Longleftrightarrow \qquad (y, x) \text{ is in the graph of } f^{-1}.$$

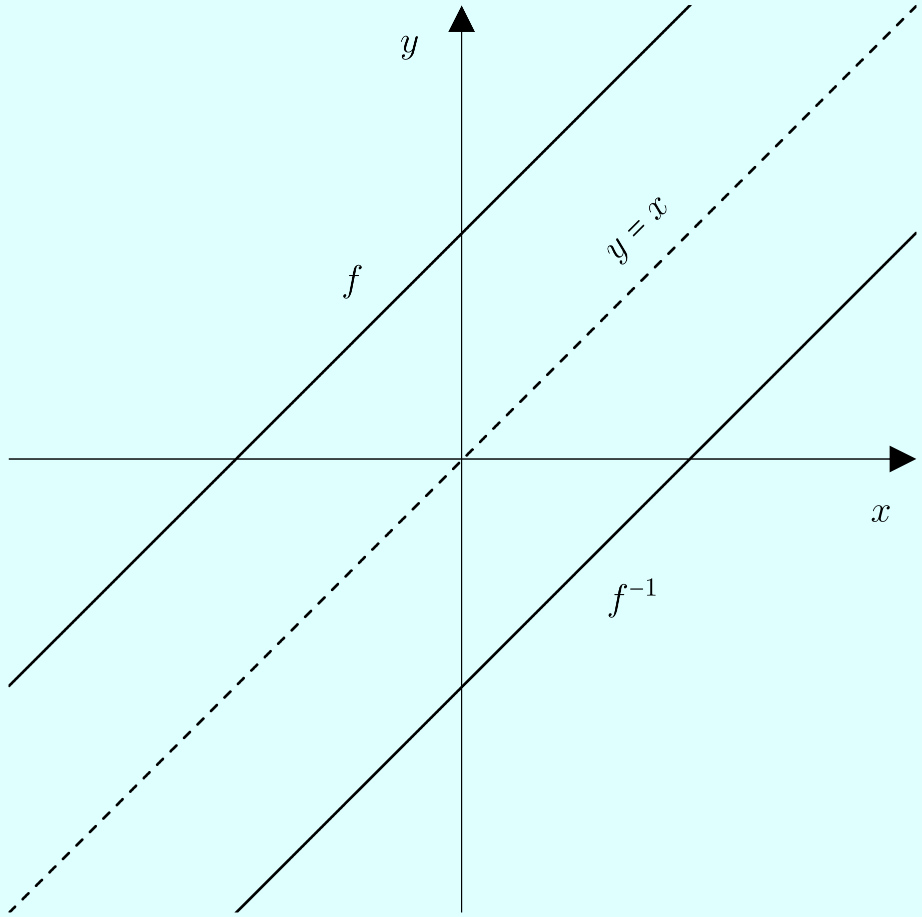
Hence, by Fact 37, the graphs of f and f^{-1} are reflections in the line $y = x$:

Fact 52. *Suppose the nice function f has inverse f^{-1}. Then the graphs of f and f^{-1} are reflections of each other in the line $y = x$.*

Example 384. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$.

The inverse of f is the function $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(x) = x - 1$.

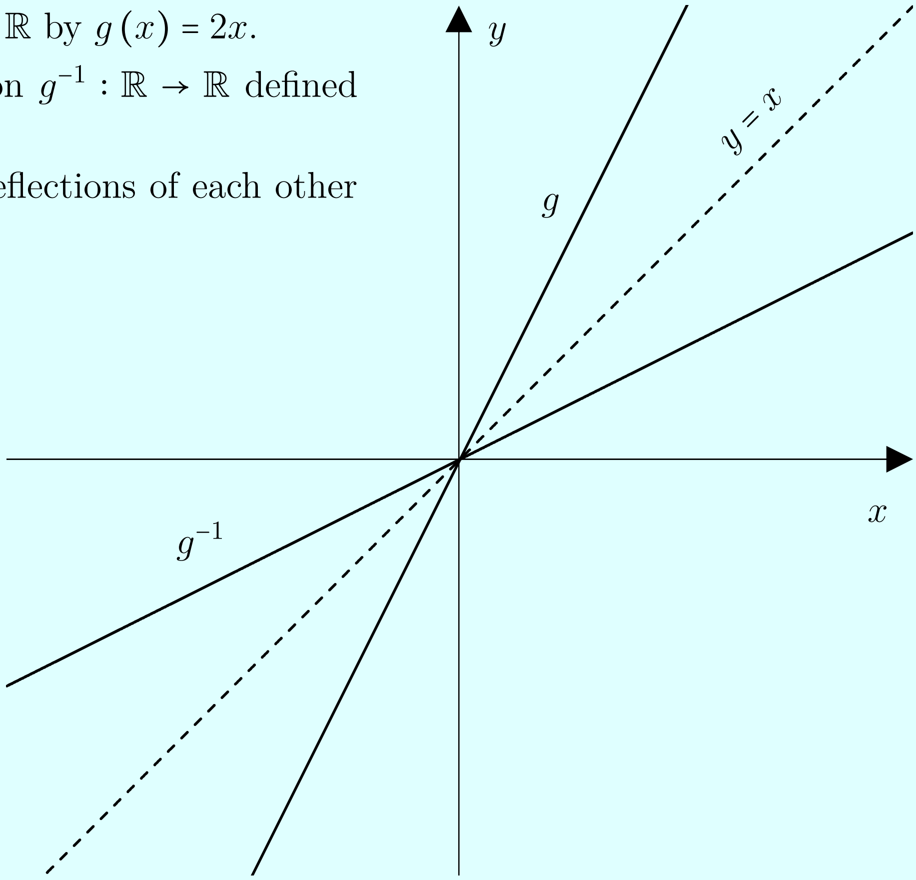
The graphs of f and f^{-1} are reflections of each other in the line $y = x$:



Example 385. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$.

The inverse of g is the function $g^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $g^{-1}(x) = x/2$.

The graphs of g and g^{-1} are reflections of each other in the line $y = x$:

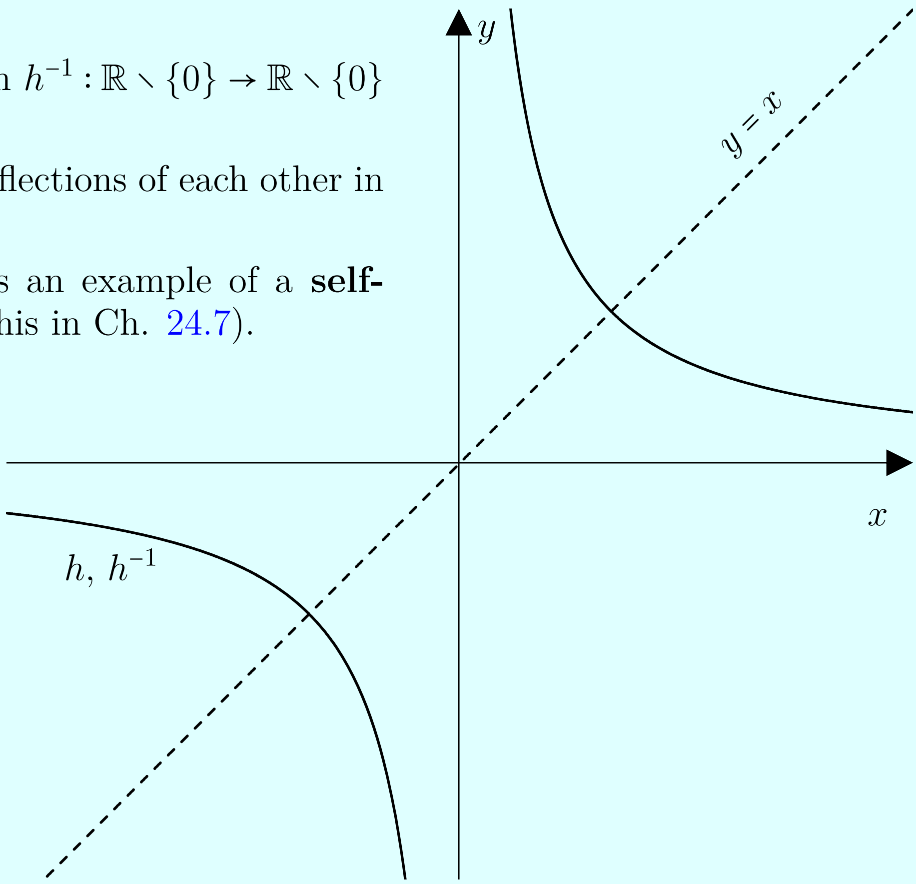


Example 386. Define $h : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ by $h(x) = 1/x$.

The inverse of h is the function $h^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by $h^{-1}(x) = 1/x$.

The graphs of h and h^{-1} are reflections of each other in the line $y = x$:

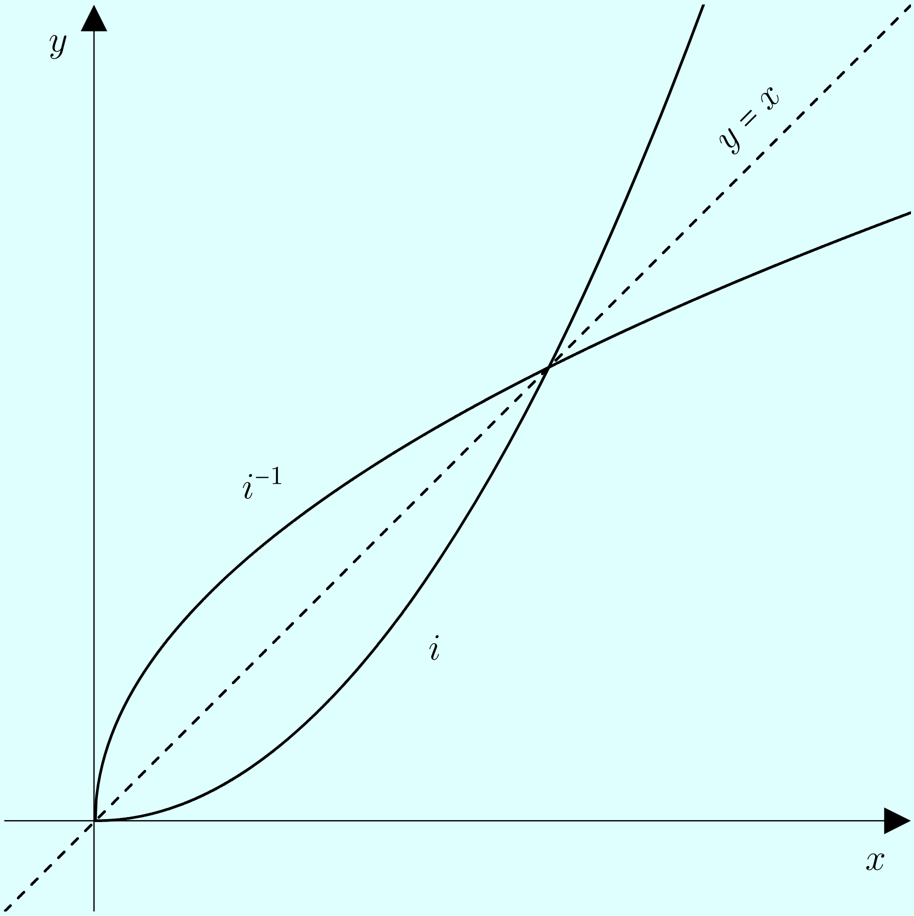
Indeed, $h = h^{-1}$. (Again, h is an example of a **self-inverse function**—more on this in Ch. 24.7).



Example 387. Define $i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by $i(x) = x^2$.

The inverse of i is the function $i^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $i^{-1}(x) = \sqrt{x}$.

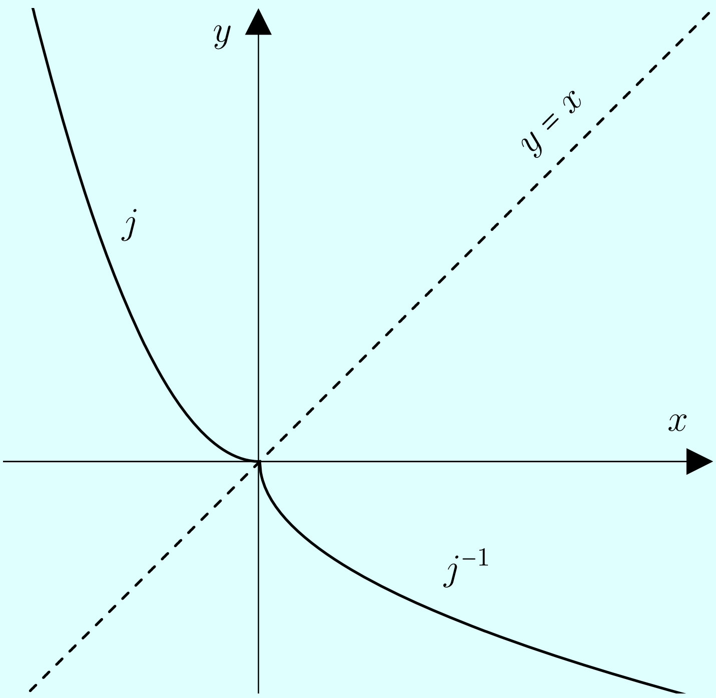
The graphs of i and i^{-1} are reflections of each other in the line $y = x$:



Example 388. Define $j : \mathbb{R}_0^- \to \mathbb{R}_0^+$ by $j(x) = x^2$.

The inverse of j is the function $j^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^-$ defined by $j^{-1}(x) = -\sqrt{x}$.

The graphs of j and j^{-1} are reflections of each other in the line $y = x$:

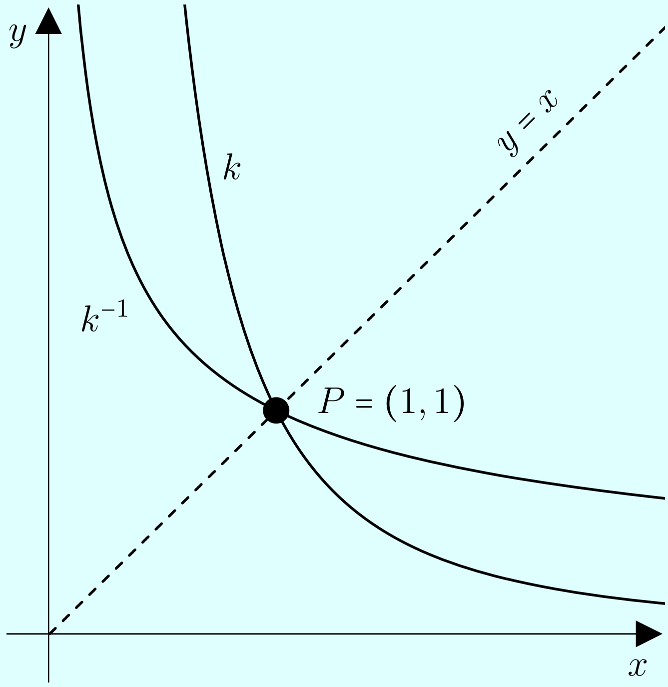


Example 389. Define $k : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$k(x) = 1/x^2.$$

Its inverse is the function $k^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $k^{-1}(x) = 1/\sqrt{x}$.

The graphs of k and k^{-1} are reflections of each other in the line $y = x$:



Some observations to help graph k and k^{-1}:

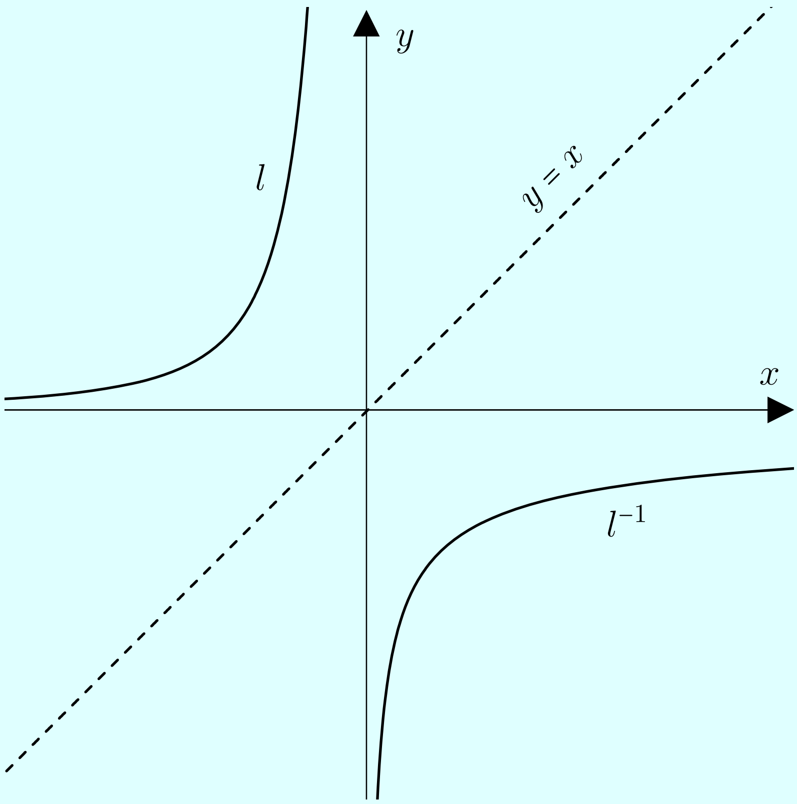
- Since $k(1) = 1/1^2 = 1$ and $k^{-1}(1) = 1/\sqrt{1} = 1$, both intersect the line $y = x$ at $P = (1, 1)$.
- For $x < 1$ (left of P), k is above[182] k^{-1}.
- For $x > 1$ (right of P), k is below[183] k^{-1}.

Example 390. Define $l : \mathbb{R}^- \to \mathbb{R}^+$ by

$$l(x) = 1/x^2.$$

Its inverse is the function $l^{-1} : \mathbb{R}^+ \to \mathbb{R}^-$ defined by $l^{-1}(x) = -1/\sqrt{x}$.

The graphs of l and l^{-1} are reflections of each other in the line $y = x$:



Sometimes, we may know a function is one-to-one and so has an inverse; however, we may be unable to find the mapping rule of the inverse. In such cases, Fact 52 is very useful:

[182]E.g. $k\left(\frac{1}{4}\right) = \frac{1}{(1/4)^2} = \frac{1}{1/16} = 16 > k^{-1}\left(\frac{1}{4}\right) = \frac{1}{\sqrt{1/4}} = \frac{1}{1/2} = 2.$

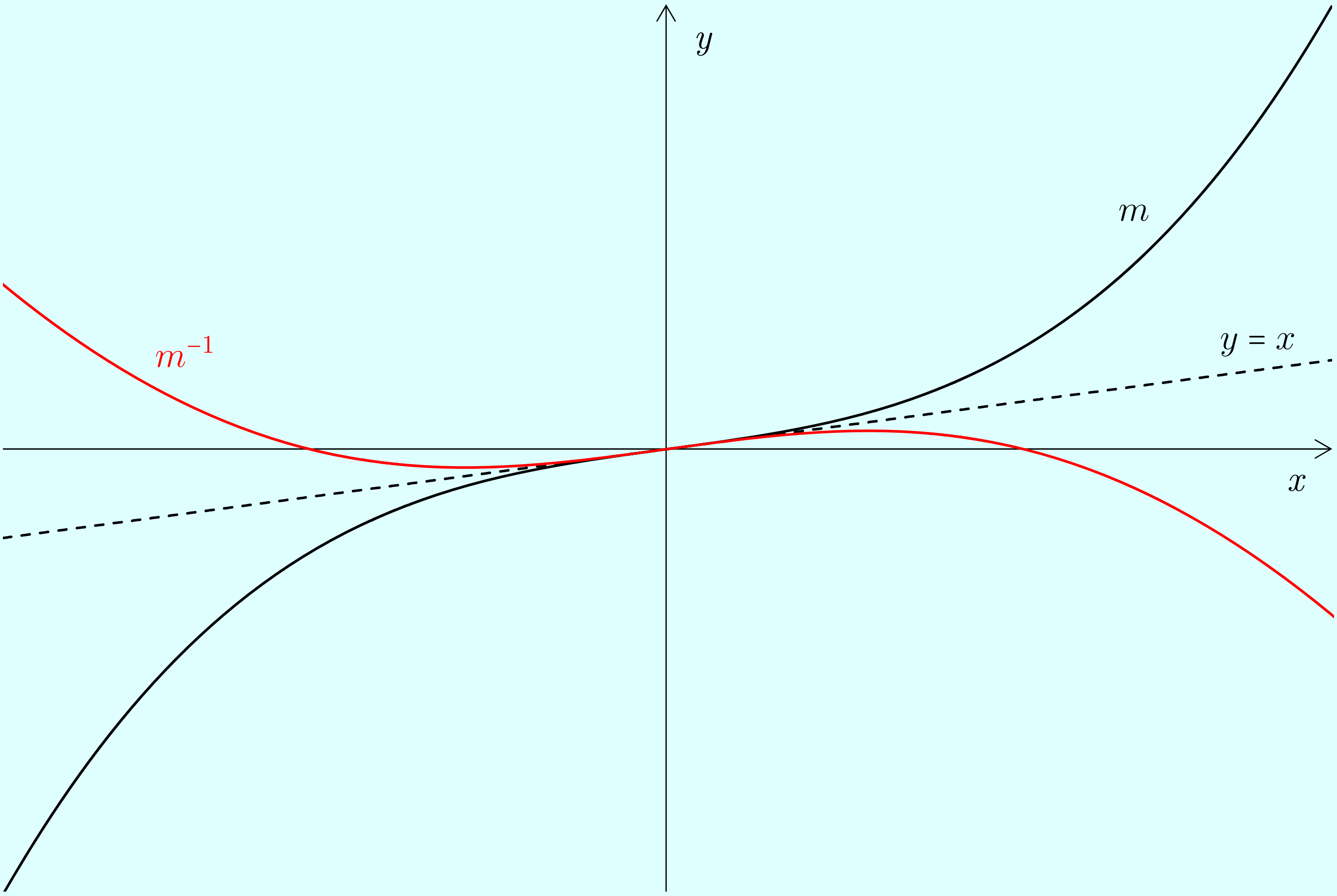
[183]E.g. $k(4) = 1/4^2 = 1/16 < k^{-1}(4) = 1/\sqrt{4} = 1/2.$

Example 391. Define $m : \mathbb{R} \to \mathbb{R}$ by $m(x) = x^3 + x$.

Since m is strictly increasing (can you show this?),[184] it is one-to-one and has an inverse $m^{-1} : \mathbb{R} \to \mathbb{R}$

Unfortunately, not having learnt to solve cubic equations, we don't know how to find $m^{-1}(x)$.

Nonetheless, even though we have no idea what $m^{-1}(x)$ is, if we already have the graph of m, we can use Fact 52 to sketch the graph of m^{-1}:



In the above example, it's actually possible to find the mapping rule of the inverse:

$$m^{-1}(x) = \sqrt[3]{\frac{x}{2} + \sqrt{\frac{x^2}{4} + \frac{1}{27}}} + \sqrt[3]{\frac{x}{2} - \sqrt{\frac{x^2}{4} + \frac{1}{27}}}.$$

We don't know how to show the above, because we haven't learnt to solve cubic equations.

In the next example, it's *impossible*. But again, we can use Fact 52 to graph the inverse.

[184] *Method 1.* Suppose $x_1 < x_2$. In n. 162, we already proved that $x_1^3 - x_2^3 < 0$, so that $x_1^3 < x_2^3$, $x_1^3 + x_1 < x_2^3 + x_2$ and m is indeed strictly increasing.

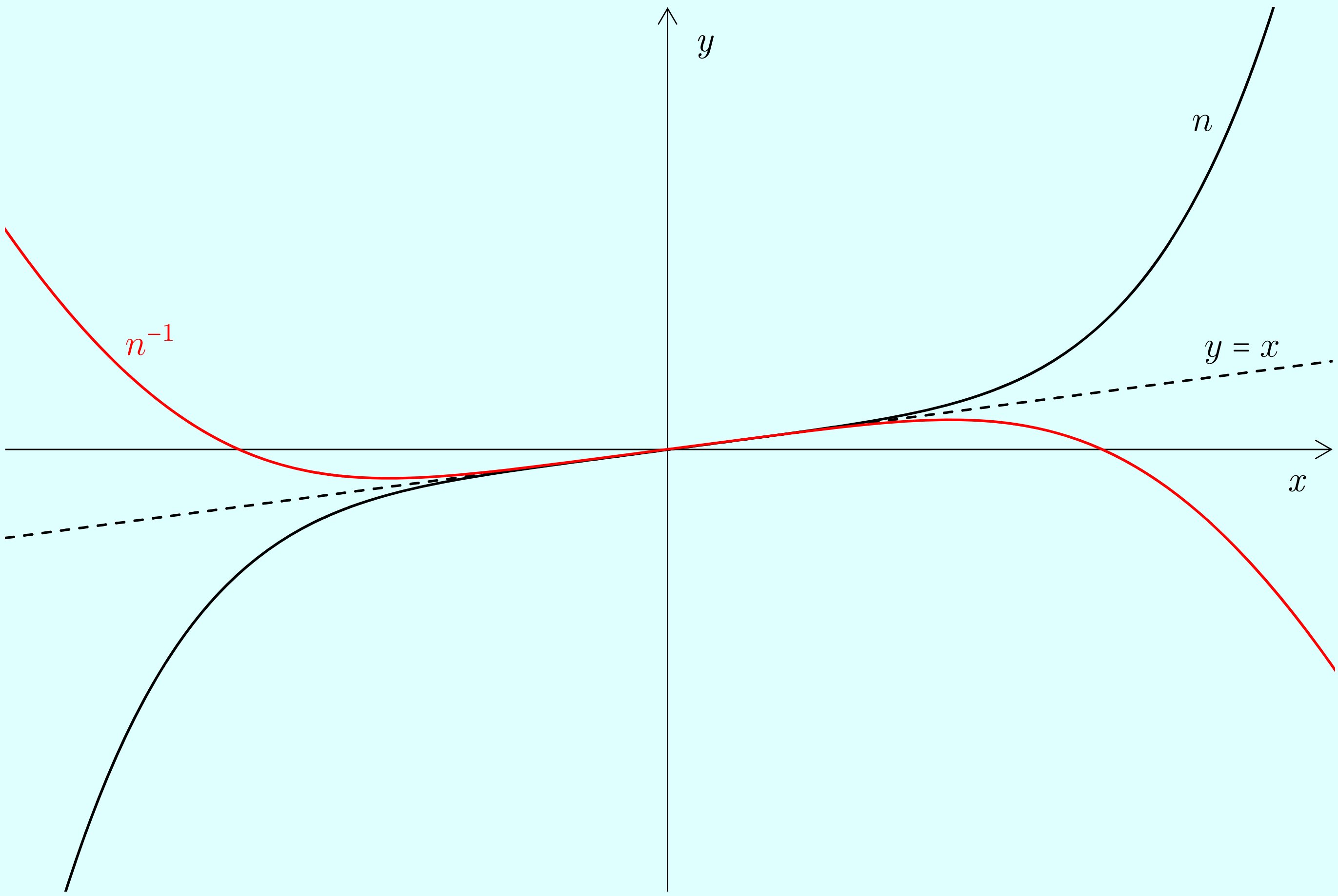
Method 2 (calculus). Since $m'(x) = 3x^2 + 1 > 0$ for all $x \in \mathbb{R}$, m is strictly increasing on $\mathbb{R}$. (We'll learn more about this method in Ch. 43 and Part V).

Example 392. Define $n : \mathbb{R} \to \mathbb{R}$ defined by $n(x) = x^5 + x$.

We can show that n is strictly increasing.[185] Hence, n is one-to-one and has an inverse $n^{-1} : \mathbb{R} \to \mathbb{R}$.

Unfortunately, it is *impossible* to write down an algebraic expression for $n^{-1}(x)$.[186]

Nonetheless, even though we have no idea what $n^{-1}(x)$ is, if we already have the graph of n, we can use Fact 52 to sketch the graph of n^{-1}:



[185] *Method 1.* Suppose $x_1 < x_2$. Observe that $x_1^5 - x_2^5 = (x_1 - x_2)\left(x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4\right)$. We shall prove that $x_1^5 - x_2^5 < 0$, so that $x_1^5 < x_2^5$, $x_1^5 + x_1 < x_2^5 + x_2$ and n is indeed strictly increasing.
Since $x_1 < x_2$, $x_1 - x_2 < 0$. Next,

$$x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4 = \frac{1}{2}\left(2x_1^4 + 2x_1^3x_2 + 2x_1^2x_2^2 + 2x_1x_2^3 + 2x_2^4\right) =$$
$$\frac{1}{2}\left[\left(x_1^2 + x_2^2\right)^2 + x_1^4 + x_2^4 + 2x_1x_2\left(x_1^2 + x_2^2\right)\right].$$

If x_1 and x_2 have opposite signs, then $x_1 < 0 < x_2$, so that $x_1^5 < 0 < x_2^5$. So, assume x_1 and x_2 do not have opposite signs. Then each of the four terms in brackets is non-negative and at least one of x_1^4 or x_2^4 is strictly positive. Hence, $x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4 > 0$. Thus, $x_1^5 - x_2^5 < 0$.
Method 2 (calculus). Since $n'(x) = 5x^4 + 1 > 0$ for all $x \in \mathbb{R}$, n is strictly increasing on $\mathbb{R}$. (We'll learn more about this method in Ch. 43 and Part V).

[186] We can find algebraic expressions for the roots of polynomial equations of degree 2 (quadratic formula), 3 (not taught in A-Level Maths), and 4 (ditto). However, it is *impossible* to do the same for polynomial equations of degree 5 or higher—this is **Abel's Impossibility Theorem**.

Exercise 126. Find each function's inverse, then graph both the function and its inverse.

(a) $f:(0,1] \to (1,2]$ defined by $f(x) = x+1$.

(b) $g:(0,1] \to (0,2]$ defined by $g(x) = 2x$.

(c) $h:(0,1] \to [1,\infty)$ defined by $h(x) = 1/x$.

(d) $i:(0,1] \to (0,1]$ defined by $i(x) = x^2$. (Answers on p. 1783.)

24.6. When Do the Graphs of f and f^{-1} Intersect?

Let f be a nice, one-to-one function. By Fact 52, f and f^{-1} are reflections of each other in the line $y = x$.

So, if f intersects the line $y = x$ at some point P, then f^{-1} must also intersect $y = x$ at P.

Hence, any point at which f intersects $y = x$ is also a point at which f and f^{-1} intersect:

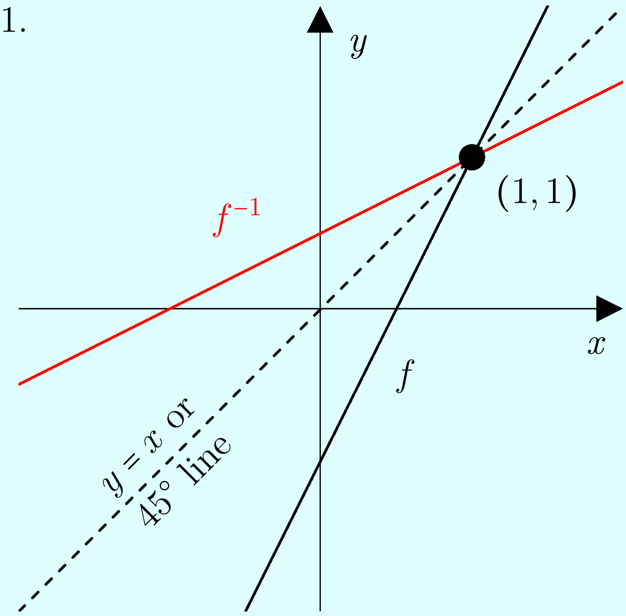
Example 393. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x - 1$.

Its inverse is $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by

$$f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}.$$

The graph of f intersects the line $y = x$ at $(1, 1)$.

So, f and f^{-1} also intersect at $(1, 1)$.



Example 394. Define $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by

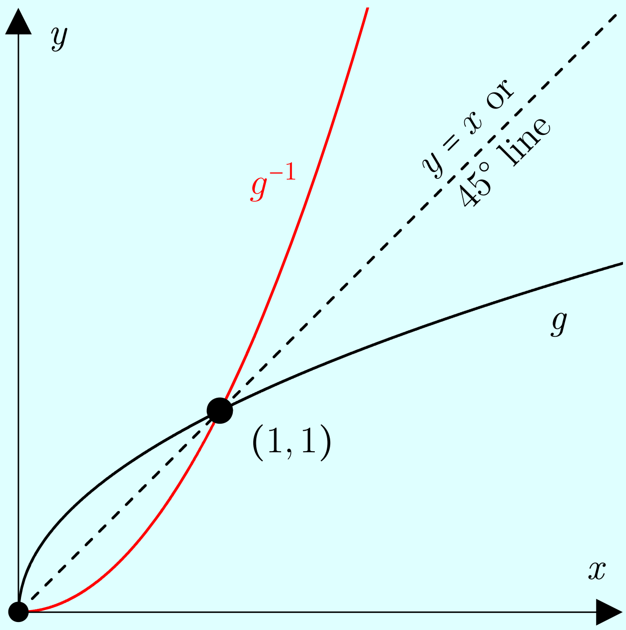
$$g(x) = x^2.$$

Its inverse is $g^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by

$$g^{-1}(x) = \sqrt{x}.$$

The graph of f intersects the line $y = x$ at $(0, 0)$ and $(1, 1)$.

So, g and g^{-1} also intersect at $(0, 0)$ and $(1, 1)$.



For future reference, let's jot down the above observation:

Fact 53. *Let f be a one-to-one function with inverse f^{-1}. If f intersects the line $y = x$ at some point P, then f also intersects f^{-1} at P.*

Or, Any point at which f intersects $y = x$ is also a point at which f intersects f^{-1}. Or,

$$f(a) = a \qquad \Longrightarrow \qquad f(a) = f^{-1}(a).$$

Now, consider the converse of Fact 53. That is, consider this next statement:

"If f intersects f^{-1} at some point P, then f also intersects the line $y = x$ at P." ✗

Or, "Any point at which f intersects f^{-1} is also a point at which f intersects $y = x$." ✗

Or, "$f(a) = f^{-1}(a) \implies f(a) = a$." ✗

The above statement sounds perfectly plausible. But unfortunately, it is false. And even more unfortunately, those writing your A-Level exams incorrectly assumed it to be true in N2008-II-4 (iv), (Exercise 563).[187]

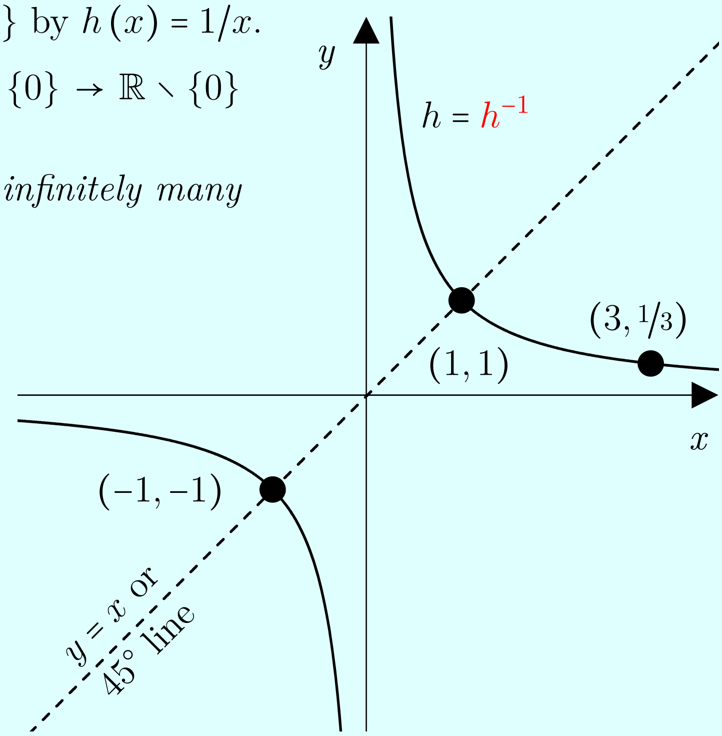
Two counterexamples to the above false statement:

Example 395. Define $h : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ by $h(x) = 1/x$.

The inverse of h is the function $h^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by $h^{-1}(x) = 1/x$.

Observe that $h = h^{-1}$. So, h and h^{-1} *share infinitely many intersection points.*

Of these, only $(-1, -1)$ and $(1, 1)$ are on the line $y = x$; every other intersection point is not. For example, h and h^{-1} intersect at the point $(3, 1/3)$, but this point is **not** on $y = x$.



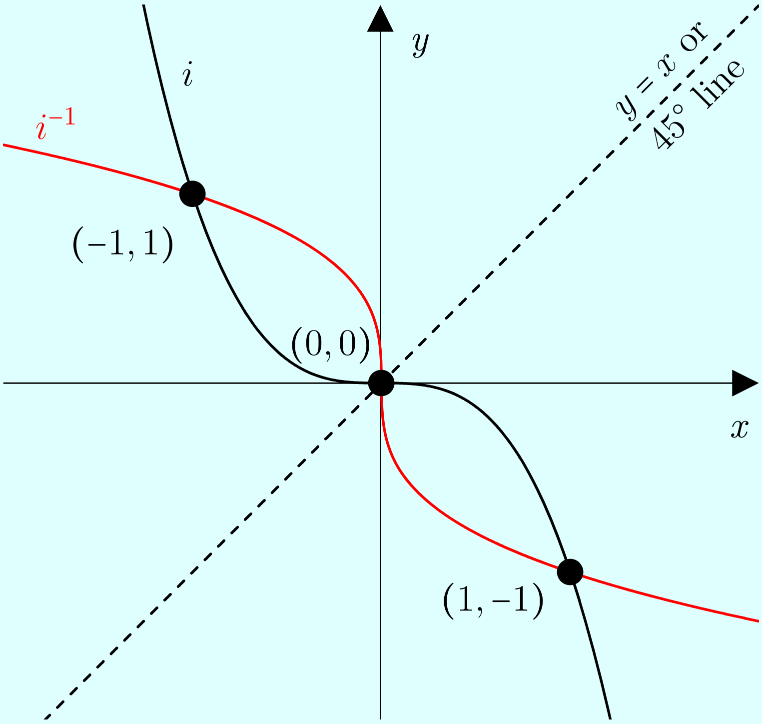
One might object that the function in the above counterexample is unusual because it is **(a)** not defined on $\mathbb{R}$; and **(b)** its own inverse. So consider the next example:

[187]Similarly, one set of published TYS answers incorrectly states, "As $y = f(x)$ is a reflection of $y = f^{-1}(x)$ about the line $y = x$, the point of intersection of the two curves must meet on $y = x$."

Example 396. The function $i : \mathbb{R} \to \mathbb{R}$ defined by $i(x) = -x^3$ is **(a)** defined on $\mathbb{R}$ (and moreover continuous everywhere); and **(b)** not its own inverse.

Its inverse is $i^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $i^{-1}(x) = -\sqrt[3]{x}$.

The graphs of i and i^{-1} intersect at $(0,0)$, $(-1,1)$, and $(1,-1)$. But only $(0,0)$ is on the line $y = x$.



So again (and as shown by the last two examples), the following statement is false:

"If f intersects f^{-1} at some point P, then f also intersects the line $y = x$ at P." ✗

Or, $$f(a) = f^{-1}(a) \quad \not\Rightarrow \quad f(a) = a.$$ ✓

Nonetheless, here are **two results** that come *kinda* close:

Fact 54. *Let D be an interval and $f : D \to \mathbb{R}$ be a continuous function with inverse f^{-1}. If f and f^{-1} intersect at least once, then* at least one *of their intersection points is on the line $y = x$.*

Proof. See p. 1582 (Appendices). □

Fact 54 is illustrated by the functions f, g, and i from the last four examples. (How?)[188]

Our second result says that by adding the weird assumption that f and f^{-1} intersect at an **even** number of points, we can obtain the stronger result that **all** of the intersection points are on the line $y = x$.

Fact 55. *Let D be an interval and $f : D \to \mathbb{R}$ be a continuous function with inverse f^{-1}. If f and f^{-1} intersect at an even number of points, then* all *of their intersection points are on the line $y = x$.*

Proof. See p. 1582 (Appendices). □

Fact 55 is illustrated by the function g (Example 394).

[188] In each case, the given function was continuous on an interval and intersected its inverse at least once. And so sure enough, in each case, the function intersected its inverse on the line $y = x$ at least once. Fact 54 may also be illustrated by the functions $h\big|_{\mathbb{R}^+}$ and $h\big|_{\mathbb{R}^-}$.

24.7. Self-Inverses

Your A-Level examiners like to write questions about **self-inverses**:[189]

Definition 80. The function f is *self-inverse* if

$$f(f(x)) = x, \qquad \text{for all } x \in \text{Domain } f.$$

Example 397. XXX

Example 398. XXX

Your A-Level examiners also like to use this result:

Fact 56. *Let f be a function and $a \in \text{Domain } f$. Suppose f has inverse f^{-1}. Then*

$$f(f(a)) = a \qquad \Longleftrightarrow \qquad f^{-1}(a) = f(a).$$

Proof. ($\Longrightarrow$) Suppose $f(f(a)) = a$.

Since $a \in \text{Range } f = \text{Domain } f$, we can apply f^{-1} to get $f^{-1}(f(f(a))) \stackrel{1}{=} f^{-1}(a)$.

But by Proposition 5(a) (Cancellation Law), we also have $f^{-1}(f(f(a))) \stackrel{2}{=} f(a)$.

So, by $\stackrel{1}{=}$ and $\stackrel{2}{=}$, $f^{-1}(a) = f(a)$.

($\Longleftarrow$) Suppose $f^{-1}(a) = f(a)$.

Since $f^{-1}(a) \in \text{Codomain } f^{-1} = \text{Domain } f$, we can apply f to get $f\left(f^{-1}(a)\right) \stackrel{3}{=} f(f(a))$.

But by Proposition 5(b) (Cancellation Law), we also have $f\left(f^{-1}(a)\right) \stackrel{4}{=} a$.

So, by $\stackrel{3}{=}$ and $\stackrel{4}{=}$, $f(f(a)) = a$. □

Example 399. XXX

Example 400. XXX

Remark 55. A synonym for *self-inverse* is *involution*, which is actually the more commonly used term. But we'll stick with the term *self-inverse*, which is simpler and more readily understood.

Example 401. XXX

Exercise 127. XXX (Answer on p. 282.)

A127.

[189] See e.g. N2018/I/5, N2017/II/3, N2016/I/10, N2012/I/7 (respectively, Exercises 531, 536, 539, 551). Indeed, N2012/I/7 explicitly uses the term *self-inverse*.

24.8. If f Is Strictly Increasing/Decreasing, Then So Is f^{-1}

In Ch. 23.2, we learnt that if a function f is strictly increasing (or decreasing), then it is also one-to-one—and so has some inverse f^{-1}. It turns out that f^{-1} must also be strictly increasing (or decreasing):

Proposition 6. (a) *A strictly increasing function's inverse is strictly increasing.*
(b) *A strictly decreasing function's inverse is strictly decreasing.*

Proof. **(a)** Let f be a strictly increasing function with inverse f^{-1}.

Pick[190] any $a, b \in \text{Domain } f$ with $a < b$.

Let $c = f^{-1}(a)$ and $d = f^{-1}(b)$, so that $f(c) = a$ and $f(d) = b$. Since f is strictly increasing and $a < b$, it must be that $c < d$.

We've just shown that $a < b$ implies $f^{-1}(a) < f^{-1}(b)$, i.e. f^{-1} is strictly increasing.

(b) Similar, omitted. □

Example 402. XXX

Example 403. XXX

Exercise 128. XXX (Answer on p. 283.)

A128.

[190] If no such a, b exist, then it is vacuously true that f^{-1} is strictly increasing.

24.9. When Is a Function the Inverse of Its Inverse?

Let f be a function with inverse f^{-1}. Suppose f^{-1} itself has inverse $\left(f^{-1}\right)^{-1}$.[191]

So, $\left(f^{-1}\right)^{-1}$ is the inverse of the inverse of f.

One question we might ask is this: How are f and $\left(f^{-1}\right)^{-1}$ related?

Here's one possible answer:

Fact 57. *Suppose the function f has inverse f^{-1} and $\left(f^{-1}\right)^{-1}$ is the inverse of f^{-1}. Then*

$$f = \left(f^{-1}\right)^{-1} \qquad \iff \qquad f \text{ is onto.}$$

Proof. See p. 1576 (Appendices). ☐

As discussed in Remark 54, we'll assume you'll only ever be asked to find the inverse of a function that is onto. And so, by Fact 57, we may assume that $f = \left(f^{-1}\right)^{-1}$ is always true (even though it's actually false if f isn't onto).

Example 404. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x - 1$. (Is f onto?)[192]

The inverse of f is the function $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(y) = (y+1)/3$.

The inverse of f^{-1} is the function $\left(f^{-1}\right)^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $\left(f^{-1}\right)^{-1}(x) = 3x - 1$.

Since f and $\left(f^{-1}\right)^{-1}$ have the same domain, codomain, and mapping rule, $f = \left(f^{-1}\right)^{-1}$.

Example 405. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 5x + 2$. (Is g onto?)[193]

The inverse of g is the function $g^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $g^{-1}(y) = (y-2)/5$.

The inverse of g^{-1} is the function $\left(g^{-1}\right)^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $\left(g^{-1}\right)^{-1}(x) = 5x + 2$.

Since g and $\left(g^{-1}\right)^{-1}$ have the same domain, codomain, and mapping rule, $g = \left(g^{-1}\right)^{-1}$.

Exercise 129. For each function given below, find **(i)** its inverse; and **(ii)** the inverse of its inverse. Also, is the function **(iii)** onto; **(iv)** and equal to the inverse of its inverse?

(a) $h : (0,1] \to [1,\infty)$ defined by $h(x) = 1/x$.

(b) $i : (0,1] \to (0,1]$ defined by $i(x) = x^2$. (Answer on p. 1784.)

[191] It turns out that $\left(f^{-1}\right)^{-1}$ *must* exist because f^{-1} must be one-to-one—Fact 254 (Appendices).

[192] Yes, because every element in Codomain $f = \mathbb{R}$ is "hit" at least once. (Or equivalently, "Yes, because Range $f = \mathbb{R} =$ Codomain f.")

[193] Yes, because every element in Codomain $g = \mathbb{R}$ is "hit" at least once. (Or equivalently, "Yes, because Range $g = \mathbb{R} =$ Codomain g.")

24.10. Domain Restriction to Create a One-to-One Function

Again, if a function is not one-to-one, then it does not have an inverse.

Nonetheless, given a function that isn't one-to-one, we can *always*[194] restrict its domain to create a new function that *is* one-to-one (and so has an inverse):

Example 406. Define $f : \mathbb{R} \to \mathbb{R}_0^+$ by $f(x) = x^2$.

We can show[195] that f is **not** one-to-one and so has no inverse.

But consider the function $f\big|_{\mathbb{R}_0^+}$ (i.e. the restriction of f to $\mathbb{R}_0^+$).

We can show that $f\big|_{\mathbb{R}_0^+}$ is strictly increasing[196] and hence one-to-one.

Figure to be
inserted here.

The inverse of $f\big|_{\mathbb{R}_0^+}$ is $f\big|_{\mathbb{R}_0^+}^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $f\big|_{\mathbb{R}_0^+}^{-1}(y) = \sqrt{y}$.

Exercise 130. Continue to define $f : \mathbb{R} \to \mathbb{R}_0^+$ by $f(x) = x^2$. (Answer on p. 1785.)

(a) In words, what is $f\big|_{\mathbb{R}_0^-}$?

(b) Show that $f\big|_{\mathbb{R}_0^-}$ is one-to-one.

(c) Find the inverse of $f\big|_{\mathbb{R}_0^-}$.

(d) Are there any other sets S such that the function $f\big|_S$ is also one-to-one?

[194] Given any function f, we can, in the worst case, consider its restriction to the empty set, i.e. the function $f\big|_{\varnothing}$. Observe that the range of $f\big|_{\varnothing}$ is the empty set. So, it is vacuously true that no element in the codomain of $f\big|_{\varnothing}$ is "hit" more than once. Hence, $f\big|_{\varnothing}$ is is one-to-one.

[195] The element 4 in its codomain is "hit" twice: $f(2) = 2^2 = 4$ and $f(-2) = (-2)^2 = 4$.

[196] Pick any distinct $a, b \in \mathbb{R}_0^+$ with $a < b$. Then $a - b < 0$ and $a + b > 0$ (because $a \geq 0$ and $b > 0$), so that $a^2 - b^2 = (a - b)(a + b) < 0$.

Example 407. Define $g : \mathbb{R} \to \mathbb{R}_0^+$ by $g(x) = |x|$.

We can show[197] that g is **not** one-to-one and so has no inverse.

But consider the function $g|_{\mathbb{R}_0^+}$ (i.e. the restriction of g to $\mathbb{R}_0^+$).

We can show that $g|_{\mathbb{R}_0^+}$ is strictly increasing[198] and hence one-to-one.

Figure to be
inserted here.

The inverse of $f|_{\mathbb{R}_0^+}$ is $g|_{\mathbb{R}_0^+}^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $g|_{\mathbb{R}_0^+}^{-1}(y) = y$.

Exercise 131. Continue to define $g : \mathbb{R} \to \mathbb{R}_0^+$ by $g(x) = |x|$. (Answer on p. 149.15.)

(a) In words, what is $g|_{\mathbb{R}_0^-}$?

(b) Show that $g|_{\mathbb{R}_0^-}$ is one-to-one.

(c) Find the inverse of $g|_{\mathbb{R}_0^-}$.

(d) Are there any other sets S such that the function $g|_S$ is also one-to-one?

Exercise 132. Define $h : \mathbb{R} \setminus \{1\} \to \mathbb{R}_0^+$ by $h(x) = \dfrac{1}{(x-1)^2}$. (Answer on p. 1785.)

(a) Explain whether h is one-to-one.

(b) In words, what is $h|_{(1,\infty)}$?

(c) Show that $h|_{(1,\infty)}$ is one-to-one.

(d) Find the inverse of $h|_{(1,\infty)}$.

(e) In words, what is $h|_{(-\infty,1)}$?

(f) Show that $h|_{(-\infty,1)}$ is one-to-one.

(g) Find the inverse of $h|_{(-\infty,1)}$.

(h) Are there any other sets S such that the function $h|_S$ is also one-to-one?

[197] The element 2 in its codomain is "hit" twice: $g(2) = |2| = 2$ and $g(-2) = |-2| = 2$.

[198] Pick any $a, b \in \mathbb{R}_0^+$ with $a < b$. Then $g|_{\mathbb{R}_0^+}(a) = |a| = a < b = |b| = g|_{\mathbb{R}_0^+}(b)$.

25. Asymptotes and an Introduction to Limit Notation

Informally, an **asymptote** is a line that a graph approaches (or "gets ever closer to").

Example 408. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = \dfrac{1}{x}$. Observe that

$$\text{As } x \text{ approaches } -\infty,\ f(x) \text{ approaches } 0 \text{ (from below).} \qquad \text{(I)}$$

Statement (I) can be rewritten in another three ways:

$$\text{The } \textbf{limit} \text{ of } f(x) \text{ as } x \text{ approaches } -\infty \text{ is } 0^-. \qquad \text{(I)}$$

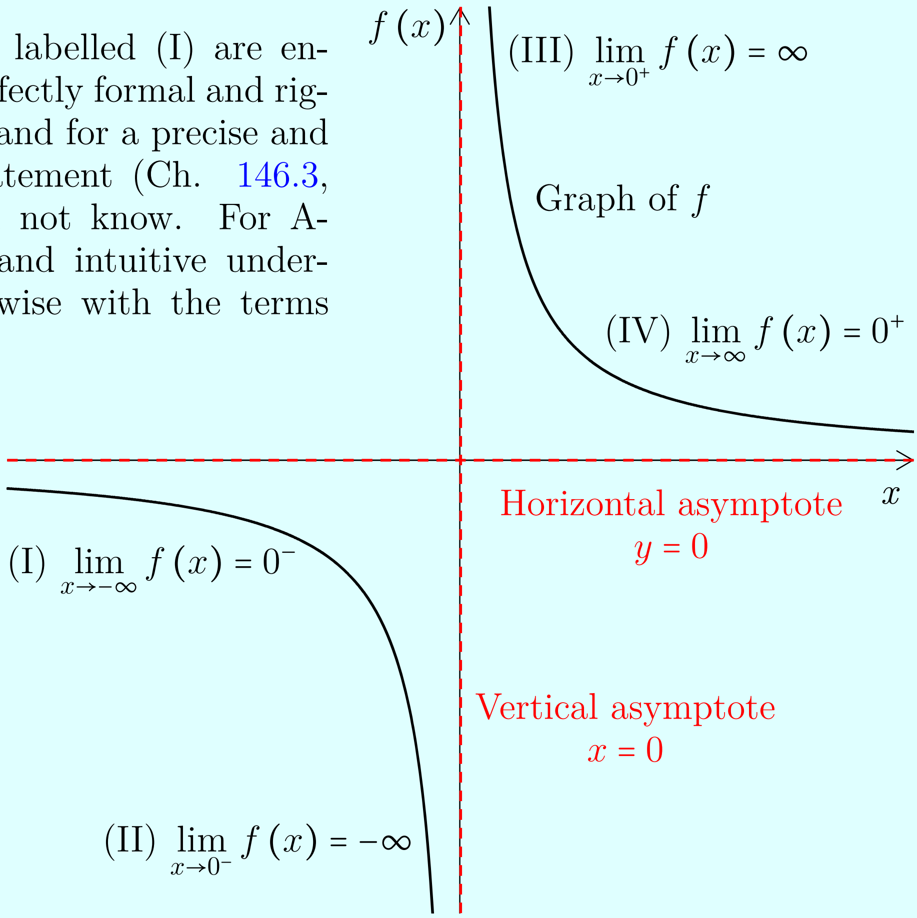
$$\text{As } x \to -\infty,\ f(x) \to 0^-. \quad \text{(I)} \qquad\qquad \lim_{x\to-\infty} f(x) = 0^-. \quad \text{(I)}$$

Statement (I) implies that

The horizontal line $y = 0$ is a **horizontal asymptote** of the function f.

Three remarks:

1. The above four statements labelled (I) are entirely equivalent and are perfectly formal and rigorous. They serve as shorthand for a precise and formal but long-winded statement (Ch. 146.3, Appendices) that you need not know. For A-Level Maths, an informal and intuitive understanding will suffice. Likewise with the terms **limits** and **asymptotes**.



2. The parenthetical "from below" and superscript minus sign are optional and may be omitted. Indeed, in A-Level Maths, they are usually omitted. Nonetheless, they should be easy to understand and so I've included them. (Rather than $f(x) \to 0^-$, others may instead write $f(x) \nearrow 0$ or $f(x) \uparrow 0$.)

3. We could've said "x approaches $-\infty$ *(from the right)*", but that would be redundant because there's only one possible way to approach $-\infty$.

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Next, similarly observe that

$$\text{As } x \text{ approaches 0 (from the left), } f(x) \text{ approaches } -\infty. \tag{II}$$

Statement (II) is equivalent to each of the following three statements:

$$\text{The } \textbf{limit} \text{ of } f(x) \text{ as } x \text{ approaches 0 (from the left) is } -\infty. \tag{II}$$

$$\text{As } x \to 0^-,\ f(x) \to -\infty. \quad \text{(II)} \qquad\qquad \lim_{x\to 0^-} f(x) = -\infty. \quad \text{(II)}$$

Statement (II) implies that

The vertical line $x = 0$ is a **vertical asymptote** of the function f.

Next, $$\text{As } x \text{ approaches 0 (from the right), } f(x) \text{ approaches } \infty. \tag{III}$$

Statement (III) is equivalent to each of the following three statements:

$$\text{The } \textbf{limit} \text{ of } f(x) \text{ as } x \text{ approaches 0 (from the right) is } \infty. \tag{III}$$

$$\text{As } x \to 0^+,\ f(x) \to \infty. \quad \text{(III)} \qquad\qquad \lim_{x\to 0^+} f(x) = \infty. \quad \text{(III)}$$

Statement (III) implies that

The vertical line $x = 0$ is a **vertical asymptote** of the function f.

Next, $$\text{As } x \text{ approaches } \infty,\ f(x) \text{ approaches 0 (from above).} \tag{IV}$$

Statement (IV) is equivalent to each of the following three statements:

$$\text{The } \textbf{limit} \text{ of } f(x) \text{ as } x \text{ approaches } \infty \text{ (from the left) is } 0^+. \tag{IV}$$

$$\text{As } x \to \infty,\ f(x) \to 0^+. \quad \text{(IV)} \qquad\qquad \lim_{x\to \infty} f(x) = 0^+. \quad \text{(IV)}$$

Statement (IV) implies that

The horizontal line $y = 0$ is a **horizontal asymptote** of the function f.

Example 409. Define $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ by $g(x) = \dfrac{1}{x-1} + 2$. Observe that

(I) As x approaches $-\infty$, $g(x)$ approaches 2 (from below).
(I) The **limit** of $g(x)$ as x approaches $-\infty$ is 2^-.
(I) As $x \to -\infty$, $g(x) \to 2^-$. (I) $\lim_{x \to -\infty} g(x) = 2^-$.

(II) As x approaches 1 (from the left), $g(x)$ approaches $-\infty$.
(II) The **limit** of $g(x)$ as x approaches 1 (from the left) is $-\infty$.
(II) As $x \to 1^-$, $g(x) \to -\infty$. (II) $\lim_{x \to 1^-} g(x) = -\infty$.

Figure to be inserted here.

(III) As x approaches 1 (from the right), $g(x)$ approaches ∞.
(III) The **limit** of $g(x)$ as x approaches 1 (from the right) is ∞.
(III) As $x \to 1^+$, $g(x) \to \infty$. (III) $\lim_{x \to 1^+} g(x) = \infty$.

(IV) As x approaches ∞, $g(x)$ approaches 2 (from above).
(IV) The **limit** of $g(x)$ as x approaches ∞ is 2^+.
(IV) As $x \to \infty$, $g(x) \to 2^+$. (IV) $\lim_{x \to \infty} g(x) = 2^+$.

The horizontal line $y = 2$ is a horizontal asymptote of g.

The vertical line $x = 1$ is a vertical asymptote of g.

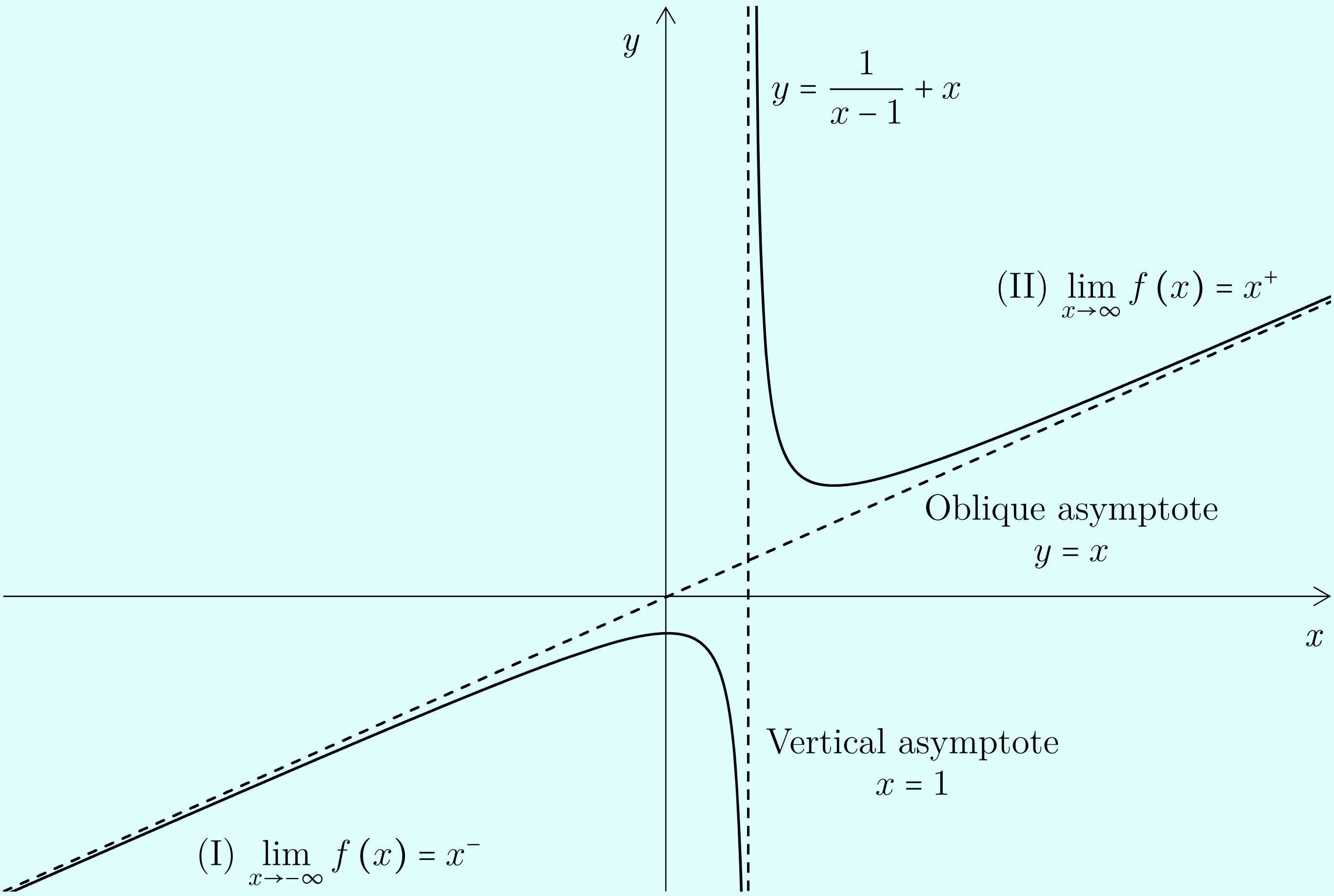
Remark 56. Just so you know, in the above two examples, some writers might refer to statements (I) and (IV) as **limits at infinity** (what happens as $x \to \pm\infty$) and statements (II) and (IV) as **infinite limits** (where $f(x) \to \pm\infty$, i.e. the limits are themselves $\pm\infty$). These two terms aren't terribly important and we won't be using them.

25.1. Oblique Asymptotes

Asymptotes need not only be vertical or horizontal—they can also be **oblique** (or slanted).

Example 410. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = \dfrac{1}{x-1} + x$.

First, $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$, so that a vertical asymptote of f is $x = 1$.



Next,	As x approaches $-\infty$, $f(x)$ approaches x (from below).	(I)
	The **limit** of $f(x)$ as x approaches $-\infty$ is x^-.	(I)
	As $x \to -\infty$, $f(x) \to x^-$. (I) $\quad \lim_{x\to-\infty} f(x) = x^-$.	(I)
Similarly,	As x approaches ∞, $f(x)$ approaches x (from above).	(II)
	The **limit** of $f(x)$ as x approaches ∞ is x^+.	(II)
	As $x \to \infty$, $f(x) \to x^+$. (I) $\quad \lim_{x\to\infty} f(x) = x^+$.	(II)

Either statement (I) or (II) (by itself) implies that

The oblique line $y = x$ is an oblique asymptote for the function f.

Remark 57. As before, you needn't know the formal definition of **oblique asymptotes** (Definition 325, Appendices). An informal and intuitive understanding will suffice.

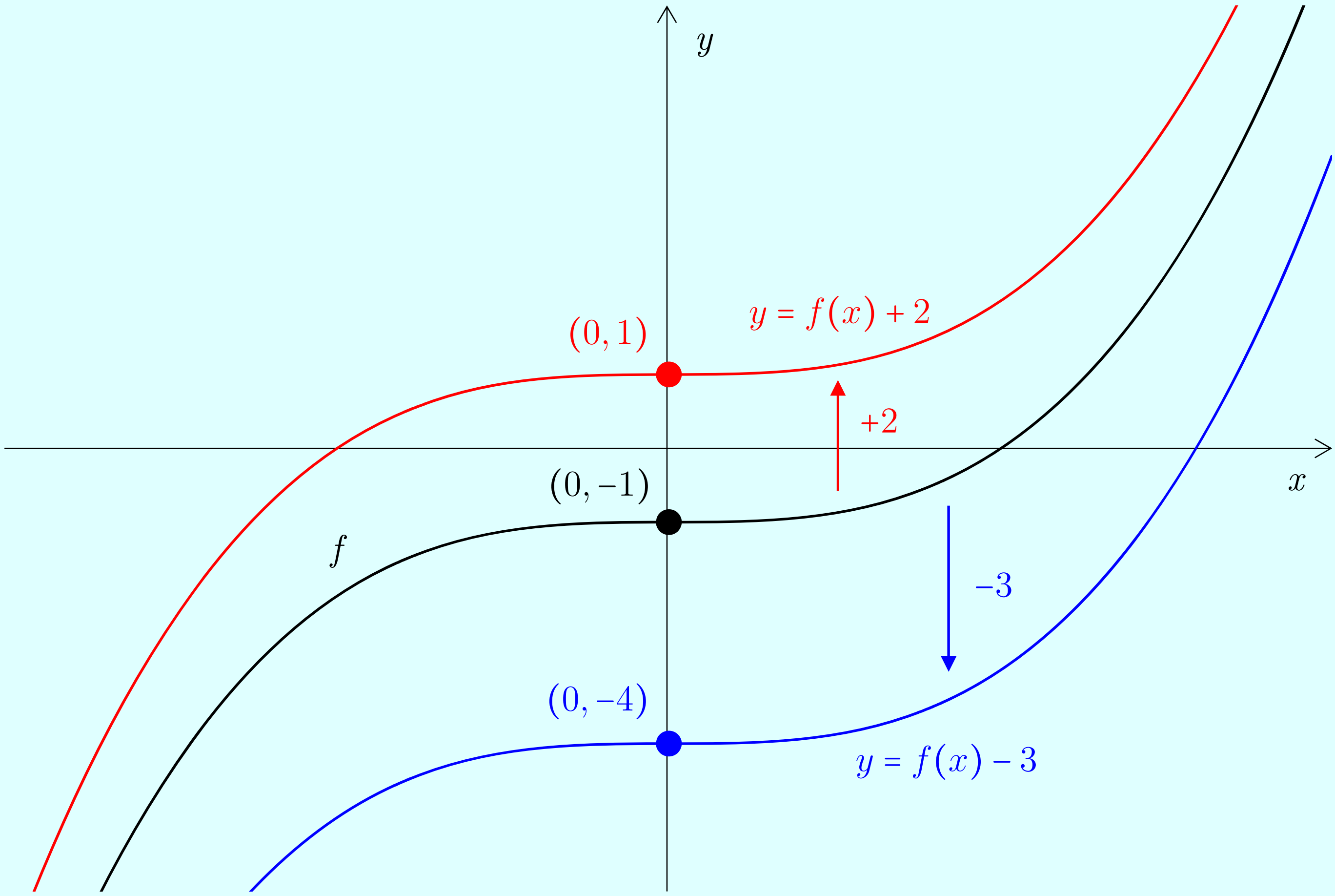
26. Transformations

26.1. $y = f(x) + a$

The graph of $y = f(x) + a$ is simply the graph of f **translated** (or **shifted**) upwards by a units. (If $a < 0$, then we have a "negative upward" shift—or more simply, a *down*ward shift.)

Example 411. Define $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 1$.

Translate (the graph of) f upwards by 2 units to get (the graph of) $y = f(x) + 2 = x^3 + 1$.



Translate f downwards by 3 units to get $y = f(x) - 3 = x^3 - 4$.

When a graph is translated upwards or downwards, any y**-intercepts**, **lines of symmetry**, **turning points**, and **asymptotes** simply shift along with it.[199]

Here, f has y-intercept $(0, -1)$. So, $y = f(x) + 2$ and $y = f(x) - 3$ simply have y-intercepts $(0, 1)$ and $(0, -4)$.

(But we can't say anything general about how the x**-intercepts** change.)

[199]This assertion is formally stated as Fact 256(a) (Appendices).

Example 412. Define $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $g(x) = 1/x$.

Translate g upwards by 1 unit to get $y = g(x) + 1 = 1/x + 1$.

Since g has horizontal asymptote $y = 0$ (the x-axis), $y = g(x) + 1$ has horizontal asymptote $y = 1$.

Figure to be inserted here.

Observe that with an upward or downward shift, *any vertical asymptotes remain unchanged*, because a vertical line translated upwards or downwards is simply the same vertical line. So, both g and $y = g(x) + 1$ have the vertical asymptote $x = 0$ (the y-axis).

The two lines of symmetry for g are $y = x$ and $y = -x$. So, the two lines of symmetry for $y = g(x) + 1$ are simply the same, but translated upwards by 1 unit—$y = x + 1$ and $y = -x + 1$.

26.2. $y = f(x+a)$

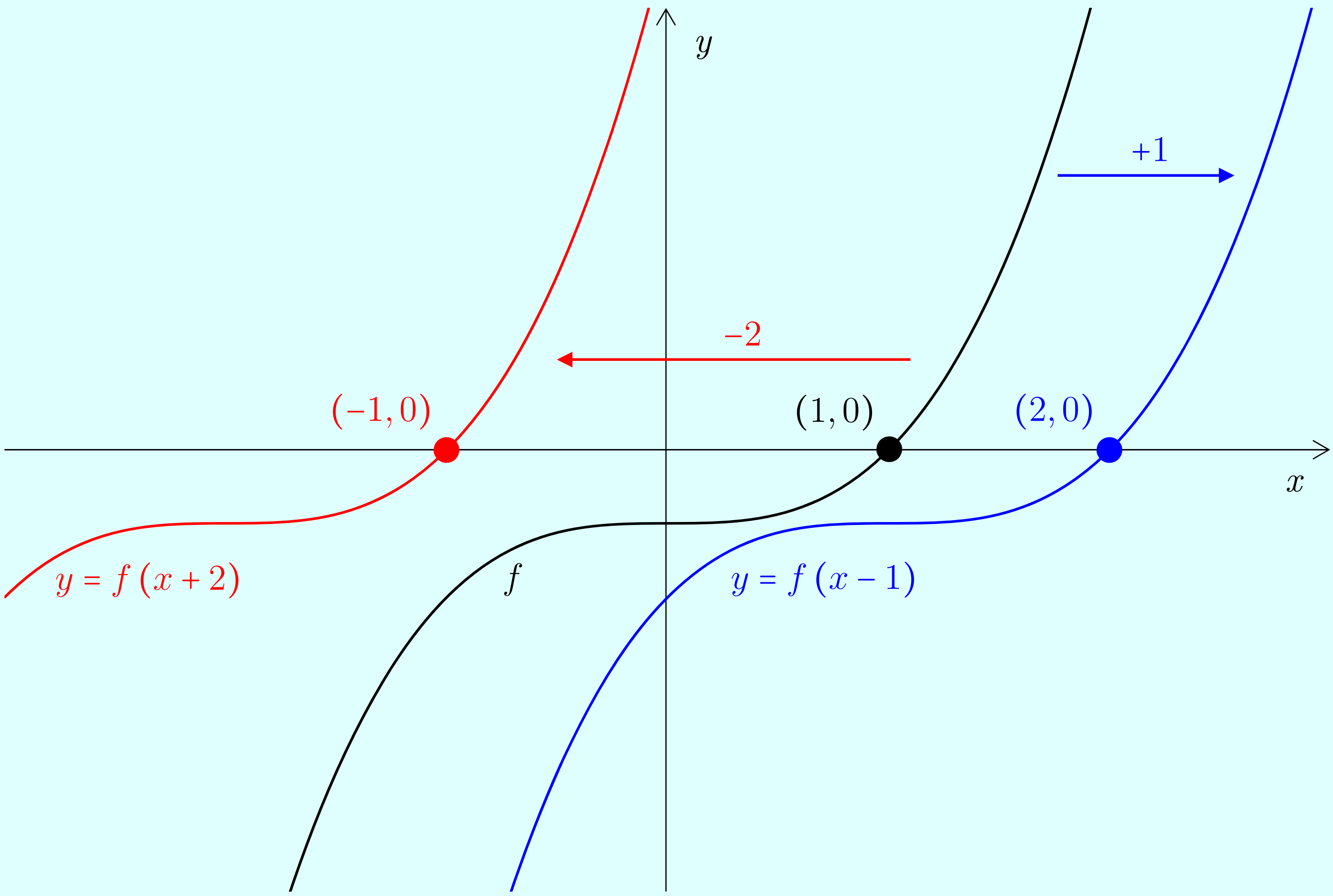
The graph of $y = f(x+a)$ is simply the graph of f translated *left*wards by a units. (If $a < 0$, then we have a "negative leftward" shift, or more simply a *right*ward shift.)

Why *left*wards (and not rightwards as one might expect)? Because for $f(x_1)$ and $f(x_2+a)$ to "hit" the same value, it must be that $x_2 = x_1 - a$. That is, x_2 must be a units to the *left* of x_1.

Example 413. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 1$.

Translate f *left*wards by 2 units to get $y = f(x+2) = (x+2)^3 - 1 = x^3 + 6x^2 + 12x + 7$.

Translate f *right*wards by 1 unit to get $y = f(x-1) = (x-1)^3 - 1 = x^3 - 3x^2 + 3x - 2$.



When a graph is translated leftwards or rightwards, any x-**intercepts**, **lines of symmetry**, **turning points**, and **asymptotes** simply shift along with it.[200]

In this example, f has x-intercept $(1,0)$. And so, $y = f(x+2)$ and $y = f(x-1)$ simply have x-intercepts $(-1,0)$ and $(2,0)$.

(But we can't say anything general about how the y-**intercepts** change.)

[200]This assertion is formally stated as Fact 256(b) (Appendices).

Example 414. Define $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $g(x) = 1/x$.

Translate g *left*wards by 1 unit to get $y = g(x+1)$.

Since g has vertical asymptote $x = 0$ (the y-axis), $y = g(x+1)$ has vertical asymptote $x = -1$.

Figure to be
inserted here.

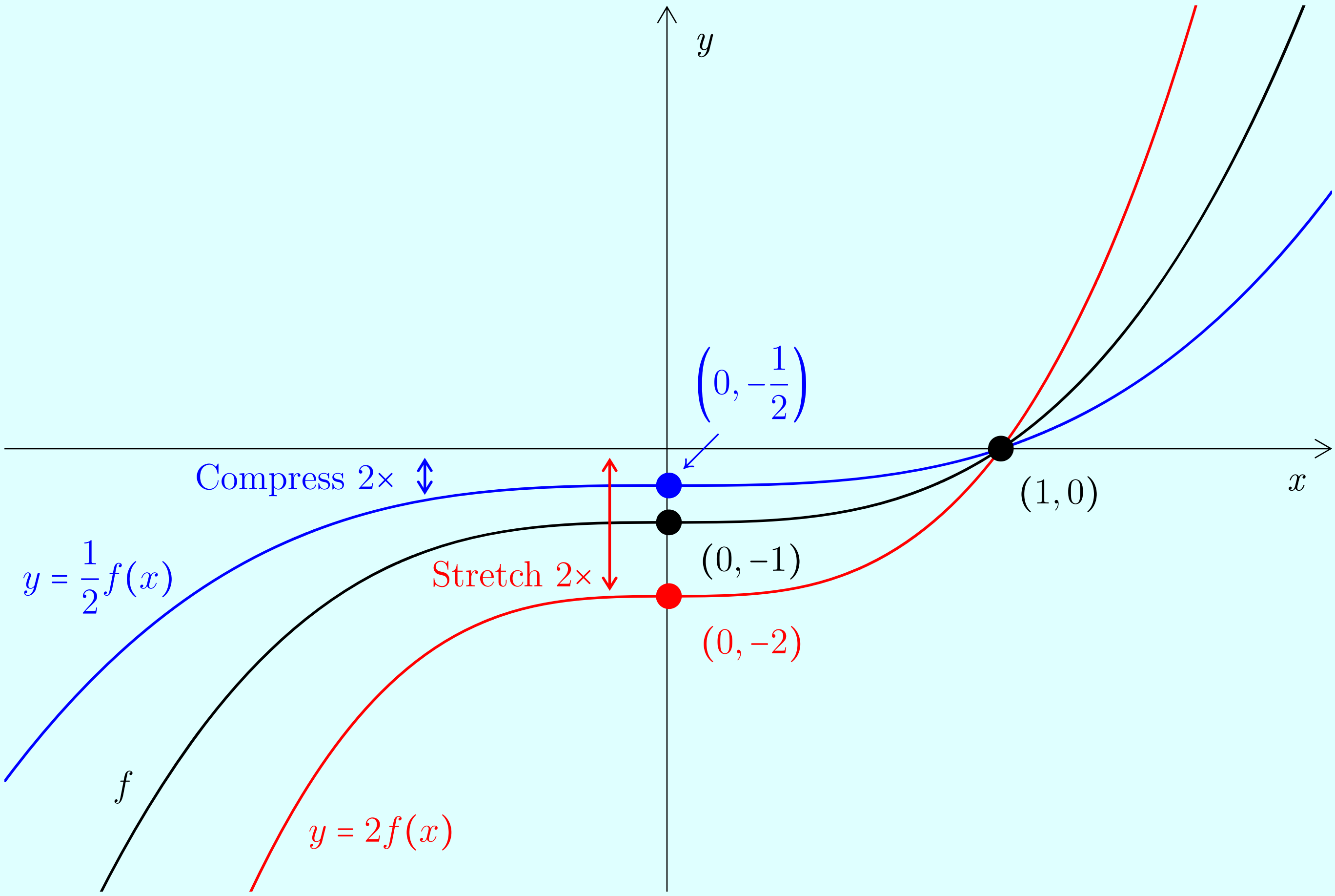
Observe that with a leftward or rightward shift, *any horizontal asymptotes remain unchanged*, because a horizontal line translated leftwards or rightwards is simply the same horizontal line. So, both g and $y = g(x+1)$ have the horizontal asymptote $y = 0$ (the x-axis).

The two lines of symmetry for g are $y = x$ and $y = -x$. So, the two lines of symmetry for $y = g(x+1)$ are simply the same, but translated leftwards by 1 unit—$y = x + 1$ and $y = -(x+1) = -x - 1$.

Let $a > 0$. The graph of $y = af(x)$ is simply that of f stretched vertically (outwards from the x-axis) by a factor of a. (If $a < 1$, then the graph is *compressed* rather than stretched.)

Example 415. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 1$.

Stretch f vertically (outwards from the x-axis) by a factor of 2 to get $y = 2f(x) = 2x^3 - 2$.



Compress f vertically (inwards towards the x-axis) by a factor of 2 to get $y = 0.5f(x) = 0.5x^3 - 0.5$.

When a graph is stretched vertically, any **y-intercepts**, **lines of symmetry**, **turning points**, and **asymptotes** stretch along with it.[201]

Here, f has y-intercept $(0, -1)$. So, $y = 2f(x)$ and $y = 0.5f(x)$ simply have y-intercepts $(0, -2)$ and $(0, -0.5)$.

Under a vertical stretch, any x-intercepts remain unchanged. Here, all three graphs have the same x-intercept $(1, 0)$.

[201]This assertion is formally stated as Fact 256(c) (Appendices).

The graph of $y = -f(x)$ is simply that of f reflected in the x-axis:

Example 416. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 3x + 2$.

Figure to be
inserted here.

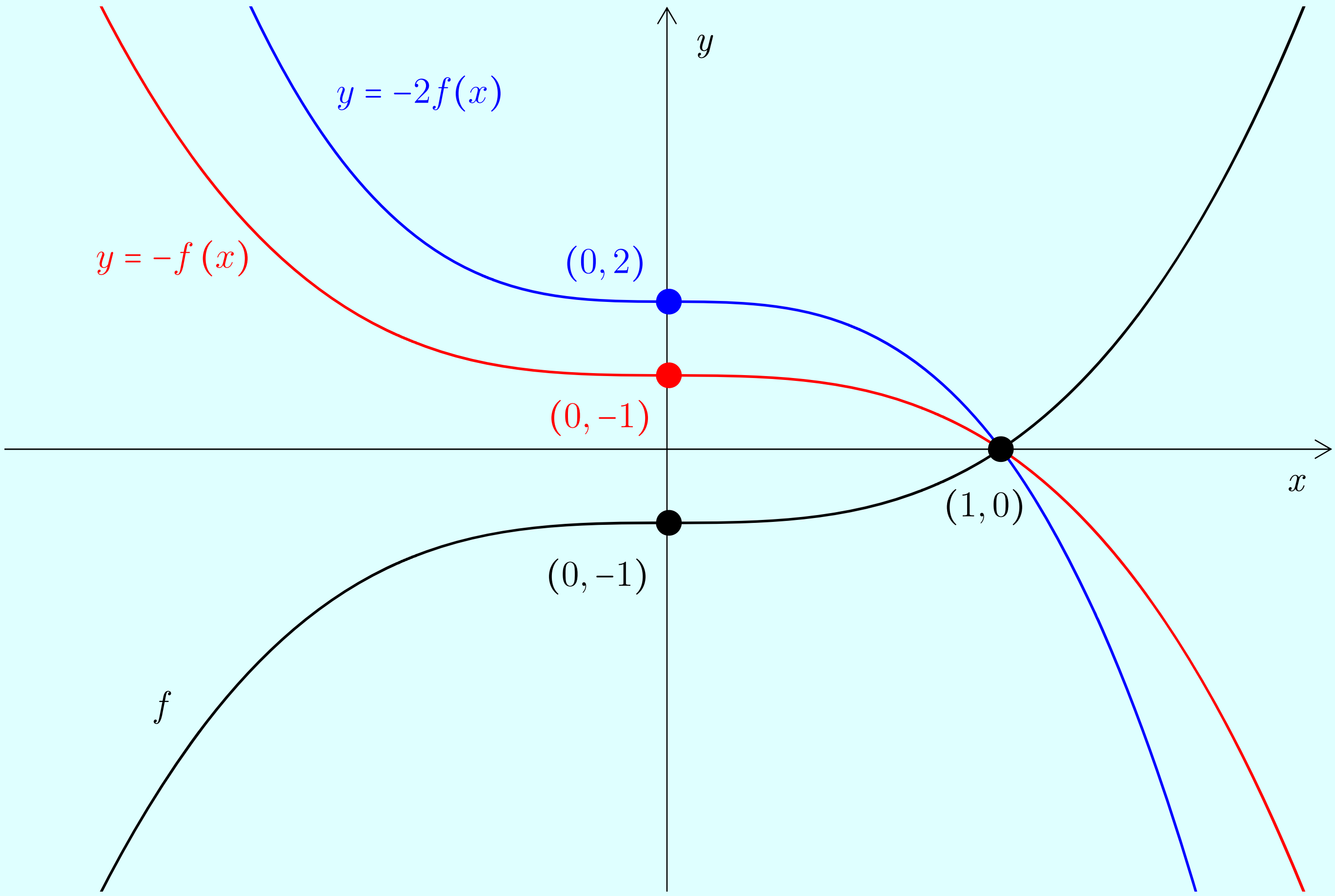
Reflect f in the x-axis to get $y = -f(x) = -\left(x^2 + 3x + 2\right) = -x^2 - 3x - 2$.

So, to get $y = -af(x)$ (where $a > 0$), first reflect f in the x-axis to get $y = -f(x)$, then stretch vertically by a factor of a:

Example 417. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 1$.

Suppose we want to graph $y = -2f(x)$.

To do so, first reflect f in the x-axis to get $y = -f(x) = -x^3 + 1$.



Then stretch vertically by a factor of 2 to get $y = -2f(x) = -2x^3 + 2$.

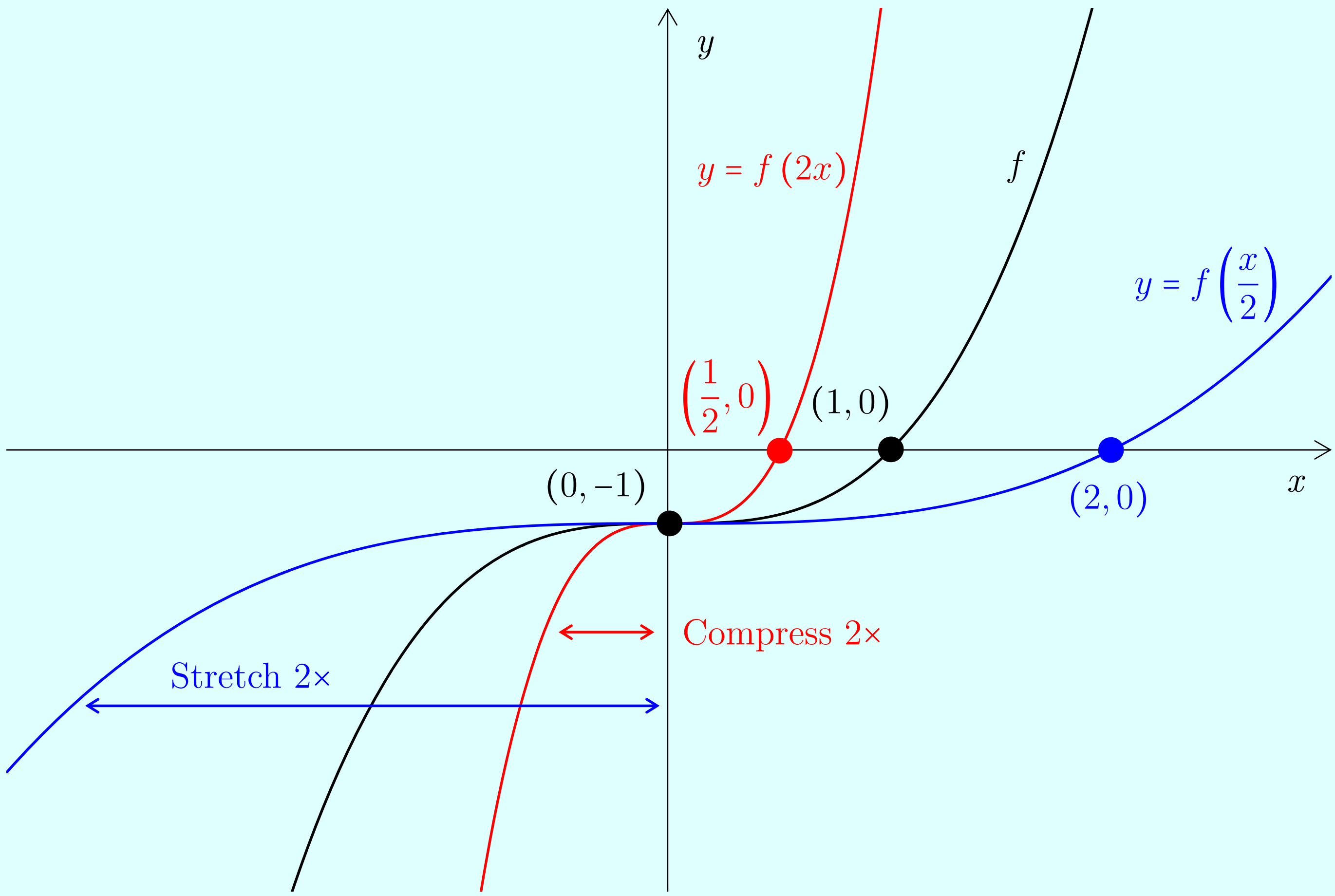
26.4. $y = f(ax)$

Let $a > 0$. The graph of $y = f(ax)$ is simply that of f *compressed* horizontally (*in*wards *towards* the y-axis) by a factor of a.

Why *compressed* *in*wards (and not stretched outwards as one might expect)? Because for $f(x_1)$ and $f(ax_2)$ to "hit" the same value, we must have $x_2 = x_1/a$.

Example 418. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 1$.

Compress f horizontally (*in*wards *towards* the y-axis) by a factor of 2 to get $y = f(2x) = (2x)^3 - 1 = 8x^3 - 1$.



Stretch f horizontally (outwards from the y-axis) by a factor of 2 to get $y = f(0.5x) = (0.5x)^3 - 1 = 0.125x^3 - 1$.

When a graph is stretched horizontally, any x-**intercepts**, **lines of symmetry**, **turning points**, and **asymptotes** stretch along with it.[202]

In this example, f has x-intercept $(1, 0)$. So, $y = f(2x)$ and $y = f(0.5x)$ simply have x-intercepts $(0.5, 0)$ and $(2, 0)$.

Under a horizontal stretch, any y-intercepts remain unchanged. Here, all three graphs have the same y-intercept $(0, -1)$.

[202] This assertion is formally stated as Fact 256(d) (Appendices).

The graph of $y = f(-x)$ is simply that of f reflected in the y-axis:

Example 419. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 3x + 2$.

Figure to be
inserted here.

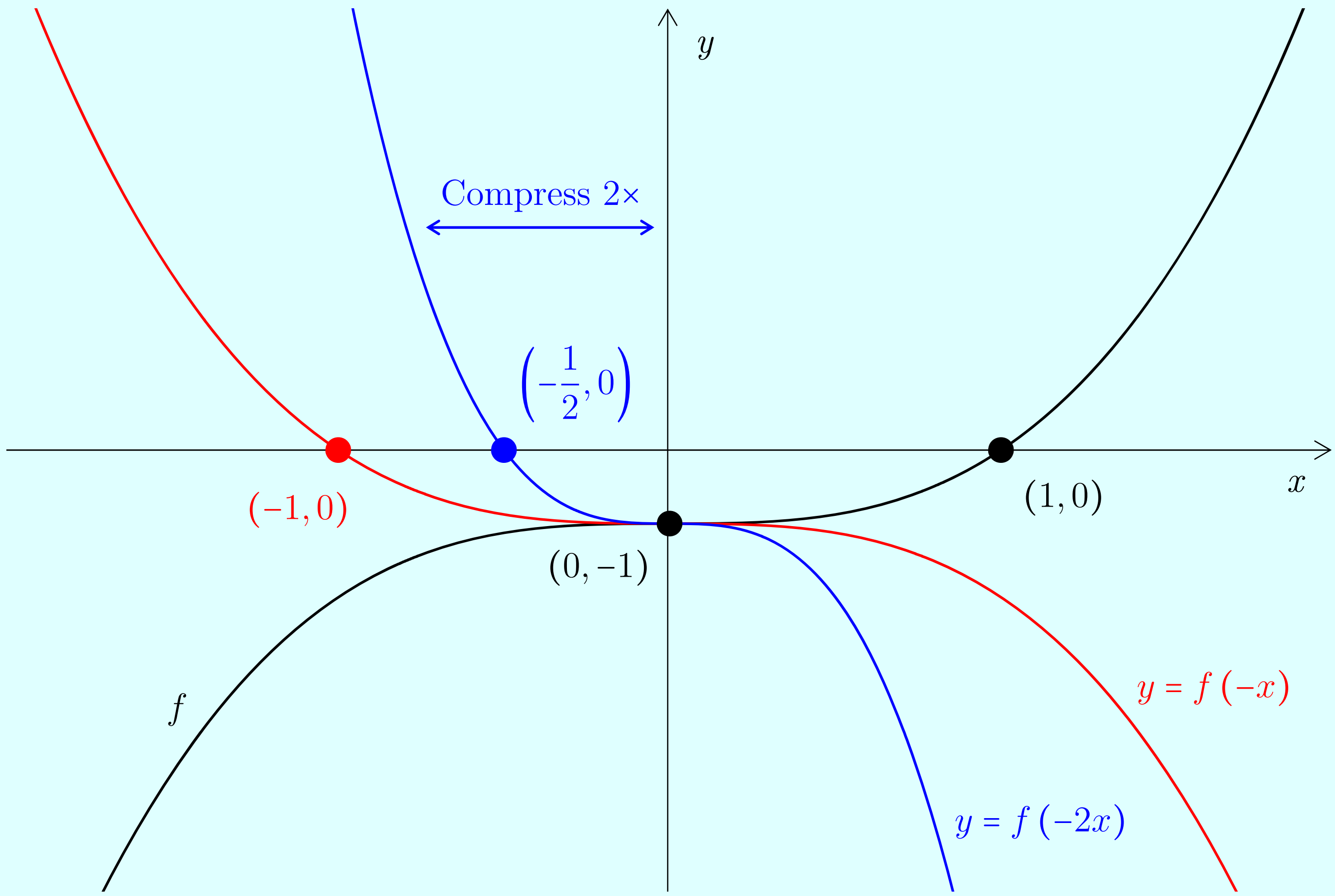
Reflect f in the y-axis to get $y = f(-x) = (-x)^2 + 3(-x) + 2 = x^2 - 3x + 2$.

So, to get $y = f(-ax)$ (where $a > 0$), first reflect f in the y-axis to get $y = f(-x)$, then *compress* horizontally by a factor of a:

Example 420. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 1$.

Say we want to graph $y = f(-2x)$.

To do so, first reflect f in the y-axis to get $y = f(-x) = (-x)^3 - 1 = -x^3 - 1$.



Then *compress* horizontally by a factor of 2, to get $y = f(-2x) = (-2x)^3 - 1 = -8x^3 - 1$.

26.5. Combinations of the Above

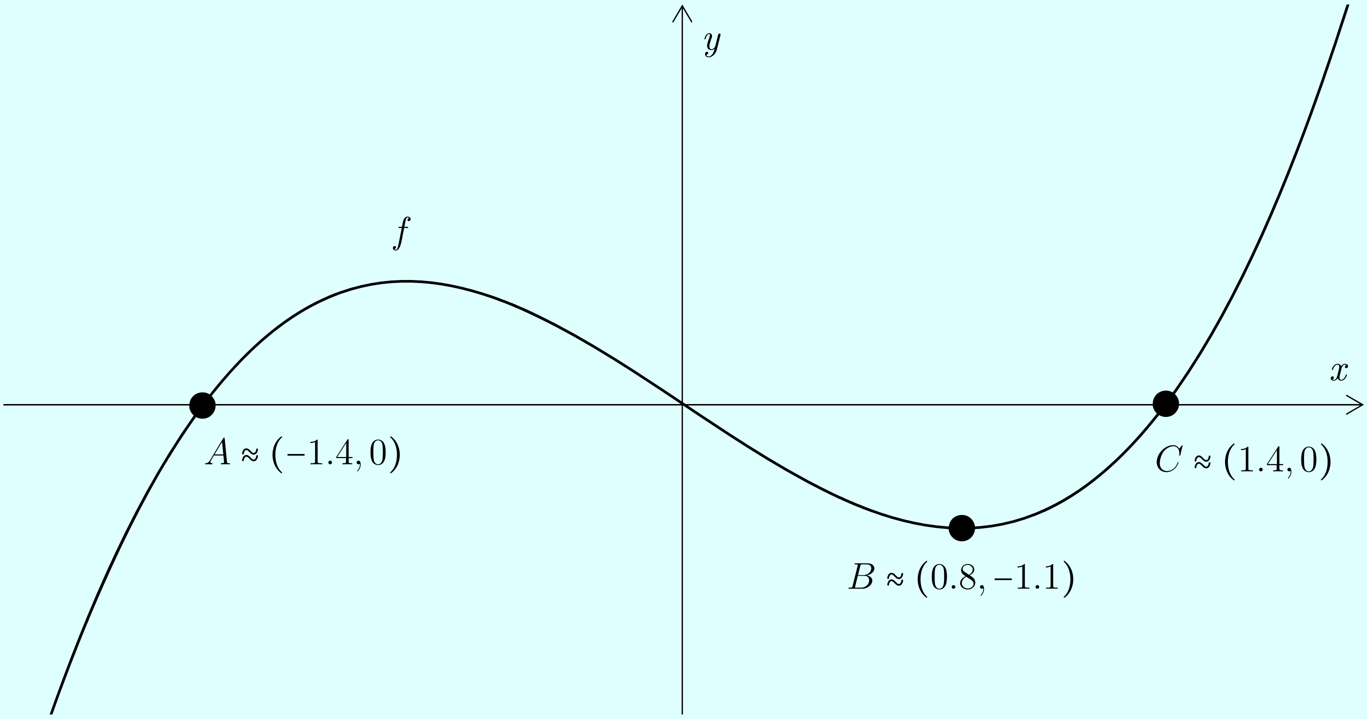
Fact 58. *Let $a, b > 0$, and $c, d \in \mathbb{R}$. Suppose f is a nice function. Then to get from the graph of f to the graph of $y = af(bx + c) + d$, follow these steps:*

1. *To get (the graph of) $y = f(x + c)$, translate* left*wards by c units.*
2. *To get $y = f(bx + c)$,* compress *inwards towards y-axis by a factor of b.*
3. *To get $y = af(bx + c)$, stretch outwards from x-axis by a factor of a.*
4. *To get $y = af(bx + c) + d$, translate upwards by d units.*

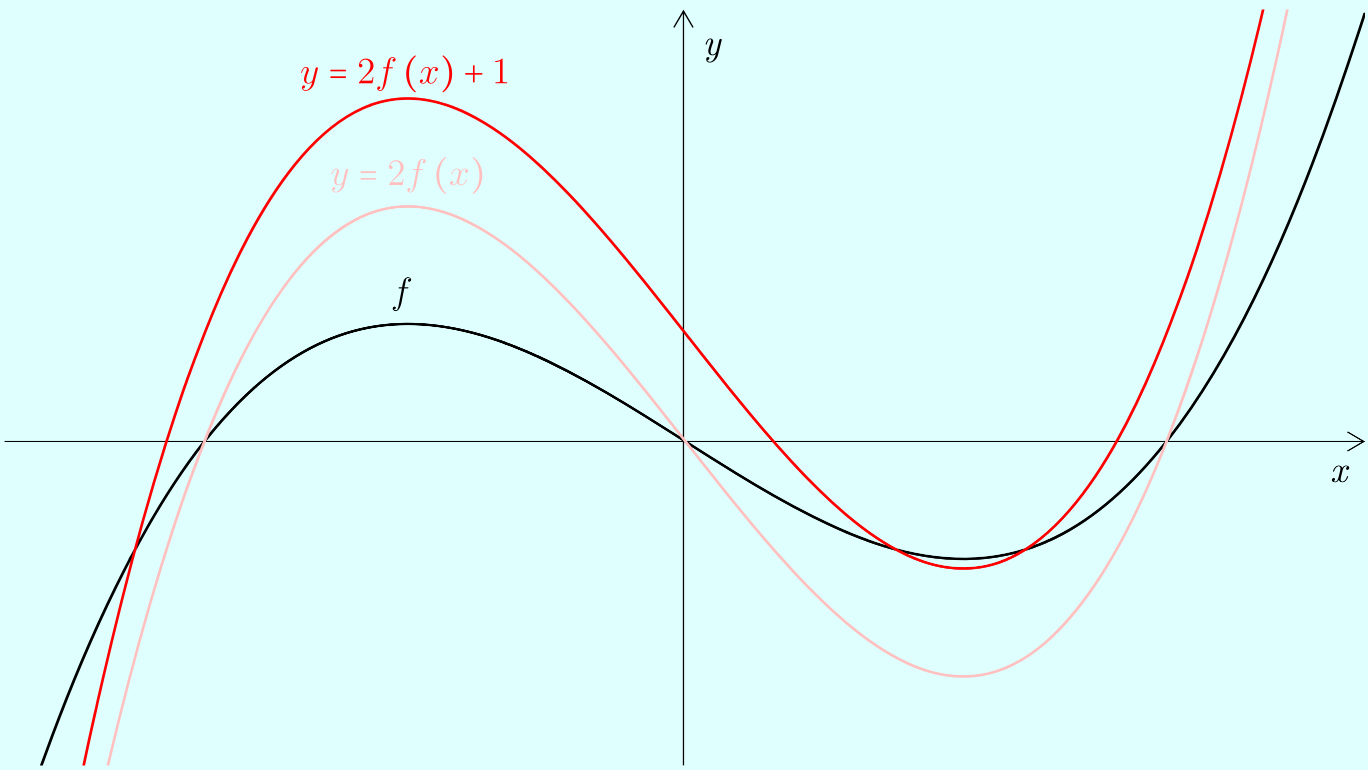
Proof. See p. 1583 (Appendices). □

Example 421. Some unknown function f is graphed below. You are told only that points A, B, and C are (approximately) $(-1.4, 0)$, $(0.8, -1.1)$, and $(1.4, 0)$.

Armed only with this knowledge, we will try to graph four equations: $y = 2f(x) + 1$, $y = 2f(x+1)$, $y = f(2x) + 1$, and $y = f(2x+1)$.



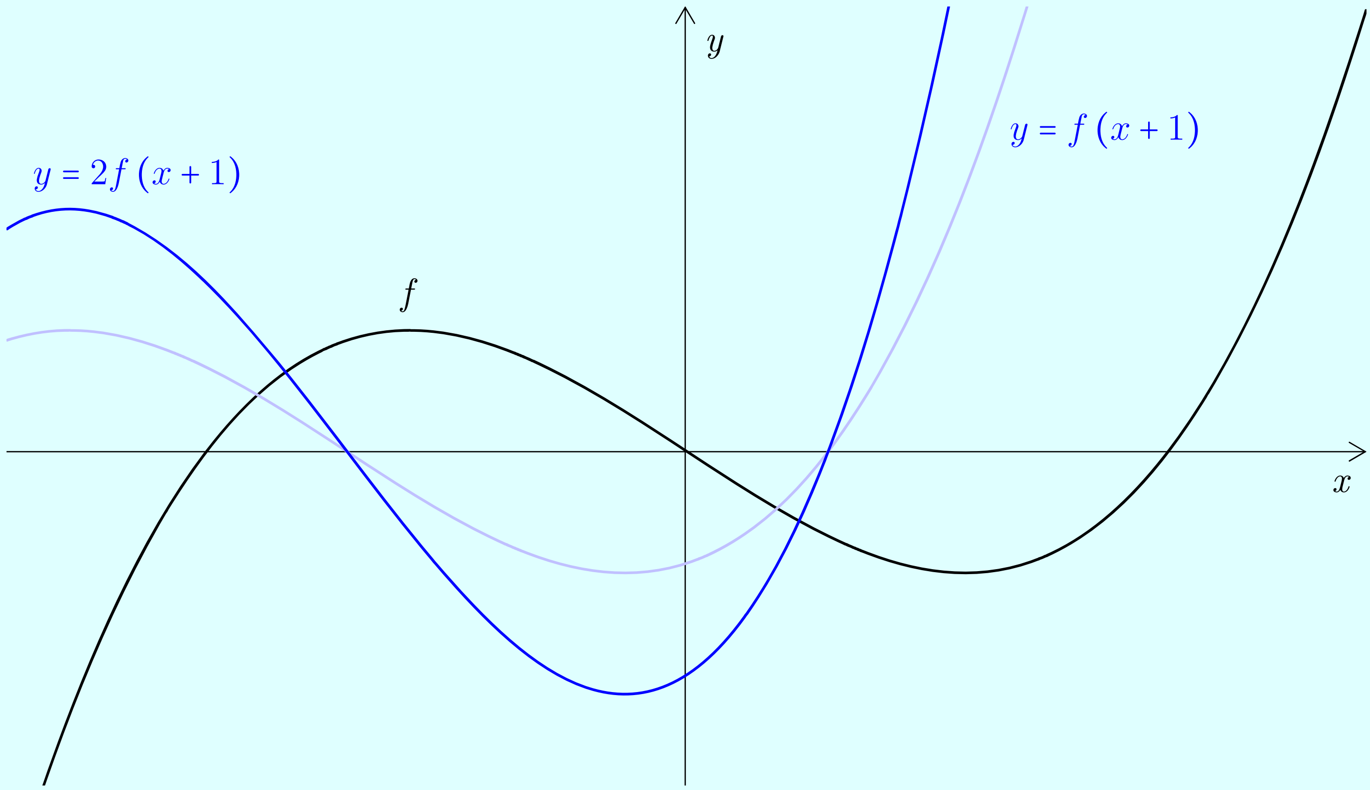
First stretch f vertically by a factor of 2 to get $y = 2f(x)$, then translate upwards by 1 unit to get $y = 2f(x) + 1$.



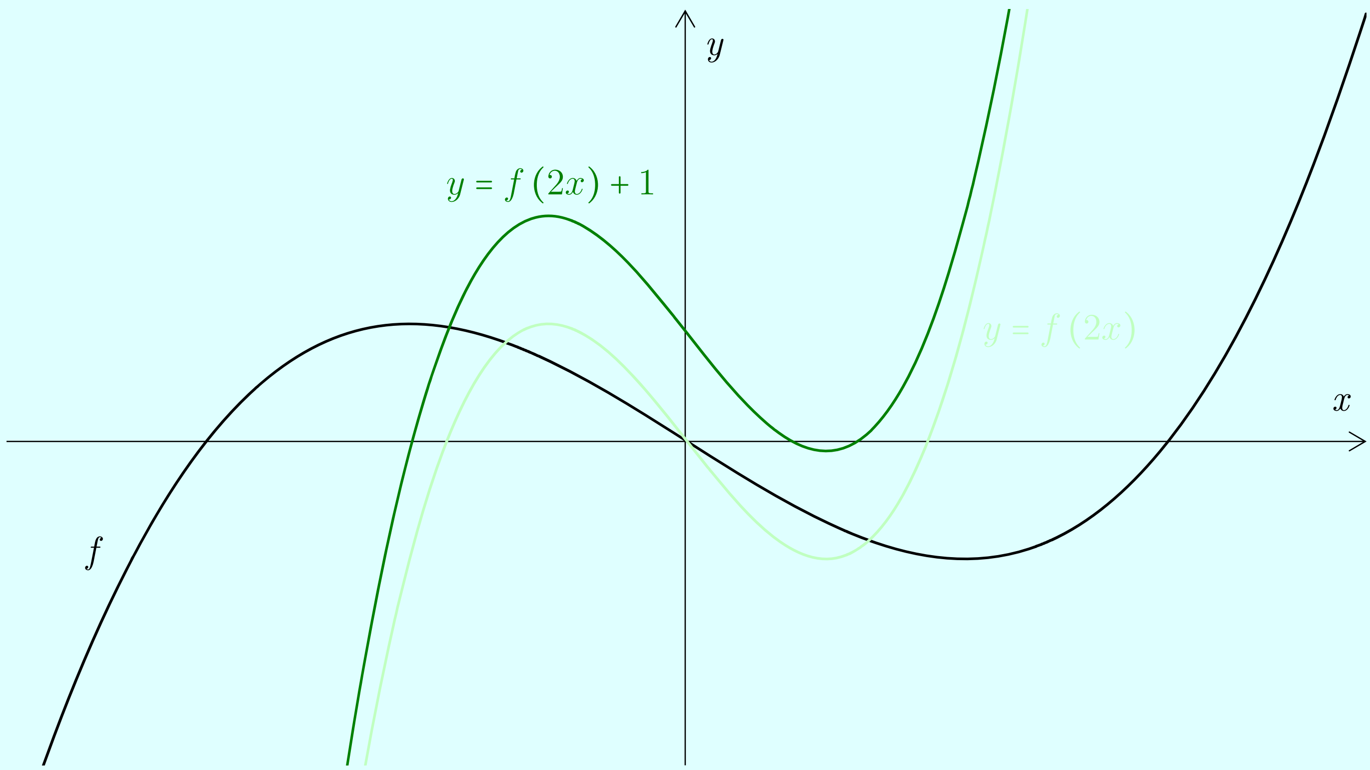
(*Example continues on the next page ...*)

(*... Example continued from the previous page.*)

First translate f *left*wards by 1 unit to get $y = f(x+1)$, then stretch vertically by a factor of 2 to get $y = 2f(x+1)$.



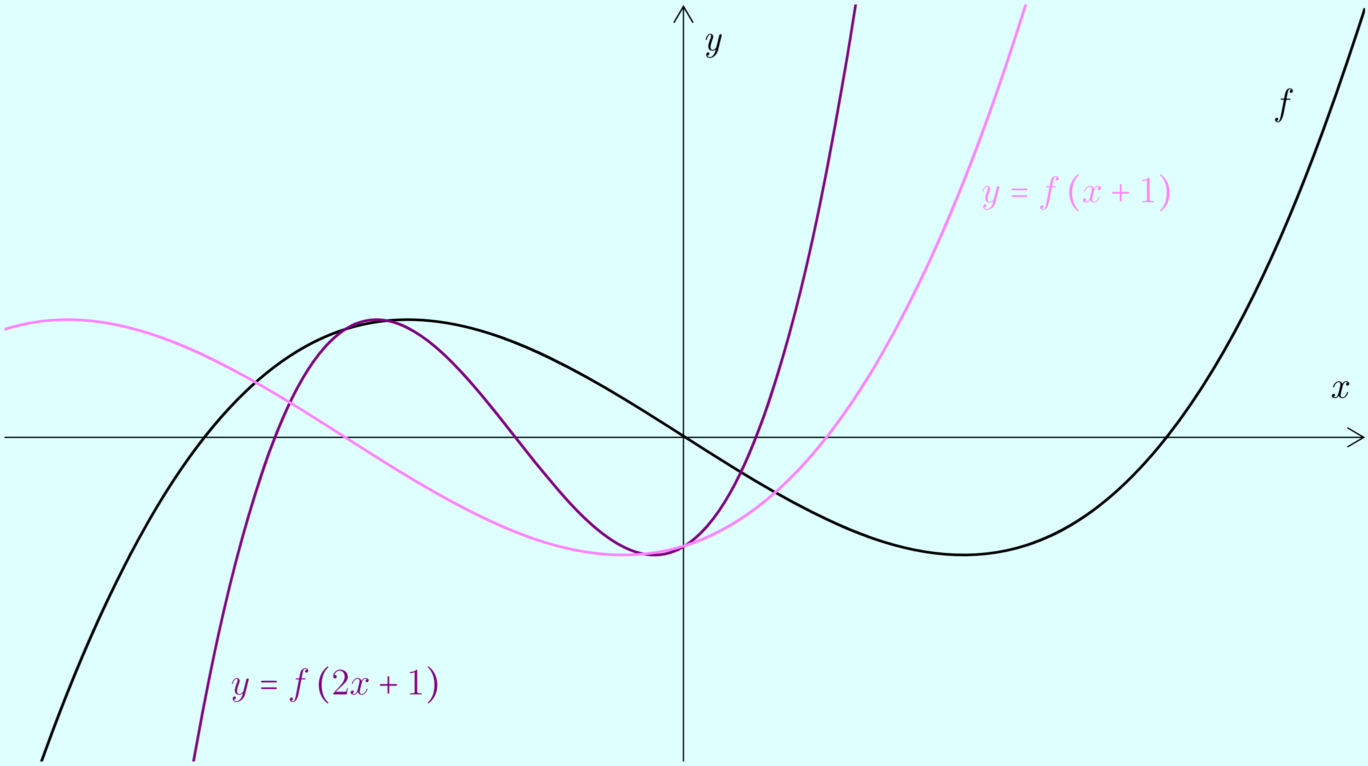
First *compress* f horizontally by a factor of 2 to get $y = f(2x)$, then translate upwards by 1 unit to get $y = f(2x) + 1$.



(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

First translate f *left*wards by 1 unit to get $y = f(x+1)$, then *compress* horizontally by a factor of 2 to get $y = f(2x+1)$.



Exercise 133. Continue with f from the last example. Graph the following eight equations. (Hint: You can make use of what was already shown in the above example.)

(a) $y = -2f(x) - 1$. **(b)** $y = 2f(-x) + 1$. (Answers on p. 1788.)

(c) $y = -2f(x+1)$. **(d)** $y = 2f(-x+1)$. (Answers on p. 1789.)

(e) $y = -f(2x) + 1$. **(f)** $y = f(-2x) + 1$. (Answers on p. 1790.)

(g) $y = -f(2x+1)$. **(h)** $y = f(-2x+1)$. (Answers on p. 1791.)

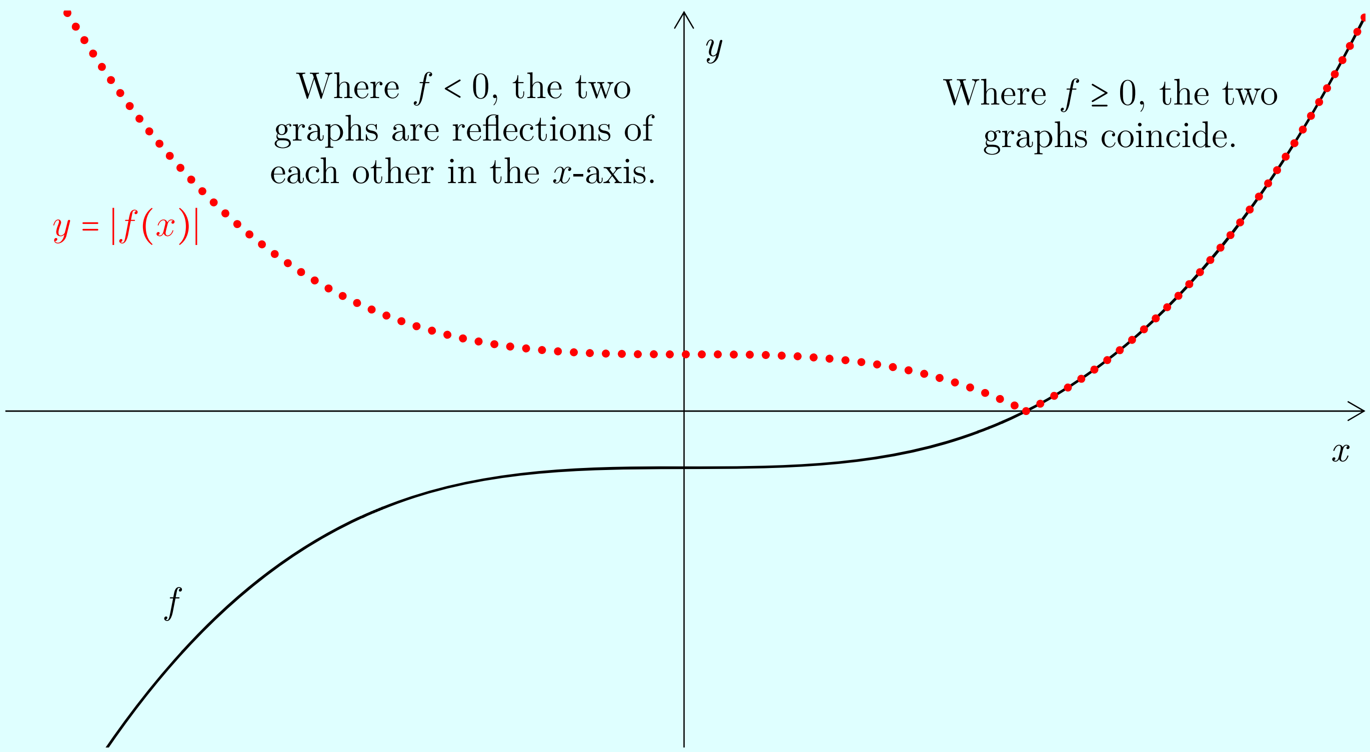
26.6. $y = |f(x)|$

Given the graph of f, we can easily graph $y = |f(x)|$:

1. Where $f \geq 0$ (i.e. above the x-axis), the two graphs coincide.
2. But where $f < 0$ (i.e. below the x-axis), they're reflections of each other in the x-axis.

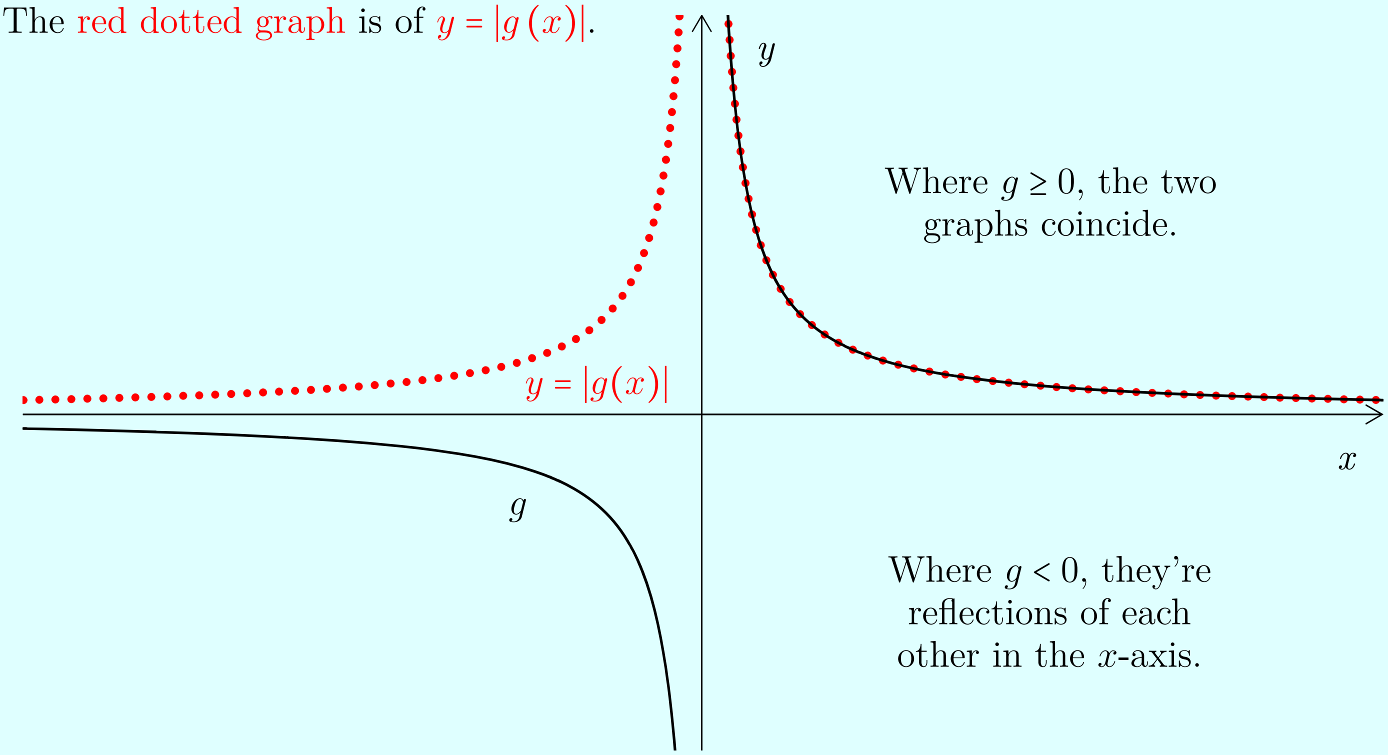
Example 422. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 1$.

The red dotted graph is of $y = |f(x)|$.



Example 423. Define $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $g(x) = 1/x$.

The red dotted graph is of $y = |g(x)|$.

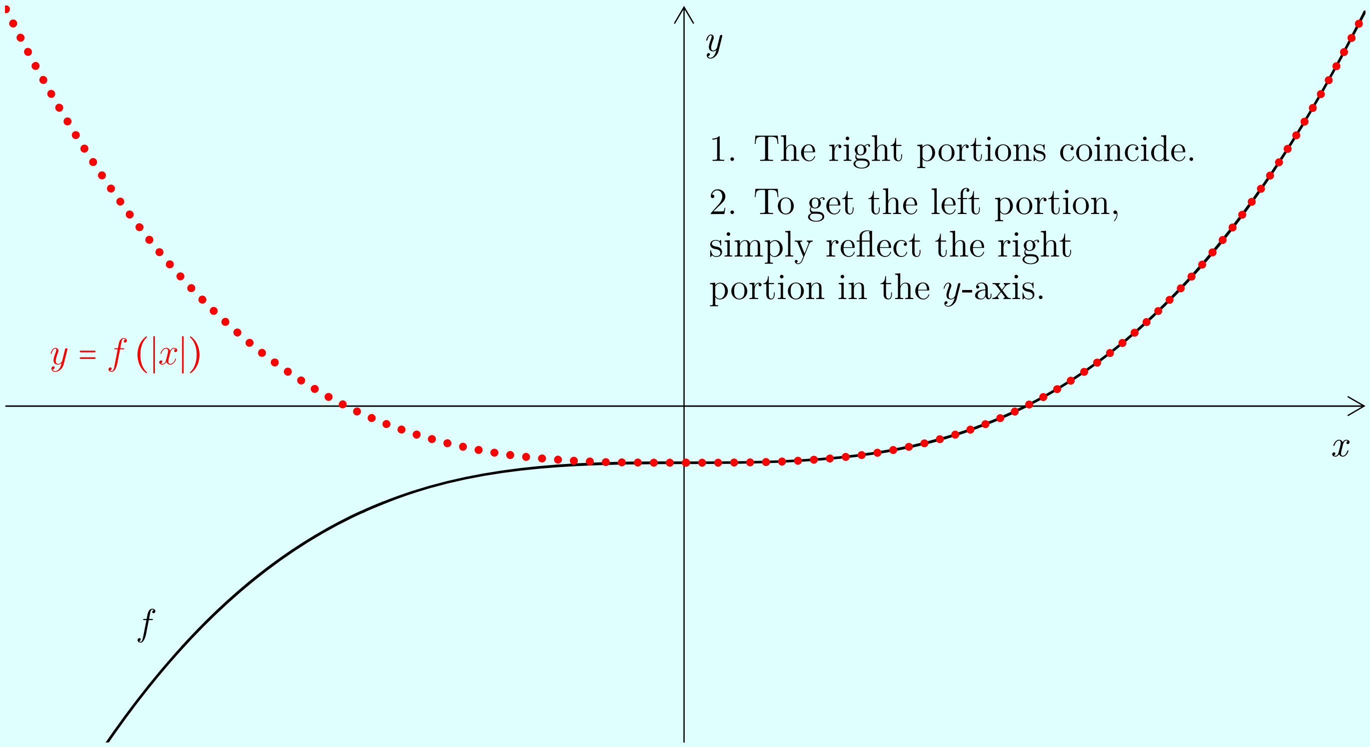


26.7. $y = f(|x|)$

Given the graph of f, we can easily graph $y = f(|x|)$:

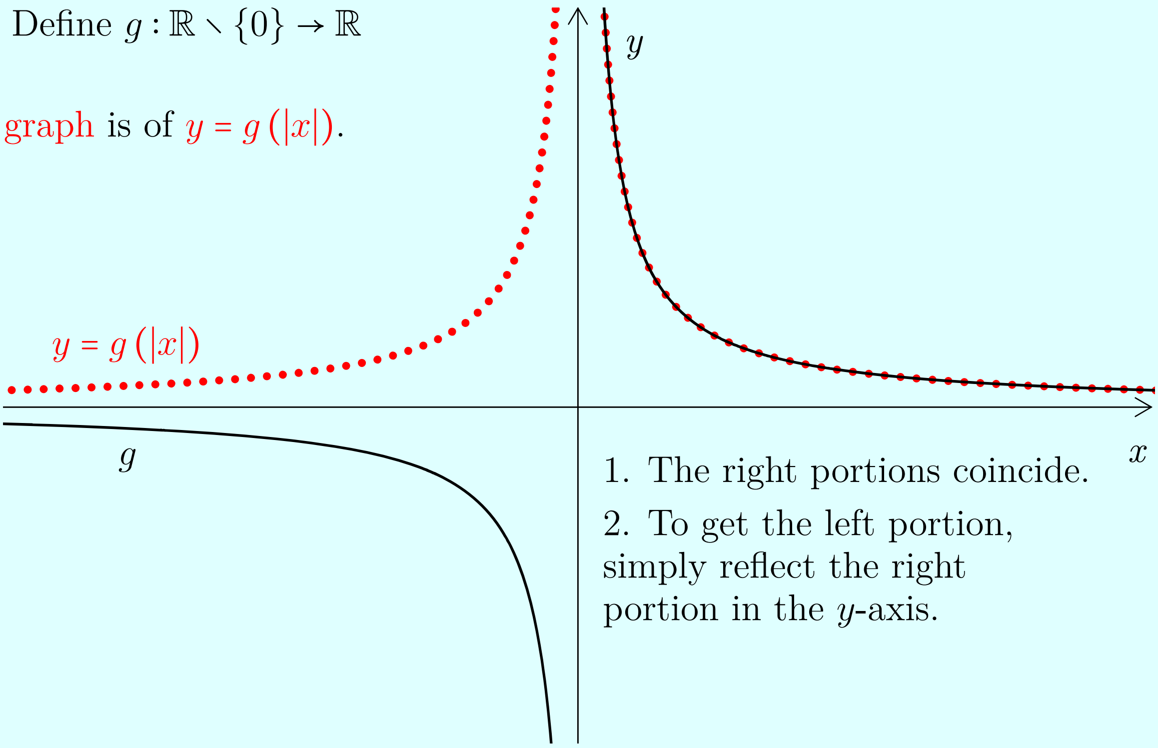
1. Where $x \geq 0$, the two graphs coincide.
2. Where $x < 0$, they're reflections of each other in the y-axis.

Example 424. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 1$. The red dotted graph is of $y = f(|x|)$.



Example 425. Define $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $g(x) = 1/x$.

The red dotted graph is of $y = g(|x|)$.



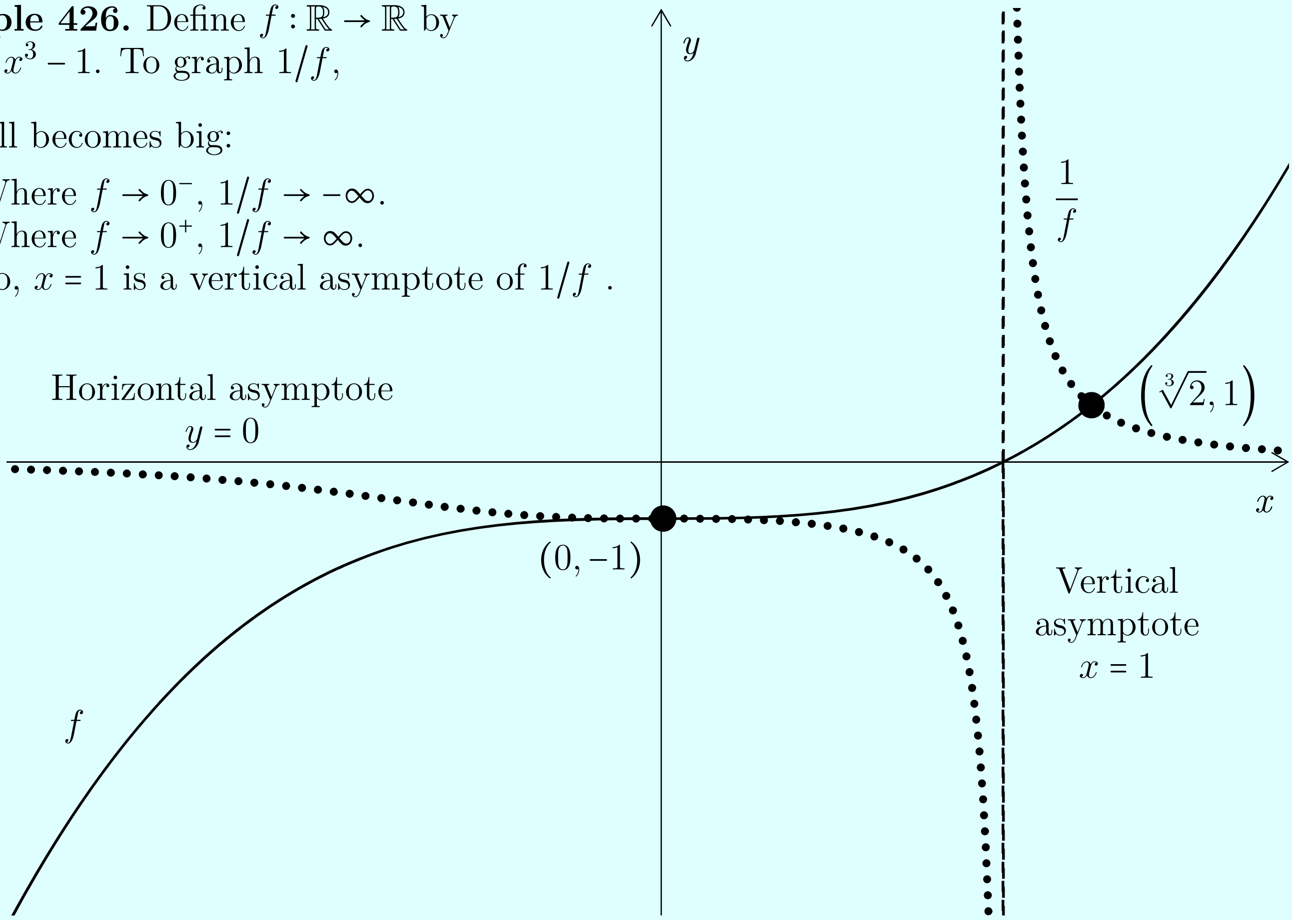
26.8. $\frac{1}{f}$

Given the graph of f, we can easily graph $1/f$:

1. Small becomes big ("0 becomes $\pm\infty$"—x-intercepts become vertical asymptotes).
2. Big becomes small ("$\pm\infty$ becomes near 0"—vertical asymptotes become x-intercepts).
3. The two graphs intersect wherever $f(x) = \pm 1$ (because $1/f(x) = \pm 1$ too).
4. y-intercept $(0, a)$ becomes y-intercept $(0, 1/a)$.

Example 426. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 1$. To graph $1/f$,

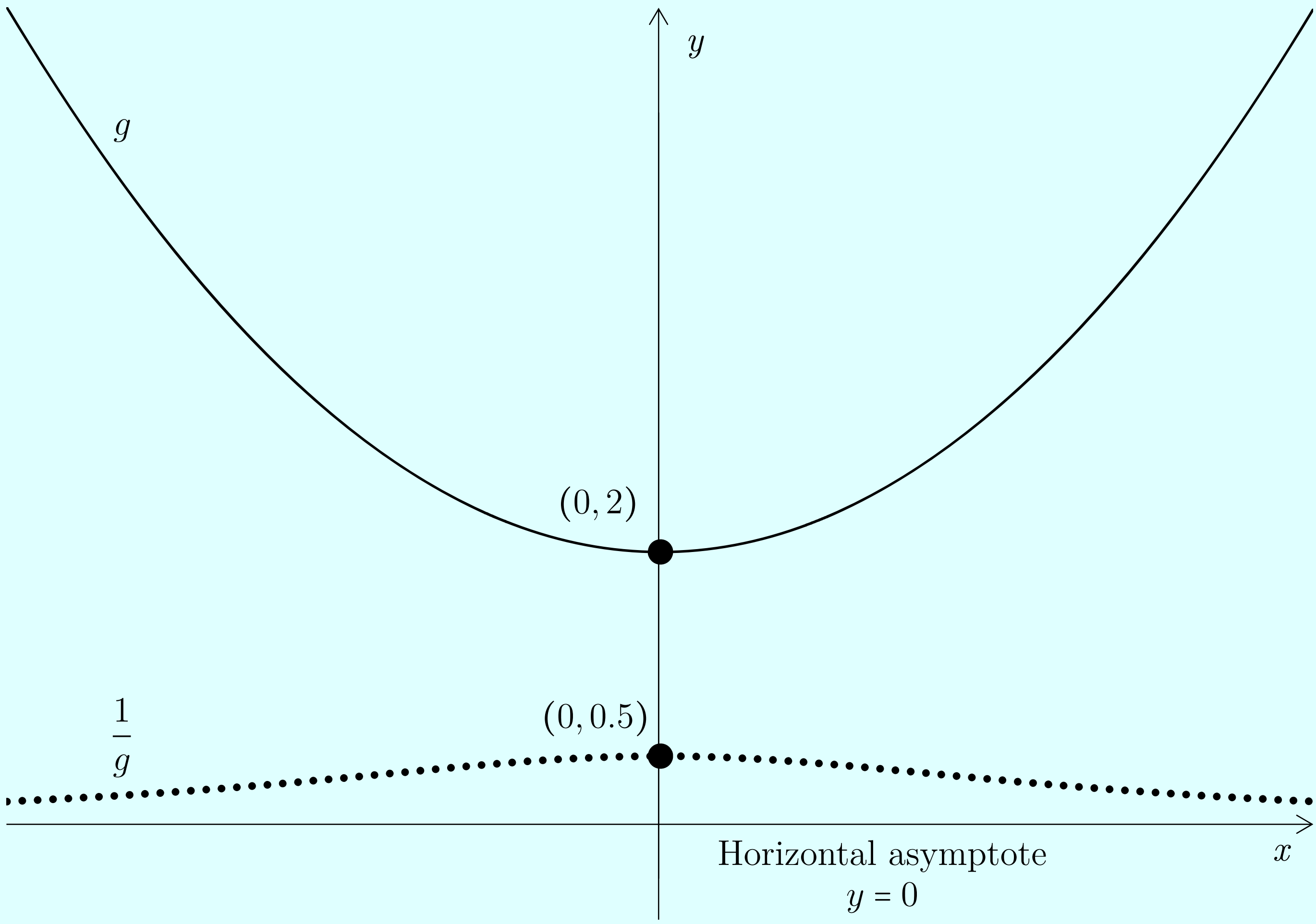
1. Small becomes big:
   - Where $f \to 0^-$, $1/f \to -\infty$.
   - Where $f \to 0^+$, $1/f \to \infty$.
   - So, $x = 1$ is a vertical asymptote of $1/f$.



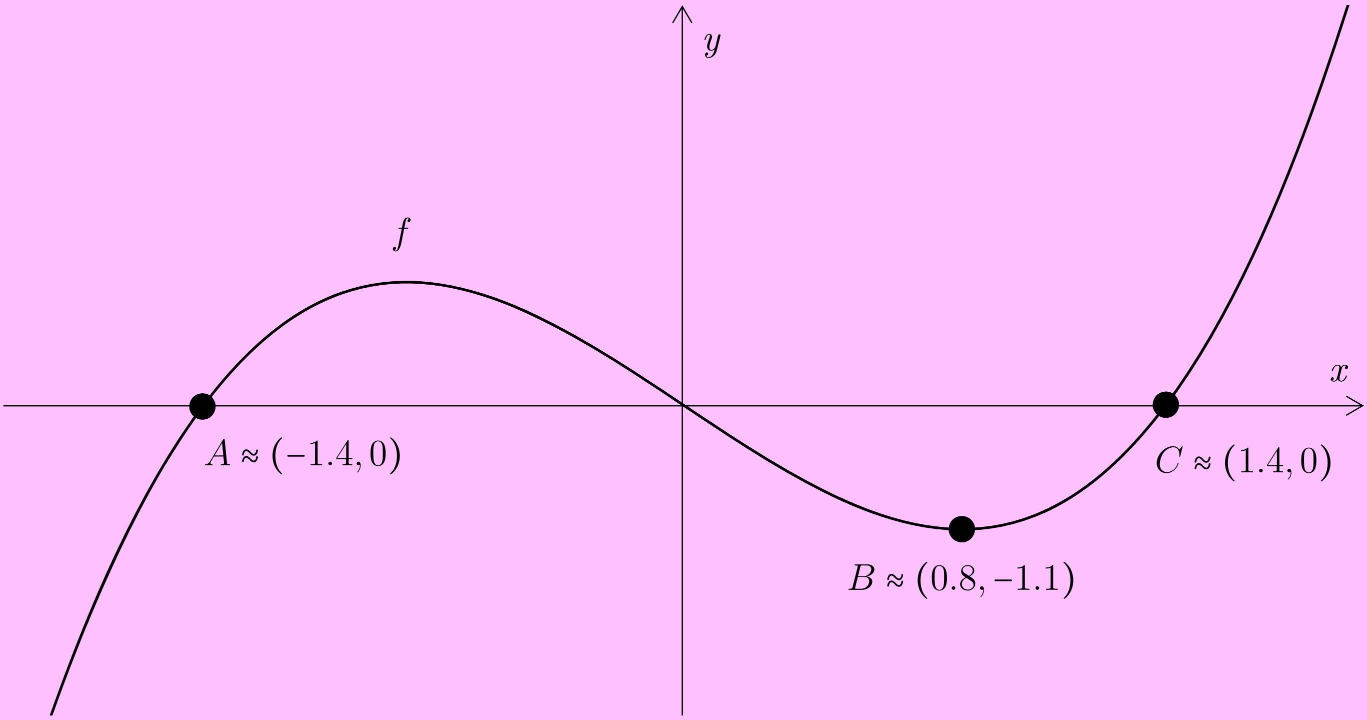
2. Big becomes small:
   (a) Where $f \to -\infty$, $1/f \to 0^-$.
   (b) Where $f \to \infty$, we have $1/f \to 0^+$.
   (c) So, $y = 0$ is a horizontal asymptote of $1/f$.
3. Intersect at $f(x) = \pm 1$: So, f and $1/f$ intersect at $\left(\sqrt[3]{2}, 1\right)$ and $(0, -1)$.
4. y-intercept $(0, -1)$ becomes y-intercept $(0, 1/-1) = (0, -1)$ (here it so happens that the y-intercepts are the same, but this won't generally be the case).

Example 427. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2 + 2$. To graph $1/g$,

1. Small becomes big: Not applicable here.
2. Big becomes small:
   - Where $g \to \infty$, $1/g \to 0^+$.
   - So, $y = 0$ is a horizontal asymptote of $1/g$.
3. Intersect at $g(x) = \pm 1$: But here there is no x at which $g(x) = \pm 1$. So, g and $1/g$ do not intersect at all.
4. y-intercept $(0, 2)$ becomes y-intercept $(0, 1/2)$.



Exercise 134. Graphed below is some unknown function f. You are told only that points A, B, and C are (approximately) $(-1.4, 0)$, $(0.8, -1.1)$, and $(1.4, 0)$. Graph **(a)** $y = |2f(2x)|$; and **(b)** $y = f(|x-1|) + 2$ (Answer on p. 1792.)



Exercise 135. Describe a sequence of transformations that would transform the graph of

$$y = \frac{1}{x} \qquad \text{onto} \qquad y = 3 - \frac{1}{5x-2}.$$

(Answer on p. 1793.)

27. Reflection and Symmetry for Functions

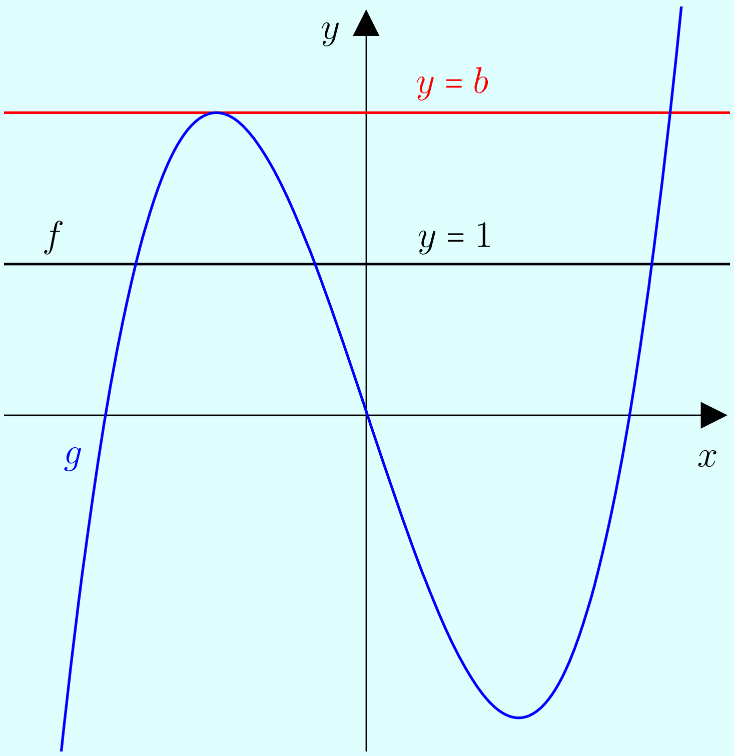
Everything we learnt in Ch. 16 about the reflection and symmetry of graphs also applies to the graphs of functions.

Definition 81. Let f be a function and G be a graph. We say that f is symmetric in G if the graph of f is symmetric in G.

27.1. (Almost) No Function Is Symmetric in a Horizontal Line

Recall that a function must pass the vertical line test. So, not surprisingly, it's (almost) impossible for a function to be symmetric in a horizontal line.

Example 428. Graphed below are two functions f and g, and two horizontal lines $y = 1$ and $y = b$.



(a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1$.

(i) "Obviously", f is symmetric in the horizontal line $y = 1$.

(ii) f is not symmetric in $y = b$ or any other horizontal line.

(b) g is not symmetric in $y = 1$, $y = b$, or any other horizontal line.

The function f illustrates the one and only (unusual) way by which a function could be symmetric in a horizontal line—when the graph of the function is identical to (or a subset of) the horizontal line.

More formally and generally,

Fact 59. *Let $a \in \mathbb{R}$ and f be a nice function.*

(a) *Suppose $f(x) \stackrel{1}{=} a$ for all $x \in \operatorname{Domain} f$. Then*

- **(i)** *f is symmetric in the horizontal line $y = a$; and*
- **(ii)** *f is not symmetric in any other horizontal line.*

(b) *Suppose f is not constant. Then f is not symmetric in any horizontal line.*

Proof. Let F be the graph of f.

(a)(i) By Fact 40, the reflection of any $(x, a) \in F$ in $y = a$ is $(x, 2a - a) = (x, a)$, which is also in F. So, by Fact 43, f is symmetric in $y = a$.

For a formal proof of **(a)(ii)** and **(b)**, see p. 1585 (Appendices). □

27.2. (Almost) No One-to-One Function Is Sym. in a Vert. Line

Recall that a one-to-one function must pass the horizontal line test. So, not surprisingly, it's (almost) impossible for a one-to-one function to be symmetric in a vertical line.

Example 429. Graphed below are two one-to-one functions f and g, and two vertical lines $x = 1$ and $x = b$.

Figure to be inserted here.

(a) Define $f : \{1\} \to \mathbb{R}$ by $f(1) = 2$.

- **(i)** "Obviously", f is symmetric in the vertical line $x = 1$.
- **(ii)** f is not symmetric in $x = b$ or any other vertical line.

(b) g is not symmetric in $x = 1$, $x = b$, or any other vertical line.

The function f illustrates the one and only (unusual) way by which a one-to-one function could be symmetric in a vertical line—when the graph of the function is a single point on the vertical line.

More formally and generally,

Fact 60. *Let $a \in \mathbb{R}$ and f be a nice, one-to-one function (whose domain is not empty).*

(a) *Suppose* Domain $f = \{a\}$. *Then*

- **(i)** *f is symmetric in the vertical line $x = a$; and*
- **(ii)** *f is not symmetric in any other vertical line.*

(b) *Suppose* Domain f *contains more than one point. Then f is not symmetric in any vertical line.*

Proof. Let F be the graph of f.

(a) Given Domain $f = \{a\}$, we have $F = \{(a, f(a))\}$—i.e., the graph of f consists of only the single point $(a, f(a))$.

(a)(i) By Fact 39, the reflection of $(a, f(a))$ in $x = a$ is $(2a - a, f(a)) = (a, f(a))$, which is also in F. So, by Fact 42, f is symmetric in $x = a$.

(a)(ii) Let $b \neq a$, so that $2b - a \neq a$. By Fact 39, the reflection of $(a, f(a))$ in the vertical line $x = b$ is $(2b - a, f(a)) \neq (a, f(a))$, which is the only point in F. So, by Fact 42, f is not symmetric in $x = b$.

(b) See p. 1586 (Appendices). □

27.3. An Even Function Is Symmetric in the Vertical Axis

Definition 82. A nice function f is called an *even function* if $f(x) = f(-x)$ for all $x \in \text{Domain } f$.

Example 430. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is even because for every $x \in \text{Domain } f = \mathbb{R}$,

$$f(x) = x^2 = (-x)^2 = f(-x).$$

Example 431. The absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ is even because for every $x \geq 0$,

$$|x| = x = |-x|.$$

And similarly, for every $x < 0$,

$$|x| = -x = |-x|.$$

Fact 61. *Suppose f is a nice function. Then f is even $\iff$ f is symmetric in the y-axis.*

Proof. See p. 1586 (Appendices). □

Example 432. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is both even and symmetric in the y-axis:

Figure to be
inserted here.

Example 433. The absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ is even and symmetric in the y-axis:

Figure to be
inserted here.

Example 434. It turns out that for every positive even integer n, the function $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^n$ is even (and hence also symmetric in the y-axis):

Since n is a positive even integer, for each $x \in \mathbb{R}$, $f_n(x) = x^n = (-1)^n x^n = (-x)^n = f_n(-x)$.

Figure to be
inserted here.

As we'll see shortly, the **cosine** function is another example of an even function.

27.4. An Odd Function Is Symmetric about the Origin

Definition 83. A nice function f is called an *odd function* if $-f(x) = f(-x)$ for all $x \in \text{Domain } f$.

Example 435. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is odd because for every $x \in \text{Domain } f = \mathbb{R}$,

$$-f(x) = -x^3 = (-x)^3 = f(-x).$$

Example 436. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x$ is odd because for every $x \in \text{Domain } g = \mathbb{R}$,

$$-f(x) = -x = f(-x).$$

Fact 62. *Suppose f is a nice function. Then f is odd $\iff$ f is symmetric about the origin.*

Proof. See p. 1586 (Appendices). □

Example 437. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is both odd and symmetric about the origin:

Figure to be
inserted here.

Example 438. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x$ is both odd and symmetric about the origin:

Figure to be
inserted here.

Example 439. It turns out that for every positive odd integer n, the function $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^n$ is odd (and hence also symmetric about the origin):

Since n is a positive odd integer, for each $x \in \mathbb{R}$, $-f_n(x) = -x^n = (-1)(-1)^{n-1} x^n = (-x)^n = f_n(-x)$.

Figure to be
inserted here.

As we'll see shortly, the **sine** and **tangent** functions are also examples of odd functions.

28. ln, exp, and e

In secondary school, your teachers probably went in this order:

1. First introduce **Euler's number** $\mathrm{e} = 2.718\,281\,828\,459\ldots$
2. *Then* define the **natural logarithm** ln as the logarithm with base e.

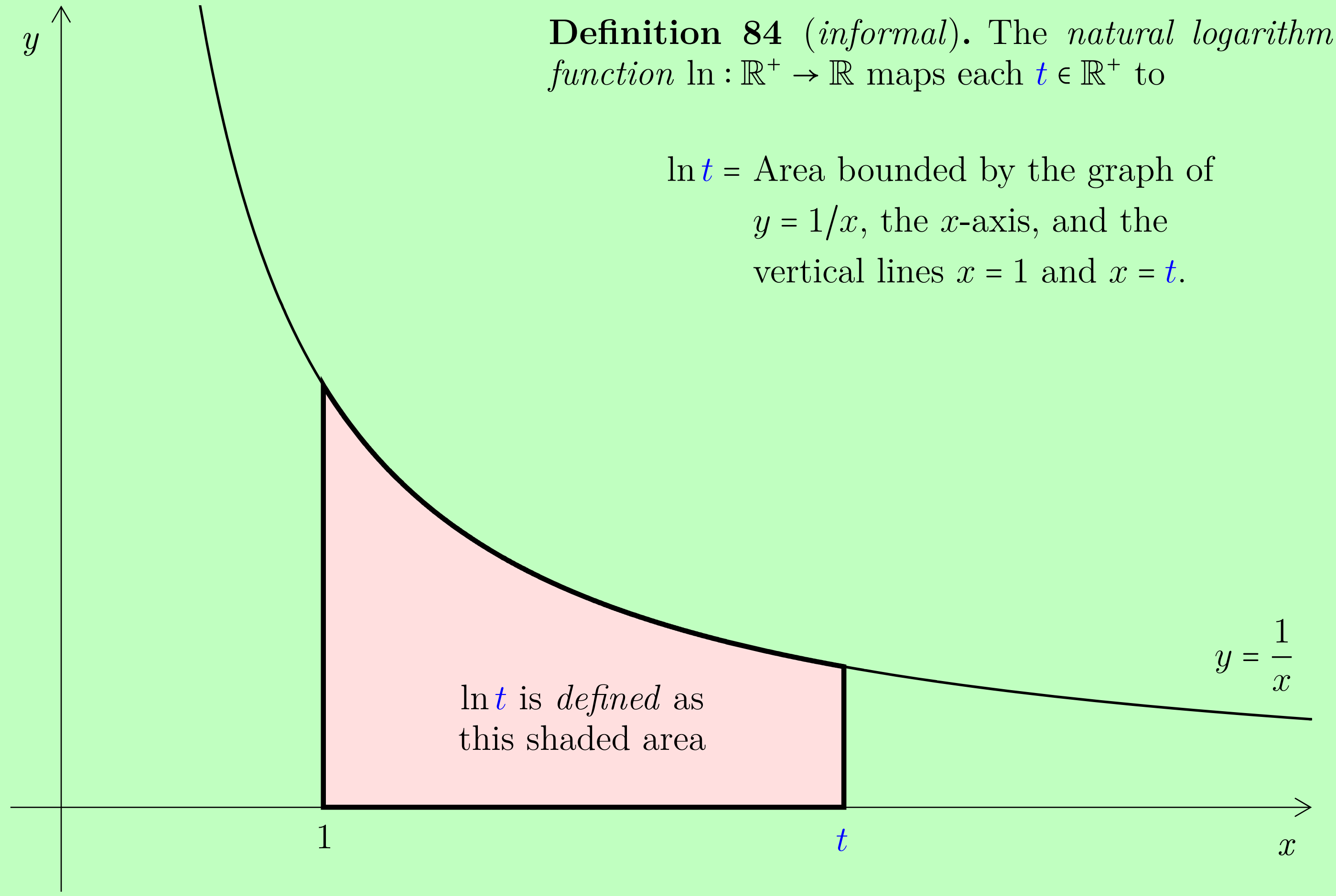
In this textbook, we'll go the other way round. We'll

1. First define the natural logarithm function ln.
2. *Then* define Euler's number e as being the number such that $\ln \mathrm{e} = 1$.

This may seem strange but will pay off in Part V (Calculus).

Definition 84 (*informal*)**.** The *natural logarithm function* $\ln : \mathbb{R}^+ \to \mathbb{R}$ maps each $t \in \mathbb{R}^+$ to

$\ln t =$ Area bounded by the graph of $y = 1/x$, the x-axis, and the vertical lines $x = 1$ and $x = t$.

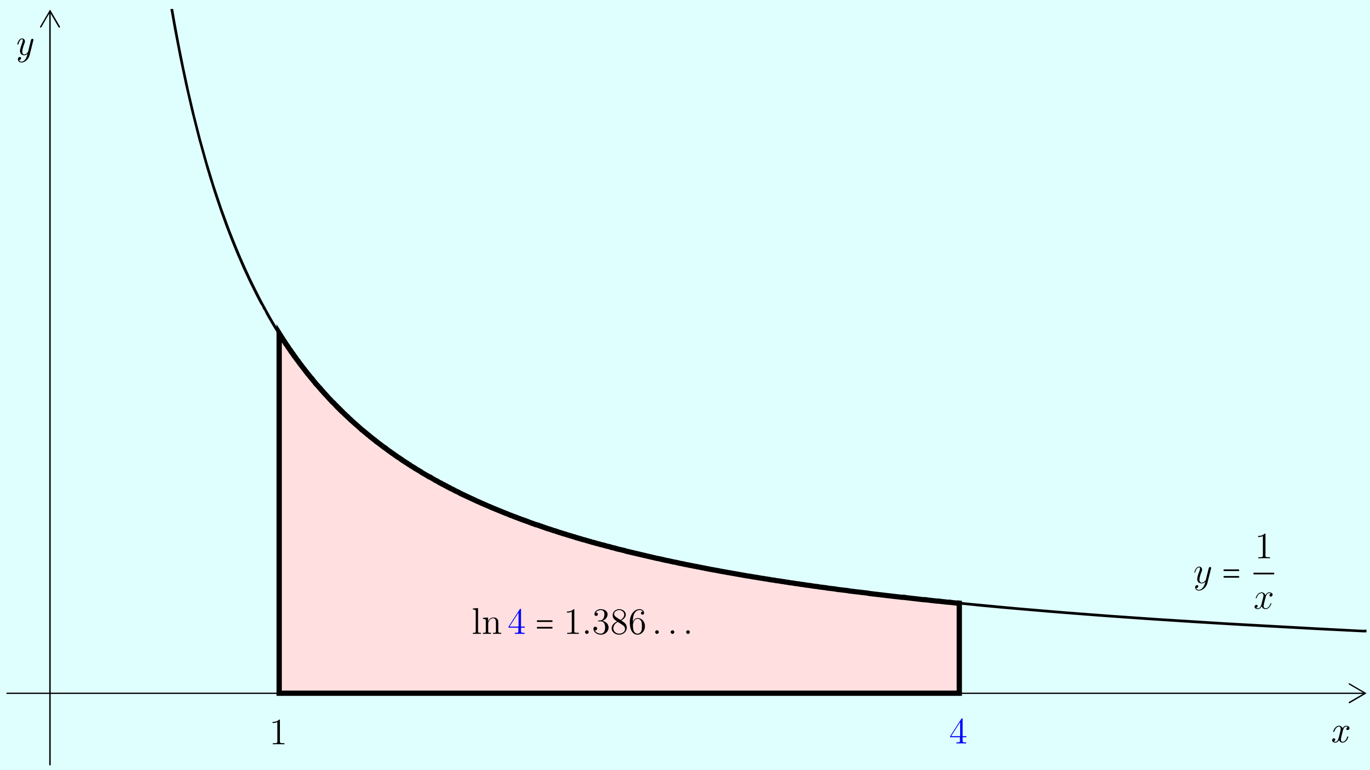


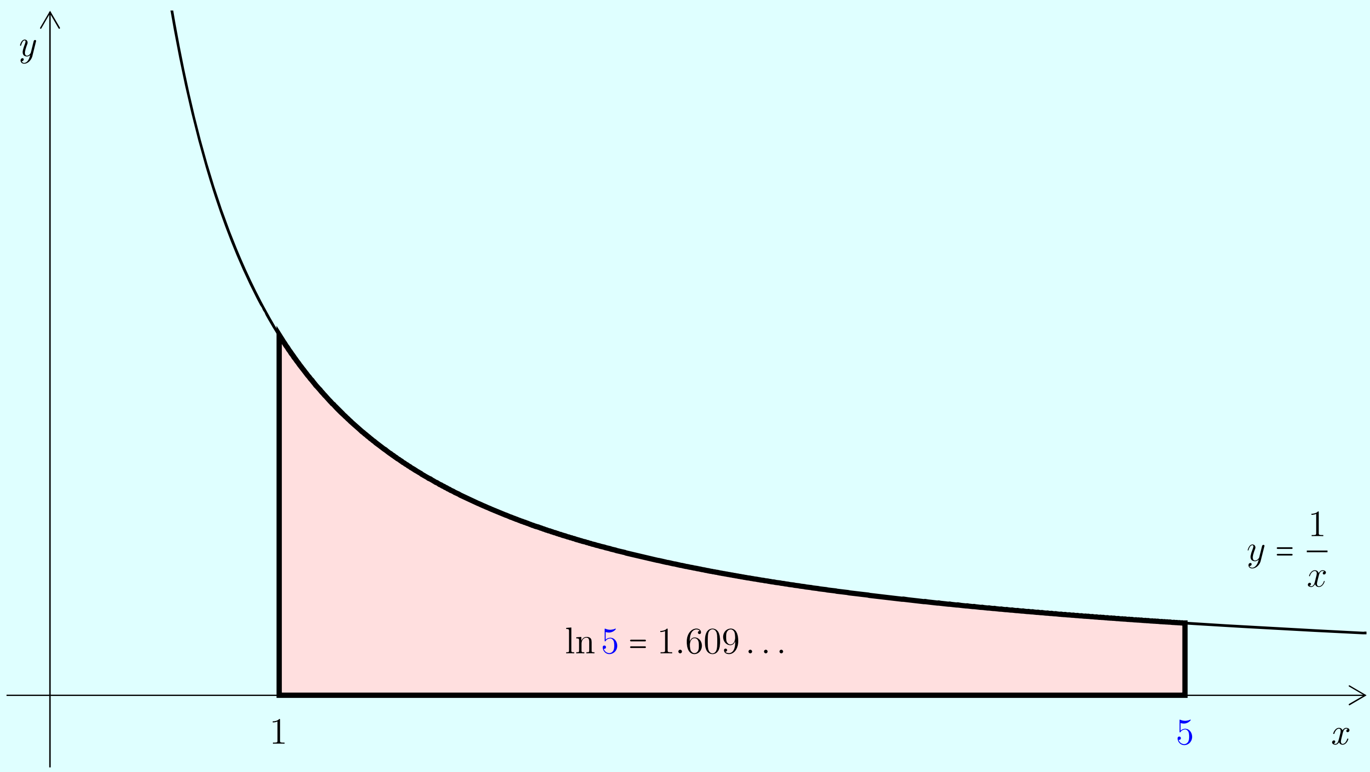
The above definition is considered informal because the mapping rule is described using geometry (and in particular, with reference to an "area" bounded by a curve and three lines). After we've learnt about the definite integral in Part V (Calculus), we'll give Definition 229—a formal definition of the natural logarithm function that supersedes the above informal definition.

Remark 58. In Singapore and some other Britishy bits of the world, ln is usually read aloud as *lawn*. In the US, it's usually read out loud as "*el en*".

The notation ln was probably first published in Steinhauser (1875, p. 277), where it stood for the Latin *Logarithmus naturalis*.

Example 440. We have $\ln 4 = 1.386\ldots$ and $\ln 5 = 1.609\ldots$

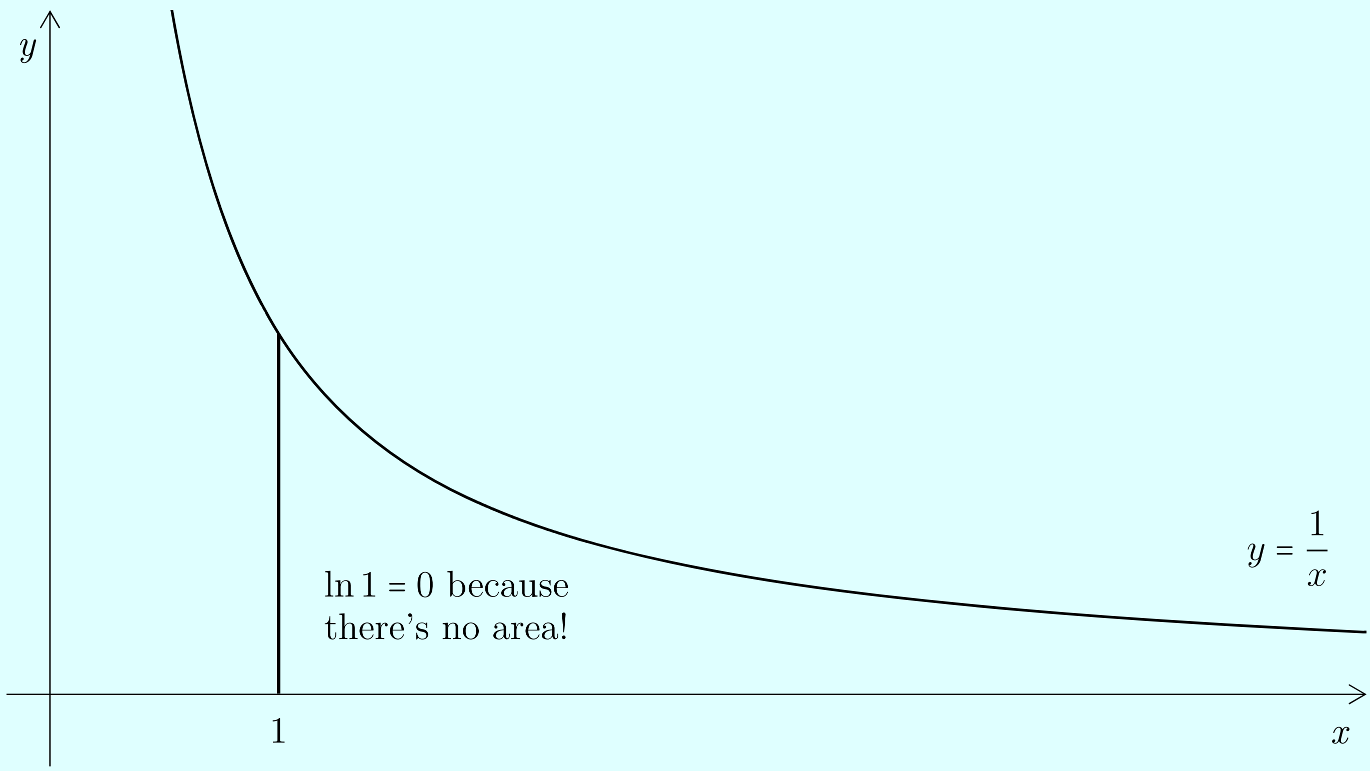




Note that right now, we don't yet know how to calculate $\ln 4$ or $\ln 5$.

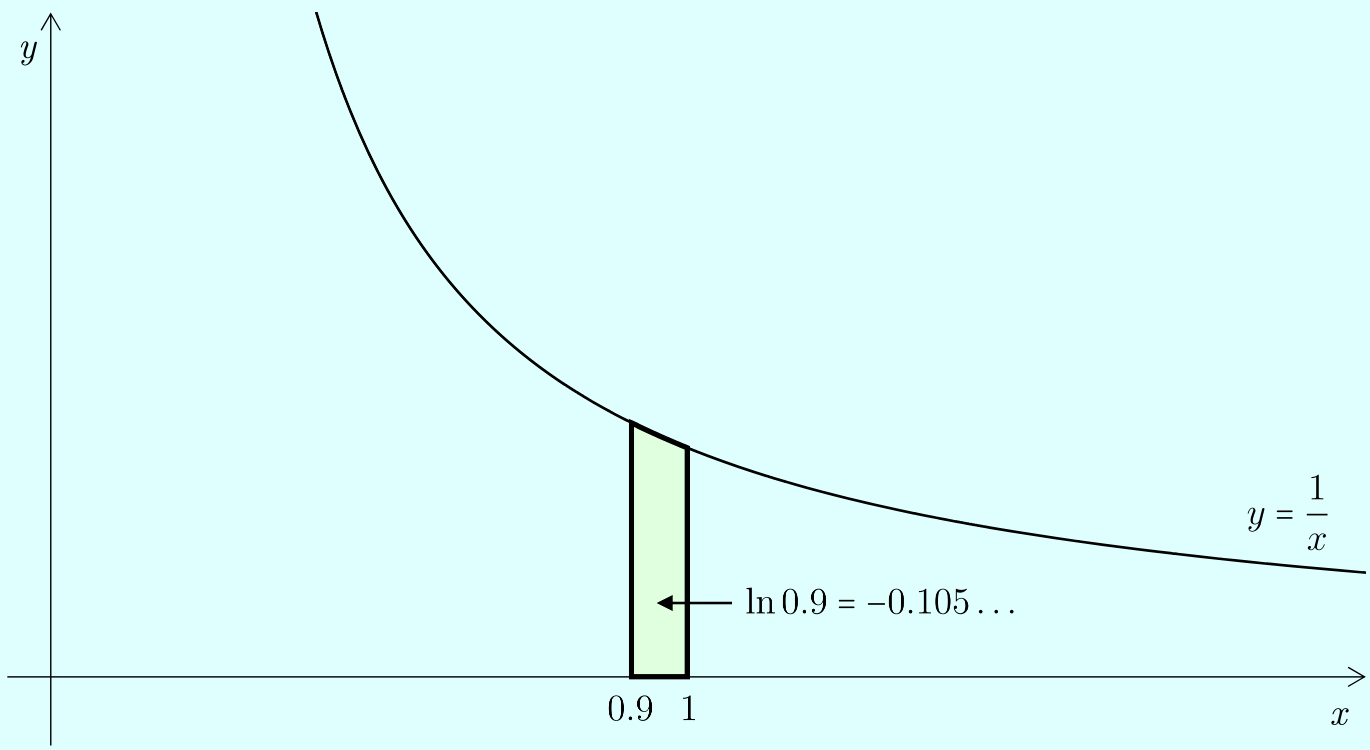
One possibility is to draw an extremely precise and large graph of $y = 1/x$ on some graph paper, then slowly count up the squares. This sounds like a ridiculous idea, but as we'll learn in Part V, it isn't too different from how integration is actually done.

Example 441. By definition, $\ln 1$ is the area under $y = 1/x$, between $x = 1$ and $x = 1$. Hence, $\ln 1 = 0$.

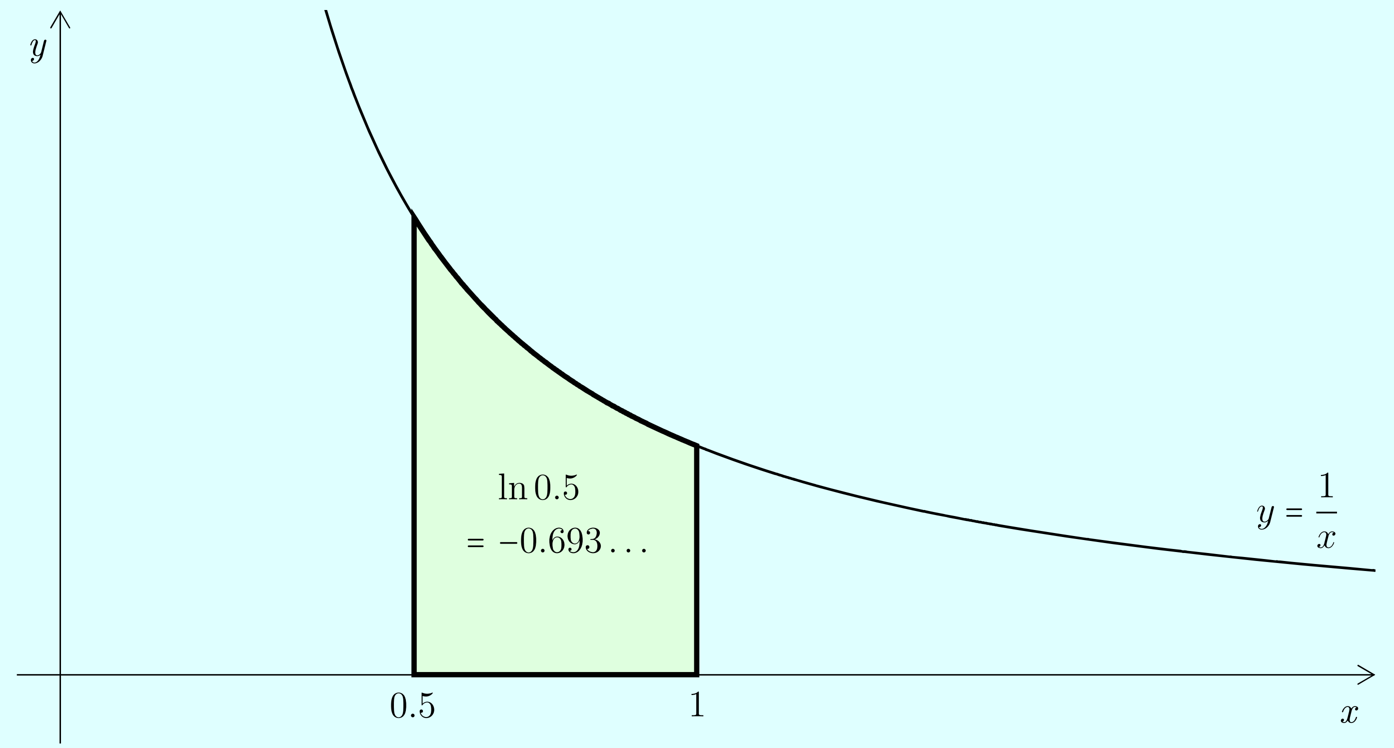


Note that if $x < 1$, then $\ln x$ is **negative**:

Example 442. $\ln 0.9 = -0.105\ldots$



Example 443. $\ln 0.5 = -0.693\ldots$

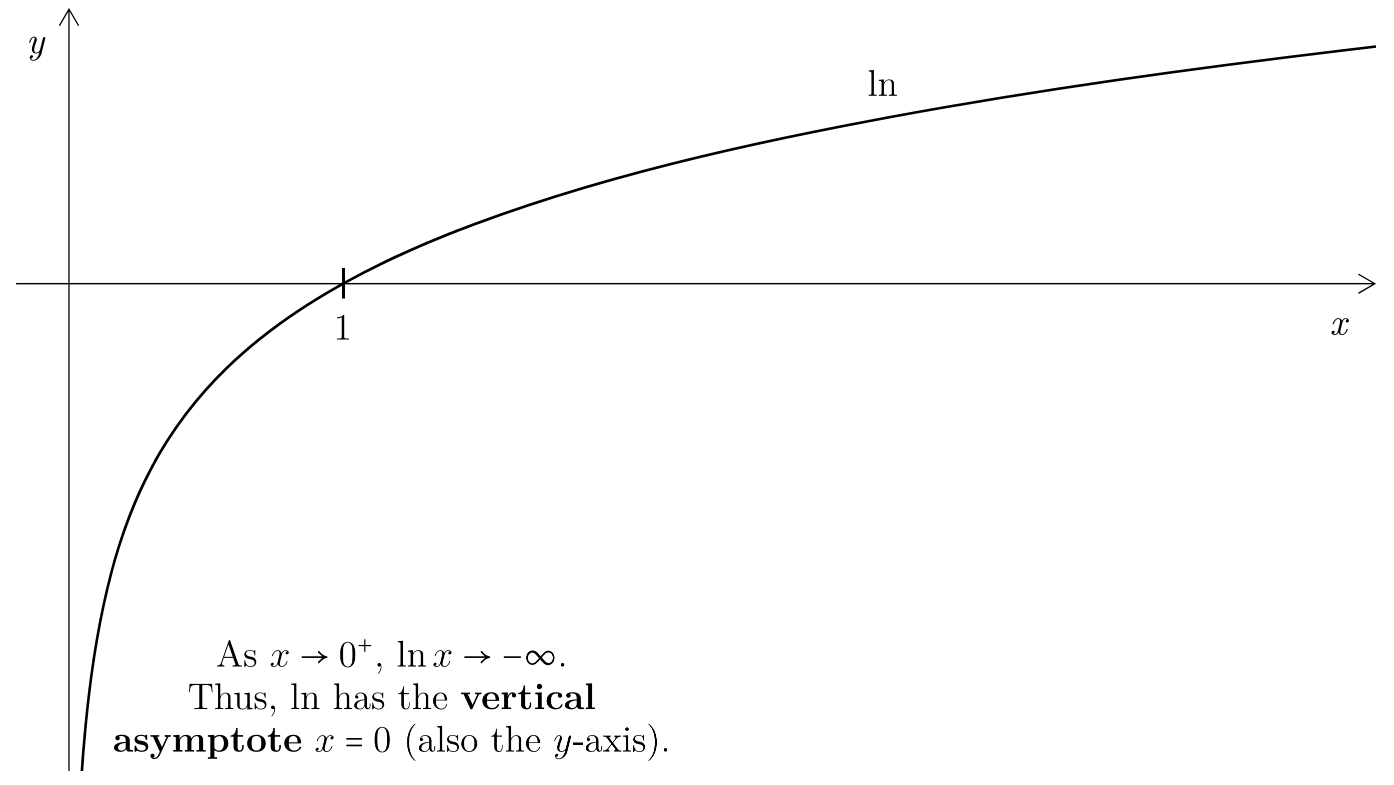


Note that since Domain $\ln = \mathbb{R}^+$, if $x \le 0$, then $\ln x$ is simply undefined:

Example 444. $\ln 0$ and $\ln(-2.5)$ are undefined.

Below is the graph of ln.

- Since $\ln 1 = 0$, the x-intercept of ln is $(1, 0)$.
- For $x \in (0, 1)$, $\ln x < 0$.
- For $x \le 0$, $\ln x$ is simply undefined.



28.1. The Exponential Function exp

For $x > 0$, the graph of $y = 1/x$ is strictly positive. But ln is defined as the area under the graph of $y = 1/x$. So, ln is strictly increasing. Hence, by Fact 49, ln is one-to-one and has an inverse.

Now, we don't know what exactly this inverse function is, but we know it exists. So, let's simply name it the **exponential function**:

Definition 85. The *exponential function*, denoted exp, is the inverse of the natural logarithm function.

By Definition 79 (of inverse functions),

1. Domain exp = Range ln = $\mathbb{R}$;
2. Codomain exp = Domain ln = $\mathbb{R}^+$;
3. Mapping rule: $y = \ln x \implies \exp y = x$.

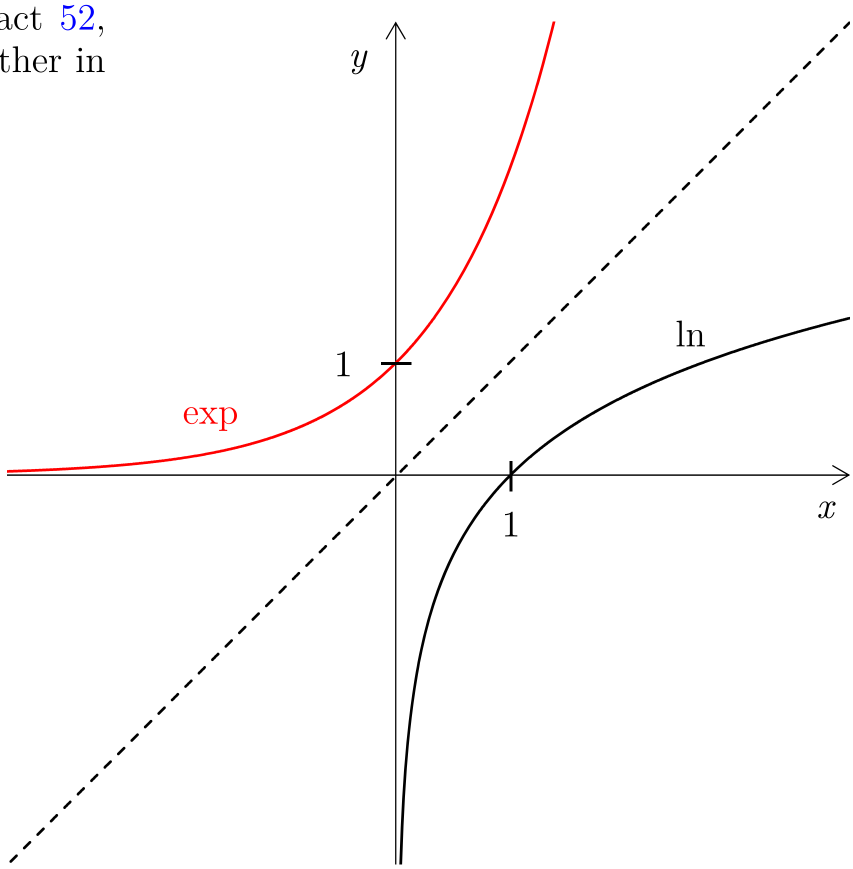
Example 445. Earlier, we saw that

$$\ln 4 \approx 1.386, \quad \ln 5 \approx 1.609, \quad \ln 1 = 0, \quad \ln 0.9 \approx -0.105, \quad \ln 0.5 \approx -0.693.$$

Since exp is the inverse of ln, we have

$$\exp 1.386 \approx 4, \quad \exp 1.609 \approx 5, \quad \exp 0 = 1, \quad \exp(-0.105) \approx 0.9, \quad \exp(-0.693) \approx 0.5.$$

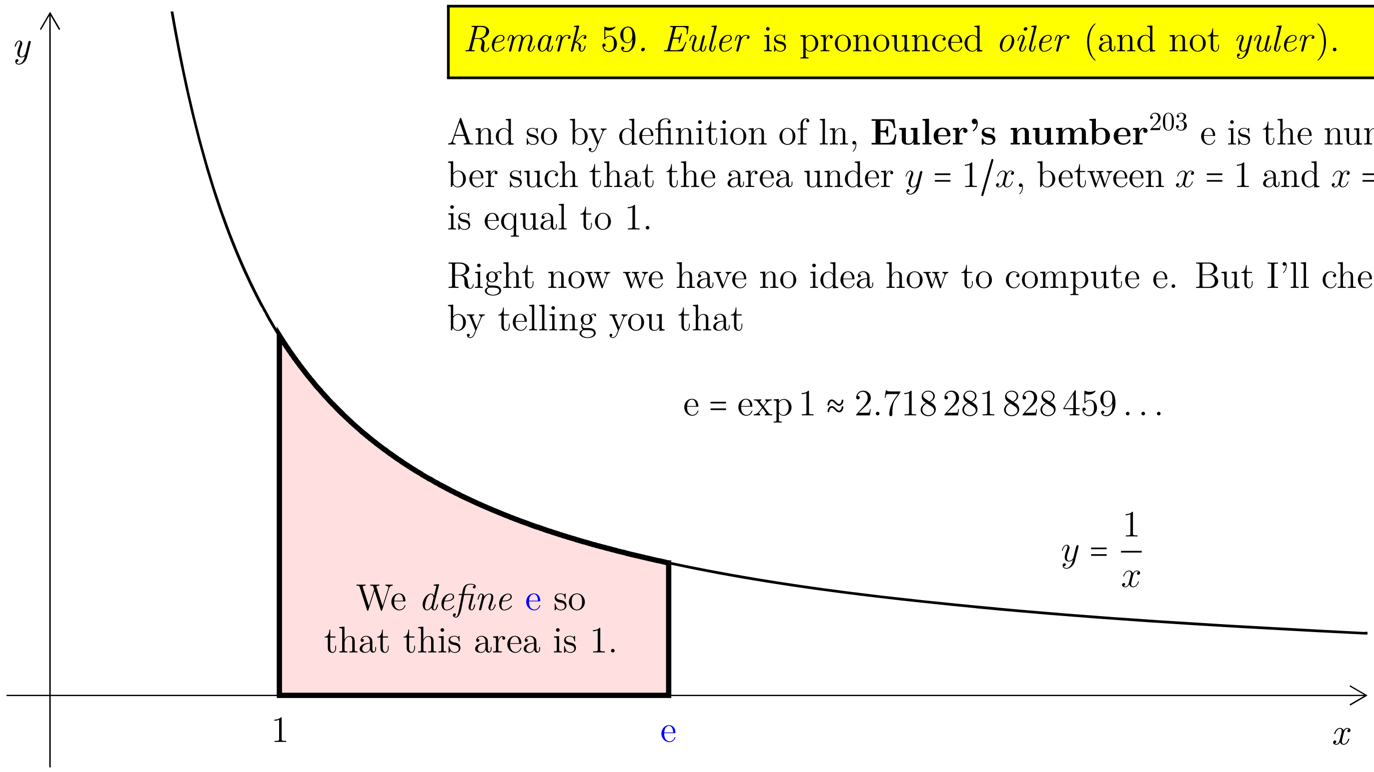
Since exp is the inverse of ln, by Fact 52, their graphs are reflections of each other in the line $y = x$:



We next introduce Euler's number e.

Definition 86. *Euler's number*, denoted e, is the number that satisfies

$$\ln \mathrm{e} = 1 \qquad \text{or equivalently,} \qquad \mathrm{e} = \exp 1.$$



Remark 59. *Euler* is pronounced *oiler* (and not *yuler*).

And so by definition of ln, **Euler's number**[203] e is the number such that the area under $y = 1/x$, between $x = 1$ and $x = \mathrm{e}$ is equal to 1.

Right now we have no idea how to compute e. But I'll cheat by telling you that

$$\mathrm{e} = \exp 1 \approx 2.718\,281\,828\,459\ldots$$

In secondary school, you were probably taught that e^x is simply another way to write $\exp x$. It turns out that this is actually a *result* that must be proven using the definitions of the exponential function (given above) and exponentiation:

Fact 63. *For every* $x \in \mathbb{R}$, $\mathrm{e}^x = \exp x$.

Proof. See p. 1726 (Appendices). □

So, it is **not** that e^x is simply another way to write $\exp x$. Instead, it takes some work to prove that $\mathrm{e}^x = \exp x$—i.e. that the number e raised to the power of x is equal to the value of the exponential function at x.

Remark 60. While interesting, Euler's number e is by itself not particularly important. What's really important are the **natural logarithm** and **exponential functions**.[204]

For this reason, we will in this textbook often write $\exp x$ rather than e^x. This is to emphasise that we're usually thinking more about the value of the **exponential function** at x than about some number e raised to the power of x.

[203] It was Leonhard Euler (1707–1783) himself who first used the letter e to denote this number. Presumably he did not do this to honour himself. Calling e *Euler's number* is simply an honour conferred by posterity. Confusingly, there's also another number called **Euler's constant** $\gamma \approx 0.577\,215\,664\ldots$ that, fortunately, we will not encounter in A-Level maths. W

[204] Indeed, one mathematician Walter Rudin (*Real and Complex Analysis*, 1966 [1987, 3e, p. 1]) wrote that the exponential function "is the most important function in mathematics".

We also have these two lovely results:

Theorem 5. $\mathrm{e} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$.

Proof. Happily, we'll learn to prove this in Part V (Calculus)—see p. 1007. □

Theorem 6. $\mathrm{e} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$.

Proof. Happily, we'll learn to prove this in Part V (Calculus)—see p. 1151. □

Although we can't prove either of the above theorems right now, we can nonetheless numerically "verify" that they are plausible:

To "verify" Theorem 5, define $f : \mathbb{Z}_0^+ \to \mathbb{R}$ by $f(n) = \frac{1}{0!} + \dots + \frac{1}{n!}$.

Then write

$$f(0) = \frac{1}{0!} = \frac{1}{1} = 1. \qquad f(3) = f(2) + \frac{1}{3!} = 2.5 + \frac{1}{6} = 2.\overline{6}.$$

$$f(1) = f(0) + \frac{1}{1!} = 1 + \frac{1}{1} = 2. \qquad f(4) = f(3) + \frac{1}{4!} = 2.\overline{6} + \frac{1}{24} = 2.71\overline{6}.$$

$$f(2) = f(1) + \frac{1}{2!} = 2 + \frac{1}{2} = 2.5. \qquad f(5) = f(4) + \frac{1}{5!} = 2.71\overline{6} + \frac{1}{120} = 2.718\dots$$

We see that f rapidly converges towards $\mathrm{e} = 2.718\,281\,828\,459\dots$ By $f(6)$, we have e correct to three decimal places. Lovely.

Similarly, to "verify" Theorem 6, define $g : \mathbb{R} \to \mathbb{R}$ by $g(n) = \left(1 + \frac{1}{n}\right)^n$.

Then write

$$g(1) = \left(1 + \frac{1}{1}\right)^1 = 2. \qquad g(10) = \left(1 + \frac{1}{10}\right)^{10} = 2.593\,742\dots$$

$$g(2) = \left(1 + \frac{1}{2}\right)^2 = 2.25. \qquad g(100) = \left(1 + \frac{1}{100}\right)^{100} = 2.704\,813\dots$$

$$g(3) = \left(1 + \frac{1}{3}\right)^3 = 2.\overline{6}. \qquad g(1\,000) = \left(1 + \frac{1}{1\,000}\right)^{1\,000} = 2.716\,923\dots$$

$$g(4) = \left(1 + \frac{1}{4}\right)^4 = 2.708\,\overline{3}. \qquad g\left(10^6\right) = \left(1 + \frac{1}{10^6}\right)^{10^6} = 2.718\,280\dots$$

$$g(5) = \left(1 + \frac{1}{5}\right)^5 = 2.71\overline{6}. \qquad g\left(10^9\right) = \left(1 + \frac{1}{10^9}\right)^{10^9} = 2.718\,281\dots$$

We see that g also converges towards $\mathrm{e} = 2.718\,281\,828\,459\dots$, though much less rapidly than f—for example, $g(100)$ gets e correct only to two decimal places, while even $g\left(10^6\right)$ gets e correct to only five decimal places.

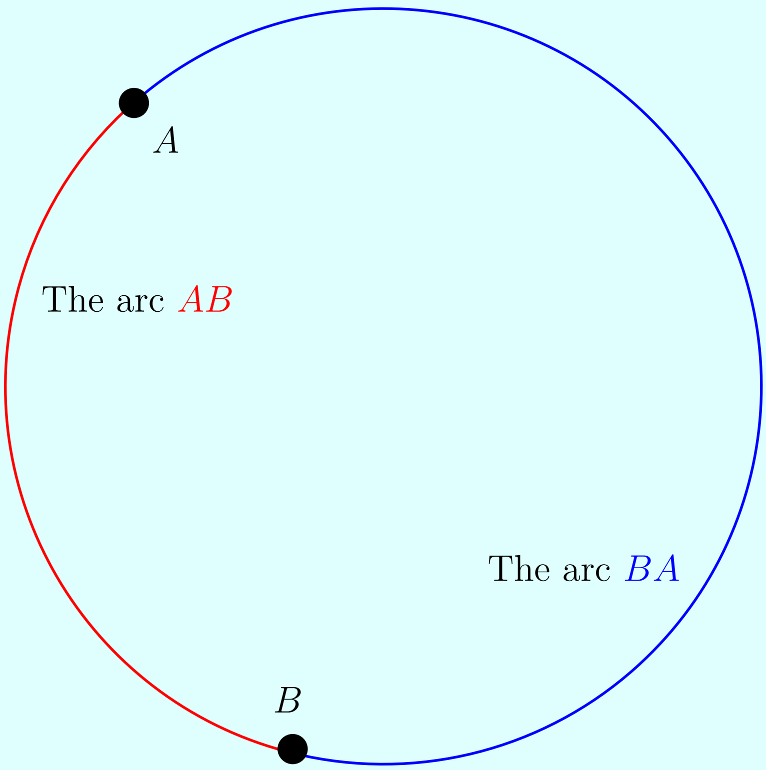
29. Trigonometry: Arcs (of a Circle)

This and the next few chapters will cover trigonometry. You'll already have seen a lot of this in secondary (and even primary) school. But as usual, we'll go into slightly greater depth.

Example 446. Graphed below is a circle containing the points A and B. (Recall that a circle is simply a set of points.)

Informally, an **arc of a circle** is any connected[205] subset of that circle.

But suppose we simply said that the **arc** AB is the subset of the circle connecting the points A and B. Then we'd be faced with an ambiguity: Are we referring to the red arc or the blue arc?



To resolve this ambiguity, we adopt the convention that we "go anticlockwise" from the first point to the second. So,

- Arc AB "goes anticlockwise" from A to B.
- Arc BA "goes anticlockwise" from B to A.

Definition 87 (*informal*)**.** Given two points A and B on a circle, the *arc AB (of the circle)* is the set of points on the circle when we "go anticlockwise" from A and B (including the two points A and B).

Remark 61. The above definition is still a little imprecise and informal (because of the phrase "go anticlockwise"). For a formal definition, see Definition 290 (Appendices).

[205] The term *connected* can actually be formally defined but as usual, an intuitive understanding will suffice.

Remark 62. Some writers use the word **arc** to mean *any* "smooth" curve.

But we'll use **arc** only in the context of circles. So, in this textbook, **arc** will *always* refer to a connected subset of a circle.

Remark 63. The usual western or European convention in most matters is to go clockwise (when viewed from above), be it with clocks (which began as sundials in the northern hemisphere)[206] or card games. (In contrast, mahjong is usually played anticlockwise.)

It may thus seem a bit puzzling that here we (or rather European mathematicians from centuries ago) go anticlockwise. Later on, when we learn about the unit circle definitions of sine and cosine, we'll see how this convention *might* have come about (see p. 350).

In any case, this is to some extent an arbitrary convention that doesn't really matter. Just like which side of the road we drive on, what matters is that we are all aware of, agree to, and stick to the given convention (Remark 23).

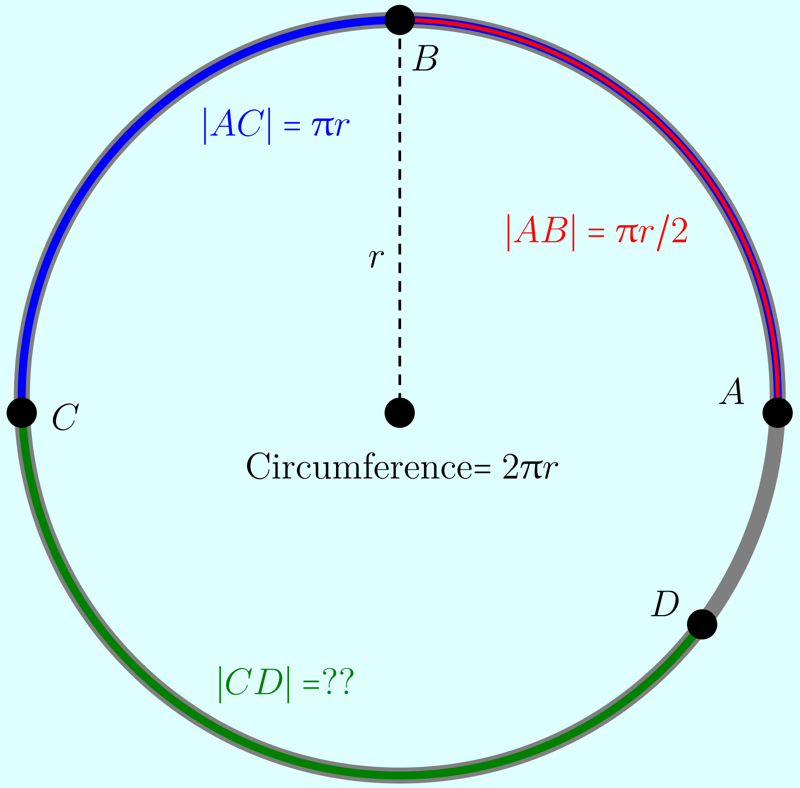
Exercise 136. Name the labelled arcs using the labelled points. (Answer on p. 323.)

Figure to be
inserted here.

A136.

[206]Sundials in the southern hemisphere go *anti*clockwise.

Example 447. By Fact 29, a circle with radius r has circumference (or length) $2\pi r$.



Note that like a line, an arc is a **one-dimensional object**. (In contrast, a point, an area, and a volume are, respectively, a zero-, two-, and three object.) And similar to line segments, given an arc AB, we'll denote by $|AB|$ its **length**.

Here we may repeat the warning given earlier in Remark 34: So far, we've neither learnt how to compute nor formally define the length of any curve (other than the special case where the curve is a line segment). So here, we don't (yet) know how to compute $|CD|$.

We can however tell that, "clearly", $|AB| = \pi r/2$ and $|AC| = \pi r$.

30. Trigonometry: Angles

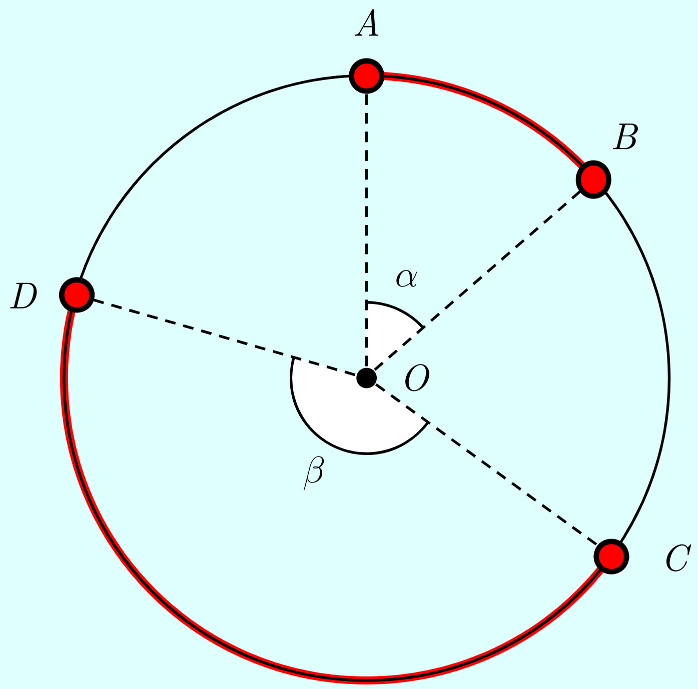
You've known about **angles** since primary school. But it turns out that formally and precisely defining angle is hard and will be done only in Part IV (Vectors) (Definition 145).

One informal and intuitive way to understand angles is to think of an angle as the anticlockwise rotation that a ray[207] must undergo to coincide with another:

Example 448. Let A, B, C, and D be points on a circle with centre O.

The *angle* α is the "amount"[208] the ray OB must rotate anticlockwise to coincide with the ray OA.

We'll also call α *the angle subtended by the arc* AB.[209]



Similarly, the angle β is the "amount" the ray OD must rotate anticlockwise to coincide with OC.

We'll also call β *the angle subtended by the arc* CD.

Example 449. The **full angle** is the minimum positive "amount" a ray must rotate anticlockwise to coincide with itself. In primary school you learned to call this 360°.

Remark 64. The **full angle** is also called the **complete angle**, **round angle**, or **perigon**. (This textbook use only the term **full angle**.)

[207]Recall that a ray is, informally, a "half-infinite line" (see Ch. 8.6).

[208]"Amount" is in scare quotes because it's an imprecise term. So too is the notion of "anticlockwise rotation".

[209]For this textbook's formal definition of the *angle subtended by an arc*, see Definition 291 (Appendices).

30.1. An Informal Definition of Angles

Definition 88 (*informal*)**.** Suppose a circle of radius r with centre O contains the points A and B, so that the length of the arc AB is $|AB|$. Then the *angle between the rays OA and OB* or the *angle subtended by the arc AB* is this number:

$$\frac{|AB|}{r}.$$

Figure to be
inserted here.

Remark 65. Following convention, we'll often denote angles by lower-case Greek letters α (*alpha*), β (*beta*), γ (*gamma*), and θ (*theta*).

We'll also follow your List of Formulae (MF26, p. 3) in also using upper-case Latin letters like A, B, P, and Q.

Example 450. Suppose $r = 2$, $|AB| = 2$, and $|CD| = 3$. Then by Definition 88, the angles α and β are

$$\alpha = \frac{|AB|}{r} = \frac{2}{2} = 1 \qquad \text{and} \qquad \beta = \frac{|CD|}{r} = \frac{3}{2}.$$

Figure to be
inserted here.

By the above definition, an angle is *defined* as the ratio of two lengths. And so,

an angle is a unitless, dimensionless, or "pure" number.

Hence, strictly speaking, angles should **not** have units. Nonetheless and perhaps slightly confusingly, angles *are* given units.

For example, in primary school, you learnt to use **degrees** (°), following the Babylonians, who assigned to the **full angle** the value of 360°.

[209] This definition is labelled "informal" because it makes use of $|AB|$, a number we don't yet know how to compute.

Figure to be
inserted here.

In secondary school, you also learnt a second unit called the **radian** (rad). This unit is what we're using in the above definition and is the SI unit for angles. It is also what we'll be using in this textbook. So, in the last example, we could also have written

$$\alpha = \frac{|AB|}{r} = \frac{2}{2} = 1\,\text{rad} \qquad \text{and} \qquad \beta = \frac{|CD|}{r} = \frac{3}{2}\,\text{rad}.$$

But again, angles are unitless. So, it's actually entirely optional whether we want to include the "rad" following the value of the angle. Indeed, going forward, we'll almost always omit it.

To repeat, the Babylonians assigned to the **full angle** the value of 360°. Let's now work out what the full angle is in radians.

By Definition 88, it is the ratio of the circle's circumference to the circle's radius, which by Fact 29 is

$$\frac{\text{Circumference}}{\text{Radius}} = \frac{2\pi r}{r} = 2\pi.$$

Hence, the full angle is 2π (radians). Equivalently, the full angle *is* the number 2π.

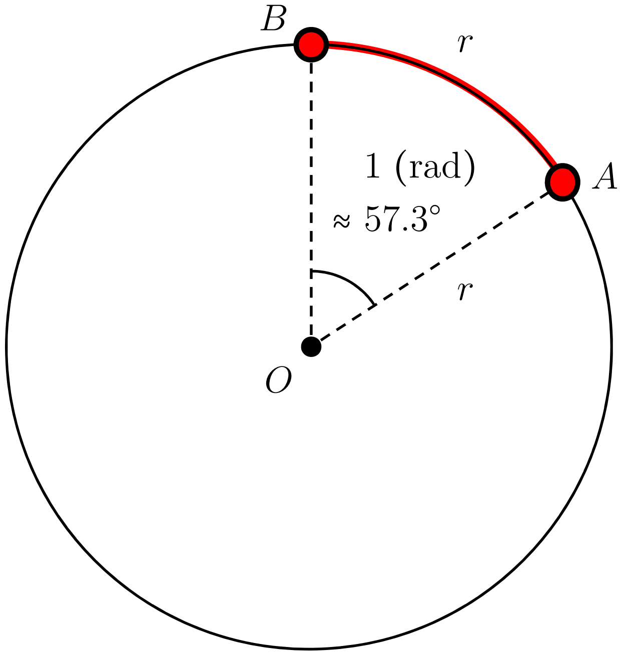
Let's now find the conversion rate between radians and degrees. Since

$$\text{Full angle} = 360° = 2\pi\ (\text{rad}),$$

we must have

$$1\ (\text{rad}) = \frac{360°}{2\pi} \approx 57.3°.$$

By Definition 88, 1 (rad) ≈ 57.3° is the angle subtended by an arc whose length is equal to the circle's radius r:



Going forward in this textbook,

we'll avoid using degrees and try to use only radians.

(Indeed and to repeat, since angles are actually unitless, we'll also almost always not even bother to write "radians" or "rad".)

One reason for this is that radians work out much more nicely in calculus:

Example 451. If x is in radians, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x.$$

But if x is in degrees, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \frac{2\pi}{360^\circ}\cos x.$$

Of course, in the everyday world, laypersons are still more likely to use the degree, probably because it's about four millennia older.[210] So, you shouldn't forget about the degree!

[210] The Babylonians have been using degrees since around 2000 BC. In contrast, the term radian was invented only in the late 19th century.

30.2. Names of Angles

Definition 89. We call the angle A

(a) The *zero angle* if $A = 0$.

(b) An *acute angle* if $A \in (0, \pi/2) = (0°, 90°)$.

(c) The *right angle* if $A = \pi/2 = 90°$.

(d) An *obtuse angle* if $A \in (\pi/2, \pi) = (90°, 180°)$.

(e) The *straight angle* if $A = \pi = 180°$.

(f) A *reflex angle* if $A \in (\pi, 2\pi) = (180°, 360°)$.

(g) The *full angle* if $A = 2\pi = 360°$.

Example 452. xxx

Figure to be
inserted here.

Definition 90. Let A and B be angles.

(a) If $A + B = \pi/2$, then A and B are *complementary* (and are each other's *complements*).

(b) If $A + B = \pi$, then A and B are *supplementary* (and are each other's *supplements*).

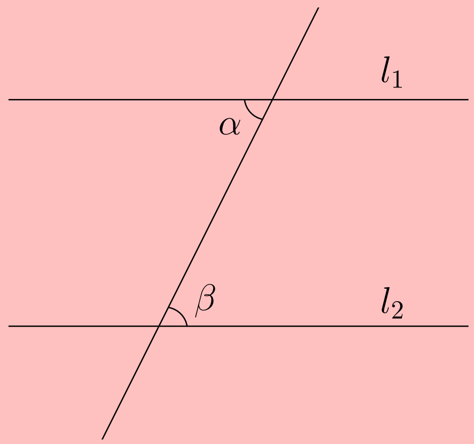
(c) If $A + B = 2\pi$, then A and B are *explementary* or *conjugate* (and are each other's *explements* or *conjugates*).

Example 453. xxx

Figure to be
inserted here.

Remark 66. The names introduced in the two definitions above aren't very important. We'll often use the terms *acute*, *right*, and *obtuse* angles. We'll sometimes use the terms *complementary* and *supplementary* angles. We'll rarely use the other terms.

Fact 64 (informal)**. ("Z" Rule)** *If l_1 and l_2 are parallel, then $\alpha = \beta$.*



This primary-school result is sometimes called the "Z" Rule (because you can see the angles α and β in the letter "Z").

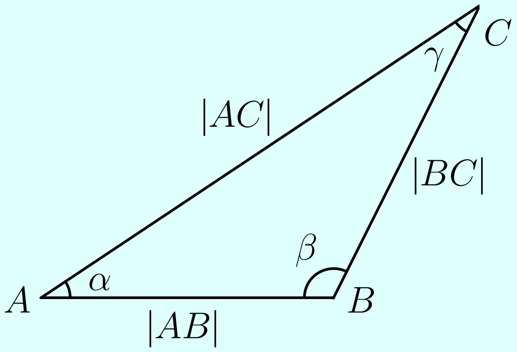
In more formal and precise jargon from geometry, the "Z" Rule says that "alternate angles formed by a transversal through parallel lines are equal".[211]

[211]For a formal statement and proof of the "Z" Rule, see Fact 257 (Appendices)

31. Trigonometry: Triangles

PSLE review: A triangle has three **vertices** (singular: **vertex**), **sides** (whose lengths are strictly positive), and **angles** (also strictly positive).[212]

Example 454. $\triangle ABC$ has vertices A, B, and C; sides AB, AC, and BC (whose lengths are, respectively, $|AB|$, $|AC|$, and $|BC|$); and angles α, β, and γ, which, respectively, *face* (or *are opposite to*) the sides BC, AC, and AB.



A vertex is simply a point. A side is simply a line segment.

Definition 91. Let α be a triangle's largest angle. We call the triangle

(a) *acute* if α is acute;

(b) *right* if α is right;

(c) *obtuse* if α is obtuse;

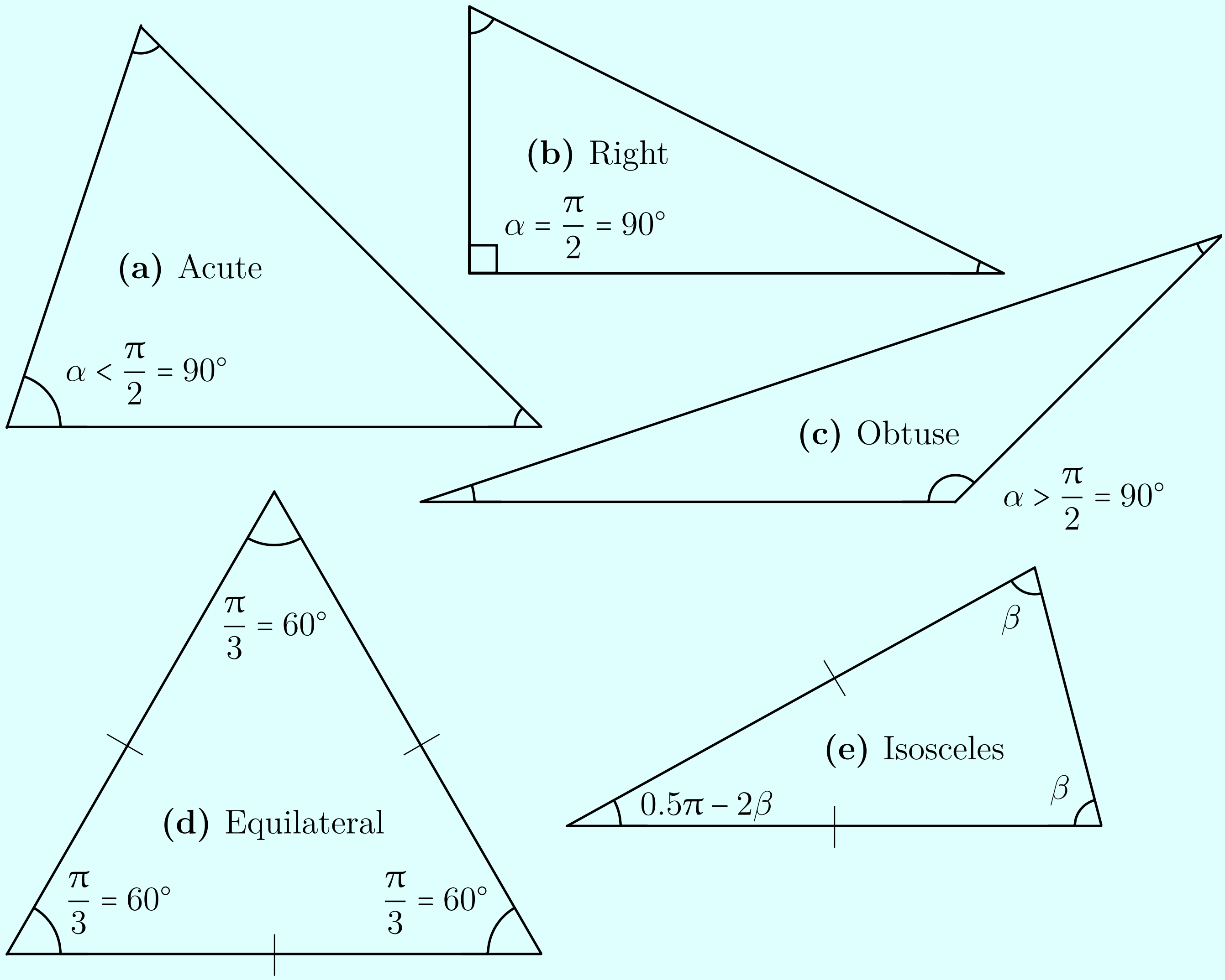
(d) *equilateral* if all three of its sides have equal length;

(e) *isosceles* if two of its sides have equal length; and

(f) *scalene* if no two of its sides have equal length.

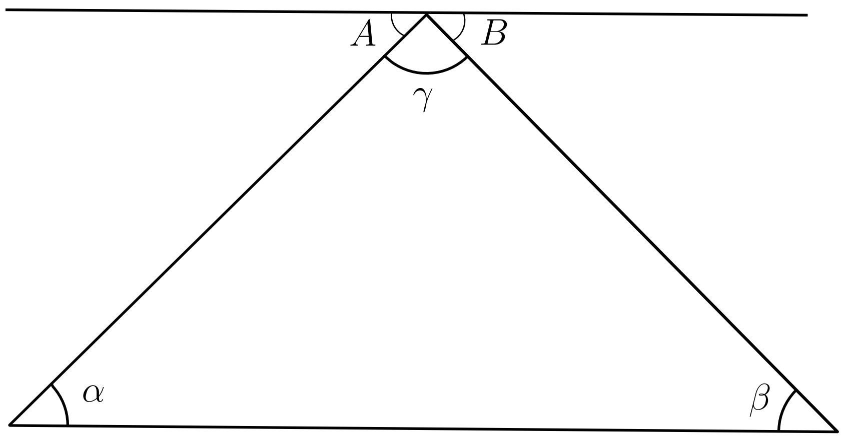
[212]For a more complete and formal treatment of triangles, see Definition 292 (Appendices) and what follows.

Example 455. Triangles **(a)**, **(b)**, and **(c)** are also scalene triangles.



Fact 65. *The sum of any triangle's three angles is* π.

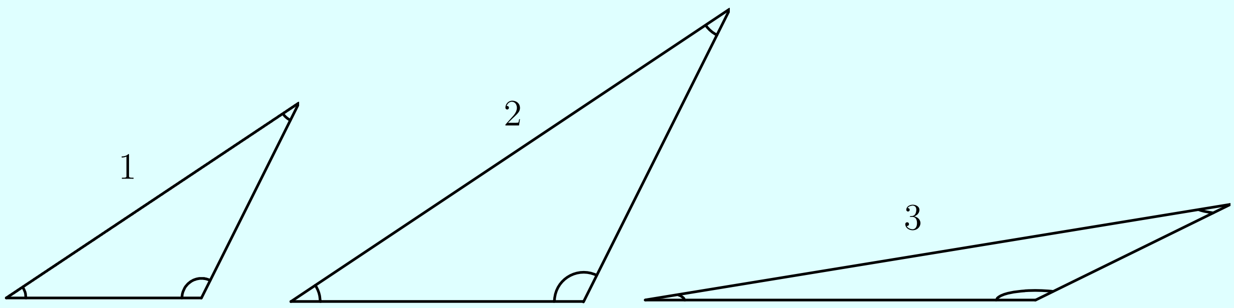
Proof. Informal proof-by-picture:[213]



By Fact 64 ("Z" Rule), $\alpha = A$ and $\beta = B$. So, $\alpha + \beta + \gamma = A + B + \gamma = \pi$. □

Definition 92. Two triangles are *similar* if they share the same three angles.

Example 456. $\triangle_1$ and $\triangle_2$ are similar. $\triangle_1$ and $\triangle_3$ are not. $\triangle_2$ and $\triangle_3$ are not.



Fact 66. *If two triangles share the same two angles, then they are similar.*

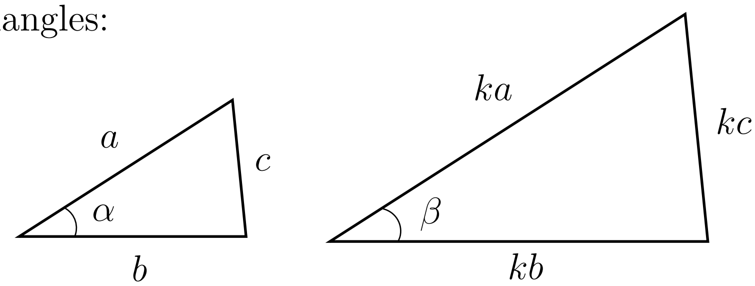
Proof. Suppose two triangles share the same two angles α and β. Then by Fact 65, both triangles' third angle must be $\pi - \alpha - \beta$. So, they share all three angles and are similar. □

Fact 67. *Two triangles are similar* $\iff$ *Suppose one triangle's sides have lengths a, b, and c. Then there exists $k > 0$ such that the other's sides' lengths' are ka, kb, and kc.*

Proof. See p. 1591 (Appendices). □

[213]For a formal proof, see p. 1589 (Appendices).

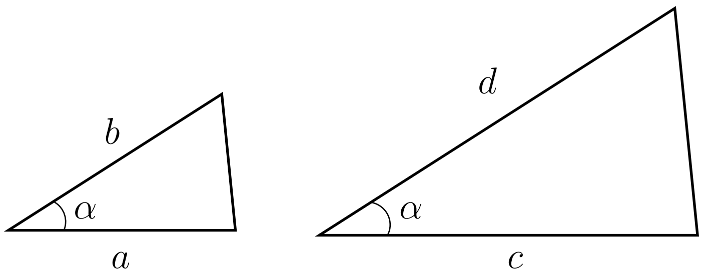
The next result says that $\alpha = \beta$ in these triangles:



Corollary 9. *Let $k > 0$. Suppose one triangle has sides of lengths a, b, and c, while another has sides of lengths ka, kb, and kc. Then the angle in the first triangle facing the side of length c is equal to the angle in the second triangle facing the side of length c.*

Proof. See the proof of Fact 67($\Longleftarrow$) on p. 1591 (Appendices). □

The next result says that if $a/b = c/d$, then these two triangles are similar:



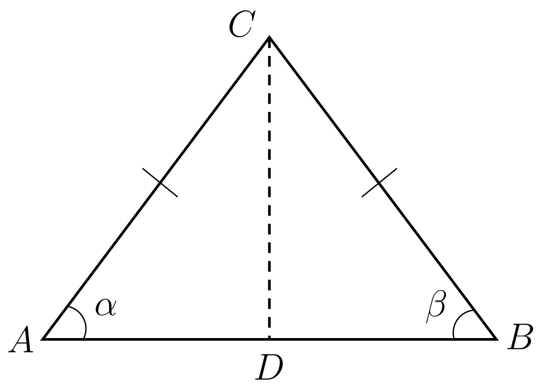
Corollary 10. *Suppose two triangles share an angle. If both triangles also share the ratio of the lengths of the sides adjacent to that angle, then they are similar.*

Proof. See p. 1592 (Appendices). □

Fact 68. *In an isosceles triangle, the angles facing the sides of equal length are equal.*

Proof. Let $\triangle ABC$ be an isosceles triangle with $|AC| = |BC|$. Let D be the midpoint of AB.

By Fact 67, $\triangle ADC$ and $\triangle BDC$ are similar (indeed identical) because they share the side CD, $|AD| = |DB|$, and $|AC| = |BC|$. Hence, by Corollary 9, $\alpha = \beta$. □



Fact 69. *The three angles of an equilateral triangle are equal.*

Proof. Apply Fact 68 twice. □

Corollary 11. *Each of the three angles of an equilateral triangle equals $\pi/3$*

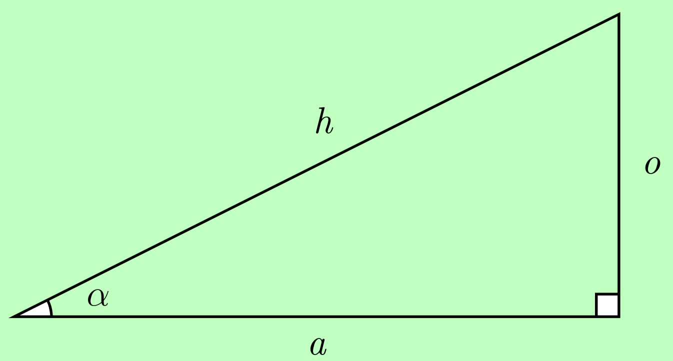
Proof. By Facts 65 and 69. □

32. Trigonometry: Sine and Cosine

The two basic **trigonometric** (or **circular**) **functions** are **sine** and **cosine**.

In secondary school, you probably learnt these **right-triangle definitions** of sine and cosine:

Definition 93 (*informal*)**.** Let $\alpha \in (0, \pi/2)$ (i.e. let α be an acute angle). Construct a right triangle with angles α, $\pi/2$, and $\pi/2 - \alpha$. Let the lengths of the three sides opposite these three angles be o, h, and a (for "Opposite", "Adjacent", and "Hypotenuse").



(a) For $\alpha \in (0, \pi/2)$, we define the *sine* and *cosine* functions (denoted sin and cos) by

$$\sin\alpha \overset{1}{=} \frac{o}{h} \qquad \text{and} \qquad \cos\alpha \overset{2}{=} \frac{a}{h}.$$

(b) For $\alpha = 0$ or $\alpha = \pi/2$, we define

$$\sin 0 \overset{3}{=} 0, \qquad \cos 0 \overset{4}{=} 1, \qquad \sin\frac{\pi}{2} \overset{5}{=} 1, \qquad \cos\frac{\pi}{2} \overset{6}{=} 0.$$

We'll discuss $\overset{3}{=}$ through $\overset{6}{=}$ in the next subchapter. But first, let's illustrate $\overset{1}{=}$ and $\overset{2}{=}$ with some simple examples:

Example 457. XXX

Example 458. XXX

Remark 67. For now, we'll stick with Definition 93. But in Ch. 32.6, it will be superseded by the unit-circle definitions of sine and cosine.

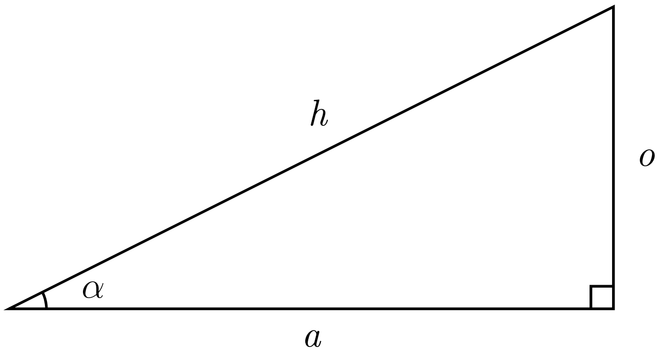
Example 459. XXX

32.1. The Values of Sine and Cosine at 0 and 1

In Definition 93, we defined

$$\sin 0 \stackrel{3}{=} 0, \qquad \cos 0 \stackrel{4}{=} 1, \qquad \sin\frac{\pi}{2} \stackrel{5}{=} 1, \qquad \cos\frac{\pi}{2} \stackrel{6}{=} 0.$$

These definitions seemed a bit arbitrary. Let's see why they might make sense: Observe that as $\alpha \to 0$,[214] $o \to 0$ and $a \to h$.



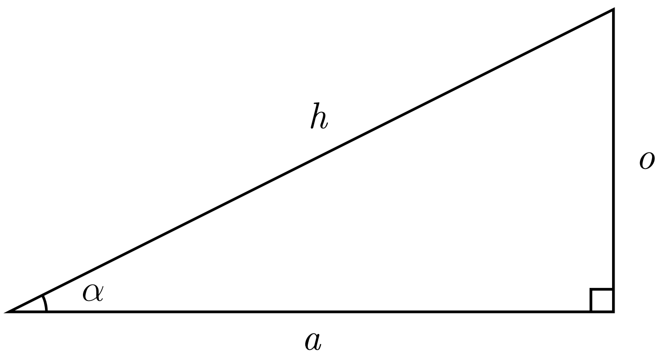
Since $\sin\alpha = o/h$ for all $\alpha \in (0, \pi/2)$, it makes sense to define

$$\sin 0 = \frac{0}{h} \stackrel{3}{=} 0.$$

And since $\cos\alpha = a/h$ for all $\alpha \in (0, \pi/2)$, it makes sense to define

$$\cos 0 = \frac{h}{h} \stackrel{4}{=} 1.$$

Similarly, observe that as $\alpha \to \pi/2$, $o \to h$ and $a \to 0$.



Since $\sin\alpha = o/h$ for all $\alpha \in (0, \pi/2)$, it makes sense to define

$$\sin\frac{\pi}{2} = \frac{h}{h} \stackrel{5}{=} 1.$$

And since $\cos\alpha = a/h$ for all $\alpha \in (0, \pi/2)$, it makes sense to define

$$\cos\frac{\pi}{2} = \frac{0}{h} \stackrel{6}{=} 0.$$

[214] As in our discussion on limits, the right arrow "→" here may be read aloud as "approaches".

Fun Fact

According to Katz (2009), *sine* means *bosom* or *breast*:

> *The English word "sine" comes from a series of mistranslations of the Sanskrit* jyā-ardha *(chord-half). Āryabhata frequently abbreviated this term to* jyā *or its synonym* jīvā. *When some of the Hindu works were later translated into Arabic, the word was simply transcribed phonetically into an otherwise meaningless Arabic word* jiba. *But since Arabic is written without vowels, later writers interpreted the consonants* jb *as* jaib*, which means bosom or breast. In the twelfth century, when an Arabic trigonometry work was translated into Latin, the translator used the equivalent Latin word* sinus*, which also meant bosom, and by extension, fold (as in a toga over a breast), or a bay or gulf. This Latin word has now become our English "sine".*

32.2. Some Values of Sine and Cosine

From Definition 93(b), we already know that

$$\sin 0 = 0 = \frac{\sqrt{0}}{2}, \qquad \cos 0 = 1 = \frac{\sqrt{4}}{2}, \qquad \sin\frac{\pi}{2} = 1 = \frac{\sqrt{4}}{2}, \qquad \cos\frac{\pi}{2} = 0 = \frac{\sqrt{0}}{2}.$$

We now also give the values of sine and cosine at $\pi/6$, $\pi/4$, and $\pi/3$:

Fact 70.

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\alpha$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
$\cos\alpha$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

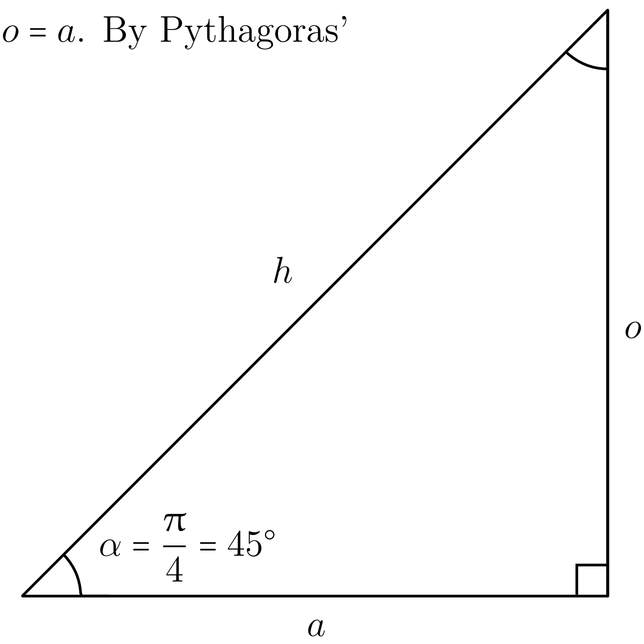
Remark 68. Of course, $\sqrt{0}/2 = 0$ and $\sqrt{4}/2 = 1$. Nonetheless, the above table is written as it is because that's how I find it easiest to remember these values—sine is half square root 0, 1, 2, 3, 4, while cosine is half square root 4, 3, 2, 1, 0.

Note: The following proof of Fact 70 is informal because it relies on geometry and our right-triangle definitions of sine and cosine:

Proof. In the right isosceles triangle, $\alpha = \pi/4$ and $o = a$. By Pythagoras' Theorem, $h = \sqrt{o^2 + a^2} = \sqrt{2}o = \sqrt{2}a$.

So, $\sin\alpha = \sin\frac{\pi}{4} = \frac{o}{h} = \frac{o}{\sqrt{2}o} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

And, $\cos\alpha = \cos\frac{\pi}{4} = \frac{a}{h} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.



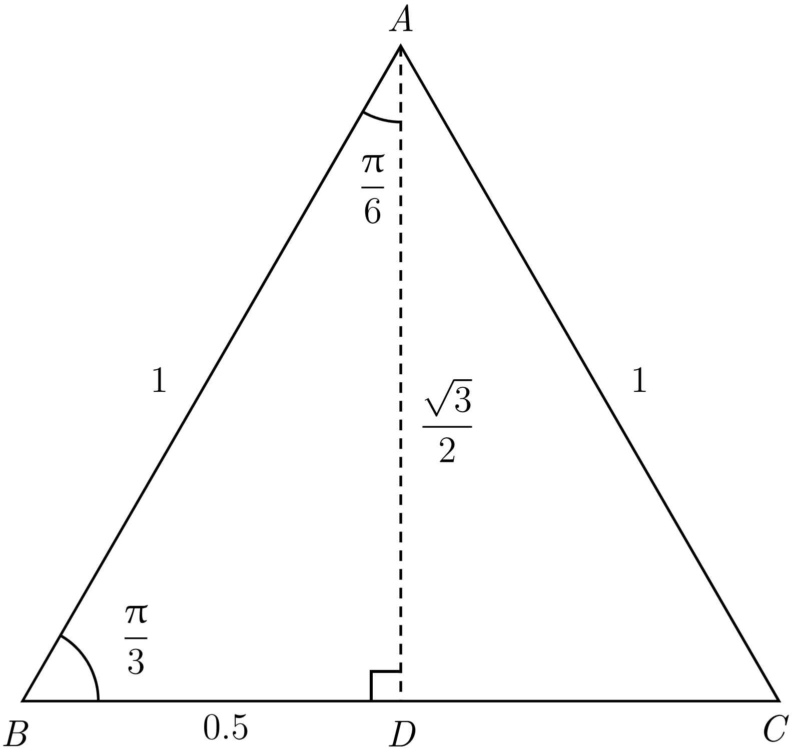
Next, $\triangle ABC$ is an equilateral triangle (below)

Each of its sides has length 1.

Let D be the midpoint of BC.

(*Proof continues below ...*)

(... *Proof continued from above.*)



By Pythagoras' Theorem,

$|AD| = \sqrt{1^2 - 0.5^2} = \sqrt{0.75} = \sqrt{3}/2$.

So, $\sin\frac{\pi}{3} = \frac{|AD|}{|AB|} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$.

And, $\cos\frac{\pi}{3} = \frac{|BD|}{|AB|} = \frac{0.5}{1} = 0.5 = \frac{\sqrt{1}}{2}$.

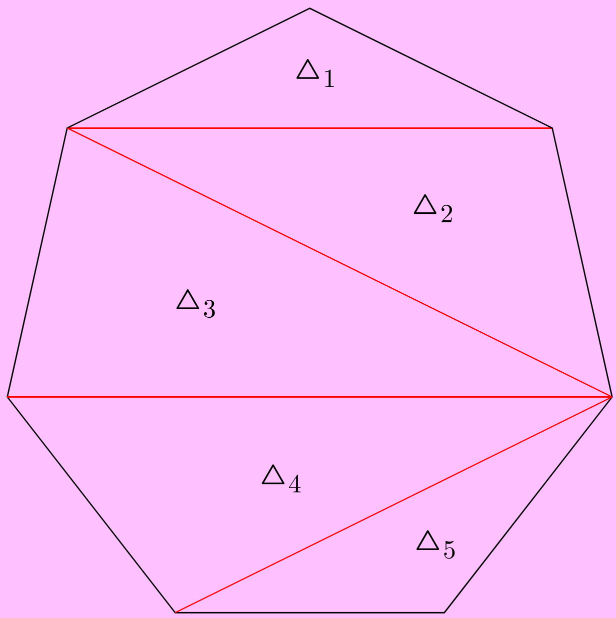
Also, $\sin\frac{\pi}{6} = \frac{|BD|}{|AB|} = \frac{0.5}{1} = 0.5 = \frac{\sqrt{1}}{2}$.

And, $\cos\frac{\pi}{6} = \frac{|AD|}{|AB|} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$. □

Exercise 137. This Exercise will prove that

$$\cos\frac{\pi}{5} \stackrel{1}{=} \frac{1+\sqrt{5}}{4}.$$

(a) Any polygon with n sides can be partitioned into $n-2$ triangles. For example, a polygon with 7 sides (called a *heptagon*) can be partitioned into 5 triangles:



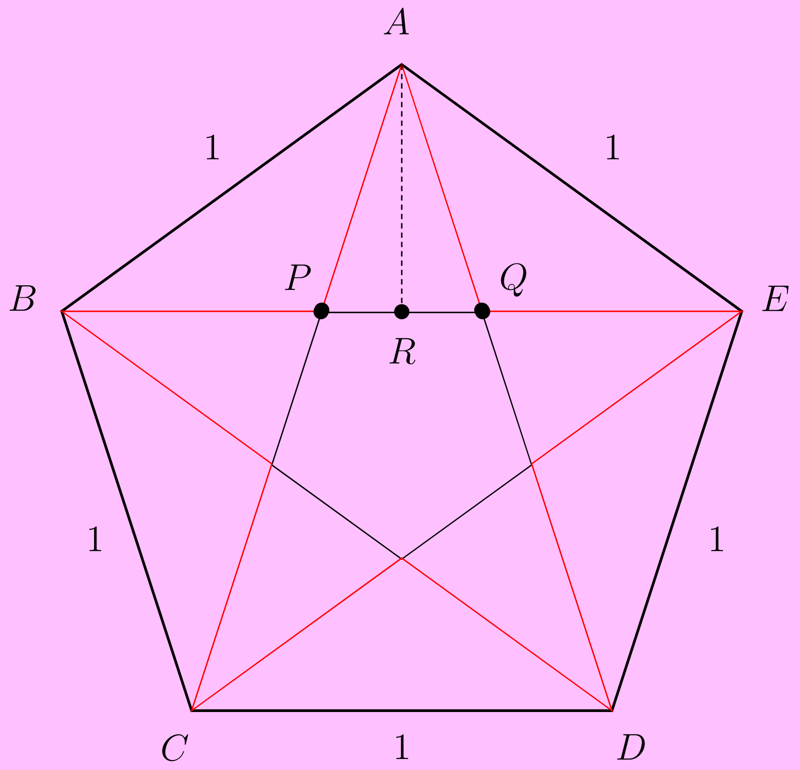
(i) What's the sum of a triangle's three angles?

(ii) Hence, what's the sum of the angles of a polygon with n sides?

(*Exercise continues below ...*)

(... *Exercise continued from above.*)

(b) Now consider the regular[215] pentagon $ABCDE$. Each of its five sides has length 1.



(i) What is the sum of the pentagon's five angles?

(ii) Find $\angle BAE$.

(iii) Find $\angle ABE$.

(iv) The midpoint of the line segment BE is R and $\triangle ARB$ is a right triangle. Show that $\cos\frac{\pi}{5} = \frac{|BE|}{2}$.

(c) We now find $|BE|$ and hence prove $\stackrel{1}{=}$:

(i) Find $\angle PAQ$. (Hint: What sort of triangle is $\triangle ABP$?)

(ii) Explain why the triangles $\triangle ABQ$ and $\triangle APQ$ are similar.

(iii) Find $|BQ|$.

(iv) Let $x = |AQ|$. Show that $\frac{1}{x} = \frac{x}{1-x}$.

(v) Solve for x.

(vi) Find $|BE|$.

(vii) Complete the proof of $\stackrel{1}{=}$.

(d) Bonus: Show that $\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$. (Answer on p. 1798.)

[215]A polygon is *regular* if its sides are of equal length.

32.3. Confusing Notation

In Ch. 22, we learnt that

- f^2 denotes the composite function $f \circ f$;
- f^3 denotes the composite function $f \circ f^2$;
- Etc.

We'd thus expect that $\sin^2$ denotes the *composite* function $\sin \circ \sin$. That is, we'd expect

$$\text{“}\sin^2 x = \sin(\sin x)\text{.”} \qquad ✗$$

But this is not the case! Very confusingly, $\sin^2$ denotes the *product* function $\sin \cdot \sin$:

$$\sin^2 x = (\sin x)^2 = (\sin x)(\sin x). \qquad ✓$$

Example 460. $\sin^2 \dfrac{\pi}{2} \neq \sin\left(\sin \dfrac{\pi}{2}\right)$ because

$$\sin^2 \frac{\pi}{2} = \left(\sin \frac{\pi}{2}\right)^2 = (1)^2 = 1, \qquad \text{but} \qquad \sin\left(\sin \frac{\pi}{2}\right) = \sin 1 \approx 0.845.$$

And in general, for any positive integer n, $\sin^n$ does **not** denote $\sin \circ \sin \circ \cdots \circ \sin$. That is,

$$\sin^n x \neq \sin(\sin(\sin(\ldots(\sin x)))).$$

Instead, $\sin^n$ denotes $\sin \cdot \sin \cdot \cdots \cdot \sin$:

$$\sin^n x = (\sin x)^n = (\sin x)(\sin x)\ldots(\sin x). \qquad ✓$$

This is yet another annoying and confusing bit of notation you'll have to learn to live with.

The above remarks also apply to cosine and the other four trigonometric functions we'll be looking at shortly (tangent, cosecant, secant, and cotangent).

32.4. The Area of a Triangle, Law of Sines, and Law of Cosines

Here's an identity you should find familiar from secondary school:

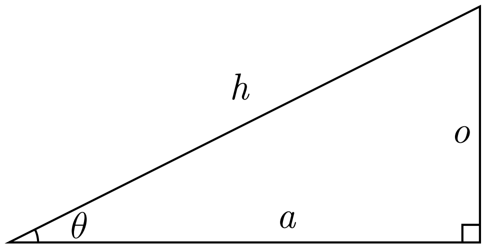
Fact 71. (First or *The* Pythagorean Identity) *Suppose $\theta \in \mathbb{R}$. Then*

$$\sin^2\theta + \cos^2\theta = 1.$$

Proof. Informal proof-by-picture:[216]

Apply Definition 93 and Pythagoras' Theorem:

$$\sin^2\theta + \cos^2\theta = \left(\frac{o}{h}\right)^2 + \left(\frac{a}{h}\right)^2 = \frac{o^2 + a^2}{h^2} = \frac{h^2}{h^2} = 1. \qquad \square$$



Proposition 7. *Suppose a triangle has sides of lengths a, b, and c with angles A, B, and C facing those sides, respectively. Then*

(a) The triangle's area is $$\frac{1}{2}ab\sin C.$$

(b) The **Law of Sines**: $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

(c) The **Law of Cosines**: $$c^2 = a^2 + b^2 - 2ab\cos C$$

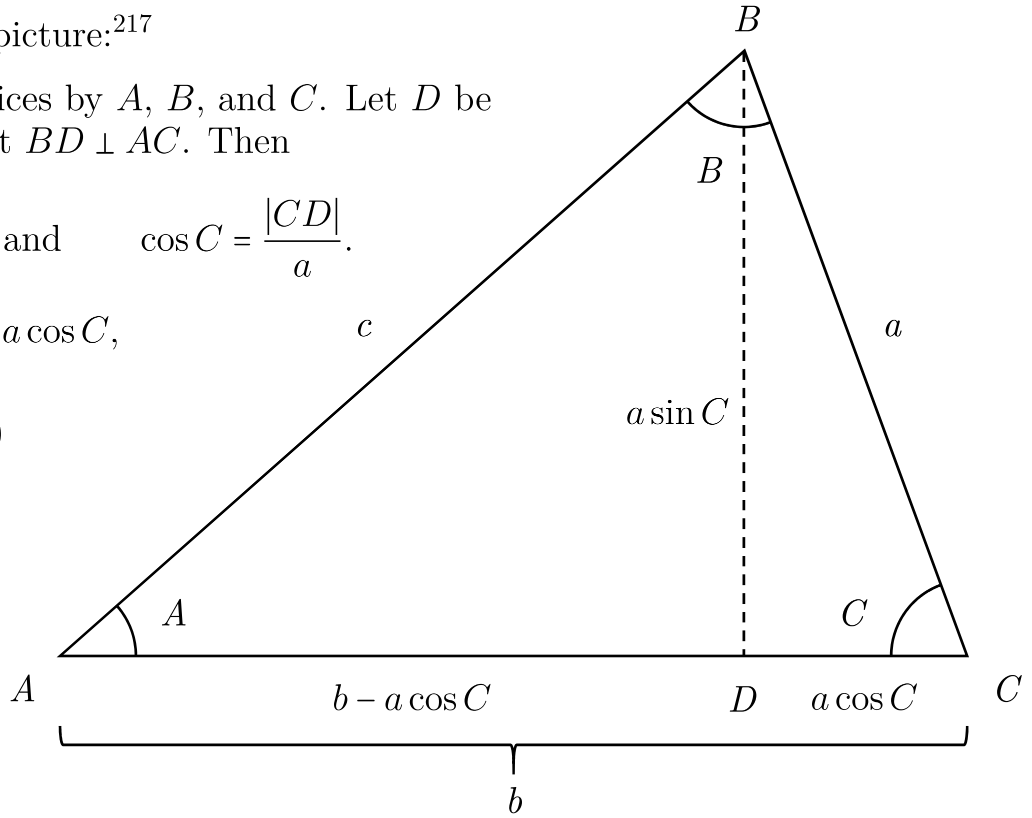
Proof. Informal proof-by-picture:[217]

Denote the triangle's vertices by A, B, and C. Let D be the point on AC such that $BD \perp AC$. Then

$$\sin C = \frac{|BD|}{a} \qquad \text{and} \qquad \cos C = \frac{|CD|}{a}.$$

So, $|BD| = a\sin C$, $|CD| = a\cos C$, and $|AD| = b - a\cos C$.

(*Proof continues below ...*)



[216]This proof is also incomplete because it covers only the case where A is acute. For a formal and general proof, see p. 1013 in Part V (Calculus).

[217]For formal proofs of **(a)** and **(c)**, see pp. 1592 and 1588 (Appendices). The proof of **(b)** given here is perfectly complete and formal.

(*... Proof continued from above.*)

(a) The triangle has base b and height $a \sin C$. Hence, its area is $0.5ab \sin C$.

(b) We can similarly show that the triangle's area is also $0.5bc \sin A$ or $0.5ac \sin B$. So,

$$\frac{bc \sin A}{2} = \frac{ac \sin B}{2} = \frac{ab \sin C}{2} \quad \stackrel{\times 2 \div abc}{\Longleftrightarrow} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

(c) Consider $\triangle ABD$. Its hypotenuse has length c. Its legs have lengths $a \sin C$ and $b - a \cos C$. Now use Pythagoras' Theorem and the First Pythagorean Identity:

$$\begin{aligned} c^2 &= (a \sin C)^2 + (b - a \cos C)^2 = a^2 \sin^2 C + b^2 - 2ab \cos C + a^2 \cos^2 C \\ &= a^2 \left(\sin^2 C + \cos^2 C\right) + b^2 - 2ab \cos C = a^2 + b^2 - 2ab \cos C. \end{aligned}$$

□

We can use the Law of Cosines to prove the **Triangle Inequality**:

Corollary 12. (Triangle Inequality) *The length of one side of a triangle is less than the sum of the lengths of the other two sides.*

Proof. Consider a triangle with sides of lengths a, b, c. Let C be the angle opposite the side of length c. Below, $\stackrel{1}{=}$, $\stackrel{2}{=}$, and $\stackrel{3}{>}$ use the Law of Cosines, Plus Zero Trick, and $a, b, \cos C < 1$:

$$\begin{aligned} c^2 &\stackrel{1}{=} a^2 + b^2 - 2ab \cos C \stackrel{2}{=} a^2 + b^2 - 2ab + 2ab - 2ab \cos C \\ &= (a - b)^2 + 2ab (1 - \cos C) \stackrel{3}{>} (a - b)^2. \end{aligned}$$

Now, $c^2 > (a - b)^2$ implies $c > a - b$ or $a < b + c$.

We can similarly prove that $b < a + c$ and $c < a + b$. □

Fact 72. (Heron's Formula) *Suppose a triangle has sides of lengths a, b, and c. Let* $s = (a + b + c)/2$.[218] *Then the triangle's area is* $\sqrt{s(s - a)(s - b)(s - c)}$.

Proof. See Exercise 138. □

Exercise 138. Prove Heron's Formula (Fact 72) using the steps below.

Let C be the angle opposite the side of length c. (Answer on p. 1799.)

(a) By rearranging the Law of Cosines, express $\cos C$ in terms of a, b, and c.

(b) Now show that $1 + \cos C = \dfrac{(a + b)^2 - c^2}{2ab}$ and $1 - \cos C = \dfrac{c^2 - (a - b)^2}{2ab}$.

(c) Show that $(a + b)^2 - c^2 = 4s(s - a)$ and $c^2 - (a - b)^2 = 4(s - b)(s - c)$.

(d) Use the First Pythagorean Identity to express $\sin C$ in terms of $\cos C$. Here you'll get a "±". Explain why you can discard the negative value. (Hint: What can C be? Hence, what can $\sin C$ be?)

(e) Now use **(b)**–**(d)** and Area $= ab \sin C/2$ to find Heron's Formula.

[218] We call s the triangle's **semiperimeter**.

32.5. Some Formulae Involving Sine and Cosine

Happily, List MF26 (p.3) contains the following formulae, so no need to *mug*:

Fact 73. Suppose $A, B \in \mathbb{R}$. Then

Addition and Subtraction Formulae for Sine and Cosine

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

Double Angle Formulae for Sine and Cosine

(c) $\sin 2A = 2 \sin A \cos A$

(d) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Proof. Informal proofs-by-picture:[219]

In the figure[220], $|PR| = 1$.[221]

Observe that $|PT| = \cos B$. So,

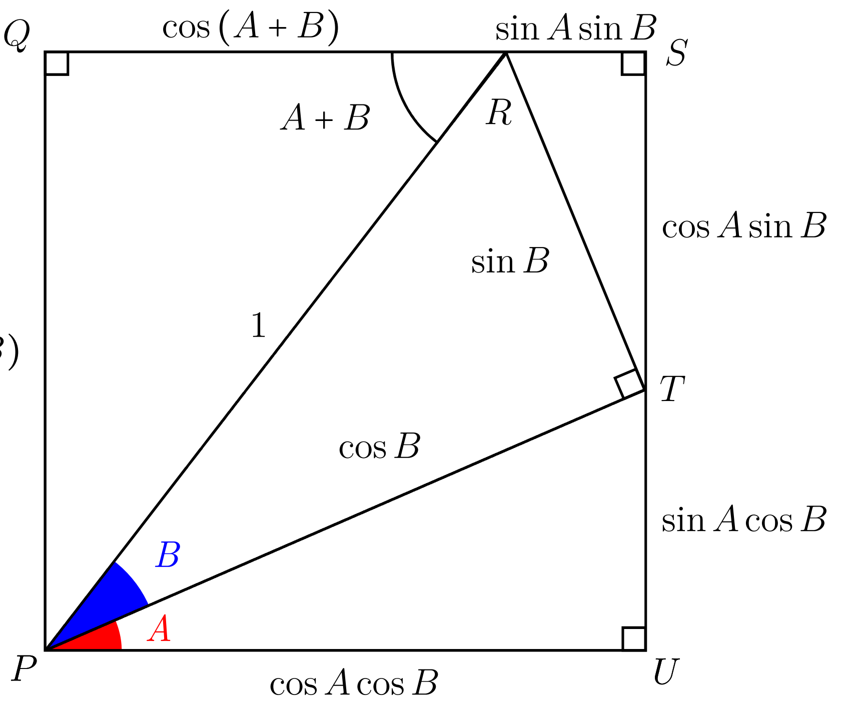
$$|PU| = \cos A \cos B;$$

$$|TU| = \sin A \cos B.$$

Next, $\angle PRQ$ and $\angle RPU$ are "Z" angles. So, $\angle PRQ = \angle RPU = A + B$. Hence,

$$|PQ| = \sin(A+B);$$

$$|QR| = \cos(A+B).$$



Next, $\angle RTS$ is complementary[222] to $\angle PTU$, which is in turn complementary to $\angle TPU$. So, $\angle RTS = A$. Since $|RT| = \sin B$, we have $|ST| = \cos A \sin B$ and $|RS| = \sin A \sin B$.

Now, $|PQ| = |TU| + |TS|$ or $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

Similarly, $|QR| = |PU| - |RS|$ or $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

Exercise 139 guides you through a similar (informal and incomplete) proof of the **Subtraction Formulae for Sine and Cosine**.

Exercise 140 asks you to prove a **Double Angle Formulae**. □

[219]This proof is incomplete because it applies only in the special case where A, B, and $A+B$ are acute. For a formal and complete proof, see p. 1593 (Appendices).

[220]Credit to Blue.

[221]*Construction details.* Let $A, B, A+B \in (0, \pi/2)$. Construct the horizontal line segment PU. Rotate Ray PU anticlockwise by A to produce Ray PT. Then rotate Ray PT anticlockwise by B to produce Ray PR. Pick R to be the point on Ray PR such that $|PR| = 1$. Pick T to be the point on Ray PT such that $RT \perp PT$. Now construct the rectangle $PQSU$ so that R and T are on the line segments QS and SU, respectively.

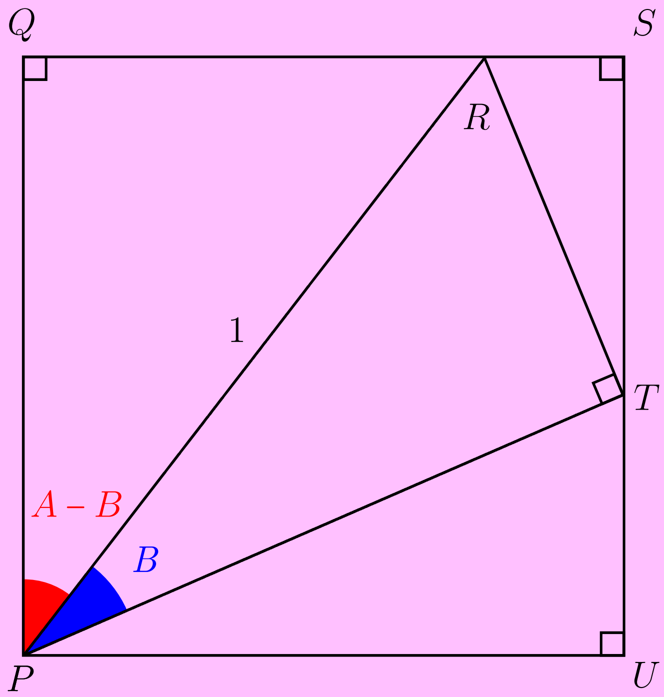
[222]See Definition 90.

Exam Tip for *Towkays*

Whenever you see a question with trigonometric functions, put MF26 (p. 3) next to you!

Exercise 139. Use the figure below[223] to prove the **Subtraction Formulae for Sine and Cosine** (in the special case where A is acute and $B < A$). (Answer on p. 1799.)

$$\sin(A-B) = \sin A\cos B - \cos A\sin B,$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$



Exercise 140. Use the Addition and Subtraction Formulae for Sine and Cosine to prove the **Double-Angle Formulae for Sine and Cosine**. (Answers on p. 1801.)

Remark 69. To repeat, the Double-Angle Formulae **are** on List MF26. So strictly speaking, you needn't memorise them. Nonetheless, it's a good idea to have them committed to memory so you can solve problems that much more quickly.

[223] The construction of this figure is very similar to before, except now we start with Ray PQ, rotate it clockwise by $A - B$ to get Ray PR, then rotate Ray PR clockwise by B to get Ray PT.

Fact 74. (Triple-Angle Formulae for Sine and Cosine) *Suppose $A \in \mathbb{R}$. Then*

(a) $\sin 3A = 3\sin A - 4\sin^3 A$;

(b) $\cos 3A = 4\cos^3 A - 3\cos A$.

Proof. See Exercise 141. □

Exercise 141. Prove Fact 74(a) and (b). (Answers on p. 1801.)

Remark 70. The Triple-Angle Formulae are **not** on List MF26. So if they ever come up on exams, you'll want to be able to either derive them from scratch or recall them.

For $\cos 3A = 4\cos^3 A - 3\cos A$, there used to be the Hokkien mnemonic "$1.30 = $4.30−$3":

箍	三	等於	四	箍	三	減	三	箍	。
khoo	*sam*		*sì*	*khoo*	*sam*		*sam*	*khoo*	.
$1.30		=		$4.30		−	$3		.

Happily, List MF26 (p.3) contains the following formulae, so no need to *mug*:

Fact 75. *Suppose* $P, Q \in \mathbb{R}$. *Then*

Sum-to-Product (S2P) or Product-to-Sum (P2S) Formulae

(a) $\sin P + \sin Q = 2\sin\dfrac{P+Q}{2}\cos\dfrac{P-Q}{2}$

(b) $\sin P - \sin Q = 2\cos\dfrac{P+Q}{2}\sin\dfrac{P-Q}{2}$

(c) $\cos P + \cos Q = 2\cos\dfrac{P+Q}{2}\cos\dfrac{P-Q}{2}$

(d) $\cos P - \cos Q = -2\sin\dfrac{P+Q}{2}\sin\dfrac{P-Q}{2}$

Proof. See Exercise 142. □

The P2S Formulae will be useful when we do integration (they let us rewrite a difficult-to-integrate product into an easy-to-integrate sum).

Fun Fact

The above S2P or P2S Formulae are also known as the **Prosthaphaeresis Formulae**. Sounds *cheem*, but that's just the combination of the Greek words for *addition* and *subtraction*—*prosthesis* and *aphaeresis*. So yea, something you can totally use to impress your friends and family.

Exercise 142. Prove Fact 75(a)–(d).[224] (Answers on p. 1801.)

Exercise 143. Rewrite each expression using the P2S Formulae: (Answer on p. 1801.)

(a) $\sin 2x \cos 5x$ **(b)** $\cos 2x \sin 5x$ **(c)** $\cos 2x \cos 5x$ **(d)** $\sin 2x \sin 5x$

[224] Hint: Write $P = \dfrac{P+Q}{2} + \dfrac{P-Q}{2}$ and $Q = \dfrac{P+Q}{2} - \dfrac{P-Q}{2}$.

32.6. The Unit-Circle Definitions

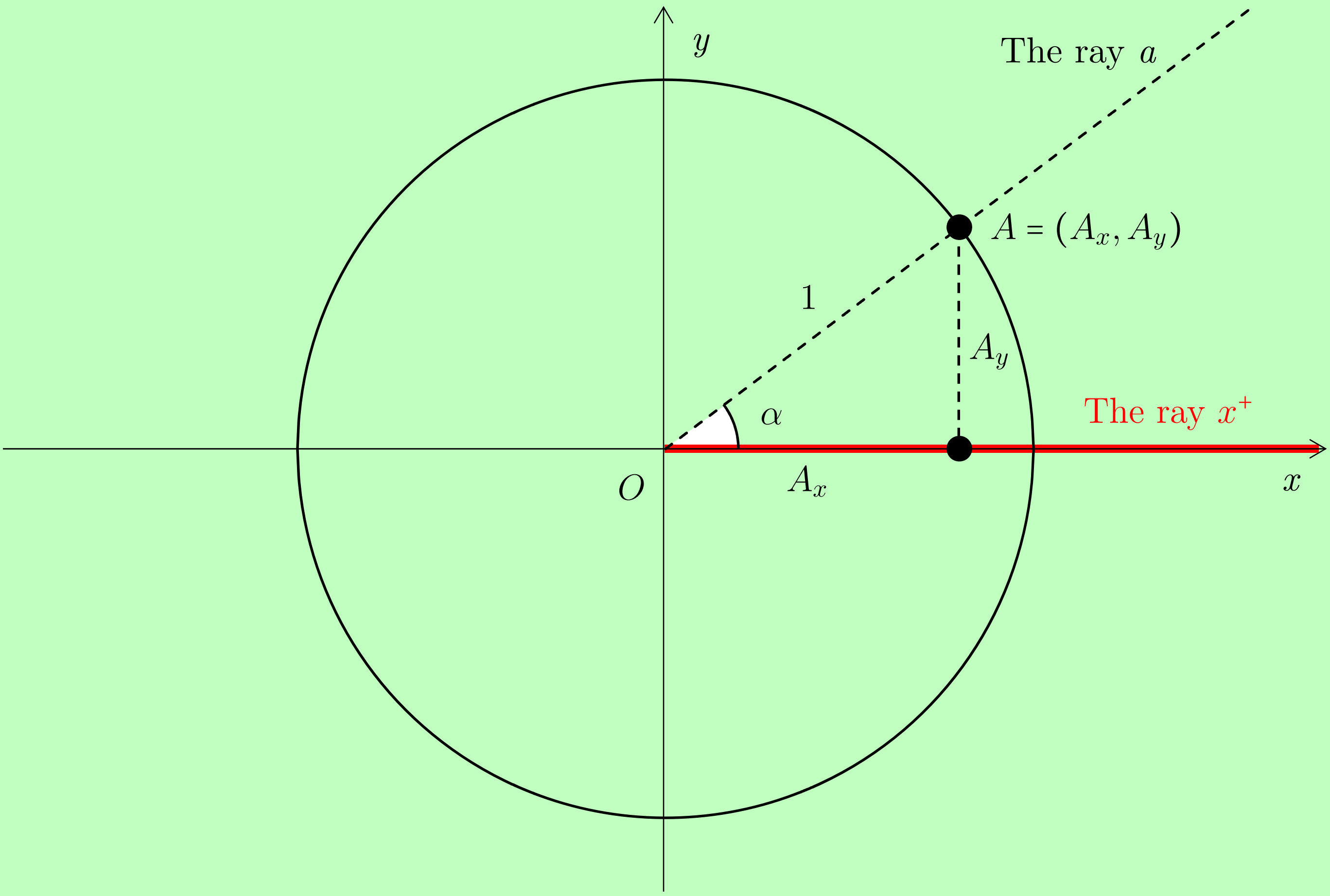
So far, we've been using the right-triangle definitions of sine and cosine (Definition 93). Unfortunately, these define sin and cos only for $\alpha \in [0, \pi]$.

To define sin and cos more generally, that is, for *all* $\alpha \in \mathbb{R}$, we turn to the **unit-circle definitions** (which supersede Definition 93)

Definition 94 (*informal*)**.** Consider the ray that is the positive x-axis (it begins at the origin O and extends eastwards). Call this ray x^+.

Let α be any angle. Rotate x^+ anticlockwise by α to produce the ray a.

Let $A = (A_x, A_y)$ be the point at which a intersects the unit circle centred on the origin.



The *sine* and *cosine* functions, $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$, are defined by

$$\sin \alpha = A_y \qquad \text{and} \qquad \cos \alpha = A_x.$$

Equivalently, $\sin \alpha$ and $\cos \alpha$ are the x- and y-coordinates of the point A.

And so, here's an entirely equivalent definition that gives us a slightly different way of thinking about sin and cos (and hopefully also a better understanding):

Definition 95 (*informal*)**.** Consider the unit circle centred on the origin. A particle A travels anticlockwise around the circle at the constant speed of 1 unit per second. At time $\alpha = 0\,\mathrm{s}$, A is at the point $(1, 0)$.

Figure to be inserted here.

We define $\sin\alpha$ and $\cos\alpha$ to be the y- and x-coordinates of A at time α.

Example 461. If $\alpha = 0$, then $A = (0, 1) = (\sin 0, \cos 0)$.

Figure to be inserted here.

Example 462. If $\alpha = \dfrac{\pi}{6}$, then by the reasoning given in the proof of Fact 70,

$$A = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\sin\frac{\pi}{6}, \cos\frac{\pi}{6}\right).$$

Figure to be inserted here.

Example 463. If $\alpha = \frac{\pi}{4}$, then by the reasoning given in the proof of Fact 70,

$$A = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left(\sin\frac{\pi}{4}, \cos\frac{\pi}{4}\right).$$

Figure to be
inserted here.

Remark 71. We can now explain how the anticlockwise convention *might* have arisen.

We already adopt this convention:

1. The x-axis points rightwards and the y-axis upwards.

Let's also adopt these conventions:

2. The zero angle corresponds to the positive x-axis.
3. Small but positive angles are in the quadrant (of the cartesian plane) where x and y are positive.

Figure to be
inserted here.

Adopting the above conventions means also adopting the anticlockwise convention.

Example 464. If $\alpha = \frac{\pi}{3}$, then by the reasoning given in the proof of Fact 70,

$$A = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(\sin\frac{\pi}{3}, \cos\frac{\pi}{3}\right).$$

Figure to be
inserted here.

Example 465. If $\alpha = \frac{\pi}{2}$, then

$$A = (1, 0) = \left(\sin\frac{\pi}{2}, \cos\frac{\pi}{2}\right).$$

Figure to be
inserted here.

The above five examples show that the following values for sine and cosine (from Fact 70) also hold under the unit-circle definitions of sine and cosine:

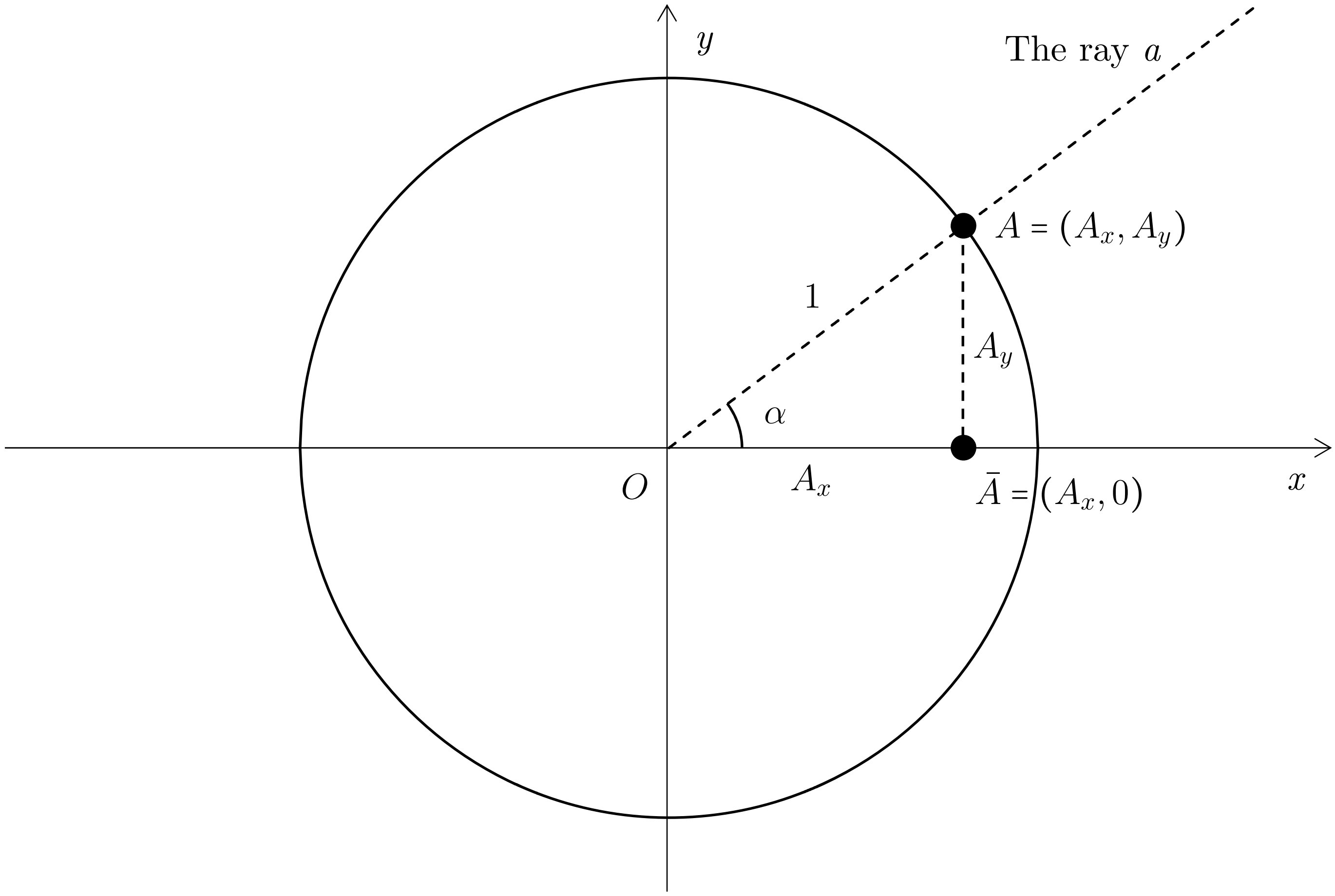
α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\alpha$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
$\cos\alpha$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

In fact,

Fact 76 (informal)**.** *At every $\alpha \in [0, \pi/2]$, the unit-circle definitions (Definition 94) and the right-triangle definitions (Definition 93) define the same values for sine and cosine.*

Remark 72. Again, for $\alpha \notin [0, \pi/2]$, the right-triangle definitions (Definition 93) simply leave sine and cosine undefined.

Proof. Continue with the figure in Definition 94. Let $\bar{A} = (A_x, 0)$.



Consider the right triangle $O\bar{A}A$. We have "Hypotenuse" of length $|OA| = 1$, "Opposite" of length $|A\bar{A}| = A_y$, and "Adjacent" of length $|O\bar{A}| = A_x$. And so, by Definition 93(a), for $\alpha \in (0, \pi/2)$,

$$\sin\alpha = \frac{o}{h} = \frac{|A\bar{A}|}{|OA|} = \frac{A_y}{1} = A_y \qquad \text{and} \qquad \cos\beta = \frac{a}{h} = \frac{|O\bar{A}|}{|OA|} = \frac{A_x}{1} = A_x,$$

which indeed coincide with Definition 94.

For $\alpha = 0$, by Definition 93(b), $\sin 0 = 0$ and $\cos 0 = 1$, which coincides with Definition 94—see Example 461.

For $\alpha = \pi/2$, by Definition 93(b), $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, which coincides with Definition 94—see Example 465. □

If $\alpha > \pi/2$, then our right-triangle definitions (Definition 93) are silent. Let's see what our unit-circle definitions say:

Example 466. If $\alpha = \frac{2\pi}{3} = \pi - \frac{\pi}{3}$, then

$$A = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \left(\sin\frac{2\pi}{3}, \cos\frac{2\pi}{3}\right).$$

Figure to be
inserted here.

Example 467. If $\alpha = \frac{3\pi}{4} = \pi - \frac{\pi}{4}$, then

$$A = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \left(\sin\frac{3\pi}{4}, \cos\frac{3\pi}{4}\right).$$

Figure to be
inserted here.

Example 468. If $\alpha = \frac{5\pi}{6} = \pi - \frac{\pi}{6}$, then

$$A = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \left(\sin\frac{5\pi}{6}, \cos\frac{5\pi}{6}\right).$$

Figure to be
inserted here.

Example 469. If $\alpha = \pi$, then

$$A = (0, -1) = (\sin \pi, \cos \pi).$$

Figure to be inserted here.

Example 470. If $\alpha = \frac{7\pi}{6} = \pi + \frac{\pi}{6}$, then

$$A = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \left(\sin \frac{7\pi}{6}, \cos \frac{7\pi}{6}\right).$$

Figure to be inserted here.

Example 471. If $\alpha = \frac{5\pi}{4} = \pi + \frac{\pi}{4}$, then

$$A = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \left(\sin \frac{5\pi}{4}, \cos \frac{5\pi}{4}\right).$$

Figure to be inserted here.

Example 472. If $\alpha = \frac{4\pi}{3} = \pi + \frac{\pi}{3}$, then

$$A = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \left(\sin\frac{4\pi}{3}, \cos\frac{4\pi}{3}\right).$$

Figure to be inserted here.

Example 473. If $\alpha = \frac{3\pi}{2}$, then

$$A = (-1, 0) = \left(\sin\frac{3\pi}{2}, \cos\frac{3\pi}{2}\right).$$

Figure to be inserted here.

Example 474. If $\alpha = \frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, then

$$A = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(\sin\frac{5\pi}{3}, \cos\frac{5\pi}{3}\right).$$

Figure to be inserted here.

Example 475. If $\alpha = \dfrac{7\pi}{4} = 2\pi - \dfrac{\pi}{4}$, then

$$A = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left(\sin\frac{7\pi}{4}, \cos\frac{7\pi}{4}\right).$$

Figure to be inserted here.

Example 476. If $\alpha = \dfrac{11\pi}{6} = 2\pi - \dfrac{\pi}{6}$, then

$$A = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\sin\frac{11\pi}{6}, \cos\frac{11\pi}{6}\right).$$

Figure to be inserted here.

Example 477. If $\alpha = 2\pi$, then we will have gone one full circle, with

$$A = (0, 1) = (\sin 2\pi, \cos 2\pi) = (\sin 0, \cos 0).$$

Figure to be inserted here.

(As we'll see in the next subchapter, each of sin and cos is **periodic** with **period** 2π.)

Let's summarise our findings from the last 17 examples (Examples 461–477) in a single table:

Fact 77.

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\alpha$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$	$-\frac{\sqrt{1}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{4}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$
$\cos\alpha$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$	$-\frac{\sqrt{1}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{4}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

Remark 73. To repeat, henceforth, we'll abandon the right-triangle definitions (Definition 93) and use only the unit-circle definitions (Definition 94 or equivalently, Definition 95).

But note that even the unit-circle definitions are considered informal, because they rely on geometry.

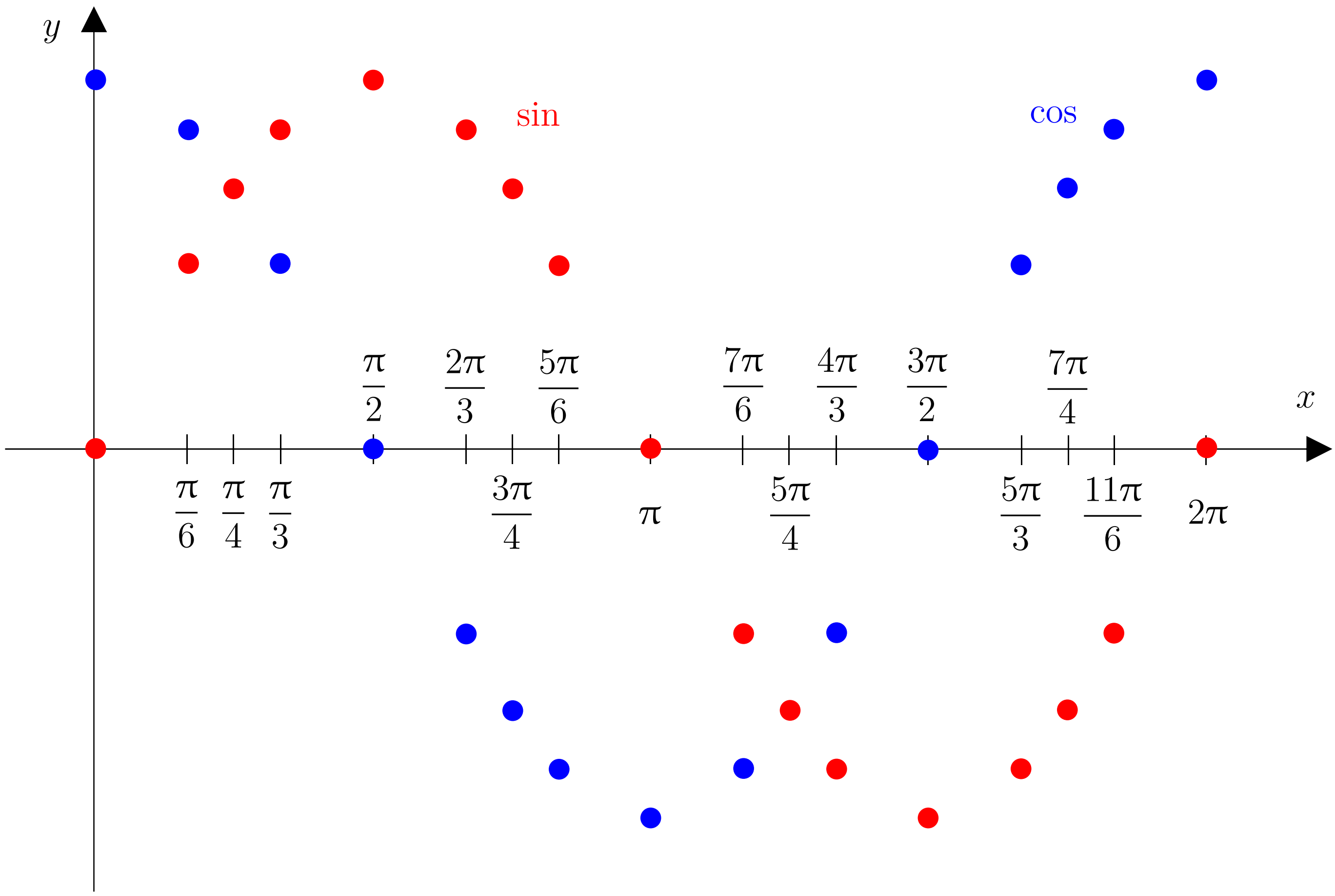
To formally and rigorously define sine and cosine, there are (at least) three common approaches:

1. Use **power series**—see Definitions 219 and 220 (in Part V (Calculus), Ch. 102.3). These will serve as this textbook's official definitions of sine and cosine.
2. Use **Euler's formula**—see Remark 123 (in Part III (Complex Numbers), Ch. 84.2).
3. Use **differential equations**—see Part V (Calculus), Ch. 110.5.

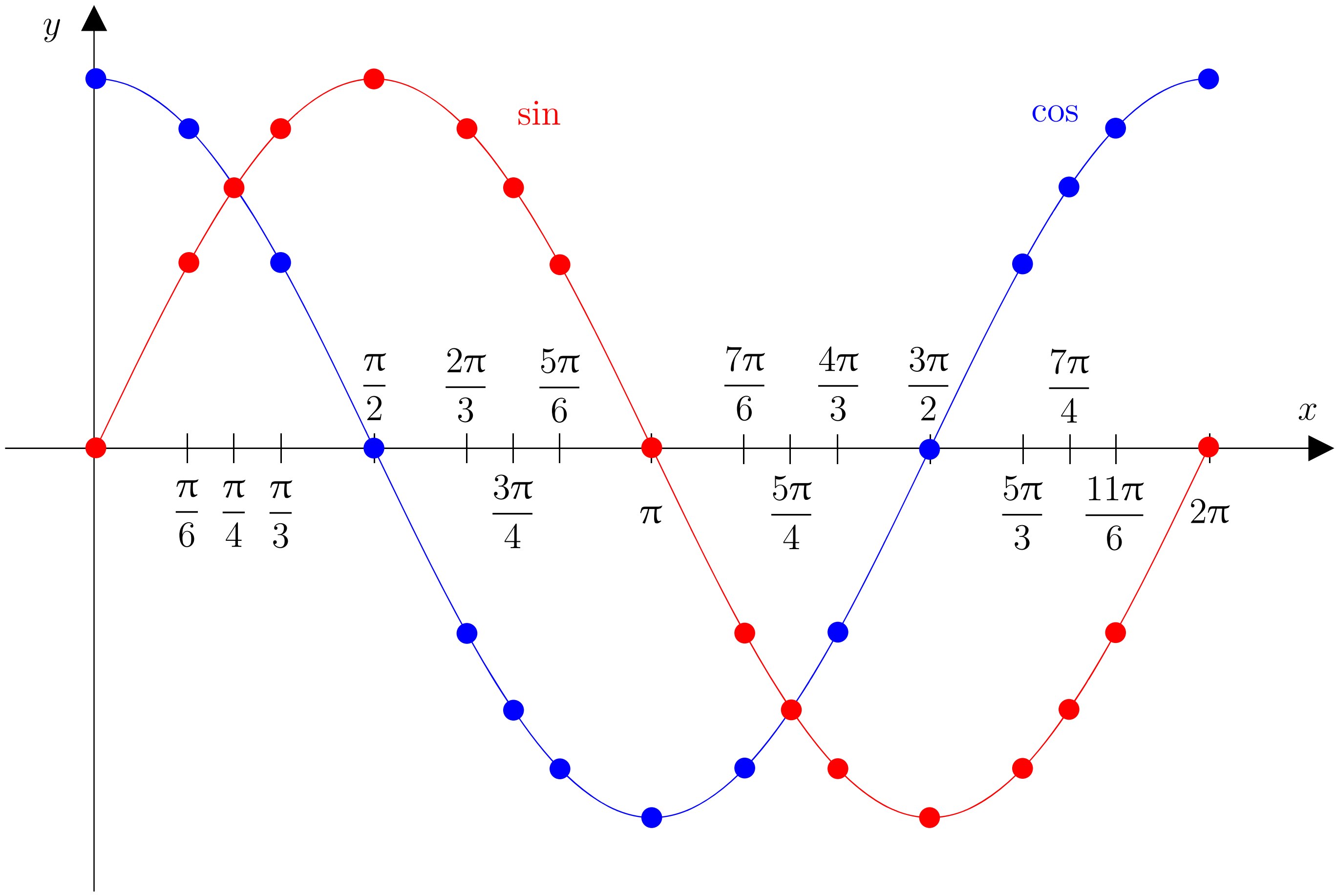
But don't worry about these formal definitions of sine and cosine. For A-Level Maths, the unit-circle definitions (Definition 94 or 95) will be more than good enough.

32.7. Graphs of Sine and Cosine

Fact 77 gave 17 points for each of sin and cos. Let's graph these points:



Hm, it looks a lot like we can simply "connect the dots" to get these graphs of sin and cos on $[0, 2\pi]$:



But is it proper for us to simply "connect the dots"? All we've done is find 17 points for each of sin and cos. How do we know that their actual graphs can be obtained by smoothly "connecting the dots"? Given the 17 points, why couldn't the actual graphs be something crazy like the following?

Figure to be
inserted here.

Here's an informal argument for why this last figure is wrong (and the previous one is correct). We'll use our alternative "travelling-particle definition" (Definition 95) of sin and cos:

1. At time $\alpha = 0$, our particle A is at $(1, 0)$. So, $\sin\alpha = 0$ and $\cos\alpha = 1$.
2. During $\alpha \in (0, \pi/2)$, A travels northwest.
   (a) Travel upwards is initially fast, but then slows down. So, throughout $\alpha \in (0, \pi/2)$, sin is increasing at a decreasing rate.
   (b) Travel leftwards is initially slow, but then speeds up. So, throughout $\alpha \in (0, \pi/2)$,cos is decreasing at an increasing rate.
3. At time $\alpha = \pi/2$, our particle A is at $(0, 1)$. So, $\sin\alpha = 1$ and $\cos\alpha = 0$.
4. During $\alpha \in (\pi/2, \pi)$, A travels southwest.

(a) Travel downwards is initially slow, but then speeds up. So, throughout $\alpha \in (0, \pi/2)$, sin is decreasing at an increasing rate.

(b) Travel leftwards is initially fast, but then slows down. So, throughout $\alpha \in (0, \pi/2)$, cos is decreasing at a decreasing rate.

5. At time $\alpha = \pi$, our particle A is at $(-1, 0)$. So, $\sin \alpha = 0$ and $\cos \alpha = -1$.

6. During $\alpha \in (\pi, 3\pi/2)$, A travels southeast.

   (a) Travel downwards is initially fast, but then slows down. So, throughout $\alpha \in (0, \pi/2)$,sin is decreasing at a decreasing rate.

   (b) Travel rightwards is initially slow, but then speeds up. So, throughout $\alpha \in (0, \pi/2)$, cos is increasing at an increasing rate.

7. At time $\alpha = 3\pi/2$, our particle A is at $(0, -1)$. So, $\sin \alpha = -1$ and $\cos \alpha = 0$.

8. During $\alpha \in (3\pi/2, 2\pi)$, A travels northeast.

   (a) Travel upwards is initially slow, but then speeds down. So, throughout $\alpha \in (0, \pi/2)$, sin is increasing at an increasing rate.

   (b) Travel rightwards is initially fast, but then slows down. So, throughout $\alpha \in (0, \pi/2)$, cos is increasing at a decreasing rate.

Figure to be
inserted here.

Lovely. So, we're fairly confident about what the graphs of sin and cos look like on $[0, 2\pi]$.

Let's now consider what the graphs of sin and cos look like on $[2\pi, 4\pi]$.

As mentioned in the last example of the previous subchapter (Example 477), we know that at 2π, we've gone a full circle. Thereafter, everything should exactly repeat. And so, the graphs of sin and cos on $[2\pi, 4\pi]$ must be exactly the same as those on $[0, 2\pi]$: ☺

Figure to be
inserted here.

Indeed, repeating the same reasoning for each $k \in \mathbb{Z}^+$, the graphs of sin and cos on $[2k\pi, (2k+2)\pi]$ must be exactly the same as those on $[0, 2\pi]$:

Figure to be inserted here.

Next, let's consider what the graphs of sin and cos look like on $[-2\pi, 0]$.

Well, what does $\sin\alpha$ or $\cos\alpha$ for **negative** α even mean? Well, we've adopted the convention that rotating the positive x-axis x^+ **anticlockwise** is the **positive** direction. So, **negative** angles should correspond to rotating x^+ **clockwise**.

Or equivalently and using our alternative definition (Definition 95), **negative** α corresponds to our particle A travelling **clockwise**.

And whether we're travelling anticlockwise or clockwise, by reasoning similar to ☺, after every 2π seconds, we will have gone one full circle. Hence, the graphs of sin and cos on $[-2\pi, 0]$ must be exactly the same as those on $[0, 2\pi]$:

Figure to be inserted here.

Indeed, repeating the same reasoning for each $k \in \mathbb{Z}^-$, the graphs of sin and cos on $[(2k-2)\pi, 2k\pi]$ must be exactly the same as those on $[0, 2\pi]$:

Figure to be inserted here.

With this last figure, we've completed our construction of the graphs of sin and cos.

To repeat, we say that each of sin and cos is **periodic** with **period** 2π. Informally, this means that each graph "repeats" every 2π.[225] A bit more formally,

Fact 78. *For all $x \in \mathbb{R}$,* **(a)** $\sin(x + 2\pi) = \sin x$; *and* **(b)** $\cos(x + 2\pi) = \cos x$.

Proof. **(a)** By the Addition Formula for Sine (Fact 73) and Fact 77,

$$\sin(x + 2\pi) = \sin x \cos 2\pi + \cos x \sin 2\pi = \sin x.$$

(b) By the Addition Formula for Cosine (Fact 73) and Fact 77,

$$\cos(x + 2\pi) = \cos x \cos 2\pi - \sin x \sin 2\pi = \cos x. \qquad \square$$

Equivalently, the translation of sin or cos by any integer multiple of 2π is itself:

Figure to be
inserted here.

[225]For a more formal definition of periodicity, see Definition 294 (Appendices).

32.8. Graphs of Sine and Cosine: Some Observations

Figure to be
inserted here.

"Clearly", sin is symmetric in the vertical line $x = \pi/2$. More generally, for any odd integer k, sin is symmetric in $x = k\pi/2$.

Similarly, cos is symmetric in the vertical line $x = \pi$. More generally, for any integer k, cos is symmetric in $x = k\pi$.

For future reference, let's jot the above observations down as a formal result:

Fact 79. *The lines of symmetry for*

(a) sin *are* $x = k\pi/2$*, for odd integers* k*;*

(b) cos *are* $x = k\pi$*, for integers* k*.*

Proof. See p. 1594 (Appendices). □

Cosine is symmetric in $x = 0$ or the y-axis; equivalently, cos is an even function):

Fact 80. *For all* $x \in \mathbb{R}$*,* $\cos x = \cos(-x)$*.*

Proof. By Fact 79, cos is symmetric in $x = 0$. By Cor. 6, for all $x \in \mathbb{R}$, $\cos x = \cos(-x)$. □

The reflection of sin in the y-axis is the reflection of sin in the x-axis. Equivalently, sin is symmetric about the origin; also equivalently, sin is an odd function:

Fact 81. *For all* $x \in \mathbb{R}$*,* $-\sin x = \sin(-x)$*.*

Proof. By Fact 75 (S2P or P2S Formulae), for all $x \in \mathbb{R}$,

$$\sin x + \sin(-x) = 2\sin\frac{x-x}{2}\cos\frac{x+x}{2} = 2\sin 0\cos x = 0.$$

Rearranging, $\sin x = -\sin(-x)$. □

The translation of sin or cos by any odd multiple of π is its own reflection in the x-axis:

Fact 82. *For all $x \in \mathbb{R}$ and all odd integers k,*

(a) $\sin(x + k\pi) = -\sin x$; **(b)** $\cos(x + k\pi) = -\cos x$.

Figure to be
inserted here.

Proof. First, observe that for all odd integers k, $\cos(k\pi/2) \stackrel{1}{=} 0$.

(a) By Fact 75 (S2P or P2S Formulae), for all $x \in \mathbb{R}$,

$$\sin(x + k\pi) + \sin x = 2\sin\frac{x + k\pi + x}{2}\cos\frac{x + k\pi - x}{2} = 2\sin\left(x + k\frac{\pi}{2}\right)\cos k\frac{\pi}{2} \stackrel{1}{=} 0.$$

Rearranging, $\sin(x + k\pi) = -\sin x$.

(b) By Fact 75 (S2P or P2S Formulae), for all $x \in \mathbb{R}$,

$$\cos(x + k\pi) + \cos x = 2\cos\frac{x + k\pi + x}{2}\cos\frac{x + k\pi - x}{2} = 2\cos\left(x + k\frac{\pi}{2}\right)\cos k\frac{\pi}{2} \stackrel{1}{=} 0.$$

Rearranging, $\cos(x + k\pi) = -\cos x$. □

Figure to be
inserted here.

The next result says that sin is cos translated right by $\pi/2$ (equivalently, cos is sin translated left by $\pi/2$):

Fact 83. *For all $x \in \mathbb{R}$,*

(a) $\sin x = \cos\left(x - \frac{\pi}{2}\right)$; **(b)** $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

Proof. **(a)** $\cos\left(x - \frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} + \sin x \sin\frac{\pi}{2} = 0 + \sin x = \sin x$.

(b) $\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2} = 0 + \cos x = \cos x$. □

32.9. About the Prefix *Co-*

You will probably have noticed that

$$cosine = co + sine.$$

In the context of trigonometric functions, the **prefix *co-*** has this meaning:
If $A + B = \pi/2$ (i.e. A and B are complementary), then

$$\sin A = \cos B.$$

Or equivalently, for any $A \in \mathbb{R}$,

$$\sin A = \cos\left(\frac{\pi}{2} - A\right).$$

Also equivalently, using what we learnt in Ch. 26 (Transformations),

Sine is cosine shifted left by $\pi/2$, then reflected in the y-axis.

And symmetrically,

Cosine is sine shifted left by $\pi/2$, then reflected in the y-axis.

As we'll see in the coming chapters, the above will also be true of the functions **tangent and *co*tangent**, and the functions **secant and *co*secant**.

Figure to be
inserted here.

33. Trigonometry: Tangent

Definition 96. The *tangent function*, denoted tan, is defined by

$$\tan = \frac{\sin}{\cos}.$$

Example 478. XXX

Observe that $\cos\alpha = 0 \iff \alpha$ is an odd-integer multiple of $\pi/2$. So, $\tan = \sin/\cos$ is undefined at odd-integer multiples of $\pi/2$:

Example 479. XXX

Exercise 144. What are the domain and codomain of tan? (Hint: Review Ch. 20.) (Answer on p. 1802.)

By the right-triangle definitions,

$$\sin = \frac{O}{H} \qquad \text{and} \qquad \cos = \frac{A}{H}.$$

So,

$$\tan = \frac{\sin}{\cos} = \frac{O/H}{A/H} = \frac{O}{A}.$$

In the past, Singaporean students were taught the Hokkien mnemonic "big leg woman":

大	跤	嫂
tuā	*kha*	*só*
TOA	**CAH**	**SOH**
$\textbf{T}\text{angent} = \dfrac{\mathbf{O}}{\mathbf{A}}$,	$\textbf{C}\text{osine} = \dfrac{\mathbf{A}}{\mathbf{H}}$,	$\textbf{S}\text{ine} = \dfrac{\mathbf{O}}{\mathbf{H}}$.

But these days, we probably prefer a more *angmoh* mnemonic like

Soccer tour—SOH-CAH-TOA.[226]

Example 480. XXX

Example 481. XXX

[226] Credit to Tom J at TheStudentRoom.co.uk (warning: that page may contain many other NSFW mnemonics).

33.1. Some Formulae Involving Tangent

Happily, List MF26 (p.3) has the following formulae, so no need to *mug*:

Fact 84. *Let $A, B \in \mathbb{R}$. Suppose A and B are not odd integer multiples of $\pi/2$.*

(a) *If $A \pm B$ is not an odd-integer multiple of $\pi/2$,*[227] *then*

Addition and Subtraction Formula for Tangent
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

(b) *If $2A$ is not an odd-integer multiple of $\pi/2$, then*

Double Angle Formula for Tangent
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}.$$

Proof. **(a)** See Exercise 145.

(b) See Exercise 146. □

Exercise 145. Prove Fact 84(a). (Answer on p. 1802.)

Exercise 146. Prove Fact 84(b). (Answer on p. 1802.)

The remaining formulae in this subchapter aren't very useful, but are conceivably things you could be asked to derive or use on an exam, and so are worth a mention.

Rearrange the above Addition and Subtraction Formulae to get the **Sum-to-Product (S2P) Formula for Tangent**:

Fact 85. (S2P Formula for Tangent) *Let $A, B \in \mathbb{R}$. Suppose A, B, and $A \pm B$ are not odd-integer multiples of $\pi/2$, then*

$$\tan A \pm \tan B = \tan(A \pm B)(1 \mp \tan A \tan B).$$

---

[227] In each of Fact 84**(a)** and **(b)**, there is the worrying possibility that the denominator is zero, in which case we'd be committing the Cardinal Sin of Dividing by Zero (see Ch. 2.2). It turns out that happily, in each case, our additional condition that $A \pm B$ or $2A$ is not an odd integer multiple of $\pi/2$, as we now show (via the contrapositives):

(a) Suppose $1 \mp \tan A \tan B = 0$. This is equivalent to

$$\tan A \tan B = \pm 1 \iff \frac{\sin A}{\cos A}\frac{\sin B}{\cos B} = \pm 1 \iff 0 = \sin A \sin B \mp \cos A \cos B = \mp \cos(A \pm B),$$

so that $A \pm B$ is not an odd integer multiple of $\pi/2$.

(b) Suppose $1 - \tan^2 A = 0$. This is equivalent to $\tan^2 A = 1 \iff \tan A = \pm 1 \iff A \in \{(k \pm 0.25)\pi : k \in \mathbb{Z}\} \iff 2A \in \{0.5k\pi : k \text{ is an odd integer}\}$, so that $2A$ is not an odd integer multiple of $\pi/2$.

Fact 86. (Triple Angle Formula for Tangent) *Let* $A, B \in \mathbb{R} \setminus \{k\pi/2 : k$ *is an odd integer*$\}$. *If* $3A$ *is not an odd-integer multiple of* $\pi/2$,[228] *then*

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Proof. See Exercise 147. □

Exercise 147. Prove Fact 86. (Answer on p. 1802.)

Earlier, we had the Laws of Sines and Cosines (Proposition 7). Similarly,

Fact 87. *Suppose a triangle has angles* A *and* B *facing sides of lengths* a *and* b. *If* $a \neq b$,[229] *then*

$$\frac{a+b}{a-b} = \frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}. \qquad \textbf{(Law of Tangents)}$$

Proof. See Exercise 148. □

Exercise 148. We'll prove Fact 87. By the Law of Sines,

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{or} \qquad b\sin A \stackrel{1}{=} a\sin B.$$

(a) By considering $\dfrac{a+b}{a-b} \times \dfrac{\sin A \sin B}{\sin A \sin B}$ (Times One Trick) and using $\stackrel{1}{=}$, show that

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

(b) Now[230] show that $\dfrac{a+b}{a-b} = \dfrac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}$. (Answer on p. 1802.)

[228] Similar to n. 227, we can show that this condition ensures $1 - 3\tan^2 A \neq 0$:
Suppose $1 - 3\tan^2 A = 0$. This is equivalent to $\tan^2 A = 1/3 \iff \tan A = \pm\sqrt{3}/3 \iff A \in \{(k \pm 1/6)\pi : k \in \mathbb{Z}\} \implies 3A \subseteq \{k\pi/2 : k \text{ is an odd integer}\}$, so that $\tan 3A$ is not defined.

[229] Suppose $a \neq b$. Then $A \neq B$ and each of the denominators of LHS and RHS is non-zero, so that both LHS and RHS are well-defined.
Observe also that the angles of a triangle are strictly positive and add up to π. So, $A + B = \pi - C \in (0, \pi)$ and $A - B \in (-\pi, \pi)$. Hence, neither $(A+B)/2 \in (0, \pi/2)$ nor $(A+B)/2 \in (-\pi/2, \pi/2)$ is ever an odd-integer multiple of $\pi/2$. Thus, both $\tan[(A+B)/2]$ and $\tan[(A-B)/2]$ are always well-defined.

[230] Hint: Whenever you see a trigonometry question, what should you do?

33.2. Graph of Tangent

Use Fact 77 (17 values for each of sin and cos) to produce these values of tan:

Fact 88.

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\tan\alpha$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	☹	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	☹	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

We already said that tan is undefined at each odd-integer multiple of $\pi/2$. We can say more:

Fact 89. *For odd integers k, $x = k\pi/2$ is a vertical asymptote for* tan.

Let's graph the points in Fact 88:

Figure to be inserted here.

Hm, it looks a lot like we can simply "connect the dots" to get this graph of tan on $[0, 2\pi]$:

Figure to be inserted here.

As before, we can make an informal argument as to why the above graph is correct and something crazy like the below is wrong. But we shall omit this.

Figure to be inserted here.

Like sine and cosine, tangent is periodic. But unlike sine and cosine, tangent has period π (instead of 2π). Informally, this means that the graph of tan "repeats" every π. More formally,

Fact 90. *For all* $x \in \mathbb{R}$, $\tan(x+\pi) = \tan x$.

Proof. By the Addition Formula for Tangent (Fact 84) and Fact 88,

$$\tan(x+\pi) = \frac{\tan x + \tan \pi}{1 - \tan x \tan \pi} = \frac{\tan x}{1-0} = \tan x. \qquad \square$$

Equivalently, the translation of tan by any integer multiple of π is itself. And so, the complete graph of tan looks like this:

Figure to be
inserted here.

33.3. Graph of Tangent: Some Observations

The reflection of tan in the y-axis is the reflection of tan in the x-axis:

Fact 91. *If $x \in \mathbb{R}$ is not an odd integer multiple of $\pi/2$, then*

$$-\tan x = \tan(-x).$$

Proof. Note that $-x + x = 0$, which is not an odd integer multiple of $\pi/2$. So, by Fact 85 (S2P Formula for Tangent),

$$\tan(-x) + \tan x = \tan(-x+x)\left[1 - \tan(-x)\tan x\right] = 0.$$

Rearranging, $-\tan x = \tan(-x)$. □

Fact 92. *The graph of* tan *has no lines of symmetry.*

Proof. See p. 1595 (Appendices). □

Figure to be
inserted here.

You may be wondering whether any single "branch" of tan might have a line of symmetry.

Figure to be
inserted here.

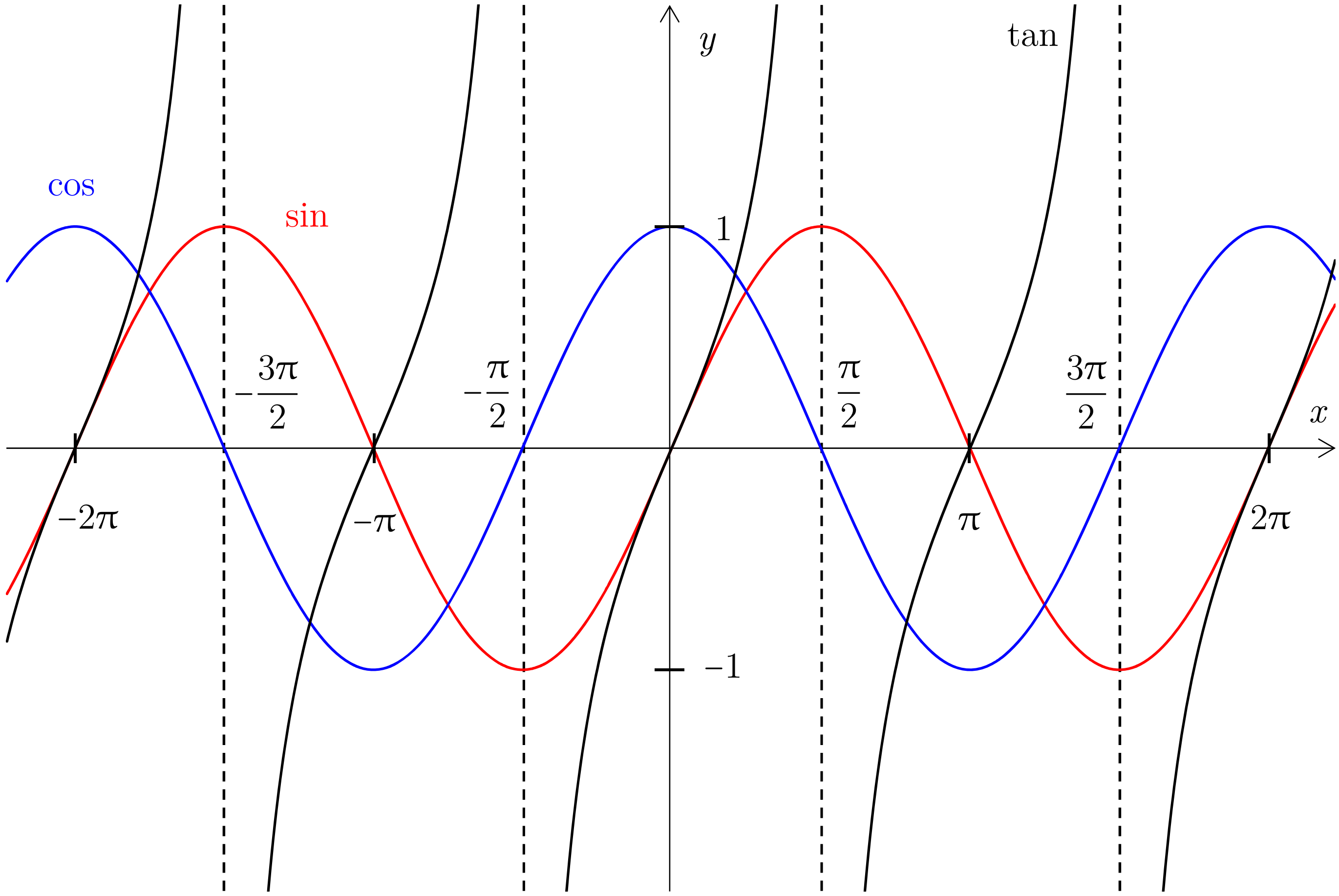
For example, might the graph of $\tan\big|_{(-\pi/2,\pi/2)}$ have a line of symmetry (perhaps some downward-sloping line)? Answer—no:

Fact 93. *The graph of* $\tan\big|_{(-\pi/2,\pi/2)}$ *has no lines of symmetry.*

Proof. See p. 1595 (Appendices). □

34. Trigonometry: Summary and a Few New Things

Graphs of sin, cos, and tan:



To remember these three graphs, here are the key things to remember:

- sin and cos both have
  - the same shapes;
  - period 2π;
  - max value 1; and
  - min value -1.
- sin and tan both go through the origin.
- cos has y-intercept $(1, 0)$.
- tan goes from $-\infty$ to ∞ and has period π.

34.1. The Signs of Sine, Cosine, and Tangent

Fact 94. (The Signs of Sine, Cosine, and Tangent)

(a) $$\sin x \begin{cases} \geq 0, & \text{for } x \in [0, \pi], \\ > 0, & \text{for } x \in (0, \pi), \\ \leq 0, & \text{for } x \in [\pi, 2\pi], \\ < 0, & \text{for } x \in (\pi, 2\pi). \end{cases}$$

(b) $$\cos x \begin{cases} \geq 0, & \text{for } x \in [-\pi/2, \pi/2], \\ > 0, & \text{for } x \in (-\pi/2, \pi/2), \\ \leq 0, & \text{for } x \in [\pi/2, 3\pi/2], \\ < 0, & \text{for } x \in (\pi/2, 3\pi/2). \end{cases}$$

(c) $$\tan x \begin{cases} \geq 0, & \text{for } x \in [0, \pi/2), \\ > 0, & \text{for } x \in (0, \pi/2), \\ \leq 0, & \text{for } x \in (-\pi/2, 0], \\ < 0, & \text{for } x \in (-\pi/2, 0). \end{cases}$$

Figure to be inserted here.

Of course, since sin and cos are periodic with period 2π, while tan is periodic with period π, we can generalise the above result as

Corollary 13. (The Signs of Sine, Cosine, and Tangent) *For every $k \in \mathbb{Z}$,*

$$\textbf{(a)}\quad \sin x \begin{cases} \geq 0, & \text{for } x \in [2k\pi, (2k+1)\pi], \\ > 0, & \text{for } x \in (2k\pi, (2k+1)\pi), \\ \leq 0, & \text{for } x \in [(2k+1)\pi, (2k+2)\pi], \\ < 0, & \text{for } x \in ((2k+1)\pi, (2k+2)\pi); \end{cases}$$

$$\textbf{(b)}\quad \cos x \begin{cases} \geq 0, & \text{for } x \in \left[\left(2k-\frac{1}{2}\right)\pi, \left(2k+\frac{1}{2}\right)\pi\right], \\ > 0, & \text{for } x \in \left(\left(2k-\frac{1}{2}\right)\pi, \left(2k+\frac{1}{2}\right)\pi\right), \\ \leq 0, & \text{for } x \in \left[\left(2k+\frac{1}{2}\right)\pi, \left(2k+\frac{3}{2}\right)\pi\right], \\ < 0, & \text{for } x \in \left(\left(2k+\frac{1}{2}\right)\pi, \left(2k+\frac{3}{2}\right)\pi\right); \end{cases}$$

$$\textbf{(c)}\quad \tan x \begin{cases} \geq 0, & \text{for } x \in \left[2k\pi, \left(2k+\frac{1}{2}\right)\pi\right), \\ > 0, & \text{for } x \in \left(2k\pi, \left(2k+\frac{1}{2}\right)\pi\right), \\ \leq 0, & \text{for } x \in \left(\left(2k-\frac{1}{2}\right)\pi, 2k\pi\right], \\ < 0, & \text{for } x \in \left(\left(2k-\frac{1}{2}\right)\pi, 2k\pi\right). \end{cases}$$

Fact 95. (Half-Angle Formulae for Sine, Cosine, and Tangent)

(a) $$\sin\frac{x}{2} = \begin{cases} \sqrt{\dfrac{1-\cos x}{2}}, & \text{for } \dfrac{x}{2} \in [0,\pi], \\ -\sqrt{\dfrac{1-\cos x}{2}}, & \text{for } \dfrac{x}{2} \in [\pi, 2\pi]. \end{cases}$$

(b) $$\cos\frac{x}{2} = \begin{cases} \sqrt{\dfrac{1+\cos x}{2}}, & \text{for } \dfrac{x}{2} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right], \\ -\sqrt{\dfrac{1+\cos x}{2}}, & \text{for } \dfrac{x}{2} \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]. \end{cases}$$

(c)(i) $$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}, \text{ for all } \frac{x}{2} \in \mathbb{R}.$$

(c)(ii) $$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}, \text{ for all } \frac{x}{2} \in \mathbb{R}.$$

(c)(iii) $$\tan\frac{x}{2} = \begin{cases} \sqrt{\dfrac{1-\cos x}{1+\cos x}}, & \text{for } \dfrac{x}{2} \in \left[0, \dfrac{\pi}{2}\right), \\ -\sqrt{\dfrac{1-\cos x}{1+\cos x}}, & \text{for } \dfrac{x}{2} \in \left(-\dfrac{\pi}{2}, 0\right]. \end{cases}$$

Proof. By the Double Angle Formulae, for all $x \in \mathbb{R}$,

$$\cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right) = 2\cos^2\frac{x}{2} - 1 = 1 - 2\sin^2\frac{x}{2}.$$

So, $$\sin^2\frac{x}{2} = \frac{1-\cos x}{2} \qquad \text{and} \qquad \cos^2\frac{x}{2} = \frac{1+\cos x}{2}.$$

Or, $$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \qquad \text{and} \qquad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}.$$

To figure out what the correct sign is for each specific $x/2$, use Corollary 13:

(a) By Corollary 13, $\sin\frac{x}{2} \geq 0$ if $\frac{x}{2} \in [0,\pi]$ and $\sin\frac{x}{2} \leq 0$ if $\frac{x}{2} \in [\pi, 2\pi]$.

Hence, $$\sin\frac{x}{2} = \begin{cases} \sqrt{\dfrac{1-\cos x}{2}}, & \text{for } \dfrac{x}{2} \in [0,\pi], \\ -\sqrt{\dfrac{1-\cos x}{2}}, & \text{for } \dfrac{x}{2} \in [\pi, 2\pi]. \end{cases}$$

(*Proof continues below ...*)

(... *Proof continued from above.*)

(b) Similarly and again by Corollary 13,

$$\cos\frac{x}{2} \geq 0 \text{ if } \frac{x}{2} \in [-\pi/2, \pi/2] \qquad \text{and} \qquad \cos\frac{x}{2} \leq 0 \text{ if } \frac{x}{2} \in [\pi/2, 3\pi/2].$$

Hence,
$$\cos\frac{x}{2} = \begin{cases} \sqrt{\dfrac{1+\cos x}{2}}, & \text{for } \dfrac{x}{2} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right], \\ -\sqrt{\dfrac{1+\cos x}{2}}, & \text{for } \dfrac{x}{2} \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]. \end{cases}$$

(c) See Exercise 149. □

Exercise 149. Prove Fact 95(c). (Answer on p. 1803.)

Of course, since sin and cos are periodic with period 2π, while tan is periodic with period π, we can generalise **(a)**, **(b)**, and **(c)(iii)** in the above result:

Corollary 14. *For every $k \in \mathbb{Z}$,*

(a)
$$\sin\frac{x}{2} = \begin{cases} \sqrt{\dfrac{1-\cos x}{2}}, & \text{for } \dfrac{x}{2} \in [2k\pi, (2k+1)\pi], \\ -\sqrt{\dfrac{1-\cos x}{2}}, & \text{for } \dfrac{x}{2} \in [(2k+1)\pi, (2k+2)\pi]; \end{cases}$$

(b)
$$\cos\frac{x}{2} = \begin{cases} \sqrt{\dfrac{1+\cos x}{2}}, & \text{for } \dfrac{x}{2} \in \left[\left(2k-\dfrac{1}{2}\right)\pi, \left(2k+\dfrac{1}{2}\right)\pi\right], \\ -\sqrt{\dfrac{1+\cos x}{2}}, & \text{for } \dfrac{x}{2} \in \left[\left(2k+\dfrac{1}{2}\right)\pi, \left(2k+\dfrac{3}{2}\right)\pi\right]; \end{cases}$$

(c)(iii)
$$\tan\frac{x}{2} = \begin{cases} \sqrt{\dfrac{1-\cos x}{1+\cos x}}, & \text{for } \dfrac{x}{2} \in \left[k\pi, (2k+1)\dfrac{\pi}{2}\right), \\ -\sqrt{\dfrac{1-\cos x}{1+\cos x}}, & \text{for } \dfrac{x}{2} \in \left((2k-1)\dfrac{\pi}{2}, k\pi\right]. \end{cases}$$

35. Trigonometry: Three More Functions

The **cosecant** and **secant** functions are defined as the reciprocals of sin and cos:

Definition 97. The *secant* and *cosecant* functions, denoted sec and cosec, are defined by

$$\sec = \frac{1}{\cos} \qquad \text{and} \qquad \operatorname{cosec} = \frac{1}{\sin}.$$

Definition 98. The *cotangent* function, denoted cot, is defined by

$$\cot = \frac{\cos}{\sin}.$$

Example 482. XXX

Remark 74. We follow your A-Level syllabus and exams in denoting the cosecant function by cosec. Be aware though that many writers instead denote it by csc.

Example 483. XXX

Remark 75. Together, sin, cos, tan, cosec, sec, and cot are the six **trigonometric functions** (or **circular**[231] **functions**) that you need to know for A-Level Maths.

Other trigonometric functions that were historically used but are no longer taught include **versine** (or **versed sine**), **coversine**, **haversine** (or **half versed sine**), **chord**, **exsecant**, and **excosecant**. (Again, these other functions may all be expressed in terms of sin and/or cos.)

In a right triangle, $\quad \sin = \frac{\text{O}}{\text{H}}, \qquad \cos = \frac{\text{A}}{\text{H}}, \qquad \tan = \frac{\text{O}}{\text{A}}.$

Hence,

$$\operatorname{cosec} = \frac{1}{\sin} = \frac{\text{H}}{\text{O}},$$

$$\sec = \frac{1}{\cos} = \frac{\text{H}}{\text{A}},$$

$$\cot = \frac{\cos}{\sin} = \frac{\text{A/H}}{\text{O/H}} = \frac{\text{A}}{\text{O}}.$$

Example 484. XXX

Example 485. XXX

Exercise 150. XXX (Answer on p. 377.)

A150.

[231] On p. 18 of your H2 Maths syllabus, these six functions are simply called the **circular functions**. However, the term **trigonometric functions** is probably more common.

The cotangent function cot is "nearly" but not quite the same as the function $1/\tan$:

Figure to be inserted here.

Fact 96. $\cot \neq 1/\tan$.

Proof. Since tan is not defined at $\pi/2$, neither is the function $1/\tan$.
In contrast, cot is defined at $\pi/2$ (with $\cot \pi/2 = 0$).
Hence, $\cot \neq 1/\tan$. □

Recall the First Pythagorean Identity (Fact 71):

$$\sin^2 A + \cos^2 A = 1 \text{ (for all } A \in \mathbb{R}).$$

We can use the above identity to prove two related identities (with equally inspiring names):

Fact 97. *Suppose* $A \in \mathbb{R}$.

(a) *If* $\cos A \neq 0$, *then* $1 + \tan^2 A = \sec^2 A$. **(Second Pythagorean Identity)**
(b) *If* $\sin A \neq 0$, *then* $1 + \cot^2 A = \operatorname{cosec}^2 A$. **(Third Pythagorean Identity)**

Proof. **(a)** $1 + \tan^2 A = 1 + \dfrac{\sin^2 A}{\cos^2 A} = \dfrac{\cos^2 A + \sin^2 A}{\cos^2 A} = \dfrac{1}{\cos^2 A} = \sec^2 A$.

(b) $1 + \cot^2 A = 1 + \dfrac{\cos^2 A}{\sin^2 A} = \dfrac{\sin^2 A + \cos^2 A}{\sin^2 A} = \dfrac{1}{\sin^2 A} = \operatorname{cosec}^2 A$. □

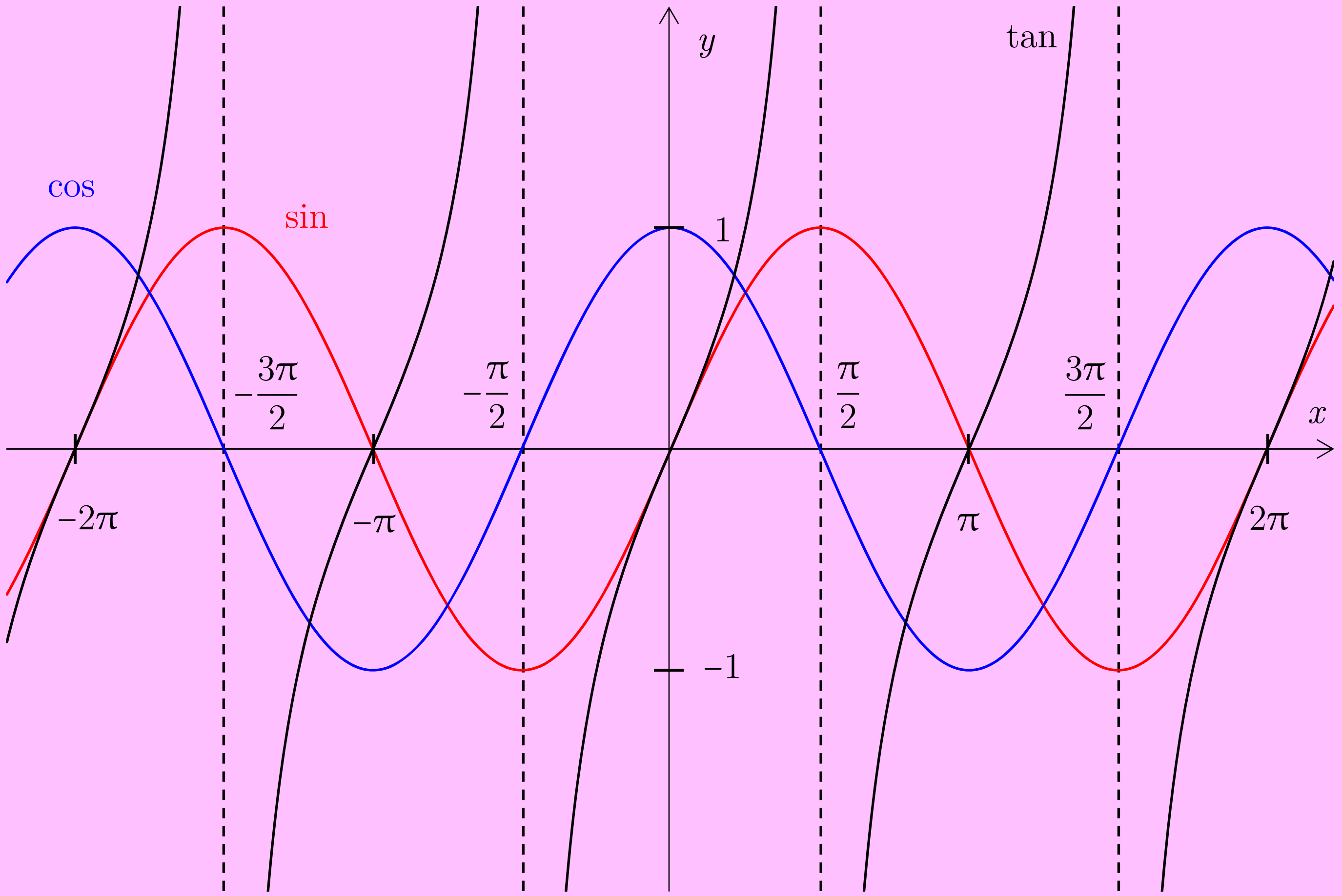
Remark 76. In **(a)** of the above result, the condition $\cos A \neq 0$ is simply to ensure that $\tan A$ and $\sec A$ are well-defined.

Similarly, in **(b)**, the condition $\sin A \neq 0$ simply ensures that $\cot A$ and $\operatorname{cosec} A$ are well-defined.

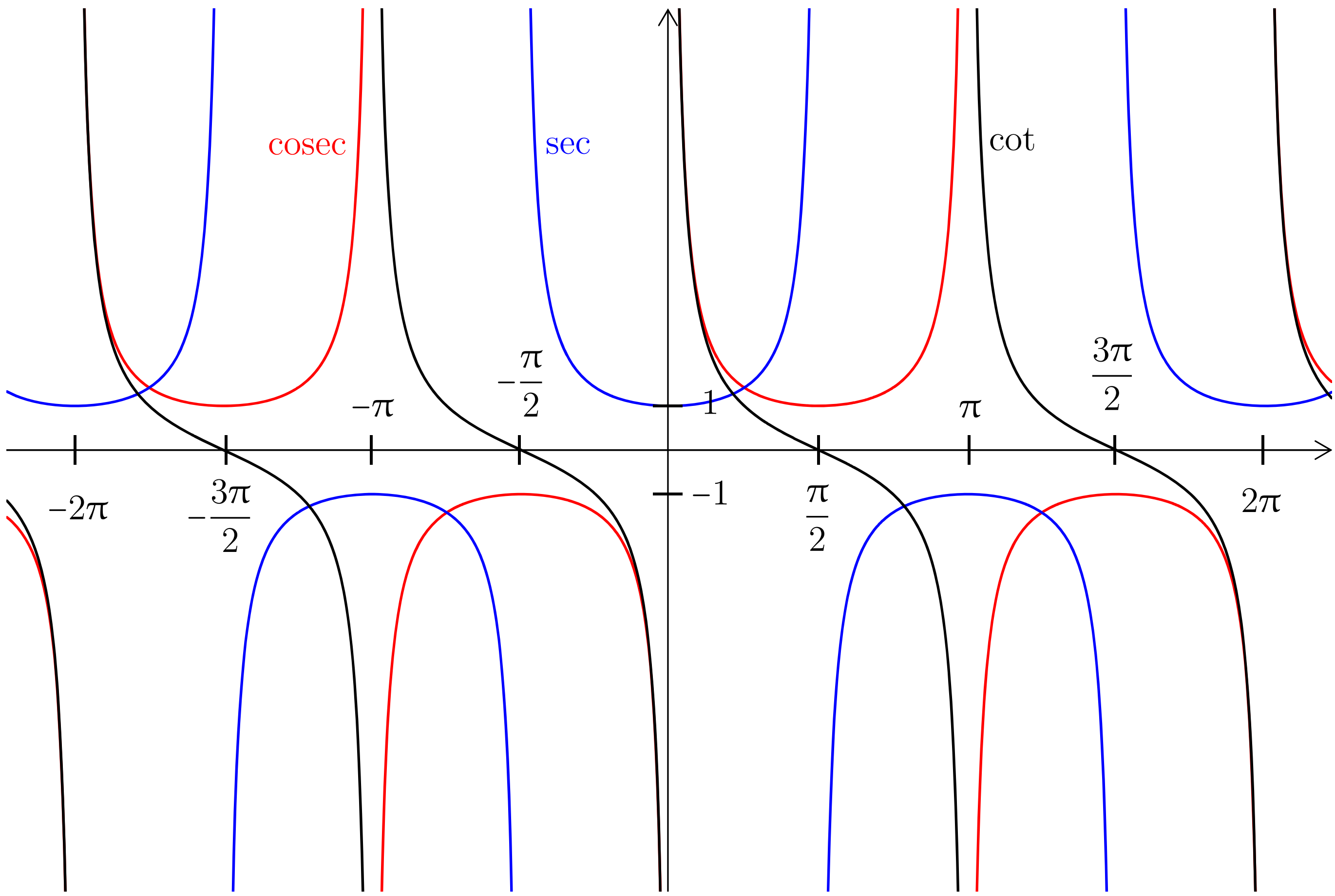
Remark 77. *Mug* the three Pythagorean Identities (they're **not** on List MF26).

In Ch. 26.8, we learnt to use the graph of f to graph $1/f$. Here's more practice:

Exercise 151. Graph cosec, sec, and cot. For your convenience, here are the graphs of sin, cos, and tan again: (Answer on the next page.)



Graphs of cosec, sec, and cot:



As discussed in Ch. 32.9, the prefix *co-* has this meaning:

Suppose $A + B = \pi/2$ (i.e. A and B are complementary). Then,

- $\sin A = \cos B$ or $\sin A = \cos(\pi/2 - A) \stackrel{1}{=} \cos(A - \pi/2)$.[232] So, cos translated $\pi/2$ rightwards is sin.
- Similarly, sec A = cosec B or cosec B = sec $(\pi/2 - B) \stackrel{2}{=}$ sec $(B - \pi/2)$.[233] So, sec translated $\pi/2$ rightwards is cosec.
- Also similarly, $\tan A = \cot B$ or $\tan A = \cot(\pi/2 - A) = \cot[-(A - \pi/2)]$. So, cot translated $\pi/2$ rightwards and then reflected in the x-axis is tan.

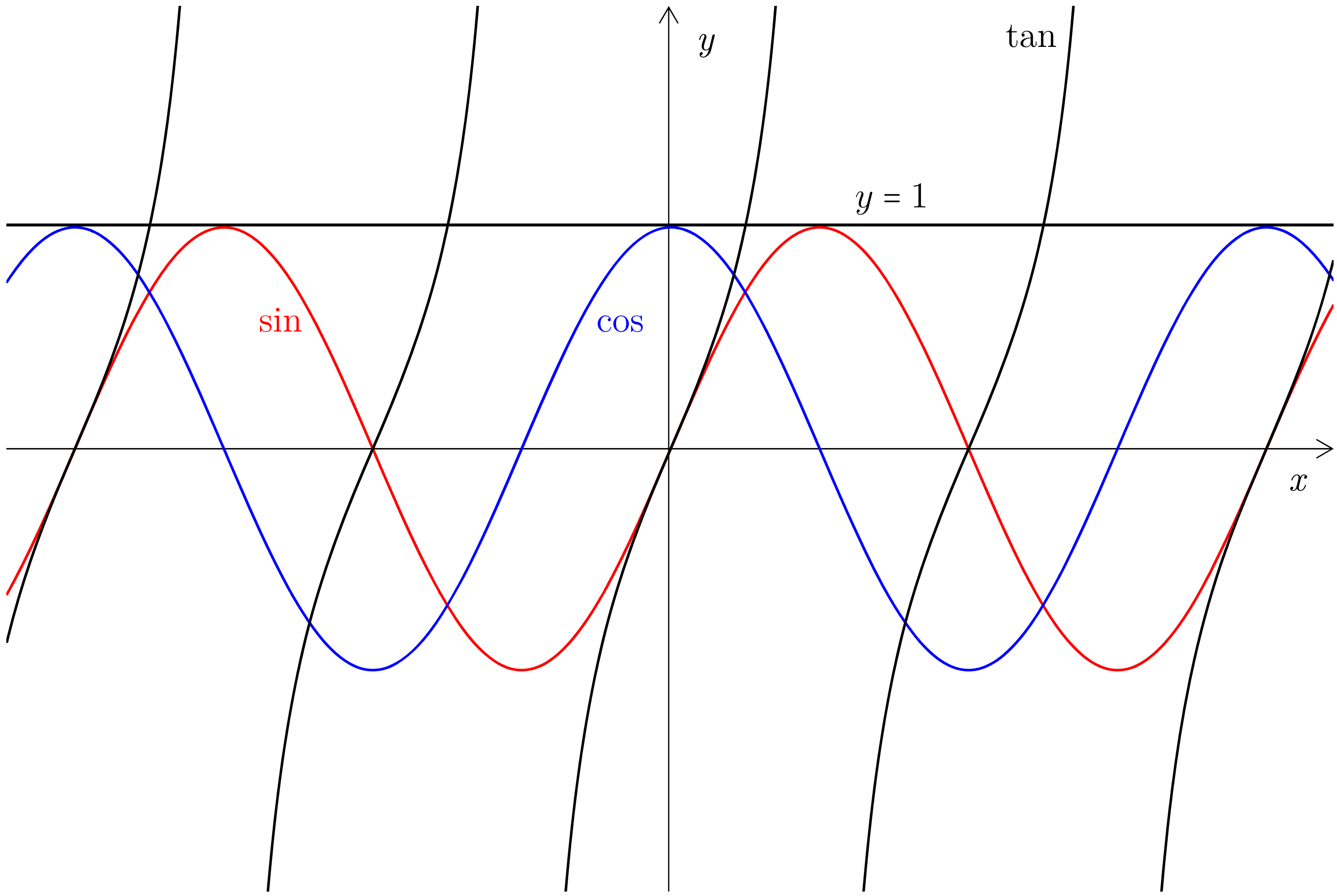
[232] $\stackrel{1}{=}$ uses Fact 80: $\cos x = \cos(-x)$ (equivalently, cos is symmetric in the y-axis).

[233] Likewise, $\stackrel{2}{=}$ uses the fact that cos and hence also sec = 1/ cos is symmetric in the y-axis.

36. Trigonometry: Three Inverse Trigonometric Functions

Suppose we want to find the **inverses** of sin, cos, and tan.

Unfortunately, none of them is one-to-one. For example, the horizontal line $y = 1$ intersects each graph more than once—so, by the Horizontal Line Test (HLT), none is one-to-one.



(Indeed, for any $c \in [-1, 1]$, the horizontal line $y = c$ intersects each graph more than once. And *any* horizontal line intersects the graph of tan more than once.)

So, sin, cos, and tan are **not** one-to-one. But as we learnt in Ch. 21, by restricting the domain of each function, we can create a brand new function that **is** one-to-one.

As usual, there are many (in fact infinitely many) ways we can do the domain restriction.

For example, the restriction of sin to $[2, 4]$ is one-to-one. So too are the restrictions of cos to $[-3, -2]$ and tan to $[4, 5]$:

Figure to be
inserted here.

But of course, as usual, we want to pick restrictions that are as "large" as possible. So, we might for example instead pick the restrictions of sin to $[\pi/2, 3\pi/2]$, cos to $[-2\pi, 0]$, and tan to $(\pi/2, 3\pi/2)$—these new functions are one-to-one:

Figure to be inserted here.

Indeed, *any* translation (or shift) of the above restrictions by any integer multiple of 2π would also work.

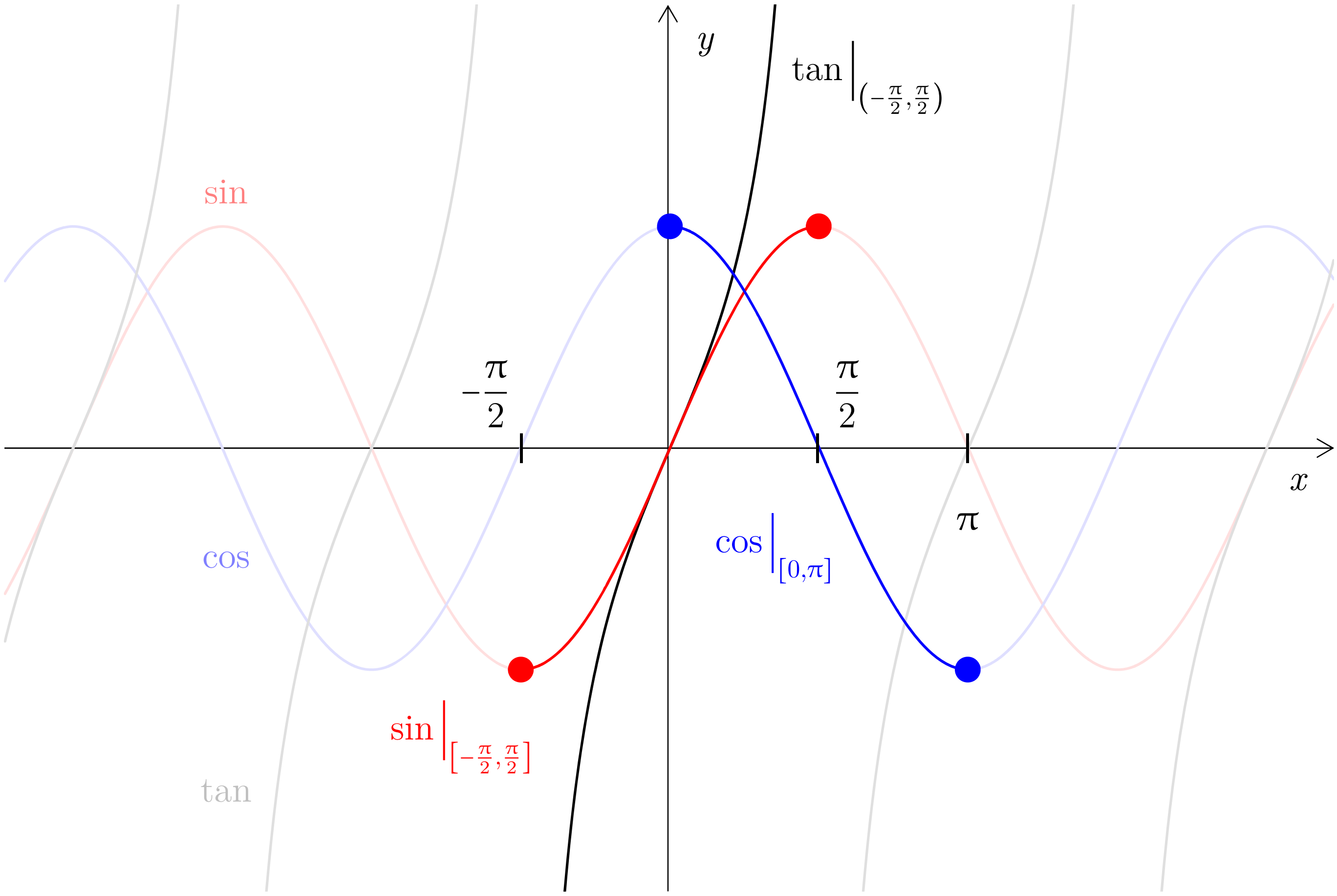
But the *standard* convention is to use these particular restrictions:

$$\sin\Big|_{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}, \qquad \cos\Big|_{[0,\pi]}, \qquad \tan\Big|_{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}.$$

Exercise 152. Graph $\sin\Big|_{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}$, $\cos\Big|_{[0,\pi]}$, $\tan\Big|_{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}$. (Answer on the next page.)

For each, explain why it is one-to-one, then write down its inverse.

No horizontal line intersects each of $\sin\big|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$, $\cos\big|_{[0,\pi]}$, $\tan\big|_{(-\frac{\pi}{2},\frac{\pi}{2})}$ more than once. So, each is one-to-one.



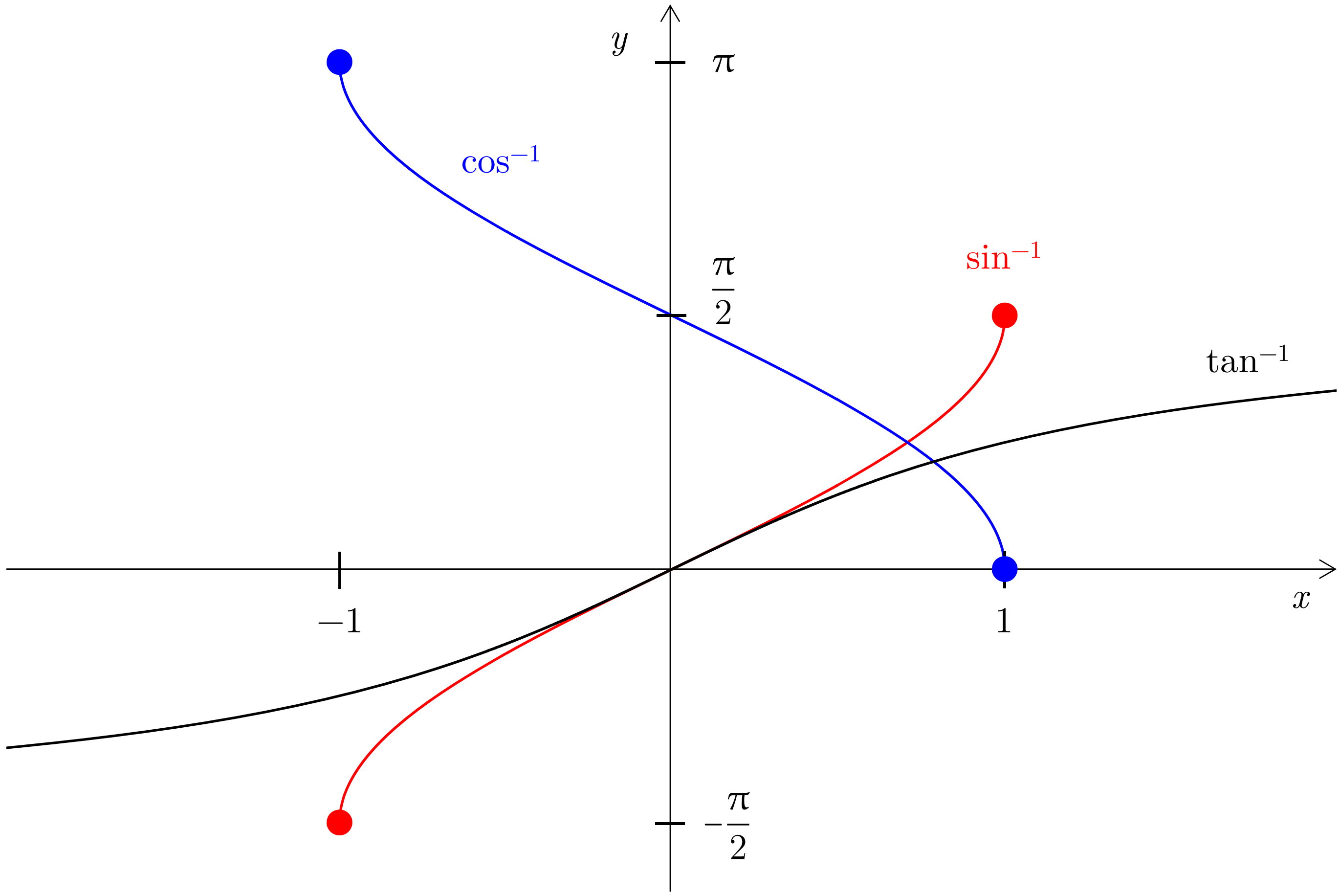
We now define the three **inverse trigonometric functions**—**arcsine**, **arccosine** and **arctangent**:

Definition 99. The inverse trigonometric functions *arcsine*, *arccosine*, and *arctangent*—denoted $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$—are the inverses of

$$\sin\big|_{[-\frac{\pi}{2},\frac{\pi}{2}]},\qquad \cos\big|_{[0,\pi]},\quad \text{and}\quad \tan\big|_{(-\frac{\pi}{2},\frac{\pi}{2})}.$$

So,

	Domain	**Codomain**	**Mapping rule**
$\sin^{-1}$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$y = \sin x \implies \sin^{-1} y = x$
$\cos^{-1}$	$[-1,1]$	$[0,\pi]$	$y = \cos x \implies \cos^{-1} y = x$
$\tan^{-1}$	$\mathbb{R}$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$y = \tan x \implies \tan^{-1} y = x$



It can be shown that every inverse function is onto;[234] equivalently, every inverse function's range and codomain coincide. And so here indeed, each of $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ is onto; equivalently, the range and codomain of each coincide.

Observe that

- $\sin^{-1}$ has endpoints $(-1, -\pi/2)$ and $(1, \pi/2)$, and is strictly increasing;
- $\cos^{-1}$ has endpoints $(-1, \pi)$ and $(1, 0)$, and is strictly decreasing; and
- $\tan^{-1}$ has **no** endpoints and is strictly increasing.

Example 486. XXX

Example 487. XXX

Example 488. XXX

Remark 78. We could, of course, also construct the **arccosecant**, **arcsecant**, and **arccotangent** functions (which are denoted $\operatorname{cosec}^{-1}$, $\sec^{-1}$, and $\cot^{-1}$ in your H2 Maths syllabus, p. 18). But these aren't in the A-Levels and so we won't bother.

Exercise 153. XXX (Answer on p. 384.)

A153.

[234] act 254 (Appendices).

36.1. Confusing Notation

There are two separate and big points of potential confusion:

1. As noted in Ch. 32.3, $\sin^2 = \sin \cdot \sin$—this is confusing and contrary to our earlier use of f^2 to denote the composite function $f \circ f$.

   Here, to add to our confusion, $\sin^{-1}$ doesn't mean $1/\sin$ (as would be logical given that $\sin^2 = \sin \cdot \sin$). Instead, $\sin^{-1} x$ denotes the inverse of the function $\sin\big|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$!

2. The notation ($\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$) and names (*inverse trigonometric functions*) may incorrectly suggest that

   $$\sin^{-1},\ \cos^{-1},\ \text{and } \tan^{-1} \quad \text{are the inverses of} \quad \sin,\ \cos,\ \text{and } \tan. \qquad ✗$$

   But the above statement is actually false! Instead,

   $$\sin^{-1},\ \cos^{-1},\ \text{and } \tan^{-1} \quad \text{are the inverses of} \quad \sin\big|_{[-\frac{\pi}{2},\frac{\pi}{2}]},\ \cos\big|_{[0,\pi]},\ \tan\big|_{(-\frac{\pi}{2},\frac{\pi}{2})}. \qquad ✓$$

Because of the above two points of confusion, other writers prefer to denote the arcsine, arccosine, and arctangent functions by

$$\text{arcsin, arccos, and arctan;}$$

or

$$\text{Sin}^{-1},\ \text{Cos}^{-1},\ \text{and Tan}^{-1};$$

or

$$\text{asin,}\quad \text{acos,}\quad \text{and atan.}$$

(The last is commonly used in computer programs.)

However and unfortunately, your A-Level exams and syllabus insist on using the confusing and misleading notation $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. And so that's what we shall do too.

36.2. Inverse Cancellation Laws, Revisited

We reproduce Proposition 5 from Ch. 24.4:

Proposition 5. (Inverse Cancellation Laws) *If the function f has inverse f^{-1}, then*

(a) $f^{-1}(f(x)) = x$ *for all* $x \in \text{Domain } f$; *and*

(b) $f\left(f^{-1}(y)\right) = y$ *for all* $y \in \text{Domain } f$.

Given the above cancellation laws, the following result is perhaps not surprising:

Fact 98. **(a)** $\sin\left(\sin^{-1} x\right) = x$ *for all* $x \in \text{Domain } \sin^{-1} = [-1, 1]$.

(b) $\cos\left(\cos^{-1} x\right) = x$ *for all* $x \in \text{Domain } \cos^{-1} = [-1, 1]$.

(c) $\tan\left(\tan^{-1} x\right) = x$ *for all* $x \in \text{Domain } \tan^{-1} = \mathbb{R}$.

Proof. See p. 1596 (Appendices). □

Example 489. For $x = 1$, we have

(a) $\sin\left(\sin^{-1} 1\right) = \sin\frac{\pi}{2} = 1$.

(b) $\cos\left(\cos^{-1} 1\right) = \cos 0 = 1$.

(c) $\tan\left(\tan^{-1} 1\right) = \tan\frac{\pi}{4} = 1$.

Example 490. For $x = 0$, we have

(a) $\sin\left(\sin^{-1} 0\right) = \sin 0 = 0$.

(b) $\cos\left(\cos^{-1} 0\right) = \cos\frac{\pi}{2} = 0$.

(c) $\tan\left(\tan^{-1} 0\right) = \tan 0 = 0$.

Of course, if $x \notin [-1, 1]$, then $\sin^{-1} x$ and $\cos^{-1} x$ are undefined, so that $\sin\left(\sin^{-1} x\right)$ and $\cos\left(\cos^{-1} x\right)$ are also undefined:

Example 491. For $x = \sqrt{3} \approx 1.73$, we have

(a) $\sin\left(\sin^{-1}\sqrt{3}\right)$ is undefined.

(b) $\cos\left(\cos^{-1}\sqrt{3}\right)$ is undefined.

(c) $\tan\left(\tan^{-1}\sqrt{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$.

The composite functions $\sin \circ \sin^{-1}$, $\cos \circ \cos^{-1}$, and $\tan \circ \tan^{-1}$ exist (why?)[235] and their graphs are these:

---

[235] The composite function $\sin \circ \sin^{-1}$ exists because $\text{Range } \sin^{-1} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \subseteq \mathbb{R} = \text{Domain } \sin$.

Figure to be
inserted here.

Perhaps surprisingly, these three claims are false:

(a) "$\sin^{-1}(\sin x) = x$ for all $x \in$ Domain $\sin = \mathbb{R}$." ✗

(b) "$\cos^{-1}(\cos x) = x$ for all $x \in$ Domain $\cos = \mathbb{R}$." ✗

(c) "$\tan^{-1}(\tan x) = x$ for all $x \in$ Domain $\tan = \mathbb{R} \setminus \{(2k+1)\pi/2 : k \in \mathbb{Z}\}$." ✗

A simple counterexample to show that the above three claims are false:

Example 492. For $x = 4\pi$, we have

(a) $\sin^{-1}(\sin 4\pi) = \sin^{-1} 0 = 0 \neq 4\pi$.

(b) $\cos^{-1}(\cos 4\pi) = \cos^{-1} 1 = \dfrac{\pi}{2} \neq 4\pi$.

(c) $\tan^{-1}(\tan 4\pi) = \tan^{-1} 0 = 0 \neq 4\pi$.

Here are the correct versions of the above three false claims:

Fact 99. **(a)** $\sin^{-1}(\sin x) = x \iff x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

(b) $\cos^{-1}(\cos x) = x \iff x \in [0, \pi]$.

(c) $\tan^{-1}(\tan x) = x \iff x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Proof. See p. 1596 (Appendices). □

Example 493. XXX

Example 494. XXX

The composite functions $\sin^{-1} \circ \sin$, $\cos^{-1} \circ \cos$, and $\tan^{-1} \circ \tan$ exist (why?).[236]

And their graphs are these:[237]

Similarly, $\cos \circ \cos^{-1}$ exists because Range $\cos^{-1} = [0, \pi] \subseteq \mathbb{R} =$ Domain cos.

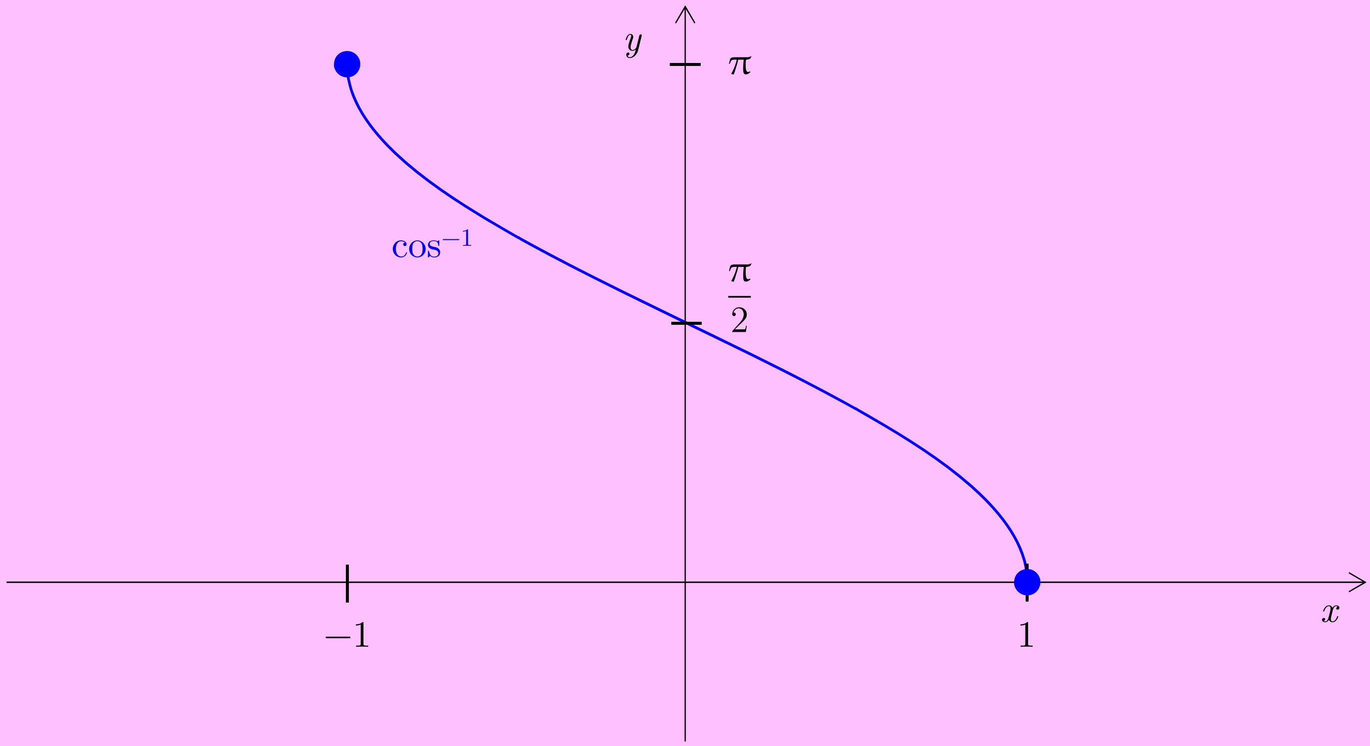
And $\tan \circ \tan^{-1}$ exists because Range $\tan^{-1} = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \subseteq \mathbb{R} \setminus \{(2k+1)\pi/2 : k \in \mathbb{Z}\} =$ Domain tan.

[236] The composite function $\sin^{-1} \circ \sin$ exists because Range $\sin = [-1, 1] \subseteq [-1, 1] =$ Domain $\sin^{-1}$.
Similarly, $\cos^{-1} \circ \cos$ exists because Range $\cos = [-1, 1] \subseteq [-1, 1] =$ Domain $\cos^{-1}$.
And $\tan^{-1} \circ \tan$ exists because Range $\tan = \mathbb{R} \subseteq \mathbb{R} =$ Domain $\tan^{-1}$.

[237] See Fact 260 (Appendices).

Figure to be
inserted here.

Exercise 154. Here's the graph of $\cos^{-1}$:



Clearly, the following claim is false:

$$\text{“If } x \in [-1, 1)\text{, then } \cos^{-1} x = 0.\text{”}$$ ✗

Nonetheless, below is a seven-step “proof” of the above false claim. Identify any error(s). (Answer on p. 1803.)

1. Suppose $x \in [-1, 1)$.
2. Let $A \overset{1}{=} \cos^{-1} x$.
3. So, $x = \cos A$.
4. But by Fact 80 (Cosine Is Symmetric in y-Axis), $\cos A = \cos(-A)$.
5. So, $x = \cos(-A)$.
6. Hence, $\cos^{-1} x \overset{2}{=} -A$.
7. Taking $\overset{1}{=} + \overset{2}{=}$, we get $2 \cos^{-1} x = 0$ and thus $\cos^{-1} x = 0$.

36.3. More Compositions

Fact 100. **(a)** $\sin\left(\cos^{-1}x\right)=\sqrt{1-x^2}$ *for all* $x\in[-1,1]$.

(b) $\cos\left(\sin^{-1}x\right)=\sqrt{1-x^2}$ *for all* $x\in[-1,1]$.

(c) $\sin\left(\tan^{-1}x\right)=\dfrac{x}{\sqrt{1+x^2}}$ *for all* $x\in\mathbb{R}$.

(d) $\cos\left(\tan^{-1}x\right)=\dfrac{1}{\sqrt{1+x^2}}$ *for all* $x\in\mathbb{R}$.

(e) $\tan\left(\sin^{-1}x\right)=\dfrac{x}{\sqrt{1-x^2}}$ *for all* $x\in(-1,1)$.

(f) $\tan\left(\cos^{-1}x\right)=\dfrac{\sqrt{1-x^2}}{x}$ *for all* $x\in[-1,1]\setminus\{0\}$.

Proof. Informal and incomplete proof-by-picture:[238]

Consider a right triangle with legs of lengths $x>0$ and 1. By Pythagoras' Theorem, its hypotenuse has length $\sqrt{1+x^2}$. Let α be the angle opposite the leg of length x.

Figure to be
inserted here.

Since $\tan\alpha=\dfrac{\text{O}}{\text{A}}=\dfrac{x}{1}=x$, we have $\alpha=\tan^{-1}x$. Hence,

$$\sin\left(\tan^{-1}x\right)=\sin\alpha=\frac{\text{O}}{\text{H}}=\frac{x}{\sqrt{1+x^2}},$$

$$\cos\left(\tan^{-1}x\right)=\cos\alpha=\frac{\text{A}}{\text{H}}=\frac{1}{\sqrt{1+x^2}}.$$

Next, consider a right triangle with hypotenuse of length 1 and a leg of length x. By Pythagoras' Theorem, its other leg has length $\sqrt{1-x^2}$.

Figure to be
inserted here.

[238]Informal because we rely on geometry and incomplete because we cover only the special case where $x>0$. For a formal and complete proof, see p. 1598 (Appendices).

Let β and γ be the angles opposite the legs of lengths x and $\sqrt{1-x^2}$, respectively.

Since $\sin\beta = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{x}{1} = x$, we have $\beta = \sin^{-1} x$. Hence,

$$\cos\left(\sin^{-1} x\right) = \cos\beta = \frac{\text{A}}{\text{H}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2},$$
$$\tan\left(\sin^{-1} x\right) = \tan\beta = \frac{\text{O}}{\text{A}} = \frac{x}{\sqrt{1-x^2}}.$$

Similarly, since $\cos\gamma = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{x}{1} = x$, we have $\gamma = \cos^{-1} x$. Hence,

$$\sin\left(\cos^{-1} x\right) = \sin\gamma = \frac{\text{O}}{\text{H}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2},$$
$$\tan\left(\cos^{-1} x\right) = \tan\gamma = \frac{\text{O}}{\text{A}} = \frac{\sqrt{1-x^2}}{x}. \qquad \square$$

The four composite functions $\sin\circ\cos^{-1}$, $\sin\circ\tan^{-1}$, $\cos\circ\sin^{-1}$, and $\cos\circ\tan^{-1}$ are well-defined. Here are their graphs:

Figure to be
inserted here.

Unfortunately, $\tan\circ\sin^{-1}$ and $\tan\circ\cos^{-1}$ are **not** well-defined. (Why?)[239]

Nonetheless, we may graph the equations $y = \tan\left(\sin^{-1} x\right)$ for $x \in (-1,1)$ and $y = \tan\left(\cos^{-1} x\right)$ for $x \in [-1,1] \setminus \{0\}$:

Figure to be
inserted here.

[239] Range $\sin^{-1} = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \nsubseteq$ Domain tan. In particular, $\pm\dfrac{\pi}{2} \notin$ Domain tan. Hence, $\tan\circ\sin^{-1}$ is not well-defined.

Range $\cos^{-1} = [0,\pi] \nsubseteq$ Domain tan. In particular, $\dfrac{\pi}{2} \notin$ Domain tan. Hence, $\tan\circ\cos^{-1}$ is not well-defined.

36.4. Addition Formulae for Arcsine and Arccosine

Fact 101. (Addition Formulae for Arcsine and Arccosine) *Let* $x \in [-1, 1]$.

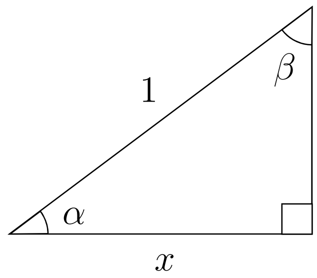
(a) $\cos^{-1} x + \sin^{-1} x = \pi/2$.

(b) $\sin^{-1} x + \sin^{-1}(-x) = 0$.

(c) $\cos^{-1} x + \cos^{-1}(-x) = \pi$.

Proof. Informal and incomplete proofs-by-picture:[240]

(a) Construct a right triangle with base $x \in (0, 1)$ and hypotenuse 1.

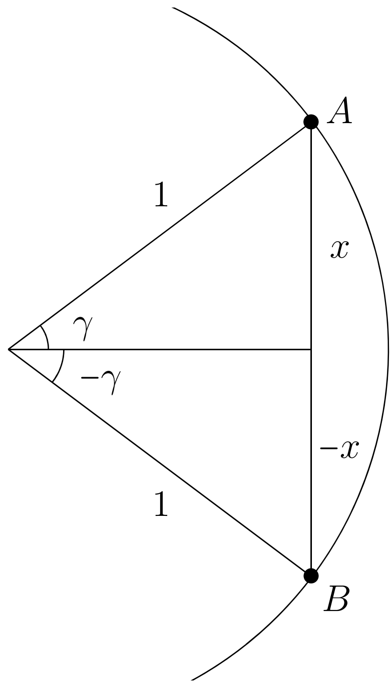


Let α and β be the labelled angles. Then

$$\cos\alpha = \frac{\mathrm{A}}{\mathrm{H}} = \frac{x}{1} = x \qquad \text{and} \qquad \sin\beta = \frac{\mathrm{O}}{\mathrm{H}} = \frac{x}{1} = x.$$

So, $\cos^{-1} x = \alpha$ and $\sin^{-1} x = \beta$. But of course, $\alpha + \beta = \pi/2$. Hence, $\cos^{-1} x + \sin^{-1} x = \pi/2$.

(b) Let $x \in (0, 1)$. Consider the unit circle centred on the origin. Let A and B be the points on its right half with y-coordinates x and $-x$.



Let γ be the angle corresponding to the point A. Then the angle corresponding to the point B is $-\gamma$.

[240]For the formal and complete proofs, see p. 1596 (Appendices).

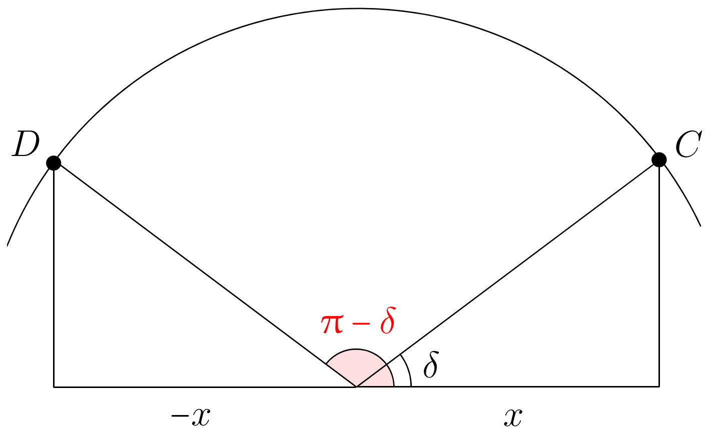
By the unit-circle definition of sine,

$$\sin\gamma = x \qquad \text{and} \qquad \sin(-\gamma) = -x.$$

Hence, $\sin^{-1} x = \gamma$, $\sin^{-1}(-x) = -\gamma$, and

$$\sin^{-1} x + \sin^{-1}(-x) = \gamma + (-\gamma) = 0.$$

(c) Let $x \in (0,1)$. Consider the unit circle centred on the origin. Let C and D be the points on its top half with x-coordinates x and $-x$.



Let δ be the angle corresponding to the point C. Then the angle corresponding to the point D is $\pi - \delta$.

By the unit-circle definition of cosine,

$$\cos\delta = x \qquad \text{and} \qquad \cos(\pi - \delta) = -x.$$

Hence, $\cos^{-1} x = \delta$, $\cos^{-1}(-x) = \pi - \delta$, and

$$\cos^{-1} x + \cos^{-1}(-x) = \delta + (\pi - \delta) = \pi. \qquad \square$$

Example 495. XXX

Example 496. XXX

Remark 79. In Ch. 32.9, we learnt that the prefix *co-* in the word *cosine* simply means that if A and B are complementary angles (i.e. $A + B = \pi/2$), then

$$\sin A = \cos B \qquad \text{and} \qquad \cos A = \sin B.$$

Fact 101(a) is simply another way of saying exactly the above. If $A + B = \pi/2$, then there must exist some x such that $A = \sin^{-1} x$ and $B = \cos^{-1} x$ (or $A = \cos^{-1} x$ and $B = \sin^{-1} x$).

36.5. Addition Formulae for Arctangent

Fact 102. (Addition Formulae for Arctangent) *Let $x \in \mathbb{R}$.*

(a) $\tan^{-1} x + \tan^{-1}(-x) = 0$.

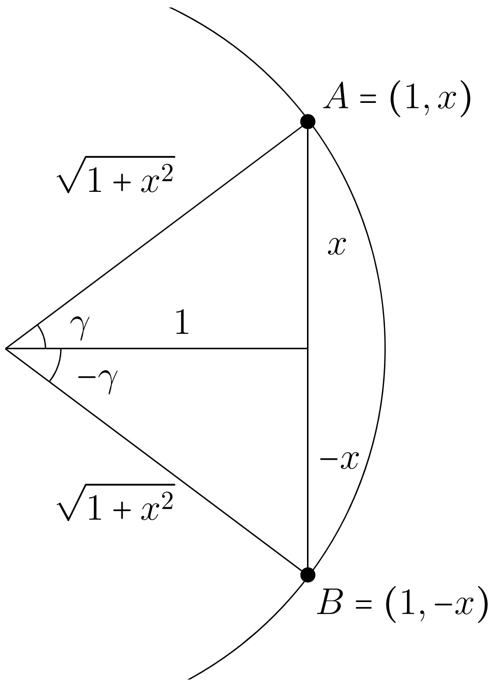
(b) $\tan^{-1} x + \tan^{-1} \dfrac{1}{x} = \begin{cases} \pi/2, & \text{for } x > 0, \\ -\pi/2, & \text{for } x < 0. \end{cases}$

(c) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \dfrac{x+y}{1-xy}, & \text{for } xy < 1, \\ \tan^{-1} \dfrac{x+y}{1-xy} + \pi, & \text{for } xy > 1 \text{ AND } x > 0, \\ \tan^{-1} \dfrac{x+y}{1-xy} - \pi, & \text{for } xy > 1 \text{ AND } x < 0. \end{cases}$

Proof. Informal and incomplete proofs-by-picture for **(a)** and **(b)**:[241]

(a) This is very similar to the earlier proof-by-picture of Fact 101(b) ($\sin^{-1} x + \sin^{-1}(-x) = 0$):

Let $x \in (0, 1)$. Consider the circle with radius $\sqrt{1+x^2}$ centred on the origin. On this circle are the points $A = (1, x)$ and $B = (1, -x)$.



Let γ be the angle corresponding to the point A. Then the angle corresponding to the point B is $-\gamma$.

By the unit-circle definitions,

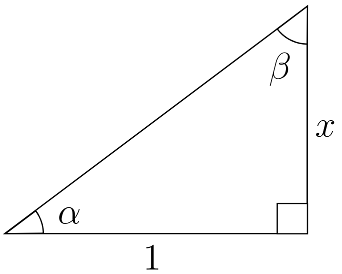
$$\tan \gamma = x \qquad \text{and} \qquad \tan(-\gamma) = -x.$$

Thus, $\tan^{-1} x = \gamma$, $\tan^{-1}(-x) = -\gamma$, and

[241]For the formal and complete proofs, see p. 1601 (Appendices).

$$\tan^{-1} x + \tan^{-1}(-x) = \gamma + (-\gamma) = 0.$$

(b) This is very similar to the earlier proof-by-picture of Fact 101(a) ($\cos^{-1} x + \sin^{-1} x = \pi/2$): Construct a right triangle with legs of lengths $x \in (0,1)$ and 1.



Let α and β be the labelled angles. Then

$$\tan\alpha = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{x}{1} = x \qquad \text{and} \qquad \tan\beta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{1}{x}.$$

So, $\tan^{-1} x = \alpha$ and $\tan^{-1}\frac{1}{x} = \beta$. But of course, $\alpha + \beta = \pi/2$. Hence, $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \pi/2$.

(c) By the Addition Formula for Tangent and Fact 98(c),

$$\tan\left(\tan^{-1} x + \tan^{-1} y\right) = \frac{\tan\left(\tan^{-1} x\right) + \tan\left(\tan^{-1} y\right)}{1 - \tan\left(\tan^{-1} x\right)\tan\left(\tan^{-1} x\right)} = \frac{x+y}{1-xy}.$$

So, $\tan^{-1}\frac{x+y}{1-xy} = \tan^{-1}\left(\tan\left(\tan^{-1} x + \tan^{-1} y\right)\right)$.

We now consider the three cases in turn:

Case 1. $xy < 1$.

Then by Lemma 1(b) (below), $\tan^{-1} x + \tan^{-1} y \in (-\pi/2, \pi/2)$.

Hence, by Fact 99(c), $\tan^{-1}\left(\tan\left(\tan^{-1} x + \tan^{-1} y\right)\right) = \tan^{-1} x + \tan^{-1} y$. Thus,

$$\tan^{-1}\frac{x+y}{1-xy} = \tan^{-1} x + \tan^{-1} y.$$

The remainder of the proof of **(c)** continues on p. 1602 (Appendices). □

Lemma 1. *Suppose $x, y \in \mathbb{R}$. Then*

(a)	$\tan^{-1} x + \tan^{-1} y = \pm\pi/2$	$\iff$	$xy = 1$;
(b)	$\in (-\pi/2, \pi/2)$	$\iff$	$xy < 1$;
(c)	$\in (\pi/2, \pi)$	$\iff$	$xy > 1$ AND $x > 0$;
(d)	$\in (-\pi, -\pi/2)$	$\iff$	$xy > 1$ AND $x < 0$.

Proof. See p. 1602 (Appendices). □

36.6. Harmonic Addition

You are supposed to have mastered the following in secondary school:

- "expression of $a\cos\theta + b\sin\theta$ in the forms $R\sin(\theta\pm\alpha)$ and $R\cos(\theta\pm\alpha)$".[242]

So, let's review how this is done.

Example 497. Q: Express $2\cos(\pi/3) + 3\sin(\pi/3)$ in the form $R\sin(\theta+\alpha)$.

A: Write $R\sin(\theta\pm\alpha) = 2\cos(\pi/3) + 3\sin(\pi/3)$. We will find R and α.

By the Addition Formulae for Sine,

$$R\sin(\theta+\pi/3) = \overbrace{R\cos\alpha}^{3}\sin\theta + \overbrace{R\sin\alpha}^{2}\cos\theta.$$

So, $\dfrac{2}{3} = \dfrac{R\sin(\alpha)}{R\cos(\alpha)} = \tan\alpha$ and $\alpha = \tan^{-1}\dfrac{2}{3}$.

Next, since $3 = R\cos\alpha = R\cos\left(\tan^{-1}\dfrac{2}{3}\right) = \dfrac{R}{\sqrt{1+(2/3)^2}} = \dfrac{3R}{\sqrt{13}}$, we have $R = \sqrt{13}$.

Example 498. XXX

Example 499. XXX

Example 500. XXX

More generally,

Theorem 7. *Let $\theta\in\mathbb{R}$. Suppose $a, b\neq 0$. Then*

(a) $a\cos\theta + b\sin\theta = \operatorname{sgn} b\sqrt{a^2+b^2}\sin\left(\theta + \tan^{-1}\dfrac{a}{b}\right)$

(b) $= \operatorname{sgn} a\sqrt{a^2+b^2}\cos\left(\theta + \tan^{-1}\dfrac{-b}{a}\right).$

(c) $= \operatorname{sgn} b\sqrt{a^2+b^2}\sin\left(\theta - \tan^{-1}\dfrac{-a}{b}\right)$

(d) $= \operatorname{sgn} a\sqrt{a^2+b^2}\cos\left(\theta - \tan^{-1}\dfrac{b}{a}\right).$

Remark 80. It's probably unwise to try to *mug* Theorem 7, which is provided only as an official reference. It is better to understand how the above examples were done and also be able to do the following exercises.

Proof. **(a)** Write $R\sin(\theta+\alpha) = a\cos\theta + b\sin\theta$. We will find R and α.

By the Addition Formulae for Sine,

[242] H2 Maths syllabus (p. 14).

$$R\sin(\theta+\alpha) = \overbrace{R\cos\alpha}^{b}\sin\theta + \overbrace{R\sin\alpha}^{a}\cos\theta.$$

Now,
$$\frac{a}{b} = \frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha.$$

So, $\alpha = \tan^{-1}\dfrac{a}{b}$.

And since $R\cos\alpha = b$, by Fact 100(d), we have

$$R = \frac{b}{\cos\alpha} = \frac{b}{\cos\left(\tan^{-1}\frac{a}{b}\right)} = \frac{b}{\frac{1}{\sqrt{1+(a/b)^2}}} = b\sqrt{1+(a/b)^2}$$

$$= b\frac{|b|}{|b|}\sqrt{1+(a/b)^2} = \frac{b}{|b|}\sqrt{b^2+a^2} = \operatorname{sgn} b\sqrt{b^2+a^2}.$$

(b) Write $R\cos(\theta+\alpha) = a\cos\theta + b\sin\theta$. We will find R and α.

By the Addition Formulae for Cosine,

$$R\cos(\theta+\alpha) = \overbrace{R\cos\alpha}^{a}\cos\theta \overbrace{-R\sin\alpha}^{b}\sin\theta.$$

Now,
$$\frac{b}{a} = \frac{-R\sin\alpha}{R\cos\alpha} = -\tan\alpha.$$

So, $\alpha = \tan^{-1}\left(-\dfrac{b}{a}\right)$.

And since $R\cos\alpha = a$, by Fact 100(d), we have

$$R = \frac{a}{\cos\alpha} = \frac{a}{\cos\left(\tan^{-1}\left(-\frac{b}{a}\right)\right)} = \frac{a}{\frac{1}{\sqrt{1+(-b/a)^2}}} = a\sqrt{1+\frac{b^2}{a^2}}$$

$$= a\frac{|a|}{|a|}\sqrt{1+\frac{b^2}{a^2}} = \frac{a}{|a|}\sqrt{a^2+b^2} = \operatorname{sgn} a\sqrt{a^2+b^2}.$$

(c) Take **(a)** and apply Fact 102(a).

(d) Take **(b)** and apply Fact 102(a). □

Exercise 155. XXX (Answer on p. 396.)

A155.

36.7. Even More Compositions (optional)

The four composite functions $\sin^{-1} \circ \cos$, $\cos^{-1} \circ \sin$, $\tan^{-1} \circ \sin$, and $\tan^{-1} \circ \cos$ are well-defined. Here are their graphs:

Figure to be
inserted here.

Unfortunately, $\sin^{-1} \circ \tan$ and $\cos^{-1} \circ \tan$ are **not** well-defined. (Why?)[243]

Nonetheless, we may graph the equations $y = \sin^{-1}(\tan x)$ and $y = \cos^{-1}(\tan x)$, for $x \in [-k\pi/4, k\pi/4]$ (for $k \in \mathbb{Z}$):

Figure to be
inserted here.

The composite functions $\sin^{-1} \circ \cos : \mathbb{R} \to \mathbb{R}$ and $\cos^{-1} \circ \sin : \mathbb{R} \to \mathbb{R}$ do exist. Their mapping rules are messy and so relegated to the Appendices (Fact 261). Here are their graphs:

Figure to be
inserted here.

However and somewhat interestingly, it's impossible to write down algebraic expressions for $\tan^{-1}(\sin x)$, $\tan^{-1}(\cos x)$, $\sin^{-1}(\tan x)$, and $\cos^{-1}(\tan x)$.

[243] Range $\tan = \mathbb{R} \nsubseteq [-1, 1] =$ Domain $\sin^{-1}$. Hence, $\sin^{-1} \circ \tan$ is not well-defined.
Range $\tan = \mathbb{R} \nsubseteq [-1, 1] =$ Domain $\cos^{-1}$. Hence, $\cos^{-1} \circ \tan$ is not well-defined.

37. Elementary Functions

This brief chapter serves as a quick review of some of the functions we've encountered so far. We'll also learn a new term: **elementary functions**.

Definition 100. A *polynomial function* is any nice function defined by $x \mapsto a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_0, a_1, \ldots a_n \in \mathbb{R}$ and $n \in \mathbb{Z}_0^+$.

Example 501. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x^3 - x + 3$ is a polynomial function. (What is $f(1)$? $f(0)$?)[244]

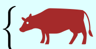
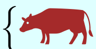
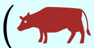
Example 502. The function $g : [1, 2] \to \mathbb{R}$ defined by $g(x) = 5x^3 - x + 3$ is a polynomial function. (What is $g(1)$? $g(0)$?)[245]

Definition 101. An *identity function* is any function defined by $x \mapsto x$.[246]

Example 503. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is an identity function. (What is $f(1)$? $f(0)$?)[247]

By the way, any nice identity function is also a polynomial function. (Why?)[248] Here for example, f is also a polynomial function.

Example 504. The function $g : [1, 2] \to \mathbb{R}$ defined by $g(x) = x$ is an identity function. (What is $g(1)$? $g(0)$?)[249]

Example 505. The function h : { ,  } → { ,  } defined by $h(x) = x$ is an identity function. (What is h()? h()? $h(1)$?)[250]

Definition 102. A *constant function* is any function defined by $x \mapsto c$, where c is any object.

[244] $f(1) = 7$, $f(0) = 3$.

[245] $g(1) = 7$, $g(0)$ is undefined.

[246] Strictly speaking, we should distinguish between an *identity function* and an *inclusion function*. Strictly speaking, Definition 101 given here is not of an identity function but of an inclusion function. See Definitions 284 and 285 (Appendices).

This is a subtle distinction that shall not matter at all for A-Level Maths. Indeed, the term *identity function* is never mentioned in your A-Level Maths syllabus or exams. But it is sufficiently convenient that we'll sometimes use Definition 101 in the main text anyway.

[247] $f(1) = 1$, $f(0) = 0$.

[248] Its mapping rule may be written as $x \mapsto a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_0, a_1, \ldots a_n \in \mathbb{R}$ and $n \in \mathbb{Z}_0^+$ (in particular, $a_0 = 0$, $a_1 = 1$, and $n = 1$).

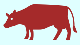
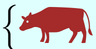
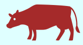
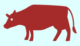
[249] $g(1) = 1$, $g(0)$ is undefined.

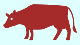
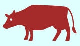
[250] h() = , h() = , $h(1)$ is undefined.

Example 506. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5$ is a constant function. (What is $f(1)$? $f(0)$?)[251]

By the way, any nice constant function is also a polynomial function. (Why?) Here for example, f is also a polynomial function.

Example 507. The function $g : [1, 2] \to \mathbb{R}$ defined by $g(x) = 5$ is a constant function. (What is $g(1)$? $g(0)$?)[252]

Example 508. The function h : { ,  } → { ,  } defined by $h(x) =$  is a constant function. (What is h()? h()? $h(1)$?)[253]

Example 509. The function $i : \mathbb{R} \to$ { ,  } defined by $i(x) =$  is a constant function. (What is i()? i()? $i(1)$? $i(0)$?)[254]

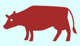
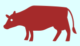
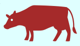
A special case of a constant function is a **zero function**:

Definition 103. A *zero function* is any function defined by $x \mapsto 0$.

Example 510. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ is a zero function. (What is $f(1)$? $f(0)$?)[255]

By the way, any zero function is also a constant function. (Why?)[256] So here for example, f is also a constant function.

Example 511. The function $g : [1, 2] \to \mathbb{R}$ defined by $f(x) = 0$ is a zero function. (What is $g(1)$? $g(0)$?)[257]

Example 512. The function h : { ,  } → { 0, ,  } defined by $h(x) = 0$ is a zero function. (What is h()? h()? $h(1)$?)[258]

Definition 104. A *power function* is any nice function defined by $x \mapsto x^k$, where $k \in \mathbb{R}$.

Example 513. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{17.3}$ is a power function. (What is $f(1)$? $f(0)$?)[259]

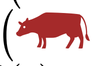
---

[250] $f(1) = 5$, $f(0) = 5$.

[251] Its mapping rule may be written as $x \mapsto a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_0, a_1, \ldots a_n \in \mathbb{R}$ and $n \in \mathbb{Z}_0^+$ (in particular, a_0 is the constant to which each x is mapped and $n = 0$).

[252] $g(1) = 5$, $g(0)$ is undefined.

[253] h() = , h() = , $h(1)$ is undefined.

[254] i() and i() are undefined, $i(1) =$ , and $i(0) =$ .

[255] $f(1) = 0$, $f(0) = 0$.

[256] It maps each x in its domain to the same object (in this case, 0).

[257] $g(1) = 0$, $g(0)$ is undefined.

[258] h() $= 0$, h() $= 0$, $h(1)$ is undefined.

[259] $f(1) = 1$, $f(0) = 0$.

Example 514. The function $g : [1, 2] \to \mathbb{R}$ defined by $g(x) = x^{-\pi}$ is a power function. (What is $g(1)$? $g(0)$?)[260]

Definition 105. A *trigonometric* (or *circular*) *function* is any nice function with the same mapping rule as sin, cos, tan, cosec, sec, or tan.

Example 515. Of course, sin is a trigonometric function.

But so too is the function $f : [1, 2] \to \mathbb{R}$ defined by $f(x) = \sin x$.

Figure to be
inserted here.

Example 516. Of course, cos is a trigonometric function.

But so too is the function $g : [2, 3] \to \mathbb{R}$ defined by $g(x) = \cos x$.

Figure to be
inserted here.

Remark 81. As stated in Remark 75, there are actually other trigonometric functions (e.g. versine, haversine, coversine). But we needn't worry about these in A-Level Maths.

Definition 106. An *inverse trigonometric* (or *circular*) *functions* is any nice function with the same mapping rule as $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$.

[260] $g(1) = 1$, $g(0)$ is undefined.

Example 517. Of course, $\sin^{-1}$ is an inverse trigonometric function.

But so too is the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sin^{-1} x$.

Figure to be
inserted here.

Example 518. Of course, $\cos^{-1}$ is an inverse trigonometric function.

But so too is the function $g : [0,0.5] \to \mathbb{R}$ defined by $g(x) = \cos^{-1} x$.

Figure to be
inserted here.

Remark 82. As stated in Remark 78, there are other inverse trigonometric functions (e.g. $\operatorname{cosec}^{-1}$, $\sec^{-1}$, $\cot^{-1}$). But we needn't worry about these in A-Level Maths.

In Ch. 28, we defined

- *The* **natural logarithm function** $\ln : \mathbb{R}^+ \to \mathbb{R}$; and
- Its inverse *the* **exponential function** $\exp : \mathbb{R} \to \mathbb{R}^+$.

We can also define

Definition 107. A *natural logarithm function* is any nice function with the mapping $x \mapsto \ln x$.

Definition 108. An *exponential function* is any nice function with the mapping $x \mapsto \exp x$.

Example 519. The function $f : [1, 2] \to \mathbb{R}$ defined by $f(x) = \ln x$ is *a* natural logarithm function.

Figure to be
inserted here.

The function $g : [1, 2] \to \mathbb{R}$ defined by $g(x) = \exp x$ is *an* exponential function.

All of the functions discussed in this brief chapter are **elementary functions**. Also, any arithmetic combination (Ch. 20) or composition (Ch. 22) of two elementary functions is also an elementary function. Formally,

Definition 109. An *elementary function* is

a polynomial function, | a trigonometric function,
an inverse trigonometric function, | a natural logarithm function,
an exponential function, | a power function,
any arithmetic combination of two elementary functions,
or any composition of two elementary functions.

Most functions you'll encounter in H2 Maths are elementary. Through arithmetic combinations and compositions, we can build ever functions that "look" ever more complicated but are nonetheless elementary:

Example 520. Define $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(x) = 1 + 2x + 3\sin\left[\cos\left(1 + x^3 - \ln x\right)^2\right].$$

While f may look complicated, it is nonetheless elementary.

Example 521. The **absolute value function** $|\cdot|:\mathbb{R}\to\mathbb{R}$ is defined by

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

At first glance, the absolute value function doesn't seem to be an elementary function. But in fact, it is. To see why, recall Fact 12:

$$|x| = \sqrt{x^2}, \text{ for all } x \in \mathbb{R}.$$

So, define $f:\mathbb{R}_0^+\to\mathbb{R}$ by $f(x)=\sqrt{x}$ and $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=x^2$.
Both f and g are elementary. Moreover, $|\cdot|$ is the composition of f and g:

$$|x| = f(g(x)) = f\left(x^2\right) = \sqrt{x^2}.$$

Hence, by Definition 109, $|\cdot|$ is an elementary function.

To repeat, most functions we'll encounter in H2 Maths are **elementary**.

Probably the only non-elementary function we'll spend significant time on is the **cumulative density function** of the **normal distribution** (Part VI, Probability and Statistics).

It is possible to prove that the derivative (if it exists) of any elementary function is also elementary. However and importantly, the integrals of many elementary functions are not. We'll learn a little about this in Part V (Calculus).

Remark 83. Unfortunately, there's no single standard definition of the term **elementary function**.[261] We'll use it anyway because it's a convenient term that includes most functions in A-Level Maths.

[261] Definition 109 is merely this textbook's and also ProofWiki's.

- Some writers consider the inverse of any elementary function to also be elementary (we do not).
- Most (including me) consider the composition of any two elementary functions to also be elementary. However, at least one person does not (davidlowryduda).

38. Factorising Polynomials

In this chapter, we'll learn methods for factorising polynomials. But first, we must learn to **long-divide polynomials**:

38.1. Long Division of Polynomials

We'll continue using the terms **dividend**, **divisor**, **quotient**, and **remainder** (Ch. 2):

Example 522. Consider $(2x+1) \div x$. We have

$$\overbrace{2x+1}^{p(x)} = \overbrace{2}^{q(x)} \cdot \overbrace{x}^{d(x)} + \overbrace{1}^{r(x)} \qquad \text{or} \qquad \frac{\overbrace{2x+1}^{p(x)}}{\underbrace{x}_{d(x)}} = \overbrace{2}^{q(x)} + \frac{\overbrace{1}^{r(x)}}{\underbrace{x}_{d(x)}}.$$

The four polynomials labelled above have the same four names as before:

The **dividend**	The **divisor**	The **quotient**	The **remainder**
$p(x) = 2x+1$	$d(x) = x$	$q(x) = 2$	$r(x) = 1$

In the above example, it was kinda obvious that the **quotient** had to be $q(x) = 2$. In the next example, it's a little less obvious and **long division** will be helpful:

Example 523. Consider $\left(x^2+3\right) \div (x-1)$. The **dividend** is $p(x) = x^2+3$ and the **divisor** is $d(x) = x-1$. This time, it's not so obvious what the quotient $q(x)$ should be. But it turns out that just like with (simple) division (of two numbers), **long division** can help us here.

With (simple) long division, going from right to left, the columns were 1s, 10s, 100s, etc. Here with polynomial long division, going from right to left, they're the constant x^0 term, the linear x^1 term, the squared x^2 term, the cubed term x^3, etc.

$$
\begin{array}{r|rrr}
\textbf{Terms:} & x^2 & x^1 & x^0 \\
 & & x & +1 \\
\hline
x-1 & x^2 & +0x & +3 \\
 & x^2 & -x & \\
\hline
 & & x & +3 \\
 & & x & -1 \\
\hline
 & & & 4
\end{array}
\qquad
\begin{array}{r}
\textbf{Explanation} \\
\\
x \cdot (x-1) = x^2 - x \\
\left(x^2+3\right) - \left(x^2-x\right) = x+3 \\
1 \cdot (x-1) = x-1 \\
(x+3)-(x-1) = 4
\end{array}
$$

The **quotient** is $q(x) = x+1$, while the **remainder** is $r(x) = 4$. Altogether,

$$\overbrace{x^2+3}^{p(x)} = \overbrace{(x-1)}^{d(x)} \cdot \overbrace{(x+1)}^{q(x)} + \overbrace{4}^{r(x)} \qquad \text{or} \qquad \frac{\overbrace{x^2+3}^{p(x)}}{\underbrace{x-1}_{d(x)}} = \overbrace{x+1}^{q(x)} + \frac{\overbrace{4}^{r(x)}}{\underbrace{x-1}_{d(x)}}.$$

Example 524. Consider $\left(3x^2 + x - 4\right) \div (2x - 3)$. The **dividend** is $p(x) = 3x^2 + x - 4$ and the **divisor** is $d(x) = 2x - 3$.

Long division:

$$
\begin{array}{rr|rrr}
\textbf{Terms:} & & x^2 & x^1 & x^0 \\
 & & & \frac{3}{2}x & +\frac{11}{4} \\ \cline{3-5}
 & 2x-3 & 3x^2 & +x & -4 \\
 & & 3x^2 & -\frac{9}{2}x & \\ \cline{4-5}
 & & & \frac{11}{2}x & -4 \\
 & & & \frac{11}{2}x & -\frac{33}{4} \\ \cline{4-5}
 & & & & \frac{17}{4}
\end{array}
$$

Explanation

$$
\begin{aligned}
\frac{3}{2}x \cdot (2x-3) &= 3x^2 - \frac{9}{2}x \\
\left(3x^2 + x - 4\right) - \left(3x^2 - \frac{9}{2}x\right) &= \frac{11}{2}x - 4 \\
\frac{11}{4} \cdot (2x-3) &= \frac{11}{2}x - \frac{33}{4} \\
\left(\frac{11}{2}x - 4\right) - \left(\frac{11}{2}x - \frac{33}{4}\right) &= \frac{17}{4}
\end{aligned}
$$

So, the **quotient** and **remainder** are

$$q(x) = \frac{3}{2}x + \frac{11}{4} \qquad \text{and} \qquad r(x) = \frac{17}{4}.$$

Hence,
$$\overbrace{3x^2 + x - 4}^{p(x)} = \overbrace{(2x-3)}^{d(x)} \cdot \overbrace{\left(\frac{3}{2}x + \frac{11}{4}\right)}^{q(x)} + \overbrace{\frac{17}{4}}^{r(x)}.$$

Or,
$$\frac{\overbrace{3x^2 + x - 4}^{p(x)}}{\underbrace{2x-3}_{d(x)}} = \underbrace{\frac{3}{2}x + \frac{11}{4}}_{q(x)} + \frac{\overbrace{17/4}^{r(x)}}{\underbrace{2x-3}_{d(x)}}.$$

Example 525. Consider $\left(4x^3+2x^2+1\right)\div\left(2x^2-x-1\right)$. The **dividend** is $p(x)=4x^3+2x^2+1$ and the **divisor** is $d(x)=2x^2-x-1$.

Long division:

$$\begin{array}{r|rrrr}
\textbf{Terms:} & x^3 & x^2 & x^1 & x^0 \\
 & & & 2x & +2 \\ \cline{2-5}
2x^2-x-1 & 4x^3 & +2x^2 & +0x & +1 \\
 & 4x^3 & -2x^2 & -2x & \\ \cline{3-5}
 & & 4x^2 & 2x & +1 \\
 & & 4x^2 & -2x & -2 \\ \cline{3-5}
 & & & 4x & +3
\end{array}$$

Explanation

$$\begin{aligned}
2x\cdot\left(2x^2-x-1\right) &= 4x^3-2x^2-2x \\
\left(4x^3+2x^2+1\right)-\left(4x^3-2x^2-2x\right) &= 4x^2+2x+1 \\
2\cdot\left(2x^2-x-1\right) &= 4x^2-2x-2 \\
\left(4x^2+2x+1\right)-\left(4x^2-2x-2\right) &= 4x+3
\end{aligned}$$

So, the **quotient** and **remainder** are

$$q(x)=2x+2 \qquad \text{and} \qquad r(x)=4x+3.$$

Hence,
$$\overbrace{4x^3+2x^2+1}^{p(x)}=\overbrace{\left(2x^2-x-1\right)}^{d(x)}\cdot\overbrace{(2x+2)}^{q(x)}+\overbrace{4x+3}^{r(x)}.$$

Or,
$$\frac{\overbrace{4x^3+2x^2+1}^{p(x)}}{\underbrace{2x^2-x-1}_{d(x)}}=\underbrace{2x+2}_{q(x)}+\frac{\overbrace{4x+3}^{r(x)}}{\underbrace{2x^2-x-1}_{d(x)}}.$$

The above examples suggest the following theorem and definitions:

Theorem 8. (Euclidean Division Theorem for Polynomials.) *Let $p(x)$ and $d(x)$ be P- and D-degree polynomials in x with $D\le P$. Then there exists a unique polynomial $q(x)$ of degree $P-D$ such that $r(x)=p(x)-d(x)q(x)$ is a polynomial with degree less than D.*

Proof. See 1605 (Appendices). □

Definition 110. Given polynomials $p(x)$ and $d(x)$ and the expression $p(x)\div d(x)$, we call $p(x)$ the *dividend*, $d(x)$ the *divisor*, the unique polynomial $q(x)$ given in the above theorem the *quotient*, and $r(x)=p(x)-d(x)q(x)$ the *remainder*.

Exercise 156. For each expression, do the long division and identify the dividend, divisor, quotient, and remainder. (Answer on p. 1804.)

(a) $\dfrac{16x+3}{5x-2}$. **(b)** $\dfrac{4x^2-3x+1}{x+5}$. **(c)** $\dfrac{x^2+x+3}{-x^2-2x+1}$.

38.2. Factorising Polynomials

Example 526. Consider the 2nd-degree (or quadratic) polynomial $x^2 + 4x + 3$.

It can be **factorised** (i.e. written as the product of lower-degree polynomials):

$$x^2 + 4x + 3 = (x + 1)(x + 3).$$

We say that each of $x + 1$ and $x + 3$ is a **factor** of the given polynomial (or equivalently, **divides** it).

Note that each of $x + 1$ and $x + 3$ is itself a 1st-degree (or linear) polynomial.

Example 527. The quadratic polynomial $6x^2 + x - 2$ can be **factorised**:

$$6x^2 + x - 2 = (2x - 1)(3x + 2).$$

Each of $2x - 1$ and $3x + 2$ is a **factor** of (or **divides**) the given polynomial.

By the way, any non-zero multiple of a factor is itself also a factor. So here,

- Since $2x - 1$ is a factor of the given polynomial, so too are $4x - 2$, $-6x + 3$, $x - 0.5$, and any $k(2x - 1)$ (for $k \neq 0$):

$$6x^2 + x - 2 = (2x - 1)(3x + 2) = \frac{1}{2}(4x - 2)(3x + 2)$$
$$= -\frac{1}{3}(-6x + 3)(3x + 2) = 2(x - 0.5)(3x + 2) = \frac{1}{k}k(2x - 1)(3x + 2).$$

- Similarly, since $3x + 2$ is a factor of the given polynomial, so too are$6x + 4$, $-9x - 6$, $1.5x + 1$, and any $k(3x + 2)$ (for $k \neq 0$):

$$6x^2 + x - 2 = (2x - 1)(3x + 2) = \frac{1}{2}(2x - 1)(6x + 4)$$
$$= -\frac{1}{3}(2x - 1)(-9x - 6) = 2(2x - 1)(1.5x + 1) = \frac{1}{k}(2x - 1)k(3x + 2).$$

Formal definitions of the terms **factor** of (or **divides**) and **factorise**:

Definition 111. Let $p(x)$ and $d(x)$ be polynomials. We say that $d(x)$ is a *factor* of (or *divides*) $p(x)$ if there exists some polynomial $q(x)$ such that $p(x) = d(x)q(x)$.

To *factorise* a polynomial is to express it as the product of polynomials, none of which is of higher degree and at least one of which is of strictly lower degree.

38.3. Compare Coefficients (plus Guess and Check)

We'll call our first method for factorising polynomials **Compare Coefficients (plus Guess and Check)**:

Example 528. To factorise $3x^2+5x+2$, write

$$3x^2+5x+2=(ax+b)(cx+d)=acx^2+(bc+ad)x+bd,$$

where a, b, c, and d are unknown constants to be found.

Comparing coefficients on the x^2, x, and constant terms, we have, respectively,

$$ac \overset{1}{=} 3, \qquad bc+ad \overset{2}{=} 5, \qquad \text{and } bd \overset{3}{=} 2.$$

Next, we guess and check:

Of course, a, b, c, and d could be *any* numbers. But if the person who wrote this problem is nice,[262] they'll probably be integers.

So, given $\overset{1}{=}$, let's guess/try $a=3$, $c=1$.

And given $\overset{3}{=}$, let's guess/try $b=1$ and $d=2$.

Unfortunately, with this set of guesses, $\overset{2}{=}$ doesn't hold:

$$bc+ad=1+6=7\neq 5. \qquad ✗$$

Well, let's instead try $b=2$ and $d=1$. With this second set of guesses, $\overset{2}{=}$ now holds:

$$bc+ad=2+3=5. \qquad ✓$$

Yay! We're done:

$$3x^2+5x+2=(3x+2)(x+1).$$

In the above example, we explicitly wrote out the unknown constants a, b, c, and d. When we look later at higher-degree polynomials, this can help reduce mistakes. But with quadratic polynomials, we can usually skip doing this and save ourselves some time:

[262] We can confirm that he is.

Example 529. To factorise $6x^2 - 11x - 35$, first try

$$(3x+7)(2x-5) = 6x^2 - x - 35.$$

Nope doesn't work. Second try:

$$(3x-5)(2x+7) = 6x^2 + 11x - 35.$$

Nope still doesn't work. Third try:

$$(3x-7)(2x+5) = 6x^2 - 11x - 35.$$

Yay works! We're done!

The **Compare Coefficients (plus Guess and Check)** method works well enough with quadratic polynomials. With higher-degree polynomials, we can sometimes get lucky and quickly arrive at the correct guess:

Example 530. To factorise $x^3 - 7x - 6$, write

$$x^3 - 7x - 6 = (ax+b)(cx+d)(ex+f).$$

Comparing coefficients, we have $ace = 1$ and $bdf = -6$.

So, let's guess $a = c = e = 1$ and $b = 2$, $d = -3$, and $f = 1$:

$$(x+2)(x-3)(x+1) = \left(x^2 - x - 6\right)(x+1) = x^3 - x^2 - 6x + x^2 - x - 6 = x^3 - 7x - 6. \quad ✓$$

Wah! So "lucky"! Success on the very first try! We're done!

But often, it will take more time (and pain), even with some lucky and intelligent guesses:

Example 531. To factorise $15x^3 - 8x^2 - 9x + 2$, write

$$15x^3 - 8x^2 - 9x + 2 = (ax + b)(cx + d)(ex + f)$$

$$= acex^3 + (acf + ade + bce)x^2 + (adf + bcf + bde)x + bdf.$$

Comparing coefficients on the x^3, x^2, x, and constant terms, we have, respectively,

$$ace \overset{1}{=} 15, \quad acf + ade + bce \overset{2}{=} -8, \quad adf + bcf + bde \overset{3}{=} -9, \quad \text{and } bdf \overset{4}{=} 2.$$

Given $\overset{1}{=}$, let's try $a = 5$, $c = 3$, and $e = 1$.

Given $\overset{4}{=}$, let's try $b = 2$, $d = 1$, and $f = 1$. Hm wait no, observe that some coefficients are negative—so, it can't be that a–f are all positive.

So, let's instead try $b = -2$, $d = -1$, and $f = 1$:

$$acf + ade + bce = 15 - 5 - 6 = 4 \neq -8. \quad ✗$$

Hm too big. Let's try switching b and d around—i.e. try $b = -1$, $d = -2$, and $f = 1$:

$$acf + ade + bce = 15 - 10 - 3 = 2 \neq -8. \quad ✗$$

A little closer but still wrong. But ah, we now notice that the numbers 15, 10, and 3 look a lot like they can be made to add up to -8. Indeed, let's see what happens if we try $b = -1$, $d = 2$, and $f = -1$:

$$acf + ade + bce = -15 + 10 - 3 = -8. \quad ✓$$

Yay! $\overset{2}{=}$ holds! Cross our fingers and check/hope that $\overset{3}{=}$ also holds:

$$adf + bcf + bde = -10 + 3 - 2 = -9. \quad ✓$$

It does! And so we're done:

$$15x^3 - 8x^2 - 9x + 2 = (5x - 1)(3x + 2)(x - 1).$$

Shortly we'll also learn of other tools to factorise higher-degree polynomials (e.g. **Factor Theorem** and **Intermediate Value Theorem** in Chs. in 38.6 and 38.8).

Exercise 157. XXX (Answer on p. 411.)

A157.

38.4. The Quadratic Formula

For quadratic polynomials, there's actually no need to use the Compare Coefficients method at all. Instead, we can simply use the **quadratic formula** (Ch. 14), which involves no guess and check.

Consider a quadratic polynomial ax^2+bx+c. Its discriminant is b^2-4ac. By Fact 34(b),

(i) If $b^2-4ac>0$, then

$$ax^2+bx+c=a\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right).$$

(ii) If $b^2-4ac=0$, then

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2.$$

(iii) If $b^2-4ac<0$, then ax^2+bx+c cannot be factorised (unless we use complex numbers, as we'll learn in Part III).

Example 532. The quadratic polynomial x^2-3x+2 has discriminant $b^2-4ac=(-3)^2-4(1)(2)=1$. So,

$$x^2-3x+2=\left(x-\frac{3-\sqrt{1}}{2}\right)\left(x-\frac{3+\sqrt{1}}{2}\right)=(x-1)(x-2).$$

Example 533. The quadratic polynomial $3x^2+5x+2$ has discriminant $b^2-4ac=5^2-4(3)(2)=1$. So,

$$3x^2+5x+2=3\left(x-\frac{-5-\sqrt{1}}{2\cdot 3}\right)\left(x-\frac{-5+\sqrt{1}}{2\cdot 3}\right)=3(x+1)\left(x+\frac{2}{3}\right)=(x+1)(3x+2).$$

Example 534. The quadratic polynomial $4x^2-12x+9$ has discriminant $b^2-4ac=(-12)^2-4(4)(9)=0$. So,

$$4x^2-12x+9=4\left(x+\frac{-12}{2\times 4}\right)^2=4\left(x-\frac{3}{2}\right)^2=(2x-3)^2.$$

Example 535. The quadratic polynomial $3x^2-2x+1$ has discriminant $b^2-4ac=(-2)^2-4(3)(1)<0$. So, $3x^2-2x+1$ cannot be factorised.

Exercise 158. XXX (Answer on p. 412.)

A158.

38.5. The Remainder Theorem

You're supposed to to have learnt this result in O-Level Additional Maths:

Theorem 9. (Remainder Theorem) *If $p(x)$ is a polynomial and a is a constant, then $p(x) \div (x-a)$ leaves a remainder of $p(a)$.*

Proof. By Definition 110, the remainder is $r(x) = p(x) - (x-a)q(x)$.

Since $x - a$ is a 1st-degree polynomial, by Theorem 8, $r(x)$ is a 0th-degree polynomial, i.e. a constant.

So, we may simply write $r(x) = R \stackrel{1}{=} p(x) - (x-a)q(x)$, where R doesn't depend on x.

Now plug $x = a$ into $\stackrel{1}{=}$ to get $R = p(a) - (a-a)q(a) = p(a)$. □

Example 536. Consider $p(x) = x^2 - 5x + 1$. By the Remainder Theorem (RT), $p(x)$ divided by $x - 3$ leaves a remainder of $p(3) = 3^2 - 5(3) + 1 = -5$.

We could've figured this out using long division (below), but clearly the RT is a lot quicker.

		Terms	
	Squared	**Linear**	**Constant**
		x	-2
$x-3$	x^2	$-5x$	$+1$
	x^2	$-3x$	
		$-2x$	$+1$
		$-2x$	$+6$
			-5.

Example 537. Consider the quintic polynomial $p(x) = 17x^5 - 5x^4 + x^2 + 1$. By the RT, $p(x)$ divided by $x - 1$ leaves a remainder of $p(1) = 17 \cdot 1^5 - 5 \cdot 1^4 + 1^2 + 1 = 14$.

We could've figured this out using long division (I didn't bother but you can try this as an exercise), but clearly the RT is a lot quicker.

Historically, the Remainder Theorem rarely featured on the A-Level exams ... which means, of course, that it made a sudden appearance in 2017 just to screw students over.[263] So yea, it's another thing you'll want to remember. An exercise to help with that:

Exercise 159. For each, find the remainder. (Answer on p. 1805.)

(a) $\left(2x^3 + 7x^2 - 3x + 5\right) \div (x - 3)$ **(b)** $\left(-2x^4 + 3x^2 - 7x - 1\right) \div (x + 2)$.

For A-Level Maths, the Remainder Theorem will have little use (except to screw over unsuspecting students). Instead, its corollary—the **Factor Theorem**—will be more useful for factorising polynomials.

[263] See Exercise 534(a) (9758 N2017/I/5).

38.6. The Factor Theorem

Another result you're supposed to have learnt in O-Level Additional Maths:

Theorem 10. (Factor Theorem) *If $p(x)$ is a polynomial and a is a constant, then*

$$x - a \text{ is a factor for } p(x) \qquad \Longleftrightarrow \qquad p(a) = 0.$$

Proof. By the Remainder Theorem (Theorem 9), $p(x) \div (x - a)$ leaves a remainder of $p(a)$. By Definition 111, $x - a$ is a factor for $p(x)$ if and only if the remainder is zero, i.e. $p(a) = 0$. □

Example 538. To factorise $p(x) = x^2 - 3x + 2$, we can use the **Factor Theorem (plus Guess and Check)** method:

First try plugging in the number 1:

$$p(1) = 1^2 - 3 \cdot 1 + 2 = 0. \qquad ✓$$

Wah! So "lucky"! Success on the very first try! Since $p(1) = 0$, by the Factor Theorem, $x - 1$ is a factor for $p(x)$.

Since $p(x)$ is quadratic (or degree 2), its other factor must be a linear (or degree-1) polynomial, i.e. of the form $ax + b$. To find this other factor, we could continue trying our luck with the Factor Theorem (i.e. try plugging other numbers into $p(x)$). But we won't do that. It's easier to write

$$x^2 - 3x + 2 \stackrel{1}{=} (x - 1)(ax + b) \stackrel{2}{=} ax^2 + (b - a)x - b,$$

where a and b are unknown constants to be found. Comparing coefficients, we have $a = 1$ and $b = -2$ (no guess and check needed). And now, we're done:

$$p(x) = x^2 - 3x + 2 = (x - 1)(x - 2).$$

Note: Above, we can actually skip writing out $\stackrel{2}{=}$ because from $\stackrel{1}{=}$ alone, we can easily tell that $a = 1$ and $b = -2$.

Example 539. Factorise $p(x) = 3x^2 + 5x + 2$

Try plugging in the number 1:

$$p(1) = 3 \cdot 1^2 + 5 \cdot 1 + 2 = 10 \neq 0. \qquad ✗$$

Aiyah, sian, doesn't work: Since $p(1) \neq 0$, by the Factor Theorem, $x - 1$ is **not** a factor for $p(x)$.

By the way, here's a tip to save you time: Above we found the exact value of $p(1)$ to be 10. But actually, we can skip finding the exact value. All we want to know is whether $p(1)$ equals zero (or not). And by observing that each of the three terms is obviously positive, we can tell that $p(1)$ is also obviously positive and, in particular, not equal to zero.

Next try -1:

$$p(-1) = 3 \cdot (-1)^2 + 5 \cdot (-1) + 2 = 0. \qquad ✓$$

Yay, works! Since $p(-1) = 0$, by the Factor Theorem, $x - (-1) = x + 1$ is a factor for $p(x)$.

As in the last example, to find the other factor, write

$$p(x) = 3x^2 + 5x + 2 = (x + 1)(ax + b).$$

Comparing coefficients (no guess and check needed), $a = 3$ and $b = 2$. So,

$$p(x) = 3x^2 + 5x + 2 = (x + 1)(3x + 2).$$

We now try using the Factor Theorem to factorise higher-degree polynomials:

Example 540. To factorise $p(x) = 15x^3 - 17x^2 - 22x + 24$, try plugging in 1:

$$p(1) = 15 \cdot 1^3 - 17 \cdot 1^2 - 22 \cdot 1 + 24 = 0. \qquad \checkmark$$

Wah! So "lucky"! Success on the very first try! Since $p(1) = 0$, by the Factor Theorem, $x - 1$ is a factor for $p(x)$.

Since $p(x)$ is cubic (degree 3), $p(x)$ divided by $x - 1$ gives us a quadratic (degree-2) polynomial $ax^2 + bx + c$ that we now want to find.

Write $$15x^3 - 17x^2 - 22x + 24 = (x - 1)\left(ax^2 + bx + c\right).$$

Comparing coefficients on x^3 and the constants, $a = 15$ and $c = -24$.

Comparing coefficients on x^2, $-17 = -a + b = -15 + b$. So, $b = -2$.

Hence, $$ax^2 + bx + c = 15x^2 - 2x - 24.$$

We next factorise $15x^2 - 2x - 24$. To do so, we can use any of the three methods introduced so far (Compare Coefficients, Quadratic Formula, and Factor Theorem). Here we'll just use the quadratic formula (which is the only one of the three that doesn't involve any guesswork):

The discriminant is $b^2 - 4ac = (-2)^2 - 4(15)(-24) = 4 + 1440 = 1444 > 0$. Moreover, $\sqrt{1444} = 38$. Hence,

$$15x^2 - 2x - 24 = 15\left(x - \frac{2 - 38}{30}\right)\left(x - \frac{2 + 38}{30}\right) = 15\left(x + \frac{6}{5}\right)\left(x - \frac{4}{3}\right).$$

Altogether, $$15x^3 - 17x^2 - 22x + 24 = 15(x - 1)\left(x + \frac{6}{5}\right)\left(x - \frac{4}{3}\right) = (x - 1)(5x + 6)(3x - 4).$$

Exercise 160. Factorise each polynomial. (Answer on p. 1805.)

(a) $2x^2 - 5x - 3$ **(b)** $7x^2 - 19x - 6$ **(c)** $6x^2 + x - 1$ **(d)** $2x^3 - x^2 - 17x - 14$.

38.7. The Integer and Rational Root Theorems

The **Integer Root Theorem (IRT)** says that given a polynomial $p(x)$ with integer coefficients, every integer solution to $p(x) = 0$ divides the constant term:

Corollary 15. (Integer Root Theorem) *Let* $p(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ *with* $p_0, p_1, \ldots, p_n \in \mathbb{Z}$. *Suppose* $a \in \mathbb{Z}$. *If* $p(a) = 0$, *then* $p_0/a \in \mathbb{Z}$.

Proof. See p. 419. □

The IRT may seem a bit abstract. But its significance is this—given a polynomial with integer coefficients, we can now find *every* factor of the form $x + a$ (where $a \in \mathbb{Z}$):

Example 541. Factorise $p(x) = x^3 + 6x^2 + 11x + 6$.

The coefficients are all integers. So, the IRT says this:

If $a \in \mathbb{Z}$ solves $p(x) = 0$, then a divides the constant term 6—i.e., a must be 1, −1, 2, −2, 3, −3, 6, or −6.

So, let's check if the value of p at any of these values is zero:

- Clearly, if $x > 0$, then $p(x) > 0$. So, $p(1)$, $p(2)$, $p(3)$, and $p(6)$ are not zero. ✗
- $p(-1) = (-1)^3 + 6(-1)^2 + 11(-1) + 6 = -1 + 6 - 11 + 6 = 0$ ✓
- $p(-2) = (-2)^3 + 6(-2)^2 + 11(-2) + 6 = -8 + 24 - 22 + 6 = 0$ ✓
- $p(-3) = (-3)^3 + 6(-3)^2 + 11(-3) + 6 = -27 + 54 - 33 + 6 = 0$ ✓
- $p(-6) = (-6)^3 + 6(-6)^2 + 11(-6) + 6 = -216 + 216 - 66 + 6 = -60$ ✗

Altogether, the IRT says that −1, −2, and −3 are the only integer solutions to $p(x) = 0$. And so, by the Factor Theorem (FT), $x + 1$, $x + 2$, and $x + 3$ are factors of $p(x)$.

Of course, since $p(x)$ is a 3rd-degree polynomial, these are also the only factors and

$$p(x) = x^3 + 6x^2 + 11x + 6 = (x + 1)(x + 2)(x + 3).$$

(Indeed, once we found three values of a for which $p(a) = 0$—as we did above with −1, −2, and −3—we needn't have bothered checking if $p(-6) = 0$ because we should already have known that $p(-6) \neq 0$.)

Example 542. Factorise $p(x) = 4x^3 + 4x^2 - 5x - 3$.

The coefficients are all integers. So, the IRT says this:

If $a \in \mathbb{Z}$ solves $p(x) = 0$, then a divides the constant term -3—i.e., a must be 1, -1, 3, or -3.

So, let's check if the value of p at any of these values is zero:

- $p(1) = 4 + 4 - 5 - 3 = 0$ ✓
- $p(-1) = -4 + 4 + 5 - 3 = 2$ ✗
- $p(3) = 108 + 36 - 15 - 3 \neq 0$ ✗
- $p(-3) = -108 + 36 + 15 - 3 \neq 0$ ✗

Altogether, the IRT says that 1 is the only integer solution to $p(x) = 0$. And so, by the Factor Theorem (FT), $x - 1$ is the only factor of $p(x)$ that is of the form $x + a$, where a is an integer.

Note though that the IRT does not rule out the possibility of other non-integer solutions to $p(x) = 0$. So, there is the possibility $p(x)$ has other factors of the form $x + a$, but where a is not an integer. This turns out to be the case:

Write $$p(x) = 4x^3 + 4x^2 - 5x - 3 = (x-1)\left(4x^2 + bx + c\right).$$

Comparing coefficients on x^2 and the constant term, we have $-4 + b = 4$ and $-c = -3$. So, $b = 8$ and $c = 3$. Hence,

$$p(x) = 4x^3 + 4x^2 - 5x - 3 = (x-1)\left(4x^2 + 8x + 3\right)$$

To factorise $4x^2 + 8x + 3$, I'll simply use the quadratic formula:

$$x = \frac{-8 \pm \sqrt{8^2 - 4(4)(3)}}{2(4)} = -\frac{1}{2}, -\frac{3}{2}.$$

Thus, $4x^2 + 8x + 3 = k\left(x + \frac{1}{2}\right)\left(x + \frac{3}{2}\right)$, where clearly $k = 4$. Altogether,

$$p(x) = (x-1)\left(4x^2 + 8x + 3\right) = 4(x-1)\left(x + \frac{1}{2}\right)\left(x + \frac{3}{2}\right).$$

So, $p(x)$ *does* have other factors of the form $x + a$, but where a is not an integer and so could not be detected by the IRT.

The **Rational Root Theorem (RRT)** is a more general version of the Integer Root Theorem. Let's first do a very quick primary-school review of what it means for a fraction to be **fully reduced**:

Example 543. The fraction $\frac{128}{24}$ is not fully reduced because we can further reduce it (i.e. cancel out common factors in the numerator and denominator): $\frac{128}{24} = \frac{64}{12} = \frac{32}{6} = \frac{16}{3}$.
The fraction $\frac{16}{3}$ is fully reduced. The fractions $\frac{64}{12}$ and $\frac{32}{6}$ are not.

More precisely and formally,

Definition 112. Let $N, D \in \mathbb{Z}$. Suppose there is no integer f greater than 1 such that N/f and D/f are both integers. Then we say that the fraction N/D is *fully reduced* (or *fully simplified*); we also say that N and D *share no common factors greater than one.*

Now, the RRT says that given a polynomial $p(x)$ with integer coefficients, *every* (fully reduced) rational solution to $p(x) = 0$ is such that its

- numerator divides the constant term; and
- denominator divides the leading coefficient.

More precisely and formally,

Theorem 11. (Rational Root Theorem) *Let* $p(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ *with* $p_0, p_1, \ldots, p_n \in \mathbb{Z}$ *and also* $N, D \in \mathbb{Z}$. *Suppose* N/D *is fully reduced.*
If $p\left(\frac{N}{D}\right) = 0$, *then* $\frac{p_0}{N}, \frac{p_n}{D} \in \mathbb{Z}$.

Proof. See p. 1606. □

(By the way, given the RRT, we can now easily prove the IRT: Simply plug in $D = 1$.)

Again, the IRT may seem a bit abstract. But its significance is this—given a polynomial with integer coefficients, we can now find *every* factor of the form $x + a$ (where $a \in \mathbb{Q}$):

Example 544. Let $p(x) = 24x^3 + 26x^2 + 9x + 1$.

The coefficients are all integers. So, the RRT says this:

If $\frac{N}{D}$ (fully reduced) solves $p(x) = 0$, then N divides the constant term 1 and D divides the leading coefficient 24—i.e. N must be 1 or −1, while D must be 1, 2, 3, 4, 6, 12, 24 (or the negative of these values). So, $\frac{N}{D}$ must be $\frac{1}{1}$, $-\frac{1}{1}$, $\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{3}$, $\frac{1}{4}$, $-\frac{1}{4}$, $\frac{1}{6}$, $-\frac{1}{6}$, $\frac{1}{12}$, $-\frac{1}{12}$, $\frac{1}{24}$, or $-\frac{1}{24}$.

So, let's check if the value of p at any of these values is zero:

- Clearly, if $x > 0$, then $p(x) > 0$. So, none of $p(1)$, $p(1/2)$, $p(1/3)$, $p(1/4)$, $p(1/6)$, $p(1/12)$, and $p(1/24)$ equals 0. ✗
- $p(-1) = -24 + 26 - 9 + 1 \neq 0$ ✗
- $p\left(-\frac{1}{2}\right) = -3 + \frac{13}{2} - \frac{9}{2} + 1 = 0$ ✓
- $p\left(-\frac{1}{3}\right) = -\frac{8}{9} + \frac{26}{9} - 3 + 1 = 0$ ✓
- $p\left(-\frac{1}{4}\right) = -\frac{6}{16} + \frac{26}{16} - \frac{9}{4} + 1 = 0$ ✓

At this point, we've already found three factors—$\left(x + \frac{1}{2}\right)$, $\left(x + \frac{1}{3}\right)$, and $\left(x + \frac{1}{4}\right)$. So, we needn't bother checking $p\left(\frac{1}{6}\right)$, $p\left(\frac{1}{12}\right)$, and $p\left(\frac{1}{24}\right)$ (we know they'll all be non-zero). We can go ahead and conclude

$$p(x) = k\left(x + \frac{1}{2}\right)\left(x + \frac{1}{3}\right)\left(x + \frac{1}{4}\right),$$

where clearly $k = 24$.

Example 545. Let $p(x) = 2x^3 + 4x^2 - x - 2$.

The coefficients are all integers. So, the RRT says this:

If $\dfrac{N}{D}$ (fully reduced) solves $p(x) = 0$, then N divides the constant term -2 and D divides the leading coefficient 2—i.e. N must be 1, 2, -1, or -2; and D must be 1, 2, -1, or -2. So, $\dfrac{N}{D}$ must be $\dfrac{1}{1}$, $-\dfrac{1}{1}$, $\dfrac{2}{1}$, $-\dfrac{2}{1}$, $\dfrac{1}{2}$, $-\dfrac{1}{2}$, $\dfrac{2}{2} = 1$, or $-\dfrac{2}{2} = -1$. (The last two possibilities are of course just repetitions of the first two.)

So, let's check if the value of p at any of these values is zero:

- $p(1) = 2 \times 1^3 + 4 \times 1^2 - 1 - 2 = 3$ ✗
- $p(-1) = 2 \times (-1)^3 + 4 \times (-1)^2 - (-1) - 2 = 1$ ✗
- $p(2) = 2 \times 2^3 + 4 \times 2^2 - 2 - 2 = 28$ ✗
- $p(-2) = 2 \times (-2)^3 + 4 \times (-2)^2 - (-2) - 2 = 0$ ✓
- $p\left(\frac{1}{2}\right) = 2 \times \left(\frac{1}{2}\right)^3 + 4 \times \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2 = \frac{1}{4} + 1 - \frac{1}{2} - 2 = -\frac{5}{4}$ ✗
- $p\left(-\frac{1}{2}\right) = 2 \times \left(-\frac{1}{2}\right)^3 + 4 \times \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 2 = -\frac{1}{4} + 1 + \frac{1}{2} - 2 = -\frac{3}{4}$ ✗

Altogether, the RRT says that -2 is the only rational solution to $p(x) = 0$. So, $x + 2$ is the only factor of $p(x)$ that is of the form $x + a$ with $a \in \mathbb{Q}$.

Note though that the RRT does **not** rule out the possibility of other irrational solutions to $p(x) = 0$. So, there is the possibility $p(x)$ has other factors of the form $x + a$ with $a \notin \mathbb{Q}$. This turns out to be the case:

Write $$p(x) = 2x^3 + 4x^2 - x - 2 = (x + 2)\left(2x^2 + bx + c\right).$$

Comparing coefficients on x^2 and the constant term, we have $4 + b = 4$ and $2c = -2$. So, $b = 0$ and $c = -1$. Hence,

$$p(x) = 2x^3 + 4x^2 - x - 2 = (x + 2)\left(2x^2 - 1\right) = 2(x + 2)\left(x^2 - \frac{1}{2}\right) = 2(x + 2)\left(x + \frac{1}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}}\right).$$

So, besides $x + 2$, $p(x)$ *does* have two other factors of the form $x + a$, but where $a \notin \mathbb{Q}$ and so could not be uncovered by the RRT.

Exercise 161. Factorise each polynomial. (Answer on p. 421.)

XXX

A161.

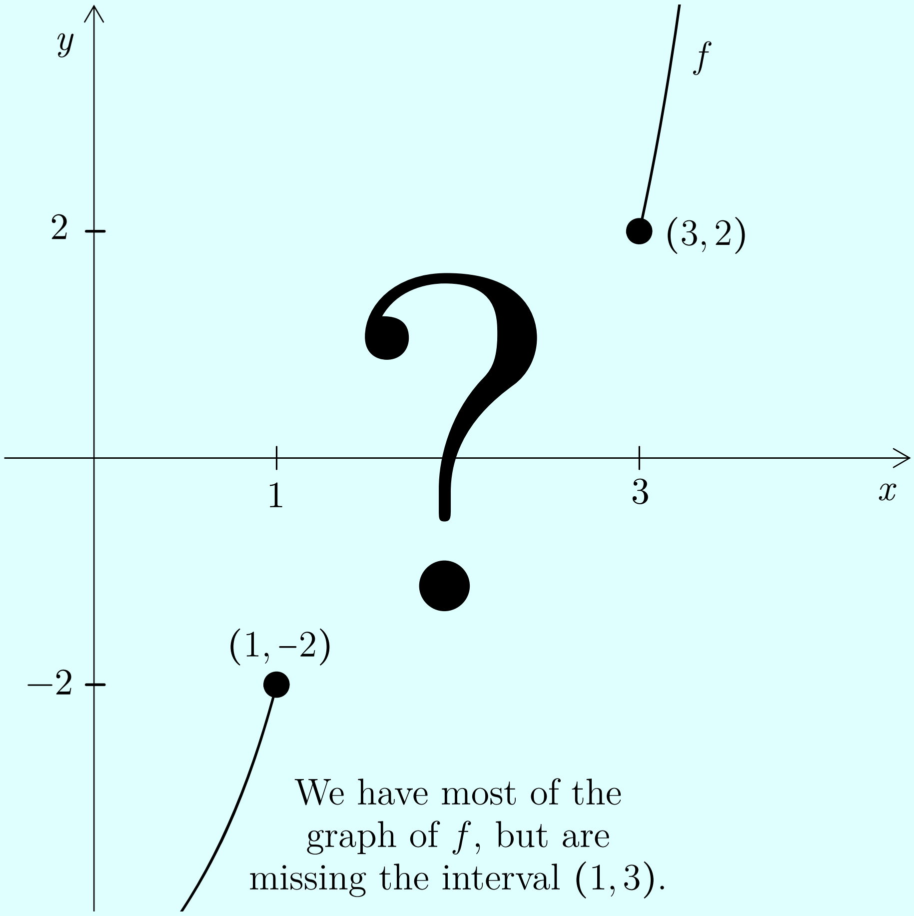
38.8. The Intermediate Value Theorem (IVT)

The **Intermediate Value Theorem (IVT)** is another useful tool for factorising polynomials. We first describe it informally with an example:

Example 546. Below is part of the graph of some continuous function $f : \mathbb{R} \to \mathbb{R}$.

We are missing the graph of f for the interval $(1, 3)$.

Say we know that $f(1) = -2$ and $f(3) = 2$. What then can we say about the missing portion of the graph?



Since f is continuous, it must be that we can draw its entire graph without lifting our pencil. In particular, we can connect the dots $(1, -2)$ and $(3, 2)$ without lifting our pencil. But obviously, the only way to do so is to have our pencil "go through" every value between -2 and 2. That is (and this is what the IVT says),

f must attain (or "hit") *every* value between -2 and 2.

And yup, that's all the IVT says—if f is continuous on the interval $[a, b]$, then f must attain (or "hit") every value between $f(a)$ and $f(b)$ in the interval (a, b). A bit more formally,

Theorem 12. (Intermediate Value Theorem) *If f is continuous on the interval $[a, b]$, then for every $y \in (f(a), f(b))$, there exists $x \in (a, b)$ such that $y = f(x)$.*

Proof. Omitted.[264] □

Let's now see how the IVT can help us factorise polynomials.

Example 547. To factorise $p(x) = 2x^2 + 9x - 5$, it's probably quicker to use either the Compare Coefficients (plus Guess and Check) method or the quadratic formula.

But here, just to illustrate how the IVT can be used, we'll start instead by using the Factor Theorem (FT). We try plugging in 1:

$$p(1) = 2 \cdot 1^2 + 9 \cdot 1 - 5 = 6 \neq 0. \qquad ✗$$

Aiyah, sian, doesn't work: Since $p(1) \neq 0$, by the FT, $x - 1$ is **not** a factor for $p(x)$.

Here we could immediately try another guess with the FT. But instead, here we can first make use of the IVT:

Observe that $p(0) = -5$. What good is this observation?

Well, since 0 is between $p(0) = -5$ and $p(1) = 6$, the IVT says there *must* be some $a \in (0, 1)$ such that $p(a) = 0$. Cool!

So, guided by this knowledge, let's try the FT again by guessing 1/2:

$$p\left(\frac{1}{2}\right) = 2 \cdot \left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 = 0 \qquad ✓$$

Yay, works! Since $p(1/2) = 0$, by the FT, $x - 1/2$ or $2x - 1$ is a factor for $p(x)$.

Write $$2x^2 + 9x - 5 = (2x - 1)(ax + b).$$

Comparing coefficients, $a = 1$ and $b = 5$. Hence,

$$p(x) = 2x^2 + 9x - 5 = (2x - 1)(x + 5).$$

Example 548. XXX

Example 549. XXX

Exercise 162. XXX (Answer on p. 423.)

A162.

[264] See e.g. O'Connor (2001).

38.9. Factorising a Quartic Polynomial

For the A-Levels, you'll often have to factorise quadratic polynomials and, sometimes, cubic polynomials.

It's unusual that they ask you to factorise polynomials of a higher degree. And if they do, the nice MOE folks will usually give you a little help.[265]

In the next example, we factorise a quartic (degree-4) polynomial using what we've learnt. It's long and tedious, but conceptually, it's not any harder than what we've already done:

Example 550. Factorise $p(x) = 6x^4 + 13x^3 - 29x^2 - 52x + 20$.

First try plugging in 1:

$$p(1) = 6 \cdot 1^4 + 13 \cdot 1^3 - 29 \cdot 1^2 - 52 \cdot 1 + 20 < 0. \qquad ✗$$

(Again, we don't need to compute the exact value of $p(1)$ to see that $p(1) < 0$.)

Aiyah, sian, doesn't work: Since $p(1) \neq 0$, by the Factor Theorem (FT), $x - 1$ is **not** a factor for $p(x)$.

But now, observe that 0 is between $p(0) = 20$ and $p(1) < 0$. And so, the IVT says there *must* be some $a \in (0, 1)$ such that $p(a) = 0$. So, let's next try 1/2:

$$p\left(\frac{1}{2}\right) = 6 \cdot \left(\frac{1}{2}\right)^4 + 13 \cdot \left(\frac{1}{2}\right)^3 - 29 \cdot \left(\frac{1}{2}\right)^2 - 52 \cdot \left(\frac{1}{2}\right) + 20 < 0. \qquad ✗$$

(Again, we don't need to compute the exact value of $p(1/2)$ to see that $p(1/2) < 0$.)

Aiyah, sian, still doesn't work: Since $p(1/2) \neq 0$, by the FT, $x - 1/2$ is **not** a factor for $p(x)$.

But again, observe that 0 is between $p(0) = 20$ and $p(1/2) < 0$. And so, the IVT says that there *must* be some $b \in (0, 1/2)$ such that $p(b) = 0$. So, let's next try 1/3:

$$\begin{aligned} p\left(\frac{1}{3}\right) &= 6 \cdot \left(\frac{1}{3}\right)^4 + 13 \cdot \left(\frac{1}{3}\right)^3 - 29 \cdot \left(\frac{1}{3}\right)^2 - 52 \cdot \left(\frac{1}{3}\right) + 20 \\ &= \frac{6}{81} + \frac{13}{27} - \frac{29}{9} - \frac{52}{3} + 20 = \frac{2}{27} + \frac{13}{27} - \frac{29}{9} + \frac{8}{3} = \frac{15}{27} - \frac{5}{9} = 0. \qquad ✓ \end{aligned}$$

Yay, works! Since $p(1/3) = 0$, by the FT, $x - 1/3$ or $3x - 1$ is a factor for $6x^4 + 13x^3 - 29x^2 - 52x + 20$.

(*Example continues on the next page ...*)

[265]See e.g. 9740 N2010/II/1(b) (Exercise 651).

(... *Example continued from the previous page.*)

Dividing $p(x)$ by $3x-1$ should yield some cubic polynomial. So, write

$$p(x) = 6x^4 + 13x^3 - 29x^2 - 52x + 20 = (3x-1)\left(2x^3 + ax^2 + bx + c\right).$$

Comparing coefficients on the constant, x^3, and x^2 terms, we have $-c = 20$, $3a - 2 = 13$, and $3c - b = -52$. So, $c = -20$, $a = 5$, and $b = -8$.

To factorise $2x^3 + 5x^2 - 8x - 20$, we try plugging in 2:

$$p(2) = 2 \cdot 2^3 + 5 \cdot 2^2 - 8 \cdot 2 - 20 = 16 + 20 - 16 - 20 = 0 \qquad \checkmark$$

Yay, works! Since $p(2) = 0$, by the FT, $x - 2$ is a factor for $2x^3 + 5x^2 - 8x - 20$.

Dividing $2x^3 + 5x^2 - 8x - 20$ by $x - 2$ should yield some quadratic polynomial. So, write

$$2x^3 + 5x^2 - 8x - 20 = (x-2)\left(2x^2 + dx + e\right).$$

Comparing coefficients on the constant and x^2 terms, we have $-2e = -20$ and $d - 4 = 5$. So, $e = 10$ and $d = 9$.

To factorise $2x^2+9x+10$, we use the Compare Coefficients (plus Guess and Check) method and try

$$(2x+5)(x+2) = 2x^2 + 9x + 10. \qquad \checkmark$$

Wah! So "lucky"! Success on the very first try! And now, at long last, we're done:

$$p(x) = 6x^4 + 13x^3 - 29x^2 - 52x + 20 = (3x-1)(x-2)(2x+5)(x+2).$$

Exercise 163. Let $p(x) = ax^4 + bx^3 - 31x^2 + 3x + 3$, where a and b are constants. You are told that **(i)** $p(x)$ divided by $x - 1$ leaves a remainder of 5; and **(ii)** $p(0.5) = 0$.

(a) Find a and b.

You are now also told that **(iii)** $p(-1/3) < 0$.

(b) Factorise $p(x)$. (Answer on p. 1806.)

38.10. Two Warnings

All of our examples have so far involved a n-degree polynomial that can be fully factorised into n 1s-degree (or linear) factors. But this need not always be the case:

Example 551. Consider the quartic polynomial $x^4 - 1$. We can write

$$x^4 - 1 = \left(x^2 + 1\right)(x + 1)(x - 1).$$

Unfortunately, $x^2 + 1$ has negative discriminant and so cannot be further factorised.[266]

Example 552. Consider the quartic polynomial $x^4 + 5x^2 + 4$. We can write

$$x^4 + 5x^2 + 2 = \left(x^2 + 1\right)\left(x^2 + 4\right).$$

Both $x^2 + 1$ and $x^2 + 4$ have negative discriminants and so cannot be further factorised.[267]

Example 553. The quartic polynomial $x^4 + x + 1$ cannot be factorised at all.[268]

The above three examples illustrate these two important warnings:

1. Not every n-degree polynomial can be fully factorised into n linear factors.
2. Indeed, an n-degree polynomial may not even have a single linear factor!

Exercise 164. XXX (Answer on p. 426.)

A164.

[266] Unless we use complex numbers, which we'll learn about in Part III. With complex numbers, we can write $x^2 + 1 = (x + \mathrm{i})(x - \mathrm{i})$ and thus $x^4 - 1 = (x + \mathrm{i})(x - \mathrm{i})(x + 1)(x - 1)$.

[267] Again, with complex numbers, we can write $x^2 + 1 = (x + \mathrm{i})(x - \mathrm{i})$, $x^2 + 4 = (x + 2\mathrm{i})(x - 2\mathrm{i})$, and thus $x^4 + 5x^2 + 2 = (x + \mathrm{i})(x - \mathrm{i})(x + 2\mathrm{i})(x - 2\mathrm{i})$.

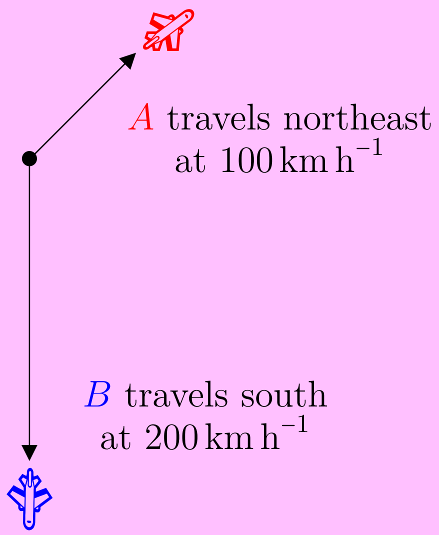
[268] Again, with complex numbers, it's actually possible to factorise $x^4 + x + 1$ into four linear factors.

39. Solving Systems of Equations

Let's start with two warm-up questions (PSLE and O-Level style):

Exercise 165. Today is Apu, Beng, and Caleb's birthday. When Apu turned 40 some years ago, Beng was twice as old as Caleb. Today, Apu is twice as old as Beng, while Caleb turns 28. What ages do Apu and Beng turn today?[269] (Answer on p. 1807.)

Exercise 166. Planes A and B leave the same point at the same time. Plane A travels (precisely) northeast at a constant speed of $100\,\mathrm{km\,h^{-1}}$. Plane B travels (precisely) south at a constant speed of $200\,\mathrm{km\,h^{-1}}$. After (precisely) 3 hours, both planes instantly turn to start flying directly towards each other (at the same speeds as before). How many hours after making this instant turn will they collide? (Answer on p. 1807.)



In Ch. 13, we learnt the terms **solution** and **solution set**, in the context of a single equation involving only one variable. These terms also carry the same meaning in a **system of equations**, which is any set of m equations in n variables:

[269]Just to be clear, we use the standard western convention where one's age on any day is an integer and increases by one on each birthday. (We do not for example use the East Asian convention.)

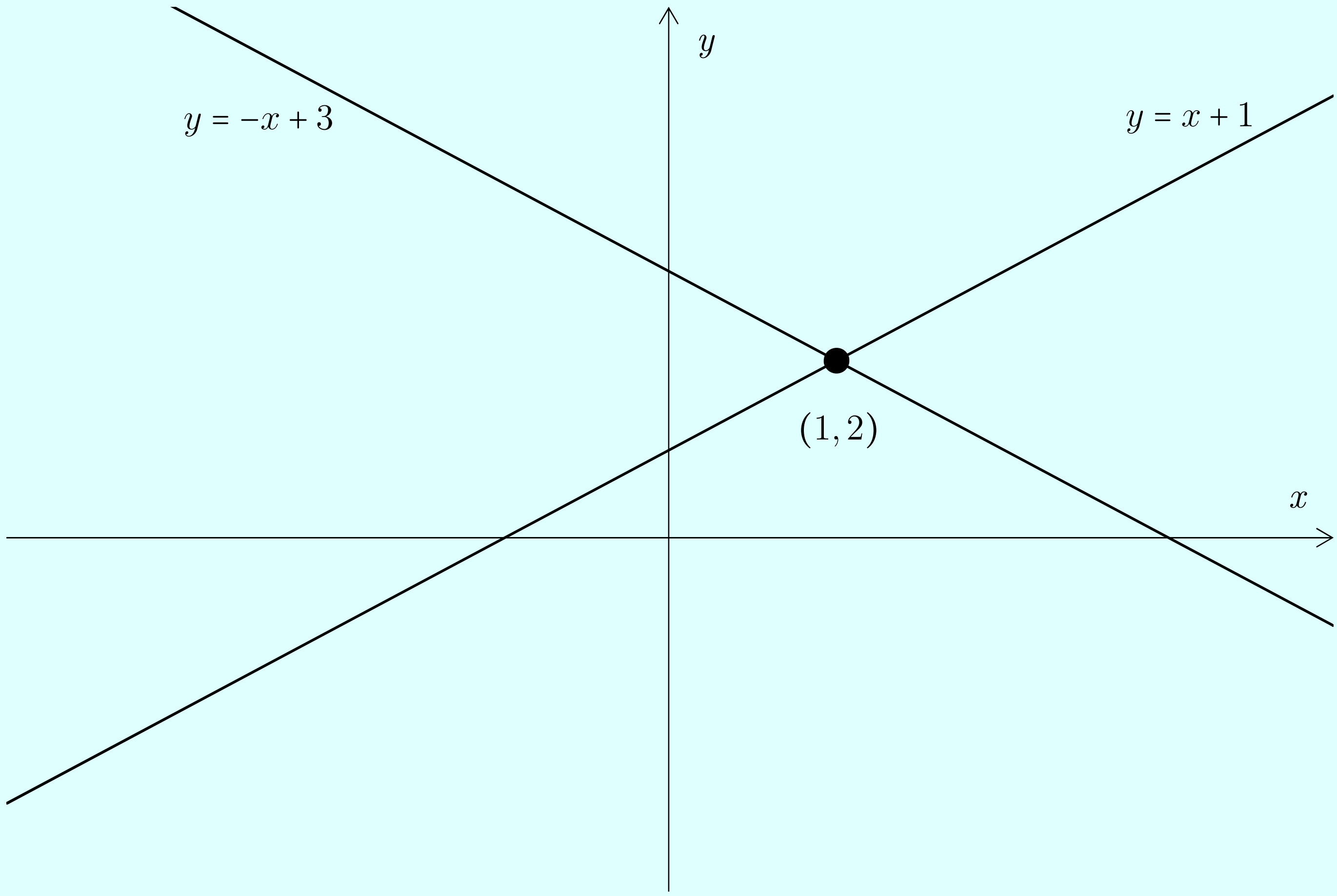
Example 554. Consider this **system of *two* equations in *two* variables**:

$$y \overset{1}{=} x + 1 \qquad \text{and} \qquad y \overset{2}{=} -x + 3 \qquad\qquad (x, y \in \mathbb{R}).$$

In secondary school, we already learnt to solve such systems of equations (it's just simple algebra):

Plug $\overset{2}{=}$ into $\overset{1}{=}$ to get $-x + 3 = x + 1$ or $2 = 2x$ or $x = 1$. Now from either $\overset{1}{=}$ or $\overset{2}{=}$, we can also get $y = 2$. Conclude,

- "The system of equations has one **solution**: $(x, y) = (1, 2)$."
- Or more simply, "The system of equations has one **solution** $(1, 2)$."
- Or, "The system of equations has **solution set** $\{(1, 2)\}$."

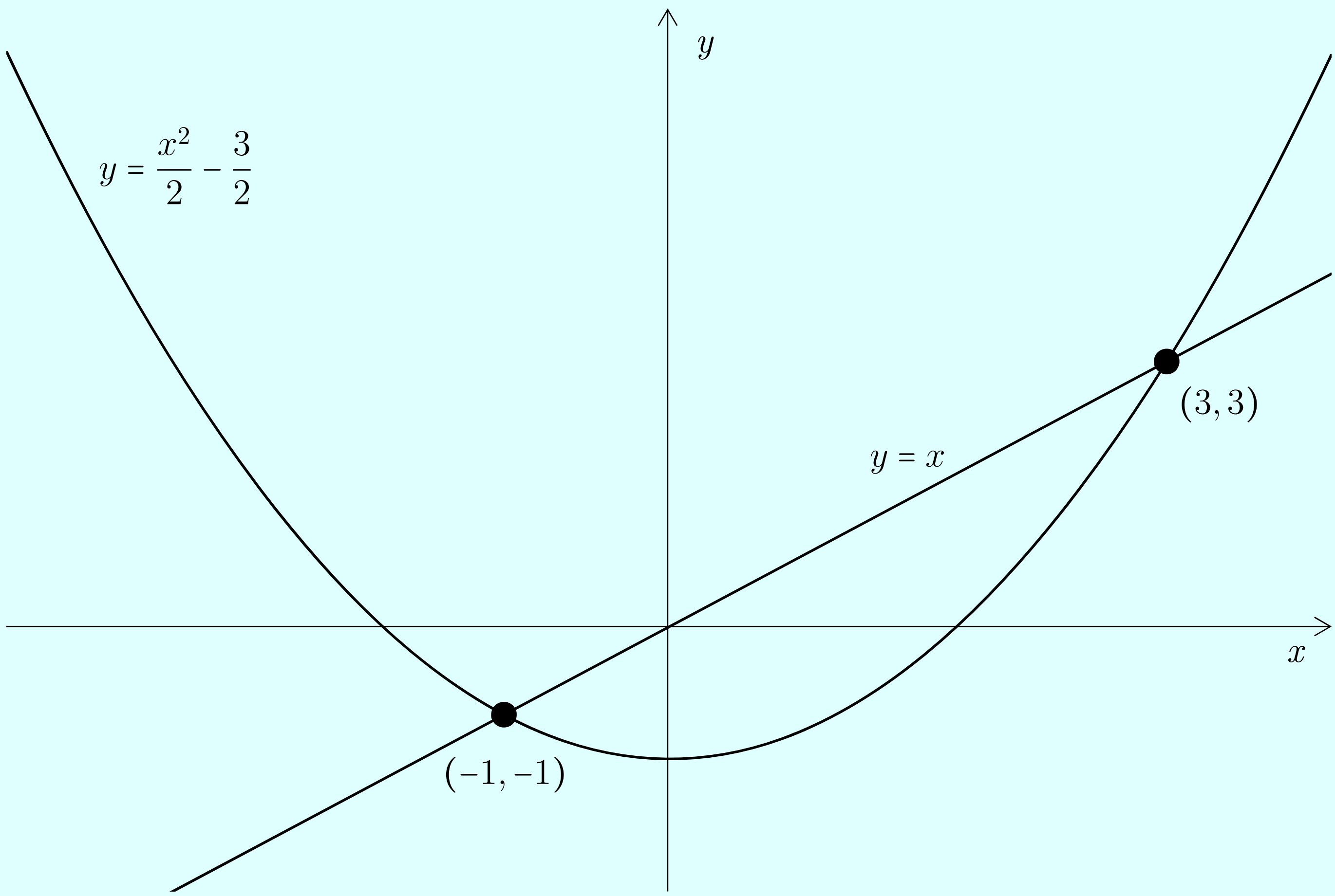


Example 555. Solve this system of two equations in two variables:

$$y \overset{1}{=} \frac{x^2}{2} - \frac{3}{2} \qquad \text{and} \qquad y \overset{2}{=} x \qquad (x, y \in \mathbb{R}).$$

Plug $\overset{2}{=}$ into $\overset{1}{=}$ to get $x \overset{3}{=} \frac{x^2}{2} - \frac{3}{2}$ or $0 = x^2 - 2x - 3 = (x-3)(x+1)$. So, $x = 3$ or $x = -1$. And correspondingly, $y = 3$ or $y = -1$. Conclude,

- "The system of equations has *two* solutions: $(x, y) = (3, 3), (-1, -1)$."
- Or more simply, "The system of equations has two solutions: $(3, 3)$ and $(-1, -1)$."
- Or, "The system of equations has solution set $\{(3, 3), (-1, -1)\}$."



Example 556. Here's a system of *three* equations in *three* variables:

$$y \overset{1}{=} 3x - 2, \qquad z \overset{2}{=} 7 - y, \qquad x \overset{3}{=} y + z \qquad (x, y, z \in \mathbb{R}).$$

Plug $\overset{1}{=}$ and $\overset{2}{=}$ into $\overset{3}{=}$ to get $x = 3x - 2 + 7 - y = 3x + 5 - y$ or $y \overset{4}{=} 2x + 5$.

Plug $\overset{4}{=}$ into $\overset{1}{=}$ to get $2x + 5 = 3x - 2$ or $7 = x$.

Plug $x = 7$ into $\overset{1}{=}$ to get $y = 19$. Then plug $y = 19$ into $\overset{2}{=}$ to get $z = -12$. Conclude

- "The system of equations has one solution: $(x, y, z) = (7, 19, -12)$."
- Or more simply, "The system of equations has one solution $(7, 19, -12)$."
- Or, "The system of equations has solution set $\{(7, 19, -12)\}$."

By the way, $(7, 19, -12)$ is this textbook's first example of an **ordered triple**. This is exactly analogous to an ordered pair, except that now there are three coordinates. As you can imagine, we also have **ordered quadruples**, **ordered quintuples**, and more generally **ordered n-tuples**.[270]

When we study vectors in Part III, we'll learn that it's actually possible to graph the above three equations. (Spoiler: Each of the three graphs will be a plane in 3-dimensional space.)

A system of equations can have **no solutions**:

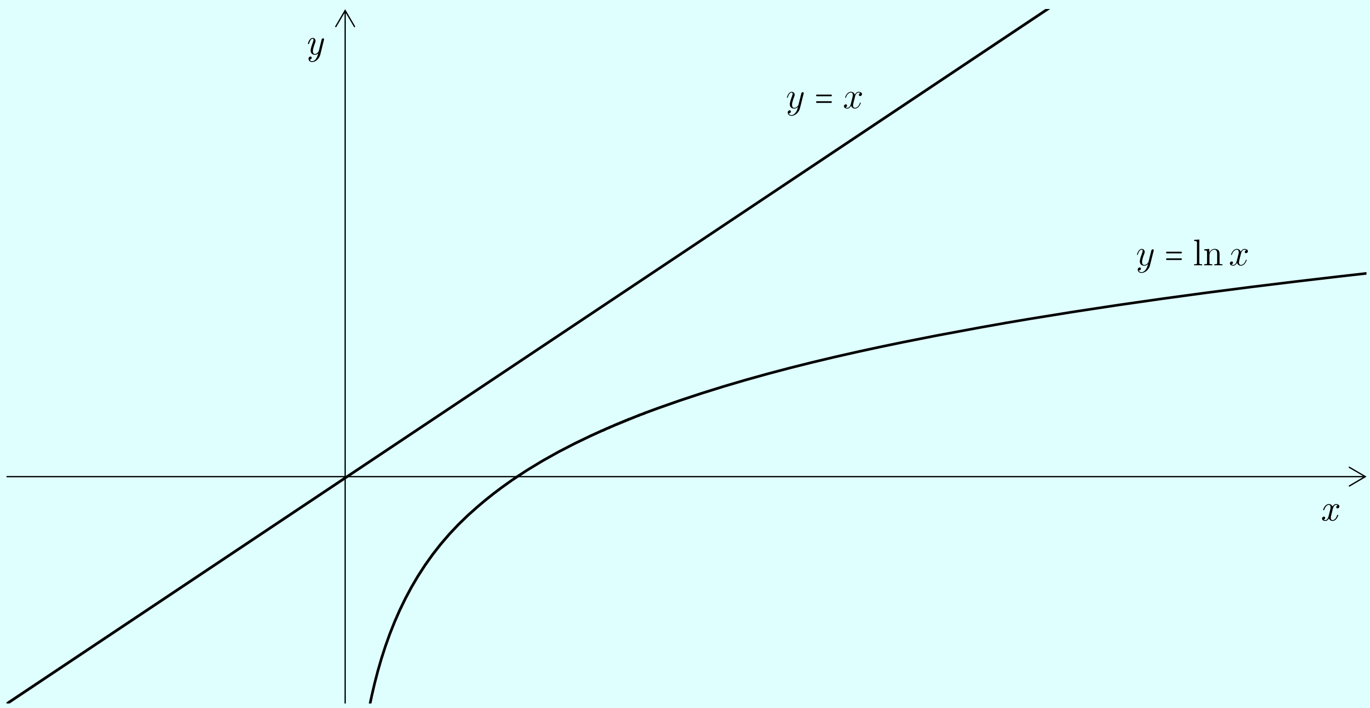
[270] For the formal definition of an n-tuple, see Definition 267 (Appendices).

Example 557. Consider this system of two equations in two variables:

$$y \overset{1}{=} \ln x \qquad \text{and} \qquad y \overset{2}{=} x \qquad\qquad (x, y \in \mathbb{R}).$$

Observe that the graph of $\overset{1}{=}$ lies below that of $\overset{2}{=}$ everywhere. Hence, no ordered pair (x, y) satisfies the above system of equations. Conclude,

- "The system of equations has *no* solutions."
- Or, "The system of equations has solution set $\varnothing$."

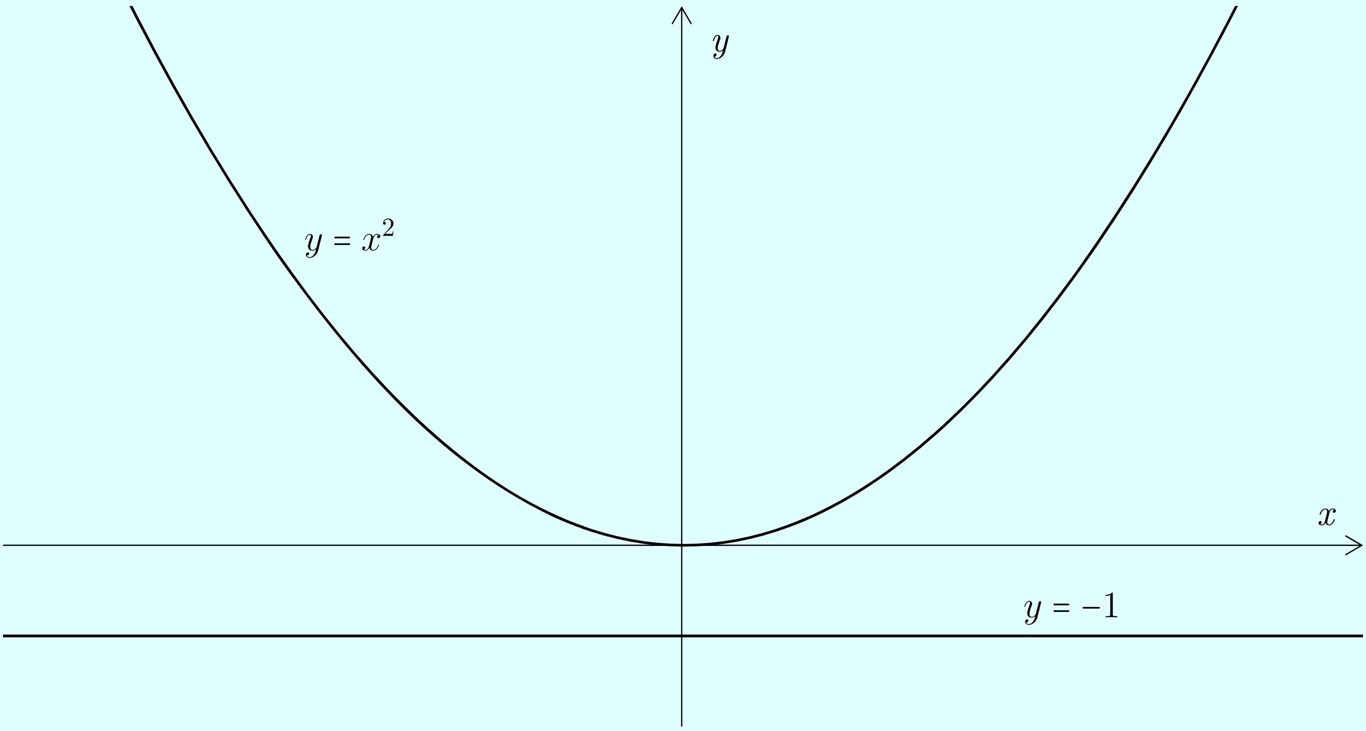


Example 558. Consider this system of two equations in two variables:

$$y \overset{1}{=} x^2 \qquad \text{and} \qquad y \overset{2}{=} -1 \qquad\qquad (x, y \in \mathbb{R}).$$

Observe that the graph of $\overset{1}{=}$ lies above that of $\overset{2}{=}$ everywhere. Hence, no ordered pair (x, y) satisfies the above system of equations. Conclude,

- "The system of equations has *no* solutions."
- Or, "The system of equations has solution set $\varnothing$."[271]



Example 559. Consider this system of four equations in three variables:

$$x \overset{1}{=} 1, \qquad y \overset{2}{=} 2, \qquad z \overset{3}{=} 3, \qquad xy \overset{4}{=} 0 \qquad (x, y, z \in \mathbb{R}).$$

There is no (x, y, z) that satisfies the above four equations. So there are zero solutions to the above system of equations. Equivalently,

- "No (x, y, z) is a solution."
- Or, "The system of equations has solution set $\varnothing$."

At the other extreme, a system of equations can have **infinitely many solutions**:

[271]But as we'll learn in Part IV, if we rewrite the system of equations so that "$x, y \in \mathbb{R}$" is replaced by "$x, y \in \mathbb{C}$", then we'd instead conclude,

- "The system of equations has *two* solutions: $(x, y) = (-\mathrm{i}, -1), (\mathrm{i}, -1)$."
- Or more simply, "The system of equations has two solutions: $(-\mathrm{i}, -1)$ and $(\mathrm{i}, -1)$."
- Or, "The system of equations has solution set $\mathrm{i}, -1$."

Example 560. Consider this system of two equations in two variables:

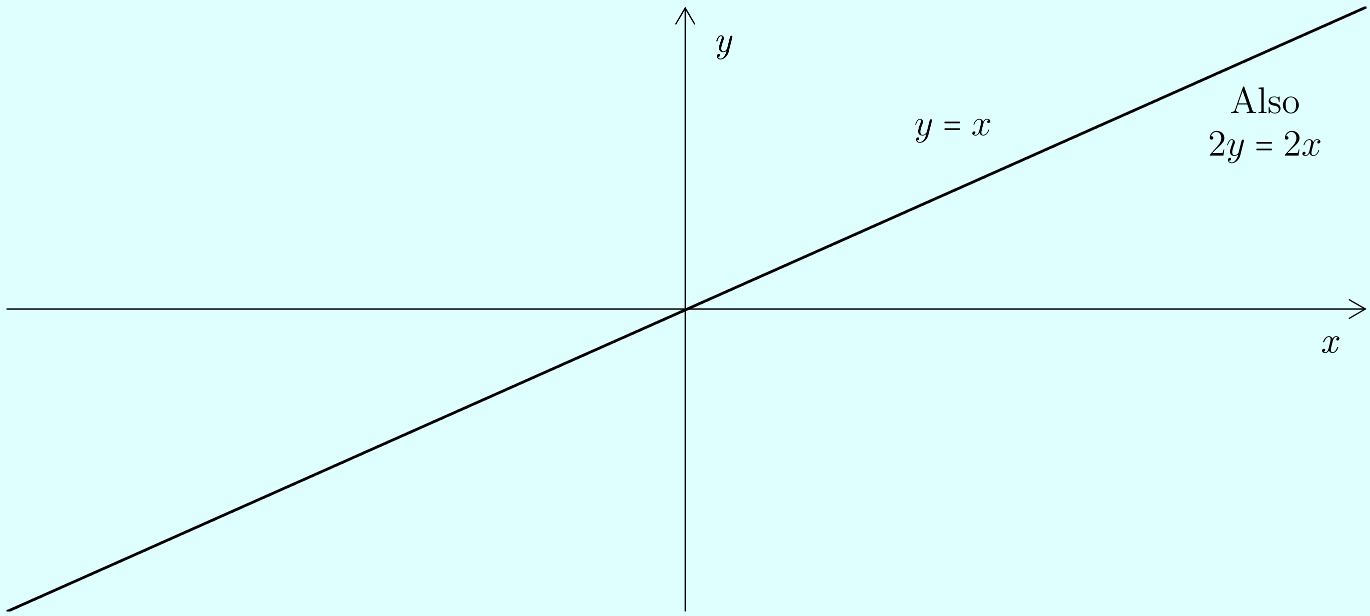
$$y \stackrel{1}{=} x \qquad \text{and} \qquad 2y \stackrel{2}{=} 2x \qquad\qquad (x, y \in \mathbb{R}).$$

Observe that $\stackrel{1}{=}$ and $\stackrel{2}{=}$ are really the same. And so, this system of equations has *infinitely many* solutions. Specifically,

- "Any (x, y) for which $y = x$ is a solution."
- Or, "The system of equations has solution set $\{(x, y) : x = y\}$."

Alternatively and equivalently,

- "For any $c \in \mathbb{R}$, (c, c) is a solution."
- Or, "The system of equations has solution set $\{(c, c) : c \in \mathbb{R}\}$."



Example 561. Consider this system of two equations in two variables:

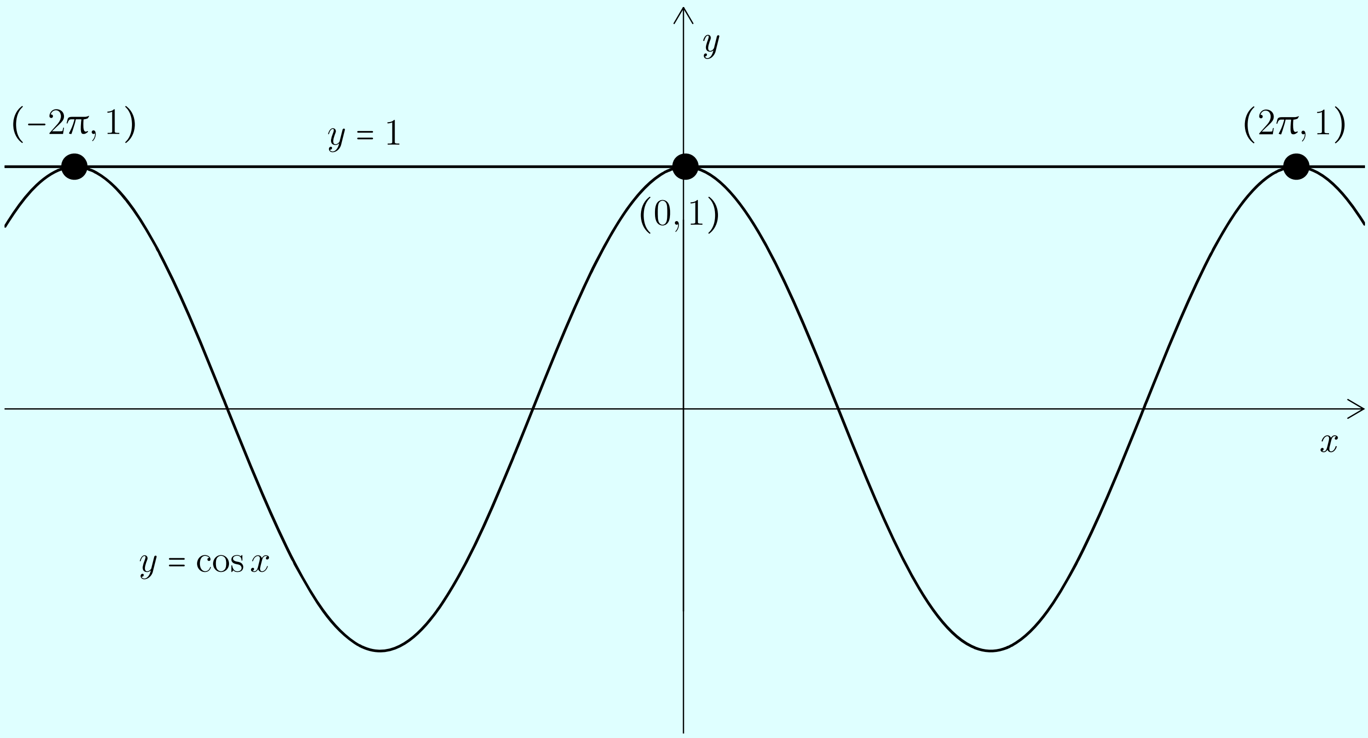
$$y \stackrel{1}{=} \cos x \qquad \text{and} \qquad y \stackrel{2}{=} 1 \qquad (x, y \in \mathbb{R}).$$

This system of equations has *infinitely many* solutions. Specifically,

- "Any (x, y) for which $x = 2k\pi$ (for any $k \in \mathbb{Z}$)and $y = 0$ is a solution."
- Or, "The system of equations has solution set $\{(x, y) : x = 2k\pi, y = 0, k \in \mathbb{Z}\}$."

Alternatively and equivalently,

- "For any $k \in \mathbb{Z}$, $(2k\pi, 0)$ is a solution."
- Or, "The system of equations has solution set $\{(2k\pi, 0) : k \in \mathbb{Z}\}$."



Example 562. Consider this system of two equations in *three* variables:

$$x \stackrel{1}{=} yz, \qquad z \stackrel{2}{=} 0 \qquad (x, y, z \in \mathbb{R}).$$

By $\stackrel{2}{=}$, any solution must have $z = 0$. Plug $\stackrel{2}{=}$ into $\stackrel{1}{=}$ to also get $x = 0$. There are no restrictions on what y can be.

So, there are infinitely many solutions. Specifically,

- "Any (x, y, z) for which $x = 0$ and $z = 0$ is a solution."
- Or, "The system of equations has solution set $\{(x, y, z) : x = 0, z = 0\}$."

Alternatively and equivalently,

- "For any $c \in \mathbb{R}$, $(0, c, 0)$ is a solution."
- Or, "The system of equations has solution set $\{(0, c, 0) : c \in \mathbb{R}\}$."

In the context of a system of equations, here are the formal definitions of a **solution** and the **solution set**:[272]

Definition 113. Given a system of equations (and/or inequalities) in $n \geq 2$ variables, we call any ordered n-tuple that satisfies the system a *solution* (or *root*). And the set of all such solutions is called the *solution set* of that system.

Exercise 167. XXX (Answer on p. 435.)

A167.

Example 563. Let $a, b, c, x, y \in \mathbb{R}$. Suppose the equation $y = ax^2 + bx + c$ has solutions $(0, 1)$, $(2, 3)$, and $(4, 5)$. Then what are a, b, and c?

Here things seem a little strange: We speak of the equation $y = ax^2 + bx + c$ and its solutions, with x and y being the variables.

Yet what we really want to do is to solve the following system of equations, where a, b, and c are the variables and we've plugged in the given values of x and y:

$$1 \stackrel{1}{=} a \cdot 0^2 + b \cdot 0 + c = c,$$
$$3 \stackrel{2}{=} a \cdot 2^2 + b \cdot 2 + c = 4a + 2b + c,$$
$$5 \stackrel{3}{=} a \cdot 4^2 + b \cdot 4 + c = 16a + 4b + c.$$

As usual, this is just simple algebra: $\stackrel{2}{=} - \stackrel{1}{=}$ yields $4a + 2b \stackrel{4}{=} 2$ and $\stackrel{3}{=} - \stackrel{1}{=}$ yields $16a + 4b \stackrel{5}{=} 4$. Now, $\stackrel{5}{=} -2\times \stackrel{4}{=}$ yields $8a = 0$ or $a = 0$. Now from $\stackrel{4}{=}$, we also have $b = 1$. Finally, from $\stackrel{2}{=}$, we also have $c = 1$. Altogether,

$$(a, b, c) = (0, 1, 1).$$

Below are two similar exercises. (See also N2011/I/2—Exercise 554.)

Exercise 168. Let $a, b, c, x, y \in \mathbb{R}$. (Answer on p. 1808.)

(a) If $y = ax^2 + bx + c$ has solutions $(x, y) = (1, 2), (3, 5), (6, 9)$, then what are a, b, and c?

(b) If $y = ax^2 + bx + c$ has the solution $(x, y) = (-1, 2)$ and the strict global minimum point $(0, 0)$, then what are a, b, and c?

[272]Note that these are exactly analogous to our earlier Definition 59.

39.1. Solving Systems of Equations with the TI84

You need to use your graphing calculator to solve systems of equations:

Example 564. Consider this system of equations:

$$y = x^4 - x^3 - 5 \qquad \text{and} \qquad y = \ln x \qquad\qquad (x > 0,\ y \in \mathbb{R}).$$

We'll solve this system using two methods.

Method 1. Graph both equations, then find the intersection points:

1. Enter the two equations (precise instructions omitted).
2. Press GRAPH to graph the two equations.
3. Press 2ND and TRACE to bring up the CALC (CALCULATE) menu.
4. Now press 5 to select the "intersect" function.

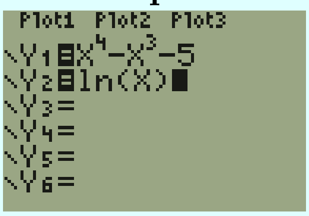
   Your TI84 now asks, "First curve?" So,
5. Press ENTER to tell the TI84 that "$y_1 = x^4 - x^3 - 5$" is indeed our first curve.

   The TI84 now asks you, "Second curve?" This time,
6. Use the left and right arrow keys ⟨ and ⟩ to move the cursor to roughly where you think there is an intersection point. For me, I've moved it to $(x, y) \approx (1.702, 0.532)$.
7. Now hit ENTER. The TI84 will now ask you "Guess?".
8. So hit ENTER again. After working furiously for a moment, the TI84 will inform you that the intersection point is $(x, y) \approx (1.87, 0.624)$.

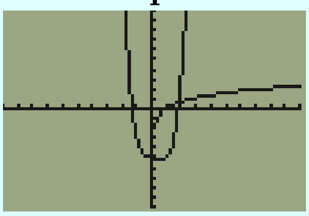
It looks though like there might be another intersection point near $x = 0$. And so, repeating the above steps but now moving the cursor left (instructions and screenshots omitted), you should find another intersection point at $(x, y) \approx (0.00674, -5)$.

Thus, the solutions are $(1.87\ldots, 0.623\ldots)$ and $(0.00674\ldots, -5)$.

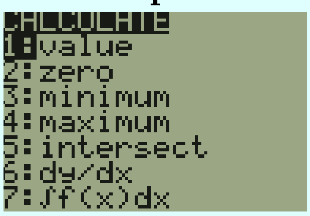
Step 1.



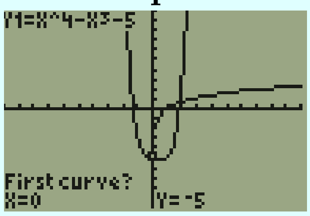
Step 2.



Step 3.



Step 4.



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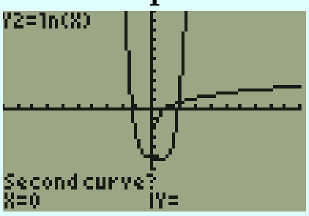
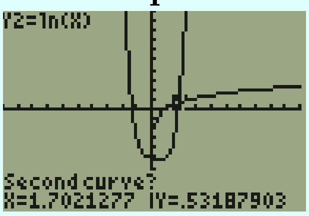
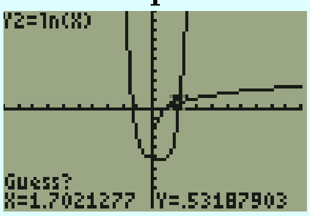


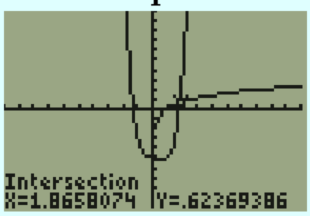
Step 6.



Step 7.



Step 8.



(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Method 2. Combine the two equations into the single equation $y = x^4 - x^3 - 5 - \ln x$. Graph it, then find the x-intercepts. Brief instructions:

1. Graph $y = x^4 - x^3 - 5 - \ln x$.

   It looks like there's an x-intercept near 2. So,

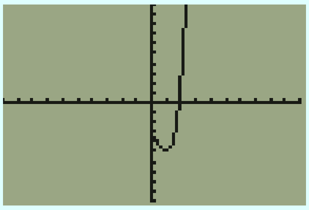
2. Use the "zero" function to find that there's indeed an x-intercept at $x \approx 1.87$.

   It's not obvious, but it looks like there might be another x-intercept near the origin. So,

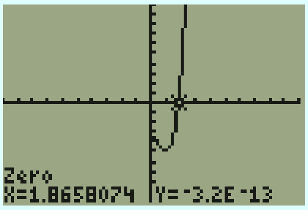
3. Use the "zero" function to find that there's indeed another x-intercept at $x \approx 0.00674$.

With Method 2, we must plug in these two values of x into either of the original two equations to get the corresponding values of y. Then as before, we'll conclude that the solutions are $(1.87\ldots, 0.623\ldots)$ and $(0.00674\ldots, -5)$.

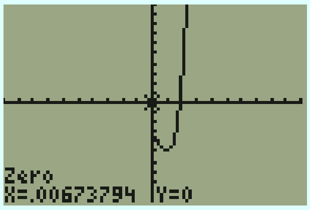
Step 1.



Step 2.



Step 3.



Exercise 169. Use your GC to solve each system of equations.

(a) $y = \dfrac{1}{1+\sqrt{x}}$, $\quad y = x^5 - x^3 + 2 \quad (x, y \in \mathbb{R},\ x \geq 0)$.

(b) $y = \dfrac{1}{1-x^2}$, $\quad y = x^3 + \sin x \quad (x, y \in \mathbb{R},\ x \neq \pm 1)$.

(c) $x^2 + y^2 = 1$, $\quad y = \sin x \quad (x, y \in \mathbb{R})$. (Answers on pp. 1808–1809.)

40. Partial Fractions Decomposition

This is yet another topic you're supposed to have mastered in secondary school.

As per p. 14 of your A-Level syllabus, you need only know how to decompose "partial fractions with cases where the denominator is no more complicated than

- $(ax+b)(cx+d)$
- $(ax+b)(cx+d)^2$
- $(ax+b)\left(x^2+c^2\right)$".

The lovely folks at MOE have kindly included the following on your List MF26 (p. 2):

Partial fractions decomposition

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$

Tip for *Towkays*

Partial fractions aren't difficult to do—it's just a bunch of tedious algebra. So the important thing is to **go slowly** and be really careful. Check and double-check that you've got everything **exactly correct** at each step of the way. This will save you time and marks, as compared to trying to do the algebra quickly and making a mistake.

Let's go through these three "categories" of partial fractions decomposition in turn:

Example 565. Rewrite or **decompose** the expression $\frac{1}{x^2+3x+2}$ into **partial fractions**.

Observe that $x^2+3x+2=(x+1)(x+2)$. So, write

$$\begin{aligned}\frac{1}{x^2+3x+2}&=\frac{1}{(x+1)(x+2)}\qquad\qquad=\frac{A}{x+1}+\frac{B}{x+2}\\&=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}=\frac{(A+B)x+2A+B}{(x+1)(x+2)}.\end{aligned}$$

From the numerator, $1=(A+B)x+2A+B$. Comparing coefficients, $A+B\overset{1}{=}0$ and $2A+B\overset{2}{=}1$. Taking $\overset{2}{=}-\overset{1}{=}$ yields $A=1$ and then $B=-1$. Hence,

$$\frac{1}{x^2+3x+2}=\frac{1}{x+1}-\frac{1}{x+2}.$$

Example 566. To decompose $\frac{5x-3}{x^2+3x+2}$, observe $x^2+3x+2=(x+1)(x+2)$ and write

$$\frac{5x-3}{x^2+3x+2}=\frac{A}{x+1}+\frac{B}{x+2}=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}=\frac{(A+B)x+2A+B}{(x+1)(x+2)}.$$

Comparing coefficients, $A+B=5$ and $2A+B=-3$. So, $A=-8$ and $B=13$. Hence,

$$\frac{5x-3}{x^2+3x+2}=\frac{-8}{x+1}+\frac{13}{x+2}.$$

Example 567. To decompose $\frac{9x-5}{-x^2+5x-6}$, observe $-x^2+5x-6=-(x-3)(x-2)$ and write

$$\frac{9x-5}{-x^2+5x-6}=\frac{A}{-(x-3)}+\frac{B}{x-2}=\frac{A(x-2)-B(x-3)}{-(x-3)(x-2)}=\frac{(A-B)x-2A+3B}{-(x-3)(x-2)}.$$

Comparing coefficients, $A-B=9$ and $-2A+3B=-5$. So, $A=22$ and $B=13$. Hence,

$$\frac{9x-5}{-x^2+5x-6}=\frac{22}{-(x-3)}+\frac{13}{x-2}=\frac{22}{3-x}+\frac{13}{x-2}.$$

Exercise 170. Decompose each expression into partial fractions. (Answer on p. 1810.)

(a) $\frac{8}{x^2+x-6}$; **(b)** $\frac{17x-5}{3x^2-8x-3}$.

Example 568. To decompose $\dfrac{x^2+x+1}{(x+1)(x-1)^2}$, write

$$\frac{x^2+x+1}{(x+1)(x-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}=\frac{A(x-1)^2+B(x+1)(x-1)+C(x+1)}{(x+1)(x-1)^2}$$
$$=\frac{A(x^2-2x+1)+B(x^2-1)+C(x+1)}{(x+1)(x-1)^2}=\frac{(A+B)x^2+(-2A+C)x+A-B+C}{(x+1)(x-1)^2}.$$

Comparing coefficients, $A+B=1$, $-2A+C=1$, $A-B+C=1$. Summing these three equations, we get $2C=3$ or $C=3/2$ and then also $A=1/4$ and $B=3/4$.

So,
$$\frac{x^2+x+1}{(x+1)(x-1)^2}=\frac{1}{4(x+1)}+\frac{3}{4(x-1)}+\frac{3}{2(x-1)^2}.$$

Example 569. To decompose $\dfrac{3x^2-x+1}{(4x-1)(x+2)^2}$, write

$$\frac{3x^2-x+1}{(4x-1)(x+2)^2}=\frac{A}{4x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$
$$=\frac{A(x+2)^2+B(4x-1)(x+2)+C(4x-1)}{(4x-1)(x+2)^2}$$
$$=\frac{A(x^2+4x+4)+B(4x^2+7x-2)+C(4x-1)}{(4x-1)(x+2)^2}$$
$$=\frac{(A+4B)x^2+(4A+7B+4C)x+4A-2B-C}{(4x-1)(x+2)^2}.$$

Comparing coefficients, $A+4B\overset{1}{=}3$, $4A+7B+4C\overset{2}{=}-1$, and $4A-2B-C\overset{3}{=}1$.

Rearrange $\overset{1}{=}$ to get $A\overset{4}{=}3-4B$. Next, $\overset{2}{=}$ minus $\overset{3}{=}$ yields $9B+5C=-2$ or $C\overset{5}{=}-(2+9B)/5$. Plug $\overset{4}{=}$ and $\overset{5}{=}$ into $\overset{3}{=}$ to get $4(3-4B)-2B+(2+9B)/5=1$ or $B=19/27$ and then also $A=5/27$ and $C=-5/3$.

So,
$$\frac{3x^2-x+1}{(4x-1)(x+2)^2}=\frac{5}{27(4x-1)}+\frac{19}{27(x+2)}-\frac{5}{3(x+2)^2}.$$

Exercise 171. Decompose $\dfrac{2x^2-x+7}{x^3-x^2-x+1}$ into partial fractions. (Answer on p. 1810.)

Example 570. To decompose $\dfrac{2x^2+x+1}{(5x-1)(x^2+1)}$, write

$$\frac{2x^2+x+1}{(5x-1)(x^2+1)}=\frac{A}{5x-1}+\frac{Bx+C}{x^2+1}=\frac{A(x^2+1)+(Bx+C)(5x-1)}{(5x-1)(x^2+1)}$$
$$=\frac{Ax^2+A+5Bx^2+(5C-B)x-C}{(5x-1)(x^2+1)}=\frac{(A+5B)x^2+(5C-B)x+A-C}{(5x-1)(x^2+1)}.$$

Comparing coefficients, $A+5B\overset{1}{=}2$, $5C-B\overset{2}{=}1$, and $A-C\overset{3}{=}1$.

Rearrange $\overset{1}{=}$ to get $A\overset{4}{=}2-5B$. Rearrange $\overset{2}{=}$ to get have $C\overset{5}{=}(B+1)/5$. Plug $\overset{4}{=}$ and $\overset{5}{=}$ into $\overset{3}{=}$ to get $2-5B-(B+1)/5=1$ or $B=2/13$ and then also $A=16/13$ and $C=3/13$.

So, $$\frac{2x^2+x+1}{(5x-1)(x^2+1)}=\frac{16}{13(5x-1)}+\frac{2x+3}{13(x^2+1)}.$$

Example 571. To decompose $\dfrac{4x^2-3x+2}{(2x+7)(x^2+9)}$, write

$$\frac{4x^2-3x+2}{(2x+7)(x^2+9)}=\frac{A}{2x+7}+\frac{Bx+C}{x^2+9}=\frac{A(x^2+9)+(Bx+C)(2x+7)}{(2x+7)(x^2+9)}$$
$$=\frac{Ax^2+9A+2Bx^2+(7B+2C)x+7C}{(2x+7)(x^2+9)}=\frac{(A+2B)x^2+(7B+2C)x+9A+7C}{(2x+7)(x^2+9)}.$$

Comparing coefficients, $A+2B\overset{1}{=}4$, $7B+2C\overset{2}{=}-3$, and $9A+7C\overset{3}{=}2$.

Rearrange $\overset{1}{=}$ to get $A\overset{4}{=}4-2B$. Rearrange $\overset{2}{=}$ to get $C\overset{5}{=}-(7B+3)/2$. Plug $\overset{4}{=}$ and $\overset{5}{=}$ into $\overset{3}{=}$ to get $9(4-2B)+7(-7B+3)/2=2$ or $B=47/85$ and then also $A=246/85$ and $C=-292/85$.

So, $$\frac{4x^2-3x+2}{(2x+7)(x^2+9)}=\frac{246}{85(2x+7)}+\frac{47x-292}{85(x^2+9)}.$$

Exercise 172. Decompose $\dfrac{-3x^2+5}{x^3-2x^2+4x-8}$ into partial fractions. (Answer on p. 1810.)

41. Solving Inequalities

41.1. Solutions and Solution Sets

Recall (Ch. 13) that to **solve an inequality** is to find *all* its **solutions** (or **roots**). Equivalently, to solve an inequality is to find its **solution set**. An inequality usually has infinitely many solutions:

Example 572. Solve $x - 1 \overset{1}{>} 0$ ($x \in \mathbb{R}$).

Three perfectly good (and equivalent) solutions to $\overset{1}{>}$:

- "$x > 1$."
- "$x \in (1, \infty)$."
- "The solution set of $\overset{1}{>}$ is $(1, \infty)$."

Example 573. Solve $x + 5 \overset{1}{\geq} 8$ ($x \in \mathbb{R}$).

Three perfectly good (and equivalent) solutions to $\overset{1}{\geq}$:

- "$x \geq 3$."
- "$x \in [3, \infty)$."
- "The solution set of $\overset{1}{\geq}$ is $[3, \infty)$."

Example 574. Solve $(x-1)(x-2) \stackrel{1}{>} 0$.

Since LHS is the product of two terms, we can use Fact 14 to make these five observations:

1. If $x < 1$, then $x-1<0$ and $x-2<0$, so that $(x-1)(x-2)>0$.

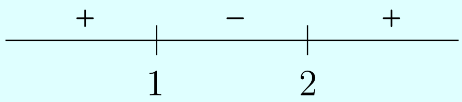
2. If $x = 1$, then $x-1=0$, so that $(x-1)(x-2)=0$.

3. If $1 < x < 2$, then $x-1>0$ and $x-2<0$, so that $(x-1)(x-2)<0$.

4. If $x = 2$, then $x-2=0$, so that $(x-1)(x-2)=0$.

5. If $x > 2$, then $x-1>0$ and $x-2>0$, so that $(x-1)(x-2)>0$.

The above observations can be concisely summarised in a single **sign diagram**:



So, here are three perfectly good (and equivalent) solutions to $\stackrel{1}{>}$:

- "$x<1$ OR $x>2$."
- "$x \in (-\infty, 1) \cup (2, \infty)$."
- "The solution set is $(-\infty, 1) \cup (2, \infty)$."

Example 575. Solve $\dfrac{x-3}{x-4} \overset{1}{>} 0$.

Again, LHS is the product of two terms: $\dfrac{x-3}{x-4} = (x-3)\dfrac{1}{x-4}$. So,

1. If $x < 3$, then $x-3<0$, $x-4<0$, and $\dfrac{1}{x-4}<0$, so that $\dfrac{x-3}{x-4}>0$.

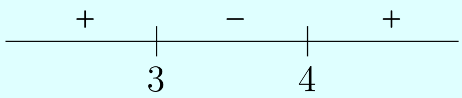
2. If $x = 3$, then $x-3=0$, so that $\dfrac{x-3}{x-4}=0$.

3. If $3<x<4$, then $x-3>0$, $x-4<0$, and $\dfrac{1}{x-4}<0$, so that $\dfrac{x-3}{x-4}<0$.

4. If $x > 4$, then $x-3>0$, $x-4>0$, and $\dfrac{1}{x-4}>0$, so that $\dfrac{x-3}{x-4}>0$.

(Note: If $x=4$, then $\overset{1}{>}$ has an undefined LHS and so can't possibly hold.)

Again, we can summarise the above observations in a single sign diagram:

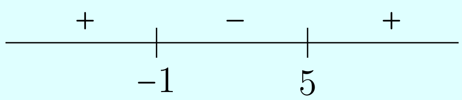


So, here are three perfectly good (and equivalent) solutions to $\overset{1}{>}$:

- "$x<3$ OR $x>4$."
- "$x \in (-\infty,3) \cup (4,\infty)$."
- "The solution set of $\overset{1}{>}$ is $(-\infty,3) \cup (4,\infty)$."

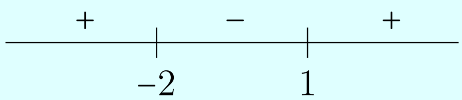
Now that we understand how sign diagrams work, we'll skip the step of explicitly listing the observations and instead jump straight to drawing the sign diagram:

Example 576. Solve $(x+1)(x-5)>0$. Sign diagram:



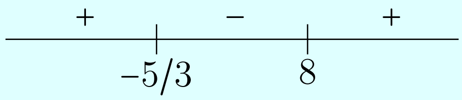
Solution: $x \in (-\infty,-1) \cup (5,\infty)$.

Example 577. Solve $(x-1)(x+2) \le 0$. Sign diagram:



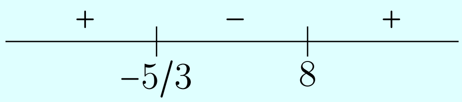
Solution: $x \in [-2,1]$.

Example 578. Solve $(3x+5)(x-8) \geq 0$. Sign diagram:



Solution: $x \in (-\infty, -5/3] \cup [8, \infty)$.

Example 579. Solve $\frac{3x+5}{x-8} \overset{1}{\geq} 0$. Sign diagram:



Solution: $x \in (-\infty, -5/3] \cup (8, \infty)$. (How and why does the solution here differ from the last example?)[273]

Exercise 173. Solve each inequality. (Answer on p. 1812.)

(a) $\frac{2x+3}{-x+7} < 9$. **(b)** $\frac{-4x+2}{x+1} > 13$.

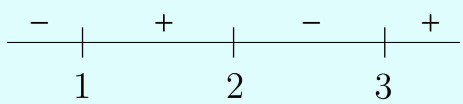
Above we looked at inequalities involving the product of two terms. Let's now look at inequalities involving the product of three or more terms:

Example 580. Solve $(x-1)(x-2)(x-3) > 0$.

We use Fact 14 to make these seven observations:

1. If $x < 1$, then $x-1 < 0$, $x-2 < 0$, and $x-3 < 0$, so that $(x-1)(x-2)(x-3) < 0$.
2. If $x = 1$, then $x-1 = 0$, so that $(x-1)(x-2)(x-3) = 0$.
3. If $1 < x < 2$, then $x-1 > 0$ $x-2 < 0$, and $x-3 < 0$, so that $(x-1)(x-2)(x-3) > 0$.
4. If $x = 2$, then $x-2 = 0$, so that $(x-1)(x-2)(x-3) = 0$.
5. If $2 < x < 3$, then $x-1 > 0$ $x-2 > 0$, and $x-3 < 0$, so that $(x-1)(x-2)(x-3) < 0$.
6. If $x = 3$, then $x-3 = 0$, so that $(x-1)(x-2)(x-3) = 0$.
7. If $x > 3$, then $x-1 > 0$ $x-2 > 0$, and $x-3 > 0$, so that $(x-1)(x-2)(x-3) > 0$.

The above observations can be summarised in a single **sign diagram**:

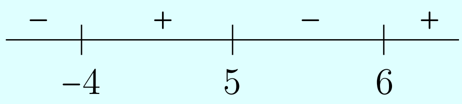


Solution: $x \in (1, 2) \cup (3, \infty)$.

Again, in the next few examples, we'll omit explicitly listing out the observations and instead jump straight to drawing the sign diagram. You should nonetheless figure out for yourself how the sign diagram is constructed:

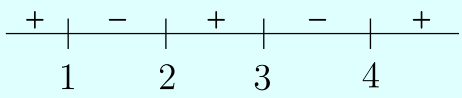
[273] In Example 579, 8 is excluded as a solution to the inequality. This is because if $x = 8$, then LHS of the inequality is undefined so that the inequality cannot hold. So, 8 cannot be a solution to the inequality.

Example 581. Solve $(x+4)(x-5)(x-6) \geq 0$. Sign diagram:



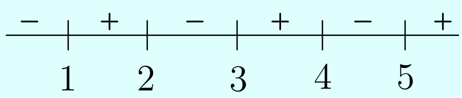
Solution: $x \in [-4,5] \cup [6,\infty)$.

Example 582. Solve $(x-1)(x-2)(x-3)(x-4) \leq 0$. Sign diagram:



Solution: $x \in [1,2] \cup [3,4]$.

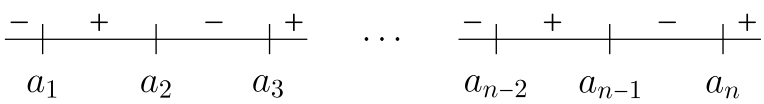
Example 583. Solve $(x-1)(x-2)(x-3)(x-4)(x-5) > 0$. Sign diagram:



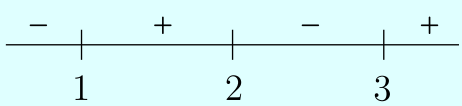
Solution: $x \in (1,2) \cup (3,4) \cup (5,\infty)$.

More generally, consider the expression $(x-a_1)(x-a_2)\ldots(x-a_n)$, where $a_1 < a_2 < \cdots < a_n$. If n is **even**, then the sign diagram is this:

If n is **odd**, then the sign diagram is this:

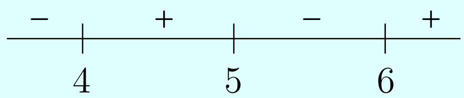


Example 584. Solve $\dfrac{(x-1)(x-2)}{(x-3)} > 0$. Sign diagram:



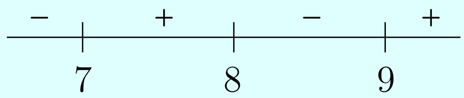
Solution: $x \in (1,2) \cup (3,\infty)$.

Example 585. Solve $\dfrac{(x-4)(x-5)}{(x-6)} \geq 0$. Sign diagram:



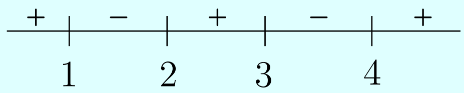
Solution: $x \in [4,5] \cup (6,\infty)$. (Note that 6 is excluded.)

Example 586. Solve $\dfrac{(x-7)(x-8)}{(x-9)} \leq 0$. Sign diagram:



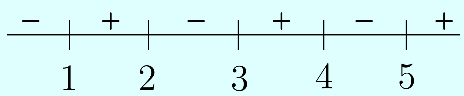
Solution: $x \in (-\infty,7] \cup [8,9)$. (Note that 9 is excluded.)

Example 587. Solve $\dfrac{(x-1)(x-2)}{(x-3)(x-4)} \geq 0$. Sign diagram:



Solution: $x \in (-\infty,1] \cup [2,3) \cup (4,\infty)$. (Note that 3 and 4 are excluded.)

Example 588. Solve $\dfrac{(x-1)(x-3)}{(x-2)(x-4)(x-5)} \leq 0$. Sign diagram:

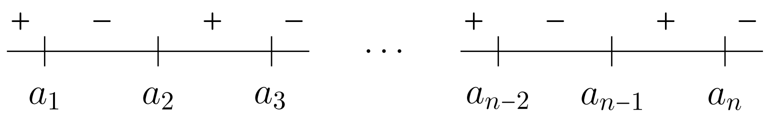


Solution: $x \in (-\infty,1] \cup (2,3] \cup (4,5)$. (Note that 2, 4, and 5 are excluded.)

More generally, consider the expression $\dfrac{(x-b_1)(x-b_2)\ldots(x-b_m)}{(x-b_{m+1})(x-b_{m+2})\ldots(x-b_n)}$, where b_1, b_2, ..., b_n are distinct.

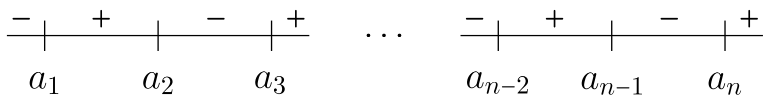
Rearrange b_1, b_2, ..., b_n in ascending order as $a_1 < a_2 < \cdots < a_n$.[274]

If n is **even**, then the sign diagram is this:



If n is **odd**, then the sign diagram is this:

[274]That is, $\{b_1, b_2, \ldots, b_n\} = \{a_1, a_2, \ldots, a_n\}$.



Exercise 174. Solve each inequality. (Answer on p. 448.)

(a) XXX

A174. XXX

41.3. Quadratic Inequalities

Let $a \neq 0$ and $b, c \in \mathbb{R}$. Recall that we call

- The expression $ax^2 + bx + c$ a **quadratic polynomial**;
- Any equation that can be rewritten as $ax^2 + bx + c = 0$ a **quadratic equation**.

We shall now also call any inequality that can be rewritten as

$$ax^2 + bx + c \overset{1}{>} 0 \qquad \text{or} \qquad ax^2 + bx + c \overset{2}{\geq} 0$$

a **quadratic inequality**.

How do we solve quadratic inequalities?

It turns out that we already learnt to do so in the previous subchapter, for those cases where $b^2 - 4ac > 0$ (positive discriminant). In such cases, the corresponding quadratic equation has two distinct roots and we may write $ax^2 + bx + c$ as the product of two distinct linear terms:

Example 589. Solve $x^2 - 3x + 2 > 0$.

Observe that $x^2 - 3x + 2 = (x-1)(x-2)$. So, as usual, by considering the five possible cases ($x < 1$, $x = 1$, $1 < x < 2$, $x = 2$, and $x > 2$), we construct this sign diagram:

Figure to be
inserted here.

Solution: $x \in (-\infty, 1) \cup (2, \infty)$.

There's actually another way to "see" where the above sign diagram comes from. Observe that in $x^2 - 3x + 2$, the coefficient on x^2 is positive. So, the graph of $y = x^2 - 3x + 2$ is $\cup$-shaped. Hence, the given inequality holds on the left of the smaller root and on the right of the larger root.

Example 590. Solve $x^2 - 2x + 3 \leq 0$.

Since $x^2 - 2x - 3 = (x+1)(x-3)$, we have this sign diagram:

Figure to be
inserted here.

Solution: $x \in [-1, 3]$.

Example 591. Solve $-x^2 - 7x + 4 \geq 0$.

Since $-x^2 - 7x + 4 = -(x+4)(2x-1)$, we have this sign diagram:

Figure to be
inserted here.

Solution: $x \in [-4, 1/2]$.

Example 592. Solve $5x^2 + 2x - 3 \geq 0$.

Since $5x^2 + 2x - 3 = (x+1)(5x-3)$, we have this sign diagram:

Figure to be
inserted here.

Solution: $x \in (-\infty, -1] \cup [3/5\infty)$.

If $b^2 - 4ac < 0$, then the graph of $y = ax^2 + bx + c$ does not intersect the x-axis at all:

Example 593. Solve $x^2 + x + 1 > 0$.

Figure to be inserted here.

The discriminant is negative $(1^2 - 4 \cdot 1 \cdot 1 = -3 < 0)$. So, the graph of $y = ax^2 + bx + c$ does not intersect the x-axis at all.

The coefficient on x^2 is positive. So, the graph is $\cup$-shaped and is entirely **above** the x-axis. Equivalently, $x^2 + x + 1 > 0$ for all x.

Solution: $x \in \mathbb{R}$.

Example 594. Solve $3x^2 - 2x + 1 \leq 0$.

Figure to be inserted here.

The discriminant is negative $((-2)^2 - 4 \cdot 3 \cdot 1 = -8 < 0)$. So, the graph of $y = ax^2 + bx + c$ does not intersect the x-axis at all.

The coefficient on x^2 is positive. So, the graph is $\cup$-shaped and is entirely **above** the x-axis. Equivalently, $x^2 + x + 1 > 0$ for all x.

Solution: No real number x satisfies the given inequality.[275]

If $b^2 - 4ac = 0$, then $y = ax^2 + bx + c$ intersects the x-axis exactly once (at its turning point):

Example 595. Solve $x^2 - 2x + 1 > 0$.

The discriminant is zero $((-2)^2 - 4 \cdot 1 \cdot 1 = 0)$. So, the graph of $y = ax^2 + bx + c$ intersects the x-axis exactly once.

Indeed, we have $x^2 - 2x + 1 = (x-1)^2$. So, the graph intersects the x-axis at $x = 1$.

The coefficient on x^2 is positive. So, the graph is $\cup$-shaped and is entirely **above** the x-axis except at $x = 1$ (where it touches the x-axis).

Solution: $x \in \mathbb{R} \setminus \{1\}$.

Example 596. Solve $4x^2 + 4x + 1 \geq 0$.

The discriminant is zero ($2^2 - 4 \cdot 1 \cdot 1 = 0$). So, the graph of $y = ax^2 + bx + c$ intersects the x-axis exactly once.

Indeed, we have $4x^2 + 4x + 1 = (2x - 1)^2$. So, the graph intersects the x-axis at $x = 1/2$.

The coefficient on x^2 is positive. So, the graph is $\cup$-shaped and is entirely **above** the x-axis except at $x = 1/2$ (where it touches the x-axis).

Solution: $x \in \mathbb{R}$.

Example 597. Solve $-x^2 + 6x - 9 \geq 0$.

The discriminant is zero ($6^2 - 4 \cdot (-1) \cdot (-9) = 0$). So, the graph of $y = ax^2 + bx + c$ intersects the x-axis exactly once.

Indeed, we have $-x^2 + 6x - 9 = -(x-3)^2$. So, the graph intersects the x-axis at $x = 3$.

The coefficient on x^2 is negative. So, the graph is $\cap$-shaped and is entirely **below** the x-axis except at $x = 3$ (where it touches the x-axis).

Solution: $x = 3$.[276] (This is this textbook's first example of an inequality that has exactly one real solution.)

For future reference, we provide the following result, which summarises our above findings and gives the general solution for the quadratic inequalities. (Do not try to mug this result! Instead, seek to understand how we solved the quadratic inequalities in the above examples. And of course, do the exercises below.)

Fact 103. *Let $a, b, c, x \in \mathbb{R}$ with $a \neq 0$. Consider the inequalities*

$$ax^2 + bx + c \overset{1}{>} 0 \qquad \text{and} \qquad ax^2 + bx + c \overset{2}{\geq} 0.$$

Let $r_1 = \dfrac{-b - \sqrt{b^2 - 4ac}}{a}$ *and* $r_2 = \dfrac{-b + \sqrt{b^2 - 4ac}}{a}$.

(a) *Suppose $a > 0$.*

- **(i)** *If $b^2 - 4ac > 0$, then the solution set for $\overset{1}{>}$ is $\mathbb{R} \setminus [r_1, r_2]$ and that for $\overset{2}{\geq}$ is $\mathbb{R} \setminus (r_1, r_2)$.*
- **(ii)** *If $b^2 - 4ac = 0$, then the solution set for $\overset{1}{>}$ is $\mathbb{R} \setminus \{-b/2a\}$ and that for $\overset{2}{\geq}$ is $\mathbb{R}$.*
- **(iii)** *If $b^2 - 4ac < 0$, then the solution set for both $\overset{1}{>}$ and $\overset{2}{\geq}$ is $\mathbb{R}$.*

(b) *Suppose $a < 0$.*

- **(i)** *If $b^2 - 4ac > 0$, then the solution set for $\overset{1}{>}$ is (r_1, r_2) and that for $\overset{2}{\geq}$ is $[r_1, r_2]$.*
- **(ii)** *If $b^2 - 4ac = 0$, then the solution set for $\overset{1}{>}$ is $\varnothing$ and that for $\overset{2}{\geq}$ is $\{-b/2a\}$.*
- **(iii)** *If $b^2 - 4ac < 0$, then the solution set for both $\overset{1}{>}$ and $\overset{2}{\geq}$ is $\varnothing$.*

Exercise 175. Solve each inequality. (Answer on p. 1813.)

(a) $-3x^2 + x - 5 > 0$. **(b)** $x^2 - 2x - 1 > 0$.

Example 598. Solve $x\mathrm{e}^x \overset{1}{\geq} x\mathrm{e}$ ($x \in \mathbb{R}$).

Beng reasons, "Divide $\overset{1}{\geq}$ by x to get $\mathrm{e}^x \geq \mathrm{e}$. Which holds for all $x \geq 1$. So, solution: $x \in [1, \infty)$." ✗

Beng's mistake is to divide $\overset{1}{\geq}$ by x and assume the inequality will be preserved.

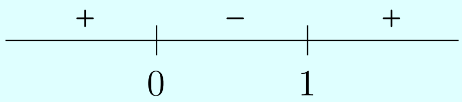
Correct solution: $x\mathrm{e}^x \overset{1}{\geq} x\mathrm{e} \iff x\mathrm{e}^x - x\mathrm{e} \geq 0 \iff x\mathrm{e}\left(\mathrm{e}^{x-1} - 1\right) \geq 0 \iff x\left(\mathrm{e}^{x-1} - 1\right) \geq 0$.

In the last step, we multiplied both sides by the positive constant $1/\mathrm{e}$—by Fact 15, the inequality is preserved.

Observe that

- $\mathrm{e}^{x-1} - 1 = 0 \iff \mathrm{e}^{x-1} = 1 \iff x = 1$;
- $\mathrm{e}^{x-1} - 1 > 0 \iff \mathrm{e}^{x-1} > 1 \iff x > 1$;
- $\mathrm{e}^{x-1} - 1 < 0 \iff \mathrm{e}^{x-1} < 1 \iff x < 1$.

So, sign diagram for $x\left(\mathrm{e}^{x-1} - 1\right)$:



Solution: $x \in (-\infty, 0] \cup [1, \infty)$.

41.5. Inequalities Involving the Absolute Value Function

Fact 104. *Suppose* $x \in \mathbb{R}$ *and* $b \geq 0$. *Then*

(a) $|x| < b \iff -b < x < b$.
(b) $|x| \leq b \iff -b \leq x \leq b$.
(c) $|x| > b \iff x < -b$ OR $x > b$.
(d) $|x| \geq b \iff x \leq -b$ OR $x \geq b$.

Proof. If $x \geq 0$, then $|x| = x$. If $x < 0$, then $|x| = -x$.

(a) Suppose $x \geq 0$. Then $x > -b$. Also, $|x| < b \iff x < b \iff -b < x < b$.[277]

Suppose $x < 0$. Then $x < b$. Also, $|x| < b \iff -x < b \iff x > -b \iff -b < x < b$.

(b) Similar, omitted.

(c) $|x| < b \iff \text{NOT-}(|x| \leq b) \overset{\textbf{(a)}}{\iff} \text{NOT-}(-b \leq x \leq b) \iff (x < -b \text{ OR } x > b)$

(d) Similar, omitted. □

Example 599. $|x| < 5 \iff -5 < x < 5$

Example 600. $|x| \leq 3 \iff -3 \leq x \leq 3$

Example 601. $|x| > 7 \iff (x > 7 \text{ OR } x < -7)$

Example 602. $|x| \geq 1 \iff (x \geq 1 \text{ OR } x \leq -1)$

Here's a more general version of the above result:

Fact 105. *Suppose* $a, x \in \mathbb{R}$, $b \geq 0$. *Then*

(a) $|x - a| < b \iff a - b < x < a + b$.
(b) $|x - a| \leq b \iff a - b \leq x \leq a + b$.
(c) $|x - a| > b \iff (x > a + b \text{ OR } x < a - b)$.
(d) $|x - a| \geq b \iff (x \geq a + b \text{ OR } x \leq a - b)$.

Proof. **(a)** By Fact 104(a), $|x - a| < b \iff -b < x - a < b \iff a - b < x < a + b$.

(b) Similar, omitted.

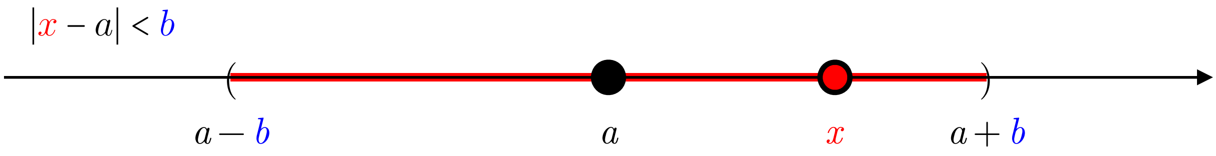
(c) By Fact 104(c), $|x - a| > b \iff (x - a > b \text{ OR } x - a < -b) \iff (x > a + b \text{ OR } x < a - b)$.

(d) Similar, omitted. □

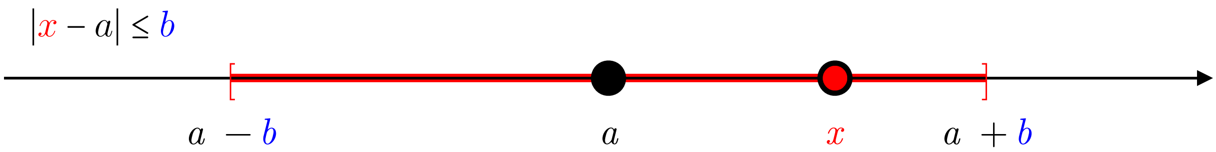
Fact 105 has a nice geometric interpretation:

(a) $|x - a| < b$ means that the distance between x and a is less than b.

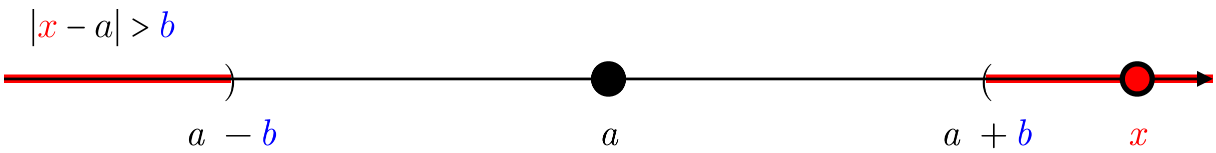
[277] The last $\iff$ follows because any statement (here $x < b$) is equivalent to the conjunction of that statement with any true statement (here $x < b$ AND $x > -b$).



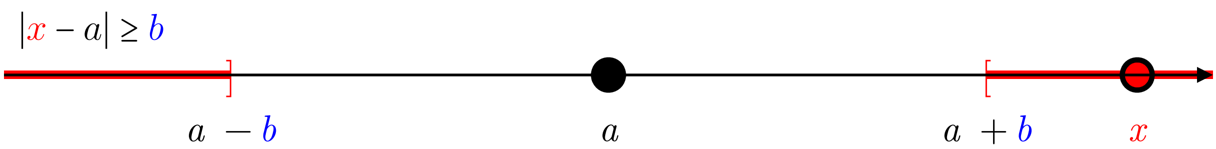
(b) $|x-a| \le b$ means that the distance between x and a is no greater than b.



(c) $|x-a| > b$ means that the distance between x and a is more than b.



(d) $|x-a| \ge b$ means that the distance between x and a is no less than b.



Example 603. Solve $|x-1| \overset{1}{<} 5$.

$$|x-1| \overset{1}{<} 5 \qquad \iff \qquad -5 < x-1 < 5 \qquad \iff \qquad -4 < x < 6.$$

Solution: $x \in (-4, 6)$.

Example 604. Solve $|x+4| \overset{2}{\le} 3$.

$$|x+4| \overset{2}{\le} 3 \qquad \iff \qquad -3 \le x+4 \le 3 \qquad \iff \qquad -7 \le x \le -1.$$

Solution: $x \in [-1, -7]$.

Example 605. Solve $|x-2| \overset{3}{>} 7$.

$$|x-2| \overset{3}{>} 7 \quad \iff \quad x-2 > 7 \text{ OR } x-2 < -7 \quad \iff \quad x > 9 \text{ OR } x < -5.$$

Solution: $x \in (\infty, -5] \cup [9, \infty)$.

Example 606. Solve $|x+1| \overset{4}{\geq} 1$.

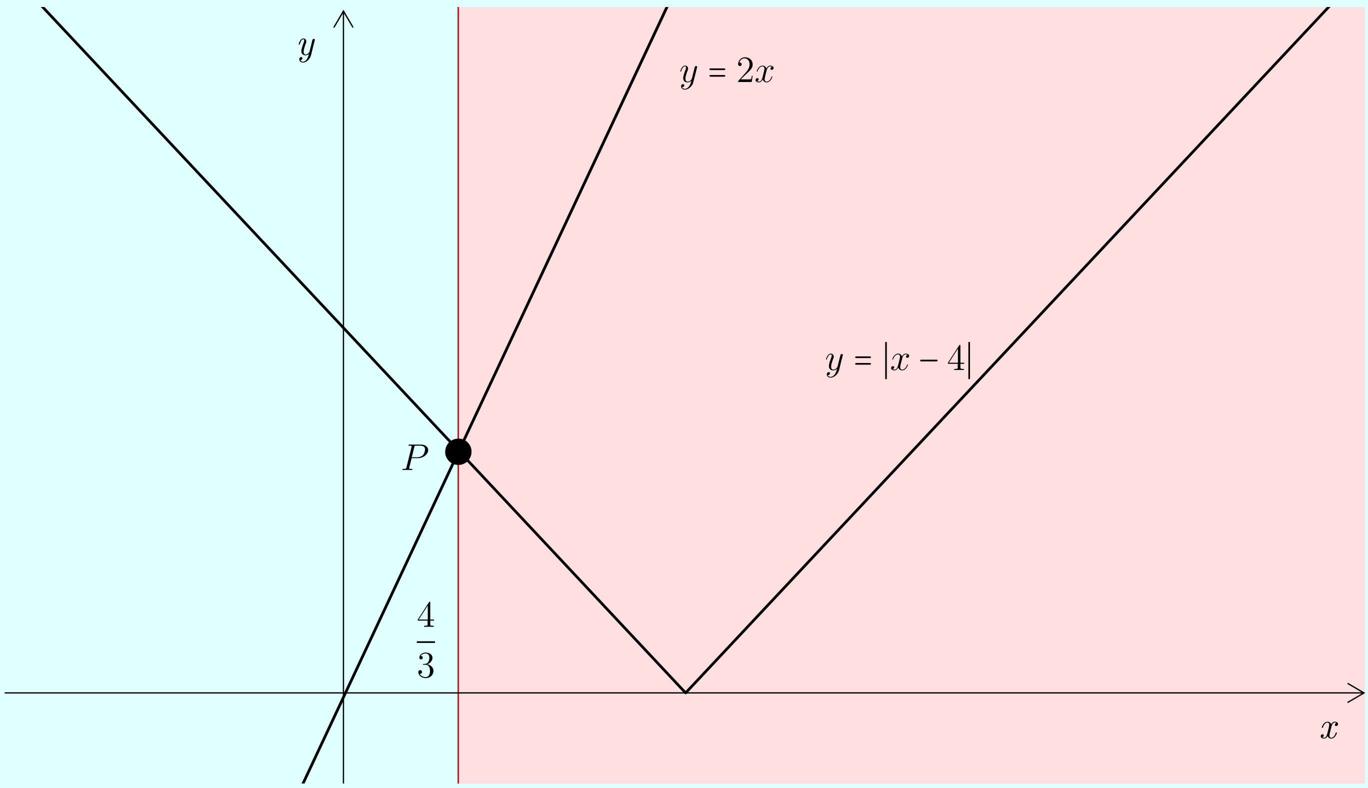
$$|x+1| \overset{4}{\geq} 1 \iff x+1 \geq 1 \text{ OR } x+1 \leq -1 \iff x \geq 0 \text{ OR } x \leq 0 \iff x \in \mathbb{R}.$$

Solution: $x \in \mathbb{R}$.

It will often be easier to solve inequalities with the aid of a graph:

Example 607. Solve $|x-4| \overset{1}{\leq} 2x$.

Let's first solve $\overset{1}{\leq}$ by graphing $y = |x-4|$ and $y = 2x$. For the former, recall (Ch. 26.6) that **(i)** above the x-axis, $y = |f(x)|$ coincides with f; but **(ii)** below, $y = |f(x)|$ is the reflection of f in the x-axis.



We "see" that the two graphs intersect at P. We also "see" that

$$\overset{1}{\leq} \text{ holds} \iff x \text{ is to the right of } P \text{ (inclusive).}$$

So, let's find P. We "see" that at P, $x - 4 < 0$. And so, at P,

$$|x-4| = 2x \iff 4 - x = 2x \iff x = 4/3.$$

Solution (to $\overset{1}{\leq}$): $x \geq 4/3$.

This should be good enough for the A-Levels. But others might consider it a little less than rigorous (due to the two "sees" above).

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

So, let's now solve $\overset{1}{\leq}$ again, but this time more rigorously (and without graphs).

We consider two possible (and mutually exhaustive) cases—$x - 4 < 0$ and $x - 4 \geq 0$:

Case 1. Suppose $x - 4 < 0$ or $x < 4$. Then

$$|x-4| \overset{1}{\leq} 2x \iff 4 - x \leq 2x \iff 4 \leq 3x \iff 4/3 \leq x.$$

So, in Case 1, $\overset{1}{\leq}$ holds $\iff$ ($x < 4$ AND $4/3 \leq x$) $\iff$ $x \in [4/3, 4)$.

Case 2. Suppose $x - 4 \geq 0$ or $x \overset{2}{\geq} 4$. Then

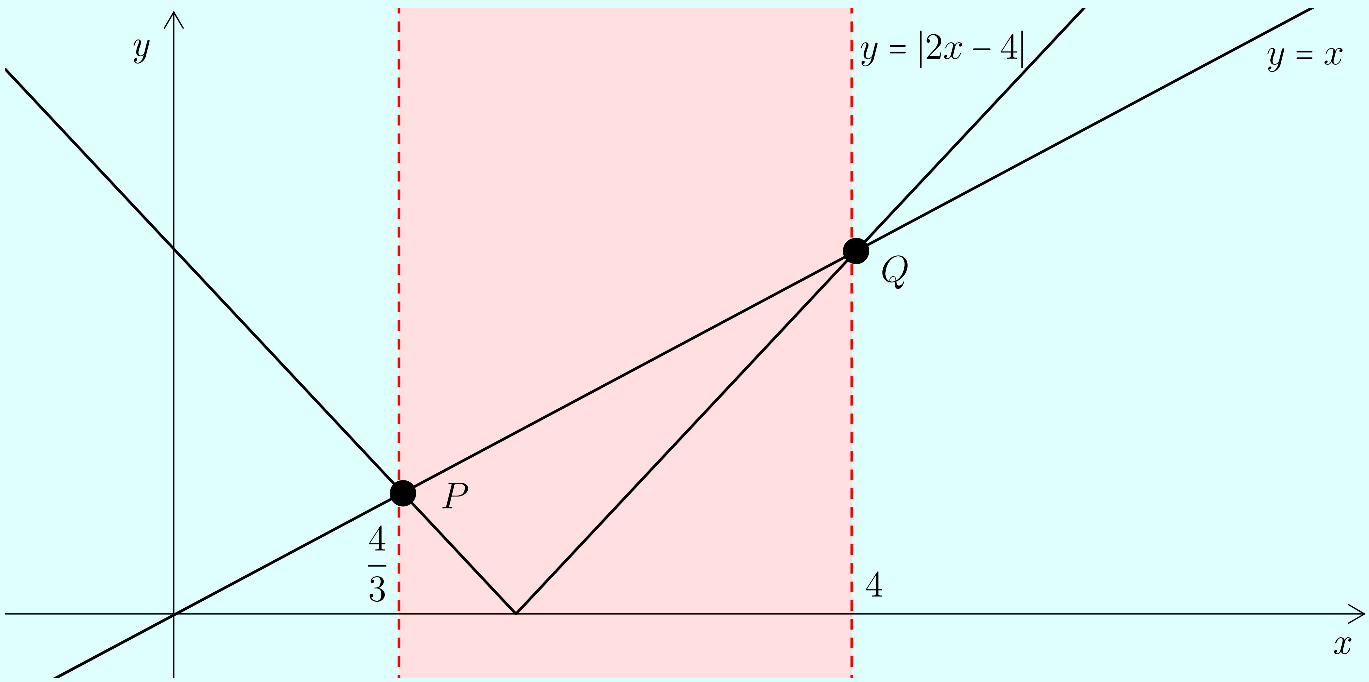
$$|x-4| \overset{1}{\leq} 2x \iff x - 4 \leq 2x \iff -4 \leq x,$$

which, of course, holds for all $x \overset{2}{\geq} 4$.

Altogether, $\overset{1}{\leq}$ holds $\iff$ $\left(x \in [4/3, 4) \text{ OR } x \overset{2}{\geq} 4\right)$ $\iff$ $x \geq 4/3$.

(Of course, this second solution is the same as the first. We'd be worried otherwise.)

Example 608. Solve $|2x - 4| \overset{1}{<} x$.



We "see" that the graphs of $y = |2x - 4|$ and $y = x$ intersect at P and Q. Also,

$$\overset{1}{<} \text{ holds} \iff x \text{ is between } P \text{ and } Q \text{ (exclusive)}.$$

So, let's find P and Q. At P, $2x - 4 < 0$. So, $|2x - 4| = -(2x - 4) = 4 - 2x$. Hence, P is given by $4 - 2x = x$ or $x = 4/3$.

At Q, $2x - 4 > 0$. So, $|2x - 4| = 2x - 4$. Hence, Q is given by $2x - 4 = x$ or $x = 4$.

Solution: $x \in (4/3, 4)$.

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Let's now solve $\overset{1}{<}$ again, but more rigorously (and without graphs).

Again, we look at the two possible (and mutually exclusive) cases—$2x-4<0$ and $2x-4\geq 0$:

Case 1. Suppose $2x-4<0$ or $x<2$. Then

$$|2x-4| \overset{1}{<} x \quad \iff \quad 4-2x<x \quad \iff \quad 4<3x \quad \iff \quad 4/3<x.$$

So, in Case 1, $\overset{1}{<}$ holds $\iff$ $(x<2 \text{ AND } 4/3<x)$ $\iff$ $x \in (4/3,2)$.

Case 2. Suppose $2x-4\geq 0$ or $x\geq 2$. Then

$$|2x-4| \overset{1}{<} x \quad \iff \quad 2x-4<x \quad \iff \quad x<4.$$

So, in Case 2, $\overset{1}{<}$ holds $\iff$ $(x\geq 2 \text{ AND } x<4)$ $\iff$ $x\in[2,4)$.

Altogether, $\overset{1}{<}$ holds $\iff$ $x\in(4/3,2)\cup[2,4)=(4/3,4)$.

Example 609. Solve $|3x-4| \overset{1}{\geq} 2x+2$.

We "see" that the graphs of $y=|3x-4|$ and $y=2x+2$ intersect at P and Q. Also,

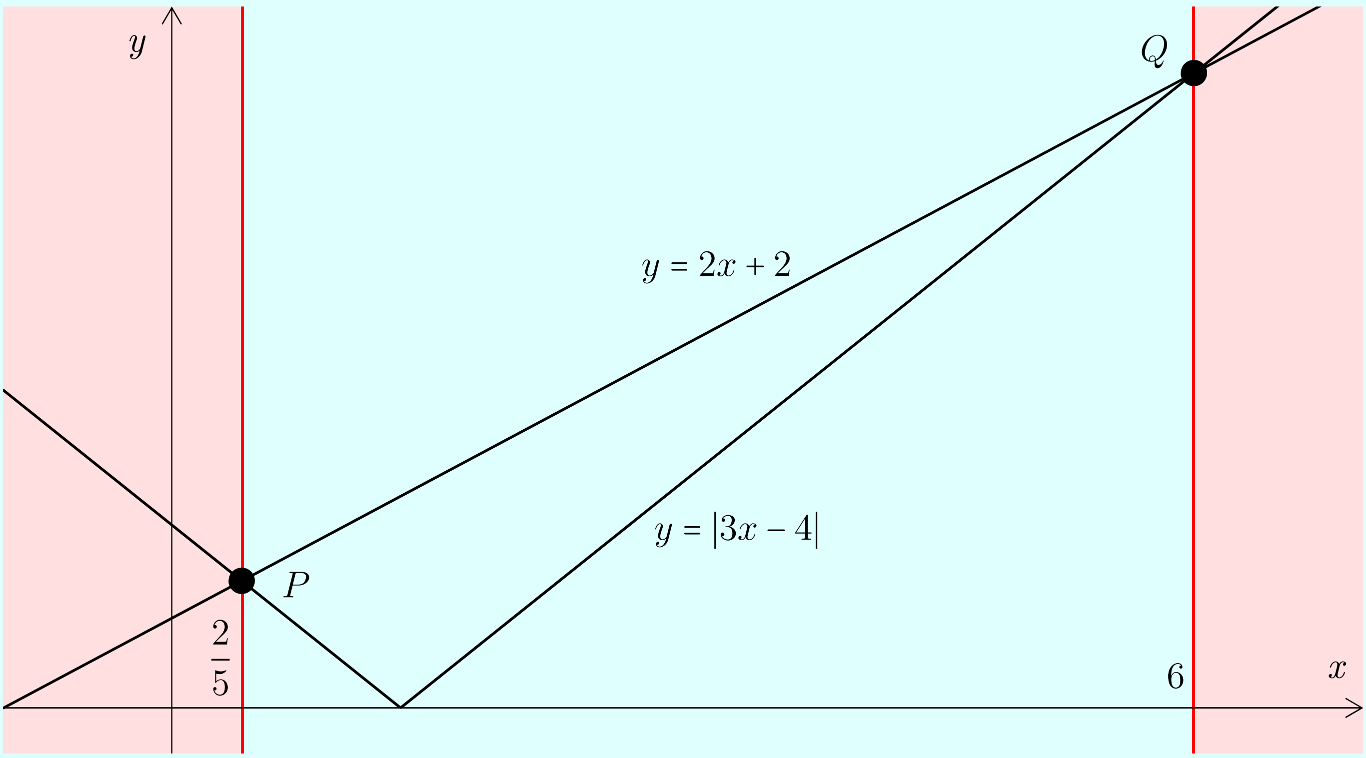
$$\overset{1}{\geq} \text{ holds} \quad \iff \quad x \text{ is to the left of } P \text{ or right of } Q \text{ (inclusive)}.$$

So, let's find P and Q. At P, $3x-4<0$. So, $|3x-4|=4-3x$. Hence, P is given by

$$4-3x=2x+2 \qquad \text{or} \qquad 2=5x \qquad \text{or} \qquad x=5/2.$$

At Q, $3x-4>0$. So, $|3x-4|=3x-4$. Hence, Q is given by $3x-4=2x+2$ or $x=6$.

Solution: $x\in\mathbb{R}\setminus[2/5,6]$.



(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Let's now solve $\overset{1}{\geq}$ again, but more rigorously (and without graphs).

Again, we look at the two possible (and mutually exclusive) cases—$3x-4<0$ and $3x-4 \geq 0$:

Case 1. Suppose $3x-4<0$ or $x<4/3$. Then

$$|3x-4| \overset{1}{\geq} 2x+2 \quad \iff \quad 4-3x \overset{1}{\geq} 2x+2 \quad \iff \quad 2 \leq 5x \quad \iff \quad 2/5 \leq x.$$

So, in Case 1, $\overset{1}{\geq}$ holds $\iff$ ($x<4/3$ AND $2/5 \leq x$) $\iff$ $x \in [2/5, 4/3)$.

Case 2. Suppose $3x-4 \geq 0$ or $x \geq 4/3$. Then

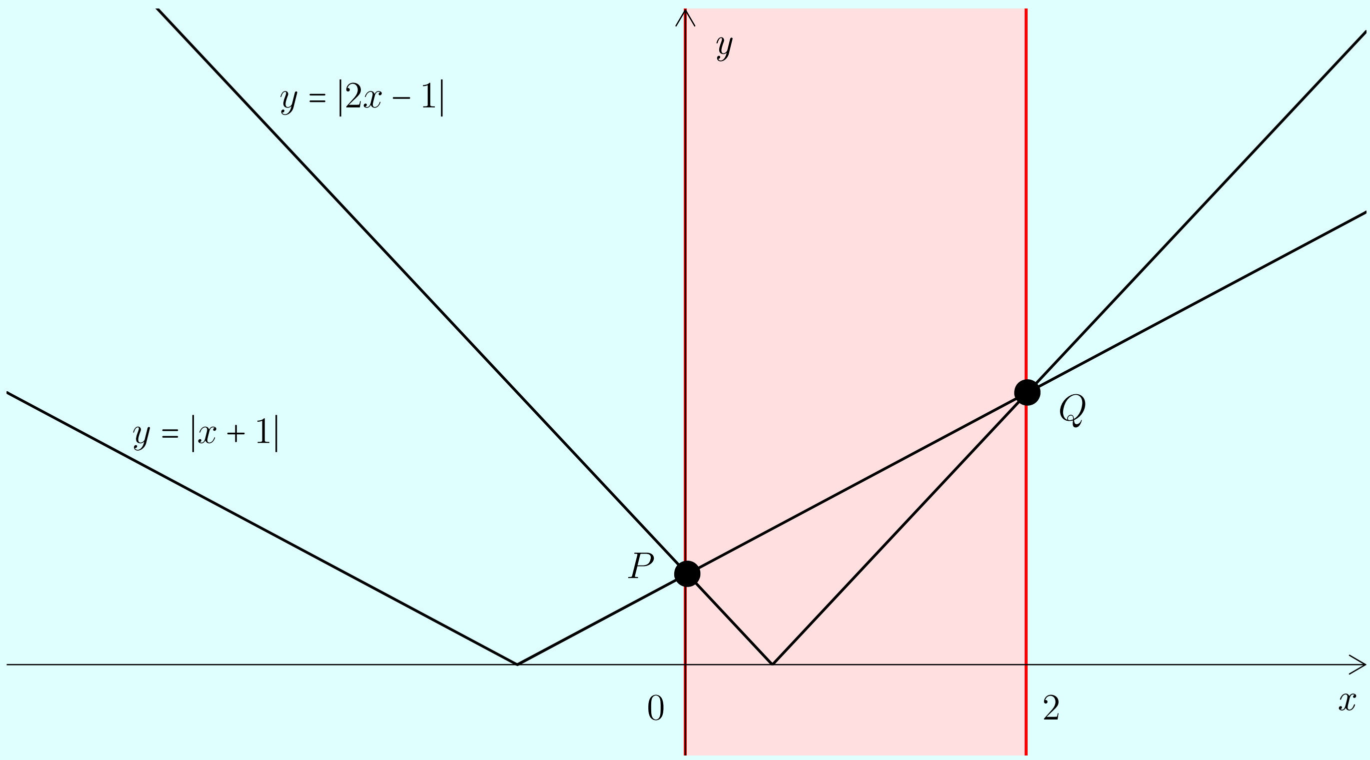
$$|3x-4| \overset{1}{\geq} 2x+2 \quad \iff \quad 3x-4 \leq 2x+2 \quad \iff \quad x \leq 6.$$

So, in Case 2, $\overset{1}{\geq}$ holds $\iff$ ($x \geq 4/3$ AND $x<6$) $\iff$ $x \in [4/3, 6]$.

Altogether, $\overset{1}{<}$ holds $\iff$ $x \in [2/5, 4/3) \cup [4/3, 6] = [2/5, 6]$.

And now for an example with two absolute value functions in the inequality:

Example 610. Solve $|x+1| \overset{1}{\geq} |2x-1|$.



We "see" that the graphs of $y=|x+1|$ and $y=|2x-1|$ intersect at P and Q. Also,

$$\overset{1}{\geq} \text{ holds} \quad \iff \quad x \text{ is between } P \text{ and } Q \text{ (inclusive)}.$$

So, let's find P and Q.

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

At P, $x+1>0$ and $2x-1<0$. So, $|x+1|=x+1$ and $|2x-1|=1-2x$. Hence, P is given by

$$x+1=1-2x \qquad \text{or} \qquad x=0.$$

At Q, $x+1>0$ and $2x-1>0$. So, $|x+1|=x+1$ and $|2x-1|=2x-1$. Hence, Q is given by

$$x+1=2x-1 \qquad \text{or} \qquad x=2.$$

Solution: $x \in [0,2]$.

Let's now solve $\overset{1}{\geq}$ again, but more rigorously (and without graphs).

This time, there are *four* possible cases:

Case 1. Suppose $x+1<0$ AND $2x-1<0$—this is equivalent to $x<-1$ AND $x<1/2$ or $x<-1$. Then

$$|x+1| \overset{1}{\geq} |2x-1| \qquad \iff \qquad -x-1 \geq 1-2x \qquad \iff \qquad x \geq 2,$$

which contradicts $x<-1$. So, in Case 1, $\overset{1}{\geq}$ never holds.

Case 2. Suppose $x+1 \geq 0$ AND $2x-1<0$—this is equivalent to $x \geq -1$ AND $x<1/2$ or $x \in [-1,1/2)$. Then

$$|x+1| \overset{1}{\geq} |2x-1| \qquad \iff \qquad x+1 \geq 1-2x \qquad \iff \qquad x \geq 0.$$

So, in Case 2, $\overset{1}{\geq}$ holds $\iff$ $x \in [0,1/2)$.

Case 3. Suppose $x+1<0$ AND $2x-1 \geq 0$—this is equivalent to $x<-1$ AND $x \geq 1/2$ or $x \in \varnothing$.

So, in Case 3, $\overset{1}{\geq}$ never holds.

Case 4. Suppose $x+1 \geq 0$ AND $2x-1 \geq 0$—this is equivalent to $x \geq -1$ AND $x \geq 1/2$ or $x \geq 1/2$. Then

$$|x+1| \overset{1}{\geq} |2x-1| \qquad \iff \qquad x+1 \geq 2x-1 \qquad \iff \qquad x \leq 2.$$

So, in Case 4, $\overset{1}{\geq}$ holds $\iff$ $x \in [1/2,2]$.

Altogether, $\overset{1}{<}$ holds $\iff$ $x \in [0,1/2) \cup [1/2,2] = [0,2]$.

Exercise 176. Solve each inequality. (Answer on p. 1813.)

(a) $|x-4| \leq 71$.

(b) $|5-x| > 13$.

(c) $|-3x+2|-4 \geq x-1$.

(d) $|x+6| > 2\,|2x-1|$.

41.6. The Triangle Inequality, Revisited

Earlier in Corollary 12, we used the Law of Cosines to prove the **Triangle Inequality** (the length of one side of a triangle is no greater than the sum of the other two). We can now prove it again now that we're better acquainted with inequalities involving the absolute value function:

Fact 106. (Triangle Inequality) *Suppose $x, y \in \mathbb{R}$. Then*

$$|x+y| \leq |x| + |y|.$$

Remark 84. Right now, it may not be obvious why "$|x+y| \leq |x| + |y|$" is the Triangle Inequality.

But later in Part IV (Vectors), we'll learn that $x+y$ can be interpreted as the third side of a triangle, $|x+y|$ its length, and $|x|$ and $|y|$ the lengths of the other two sides. Given these interpretations, we see that "$|x+y| \leq |x| + |y|$" corresponds to " the length of one side of a triangle is no greater than the sum of the other two".

Proof. We have **(a)** $-|x| \leq x \leq |x|$; and **(b)** $-|y| \leq y \leq |y|$.

Add up **(a)** and **(b)**: $-(|x|+|y|) \leq x+y \leq |x|+|y|$.

Now, $|x+y| \leq \left||x|+|y|\right| = |x|+|y|$. □

Some close relatives of the Triangle Inequality (indeed each of these results are often simply also called the Triangle Inequality):

Corollary 16. *Suppose $x, y \in \mathbb{R}$. Then*

$$|x-y| \leq |x| + |y|.$$

Proof. By the Triangle Inequality (Fact 106), $|x-y| = |x+(-y)| \leq |x| + |-y| = |x| + |y|$. □

Corollary 17. *Suppose $a_1, a_2, \dots, a_n \in \mathbb{R}$. Then*

$$|a_1 \pm a_2 \pm \cdots \pm a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

Proof. Repeatedly apply the Triangle Inequality (Fact 106) and Corollary 16:

$$|a_1 \pm a_2 \pm \cdots \pm a_n| \leq |a_1| + |a_2 \pm \cdots \pm a_n| \leq |a_1| + |a_2| + |a_3 \pm \cdots \pm a_n| \leq \cdots \leq |a_1| + |a_2| + \cdots + |a_n|.$$ □

Corollary 18. *Suppose $x, y \in \mathbb{R}$. Then*

$$|x+y| \geq |x| - |y|.$$

Proof. By Corollary 16, $|x| = |x+y-y| \leq |x+y| + |y|$. Rearranging, $|x+y| \geq |x| - |y|$. □

See also the Reverse Triangle Inequality (Fact 266, Appendices).

The Triangle Inequality looks simple and humble, but is often useful. In particular, it shows up surprisingly often in proofs of results in calculus.

Example 611. Q. Show that if $a, b, c \in \mathbb{R}$, then $|a-b| \le |a-c| + |b-c|$.

A. $|a-b| \stackrel{1}{=} |a-c+c-b| = |a-c-(b-c)| \stackrel{2}{\le} |a-c| + |b-c|$,

where $\stackrel{1}{=}$ uses the Plus Zero Trick and $\stackrel{2}{\le}$ uses Corollary 16.

Exercise 177. (*Hard*) Show that if $x, y \in \mathbb{R}$, then (Answer on p. 1814.)

$$\frac{|x+y|}{1+|x+y|} \le \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

41.7. $\dfrac{ax^2+bx+c}{dx^2+ex+f}>0$

Your syllabus says you need to solve the above inequality in those cases where both numerator and denominator are either **always positive** or **factorisable**.

Let's first look at those cases where either numerator or denominator is always positive:

Fact 107. *Consider the inequality* $\dfrac{ax^2+bx+c}{dx^2+ex+f}\overset{1}{>}0.$

(a) *If ax^2+bx+c is always positive, then $\overset{1}{>}$ is equivalent to $dx^2+ex+f>0$.*

(b) *If dx^2+ex+f is always positive, then $\overset{1}{>}$ is equivalent to $ax^2+bx+c>0$.*

Proof. By Fact 15(a), we can

(a) Multiply both sides of $\overset{1}{>}$ by $1/\left(ax^2+bx+c\right)>0$ to get $1/\left(dx^2+ex+f\right)>0$, which is equivalent to $dx^2+ex+f>0$; and

(b) Multiply both sides of $\overset{1}{>}$ by $dx^2+ex+f>0$ to get $ax^2+bx+c>0$. □

Recall (Ch. 14): A quadratic expression is always positive if and only if its coefficient on x^2 is positive AND its discriminant is positive (so that the corresponding graph is ∪-shaped AND everywhere above the x-axis).

Example 612. Solve $\dfrac{N}{D}=\dfrac{x^2+x+1}{3x^2-2x-5}\overset{1}{>}0.$

First, N is always positive, because its coefficient on x^2 is positive and its discriminant $1^2-4(1)(1)=-3$ is negative. So, by Fact 107(a), $\overset{1}{>}$ is simply equivalent to D being positive: $3x^2-2x-5\overset{2}{>}0$.

Next, D is ∪-shaped[278] and intersects the x-axis at

$$x=\frac{2\pm\sqrt{(-2)^2-4(3)(-5)}}{2\cdot3}=\frac{2\pm\sqrt{64}}{2\cdot3}=\frac{-6}{6},\frac{10}{6}=-1,\frac{5}{3}.$$

Altogether, $\overset{1}{>}$ holds $\iff$ $\overset{2}{>}$ holds $\iff$ $x\in(-\infty,-1)\cup(5/3,\infty)$.

[278] More correctly and pedantically, we should say that the graph of $y=D$ is ∪-shaped.

Example 613. Solve $\dfrac{N}{D} = \dfrac{-x^2 + 7x + 1}{2x^2 - x + 1} \overset{1}{>} 0$.

First, D is always positive, because its coefficient on x^2 is positive and its discriminant $(-1)^2 - 4(2)(1) = -7$ is negative. And so by Fact 107(b), $\overset{1}{>}$ is simply equivalent to N being positive: $-x^2 + 7x + 1 \overset{2}{>} 0$.

Next, N is $\cap$-shaped and intersects the x-axis at

$$x = \frac{-7 \pm \sqrt{7^2 - 4(-1)(1)}}{2 \cdot (-1)} = \frac{-7 \pm \sqrt{53}}{2 \cdot (-1)} = \frac{7 \mp \sqrt{53}}{2}.$$

Altogether, $\overset{1}{>}$ holds $\iff$ $\overset{2}{>}$ holds $\iff$ $x \in \left(\left(7 - \sqrt{53}\right)/2, \left(7 + \sqrt{53}\right)/2\right)$.

We next look at the case where *both* $ax^2 + bx + c$ and $dx^2 + ex + f$ are factorisable.

Example 614. Solve $\dfrac{N}{D} = \dfrac{x^2 + 3x + 2}{2x^2 - 7x + 6} > 0$.

First, $N = x^2 + 3x + 2 = (x+1)(x+2)$. Next, $D = 2x^2 - 7x + 6 = (2x-3)(x-2)$.

$$\begin{array}{ccccccccc} + & & - & & + & & - & & + \\ & -2 & & -1 & & 3/2 & & 2 & \end{array}$$

Solution: $x \in (-\infty, -2) \cup (-1, 3/2) \cup (2, \infty)$

Example 615. Solve $\dfrac{N}{D} = \dfrac{-x^2 + 5x - 4}{3x^2 - 2x - 5} > 0$.

First, $N = -x^2 + 5x - 4 = -(x-1)(x-4)$. Next, $D = 3x^2 - 2x - 5 = (3x-5)(x+1)$.

$$\begin{array}{ccccccccc} - & & + & & - & & + & & - \\ & -1 & & 1 & & 5/3 & & 4 & \end{array}$$

Solution: $x \in (-\infty, -2) \cup (-1, 3/2) \cup (2, \infty)$

Exercise 178. Solve each inequality. (Answers on pp. 1815, 1816, and 1817.)

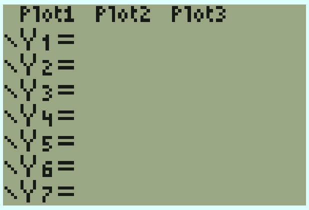
(a) $\dfrac{x^2 + 2x + 1}{x^2 - 3x + 2} > 0$. **(b)** $\dfrac{x^2 - 1}{x^2 - 4} > 0$. **(c)** $\dfrac{x^2 - 3x - 18}{-x^2 + 9x - 14} > 0$.

Example 616. Solve $x > \sin(0.5\pi x)$.

Graph $y = x - \sin(0.5\pi x)$ on your TI84. Our goal is to find the roots of this equation, i.e. the values of x for which $x - \sin(0.5\pi x) = 0$.

Step 1.

Step 2.



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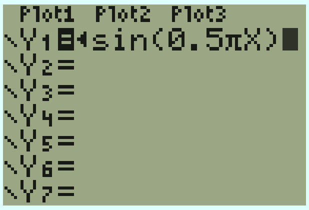

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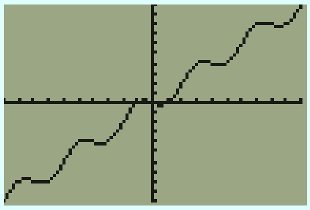

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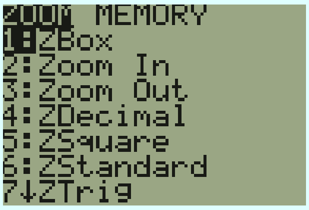
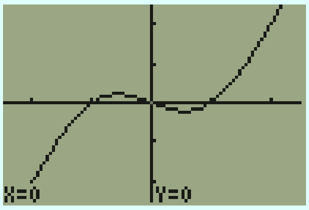
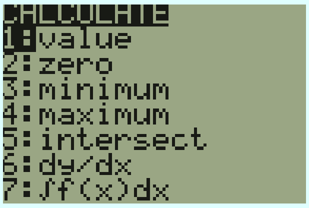


Step 6.



Step 7.



1. Press ON to turn on your TI84.
2. Press Y= to bring up the Y= editor.
3. Press X,T,θ,n − SIN 0 . 5. Next press 2ND and then ^ to enter π. Now press X,T,θ,n) and altogether you will have entered "$x - \sin(0.5\pi x)$".
4. Now press GRAPH and the TI84 will graph $y = x - \sin(0.5\pi x)$.

It looks there may be some x-intercepts close to the origin. Let's zoom in to see better.

5. Press the ZOOM button to bring up a menu of ZOOM options.
6. Press 2 to select the Zoom In option. Nothing seems to happen. But now press ENTER and the TI84 will zoom in a little for you.

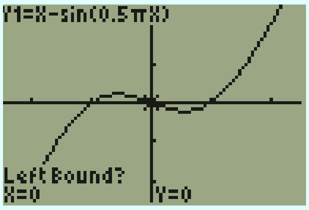
It looks like there are 3 horizontal intercepts. To find out what precisely they are, we'll use the TI84's "zero" option.

7. Press 2ND and then TRACE to bring up the CALC (CALCULATE) menu.
8. Press 2 to select the **zero function**. This brings you back to the graph, with a cursor flashing at the origin. Also, the TI84 prompts you with the question: "Left Bound?"

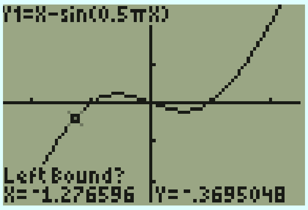
(*Example continues on the next page ...*)

(*... Example continued from the previous page.*)

Step 8.



Step 9.



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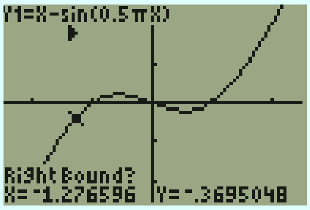
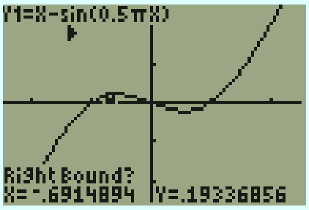
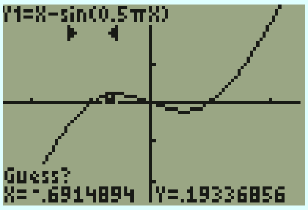


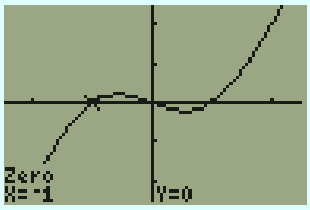
Step 11.



Step 12.



Step 13.



Recall[279] that *zero* is another word for *root*. So what TI84's zero function will do here is find the roots of the given equation (i.e. the values of x for which $y = 0$). Those of you accustomed to newfangled inventions like the world wide web and the wireless telephone will probably be expecting that the TI84 simply and immediately tells you what all the roots are. But *alas*, the TI84 is an ancient device, which means there's plenty more work you must do to find the three roots.

To find a root, you must first specify a "Left Bound" and a "Right Bound" for x. The TI84 will then check to see if there are any values of x for which $y = 0$ **between those two bounds**.

9. Using the ⟨ and ⟩ arrow keys, move the blinking cursor until it is where you want your first "Left Bound" to be. For me, I have placed it a little to the left of where I believe the leftmost horizontal intercept to be.
10. Press ENTER and you will have just entered your first "Left Bound".

TI84 now prompts you with the question: "Right Bound?".

11. So now just repeat. Using the ⟨ and ⟩ arrow keys, move the blinking cursor until it is where you want your first "Right Bound" to be. For me, I have placed it a little to the right of where I believe the leftmost horizontal is.
12. Again press ENTER and you will have just entered your first "Right Bound".

TI84 now asks you: "Guess?" This is just asking if you want to proceed and get TI84 to work out where the horizontal intercept is. So go ahead and

13. Press ENTER. TI84 now informs you that there is a "Zero" at "$x = -1$", "$y = 0$" and places the blinking cursor at that point. So, $x = -1$ is the first root we've found.

To find the other two roots, "simply" repeat steps 7 through 13—*two more times*. You should find that the other two roots are $x = 0$ and $x = 1$. Altogether, the three roots are $x = -1, 0, 1$. Based on these and what the graph looks like, we conclude:

$$x > \sin\frac{\pi x}{2} \iff x - \sin\frac{\pi x}{2} > 0 \iff x \in (-1, 0) \cup (1, \infty).$$

[279] Remark 29.

Example 617. Solve $x > \mathrm{e} + \ln x$.

1. Graph $y = x - \mathrm{e} - \ln x$ on your TI84 (precise instructions omitted).

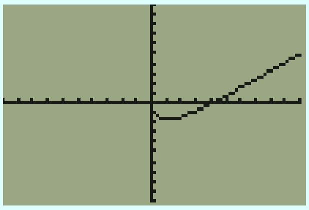
   We see that there's clearly an x-intercept at around $x \in (4, 5)$. (Note that by default, each of the little tick marks shown on your TI84 marks 1 unit.)
2. Zoom in (precise instructions omitted).

   Now we see that there's probably also an x-intercept near the origin. But unfortunately, now we can no longer see the other x-intercept. To fix this:
3. Press WINDOW to bring up the WINDOW menu.

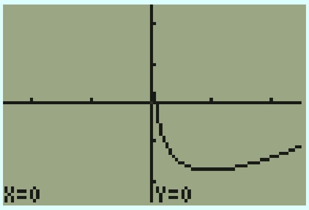
   We will adjust Xmin and Xmax:
4. Press 0. We have adjusted Xmin to 0. Next,
5. Press ENTER 5. We have adjusted Xmax to 5.
6. Now press GRAPH. We can now see the portion of the graph between $x = 0$ and $x = 5$.
7. To find the two roots, "simply" go through the steps described in the previous example—*twice* (precise instructions omitted). You should find that the two roots are $x \approx 0.708, 4.139$.

Based on these roots and what the graph looks like, we conclude that the inequality's solution set is $x \in (0, 0.708\ldots) \cup (4.139\ldots, \infty) = \mathbb{R}^+ \setminus (0.708\ldots, 4.139\ldots)$.

Step 1.



Step 2.



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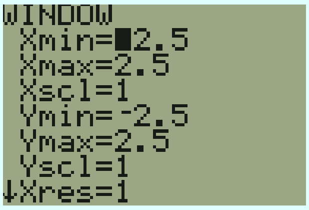

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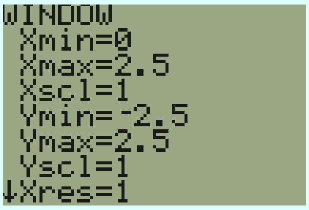
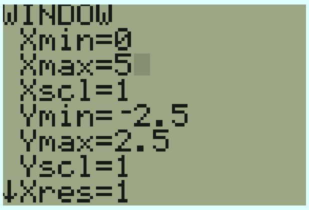
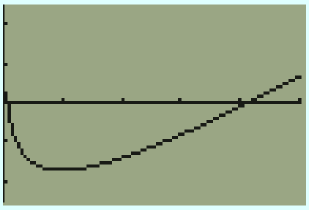


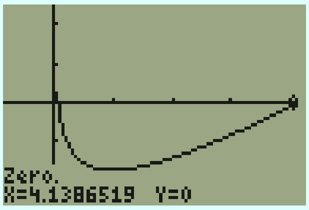
Step 5.



Step 6.



Step 7.



Exercise 179. Use a GC to solve each inequality: (Answers on p. 1818.)

(a) $x^3 - x^2 + x - 1 > \mathrm{e}^x$. **(b)** $\sqrt{x} > \cos x$. **(c)** $\dfrac{1}{1 - x^2} > x^3 + \sin x$.

42. Extraneous Solutions

42.1. Squaring

Example 618. To solve $x \overset{1}{=} \sqrt{2-x}$ $(x \leq 2)$, we try these three steps:

1. Square both sides: $x^2 = 2 - x$.
2. Rearrange and factorise: $x^2 + x - 2 = (x-1)(x+2) = 0$.
3. Conclude: $x \overset{2}{=} 1$ or $x \overset{3}{=} -2$.

Now, $x \overset{2}{=} 1$ does indeed solve $\overset{1}{=}$: $1 \overset{1}{=} \sqrt{2-1} = \sqrt{1} = 1$. ✓

But $x \overset{3}{=} -2$ does not: $-2 \neq \sqrt{2-(-2)} = \sqrt{4} = 2$. ✗

So, the above steps are wrong because they produce the **extraneous solution** $x \overset{3}{=} -2$.

Where was this extraneous solution introduced? It turns out it was introduced in Step 1, where we applied the **squaring operation**.

To see this more clearly, let's be more explicit about our above chain of reasoning. In particular, let us use the logical operators $\iff$ ("is equivalent to") and $\implies$ ("implies"):

$$\begin{aligned} & x \overset{1}{=} \sqrt{2-x} \\ \overset{\mathrm{I}}{\implies} \quad & x^2 = 2 - x \\ \overset{\mathrm{II}}{\iff} \quad & (x-1)(x+2) = 0 \\ \overset{\mathrm{III}}{\iff} \quad & x \overset{2}{=} 1 \quad \text{or} \quad x \overset{3}{=} -2. \end{aligned}$$

We now see clearly how Step I differs from Steps II and III. Step I is a " $\implies$ " statement, while Steps II and III are " $\iff$ " statements. Or in plainer English, the **squaring operation** in Step I is an **irreversible** operation.

It is always true that $\quad a = b \quad \implies \quad a^2 = b^2.$

But, the converse is false: $\quad a^2 = b^2 \quad \not\Longrightarrow \quad a = b.$

For example, $(-1)^2 = 1^2$, but $-1 \neq 1$. So, squaring is an example of an irreversible operation; it produces an " $\implies$ " statement and not a " $\iff$ " statement.

So, our above chain of reasoning produces this (true) implication:

$$x \overset{1}{=} \sqrt{2-x} \quad \implies \quad x \overset{2}{=} 1 \text{ OR } x \overset{3}{=} -2.$$

We must now check, on a case-by-case basis, whether each of $x \overset{2}{=} 1$ OR $x \overset{3}{=} -2$ solves the original equation $x \overset{1}{=} \sqrt{2-x}$. And when we do check, we find that $x \overset{2}{=} 1$ does, while $x \overset{3}{=} -2$ does not and is thus an extraneous solution that must be discarded.

The following (very) informal Theorem is our **moral of the story**. It describes *how*, *when*, and *why* extraneous solutions **may** arise:

Theorem 13 (informal)**. (Extraneous Solutions Theorem)** *If our chain of reasoning contains only* $\Longleftrightarrow$ *'s, then all is well. However, if it contains even one* $\Longrightarrow$ *(i.e. an irreversible step), then extraneous solutions* ***may*** *arise and we must be careful to check for them.*

Note the emphasis on the word **may**. Extraneous solutions **may** arise but might not.

The operation of **squaring** is merely one example of when **extraneous solutions** may be introduced. Two others are **multiplying by zero** and **removing logarithms**:

Example 619. To solve $\dfrac{x^2-3x}{x^2-1}+2+\dfrac{1}{x-1}\stackrel{1}{=}0$, we try these three steps:[280]

1. Multiply by x^2-1: $x^2-3x+2\left(x^2-1\right)+(x+1)=0$.
2. Rearrange and factorise: $3x^2-2x-1=(x-1)(3x+1)=0$.
3. Conclude: $x\stackrel{2}{=}1$ or $x\stackrel{3}{=}-1/3$.

Exercise: Verify that $x\stackrel{3}{=}-1/3$ solves $\stackrel{1}{=}$, while $x\stackrel{2}{=}1$ does not.[281]

So, $x\stackrel{2}{=}1$ is an extraneous solution. Where was it introduced?

Again, to clearly detect the error, let us write out our chain of reasoning more explicitly with $\Longrightarrow$ and $\Longleftrightarrow$:

$$\frac{x^2-3x}{x^2-1}+2+\frac{1}{x-1}\stackrel{1}{=}0$$
$$\stackrel{\mathrm{I}}{\Longrightarrow}\quad x^2-3x+2\left(x^2-1\right)+(x+1)=0$$
$$\stackrel{\mathrm{II}}{\Longleftrightarrow}\quad 3x^2-2x-1=(x-1)(3x+1)=0$$
$$\stackrel{\mathrm{III}}{\Longleftrightarrow}\quad x\stackrel{2}{=}1 \qquad \text{or} \qquad x\stackrel{3}{=}-1/3.$$

Now, why is $\stackrel{\mathrm{I}}{\Longrightarrow}$ an irreversible operation? Because we're multiplying by some unknown quantity x^2-1 which *might* be zero.

And **multiplying by zero** is an irreversible operation:

In general, $\qquad y=z \quad\Longrightarrow\quad 0\cdot y=0\cdot z.$

But, $\qquad 0\cdot y=0\cdot z \quad\not\Longrightarrow\quad y=z$

For example, $\quad 1=1\Longrightarrow 0\cdot 1=0\cdot 1, \qquad$ but $\qquad 0\cdot 2=0\cdot 3\not\Longrightarrow 2=3.$

So, our above chain of reasoning yields this (true) implication:

$$\frac{x^2-3x}{x^2-1}+2+\frac{1}{x-1}\stackrel{\mathrm{a}}{=}0 \qquad\Longrightarrow\qquad x\stackrel{2}{=}1 \text{ or } x\stackrel{3}{=}-1/3.$$

We must then check, case-by-case, whether each of $x\stackrel{2}{=}1$ or $x\stackrel{3}{=}-1/3$ actually solves the original equation $\stackrel{1}{=}$. In this example, we find that $x\stackrel{3}{=}-1/3$ solves $\stackrel{1}{=}$, while $x\stackrel{2}{=}1$ does not and is an extraneous solution that must be discarded.

[280] This example was stolen from Manning (1970, p. 170).

[281] Plug $x\stackrel{3}{=}-1/3$ into $\stackrel{1}{=}$: $\dfrac{(-1/3)^2-3(-1/3)}{(-1/3)^2-1}+2+\dfrac{1}{-1/3-1}=\dfrac{1/9+1}{1/9-1}+2-\dfrac{3}{4}=-\dfrac{5}{4}+2-\dfrac{3}{4}\stackrel{1}{=}0.$ ✓

In contrast, $x\stackrel{2}{=}1$ does not solve $\stackrel{1}{=}$ because $x-1=0$, so that some terms in $\stackrel{1}{=}$ are undefined.

By the way, a mathematician might say that the problem of **multiplying by zero** is *dual* to the problem of **dividing by zero** (Ch. 2.2):

- Dividing by zero may cause us to lose (valid) solutions; while
- Multiplying by zero may introduce extraneous (or invalid) solutions.

Example 620. To solve $\log x + \log(x+1) \overset{1}{=} \log(2x+2)$, we try these four steps:

1. Use a Logarithm Law: $\log x + \log(x+1) = \log\left(x^2+x\right) \overset{1}{=} \log(2x+2)$.
2. Remove logs: $x^2 + x = 2x + 2$.
3. Rearrange and factorise: $x^2 - x - 2 = (x+1)(x-2) = 0$.
4. Conclude: $x \overset{2}{=} -1$ or $x \overset{3}{=} 2$.

Exercise: Verify that $x \overset{3}{=} 2$ solves $\overset{1}{=}$ but $x \overset{2}{=} -1$ does not.[282]

So, $x \overset{2}{=} 1$ is an extraneous solution. Where was it introduced?

Again, to clearly detect the error, let us write out our chain of reasoning more explicitly:

$$
\begin{aligned}
& \log x + \log(x+1) \overset{1}{=} \log(2x+2) \\
\overset{\text{I}}{\iff} \quad & \log x + \log(x+1) = \log\left(x^2+x\right) \overset{1}{=} \log(2x+2) \\
\overset{\text{II}}{\implies} \quad & x^2 + x = 2x - 2 \\
\overset{\text{III}}{\iff} \quad & x^2 - x + 2 = (x+1)(x-2) = 0 \\
\overset{\text{IV}}{\iff} \quad & x \overset{2}{=} -1 \quad \text{or} \quad x \overset{3}{=} 2.
\end{aligned}
$$

This time, it's Step II that's irreversible. In general, we have

In general, $\qquad \log a = \log b \quad \implies \quad a = b$

But, $\qquad a = b \quad \not\Longrightarrow \quad \log a \overset{5}{=} \log b.$

Because if $a = b$ is non-positive, then $\log a$ or $\log b$ is undefined, so that $\overset{5}{=}$ is necessarily false.

So, our above chain of reasoning yields this (true) implication:

$$\log x + \log(x+1) \overset{1}{=} \log(2x+2) \quad \implies \quad x \overset{2}{=} -1 \text{ or } x \overset{3}{=} 2.$$

We must then check, case-by-case, whether each of $x \overset{2}{=} -1$ or $x \overset{3}{=} 2$ actually solves the original equation $\overset{1}{=}$. In this example, we find that $x \overset{3}{=} -1/3$ solves $\overset{1}{=}$, while $x \overset{2}{=} 1$ does not and is an extraneous solution that must be discarded.

[282] Plug $x \overset{3}{=} 2$ into $\overset{1}{=}$: $\log 2 + \log(2+1) = \log 2 + \log 3 = \log 6 \overset{1}{=} \log(2 \cdot 2 + 2) = \log 6$. ✓

In contrast, $x \overset{2}{=} -1$ does not solve $\overset{1}{=}$ because $\log -1$ is undefined.

This brief chapter merely examined three examples of operations by which **extraneous solutions** *may* be introduced—namely **squaring**, **multiplying by zero**, and **removing logarithms**. These are not exhaustive and you will likely encounter more of such operations as your maths education progresses.

The important thing is to remember *how* and *why* extraneous solutions arise. In particular, you should remember and understand the **Extraneous Solutions Theorem**.

Exercise 180. To solve $\sin x + \cos x \overset{1}{=} 1$ ($x \in [0, 2\pi)$), we try these steps:[283] (Answer on p. 1819.)

1. Square both sides: $(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x \cos x \overset{1}{=} 1^2 = 1$.
2. Apply the identity $\sin^2 x + \cos^2 x = 1$ to get $2\sin x \cos x = 0$.
3. So, $\sin x = 0$ or $\cos x = 0$.
4. Conclude: $x \overset{2}{=} 0$ or $x \overset{3}{=} \pi$ or $x \overset{4}{=} \pi/2$ or $x \overset{5}{=} 3\pi/2$.

Is/are there any extraneous solution(s)? At which step(s) did it/they arise?

Exercise 181. To solve $x^{1/3} + x^{1/6} - 2 \overset{1}{=} 0$ ($x \in \mathbb{R}$), we try these steps:[284]

1. Factorise: $\left(x^{1/6} - 1\right)\left(x^{1/6} + 2\right) = 0$.
2. So, $x^{1/6} = 1$ or $x^{1/6} = -2$.
3. Conclude: $x = \left(x^{1/6}\right)^6 = 1^6 \overset{2}{=} 1$ or $x = \left(x^{1/6}\right)^6 = (-2)^6 \overset{3}{=} 64$.

Is/are there any extraneous solution(s)? At which step(s) did it/they arise? (Answer on p. 1819.)

Exercise 182. To solve $x^2 + x + 1 \overset{1}{=} 0$ ($x \in \mathbb{R}$), we try these steps:[285]

1. Rearrange: $x^2 \overset{2}{=} -x - 1$.
2. Since $x = 0$ doesn't solve $\overset{1}{=}$ anyway, we know that $x \neq 0$.
   So, we can divide $\overset{2}{=}$ by x to get $x \overset{3}{=} -1 - \dfrac{1}{x}$.
3. Now plug $\overset{3}{=}$ into $\overset{1}{=}$ to get $x^2 + \left(-1 - \dfrac{1}{x}\right) + 1 \overset{4}{=} 0$ or $x^2 = \dfrac{1}{x}$ or $x^3 = 1$.
4. Conclude: $x \overset{5}{=} 1$.

Is/are there any extraneous solution(s)? At which step(s) did it/they arise? (Answer on p. 1819.)

[283]Stolen from Sullivan (*Precalculus*, 10e, 2017, p. 519), hat tip to ✿.
[284]Stolen from ✿.
[285]Stolen from ✿.

43. O-Level Review: The Derivative

This chapter is a short review of what you (supposedly) mastered in secondary school. In Part V (Calculus), we'll revisit these concepts but at greater depth.

43.1. The Derivative Is the "Gradient Function"

Example 621. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x$.

Figure to be
inserted here.

This graph's gradient is 5 at *every* point.

So, let's define a new function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 5$ and call g the "gradient function" of f. That is, g is very simply the function that tells us what the gradient of the graph of f is at each point.

But we don't use the term "gradient function".[286] Instead, we call g the **derivative** of f.

Of course, in this example, g isn't very interesting and simply tells us that the gradient of the graph of f is 5 at every point. For example, at $x = -2$, the graph's gradient is $g(x) = 5$; at $x = 0$, it's $g(x) = 5$; and at $x = 3$, it's $g(x) = 5$. Indeed, at any point $x \in \mathbb{R}$, $g(x) = 5$.

[286]The word *gradient* is actually used in another closely related context in multivariable calculus (beyond A-Level Maths).

Example 622. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^2+1$.

Figure to be
inserted here.

We don't yet know how, but it's possible to show that this graph's gradient is equal to $2x$ at every point.

So again, we can define a new function $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=2x$ and call g the **derivative** of f.

Again, g simply tells us what the gradient of the graph of f is at every point. For example, at $x=-2$, the graph's gradient is $g(-2)=2\times(-2)=-4$; at $x=0$, it's $g(0)=2\times 0=0$; and at $x=3$, it's $g(3)=2\times 3=6$.

Example 623. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^3+2x$.

Figure to be
inserted here.

We don't yet know how, but it's possible to show that this graph's gradient is equal to $3x^2+2$ at every point.

So again, we can define a new function $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=3x^2+2$ and call g the **derivative** of f.

Again, g simply tells us what the gradient of the graph of f is at every point. For example, at $x=-2$, the graph's gradient is $g(-2)=3\times(-2)^2+2=14$; at $x=0$, it's $g(0)=3\times 0^2+2=2$; and at $x=3$, it's $g(3)=3\times 3^2+2=29$.

Definition 114 (*informal*)**.** Given a function f, its *derivative* is the function that tells us what the gradient of the graph of f is at each point.

To stress, repeat, and emphasize,

The derivative is itself simply a function.

For the formal and precise definition of the derivative, see Definition 203 (Part V, Calculus).

43.2. The Confusing and Varied Notation for the Derivative

In the last three examples, given a function f, we simply denoted its derivative by g.

But given a function f, writers usually denote

- Its **derivative** (a function) by

$$\underset{\textbf{(Lagrange)},}{f'} \qquad \text{or} \qquad \underset{\textbf{(Newton)},}{\dot{f}} \qquad \text{or} \qquad \underset{\textbf{(Leibniz)}.}{\frac{\mathrm{d}f}{\mathrm{d}x}}$$

- Its **derivative at a point** (a number that we interpret as the gradient of the graph of f at that point) by

$$\underset{\textbf{(Lagrange)},}{f'(a)} \qquad \text{or} \qquad \underset{\textbf{(Newton)},}{\dot{f}(a)} \qquad \text{or} \qquad \underset{\textbf{(Leibniz)}.}{\frac{\mathrm{d}f}{\mathrm{d}x}(a) \quad \text{or} \quad \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}}$$

Example 624. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

The **derivative** of f is the function with domain $\mathbb{R}$, codomain $\mathbb{R}$, and mapping rule $x \mapsto 2x$.

This last sentence is equivalent to each of these next three sentences:

Lagrange: The derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 2x$.

Newton: The derivative of f is the function $\dot{f} : \mathbb{R} \to \mathbb{R}$ defined by $\dot{f}(x) = 2x$.

☺ Leibniz: The derivative of f is the function $\frac{\mathrm{d}f}{\mathrm{d}x} : \mathbb{R} \to \mathbb{R}$ defined by $\frac{\mathrm{d}f}{\mathrm{d}x}(x) \overset{1}{=} 2x$.

The derivative of f at $x = 5$ is the number 10 (and this is the gradient of the graph of f at $x = 5$):

$$\overbrace{f'(5)}^{\text{Lagrange}} = \underbrace{\dot{f}(5)}_{\text{Newton}} = \overbrace{\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=5} = \frac{\mathrm{d}f}{\mathrm{d}x}(5)}^{\text{Leibniz}} = 2 \times 5 = 10.$$

Lagrange places a prime symbol $'$ to the right of f.

Newton's notation is very similar—instead of a prime symbol $'$ to the right of f, we have a dot $\cdot$ above.

Leibniz's notation is the oddball.

Remarks on Leibniz's notation (references to above example):

1. In $\stackrel{1}{=}$, the "(x)" is superfluous and can be omitted, because the dummy variable x is already specified in the symbol $\dfrac{\mathrm{d}f}{\mathrm{d}x}$. And so, line ☺ may be written more simply as

$$\text{Leibniz: The derivative of } f \text{ is the function } \frac{\mathrm{d}f}{\mathrm{d}x} : \mathbb{R} \to \mathbb{R} \text{ defined by } \frac{\mathrm{d}f}{\mathrm{d}x} \stackrel{1}{=} 2x.$$

2. We've already explained the concept of dummy variables several times in this textbook.[287] In line ☺, our dummy variable is "x"—this can be replaced by any other symbol, such as y, t, or even $\star$. So, line ☺ is exactly equivalent to each of these three lines:

$$\text{Leibniz: The derivative of } f \text{ is the function } \frac{\mathrm{d}f}{\mathrm{d}y} : \mathbb{R} \to \mathbb{R} \text{ defined by } \frac{\mathrm{d}f}{\mathrm{d}y} \stackrel{1}{=} 2y.$$

$$\text{Leibniz: The derivative of } f \text{ is the function } \frac{\mathrm{d}f}{\mathrm{d}t} : \mathbb{R} \to \mathbb{R} \text{ defined by } \frac{\mathrm{d}f}{\mathrm{d}t} \stackrel{1}{=} 2t.$$

$$\text{Leibniz: The derivative of } f \text{ is the function } \frac{\mathrm{d}f}{\mathrm{d}\star} : \mathbb{R} \to \mathbb{R} \text{ defined by } \frac{\mathrm{d}f}{\star x} \stackrel{1}{=} 2\star.$$

3. Rather than $\dfrac{\mathrm{d}f}{\mathrm{d}x}$, you may be more used to seeing $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. This is perfectly fine so long as y is a well-defined function.
The thing is, your teachers and A-Level examiners will often be sloppy and write something like this: "Let $y = x^2$. Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$."
This isn't outright wrong, but is a bit sloppy, because we should really be careful to specify that y is a function of x and also specify the domain and codomain of y.

We'll have more to say about Leibniz's notation in Ch. 89.

Remark 85. Lagrange's and Leibniz's notation are widely used.

In contrast, Newton's notation is rarely used. Indeed, Newton's notation does not appear in any of your recent years' A-Level exams and we won't use it in this textbook. Nonetheless, Newton's notation is sometimes used in physics (especially when the independent variable is time). Moreover, it appears on p. 18 of your syllabus. So, it's probably worth a quick mention.

Remark 86. There's actually a fourth piece of notation due to Euler. But happily, Euler's notation does not appear on your syllabus (or exams) at all and so we shall say no more about it.

Exercise 183. For each function (recycled from Exercise 335), write down its derivative and its derivative at 2 in Newton's and Leibniz's notation. (Answer on p. 1896.)

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7$.

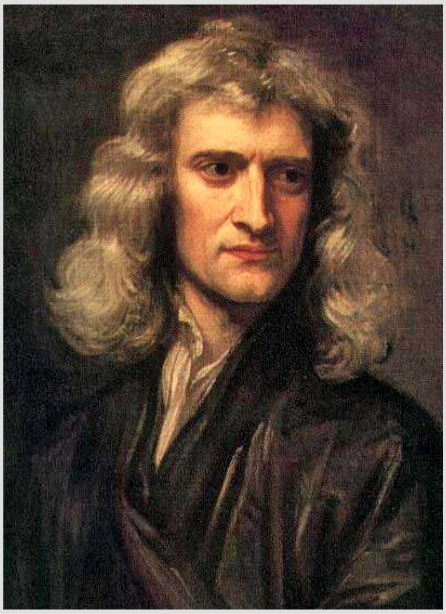
(b) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 5x + 7$.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = 2x^2 + 5x + 7$.

[287] Example 93, Ch. 5.8, p. 211. See also Ch. 46.1.

Fun Fact

Isaac Newton was both one of the greatest physicists ever *and* one of the greatest mathematicians ever. Which is why in one ranking of the most influential persons in history,[288] Newton was ranked second (and the only among the top six who was a non-religious figure).



Isaac Newton (1643–1727)

Gottfried Wilhelm von Leibniz (1646–1716)

Gottfried Wilhelm von Leibniz[289] was likewise a first-rate genius and a polymath. Indeed, he has sometimes been called "**the last man to know everything**", the rationale being that

> *Since his time the growth of knowledge has resulted in, and indeed necessitated, specialization. The horizon for the individual is now restricted, for few can hope to attain proficiency in more than one subject.*
>
> — A. L. Leigh Silver (1962).

Newton and Leibniz are often dubbed the "inventors" of the calculus. Indeed, their dispute over who "invented" calculus is perhaps history's most famous academic dispute. (Even history's greatest geniuses are not above some petty bickering.)[290] But as has been well said by the historian of mathematics Carl B. Boyer (1949),

> *Few new branches of mathematics are the work of single individuals.*

---

[288] Michael Hart, in *The 100: A Ranking of the Most Influential Persons in History* (1978, 1992). In case you're wondering, Muhammad was #1 and Jesus #3. Full rankings plus summary here. Book here.

[289] Sometimes spelt Leibnitz.

[290] For a popular account of this dispute, see *The Calculus Wars: Newton, Leibniz, and the Greatest Mathematical Clash of All Time* (2007).

43.3. The Derivative as Rate of Change

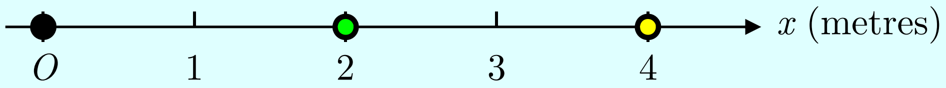
So far, we've interpreted $\frac{\mathrm{d}x}{\mathrm{d}t}$ only as the **gradient**.

But as you may recall, we can also interpret $\frac{\mathrm{d}x}{\mathrm{d}t}$ as

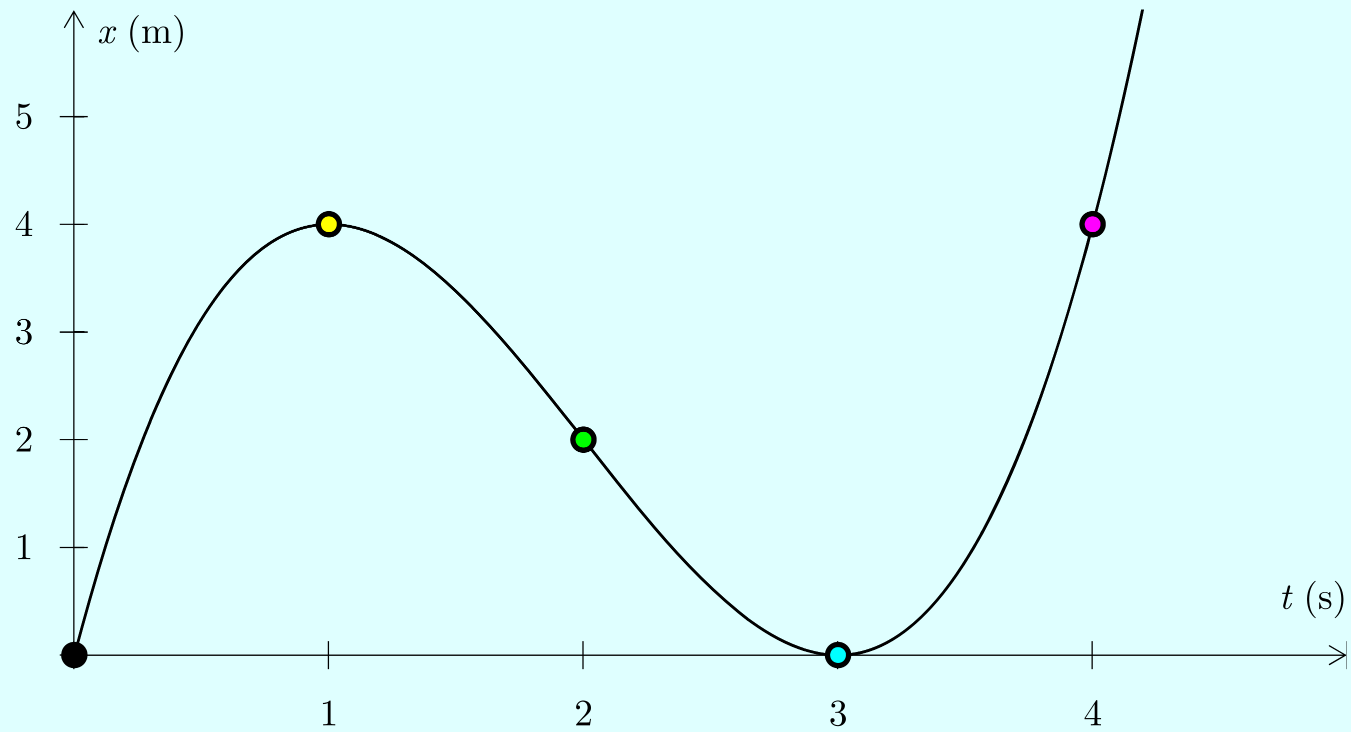
- The **change in x resulting from a small unit change in** t; and
- The **rate of change of x with respect to time** t.

Example 625. A particle P travels along a line. Its eastward displacement $x : \mathbb{R} \to \mathbb{R}$ (metres, m) from the point O at time t (seconds, s) is given by

$$x(t) = t^3 - 6t^2 + 9t.$$



- At $t = 0\,\text{s}$, P starts $x = 0^3 - 6 \cdot 0^2 + 9 \cdot 0 = 0\,\text{m}$ east of O. In other words, it is at O. And during $t \in [0, 1)$, P travels eastwards away from O.
- At $t = 1\,\text{s}$, P stops and is $x = 1^3 - 6 \cdot 1^2 + 9 \cdot 1 = 4\,\text{m}$ east of O.
- During $t \in (1, 3)$, it travels *west*wards, i.e. back towards O. For example, at $t = 2\,\text{s}$, P is $x = 2^3 - 6 \cdot 2^2 + 9 \cdot 2 = 2\,\text{m}$ east of O.
- At $t = 3\,\text{s}$, P has returned to O ($x = 3^3 - 6 \cdot 3^2 + 9 \cdot 3 = 0$) and stops.
- During $t > 3\,\text{s}$, P keeps travelling eastwards away from O. For example, at $t = 4\,\text{s}$, P is $x = 4^3 - 6 \cdot 4^2 + 9 \cdot 4 = 4\,\text{m}$ east of O. And at $t = 10\,\text{s}$ (not depicted in the figures), P is $x = 10^3 - 6 \cdot 10^2 + 9 \cdot 10 = 490\,\text{m}$ east of O.



(*Example continues on the next page ...*)

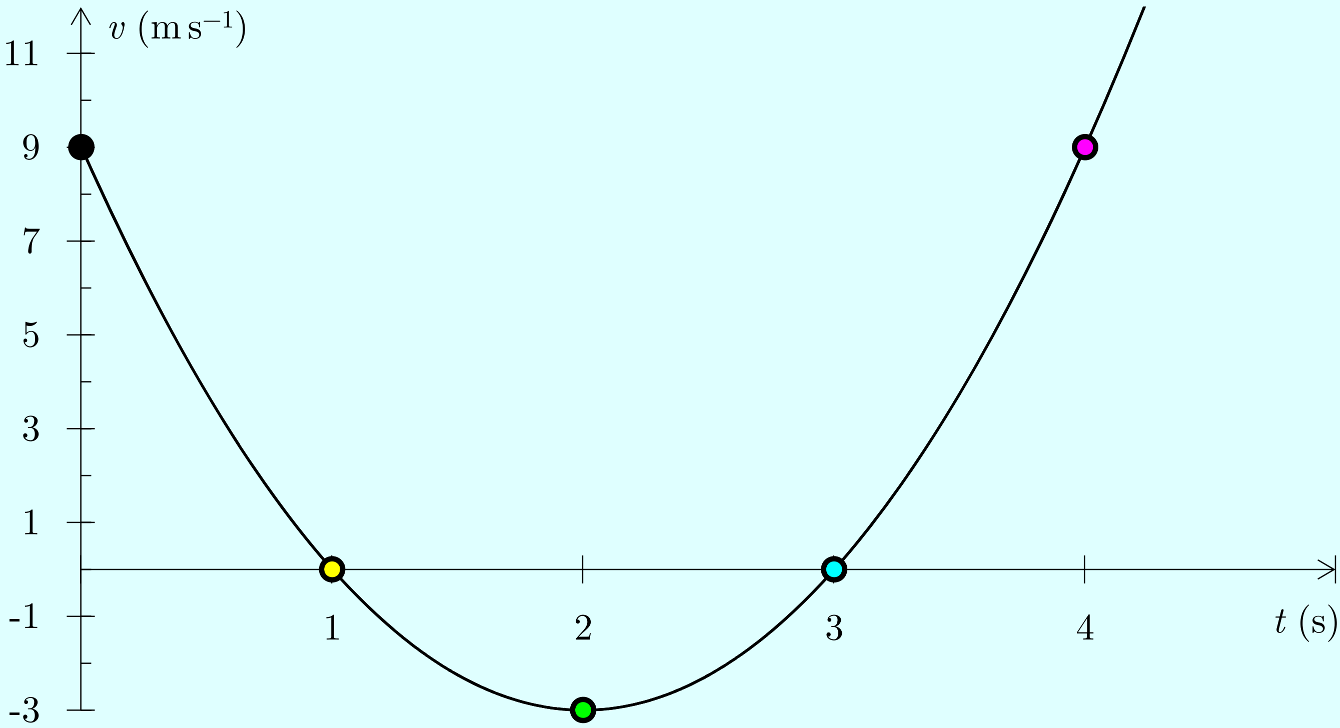
(... *Example continued from the previous page.*)

We define **velocity** v (metres per second, $\mathrm{m\,s^{-1}}$) to be the rate of change of displacement with respect to (w.r.t.) time. In other words, velocity is the (first) derivative of displacement w.r.t. time:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Using rules of differentiation that we'll review in Ch. 43.6, it's possible to work out that P's (eastward) velocity is given by

$$v(t) = 3t^2 - 12t + 9.$$



At each instant of time, v tells us what P's velocity is—in other words, the rate at which x is changing per "infinitesimally small" unit of time t. If $v > 0$, then P is travelling eastwards. While if $v < 0$, then P is travelling *west*wards.

From the above graph, we can tell that

- During $t \in [0, 1)$, P travels eastwards.
- At $t = 1$, P stops.
- During $t \in (1, 3)$, P travels westwards.
- At $t = 3$, P stops.
- During $t > 3$, P travels eastwards.

(*Example continues on the next page ...*)

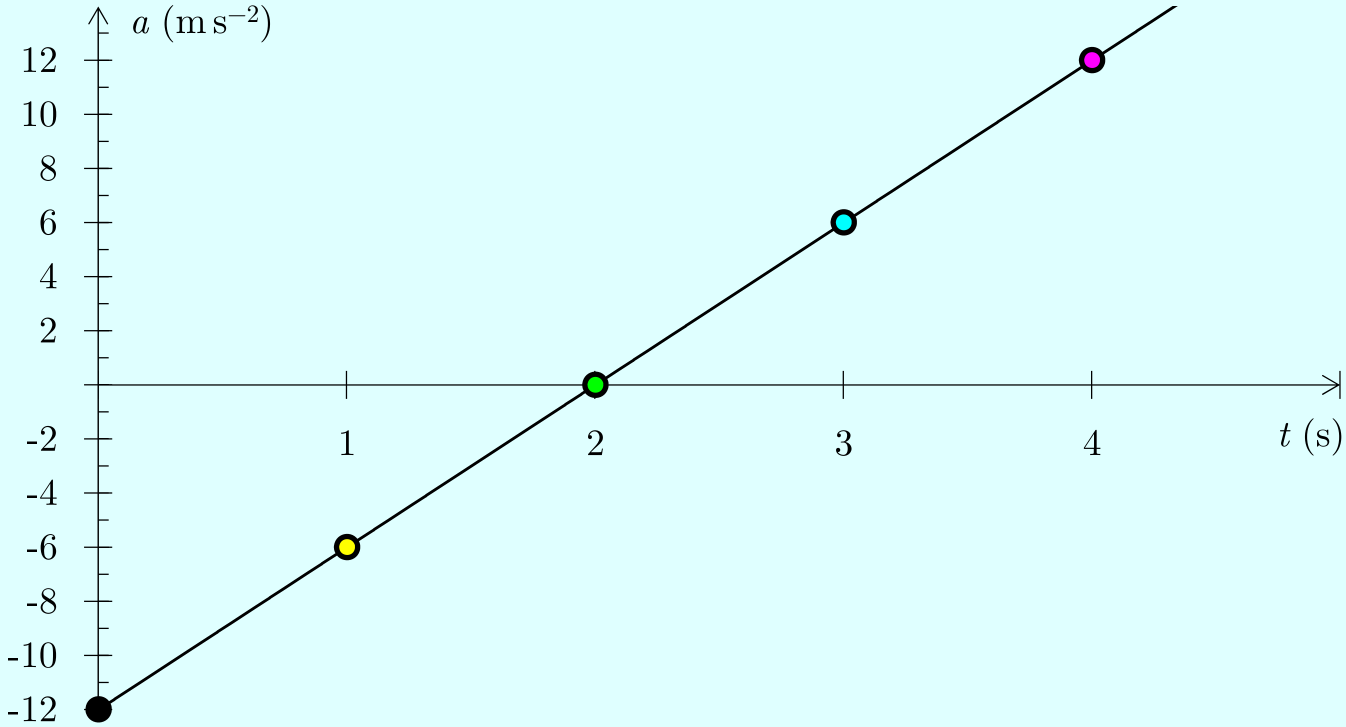
(... *Example continued from the previous page.*)

We define acceleration a (metres per second per second or metres per second squared, $\mathrm{m\,s^{-2}}$) to be the rate of change of velocity w.r.t. time. In other words, acceleration is the (first) derivative of velocity w.r.t. time and thus the *second* derivative of displacement w.r.t. time:

$$v = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}.$$

Again, it is possible to work out that P's (eastward) acceleration is given by

$$a(t) = 6t - 12.$$



At each instant of time, a tells us what P's acceleration is—in other words, the rate at which v is changing per "infinitesimally small" unit of time t. If $a > 0$, then P's eastwards velocity is increasing (or equivalently, its westwards velocity is decreasing). And if $a < 0$, then P's eastwards velocity is *de*creasing (or equivalently, its westwards velocity is *in*creasing)

From the above graph, we can tell that

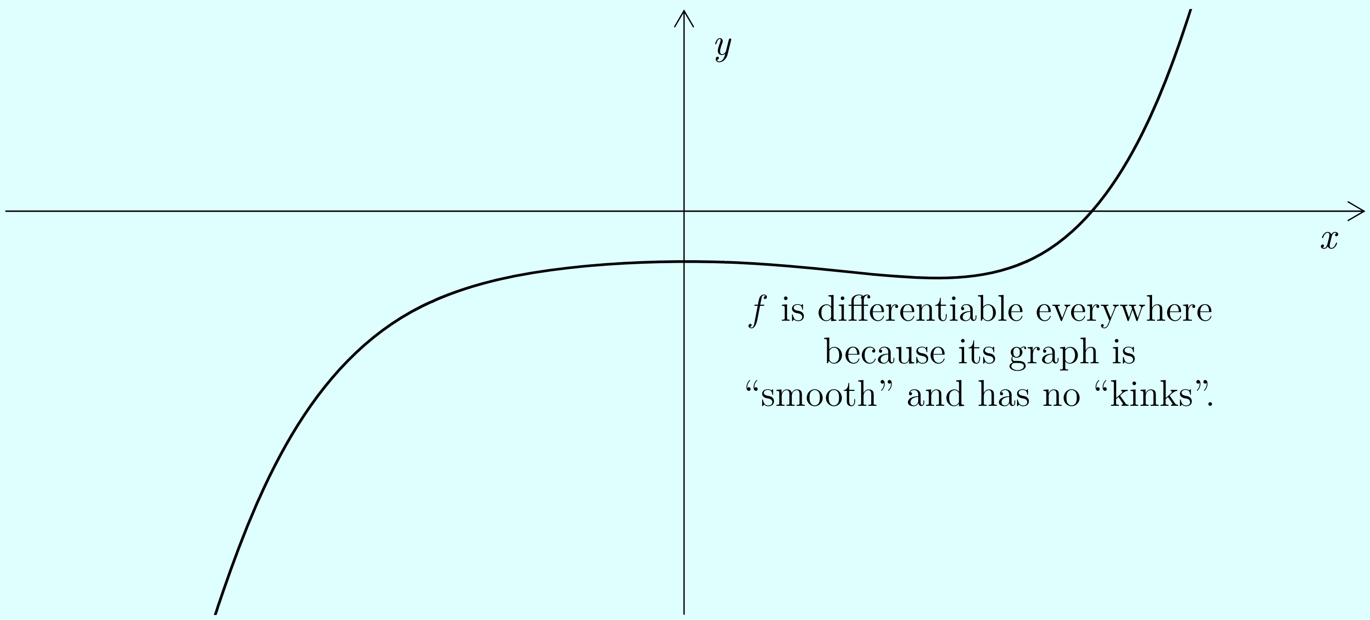
- During $t \in [0, 2)$, P's eastwards velocity is increasing (or equivalently, its westwards velocity is decreasing).
- During $t > 2$, P's eastwards velocity is *de*creasing (or equivalently, its westwards velocity is *in*creasing)

43.4. Differentiability vs Continuity

In Ch. 18, we learnt that informally, **continuity** means you can draw the graph without lifting your pencil. We now introduce the concept of **differentiability**, which turns out to be a stronger condition than continuity. Informally, a function is **differentiable** if its graph is "smooth" and, in particular, doesn't have "kinks".[291]

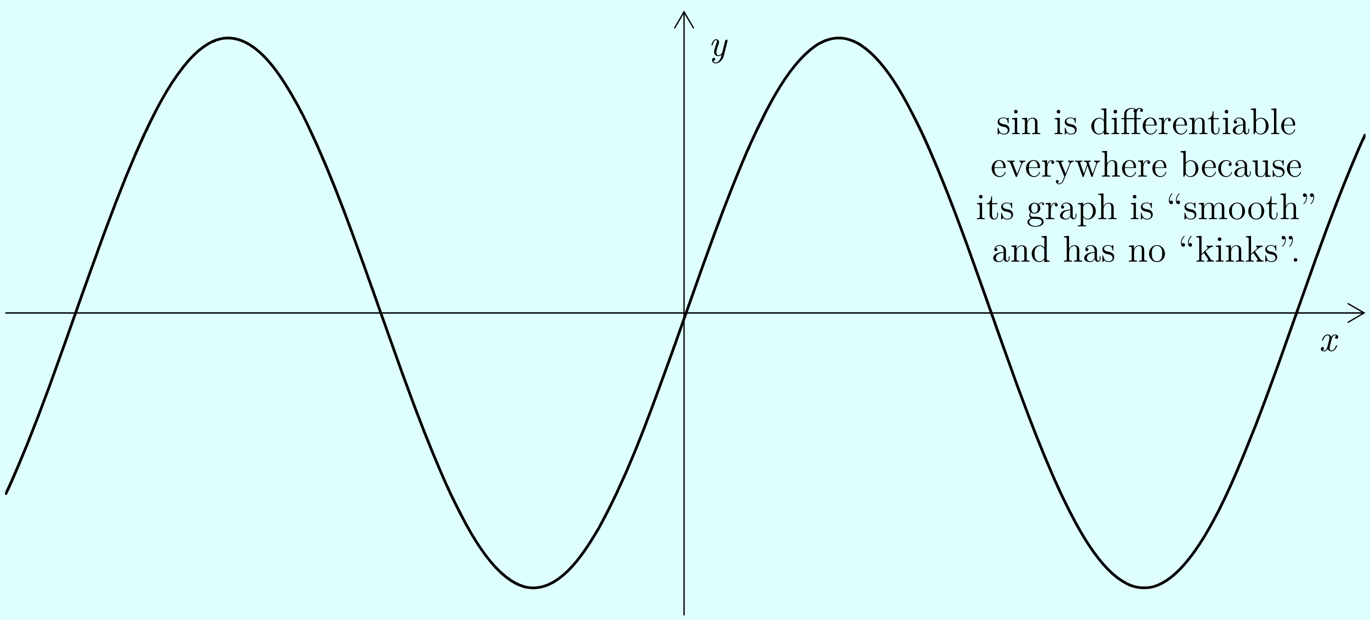
First, three examples of functions that are both continuous and differentiable:

Example 626. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^5 - x^2 - 1$ is continuous everywhere, because you can draw its entire graph without lifting your pencil.



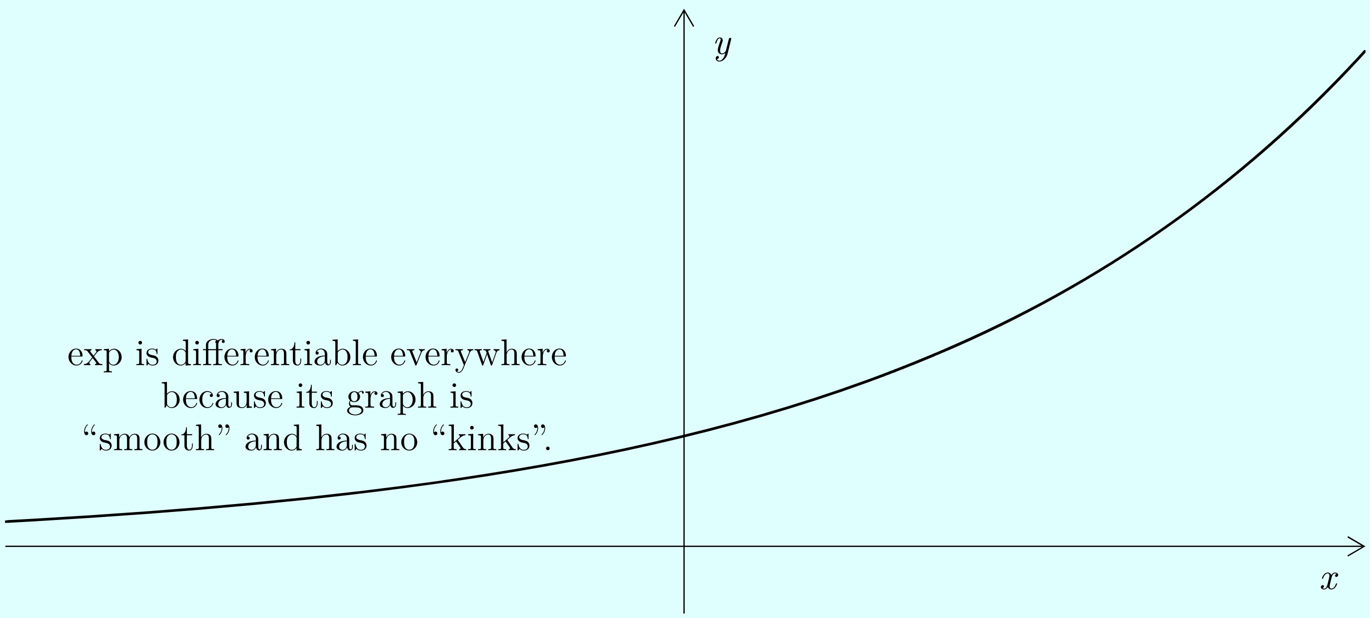
The function f is also differentiable everywhere, because it is "smooth" everywhere and has no "kinks".

Example 627. The sine function sin is both continuous everywhere and differentiable everywhere.



[291]For the formal definition of differentiability, see Definition 204 in Part V (Calculus).

Example 628. The exponential function exp is both continuous everywhere and differentiable everywhere.

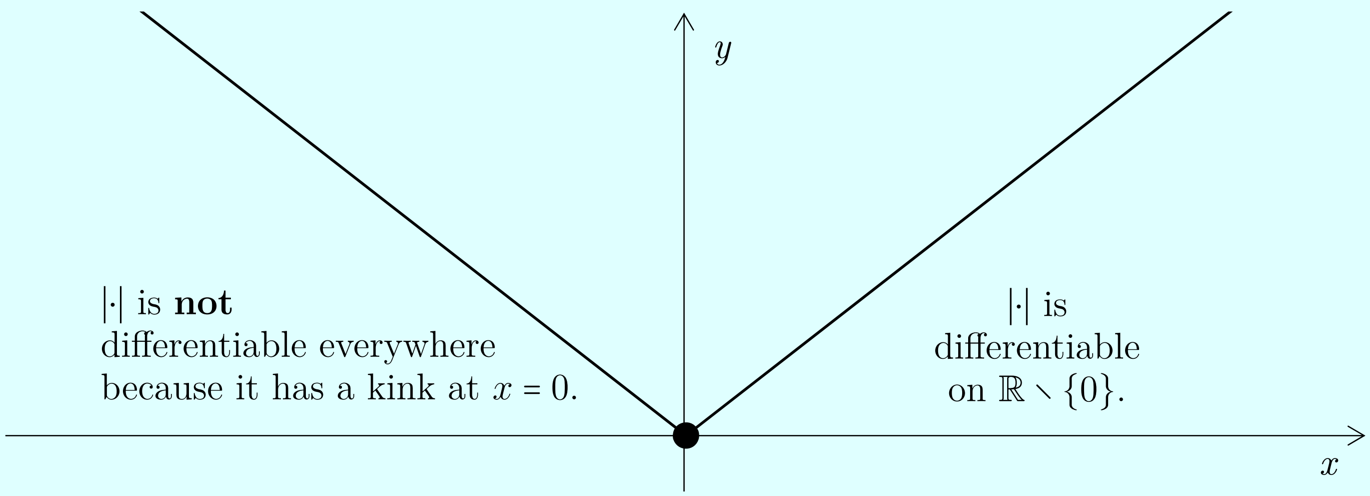


Next, two examples of functions that are continuous but not differentiable:

Example 629. The absolute value function $|\cdot|$ is continuous everywhere, because you can draw its entire graph without lifting your pencil.

However, it is **not** differentiable everywhere, because it is **not** "smooth" everywhere. In particular, it has a "kink" at $x = 0$.

We can say though that the absolute value function is differentiable everywhere *except* at $x = 0$. Or equivalently, it is differentiable on $\mathbb{R} \setminus \{0\}$.

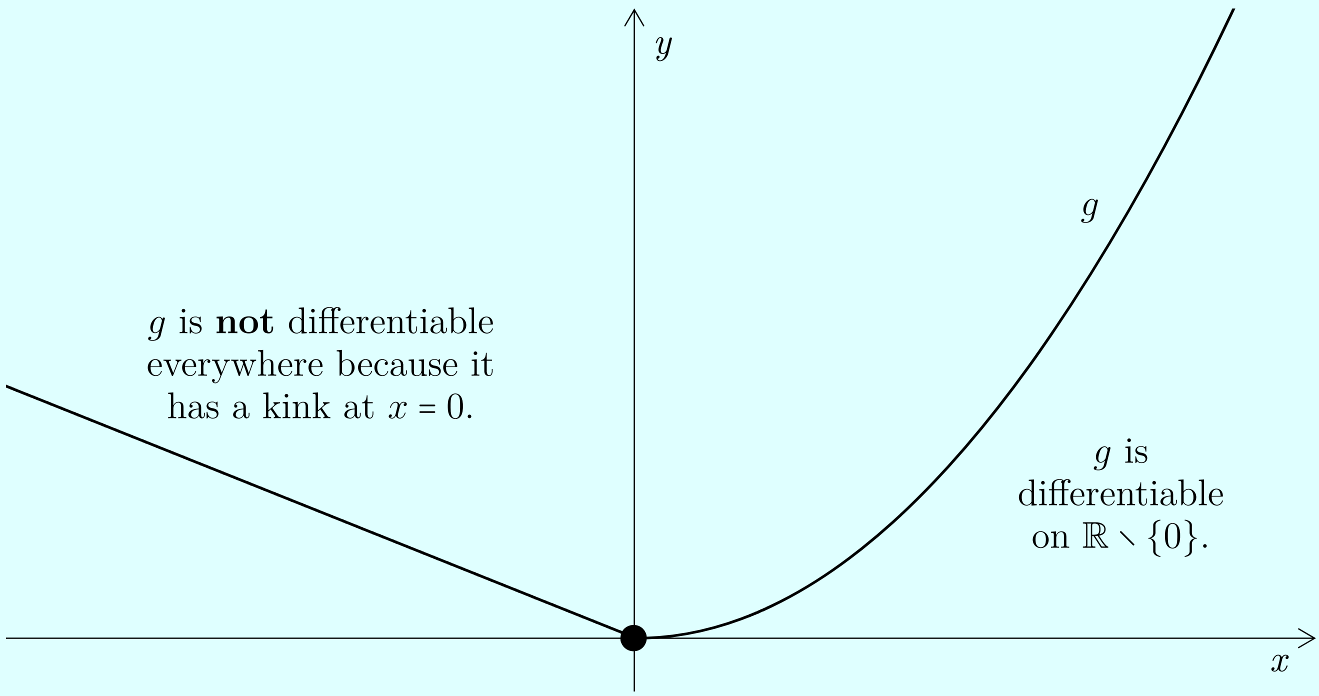


Example 630. Define the piecewise function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} -x, & \text{for } x \leq 0 \\ x^2, & \text{for } x > 0. \end{cases}$$

The function g is continuous everywhere, because you can draw its entire graph without lifting your pencil.

However, it is not differentiable everywhere because, like $|\cdot|$, it has a "kink" at $x = 0$. Nonetheless, again, we can say that g is differentiable everywhere *except* at $x = 0$. Or equivalently, g is differentiable on $\mathbb{R}\backslash\{0\}$.

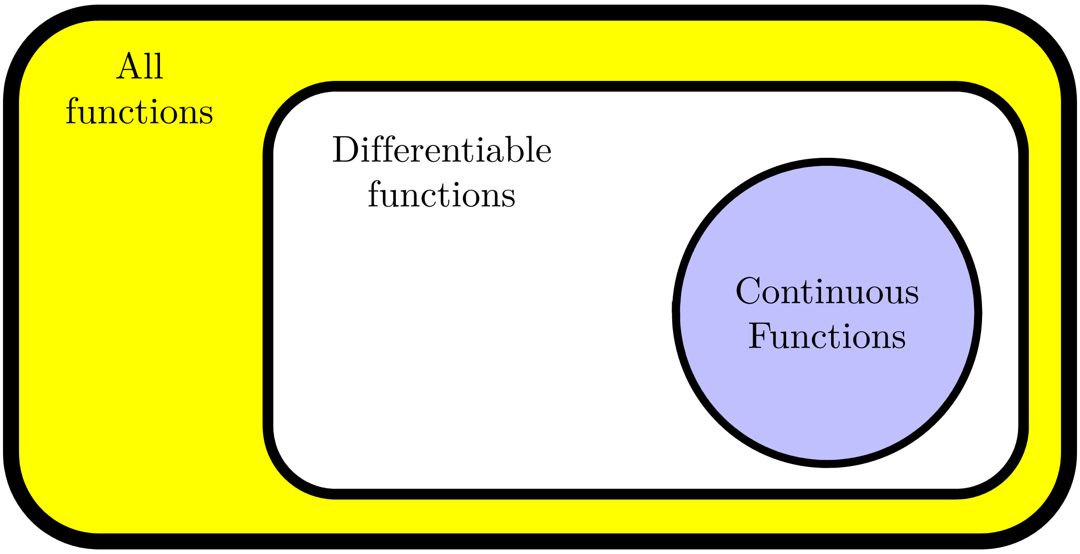


It turns out that **every differentiable function must also be continuous**.[292] (This is exactly what we meant when we said that differentiability is a stronger condition than continuity.)

However, the converse is false—not every continuous function is differentiable; that is, continuity does **not** imply differentiability. This was illustrated by the last two examples.

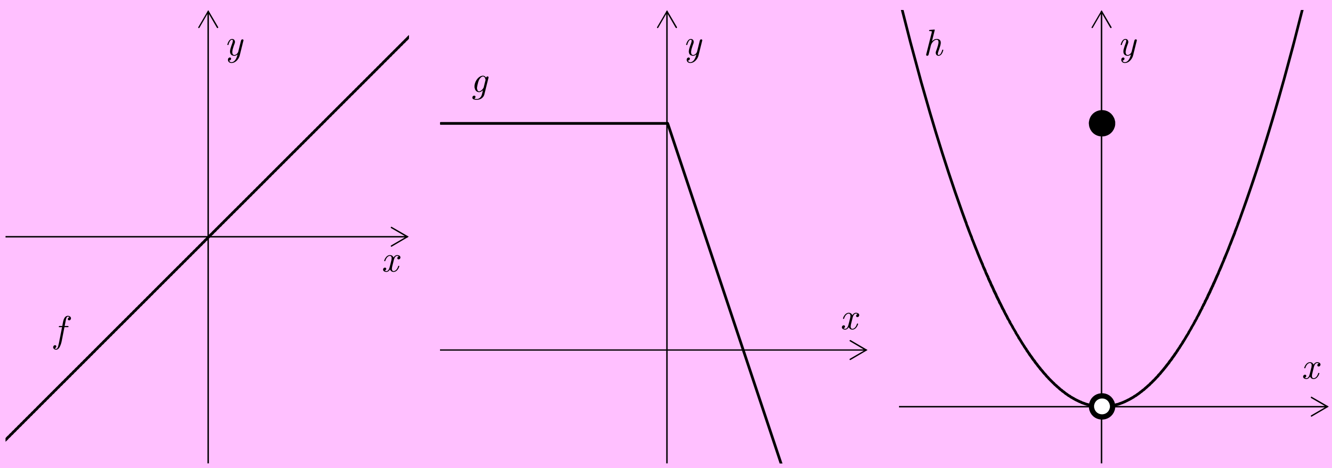
Differentiable functions are thus a subset of continuous functions. Beautiful Venn diagram drawn by an artistic genius:

[292]This assertion is formally stated and proven as Theorem 29 (Appendices).



Happily, most functions we'll encounter in A-Level maths will be both continuous and differentiable. There are however exceptions, as we've seen here and in Ch. 18.

Exercise 184. Graphed below are three functions f, g, and h. State if each is continuous and/or differentiable. If not, state the set of points on which each function is continuous or differentiable. (Answer on p. 1794.)



43.5. A Subtle Point about Differentiability

This subchapter parallels Ch. 18.1 ("A Subtle Point about Continuity"):

We'll formally define differentiability only in Part V (Calculus).

It turns out that given a function f, we say that f is

- Either differentiable or not (and not both) at each $x \in \text{Domain } f$ (Definition 198);
- Differentiable if it is differentiable at every $x \in \text{Domain } f$ (Definition 199);
- Neither differentiable nor differentiable at every $a \notin \text{Domain } f$

The last bullet point is somewhat subtle and surprising. We recycle the same three examples from Ch. 18.1 to illustrate it:

Example 631. Define $f : [1, 2] \cup [3, 4] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{for } x \in [1, 2], \\ 2, & \text{for } x \in [3, 4]. \end{cases}$$

Figure to be
inserted here.

It turns out that f is differentiable at every $x \in \text{Domain } f = [1, 2] \cup [3, 4]$.

And so, perhaps surprisingly, f is a differentiable function.

Example 632. Define $g : (0, 1) \cup (1, 2) \cup (2, 3) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1, & \text{for } x \in (0, 1), \\ 2 & \text{for } x \in (1, 2), \\ 3, & \text{for } x \in (2, 3). \end{cases}$$

Figure to be
inserted here.

It turns out that g is differentiable at every $x \in \text{Domain } f = (0, 1) \cup (1, 2) \cup (2, 3)$.

And so, perhaps surprisingly, g is a differentiable function.

Example 633. Again, it turns out that tan is differentiable at every $x \in$ Domain tan $= \mathbb{R} \setminus \{\text{Odd integer multiples of } \pi/2\}$.

Figure to be
inserted here.

And so, perhaps surprisingly, tan is a differentiable function.

43.6. Rules of Differentiation

You should find these familiar from secondary school:

Theorem 14. **(Rules of Differentiation)** *Let c be a constant and x, y, and z be variables. Then*

Constant Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}c \stackrel{\mathrm{C}}{=} 0$$

Constant Factor Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(cy) \stackrel{\mathrm{F}}{=} c\frac{\mathrm{d}y}{\mathrm{d}x}$$

Power Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}x^c \stackrel{\mathrm{P}}{=} cx^{c-1}$$

Sum and Difference Rules

$$\frac{\mathrm{d}}{\mathrm{d}x}(y \pm z) \stackrel{\pm}{=} \frac{\mathrm{d}y}{\mathrm{d}x} \pm \frac{\mathrm{d}z}{\mathrm{d}x}$$

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(yz) \stackrel{\times}{=} z\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}z}{\mathrm{d}x}$$

Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{y}{z} \stackrel{\div}{=} \frac{z\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\mathrm{d}z}{\mathrm{d}x}}{z^2}$$

Sine

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

Cosine

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

Natural Logarithm

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$

Exponential

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$$

(The **Chain Rule** will be covered in the next subchapter.)

Common mnemonic for the Quotient Rule:

Lo-D-Hi *minus* Hi-D-Lo,
Cross over and square the Lo.

In Part V (Calculus), we'll explain where these rules come from (and even learn to derive them). For now, you need merely "know" these rules and how to use them to "solve" differentiation problems:

Example 634. $\dfrac{\mathrm{d}}{\mathrm{d}x}5 \overset{C}{=} 0.$ **(Constant Rule)**

Example 635. $\dfrac{\mathrm{d}}{\mathrm{d}x}500 \overset{C}{=} 0.$ **(Constant Rule)**

Example 636. $\dfrac{\mathrm{d}}{\mathrm{d}x}(-200) \overset{C}{=} 0.$ **(Constant Rule)**

Example 637. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(5x^3\right) \overset{F}{=} 5\left(\dfrac{\mathrm{d}}{\mathrm{d}x}x^3\right) = 5\left(3x^2\right) = 15x^2.$ **(Constant Factor Rule)**

Example 638. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(500x^{0.3}\right) \overset{F}{=} 500\left(\dfrac{\mathrm{d}}{\mathrm{d}x}x^{0.3}\right) = 500\left(0.3x^{-0.7}\right) = 150x^{-0.7}.$ **(CFR)**

Example 639. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(-200x^{-1}\right) \overset{F}{=} -200\left(\dfrac{\mathrm{d}}{\mathrm{d}x}x^{-1}\right) = -200\left(-x^{-2}\right) = 200x^{-2}.$ **(CFR)**

Example 640. $\dfrac{\mathrm{d}}{\mathrm{d}x}x^3 \overset{P}{=} 3x^2.$ **(Power Rule)**

Example 641. $\dfrac{\mathrm{d}}{\mathrm{d}x}x^{0.3} \overset{P}{=} 0.3x^{-0.7}.$ **(Power Rule)**

Example 642. $\dfrac{\mathrm{d}}{\mathrm{d}x}x^{-1} \overset{P}{=} -x^{-2}.$ **(Power Rule)**

Example 643. Sum and Difference Rules:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 + 500x^{0.3}\right) &\overset{\pm}{=} \frac{\mathrm{d}}{\mathrm{d}x}x^3 + \frac{\mathrm{d}}{\mathrm{d}x}\left(500x^{0.3}\right) = 3x^2 + 150x^{-0.7}.\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 - 500x^{0.3}\right) &\overset{\pm}{=} \frac{\mathrm{d}}{\mathrm{d}x}x^3 - \frac{\mathrm{d}}{\mathrm{d}x}\left(500x^{0.3}\right) = 3x^2 - 150x^{-0.7}.
\end{aligned}$$

Example 644. Product Rule and Sine:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 \sin x\right) \overset{\times}{=} 3x^2 \sin x + x^3 \cos x.$$

Example 645. Quotient Rule and Cosine:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{\cos x} \overset{\div}{=} \frac{3x^2 \cos x - x^3(-\sin x)}{\cos^2 x} = \frac{x^2}{\cos^2 x}(3\cos x + x \sin x).$$

Example 646. Combination of various rules:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^{2x}}{\ln x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^x\mathrm{e}^x}{2\ln x} \overset{\mathrm{F}}{=} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^x\mathrm{e}^x}{\ln x} \\ &\overset{\div}{=} \frac{1}{2}\frac{\ln x\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x\mathrm{e}^x\right)-\left(\mathrm{e}^x\mathrm{e}^x\right)\frac{\mathrm{d}}{\mathrm{d}x}\ln x}{\left(\ln x\right)^2} \\ &\overset{\mathrm{P}}{=} \frac{1}{2}\frac{\ln x\left(\mathrm{e}^x\mathrm{e}^x+\mathrm{e}^x\mathrm{e}^x\right)-\left(\mathrm{e}^x\mathrm{e}^x\right)\frac{1}{x}}{\left(\ln x\right)^2} \\ &= \frac{\mathrm{e}^{2x}}{2}\frac{2\ln x-\frac{1}{x}}{\left(\ln x\right)^2} = \frac{\mathrm{e}^{2x}}{\ln x}\left(1-\frac{1}{2x\ln x}\right).\end{aligned}$$

Exercise 185. For each, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}$. (Answer on p. 1794.)

(a) $y = x^2$.

(b) $y = 3x^5 - 4x^2 + 7x - 2$.

(c) $y = \left(x^2+3x+4\right)\left(3x^5-4x^2+7x-2\right)$.

Exercise 186. For each, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ **without using the chain rule**.

(a) $y = \mathrm{e}^x \ln x$.

(b) $y = x^2\mathrm{e}^x\ln x$.

(c) $y = \frac{\sin x}{x}$.

(d) $y = \tan x$, given that $\tan x = \frac{\sin x}{\cos x}$ and $\sin^2 x + \cos^2 x = 1$.

(e) $y = \frac{1}{z}$, where z is a variable that can be expressed in terms of x. (Leave your answer in terms of z and $\frac{\mathrm{d}z}{\mathrm{d}x}$.)

Use **(e)** to solve **(f)**, **(g)** and **(h)**:

(f) $y = \mathrm{cosec}x$, where $\mathrm{cosec}x = \frac{1}{\sin x}$.

(g) $y = \sec x$, where $\sec x = \frac{1}{\cos x}$.

(h) $y = \cot x$, where $\cot x = \frac{1}{\tan x}$. (Answers on p. 1794.)

43.7. The Chain Rule

The **Chain Rule** informally stated:[293]

Theorem 15 (informal). **(Chain Rule)** *Suppose y and z are functions. Then*

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

One mnemonic is to think of the derivatives on the RHS as fractions—in which case, the dy's get cancelled out and we're left with dz/dx.

In Part V (Calculus), we'll explain **(a)** why this is merely a mnemonic; **(b)** strictly speaking, it's **wrong** to think of derivatives as fractions; **(c)** nonetheless, such thinking can be a helpful aid to intuition.

We can give the Chain Rule this informal interpretation:

$$\begin{matrix}\text{The change in } z \text{ caused by} \\ \text{a small unit change in } x\end{matrix} = \begin{matrix}\text{The change in } z \text{ caused by} \\ \text{a small unit change in } y\end{matrix} \times \begin{matrix}\text{The change in } y \text{ caused by} \\ \text{a small unit change in } x.\end{matrix}$$

This interpretation makes (common) sense:

Example 647. Suppose that when I add to a cup of water 1 g of Milo, its water volume increases by $2\,\text{cm}^3$. And when the water volume increases by $1\,\text{cm}^3$, the water level rises by 0.3 cm.

Then by common sense, when I add 1 g of Milo, the water level should rise by $2 \times 0.3 = 0.6\,\text{cm}$.

Let's now rewrite the above common-sense observations more formally:

Let x be the mass (g) of Milo in a cup of water, y be the total volume $\left(\text{cm}^3\right)$ of water in the cup, and z be the water level (cm) in the cup.

- When x increases by 1 g, y increases by $2\,\text{cm}^3$.

More formally,
$$\frac{dy}{dx} = 2\,\text{cm}^3\,\text{g}^{-1}.$$

- When y increases by $1\,\text{cm}^{-3}$, z increases by 0.3 cm.

More formally,
$$\frac{dz}{dy} = 0.3\,\text{cm}\,\text{cm}^{-3} = 0.3\,\text{cm}^{-2}.$$

- And so, by the Chain Rule, when x increases by 1 g, z increases by $2 \times 0.3 = 0.6\,\text{cm}$.

More formally,
$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 0.3\,\text{cm}^{-2} \times 2\,\text{cm}^3\,\text{g}^{-1} = 0.6\,\text{cm}\,\text{g}^{-1}.$$

[293] For a formal statement, see Theorem 32 (Part V).

Examples of how to "use" the chain rule:

Example 648. If $y = e^{\sin x}$, then

$$\frac{dy}{dx} = \frac{de^{\sin x}}{dx} \stackrel{\text{Ch}}{=} \frac{de^{\sin x}}{d\sin x}\frac{d\sin x}{dx} = e^{\sin x}\cos x.$$

Example 649. If $y = \sqrt{4x-1}$, then

$$\frac{dy}{dx} = \frac{d\sqrt{4x-1}}{dx} \stackrel{\text{Ch}}{=} \frac{d\sqrt{4x-1}}{d(4x-1)}\frac{d(4x-1)}{dx} = \frac{1}{2\sqrt{4x-1}} \times 4 = \frac{2}{\sqrt{4x-1}}.$$

A slightly more complicated example, where we use the Chain Rule more than once:

Example 650. If $y = [\sin(2x-3) + \cos(5-2x)]^3$, then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d[\sin(2x-3) + \cos(5-2x)]^3}{dx} \\
&\stackrel{\text{Ch}}{=} \frac{d[\sin(2x-3) + \cos(5-2x)]^3}{d[\sin(2x-3) + \cos(5-2x)]}\frac{d[\sin(2x-3) + \cos(5-2x)]}{dx} \\
&= 3[\sin(2x-3) + \cos(5-2x)]^2\left[\frac{d\sin(2x-3)}{dx} + \frac{d\cos(5-2x)}{dx}\right] \\
&\stackrel{\text{Ch}}{=} 3[\sin(2x-3) + \cos(5-2x)]^2\left[\frac{d\sin(2x-3)}{d(2x-3)}\frac{d(2x-3)}{dx} + \frac{d\cos(5-2x)}{d(5-2x)}\frac{d(5-2x)}{dx}\right] \\
&= 3[\sin(2x-3) + \cos(5-2x)]^2[\cos(2x-3) \times 2 + (-\sin(5-2x)) \times (-2)] \\
&= 6[\sin(2x-3) + \cos(5-2x)]^2[\cos(2x-3) + \sin(5-2x)].
\end{aligned}$$

Exercise 187. For each, find $\frac{dy}{dx}$ and $\left.\frac{dy}{dx}\right|_{x=0}$. (Answer on p. 1795.)

(a) $y = 1 + [x - \ln(x+1)]^2$. **(b)** $y = \sin \frac{x}{1 + [x - \ln(x+1)]^2}$.

Exercise 188. Let **F**, m, **v**, t, and **p** denote **force**, **mass**, **velocity**, **time**, and **momentum**. Momentum is defined as the product of mass and velocity.

(a) Newton's Second Law of Motion states that the rate of change of momentum (of an object) is equal to the force applied (to that object). Write this down as an equation.

(b) Acceleration **a** is defined as the rate of change of momentum. Explain why Newton's Second Law simplifies to $\mathbf{F} = m\mathbf{a}$ if mass is constant. (Answer on p. 1795.)

Exercise 189. In Part V (Calculus), Fact 221, we will formally state and prove that

$$\frac{d}{dx} \ln x \stackrel{1}{=} \frac{1}{x}.$$

But assuming $\stackrel{1}{=}$ is true, we can quite easily prove that $\frac{d}{dx} \exp x = \exp x$:

(a) Use the Chain Rule to write down an expression for $\frac{d}{dx} \ln(\exp x)$.

(b) What do you observe about the expression $\ln(\exp x)$? Use this observation to write down another expression for $\frac{d}{dx} \ln(\exp x)$.

(c) Then conclude that $\frac{d}{dx} \exp x = \exp x$. (Answer on p. 1796.)

Remark 87. By the way, no need to *mug* the five derivatives below because they're in List MF26 (p. 3). (We'll review the inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ in Ch. 36. Also, we'll restate the following as Fact 207 in Part V.)

$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$

43.8. Stationary and Turning Points

Earlier Definition 56 gave an informal definition of **stationary points**.

We can now formally define a **stationary point** (of a function) as any point at which that function's derivative equals zero:

Definition 115. A point x is a *stationary point* of a function f if $f'(x) = 0$.

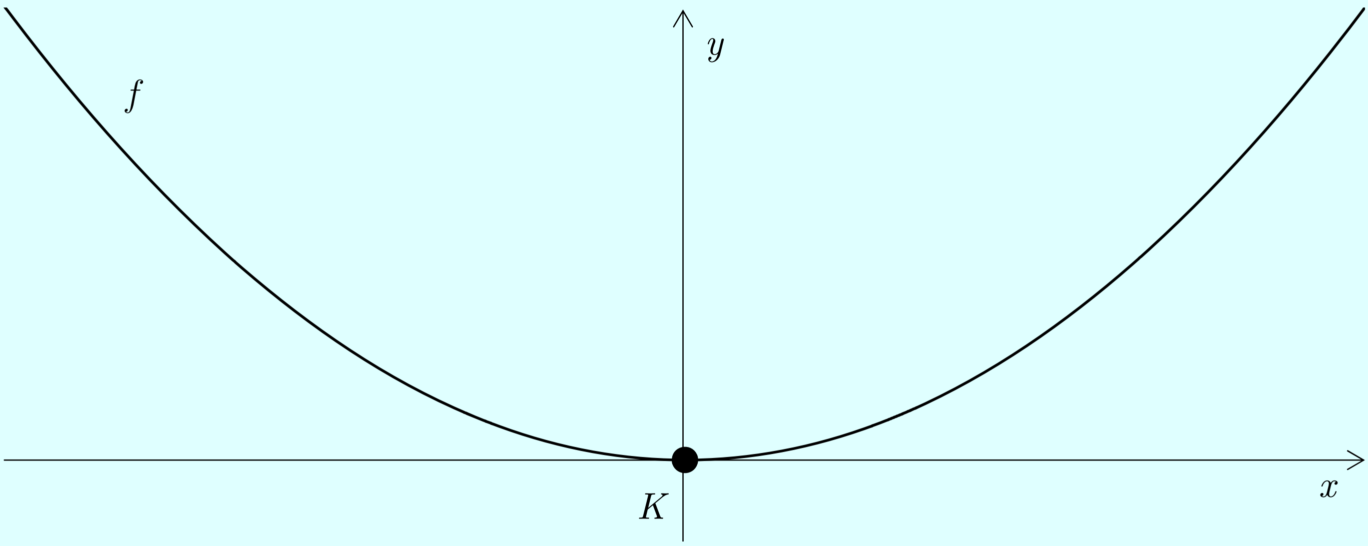
Example 651. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

The derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 2x$.

We now check if f has any stationary points. By the above definition, a is a stationary point of f if and only if

$$f'(a) \stackrel{\star}{=} 0 \qquad \Longleftrightarrow \qquad 2a = 0 \qquad \Longleftrightarrow \qquad a = 0.$$

Hence, f has only one stationary point, namely 0. (We've labelled this point as K in the graph below.)



When searching for stationary points, we will often write

$$f'(a) \stackrel{\star}{=} 0.$$

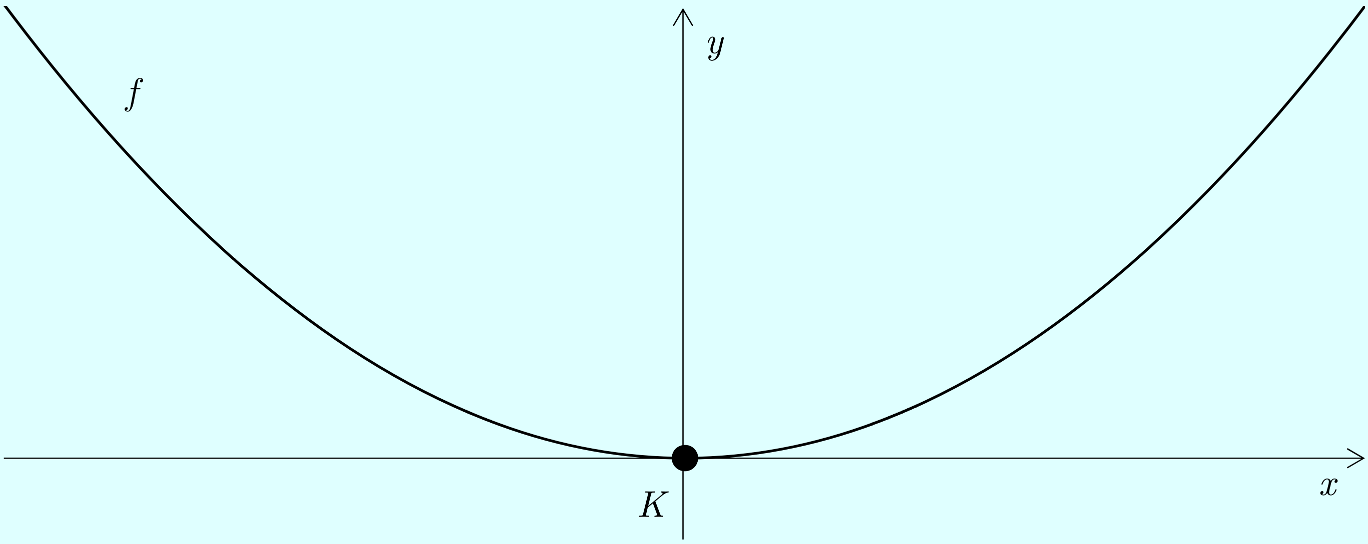
Because $\stackrel{\star}{=}$ appears so often, it has a name: the **First-Order Condition (FOC)**.

We reproduce from Ch. 12.4 our definition of a **turning point**:

Definition 58. A *turning point* is a point that's both a stationary point and a strict local maximum or minimum.

Example 652. Continue to define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^2$.

Above we showed that f has only one stationary point, namely 0 (labelled as K in the graph).



Observe that K is also a strict local minimum of f.

Since K is both a stationary point and a strict local minimum of f, by the above definition, K is a turning point of f.

Remark 88. The term *turning point* is rarely used by mathematicians (and indeed any writers). However, it appears on your O- and A-Level syllabuses and exams.[294] We shall therefore have to use it.

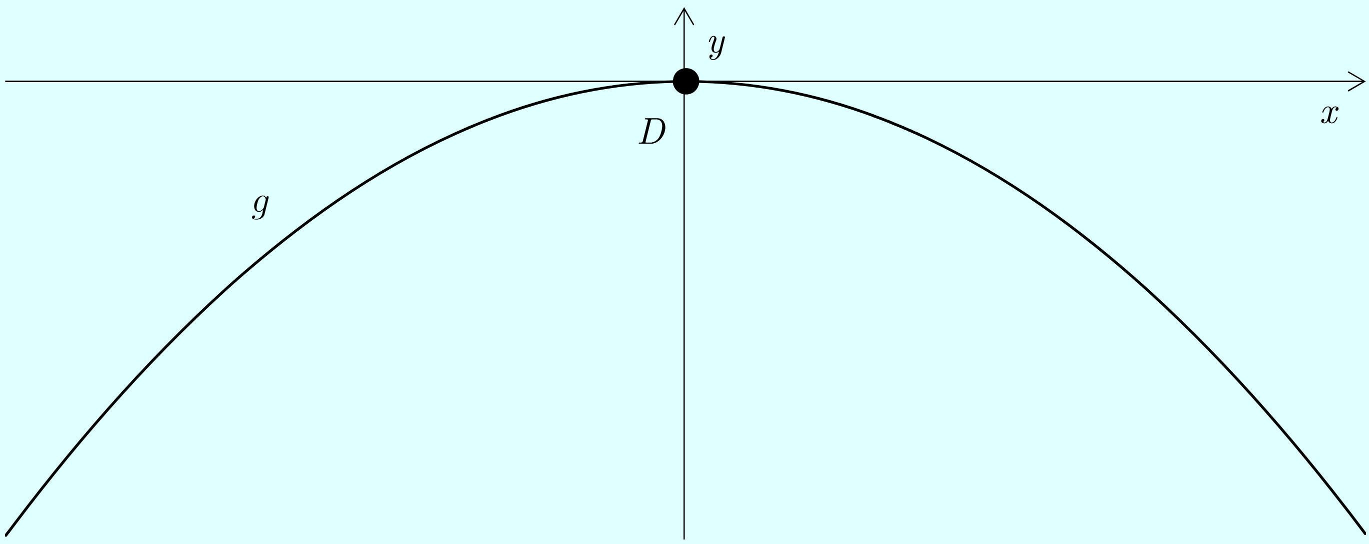
Unfortunately, I'm not sure what precisely the MOE or your examiners mean by *turning point.* They frequently use it but I'm unable to find any precise definition from them. Definition 58 is merely my best guess of what I think they mean.

[294]The term *turning point* appears on your H2 Maths syllabus (pp. 5, 15) and these A-Level exam questions: N2016/I/3, N2014/I/4, N2008/I/9, N2015/I/11, N2012/I/8, N2010/I/6, N2010/II/3, N2009/1/11.

Example 653. Consider $g:\mathbb{R}\to\mathbb{R}$ defined by $g(x)=-x^2$.

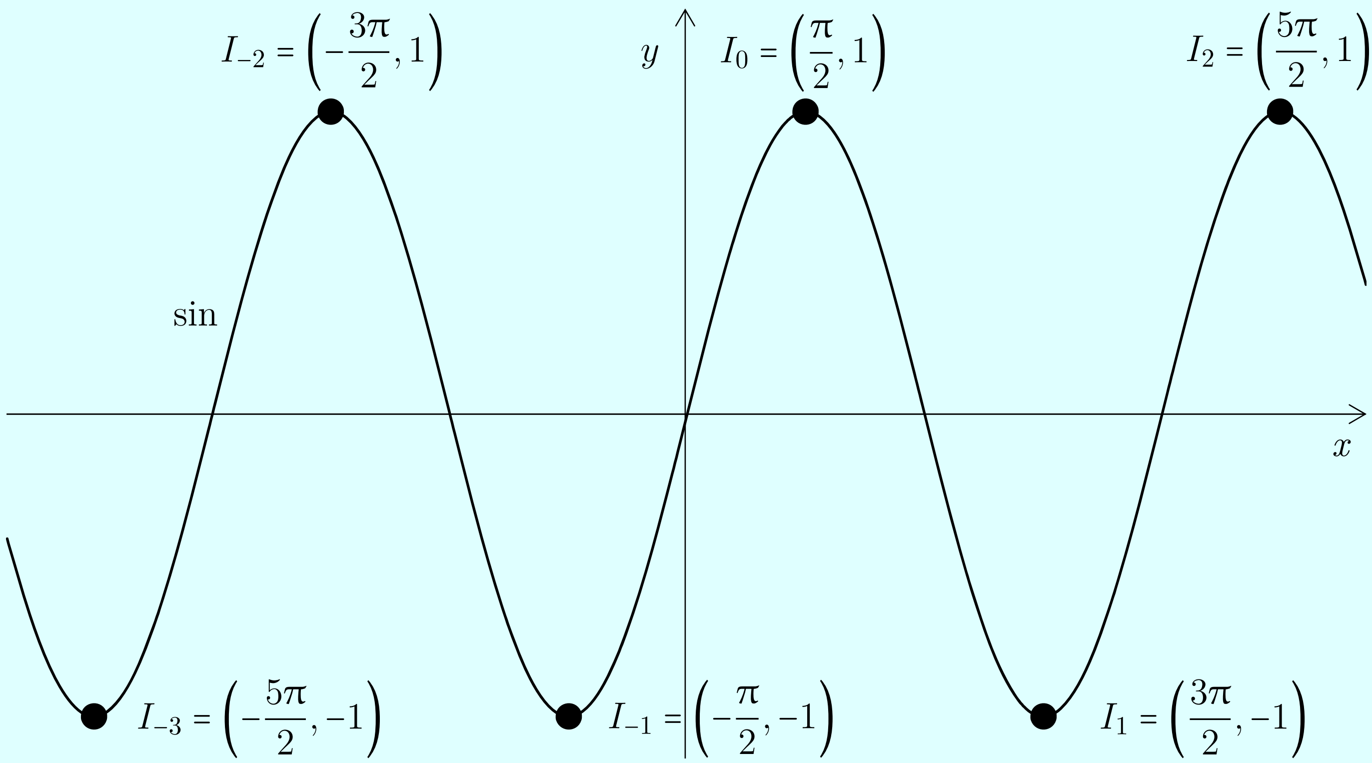
Then $D=(0,0)$ is a stationary point the derivative (or gradient) of g at K is zero:

$$g'(0)=0.$$



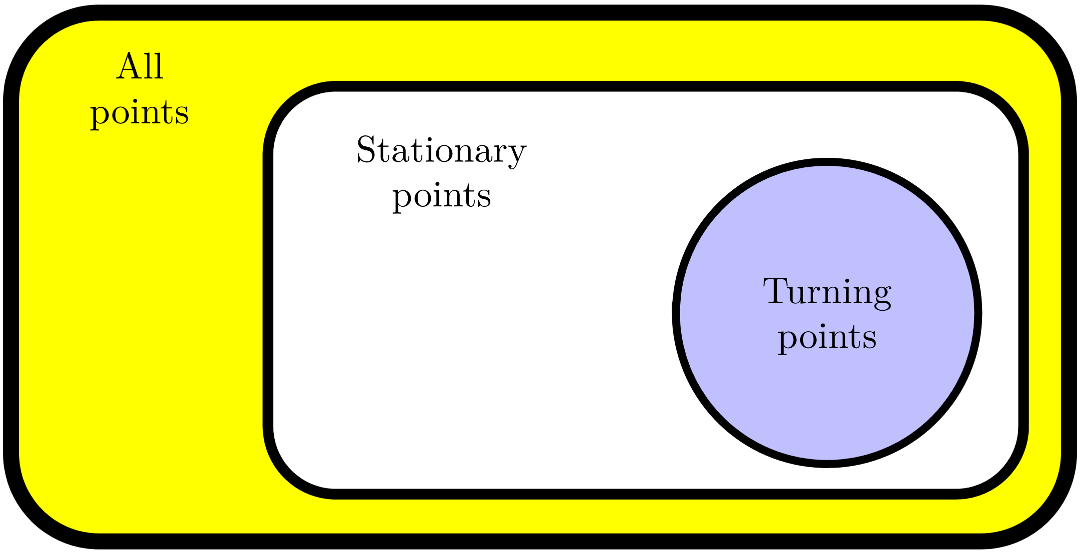
Moreover, D is also a turning point because it is both a stationary point *and* a strict extremum (in particular, it is a strict local maximum.)

Example 654. The sine function has infinitely many stationary and turning points. For every integer k, the point $I_k=((0.5+k)\pi,1)$ is **both** a stationary and a turning point.



Note that at every even integer k, the point $I_k=((0.5+k)\pi,1)$ is a strict local maximum. While at every odd integer l, the point $I_l=((0.5+2l)\pi,1)$ is a strict local minimum.

By definition, every turning point is a stationary point.

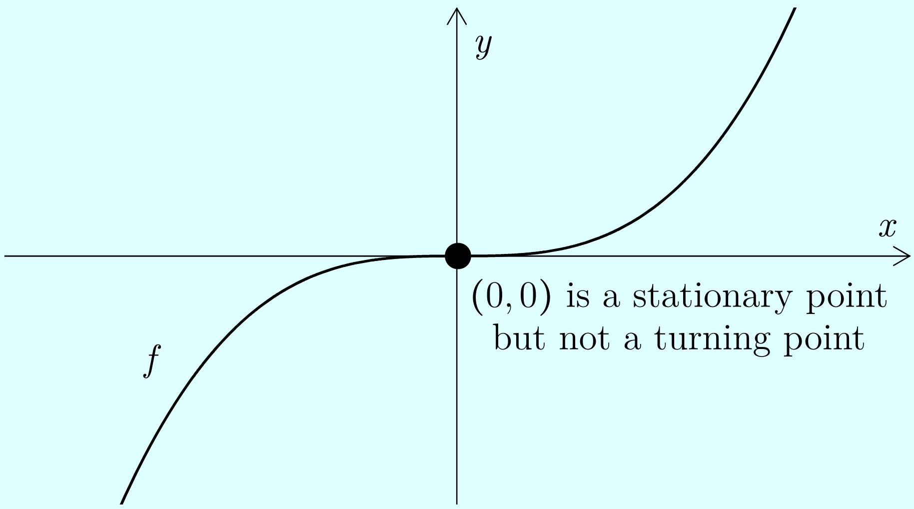


But, subtle point—the converse is false. That is, a stationary point need not be a turning point:

Example 655. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.

The origin $(0,0)$ is a stationary point of f, because the derivative at that point is zero.

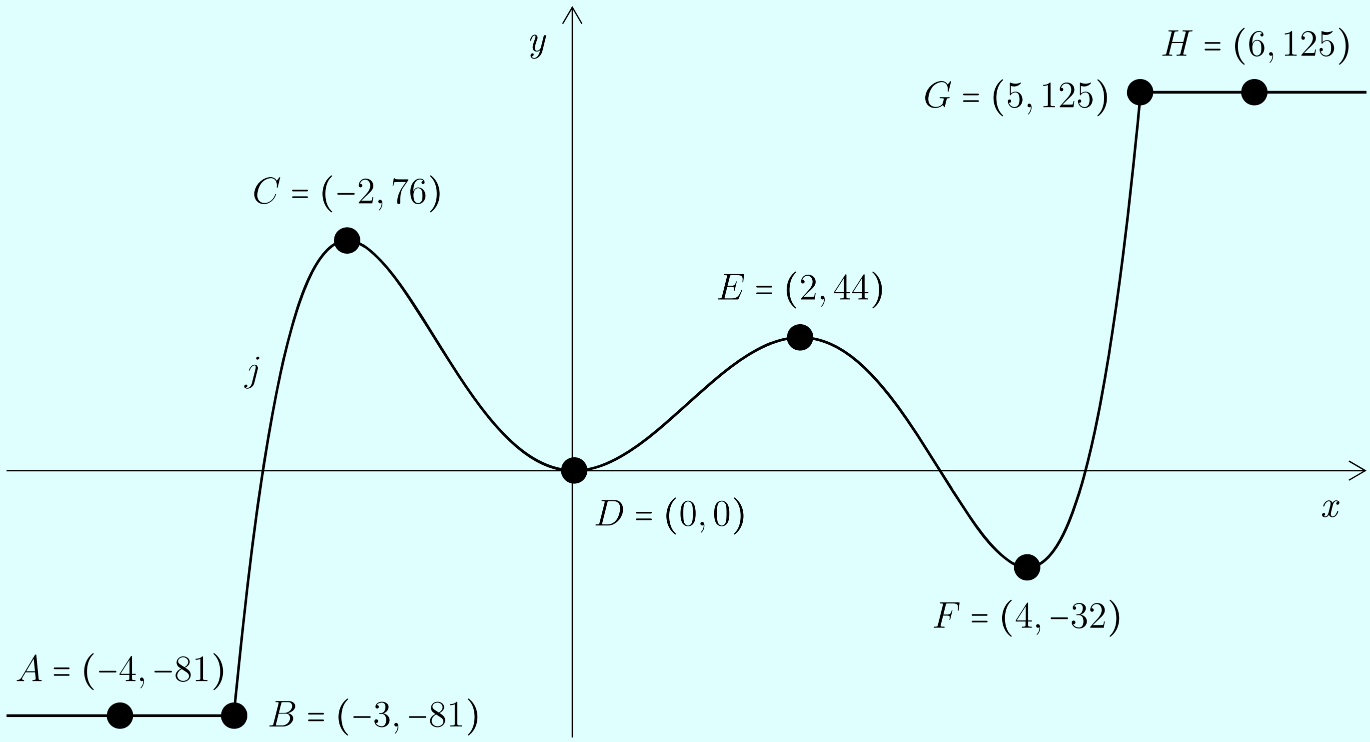
However, $(0,0)$ is not a turning point because it is not a strict extremum.



As we'll learn in Part V (Calculus), $(0,0)$ is actually an example of an **inflexion point**.

Example 656. Define $j : \mathbb{R} \to \mathbb{R}$ by

$$j(x) = \begin{cases} -81, & \text{for } x \le -3, \\ 0.75x^5 - 3.75x^4 - 5x^3 + 30x^2, & \text{for } -3 < x < 5, \\ 125, & \text{for } x \ge 5. \end{cases}$$



The derivative (or gradient) of j at each of the points A, C, D, E, F, and H is zero. Hence, each of these points is a stationary point.

Observe that moreover, C and E are strict maximum points, while D and F are strict local minimum points. Hence, the stationary points C, D, E, and F are also turning points. In contrast, A and H, which are not strict extrema, are **not** turning points.

The derivative of j at each of the points B and G is not equal to zero. Indeed and as we'll learn later on in Part V (Calculus), the derivative of j at each of those points does not even exist! Hence, neither B nor G is a stationary point. Since B and G are not stationary points, they are not turning points either.

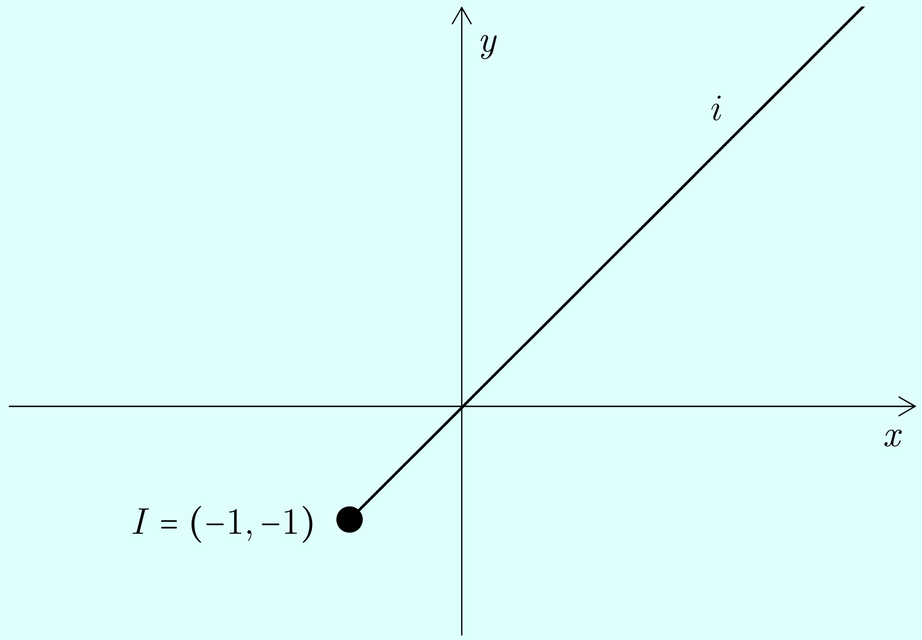
(By the way, is j continuous everywhere? Differentiable everywhere?)[295]

Every turning point is a strict local extremum.

But, second subtle point—the converse is false. That is, a strict local extremum need **not** be a turning point:

[295]The function j is continuous everywhere, but not differentiable everywhere. It is differentiable everywhere *except* at B and G.

Example 657. The function $i : [-1, \infty) \to \mathbb{R}$ defined by $i(x) = x$ has a strict local minimum at $I = (-1, -1)$. (Indeed, I is also a strict *global* minimum.)



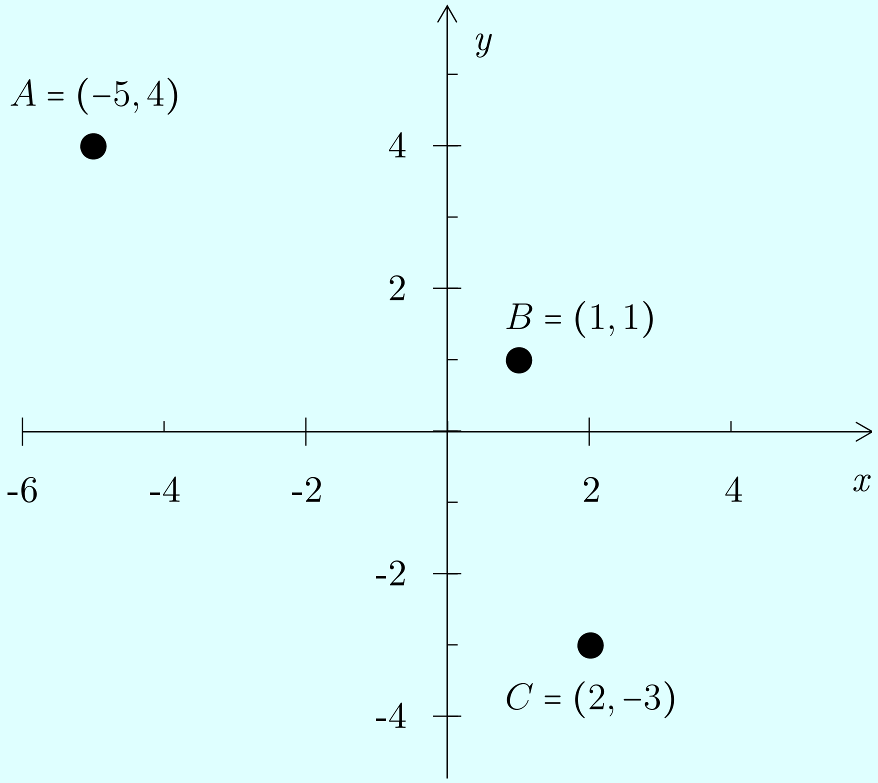
However, it can be shown that the derivative of i at I is not equal to zero.[296] So, I is not a stationary point and cannot be a turning point either.

[296]Some writers define differentiability so that i is **not** differentiable at I. In contrast, others (including this textbook) define differentiability so that i **is** differentiable at I and specifically, $i'(-1) = 1$. Either way, $i'(-1) \neq 0$.

Example 658. Define $f : \{-5, 1, 2\} \to \mathbb{R}$ by $f(-5) = 4$, $f(1) = 1$, and $f(2) = -3$.

Let $A = (-5, 4)$, $B = (1, 1)$, and $C = (2, -3)$ be points. The graph of f is simply the set $\{A, B, C\}$ of three (isolated) points.

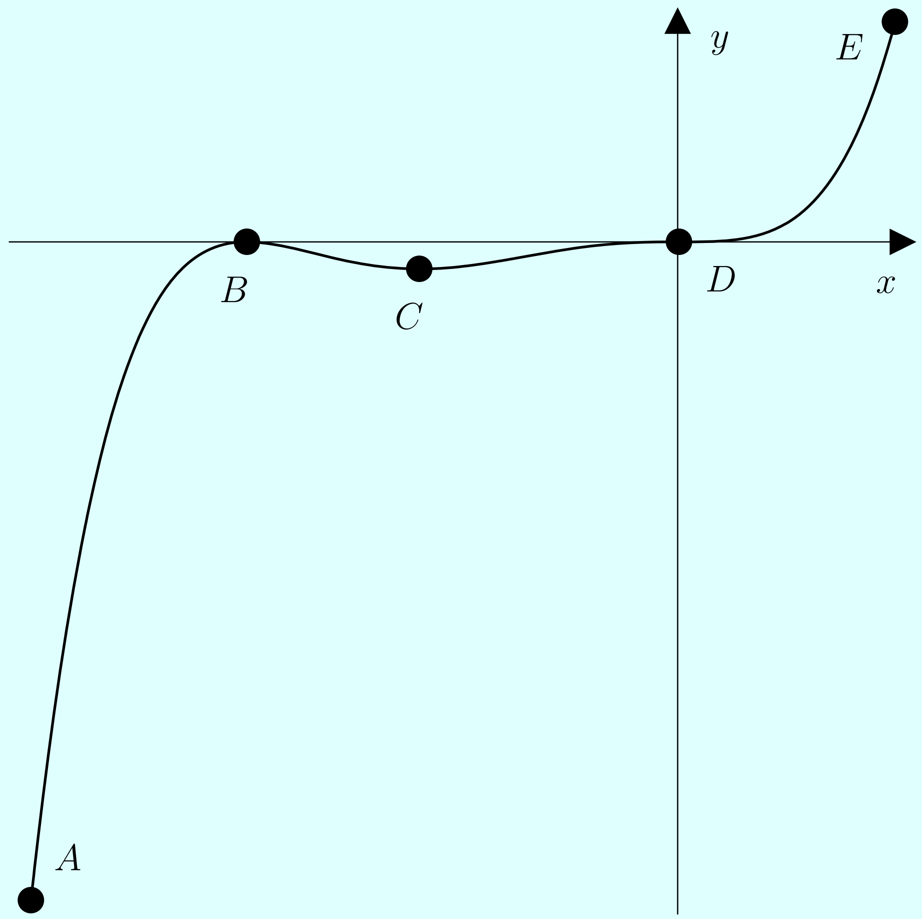
As explained earlier in Example 243, each of A, B, and C is **both** a strict local maximum **and** a strict local minimum of f.



However, it turns out that differentiability is formally defined so that a function is never differentiable at an isolated point. So here, f is not differentiable at A, B, or C (all isolated points).

Hence, f is nowhere-differentiable. Thus, f has neither stationary points nor turning points.

Example 659. Define $f : \left[-\frac{3}{2}, \frac{1}{2}\right] \to \mathbb{R}$ by $f(x) = x^5 + 2x^4 + x^3$.



	A	B	C	D	E
Global maximum					✓
Strict global maximum					✓
Local maximum		✓			✓
Strict local maximum		✓			✓
Global minimum	✓				
Strict global minimum	✓				
Local minimum	✓		✓		
Strict local minimum	✓		✓		
Turning point		✓	✓		
Stationary point		✓	✓	✓	

D is a stationary but not a turning point. (As we'll learn in Ch. 96, D is an example of an **inflexion point**.)

A and E are extrema but not turning points.

Exercise 190. For each of the following functions, identify all turning points.

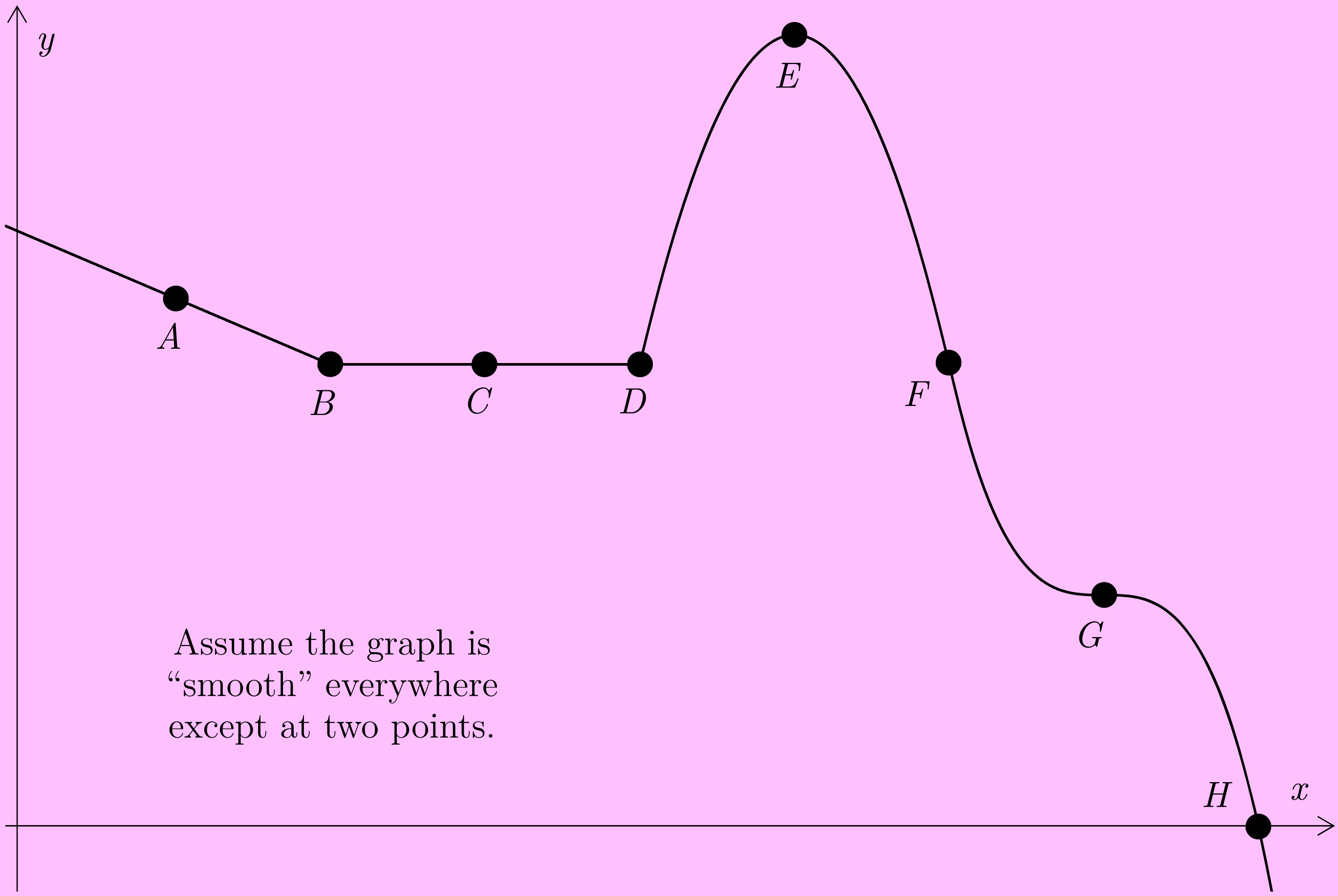
(a) $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = x^2+1$.

(b) $g:[-1,1]\to\mathbb{R}$ " $g(x) = x^2+1$

(c) $h:\mathbb{R}\to\mathbb{R}$ " $h(x) = \cos x$.

(d) $i:[-1,1]\to\mathbb{R}$ " $i(x) = \cos x$. (Answer on p. 1796.)

Exercise 191. Explain whether each of the points A–H in the graph below is a turning and/or stationary point. (Answer on p. 1796.)



Exercise 192. Is each of the following statements true or false? If true, explain why. If false, give a counterexample from Example 659.) (Answer on p. **1797**.)

(a) Every maximum point or minimum point is a stationary point.

(b) Every maximum point or minimum point is a turning point.

(c) Every stationary point is a maximum point or minimum point.

(d) Every turning point is a maximum point or minimum point.

(e) Every turning point is a stationary point.

(f) Every stationary point is a turning point.

44. Conic Sections

In this chapter, we'll study the graphs of the following eight equations. All eight are examples of **conic sections**.[297]

We first study the unit circle and ellipse centred on the origin:

$$\boxed{x^2 + y^2 = 1} \qquad \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

We then study six types of **hyperbolae**:

$$\boxed{y = \frac{1}{x}} \qquad \boxed{x^2 - y^2 = 1} \qquad \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

$$\boxed{\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1} \qquad \boxed{y = \frac{bx + c}{dx + e}} \qquad \boxed{y = \frac{ax^2 + bx + c}{dx + e}}$$

Fun Fact

Hyperbola has the same etymology as *hyperbole* (an exaggeration or overstatement).

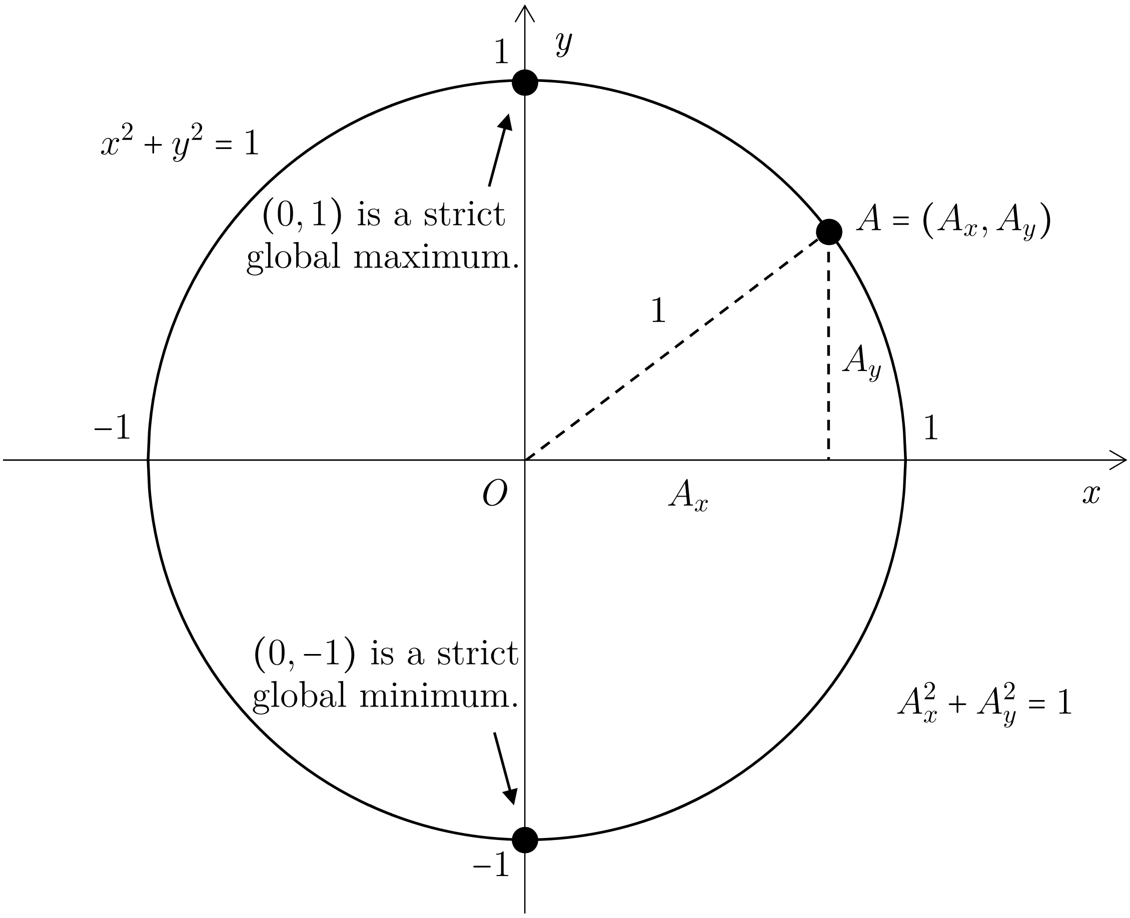
By the way, we've actually already studied one example of a conic section—this was the graph of the quadratic equation $y = ax^2 + bx + c$, which is a type of conic section called the **parabola**.

[297]For why these are called conic sections, see Ch. 142.20 (Appendices).

44.1. The Ellipse $x^2 + y^2 = 1$ (The Unit Circle)

We already discussed the **circle** in Ch. 10. It turns out that the circle is an example of a conic section. And so, for completeness, here we'll just do a quick recap.

The graph of $x^2 + y^2 = 1$ is the unit circle centred on the origin:



Some characteristics of the above graph:

1. **Intercepts.** The y-intercepts are $(0, -1)$ and $(0, 1)$. The x-intercepts are $(-1, 0)$ and $(1, 0)$.
2. **Turning points.** The two turning points[298] are $(0, 1)$ (strict global maximum) and $(0, -1)$ (strict global minimum). (This is merely by observation. We haven't actually proven that these two points are the graph's turning points.)
3. **Asymptotes.** None. (Again, this is merely by observation. We haven't actually proven that the above graph has no asymptotes.)
4. **Symmetry.** *Every* line that passes through the origin is a line of symmetry. No other line is a line of symmetry. (Again, these two assertions are merely by observation. We haven't actually proven them.)

[298] Actually, we haven't defined turning points for graphs yet. See Ch. 142.7 (Appendices) for a possible definition.

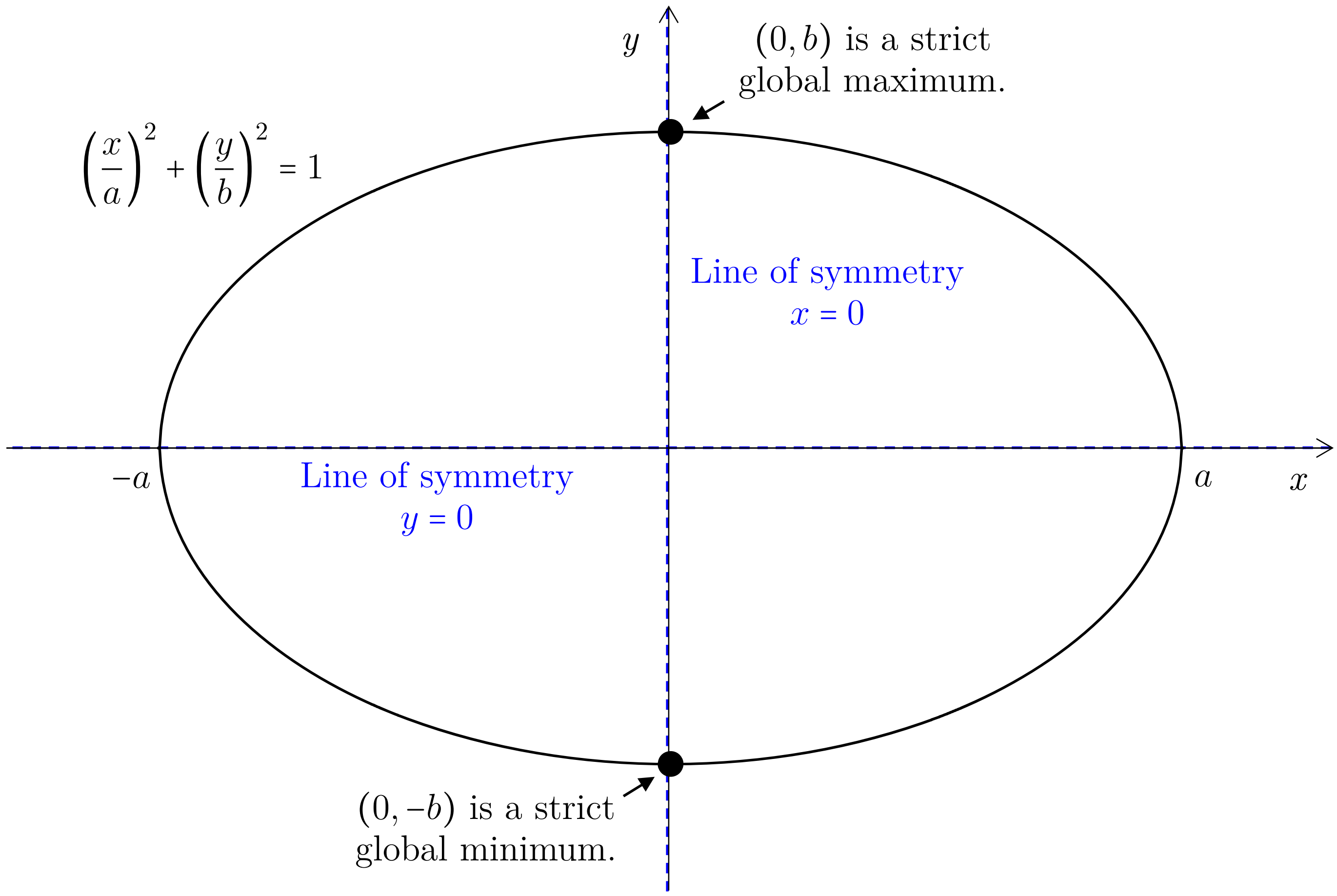
44.2. The Ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

To get from $x^2+y^2=1$ to $(x/a)^2+(y/b)^2=1$, perform these two transformations:[299]

1. To get $(x/a)^2+y^2=1$, stretch $x^2+y^2=1$ horizontally, outwards from the y-axis, by a factor of a.
2. Next, to get $(x/a)^2+(y/b)^2=1$, stretch $(x/a)^2+y^2=1$ vertically, outwards from the x-axis, by a factor of b.

Thus, $(x/a)^2+(y/b)^2=1$ is simply the unit circle stretched horizontally and vertically by factors of a and b. We call this "elongated" or "imperfect" circle an **ellipse**.

This ellipse's centre is $(0,0)$.[300]



1. **Intercepts.** The y-intercepts are $(0,-b)$ and $0,b$. The x-intercepts are $(-a,0)$ and $(a,0)$.
2. **Turning points.** By observation, the two turning points are $(0,b)$ (a strict global maximum) and $(0,-b)$ (a strict global minimum).
3. **Asymptotes.** By observation, none.

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[299] Read Ch. 26 if you haven't already.

[300] Here's one way we can define the **centre** of an ellipse: First, the ellipse has two lines of symmetry—one intersects the ellipse at the points A and B while the other intersects at C and D. (By the way, A, B, C, and D are called the ellipse's **vertices**.) If $|AB|<|CD|$, then we call the lines AB and CD the ellipse's **minor axis** and **major axis**, respectively. And vice versa. (If $|AB|=|CD|$, then we have a circle.) The centre of the ellipse may then be defined as the point at which the two axes intersect.

4. **Symmetry.** By observation, if $a \neq b$, then there are only two lines of symmetry, namely $y = 0$ (the x-axis) and $x = 0$ (the y-axis). (Note that if $a = b$, then this ellipse is in fact a circle and there are again infinitely many lines of symmetry.)

Exercise 193. Given $a, b, c, d \in \mathbb{R}$ and $a, b \neq 0$, graph the equation below. Label any turning points, asymptotes, lines of symmetry, and intercepts. (Hint in footnote.)[301]

$$\frac{(x+c)^2}{a^2} + \frac{(y+d)^2}{b^2} = 1.$$

(Answer on p. 1820.)

The rest of this chapter will look at six examples of **hyperbolae**. Our first and also the simplest example of a hyperbola is $y = 1/x$:

[301] To find the y-intercepts, plug in $x = 0$. To find the x-intercepts, plug in $y = 0$.

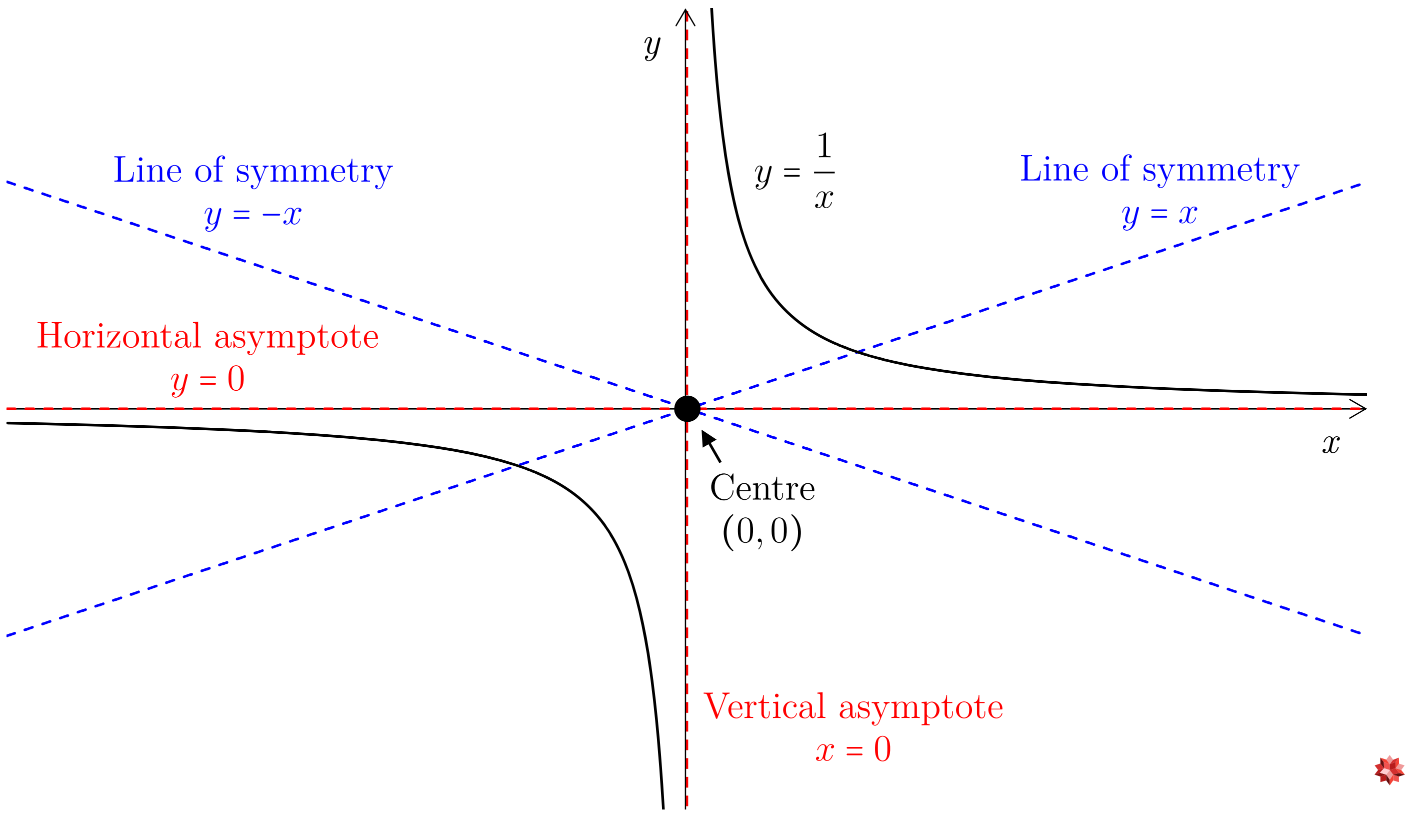
44.3. The Hyperbola $y = \dfrac{1}{x}$

All hyperbolae we'll study will share some common features:

1. There'll be **two branches**—$y = 1/x$ has a bottom-left branch and a top-right branch.
2. There may or may not be x**-** and y**-intercepts**—$y = 1/x$ has neither.
3. There may or may not be **turning points**—$y = 1/x$ has none.
4. There'll[302] be **two asymptotes**—$y = 1/x$ has horizontal asymptote $y = 0$, because as $x \to -\infty$, $y \to 0^-$ and as $x \to \infty$, $y \to 0^+$. Also, $y = 1/x$ has the vertical asymptote $x = 0$, because as $x \to 0^-$, $y \to -\infty$ and as $x \to 0^+$, $y \to \infty$.

   A **rectangular hyperbola** is any hyperbola whose two asymptotes are perpendicular—thus, $y = 1/x$ is an example of a rectangular hyperbola.
5. The hyperbola's **centre** is the point at which the two asymptotes intersect[303]—$y = 1/x$ has centre $(0, 0)$.
6. There'll be **two lines of symmetry**—each **(a)** passes through the centre; and **(b)** bisects an angle formed by the two asymptotes.

   For $y = 1/x$, they're $y = x$ and $y = -x$. Observe that indeed, each **(a)** passes through the centre; and **(b)** bisects an angle formed by the two asymptotes.

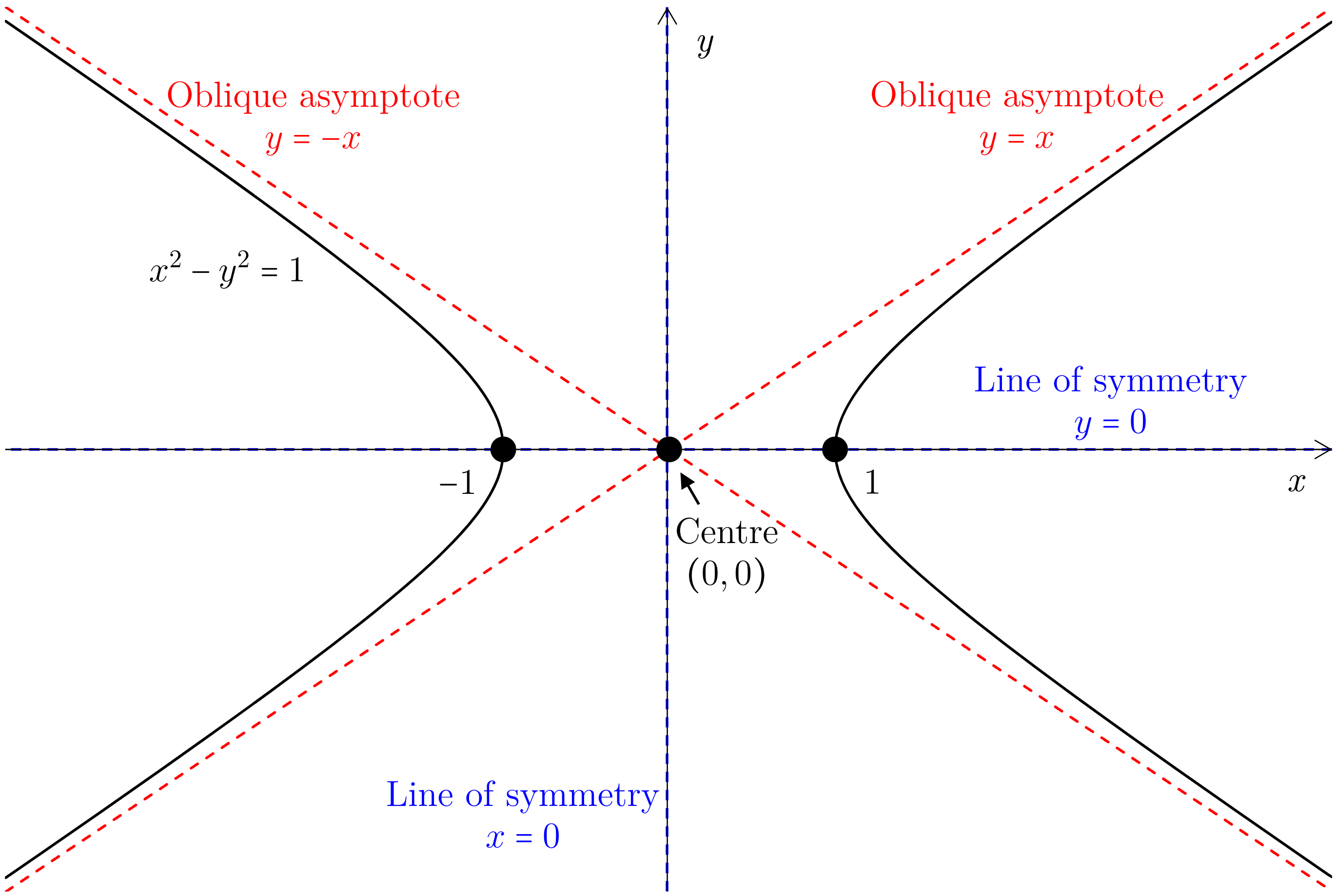


[302] Actually, we haven't defined asymptotes for graphs yet. See Ch. 146.5 (Appendices) for a possible definition.

[303] For simplicity, this shall be this textbook's *definition* of a hyperbola's **centre**.
Note though that in the usual and proper study of conic sections, the centre is instead defined as the midpoint of the line segment connecting the two **foci**. That the two asymptotes intersect at the centre is then a result rather than a definition. However, in H2 Maths, there is no mention of foci and so I thought it better to simply define the centre as where the two asymptotes intersect.

44.4. The Hyperbola $x^2 - y^2 = 1$

Consider the graph of $x^2 - y^2 = 1$. Observe that no $x \in (-1, 1)$ satisfies the equation. So, the graph contains no points for which $x \in (-1, 1)$.



1. **Two branches**—one on the left and another on the right.
2. **Intercepts**. The x-intercepts are $(-1, 0)$ and $(1, 0)$. There are no y-intercepts.
3. **Turning points**: None.
4. As $x \to -\infty$, $y \to \pm x$. And as $x \to \infty$, $y \to \pm x$. So, $x^2 - y^2 = 1$ has **two oblique asymptotes** $y = \pm x$.

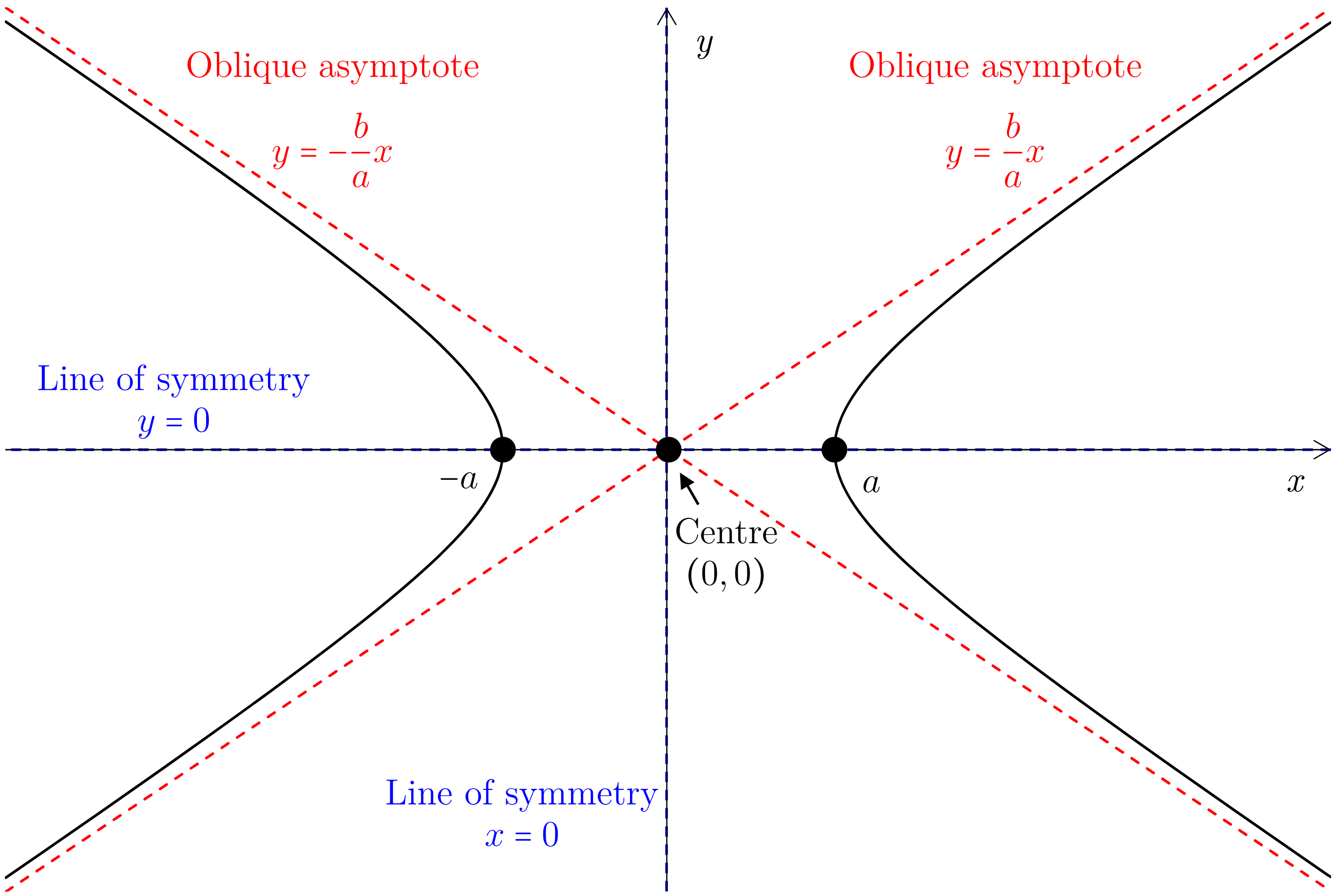
   Since the two asymptotes $y = x$ and $y = -x$ are perpendicular, this is again a **rectangular hyperbola**. (In fact, we call this an **"east-west" rectangular hyperbola**.)
5. **Centre**: $(0, 0)$.
6. **Two lines of symmetry**: $y = 0$ (the x-axis) and $x = 0$ (the y-axis). Observe that each (a) passes through the centre; and (b) bisects an angle formed by the two asymptotes.

44.5. The Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Consider the graph of $(x/a)^2 - (y/b)^2 = 1$. Again, no $x \in (-a, a)$ satisfies this equation. So, the graph contains no points for which $x \in (-a, a)$.

To get from $x^2 - y^2 = 1$ to $(x/a)^2 - (y/b)^2 = 1$, perform these two transformations:

1. To get $(x/a)^2 - y^2 = 1$, stretch $x^2 - y^2 = 1$ horizontally, outwards from the y-axis, by a factor of a.
2. Next, to get $(x/a)^2 - (y/b)^2 = 1$, stretch $(x/a)^2 - y^2 = 1$ vertically, outwards from the x-axis, by a factor of b.



1. **Two branches**—one on the left and another on the right.
2. **Intercepts**. The x-intercepts are $(-a, 0)$ and $(a, 0)$. There are no y-intercepts.
3. **Turning points**: None.
4. As $x \to -\infty$, $y \to \pm bx/a$. And as $x \to \infty$, $y \to \pm bx/a$. So, $(x/a)^2 - (y/b)^2 = 1$ has **two oblique asymptotes** $y = \pm bx/a$.

   Since the two asymptotes $y = bx/a$ and $y = -bx/a$ are perpendicular, this is again a **rectangular hyperbola**. (This is again an **"east-west" rectangular hyperbola**.)
5. **Centre**: $(0, 0)$.
6. **Two lines of symmetry** : $y = 0$ (the x-axis) and $x = 0$ (the y-axis). Observe that each (a) passes through the centre; and (b) bisects an angle formed by the two asymptotes.

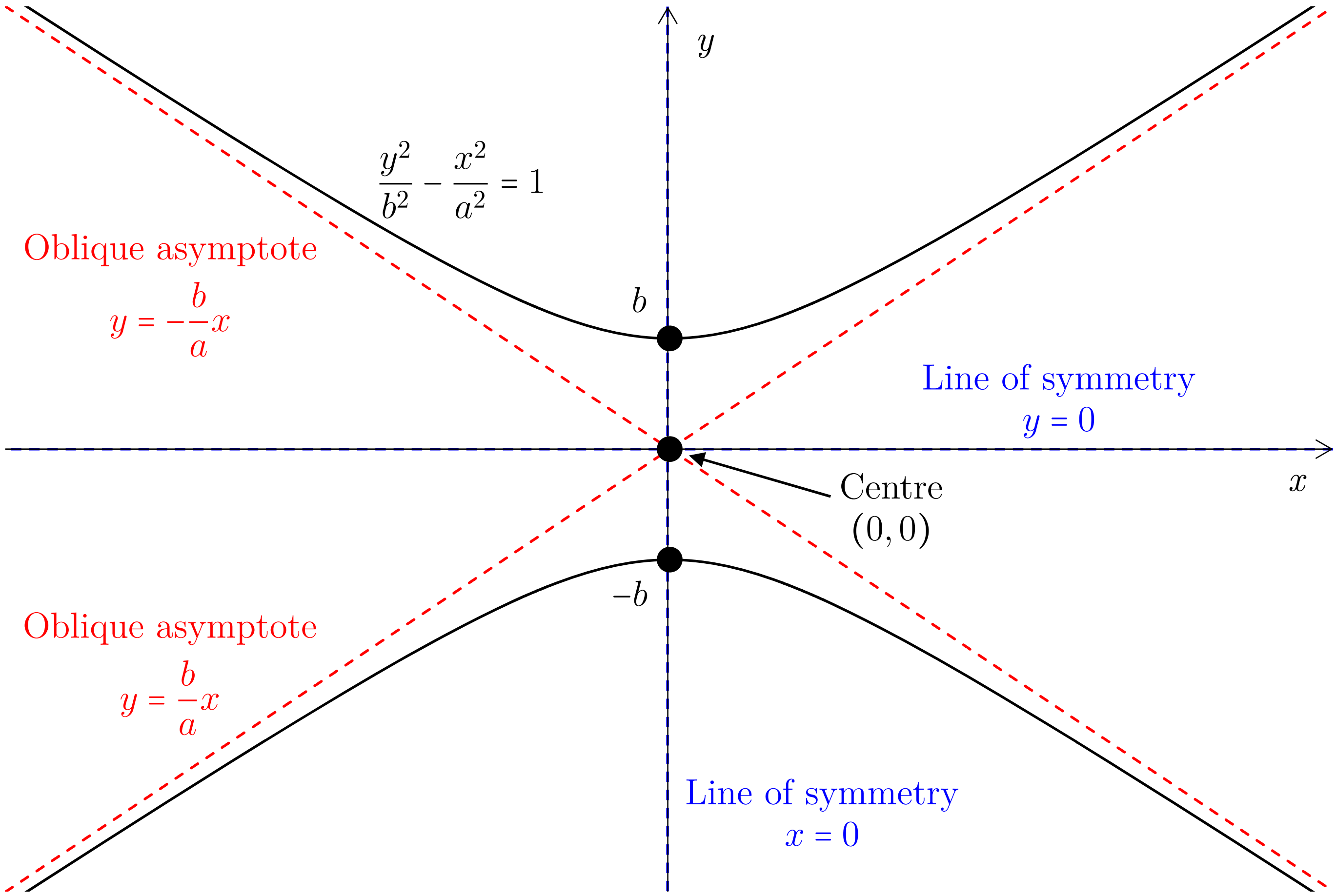
44.6. The Hyperbola $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

Take $(x/a)^2 - (y/b)^2 = 1$ (from the last subchapter), but switch a and b to get $x^2/b^2 - y^2/a^2 = 1$. Note that this is also an east-west rectangular hyperbola, but with x-intercepts $(\pm b, 0)$ instead of $(\pm a, 0)$.

To get from $x^2/b^2 - y^2/a^2 = 1$ to $y^2/b^2 - x^2/a^2 = 1$, apply any one of these transformations:

Rotate $\dfrac{\pi}{2}$ clockwise. Rotate $\dfrac{\pi}{2}$ anticlockwise. Reflect in the line $y = x$.

Observe that no $y \in (-b, b)$ satisfy $y^2/b^2 - x^2/a^2 = 1$. So, this graph contains no points for which $y \in (-b, b)$.



1. **Two branches**—one above and another below.
2. **Intercepts**. The y-intercepts are $(-b, 0)$ and $(b, 0)$. There are no x-intercepts.
3. Two **turning points**: $(0, b)$ and $(0, -b)$—the former is a strict local minimum, while the latter is a strict local maximum.
4. As $x \to -\infty$, $y \to \pm bx/a$. And as $x \to \infty$, $y \to \pm bx/a$. So, $y^2/b^2 - x^2/a^2 = 1$ has **two oblique asymptotes** $y = \pm bx/a$.

   Since the two asymptotes $y = bx/a$ and $y = -bx/a$ are perpendicular, this is again a **rectangular hyperbola**. (In fact, we call this a **"north-south" rectangular hyperbola**.)
5. **Centre**: $(0, 0)$.
6. **Two lines of symmetry**: $y = 0$ (the x-axis) and $x = 0$ (the y-axis). Each (a) passes through the centre; and (b) bisects an angle formed by the two asymptotes.

44.7. The Hyperbola $y = \dfrac{bx+c}{dx+e}$

In the next subchapter, we'll study $y = \dfrac{ax^2+bx+c}{dx+e}$.

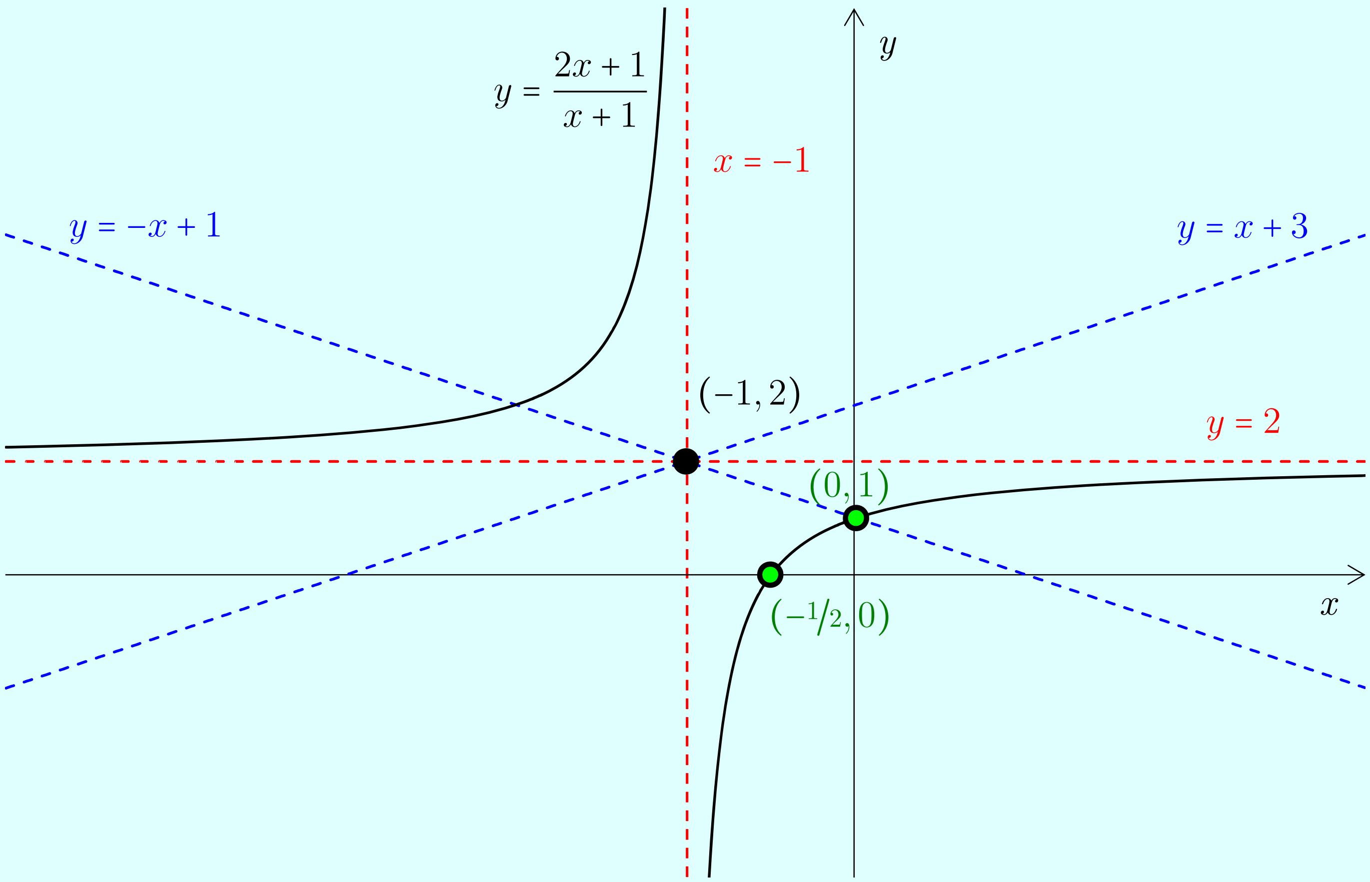
But to warm up, let's first study the simpler case where $a = 0$—i.e. $y = \dfrac{bx+c}{dx+e}$.

We'll assume that $d \neq 0$ and $cd - be \neq 0$. This is because

- If $d = 0$, then this is simply a linear equation; and
- If $cd - be = 0$, then as we'll show below, this is simply the horizontal line $y = b/d$.

Example 660. Consider $y = \dfrac{2x+1}{x+1}$. Do the long division:

$$\begin{array}{r|ll} & 2 & \\ \cline{2-3} x+1 & 2x & +1 \\ & 2x & +2 \\ \cline{2-3} & & -1 \end{array} \quad \Longrightarrow \quad y = \frac{2x+1}{x+1} = 2 - \frac{1}{x+1}.$$



1. **Two branches**—one on the top-left and another on the bottom-right.
2. **Intercepts.** Plug in $x = 0$ to get $y = (2 \cdot 0 + 1)/(0+1) = 1$—the y-intercept is $(0,1)$.

   Plug in $y = 0$ to get $2x + 1 = 0$ or $x = -1/2$—the x-intercept is $(-1/2, 0)$.
3. **Turning points**: None.
4. As $x \to -1^-$, $y \to \infty$. And as $x \to -1^+$, $y \to -\infty$. So, $y = (2x+1)/(x+1)$ has vertical asymptote $x = -1$. (Not coincidentally, this is the x-value for which $x + 1 = 0$.)

   As $x \to -\infty$, $y \to 2^+$. And as $x \to \infty$, $y \to 2^-$. So, $y = (2x+1)/(x+1)$ has horizontal asymptote $y = 2$. (Not coincidentally, this is the quotient in the above long division.)

   Since the two asymptotes $y = 2$ and $x = -1$ are perpendicular, this is again a **rectangular hyperbola**.
5. **Centre**: $(-1, 2)$. (The centre's coordinates are simply given by the vertical and horizontal asymptotes.)
6. **Two lines of symmetry**: $y = x + 3$ and $y = -x + 1$.

In general, given the hyperbola $y = (bx + c)/(dx + e)$, here's how to find its **intercepts**, **asymptotes**, and **centre**:

- **Intercepts**

Plug in $x = 0$ to get $y = (b \cdot 0 + c)/(d \cdot 0 + e) = c/e$ and thus the y-intercept $(0, c/e)$. (Note that if $e = 0$, then c/e is undefined and there is no y-intercept.)

Plug in $y = 0$ to get $bx + c = 0$ or $x = -c/b$. And thus, the x-intercept is $(-c/b, 0)$. (Note that if $b = 0$, then $-c/b$ is undefined and there is no x-intercept.)

- **Asymptotes**

The value of x for which $dx + e = 0$ is $x = -e/d$. Thus, the **vertical asymptote** is $x = -e/d$.

To find the **horizontal asymptote**, do the long division:

$$\begin{array}{r|ll} & b/d & \\ \hline dx+e & bx & +c \\ & bx & +be/d \\ \hline & & c-be/d \end{array}$$

Thus, $$\frac{bx+c}{dx+e} = \frac{b}{d} + \frac{c - \frac{be}{d}}{dx+e} = \frac{b}{d} + \frac{cd-be}{d^2}\frac{1}{x+e/d}.$$

The quotient b/d gives us the **horizontal asymptote** $y = b/d$.

Note that since we always have two perpendicular asymptotes (one horizontal and another vertical), $y = (bx + c)/(dx + e)$ is a **rectangular hyperbola**.

(By the way, note that if $cd - be = 0$, then this hyperbola is simply the horizontal line $y = b/d$. This is why above we imposed the condition that $cd - be \neq 0$.)

- We've defined the **centre** to be the point at which the two asymptotes intersect. And so, the centre is simply $(-e/d, b/d)$. (These coordinates are simply given by the vertical and horizontal asymptotes.)
- The **two lines of symmetry** are $y = \pm x + (b + e)/d$.

You need not know where the above two lines of symmetry come from (but see the proof of Fact 108 if you're interested). Indeed, you need not even *mug* them. All you need remember are the following two points: The two lines of symmetry

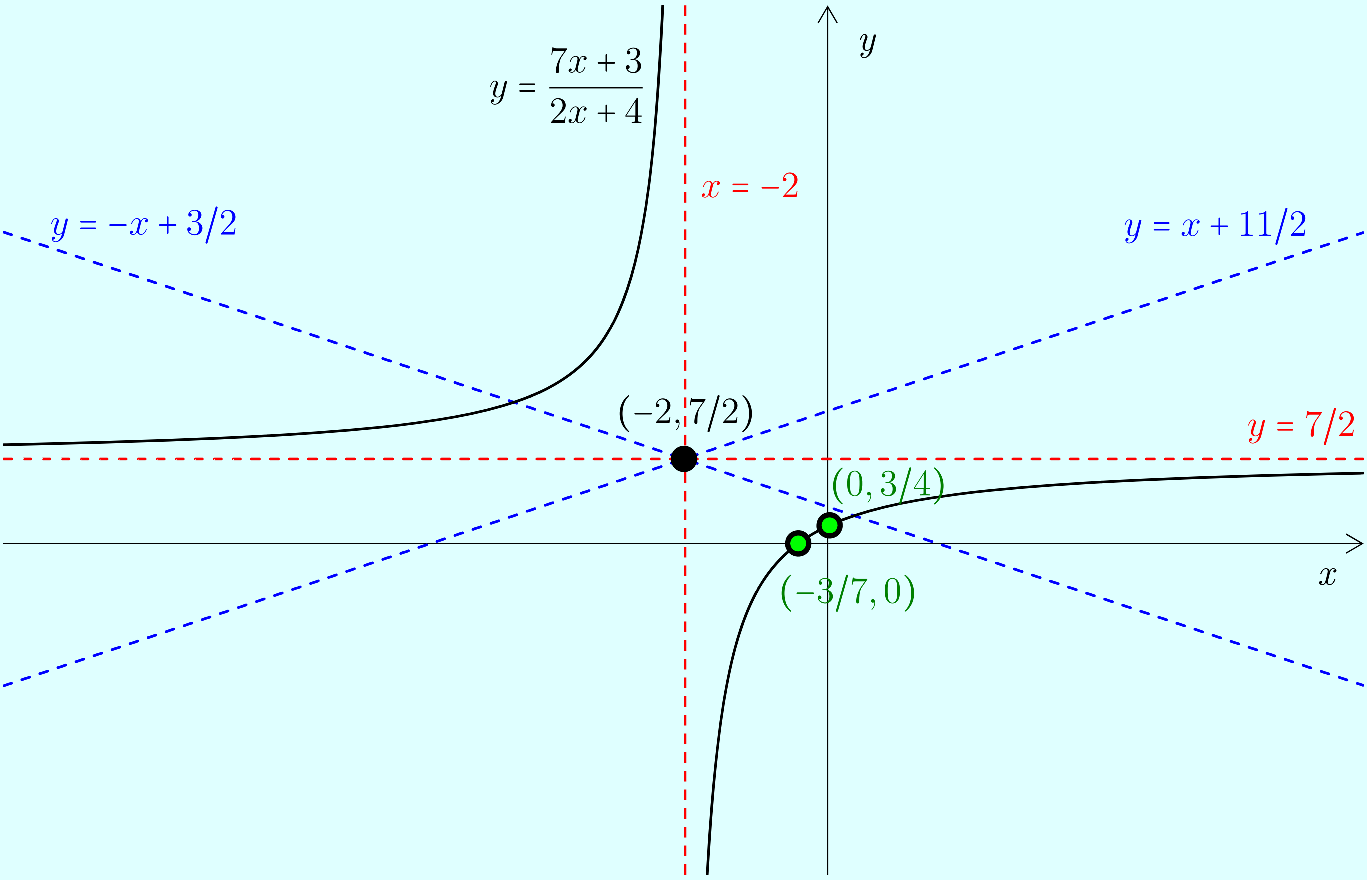
(a) Pass through the centre; and

(b) Have gradient ± 1.

The second point **(b)** implies that the two lines of symmetry may be written as $y = x + \alpha$ and $y = -x + \beta$, where $\alpha$and β are constants you can easily find. Examples:

Example 661. Consider $y = \dfrac{7x+3}{2x+4}$. Do the long division:

$$\begin{array}{r|ll} & 3.5 & \\ \cline{2-3} 2x+4 & 7x & +3 \\ & 7x & +14 \\ \cline{2-3} & & -11 \end{array} \implies y = \frac{7x+3}{2x+4} = \frac{7}{2} - \frac{11}{2x+4}.$$



1. **Two branches**—one on the top-left and another on the bottom-right.
2. **Intercepts.** Plug in $x = 0$ to get $y = 3/4$. So, the y-intercept is $(0, 3/4)$.

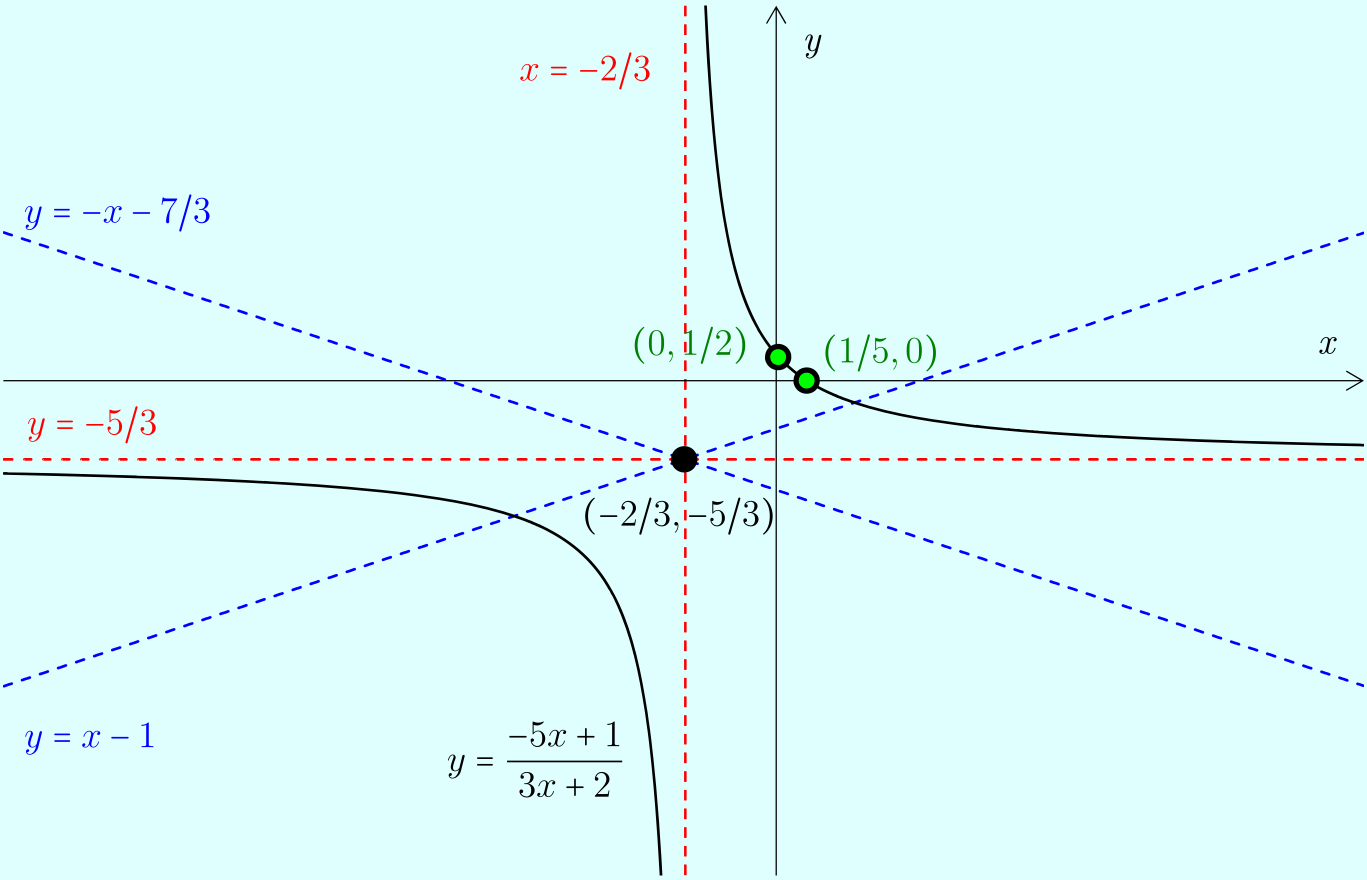
   Plug in $y = 0$ to get $7x + 3 = 0$ or $x = -3/7$. So, the x-intercept is $(-3/7, 0)$.
3. **Turning points**: None.
4. **Asymptotes**. The value of x that makes the denominator 0 is -2—hence, the vertical asymptote is $x = -2$.

   The quotient in the long division is $7/2$—hence, the horizontal asymptote is $y = 7/2$.

   Since the two asymptotes $x = -2$ and $y = 7/2$ are perpendicular, this is again a **rectangular hyperbola**.
5. **Centre**: $(-2, 7/2)$. (These coordinates are given by the vertical and horizontal asymptotes.)
6. The **two lines of symmetry** may be written as $y = x + \alpha$ and $y = -x + \beta$ and pass through the centre $(-2, 7/2)$. Plugging in the numbers, we find that $\alpha = 11/2$ and $\beta = 3/2$. Thus, the two lines of symmetry are $y = x + 11/2$ and $y = -x + 3/2$.

Example 662. Consider $y = \dfrac{-5x+1}{3x+2}$. Do the long division:

$$\begin{array}{r|ll} & -5/3 & \\ \cline{2-3} 3x+2 & -5x & +1 \\ & -5x & -10/3 \\ \cline{2-3} & & 13/3 \end{array} \implies y = \frac{-5x+1}{3x+2} = -\frac{5}{3} + \frac{13/3}{3x+2}.$$



1. **Two branches**—one on the top-left and another on the bottom-right.
2. **Intercepts.** Plug in $x = 0$ to get $y = 1/2$. So, the y-intercept is $(0, 1/2)$.

   Plug in $y = 0$ to get $-5x + 1 = 0$ or $x = 1/5$. So, the x-intercept is $(1/5, 0)$.
3. **Turning points**: None.
4. **Asymptotes**. The value of x that makes the denominator 0 is $-2/3$—hence, the vertical asymptote is $x = -2/3$.

   The quotient in the long division is $-5/3$—hence, the horizontal asymptote is $y = -5/3$.

   Since the two asymptotes $x = -2/3$ and $y = -5/3$ are perpendicular, this is again a **rectangular hyperbola**.
5. **Centre**: $(-2/3, -5/3)$. (These coordinates are given by the vertical and horizontal asymptotes.)
6. The **two lines of symmetry** may be written as $y = x + \alpha$ and $y = -x + \beta$ and pass through the centre $(-2/3, -5/3)$. Plugging in the numbers, we find that $\alpha = -1$ and $\beta = -7/3$. Thus, the two lines of symmetry are $y = x - 1$ and $y = -x - 7/3$.

The following Fact summarises the features of the hyperbola $y = \dfrac{bx + c}{dx + e}$:

Fact 108. *Let $b, c, d, e \in \mathbb{R}$ with $d \neq 0$ and $cd - be \neq 0$. Consider the graph of*

$$y = \frac{bx + c}{dx + e}.$$

(a) **Intercepts**. If $e \neq 0$, then there is one y-intercept $(0, c/e)$. (If $e = 0$, then there are no y-intercepts.) And if $b \neq 0$, then there is one x-intercept $(-c/b, 0)$. (If $b = 0$, then there are no x-intercepts.)

(b) There are no **turning points**.

(c) There is the **horizontal asymptote** $y = b/d$ and the **vertical asymptote** $x = -e/d$. (The asymptotes are perpendicular and so, this is a **rectangular hyperbola**.)

(d) The hyperbola's **centre** is $(-e/d, b/d)$.

(e) The **two lines of symmetry** are $y = \pm x + (b + e)/d$.

Proof. We proved **(a)**, **(c)**, and **(d)** above. For **(b)** and **(e)**, see p. 1609 (Appendices). □

Exam Tip for *Towkays*

For the hyperbola $y = (bx + c)/(dx + e)$, you should know how to find **(a)** the x- and y-intercepts; and **(c)** the horizontal and vertical asymptotes.

You're not required to know what **(d)** the hyperbola's centre is, but since this is simply the intersection of the two asymptotes (which you already know how to find), you may as well know about it, as it will help you sketch better graphs.

You're also not required to know how to find **(e)** the equations of the two lines of symmetry. But as we've shown in the above examples, it's not very difficult to figure out their equations. It's certainly not very difficult for you to at least sketch them.

After you are done with Part V (Calculus), there is a small possibility that you are required to prove that **(b)** this hyperbola has no turning points. (And so you may or may not be interested in reading the proof of **(b)** (Appendices).)

Exercise 194. Graph and describe the features of each equation.

(a) $y = \dfrac{3x + 2}{x + 2}$. (Answer on p. 1822.)

(b) $y = \dfrac{x - 2}{-2x + 1}$. (Answer on p. 1823.)

(c) $y = \dfrac{-3x + 1}{2x + 3}$. (Answer on p. 1824.)

44.8. The Hyperbola $y = \dfrac{ax^2+bx+c}{dx+e}$

We now study the equation $y = \left(ax^2+bx+c\right)/(dx+e)$. We'll assume that $a \neq 0$, $d \neq 0$, and either $c \neq 0$ or $e \neq 0$. This is because

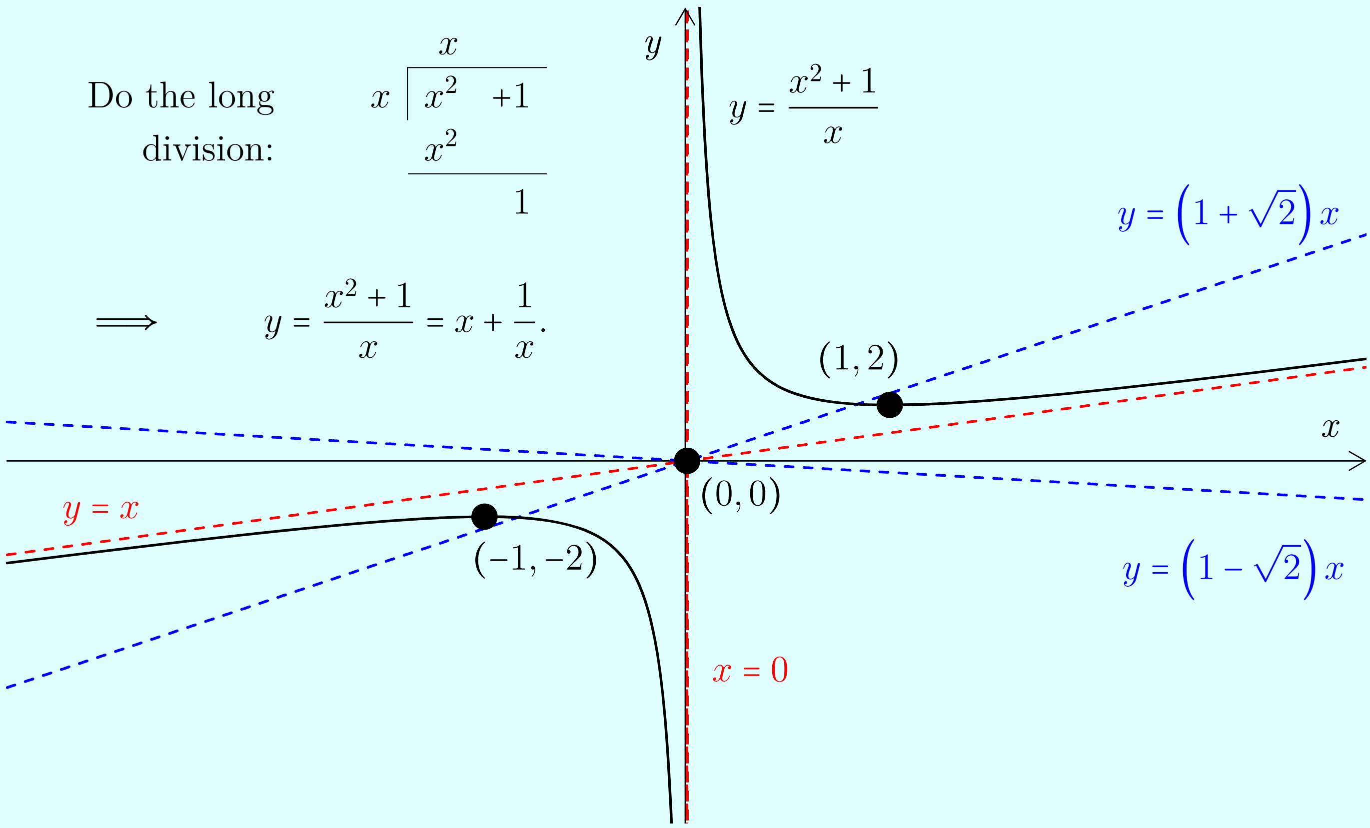
- If $a = 0$, then this is simply the equation we studied in the last subchapter.
- If $d = 0$, then the equation is quadratic and we've already studied that.
- If $c = 0 = e$, then the equation is linear and we've already studied that.

Example 663. Consider $y = \left(x^2+1\right)/x$.

Do the long division:

$$\begin{array}{r|ll} & x & \\ \hline x & x^2 & +1 \\ & x^2 & \\ \hline & & 1 \end{array}$$

$$\implies \quad y = \frac{x^2+1}{x} = x + \frac{1}{x}.$$



1. **Two branches**—one on the bottom-left and another on the top-right.
2. **Intercepts**. If we plug in $x = 0$, then y is undefined. Thus, there are no y-intercepts. And if we plug in $y = 0$, then $x^2 + 1 = 0$, an equation for which there are no (real) solutions. Thus, there are no x-intercepts.
3. Two **turning points**: $(-1,-2)$ is a strict local maximum and $(1,2)$ is a strict local minimum.
4. **Asymptotes**. The value of x that makes the denominator 0 is 0—hence, the vertical asymptote is $x = 0$. The quotient in the long division is x—hence, the oblique asymptote is $y = x$. (By the way, here for the first time, the asymptotes here are **not** perpendicular and so, this is a **non-rectangular hyperbola**.)
5. **Centre**: $(0,0)$. Recall that the centre is simply the point at which the two asymptotes intersect. In this example, the intersection of the asymptotes $x = 0$ and $y = x$ is $(0,0)$,
6. **Two lines of symmetry**: $y = \left(1 \pm \sqrt{2}\right)x$.

Here are the features of the hyperbola $y = \left(ax^2 + bx + c\right) / (dx + e)$.

- **Intercepts.** Plug in $x = 0$ to get $y = c/e$ and thus the y-intercept $(0, c/e)$. (Note that if $e = 0$, then c/e is undefined and there is no y-intercept. This was the case in the last example.) The x-intercepts are given by the values of x for which $ax^2 + bx + c = 0$:

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0\right).$$

  Note that if $b^2 - 4ac < 0$, then there are no x-intercepts (this was the case in the last example). And if $b^2 - 4ac = 0$, then there is exactly one x-intercept, namely $(-b/(2a), 0)$

- **Asymptotes.** The value of x that makes the denominator $dx + e$ zero is $x = -e/d$ and gives us the **vertical asymptote** $x = -e/d$. To find the other **oblique asymptote**, do the long division:

$$\begin{array}{r|lll}
 & ax/d & +(bd-ae)/d^2 & \\
\hline
dx+e & ax^2 & +bx & +c \\
 & ax^2 & +aex/d & \\
\cline{2-3}
 & & (bd-ae)x/d & +c \\
 & & (bd-ae)x/d & +\left(bde-ae^2\right)/d^2 \\
\cline{3-4}
 & & & \left(cd^2+ae^2-bde\right)/d^2.
\end{array}$$

$$\Longrightarrow \qquad y = \underbrace{\frac{a}{d}x + \frac{bd - ae}{d^2}}_{\text{Quotient}} + \underbrace{\frac{cd^2 + ae^2 - bde}{d^2}}_{\text{Remainder}} \cdot \frac{1}{dx + e}.$$

  The quotient gives us the **oblique asymptote** $y = ax/d + (bd - ae)/d^2$. (Since the asymptotes are **not** perpendicular, this hyperbola is **not** rectangular.)

- The **centre** is the point at which the two asymptotes intersect. Its x-coordinate is given by the vertical asymptote $x = -e/d$. For its y-coordinate, plug $x = -e/d$ into the equation of the oblique asymptote:

$$y = \frac{a}{d}\left(-\frac{e}{d}\right) + \frac{bd - ae}{d^2} = \frac{-ae + bd - ae}{d^2} = \frac{bd - 2ae}{d^2}.$$

  Thus, the centre is $$\left(-e/d, \frac{bd - 2ae}{d^2}\right).$$

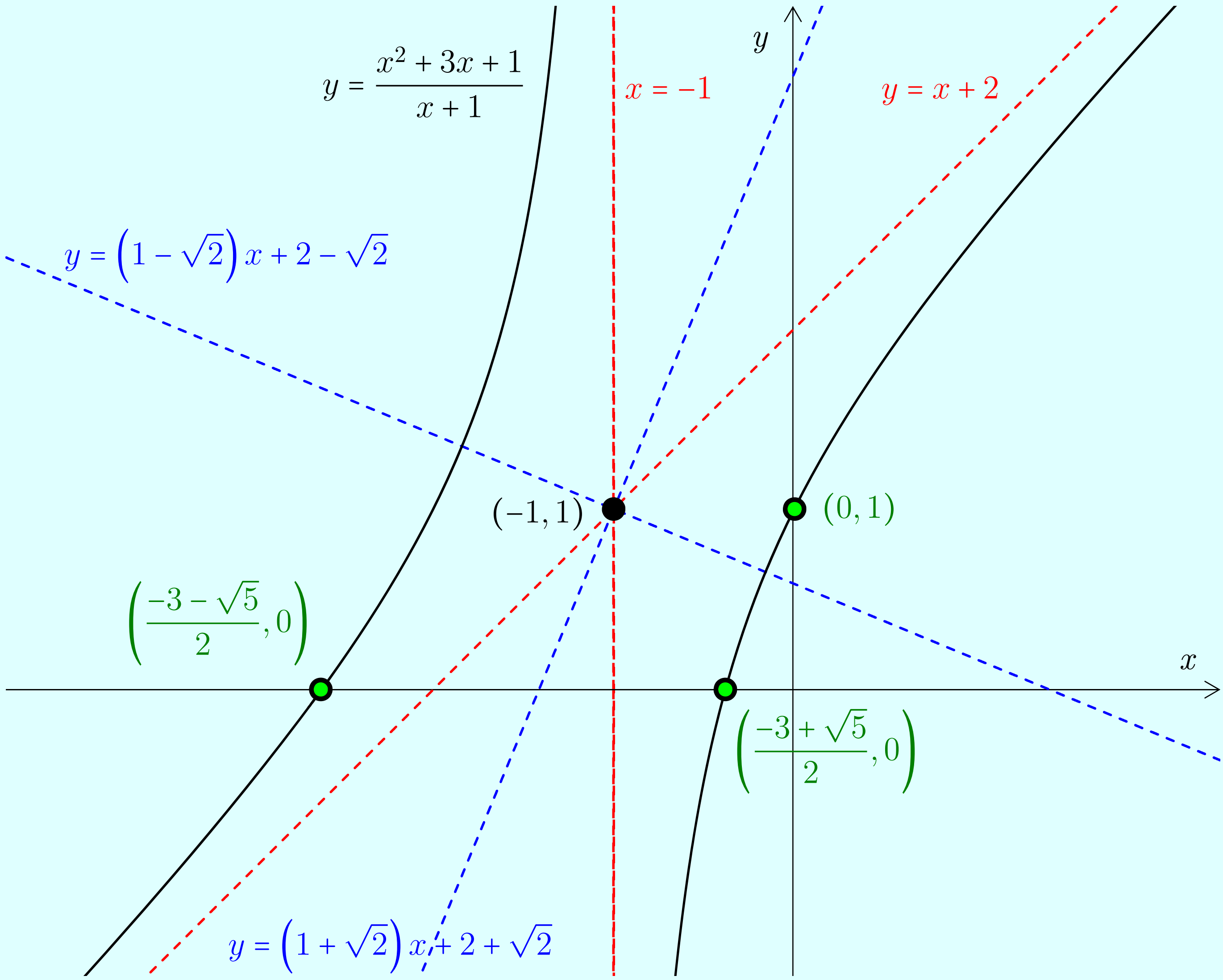
- You need **not** know how to find the equations of the **lines of symmetry**.

  You should however know how to roughly sketch them. So, just remember that they **(a) pass through the centre**; and **(b) bisect the angles formed by the two asymptotes**.

Example 664. Consider $y = \dfrac{x^2+3x+1}{x+1}$. Do the long division:

$$y = \frac{x^2+3x+1}{x+1} = x+2-\frac{1}{x+1}.$$

1. **Two branches**—one on the left and another on the right.
2. **Intercepts**. Plug in $x=0$ to get $y=1/1=1$. Thus, the y-intercept is $(0,1)$. Plug in $y=0$ to get $x^2+3x+1=0$—thus, the two x-intercepts are $\left(0.5\left(-3\pm\sqrt{5}\right),0\right)$.
3. **Turning points**: None.
4. **Asymptotes**. The value of x that makes the denominator 0 is -1—hence, the vertical asymptote is $x=-1$. The quotient in the long division is $x+2$—hence, the oblique asymptote is $y=x+2$.
5. The **centre's** x-coordinate is given by the vertical asymptote $x=-1$. For its y-coordinate, plug $x=-1$ into the oblique asymptote to get $y=-1+2=1$. Hence, the centre is $(-1,1)$.
6. **Two lines of symmetry**: $y=\left(1\pm\sqrt{2}\right)x+2\pm\sqrt{2}$.

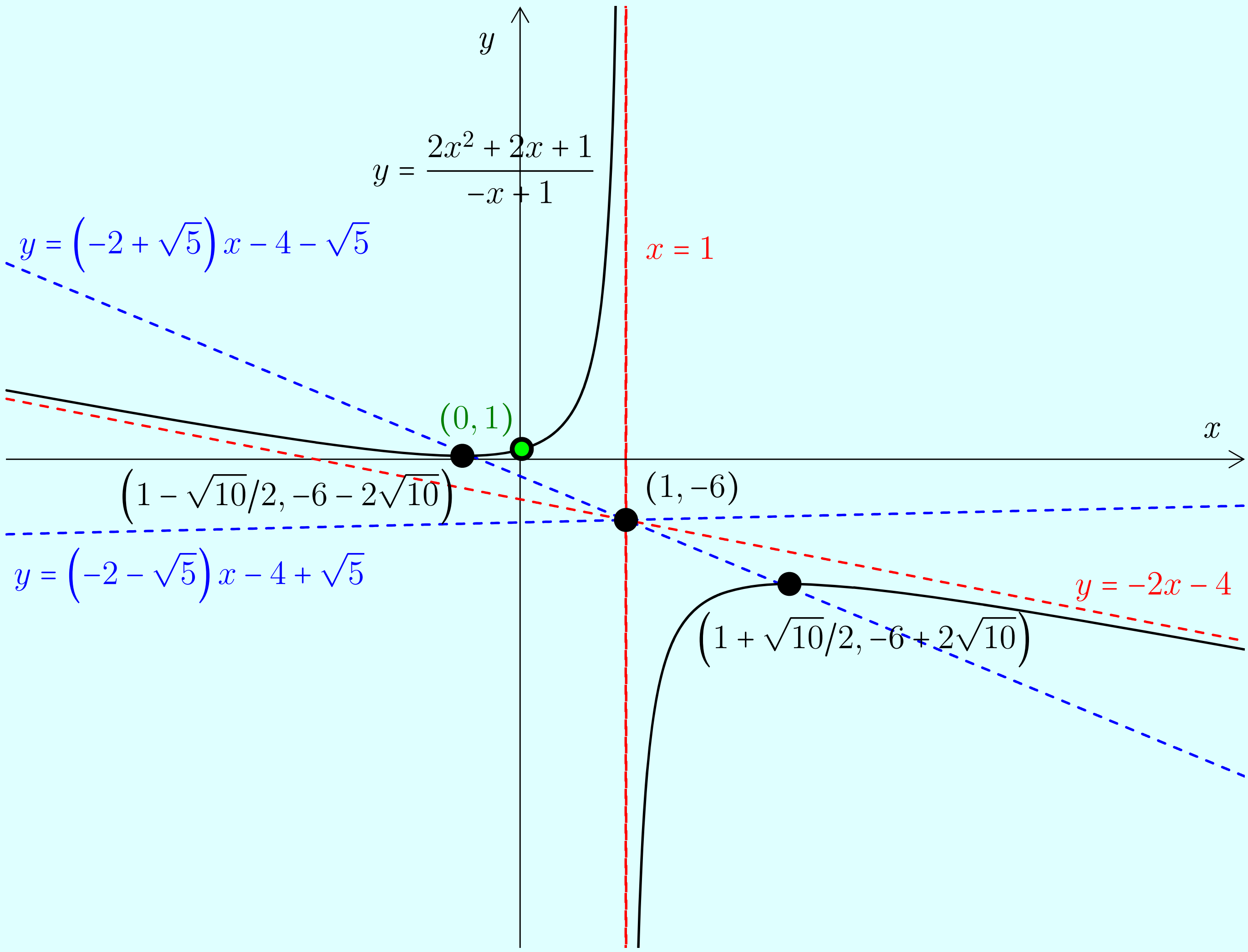


Example 665. Consider $y = \dfrac{2x^2+2x+1}{-x+1}$. Do the long division:

$$\frac{2x^2+2x+1}{-x+1} = -2x-4+\frac{5}{-x+1} = -2x-4+\frac{5}{-x+1}.$$

1. **Two branches**—one on the top-left and another on the bottom-right.
2. **Intercepts**. Plug in $x=0$ to get $y=1/1=1$. Thus, the y-intercept is $(0,1)$.

   Plug in $y=0$ to get $2x^2+2x+1=0$, an equation for which there are no (real) solutions. Thus, there are no x-intercepts.
3. Two **turning points**: $\left(1\pm 0.5\sqrt{10}, -6\pm 2\sqrt{10}\right)$.
4. **Asymptotes**. The value of x that makes the denominator 0 is 1—hence, the vertical asymptote is $x=1$. The "quotient" in the long division is $-2x-4$—hence, the oblique asymptote is $y=-2x-4$.
5. The **centre's** x-coordinate is given by the vertical asymptote $x=1$. For its y-coordinate, plug $x=1$ into the oblique asymptote to get $y=-2(1)-4=-6$. Hence, the centre is $(1,-6)$.
6. **Two lines of symmetry**: $y=\left(-2\pm\sqrt{5}\right)x-4\pm\sqrt{5}$.



You now know how to find its **intercepts**, **asymptotes**, and **centre** of the hyperbola[304]

$$y = \frac{ax^2 + bx + c}{dx + e}.$$

And after we've done Calculus (Part V), you'll also be able to find the **turning points**.

To repeat, you do **not** need to know how to find the equations of the **two lines of symmetry**. However, you should at least be able to roughly sketch them.

Exercise 195. Graph the equations below. Label any intercepts, asymptotes, and centre. Roughly indicate or sketch any turning points and lines of symmetry.

(a) $y = \dfrac{x^2 + 2x + 1}{x - 4}$. (Answer on p. 1825.)

(b) $y = \dfrac{-x^2 + x - 1}{x + 1}$. (Answer on p. 1826.)

(c) $y = \dfrac{2x^2 - 2x - 1}{x + 4}$. (Answer on p. 1827.)

[304]Fact 264 (Appendices) summarises the features of this hyperbola.

45. Simple Parametric Equations

We can sometimes describe a graph (i.e. a set of points) using an equation. We can sometimes also describe a graph using **parametric equations**:

Example 666. Let $S = \{(x, y) : x^2 + y^2 = 1\}$ be the unit circle centred on the origin.

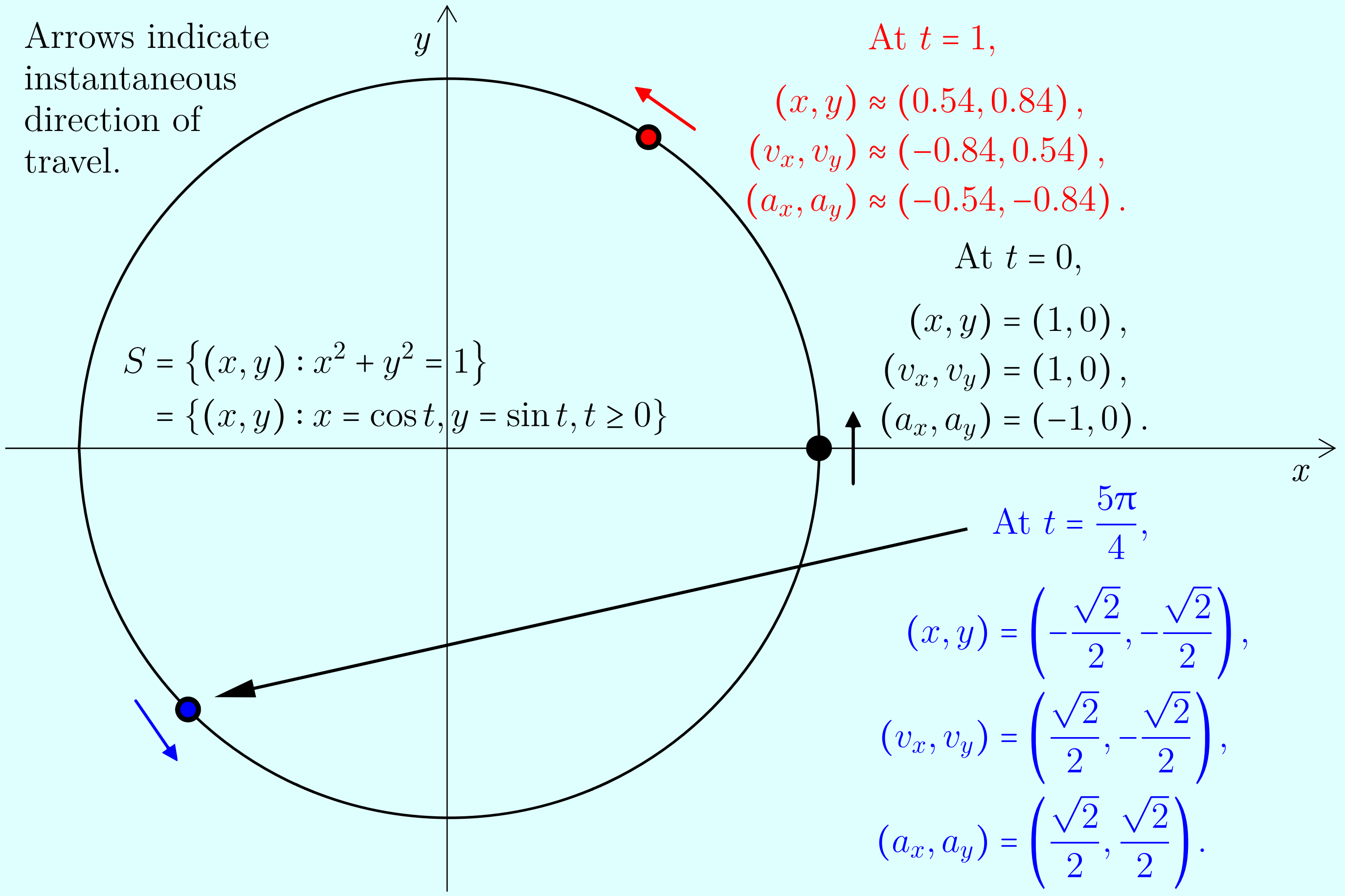
By Fact 97, $\sin^2 t + \cos^2 t = 1$ for all $t \in \mathbb{R}$. So, if we let $x = \cos t$ and $y = \sin t$, then $x^2 + y^2 = 1$. Hence, we have a second way to write down the set S:

$$S = \{(x, y) : x = \cos t, y = \sin t, t \geq 0\}.$$

In words, S is the set of points such that $x = \cos t$, $y = \sin t$, and $t \geq 0$. We call the variable t a **parameter** (hence the name **parametric equations**).

As t increases from 0 to 2π, we trace out, anti-clockwise, the unit circle:

$$\begin{aligned} t = 0 &\implies (x, y) = (1, 0), & t = \pi &\implies (x, y) = (-1, 0), \\ t = \pi/4 &\implies (x, y) = \left(\sqrt{2}/2, \sqrt{2}/2\right), & t = 5\pi/4 &\implies (x, y) = \left(-\sqrt{2}/2, -\sqrt{2}/2\right), \\ t = \pi/2 &\implies (x, y) = (0, 1), & t = 3\pi/2 &\implies (x, y) = (0, -1), \\ t = 3\pi/4 &\implies (x, y) = \left(-\sqrt{2}/2, \sqrt{2}/2\right), & t = 7\pi/4 &\implies (x, y) = \left(\sqrt{2}/2, -\sqrt{2}/2\right). \end{aligned}$$



One natural interpretation of S is as the movement of a particle over time t. For example, at $t = \pi$ s, the particle is 1 m west and 0 m north of the origin.

Example 667. Let $a, b > 0$. The graph of the equation $x^2/a^2 + y^2/b^2 = 1$ is the set

$$U = \left\{(x,y) : x^2/a^2 + y^2/b^2 = 1\right\},$$

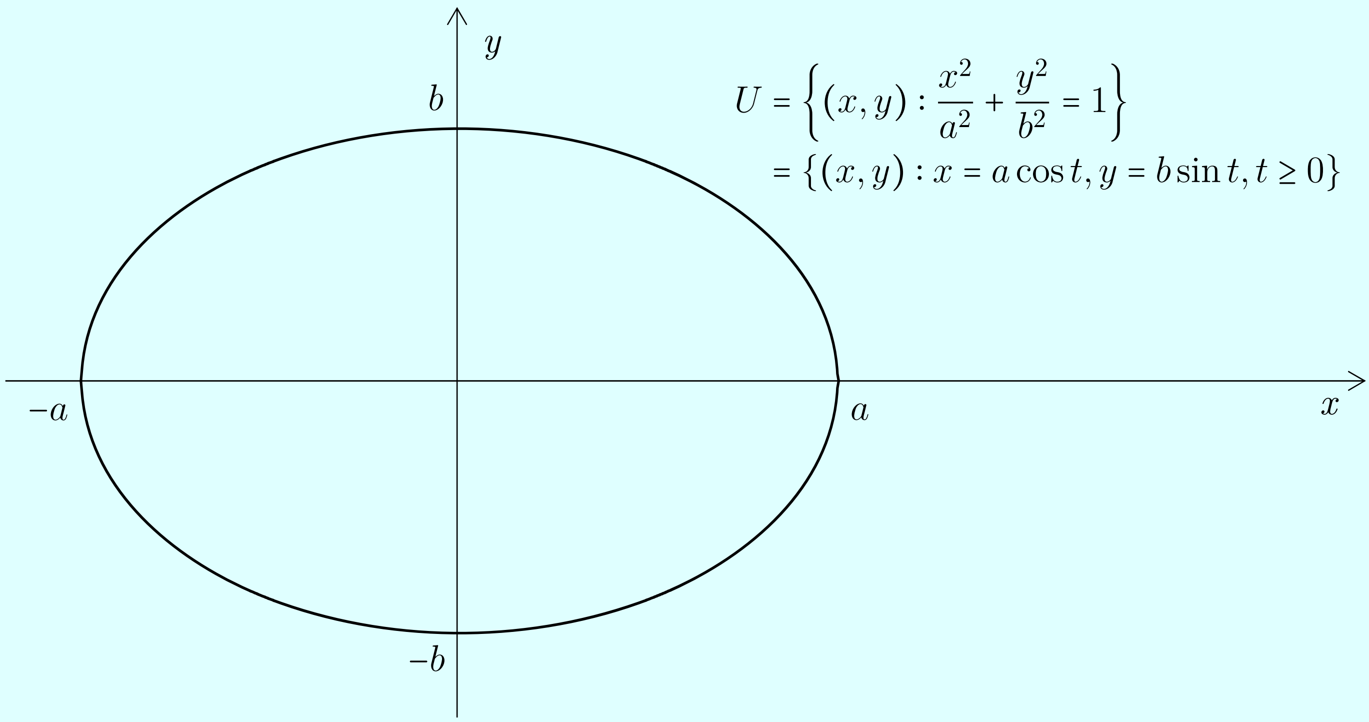
which is an **ellipse centred on the origin**, with x-intercepts $(\pm a, 0)$ and y-intercepts $(0, \pm b)$.

Observe that by letting $x = a\cos t$ and $y = b\sin t$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1.$$

So, we can use parametric equations to rewrite the set U as

$$U = \{(x,y) : x = a\cos t, y = b\sin t, t \geq 0\}.$$



Again, we can interpret U as the movement of a particle over time t. And again, the particle is travelling anticlockwise.

Example 668. The graph of $x^2 - y^2 = 1$ is the set

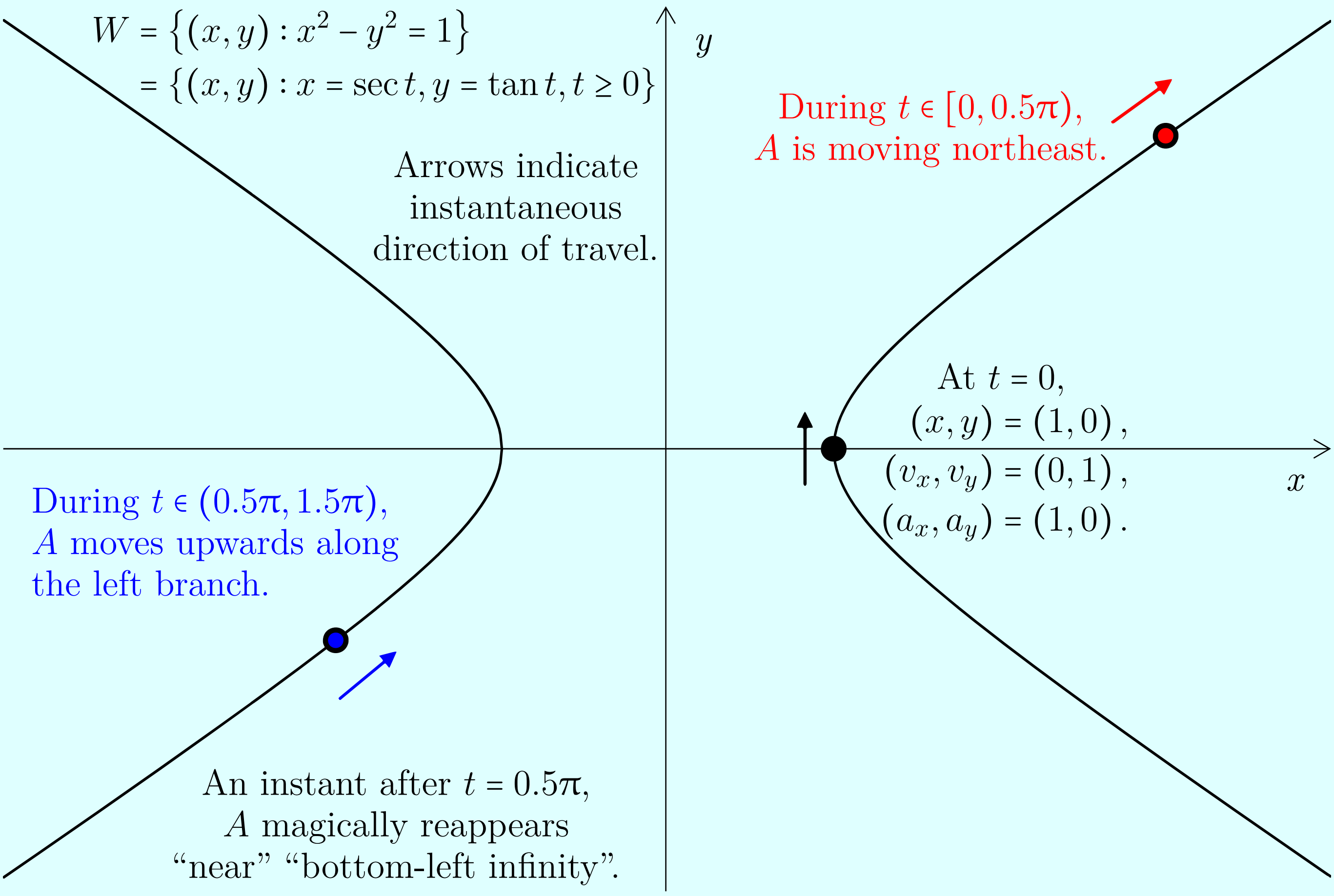
$$W = \left\{(x,y) : x^2 - y^2 = 1\right\},$$

which is an "east-west" hyperbola **centred on the origin**, with x-intercepts $(\pm 1, 0)$.

By Fact 97(a), $\sec^2 t - \tan^2 t = 1$ for all $t \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$. So, if we let $x = \sec t$ and $y = \tan t$, then $x^2 - y^2 = 1$. Hence, this gives us another way of writing the set W:

$$W = \{(x,y) : x = \sec t, y = \tan t, t \geq 0\}.$$

Again, we can interpret W as the movement of a particle A over time t.



At each instant t, A's velocity in the x- and y-directions is given by

$$(v_x, v_y) = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}\sec t}{\mathrm{d}t}, \frac{\mathrm{d}\tan t}{\mathrm{d}t}\right) = \left(\sec t \tan t, \sec^2 t\right).$$

(*Example continues on the next page ...*)

Exercise 196. Continuing with the above example, write down A's acceleration in the x- and y-directions at the instant t. (Answer on the next page.)

(... *Example continued from the previous page.*)

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\sec t \tan t) \stackrel{\times}{=} \sec t \tan t \tan t + \sec t \sec^2 t$$
$$= \sec t \left(\tan^2 t + \sec^2 t\right) = \sec t \left(2\sec^2 t - 1\right),$$

$$a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sec^2 t\right) \stackrel{\text{Ch.}}{=} 2\sec t \cdot \sec t \tan t = 2\sec^2 t \tan t.$$

Thus, $$(a_x, a_y) = \left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right) = \left(\frac{\mathrm{d}v_x}{\mathrm{d}t}, \frac{\mathrm{d}v_y}{\mathrm{d}t}\right) = \left(\sec t\left(2\sec^2 t - 1\right), 2\sec^2 t \tan t\right).$$

Note that at $t = 0.5\pi, 1.5\pi, 2.5\pi, , \ldots,$ both $\sec t$ and $\tan t$ are undefined. And so, we'll say that at these instants in time, the particle A's position, velocity, and acceleration are simply undefined.

Observe that interestingly, $v_y = \sec^2 t > 0$ for all t (for which $\sec t$ is well-defined). Hence, the particle A is *always* moving upwards (except during the aforementioned instants in time when its velocity is undefined).

At $t = 0$, $$(x, y) = (\sec 0, \tan 0) = (1, 0),$$
$$(v_x, v_y) = \left(\sec 0 \tan 0, \sec^2 0\right) = (0, 1),$$
$$(a_x, a_y) = \left(\sec 0\left(2\sec^2 0 - 1\right), 2\sec^2 0 \tan 0\right) = (1, 0).$$

So, A starts at the midpoint of the right branch of the hyperbola, is moving upwards at $1\,\mathrm{m\,s^{-1}}$, and is accelerating rightwards at $1\,\mathrm{m\,s^{-1}}$.]

During $t \in [0, \pi/2)$, the particle A is moving northeast. As $t \to \pi/2$, it "flies off" towards the "top-right infinity" (∞, ∞) and

$$x, y, v_x, v_y, a_x, a_y \to \infty.$$

An instant after $t = \pi/2$, A magically reappears "near" "bottom-left infinity" $(-\infty, -\infty)$.

During $t \in (0.5\pi, 1.5\pi)$, the particle travels upwards along the left branch of the hyperbola. And again, as $t \to 1.5\pi$, it "flies off" towards "top-left infinity" $(-\infty, \infty)$ and we have

$$x, v_x, a_x \to -\infty \qquad \text{and} \qquad y, v_y, a_y \to \infty.$$

Exercise 197. Continue with the above example. (Answer on p. 1829.)

(a) What happens to particle A an instant after $t = 1.5\pi$?

(b) Describe A's movement during $t \in (1.5\pi, 2.5\pi)$.

45.1. Eliminating the Parameter t

Above, we rewrote a single (cartesian) equation into a pair of (parametric) equations and in the process introduced the parameter t. Now we'll go in reverse—we'll rewrite a pair of (parametric) equations into a single (cartesian) equation and in the process eliminate the parameter t.

Example 669. Consider the set $S = \left\{(x,y) : x \stackrel{1}{=} t^2 - 4t, y \stackrel{2}{=} t - 1, t \geq 0\right\}$.

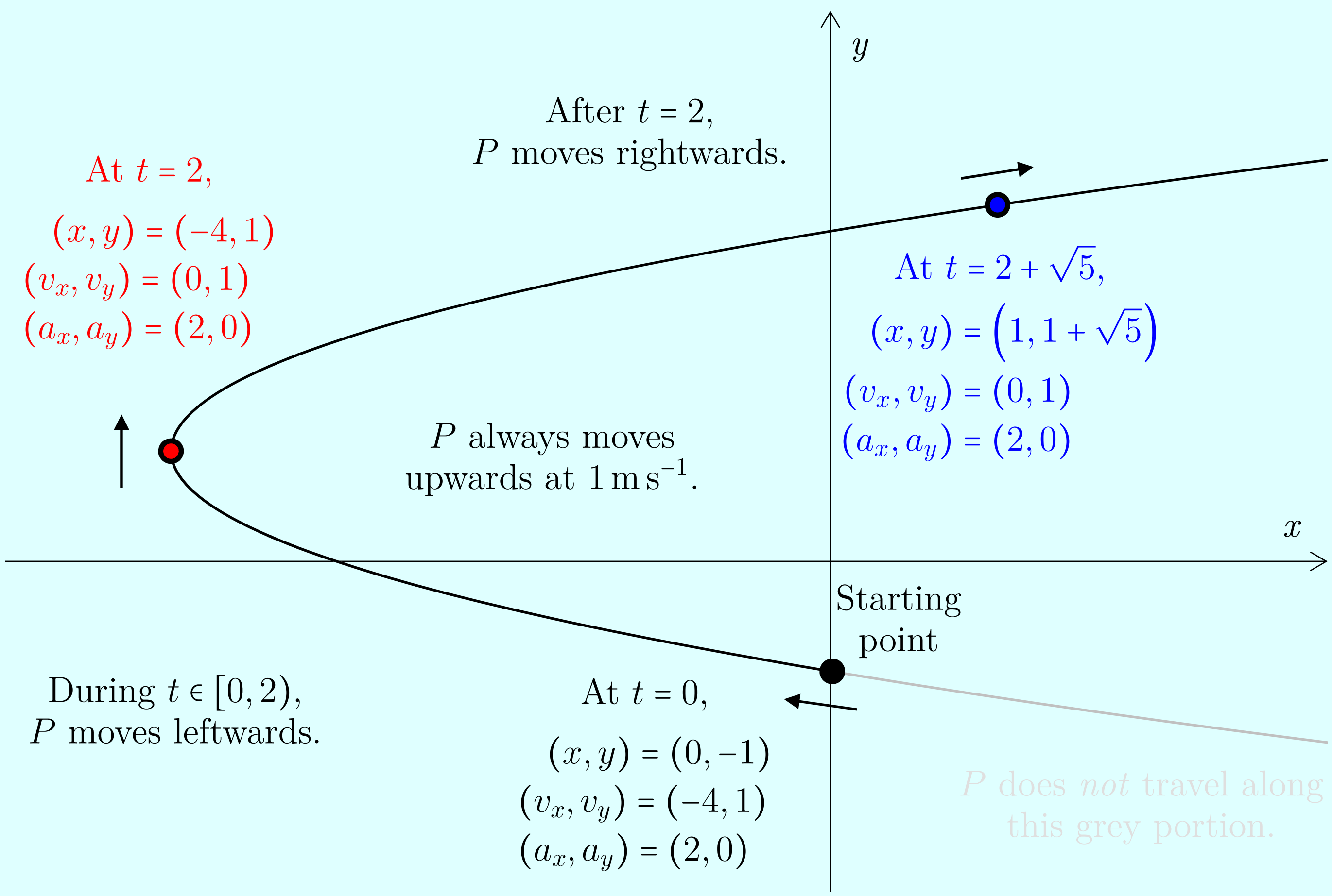
As usual, we can interpret S as the motion of a particle P in the plane, where t is time (seconds), while x and y are P's rightward and upward displacements (metres) from the origin. With a little algebra, we can combine $\stackrel{1}{=}$ and $\stackrel{2}{=}$ into a single equation and eliminate the parameter t:

- First rewrite $y \stackrel{2}{=} t - 1$ as $t \stackrel{3}{=} y + 1$.
- Then plug $\stackrel{3}{=}$ into $\stackrel{1}{=}$ to get $x = (y+1)^2 - 4(y+1) = y^2 - 2y - 3$.
- Observe also that $t \geq 0 \iff y \geq 0 - 1 = -1$.

Altogether, $S = \left\{(x,y) : x = y^2 - 2y - 3, y \geq -1\right\}$.

Note that here we have a quadratic equation. Our quadratic equations have usually been the variable x, but in this case, it is in the variable y.

The set S is the black graph below and does *not* include the grey portion. At $t = 0$, P starts at the position $(x,y) = (0,-1)$ and is travelling northwest.



(*Example continues on the next page ...*)

(*... Example continued from the previous page.*)

Here are P's velocity and acceleration, decomposed into the x- and y-directions:

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = 2t - 4, \qquad v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = 1, \qquad a_x = \frac{\mathrm{d}a_x}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2, \qquad a_y = \frac{\mathrm{d}a_y}{\mathrm{d}t} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 0.$$

In the y- or upwards-direction, P moves at a constant velocity of $1\,\mathrm{m\,s^{-1}}$ and does not accelerate.

In the x- or rightwards-direction, P moves at velocity $2t - 4\,\mathrm{m\,s^{-1}}$ and accelerates at a constant rate of $2\,\mathrm{m\,s^{-2}}$. In particular,

- When $t < 2$, we have $2t - 4 < 0$ and so P is moving *left*wards.
- At $t = 2$, the particle is at the leftmost point of the parabola and its velocity in the x-direction is $0\,\mathrm{m\,s^{-1}}$.
- And when $t > 2$, we have $2t - 4 > 0$ and so P is moving rightwards.

Example 670. Consider the set $U = \left\{(x,y) : x \overset{1}{=} 2\cos t - 4, y \overset{2}{=} 3\sin t + 1, t \geq 0\right\}$.

Again, we can interpret U as the motion of a particle Q in the plane. With a little algebra, we can combine $\overset{1}{=}$ and $\overset{2}{=}$ into a single equation and eliminate the parameter t:

- First rewrite $x \overset{1}{=} 2\cos t - 4$ as $\cos t = \dfrac{x+4}{2}$.
- Next rewrite $y \overset{2}{=} 3\sin t + 1$ as $\sin t = \dfrac{y-1}{3}$.
- Now recall that by Fact 97(a), $\sin^2 t + \cos^2 t = 1$ for all $t \in \mathbb{R}$.

So, we can rewrite the set U as

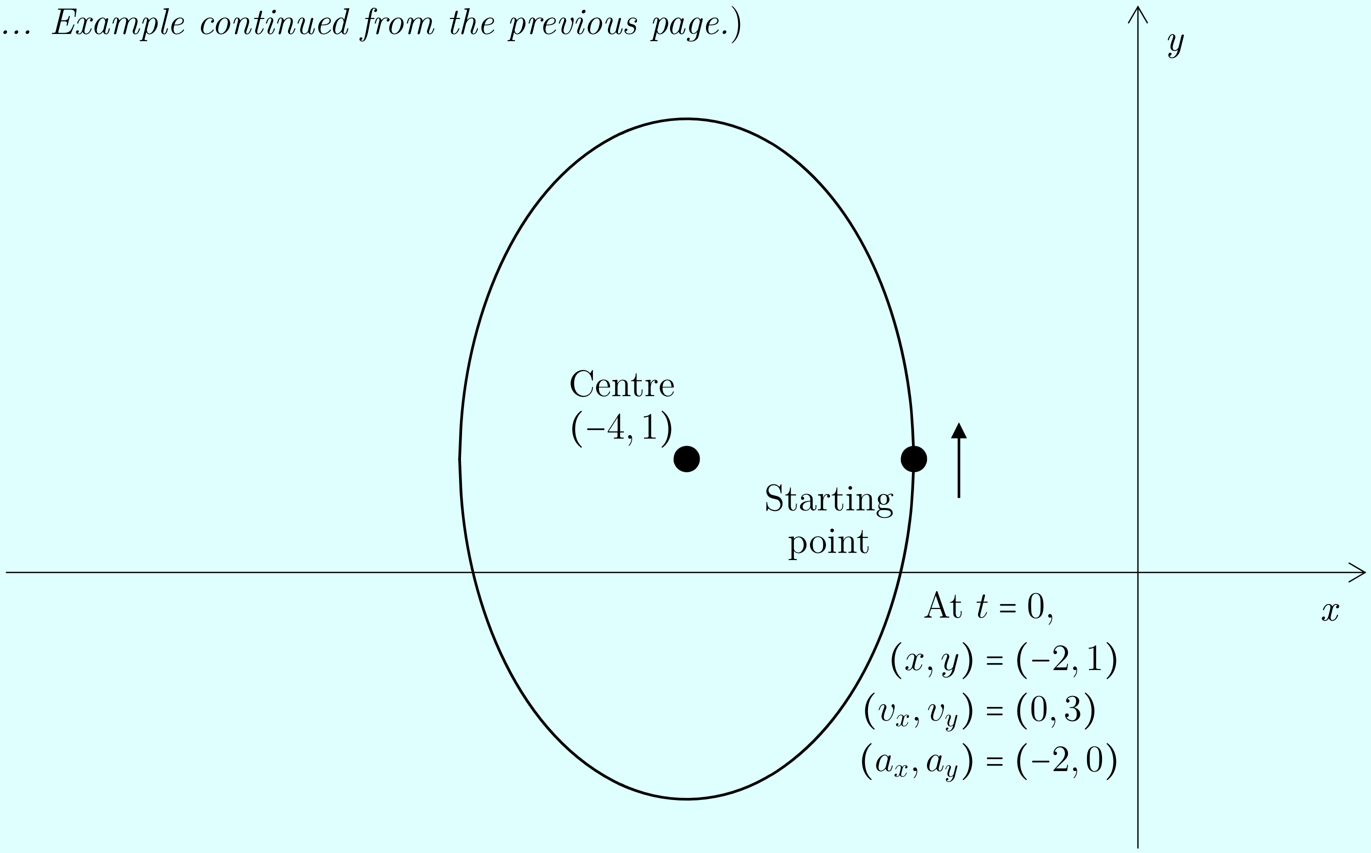
$$U = \left\{(x,y) : \left(\frac{x+4}{2}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1\right\}.$$

This describes an ellipse centered on $(-4, 1)$.

Since x and y can be expressed in terms of trigonometric functions, they must be periodic. That is, the particle Q will keep repeating its movement along a certain path. And so, we need **not** include any constraints for x and y.

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)



The velocity and acceleration of Q, decomposed into the x- and y-directions, are

$$v_x = \frac{dx}{dt} = -2\sin t, \quad v_y = \frac{dy}{dt} = 3\cos t, \quad a_x = \frac{da_x}{dt} = \frac{d^2x}{dt^2} = -2\cos t,$$
$$a_y = \frac{da_y}{dt} = \frac{d^2y}{dt^2} = -3\sin t.$$

At $t=0$, Q's starting position and velocity are

$$(x,y) = (2\cos 0 - 4, 3\sin 0 + 1) = (-2,1) \qquad \text{and} \qquad (v_x,v_y) = (-2\sin 0, 3\cos 0) = (0,3).$$

So, it starts at the rightmost point of the ellipse and is moving upwards at $3\,\mathrm{m\,s^{-1}}$. Thus, its direction of travel is anticlockwise around the ellipse. Every 2π s, Q completes one full revolution around the ellipse.

Exercise 198. The sets A, B, and C below describe the positions (metres) of particles A, B, and C at time t (seconds), relative to the origin. For each set,

(i) Rewrite the set so that the parameter t is eliminated.

(ii) Sketch the graph.

(iii) Describe the particle's position and velocity as time progresses.

(a) $A = \{(x,y) : x = t-1, y = \ln(t+1), t \geq 0\}$ (Answer on p. 1831.)

(b) $B = \left\{(x,y) : x = \dfrac{1}{t+1}, y = t^2+1, t \geq 0\right\}$. (Answer on p. 1832.)

(c) $C = \left\{(x,y) : x = 2\sin t - 1, y = 3\cos^2 t, t \geq 0\right\}$. (Answer on p. 1833.)

Part II.
Sequences and Series

[T]here is nothing as dreamy and poetic, nothing as radical, subversive, and psychedelic, as mathematics.

— Paul Lockhart (2002, 2009).

[M]athematics is capable of an artistic excellence as great as that of any music, perhaps greater; not because the pleasure it gives (although very pure) is comparable, either in intensity or in the number of people who feel it, to that of music, but because it gives in absolute perfection that combination, characteristic of great art, of godlike freedom, with the sense of inevitable destiny; because, in fact, it constructs an ideal world where everything is perfect and yet true.

— Bertrand Russell (1902).

Beauty is Truth, Truth Beauty.–That is all
Ye know on Earth, and all ye need to know.

— John Keats (1819).

46. Sequences

Informally, a **sequence** is simply an ordered list of real numbers.

Example 671. The **Fibonacci sequence** is

$$(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots)$$

We call the numbers in a sequence its **terms**.

So the Fibonacci sequence's 1st **term** is 1, 2nd is 1, 3rd is 2, etc.

The first two terms of the Fibonacci sequence are fixed as 1. Each subsequent term is then simply the sum of the previous two terms. So,

$$\begin{array}{ll} \text{the 3rd term is } 1+1=2; & \text{the 7th term is } 5+8=13; \\ \text{the 4th term is } 1+2=3; & \text{the 8th term is } 8+13=21; \\ \text{the 5th term is } 2+3=5; & \text{the 9th term is } 13+21=34; \\ \text{the 6th term is } 3+5=8; & \text{etc.} \end{array}$$

Letting $f(n)$ denote the nth term, the Fibonacci sequence may be defined by

$$f(n) = \begin{cases} 1, & \text{for } n = 1, 2, \\ f(n-2) + f(n-1), & \text{for } n \geq 3. \end{cases}$$

Example 672. The **sequence of square numbers** is

$$(1, 4, 9, 16, 25, 36, 49, 64, \ldots)$$

This sequence's 1st **term** is 1, 2nd is 4, 3rd term is 9, etc.

Letting $s(n)$ denote the nth term, this sequence may be defined by $s(n) = n^2$.

Example 673. The **sequence of triangular numbers** is

$$(1, 3, 6, 10, 15, 21, 28, 36, \ldots)$$

This sequence's 1st **term** is 1, 2nd is 3, 3rd term is 6, etc.

Letting $t(n)$ denote the nth term, this sequence may be defined by $t(n) = 1 + 2 + \cdots + n$.

Remark 89. For clarity, it is wise and indeed customary to enclose a sequence in parentheses.[305] And so, even though your A-Level exams do not, I will insist on doing so.

[305] Note though that some writers prefer using braces $\{\}$ or angle brackets $\langle\rangle$.

The above sequences were **infinite**. But of course, sequences can also be **finite**:

Example 674. The finite sequence of the first six Fibonacci numbers is

$$(1, 1, 2, 3, 5, 8).$$

We call this a finite sequence of length 6.

Example 675. The finite sequence of the first seven square numbers is

$$(1, 4, 9, 16, 25, 36, 49).$$

We call this a finite sequence of length 7.

Example 676. The finite sequence of the first four triangular numbers is

$$(1, 3, 6, 10).$$

We call this a finite sequence of length 4.

Remark 90. We'll generally be more interested in infinite sequences than finite sequences. And so, when we simply say *sequence*, it may be assumed that we're talking about an *infinite* sequence. And when we want to talk about a finite sequence, we'll clearly and explicitly include the word *finite*.

46.1. Sequences *Are* Functions

A little more formally, sequences *are* functions:

Definition 116. A *finite sequence* of length k is a function with domain $\{1, 2, \ldots, k\}$.

Definition 117. An *(infinite) sequence* is a function with domain $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

Remark 91. For A-Level Maths, the objects in a sequence will *always* be real numbers. (But in general, they could be anything whatsoever and not just real numbers.)

So, for A-Level Maths, we can simply fix the codomain of any sequence (which is a function) as $\mathbb{R}$. (But in general, the codomain could be any set whatsoever.)

We now formally rewrite our earlier examples of sequences as functions:

Example 677. Formally, the Fibonacci sequence *is* the function $f : \mathbb{Z}^+ \to \mathbb{R}$ defined by

$$f(n) = \begin{cases} 1, & \text{for } n = 1, 2, \\ f(n-2) + f(n-1), & \text{for } n \geq 3. \end{cases}$$

Example 678. Formally, the sequence of square numbers *is* the function $s : \mathbb{Z}^+ \to \mathbb{R}$ defined by $s(n) = n^2$.

Example 679. Formally, the sequence of triangular numbers *is* the function $t : \mathbb{Z}^+ \to \mathbb{R}$ defined by $t(n) = 1 + 2 + \cdots + n$.

Example 680. Formally, the finite sequence of the first six Fibonacci numbers *is* the function $f_6 : \{1, 2, 3, 4, 5, 6\} \to \mathbb{R}$ defined by

$$f_6(n) = \begin{cases} 1, & \text{for } n = 1, 2, \\ f_6(n-2) + f_6(n-1), & \text{for } n = 3, 4, 5, 6. \end{cases}$$

Example 681. Formally, the finite sequence of the first seven square numbers *is* the function $s_7 : \{1, 2, 3, 4, 5, 6, 7\} \to \mathbb{R}$ defined by $s_7(n) = n^2$.

Example 682. Formally, the finite sequence of the first four triangular numbers *is* the function $t_4 : \{1, 2, 3, 4\} \to \mathbb{R}$ defined by $t_4(n) = 1 + 2 + \cdots + n$.

Remark 92. As repeatedly emphasised, the letter x we often use with functions is merely a dummy or placeholder variable that can be replaced by any another.

Indeed, in the context of sequences, we'll often prefer using n rather than x as our dummy variable.

More examples:

Example 683. The function $e : \mathbb{Z}^+ \to \mathbb{R}$ defined by $e(n) = 2n$ *is* the sequence of (positive) even numbers $(2, 4, 6, 8, 10, 12, \ldots)$.

Example 684. The function $g : \mathbb{Z}^+ \to \mathbb{R}$ defined by $g(n) = 2n^2 - 3n + 3$ *is* the following sequence $(2, 5, 12, 23, 38, 57, 80, 107, 138, 173, \ldots)$.

Recall that a function need not "follow any formula" or "make any sense". The same is true of sequences (since sequences are simply functions):

Example 685. The function $h : \{1, 2, 3, 4\} \to \{\text{Cow, Chicken}\}$ is defined by

$$h(1) = \text{Cow}, \quad h(2) = \text{Cow}, \quad h(3) = \text{Chicken}, \quad \text{and} \quad h(4) = \text{Cow}.$$

Although h does not seem to "make any sense", it is a (perfectly) well-defined function. Indeed, the function h *is* also the following finite sequence of length 4:

$$(\text{Cow, Cow, Chicken, Cow}).$$

Although this sequence does not seem to "make any sense", it is a (perfectly) well-defined finite sequence of length 4, simply because it is a function with domain $\{1, 2, 3, 4\}$.

Example 686. The function $j : \{1, 2, 3\} \to \{\uparrow, \downarrow, \to, \leftarrow, \text{Punch, Kick}\}$ is defined by

$$j(1) = \downarrow, \quad j(2) = \to, \quad \text{and} \quad j(3) = \text{Punch}$$

Although j does not seem to "make any sense", it is a (perfectly) well-defined function. Indeed, the function j *is* also the following finite sequence of length 3:

$$(\downarrow, \to, \text{Punch}).$$

Although this sequence does not seem to "make any sense", it is a (perfectly) well-defined finite sequence of length 3, simply because it is a function with domain $\{1, 2, 3\}$.

Exercise 199. Formally rewrite each sequence as a function. (Answer on p. 1834.)

(a) $(1, 4, 9, 16, 25, 36, 49, 64, 81, 100)$.

(b) $(2, 5, 8, 11, 14, 17, 20, \ldots, 299)$.

(c) $(1, 8, 27, 64, 125, 216, 343)$.

(d) $(2, 6, 6, 12, 10, 18, 14, 24, 18, 30, 22, 36, 26, 42, \ldots)$.

(e) $(5, 0, 99)$.

(f) $(1, 2, 6, 24, 120, 720, 5\,040, 40\,320, \ldots)$.

(g) $(1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, \ldots)$.

46.2. Notation

Let $(a_1, a_2, \dots, a_k)$ be a finite sequence of length k. This sequence can also be denoted as any of the following:

$$(a_n)_{n=1}^{k} \quad \text{or} \quad (a_n)_{n=1,2,\dots,k} \quad \text{or} \quad (a_n)_{n\in\{1,2,\dots,k\}} \quad \text{or} \quad (a_n)_{1\leq n\leq k}.$$

Example 687. Let $s_1 = 1$, $s_2 = 4$, $s_3 = 9$, $s_4 = 16$, and $s_5 = 25$. Then we can denote the finite sequence of the first five square numbers by

$$(s_1, s_2, s_3, s_4, s_5) = (s_n)_{n=1}^{5} = (s_n)_{n=1,2,3,4,5} = (s_n)_{n\in\{1,2,3,4,5\}} = (s_n)_{1\leq n\leq 5}.$$

Again, s and n are merely dummy or placeholder variables. And here, n in particular may also be called an **index variable**, because it **indexes** or **indicates** which term in the sequence we're referring to.

We could replace s or n with any other symbol, like ☺ or $\star$. And so, we could rewrite the above example as

Example 688. Let $☺_1 = 1$, $☺_2 = 4$, $☺_3 = 9$, $☺_4 = 16$, and $☺_5 = 25$. Then we can *also* denote the finite sequence of the first five square numbers by

$$(☺_1, ☺_2, ☺_3, ☺_4, ☺_5) = (☺_\star)_{\star=1}^{5} = (☺_\star)_{\star=1,2,3,4,5} = (☺_\star)_{\star\in\{1,2,3,4,5\}} = (☺_\star)_{1\leq\star\leq 5}.$$

Of course, it's a bit strange to use symbols like ☺ or $\star$. The point here is simply to illustrate that once again, these are mere symbols that can be replaced by any other. We'll usually stick to using boring symbols like letters from the Latin alphabet.

Next, let $(a_1, a_2, \dots)$ be an (infinite) sequence. This sequence can also be denoted as any of the following:

$$(a_n)_{n=1}^{\infty} \quad \text{or} \quad (a_n)_{n=1,2,\dots} \quad \text{or} \quad (a_n)_{n\in\mathbb{Z}^+} \quad \text{or} \quad (a_n)_{n\in\{1,2,\dots\}} \quad \text{or} \quad (a_n).$$

Example 689. Let $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10$, $t_5 = 15$, etc. Then we can denote the (infinite) sequence of triangular numbers by

$$(t_1, t_2, \dots) = (t_n)_{n=1}^{\infty} = (t_n)_{n=1,2,\dots} = (t_n)_{n\in\mathbb{Z}^+} = (t_n)_{n\in\{1,2,\dots\}} = (t_n).$$

Again, we can replace t and n with any other symbols:

Example 690. Let $☺_1 = 1$, $☺_2 = 3$, $☺_3 = 6$, $☺_4 = 10$, $☺_5 = 15$, etc. Then we can *also* denote the (infinite) sequence of triangular numbers by

$$(☺_1, ☺_2, \dots) = (☺_\star)_{\star=1}^{\infty} = (☺_\star)_{\star=1,2,\dots} = (☺_\star)_{\star\in\mathbb{Z}^+} = (☺_\star)_{\star\in\{1,2,\dots\}} = (☺_\star).$$

46.3. Arithmetic Combinations of Sequences

In Ch. 20, we learnt to create arithmetic combinations of functions. Since sequences *are* functions, we can likewise create **arithmetic combinations of sequences**:

Example 691. Suppose $(a_n) = (1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ is the Fibonacci sequence; $(b_n) = (2, 4, 6, 8, 10, 12, 14, 16, 18, \dots)$ is the sequence of even numbers; and $k = 10$. Then

$$
\begin{aligned}
(a_n + b_n) &= (3, 5, 8, 11, 15, 20, 27, 37, 52, \dots), \\
(a_n - b_n) &= (-1, -3, -4, -5, -5, -4, -1, 5, 16, \dots), \\
(a_n b_n) &= (2, 4, 12, 24, 50, 96, 182, 336, 612, \dots), \\
\left(\frac{a_n}{b_n}\right) &= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{1}{2}, \frac{2}{3}, \frac{13}{14}, \frac{21}{16}, \frac{17}{9}, \dots\right), \\
(k a_n) &= (10, 10, 20, 30, 50, 80, 130, 210, 340, \dots), \\
(k b_n) &= (20, 40, 60, 80, 100, 120, 140, 160, 180, \dots).
\end{aligned}
$$

Exercise 200. Let (c_n) be the sequence of *negative* odd numbers, (d_n) the sequence of cube numbers, and $k = 2$. Write out the first five terms of **(a)** (c_n); and **(b)** (d_n). Then write out the first five terms of each of **(c)**–**(h)**. (Answer on p. 1834.)

(c) $(c_n + d_n)$ **(d)** $(c_n - b_n)$ **(e)** $(c_n d_n)$

(f) $\left(\frac{c_n}{d_n}\right)$ **(g)** $(k c_n)$ **(h)** $(k d_n)$

47. Series

Definition 118. Given a finite sequence $(a_n)_{n\in\{1,2,\ldots,k\}}$, its *series* is the expression

$$a_1 + a_2 + a_3 + \cdots + a_k.$$

And the *sum* of this series is the number that equals the above expression.

Example 692. Consider the finite sequence of the first five square numbers: $(1, 4, 9, 16, 25)$. Its **series** is the expression $1 + 4 + 9 + 16 + 25$, while the **sum** of this series is the number 55.

Example 693. Consider the finite sequence of the first six even numbers: $(2, 4, 6, 8, 10, 12)$. Its series is the expression $2 + 4 + 6 + 8 + 10 + 12$, while the sum of this series is the number 42.

It may seem strange and unnecessary to distinguish between a **series** and its **sum**. Aren't they exactly the same thing?

It turns out that expressions like $a_1 + a_2 + a_3 + \cdots + a_k$ play an important role in maths. And so, we want to reserve a special name for the expression itself, in order to distinguish it from the number that is the sum of the series.

Example 694. Given the sequence $(1, 3, 5, 7)$, we might be specifically interested in the expression $1 + 3 + 5 + 7$, rather than just the number 16.

It is thus convenient to have separate names for them—we call the expression $1 + 3 + 5 + 7$ the **series** and the number 16 the **sum** (of the series).

47.1. Convergent and Divergent Series

Definition 119. Given an (infinite) sequence (a_n), its *series* is the expression

$$a_1 + a_2 + a_3 + \dots$$

As we saw on the previous page, every finite series has a well-defined sum—simply add up all the numbers!

With an infinite series, things get a little trickier. It may sometimes be that an infinite series **diverges** and its **limit does not exist**.

Example 695. Let $(1, 1, 1, 1, 1, \dots)$ be the (infinite) sequence that consists solely of 1s.

Its series is the expression $1 + 1 + 1 + 1 + 1 + \dots$

"Clearly", this expression is not equal to any number. And so formally, we say that this series diverges and that its limit does not exist.

Also, observe that this expression "grows ever larger". As shorthand, we write[306]

$$1 + 1 + 1 + 1 + 1 + \dots = \infty.$$

Remark 93. As was the case with sequences, we'll generally be more interested in infinite series than finite series. And so, when we simply say *series*, it may safely be assumed that we're talking about an *infinite* series. And when we want to talk about a finite series, we'll clearly and explicitly include the word *finite*.

Example 696. Let $(2, 4, 6, 8, 10, \dots)$ be the sequence of (positive) even numbers.

Its series is the expression $2 + 4 + 6 + 8 + 10 + \dots$

"Clearly", this expression is not equal to any number. And so formally, we say that this series diverges and that its limit does not exist.

Also, observe that this expression "grows ever larger". As shorthand, we write

$$2 + 4 + 6 + 8 + 10 + \dots = \infty.$$

[306]Pedantic point: this "equation" is not really an equation. Instead, it is merely shorthand for this informal statement:

"The expression $1 + 1 + 1 + 1 + 1 + \dots$ grows ever larger."

"Grows ever larger" is, in turn, a vague and informal phrase that we clarify only in Ch. 143 (Appendices).

We just looked at two examples of series that diverge. We now look at examples of series that **converge** to some **limit**:

Example 697. Consider the **zero sequence** $(0, 0, 0, 0, 0, \dots)$.

Its series is the expression $0 + 0 + 0 + 0 + 0 + \dots$

This series **converges** to 0. We call 0 its **limit**. And as shorthand, we write

$$0 + 0 + 0 + 0 + 0 + \cdots = 0.$$

Note that what was called the **sum** (in the previous context of finite series) is now called the **limit** (in the current context of infinite series).

Example 698. Consider the sequence $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots\right)$.

Its series is the expression $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$

As we'll soon learn, it turns out that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1.$$

That is, this series converges to 1. (And we call 1 the limit of this series.)

Here we should remark that whenever we are dealing with infinite series, we must be very careful. Here the = sign in the above equation is not the usual one. Instead, the above equation is merely shorthand for

$$\text{“the expression } \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \textbf{ converges to} \text{ the number 1”.}$$

In A-Level Maths, an intuitive and informal understanding of the phrase **converges to** will suffice. But you should be aware that it does actually have a clear and precise meaning—see Ch. 143 (Appendices).

Example 699. Consider the sequence of reciprocals of squares $\left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots\right)$.

It series is the expression $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = 1 + \frac{1}{4} + \frac{1}{9} + \dots$

It's not at all obvious, but it turns out that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}.$$

That is, this series converges to $\pi/6$. (And we call $\pi/6$ the limit of this series.)

The problem of finding the above limit is called the Basel Problem and was first solved by Euler (albeit not quite rigorously) in 1734. We'll revisit it in Ch. XXX.

Now, when does a series **converge** or **diverge**? Or equivalently, when does its **limit** exist?

The precise definitions of these terms are beyond the scope of A-Level Maths.[307] You need only know—roughly and intuitively—what **convergence**, **divergence**, and **limits** are.

It turns out that what exactly these terms mean is no simple matter. On the next page are two fun examples to give you a glimpse of the difficulties involved:

Example 700. Consider **Grandi's series**:[308]

$$1 - 1 + 1 - 1 + 1 - 1 + \dots$$

Does this series converge or diverge? Or equivalently, does its limit exist?

Remarkably, we can "prove" that this series is equal to 0, 1, *and* 1/2.

- To "prove" that it equals 0, pair off the terms:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \underbrace{(1-1)}_{0} + \underbrace{(1-1)}_{0} + \underbrace{(1-1)}_{0} + \cdots = 0 + 0 + 0 + \cdots = 0.$$

- To "prove" that it equals 1, pair off the terms after the first:

$$\begin{aligned} 1 - 1 + 1 - 1 + 1 - 1 + \dots &= 1 + \underbrace{(-1+1)}_{0} + \underbrace{(-1+1)}_{0} + \underbrace{(-1+1)}_{0} + \dots \\ &= 1 + 0 + 0 + 0 + \cdots = 1. \end{aligned}$$

- To "prove" that it equals $1/2$, let $S = 1-1+1-1+1-1+\dots$, then "show" that $1 - S = S$:

$$1 - S = 1 - (1 - 1 + 1 - 1 + 1 - 1 + \dots) = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = S.$$

  Since $1 - S = S$, simple algebra yields $1 = 2S$ or $S = 1/2$.

We've just "proven" that the expression $1 - 1 + 1 - 1 + 1 - 1 + \dots$ equals 0, 1, *and* 1/2.

Well, which is it then? It turns out that the series $1-1+1-1+1-1+\dots$ is not equal to 0, 1, or $1/2$. Instead, it is **divergent** and its limit does not exist.[309]

Here's a series whose convergence is unknown:

Example 701. Consider this series:

$$\frac{1}{2} - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \frac{5}{11} - \frac{6}{13} + \frac{7}{17} - \frac{8}{19} + \frac{9}{23} - \frac{10}{29} + \dots$$

The terms are fractions, with the numerators being the (positive) integers and the denominators being the prime numbers. This series looks "simple" enough. But remarkably, mathematicians still do not know whether it converges or diverges![310]

[307] If you're interested, see Ch. 143 (Appendices).

[308] The Italian priest-mathematician Luigi Guido Grandi (1671–1742) thought he had succeeded in proving that the sum of this series equalled both 0 and 1/2, and thus that "God could create the word out of nothing" (source).

[309] We prove this in Example 1577 (Appendices).

[310] This problem is listed as equation (8) on this Wolfram Mathworld page and as Problem E7 in Richard

48. Summation Notation $\sum$

The symbol Σ is the upper-case Greek letter *sigma* (σ is lower case).

An enlarged version of that symbol is $\sum$, which we'll read aloud as "sum". We use the symbol $\sum$ to write down series more compactly, in what is called **summation** or **sigma notation**.

Example 702. $\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25.$

- The variable n is the dummy, placeholder, or index **variable**. (As usual, we can replace all instances of n with any other symbol without changing the meaning of the above sentence.)
- The integer below $\sum$ is the **starting point**. So here, 1 tells us to start counting (the index variable n) from $n = 1$.
- The integer above $\sum$ is the **stopping point**. So here, 5 tells us to stop at $n = 5$.
- The expression to the right of $\sum$ describes the nth term to be added up. So here, the nth term to be added up is n^2.

Altogether then, $\sum_{n=1}^{5} n^2$ tells us to add up the terms 1^2, 2^2, 3^2, 4^2, and 5^2.

Example 703. $\sum_{n=1}^{3} (2n + 3) = (2 \cdot 1 + 3) + (2 \cdot 2 + 3) + (2 \cdot 3 + 3) = 5 + 7 + 9 = 21.$

Example 704. $\sum_{n=1}^{4} n = 1 + 2 + 3 + 4 = 10.$

Example 705. $\sum_{n=1}^{6} 2n = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 + 2 \cdot 6 = 2 + 4 + 6 + 8 + 10 + 12 = 42.$

Example 706. $\sum_{n=1}^{7} 2^n = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 = 2 + 4 + 8 + 16 + 32 + 64 + 128 = 254.$

Example 707. $\sum_{n=1}^{5} 1 = 1 + 1 + 1 + 1 + 1 = 5.$ Here each term to be added up is simply the constant 1. And so, $\sum_{n=1}^{5} 1$ is simply the sum of five 1s.

Example 708. $\sum_{n=1}^{3} (10 - 2n) = (10 - 2 \cdot 1) + (10 - 2 \cdot 2) + (10 - 2 \cdot 3) = 8 + 6 + 4.$

Guy's *Unsolved Problems in Number Theory* (3e, p. 316), where it is attributed to Paul Erdős.

Example 709. $\sum_{n=1}^{4} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$.

Example 710. $\sum_{n=1}^{4} \frac{1}{(n+1)^2} = \frac{1}{(1+1)^2} + \frac{1}{(2+1)^2} + \frac{1}{(3+1)^2} + \frac{1}{(4+1)^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$.

Example 711. Suppose $x \in \mathbb{R}$. Then

$$\sum_{n=1}^{5} x^{n-1} = x^{1-1} + x^{2-1} + x^{3-1} + x^{4-1} + x^{5-1} = x^0 + x^1 + x^2 + x^3 + x^4 = 1 + x + x^2 + x^3 + x^4.$$

It's nice to have 1 as the starting point and that's what we'll usually do. But there's no reason why the starting point must always be 1. Examples:

Example 712. Starting point 3:

$$\sum_{n=3}^{5} n^3 = 3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 216.$$

Example 713. Starting point 0:

$$\sum_{n=0}^{4} \sqrt{n} = \sqrt{0} + \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} = 0 + 1 + \sqrt{2} + \sqrt{3} + 2.$$

The starting point can even be negative:

Example 714. Starting point -2:

$$\sum_{n=-2}^{3} \frac{n}{4} = \frac{-2}{4} + \frac{-1}{4} + \frac{0}{4} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} = -\frac{1}{2} - \frac{1}{4} + 0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} = \frac{3}{4}.$$

We'll follow this fairly standard (though not universal) convention—if the stopping point is smaller than the starting point, then the sum is simply 0:

Example 715. $\sum_{n=3}^{2} n = 0$.

Example 716. $\sum_{n=-2}^{-5} n^2 = 0$.

Exercise 201. Rewrite each series in summation notation. (Answer on p. 1835.)

(a) $1+2+6+24+120+720+5\,040$.

(b) $2+5+8+11+14+17+20+23$.

(c) $1/2+1+3/2+2+5/2+3+7/2$.

(d) $8+7+6+5+4+3$.

Exercise 202. Redo the last exercise, but with starting point 0. (Answer on p. 1835.)

Exercise 203. Find the sum of each series. (Observe that here the dummy or index variables are not the usual n. Instead, they are i, $\star$, and x.) (Answer on p. 1835.)

(a) $\displaystyle\sum_{i=1}^{4}(2-i)^i$. **(b)** $\displaystyle\sum_{\star=16}^{17}(4\star+5)$. **(c)** $\displaystyle\sum_{x=31}^{33}(x-3)$.

Example 717. In the following series, the "$n = 1$" below the $\sum$ symbol indicates that it has starting point 1.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

The ∞ above the $\sum$ symbol indicates that there is no stopping point. This is thus an **infinite series**.

As already mentioned and as we'll soon learn, this series **converges** to 1. We may write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

If it's very clear from the context what the starting and stopping points are, then we'll sometimes be lazy/sloppy and omit them. And so here, we may also write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum \frac{1}{2^n} = 1.$$

Example 718. "Clearly", the series $\sum_{n=1}^{\infty} n = \sum n = 1 + 2 + 3 + \ldots$ **diverges**.

So, we may write $\sum_{n=1}^{\infty} n = \sum n = \infty$.

Example 719. Consider $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = 1 + \frac{1}{4} + \frac{1}{9} + \ldots$.

As mentioned in Example 699, this series converges to $\pi/6$.

So, we may write $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum \frac{1}{n^2} = \frac{\pi}{6}$.

Example 720. Consider $\sum_{n=1}^{\infty} nx^2 = \sum nx^2 = 1x^2 + 2x^2 + 3x^2 + 4x^2 + \dots$.

This infinite series has starting point 1 and each term to be added up is nx^2.

Thus, $\sum_{n=1}^{\infty} nx^2$ is the sum of infinitely many terms, namely x^2, $2x^2$, $3x^2$, $4x^2$ ...

By the way, this series diverges for all $x \neq 0$. That is, for all $x \neq 0$, we have

$$\sum_{n=1}^{\infty} nx^2 = \sum nx^2 = \infty.$$

And "clearly", if $x = 0$, then this series converges to 0:

$$\sum_{n=1}^{\infty} n \cdot 0^2 = \sum n \cdot 0^2 = 0.$$

Example 721. Consider the **harmonic series**:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$$

We have

$$\sum_{n=1}^{1} \frac{1}{n} = \frac{1}{1} = 1, \qquad \sum_{n=1}^{2} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} = 1.5, \qquad \sum_{n=1}^{3} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = 1.8\overline{3},$$

$$\sum_{n=1}^{100} \frac{1}{n} = 5.187\dots, \qquad \sum_{n=1}^{200} \frac{1}{n} = 5.878\dots, \qquad \sum_{n=1}^{300} \frac{1}{n} = 6.282\dots,$$

$$\sum_{n=1}^{1\,000} \frac{1}{n} = 7.485\dots, \qquad \sum_{n=1}^{10^4} \frac{1}{n} = 9.787\dots, \qquad \sum_{n=1}^{10^5} \frac{1}{n} = 12.090\dots.$$

Does the harmonic series converge or diverge? From the above, it's not obvious.

It turns out that it diverges. Here's a heuristic (i.e. not totally rigorous) "proof":

First, consider the series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$—"clearly", this series diverges.

We'll show below that this series is "smaller than" the harmonic series.[311] and hence, "clearly", the harmonic series must also diverge:

$$\begin{aligned}
& 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \\
&= \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{1/2} + \underbrace{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}}_{1/2} + \dots \\
&< \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \dots \\
&= \sum_{n=1}^{\infty} \frac{1}{n}.
\end{aligned}$$

[311] Again, here we must be careful to define what we mean by one infinite series being "smaller than" another.

As before, it's nice to have 1 as the starting point, but this need not always be so:

Example 722. Consider, $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

By the way, since $\sum_{n=1}^{\infty} 1/2^n = 1$, it "clearly" follows that

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + 1 = 2.$$

Example 723. $\sum_{n=-2}^{\infty} \frac{1}{n+5} = \frac{1}{-2+5} + \frac{1}{-1+5} + \frac{1}{0+5} + \frac{1}{1+5} + \dots = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$

By the way, since the harmonic series diverges—i.e. $\sum_{n=1}^{\infty} 1/n = \infty$—we have

$$\sum_{n=-2}^{\infty} \frac{1}{n+5} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots = -\frac{1}{1} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

That is, since the harmonic series diverges, this series also diverges.

Exercise 204. Rewrite each series in summation notation. (Answer on p. 1835.)

(a) $1 + 2 + 6 + 24 + 120 + 720 + 5\,040 + \dots.$

(b) $2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + \dots$

(c) $1/2 + 1 + 3/2 + 2 + 5/2 + 3 + 7/2 + \dots$

(d) $8 + 7 + 6 + 5 + 4 + 3 + \dots$

Exercise 205. Redo the last exercise, but with starting point 0. (Answer on p. 1835.)

49. Arithmetic Sequences and Series

Definition 120. An *arithmetic sequence* (or *progression*) is a sequence where the difference between any two consecutive terms is constant (and called the *common difference*).

Example 724. Below are six **arithmetic sequences**—three finite (left) and three infinite (right).

$$(a_n)_{n=1}^5 = (1, 3, 5, 7, 9), \qquad (a_n) = (1, 3, 5, 7, 9, \dots),$$
$$(b_n)_{n=1}^7 = (4, 7, 10, 13, 16, 19, 22) \qquad (b_n) = (4, 7, 10, 13, 16, 19, 22, \dots)$$
$$(c_n)_{n=1}^3 = (0, \pi, 2\pi) \qquad (c_n) = (0, \pi, 2\pi, \dots)$$

In each sequence, the difference between any two consecutive terms is a constant.

In $(a_n)_{n=1}^5$ and (a_n), the common difference is $d = 2$.

In $(b_n)_{n=1}^7$ and (b_n), the common difference is $d = 3$.

In $(c_n)_{n=1}^3$ and (c_n), the common difference is $d = \pi$.

Definition 121. Given an arithmetic sequence, its series is called an *arithmetic series*.

Example 725. The six arithmetic sequences in the above example have these corresponding **arithmetic series**:

$$1 + 3 + 5 + 7 + 9, \qquad 1 + 3 + 5 + 7 + 9 + \dots,$$
$$4 + 7 + 10 + 13 + 16 + 19 + 22, \qquad 4 + 7 + 10 + 13 + 16 + 19 + 22 + \dots,$$
$$0 + \pi + 2\pi, \qquad 0 + \pi + 2\pi + \dots$$

Fact 109. *If $(a_n)_{n=1}^k$ is a finite arithmetic sequence with $d = a_2 - a_1$, then*

(a) *The nth term is* $\quad a_n = a_1 + (n-1)\,d;$

(b) *The number of terms is* $\quad k = \dfrac{a_k - a_1}{d} + 1;$

(c) $$\sum_{n=1}^k a_n = \sum_{n=1}^k [a_1 + (n-1)\,d].$$

Proof. **(a)** Since a_n is $(n-1)$ terms "after" a_1, we must have $a_n = a_1 + (n-1)\,d$.

(b) By **(a)**, $a_k = a_1 + (k-1)\,d$. Rearranging, $k = (a_k - a_1)/d + 1$.

(c) Immediate from **(a)**. □

Example 726. You've probably heard the apocryphal story where a seven-year-old[312] Gauss added up the numbers from 1 to 100 in an instant. His trick was to pair the first number with the last, the second with the second last, etc., then multiply:

$$1+2+3+4+\cdots+100 = \overbrace{\underbrace{(1+100)}_{101}+\underbrace{(2+99)}_{101}+\underbrace{(3+98)}_{101}+\cdots+\underbrace{(50+51)}_{101}}^{50 \text{ pairs}}$$
$$= 101\times 50 = 5\,050.$$

More generally, the sum of a finite arithmetic series is

$$(\text{First Term}+\text{Last Term})\times\frac{\text{Number of Terms}}{2}.$$

A bit more formally,

Fact 110. *Suppose $(a_n)_{n=1}^k$ is a finite arithmetic sequence. Then*

$$\sum_{n=1}^{k} a_n = (a_1+a_k)\,\frac{k}{2}.$$

Our proof of this formula is simply a formalisation of Gauss's apocryphal idea:

Proof. First, suppose k is even. Then

$$a_1+a_2+\cdots+a_k = (a_1+a_k)+(a_2+a_{k-1})+\cdots+\left(a_{k/2}+a_{k/2+1}\right).$$

Note that RHS has $k/2$ pairs of terms.

Next, $$a_1+a_k = a_2+a_{k-1} = a_3+a_{k-3} = \cdots = a_{k/2}+a_{k/2+1}.$$

So, $$\sum_{n=1}^{k} a_n \stackrel{1}{=} (a_1+a_k)\,\frac{k}{2}. \quad ✓$$

Next, suppose k is odd. Then $a_{k-1} = a_k - d \stackrel{2}{=} a_k - \left(\dfrac{a_k-a_1}{k-1}\right)$ and

$$\sum_{n=1}^{k} a_n = \sum_{n=1}^{k-1} a_n + a_k \stackrel{1}{=} (a_1+a_{k-1})\,\frac{k-1}{2}+a_k \stackrel{2}{=} \left[a_1+a_k-\left(\frac{a_k-a_1}{k-1}\right)\right]\frac{k-1}{2}+a_k$$
$$= (a_1+a_k)\,\frac{k-1}{2}+\frac{a_1-a_k}{2}+a_k = (a_1+a_k)\,\frac{k-1}{2}+\frac{a_1+a_k}{2} = (a_1+a_k)\,\frac{k}{2}. \qquad \square$$

[312]Or eight- or nine- or ten-year-old. Brian Hayes has collected 145 "Versions of the Gauss Schoolroom Anecdote"—"the earliest such instance I have found" is "a 1906 pamphlet authored by Franz Mathé".

Example 727. Consider the arithmetic sequence $(a_n)_{n=1}^{k} = (7, 17, 27, 37, \ldots, 837)$.
The first and last terms are $a_1 = 7$ and $a_k = 837$. The common difference is 10.
So, by Fact 109(b), the total number of terms is $k = (837 - 7)/10 + 1 = 83 + 1 = 84$.
And now by Fact 110,

$$\sum_{n=1}^{k} a_n = (a_1 + a_k)\frac{k}{2} = (7 + 837)\frac{84}{2} = 35\,448.$$

Example 728. Consider the arithmetic sequence $(b_n)_{n=1}^{k} = (1, 5, 9, 13, 17, \ldots, 393)$.
The first and last terms are $b_1 = 1$ and $b_k = 393$. The common difference is 4.
So, by Fact 109(b), the total number of terms is $k = (393 - 1)/4 + 1 = 98 + 1 = 99$.
And now by Fact 110,

$$\sum_{n=1}^{k} b_n = (b_1 + b_k)\frac{k}{2} = (1 + 393)\frac{99}{2} = 19\,503.$$

Here's another formula for the sum of a finite arithmetic series:

Corollary 19. *Suppose $(a_n)_{n=1}^{k}$ is a finite arithmetic sequence with $d = a_2 - a_1$. Then*

$$\sum_{n=1}^{k} a_n = ka_1 + \frac{k(k-1)}{2}d.$$

Proof. Use Fact 110, then Fact 109:

$$\sum_{n=1}^{k} a_n = (a_1 + a_k)\frac{k}{2} = [a_1 + a_1 + (k-1)d]\frac{k}{2} = ka_1 + \frac{k(k-1)}{2}d. \qquad \square$$

Example 729. XXX

Example 730. XXX

Exercise 206. Rewrite each series in summation notation, then compute its sum.

(a) $2 + 7 + 12 + 17 + 22 + 27 + 32 + \cdots + 997$.

(b) $3 + 20 + 37 + 54 + 71 + \cdots + 1\,703$.

(c) $81 + 89 + 97 + 105 + 113 + \cdots + 8\,081$. (Answer on p. 1836.)

49.2. Infinite Arithmetic Series

"Clearly", the infinite arithmetic series $4 + 7 + 10 + 13 + 16 + \dots$ does not converge. And more generally,

Fact 111. *Other than the zero series $0 + 0 + 0 + \dots$, every (infinite) arithmetic series diverges.*

Proof. See p. 1614 (Appendices). □

50. Geometric Sequences and Series

Definition 122. A *geometric sequence* (or *progression*) is a sequence where the ratio between any two consecutive terms is constant (and called the *common ratio*).

Example 731. Here are six **geometric sequences**, three finite (left) and three infinite (right):

$$(a_n)_{n=1}^{5} = (1,2,4,8,16), \qquad (a_n) = (1,2,4,8,16,\dots),$$

$$(b_n)_{n=1}^{7} = \left(1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},\frac{1}{64}\right), \qquad (b_n) = \left(1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},\frac{1}{64},\dots\right),$$

$$(c_n)_{n=1}^{3} = \left(7,7\pi,7\pi^2\right), \qquad (c_n) = \left(7,7\pi,7\pi^2,\dots\right).$$

In each sequence, the ratio between any two consecutive terms is a constant.

In $(a_n)_{n=1}^{5}$ and (a_n), the common ratio is $r = 2$.

In $(b_n)_{n=1}^{7}$ and (b_n), the common ratio is $r = 1/2$.

In $(c_n)_{n=1}^{3}$ and (c_n), the common ratio is $r = \pi$.

Definition 123. Given a geometric sequence, its series is called a *geometric series*.

Example 732. In the above example, the six geometric sequences have these corresponding **geometric series**:

$$1+2+4+8+16, \qquad 1+2+4+8+16+\dots,$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64} \qquad 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\dots$$

$$7+7\pi+7\pi^2 \qquad 7+7\pi+7\pi^2+\dots$$

Fact 112. *If $(a_n)_{n=1}^{k}$ is a finite geometric sequence with common ratio $r = a_2/a_1$, then*

(a) *The nth term is* $a_n = a_1 r^{n-1}$;

(b) *The number of terms is* $k = \log_r(a_k/a_1) + 1$;

(c) $$\sum_{n=1}^{k} a_n = \sum_{n=1}^{k} \left(a_1 r^{n-1}\right).$$

Proof. **(a)** Since a_n is $(n-1)$ terms "after" a_1, we must have $a_n = a_1 r^{n-1}$.

(b) By **(a)**, $a_k = a_1 r^{k-1}$, or $a_k/a_1 = r^{k-1}$, or $\log_r(a_k/a_1) = k-1$, or $k = \log_r(a_k/a_1)+1$.

(c) Immediate from **(a)**. □

50.1. Finite Geometric Sequences and Series

The sum of a finite geometric series is

$$\text{First Term} \times \frac{1 - \text{Common Ratio}^{\text{Number of Terms}}}{1 - \text{Common Ratio}}.$$

A bit more formally,

Fact 113. *Let $a, r \in \mathbb{R}$ with $r \neq 1$. If $S = a + ar + ar^2 + ar^3 + \cdots + ar^{k-1}$, then*

$$S = a\frac{1 - r^k}{1 - r}.$$

Proof. $(1-r)\,S = \left(a + ar + ar^2 + ar^3 + \cdots + ar^{k-1}\right) - r\left(a + ar + ar^2 + ar^3 + \cdots + ar^{k-1}\right) = a - ar^k$.

Rearranging, $S = a\left(1 - r^k\right)/(1-r)$. □

Remark 94. By the way, the mass cancellation trick used in the above proof is called the **method of differences** (which is the topic of Ch. 52).

Example 733. Consider the geometric series $1 + 2 + 2^2 + 2^3 + \cdots + 2^{20}$.

The first term is 1, the common ratio is 2, and there are 21 terms. So,

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{20} = 1\frac{1 - 2^{21}}{1 - 2} = \frac{2^{21} - 1}{1} = 2\,097\,152 - 1 = 2\,097\,151.$$

Example 734. Consider the geometric series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{20}}$.

The first term is 1, the common ratio is $1/2$, and there are 21 terms. So,

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{20}} = 1\frac{1 - 0.5^{21}}{1 - 0.5} = \frac{1 - 0.5^{21}}{0.5} = \frac{2^{21} - 1}{2^{20}} = \frac{2\,097\,151}{1\,048\,576}.$$

Here's a second formula for the sum of a finite geometric series:

$$\frac{\text{First Term} - \text{Common Ratio} \times \text{Last Term}}{1 - \text{Common Ratio}}.$$

A bit more formally,

Corollary 20. *If $(a_n)_{n=1}^{k}$ is a finite geometric sequence with $r = a_2/a_1$, then*

$$\sum_{n=1}^{k} a_n = \frac{a_1 - ra_k}{1-r}.$$

Proof. From Fact 113, $S = a\left(1 - r^k\right)/(1-r)$. Now, simply plug in $ar^k = rar^{k-1} = ra_k$. □

Example 735. Consider the geometric series $1 + 2 + 4 + 8 + 16 + \cdots + 1\,024$.

The first and last terms are 1 and 1 024, and the common ratio is $r = 2$. So,

$$1 + 2 + 4 + 8 + 16 + \cdots + 1\,024 = \frac{1 - 2 \cdot 1\,024}{1-2} = \frac{1 - 2\,048}{-1} = 2\,047$$

Example 736. Consider the geometric series $4 + 12 + 36 + 108 + \cdots + 8\,748$.

The first and last terms are 4 and 8 748, and the common ratio is 3. So,

$$4 + 12 + 36 + 108 + \cdots + 8\,748 = \frac{4 - 3 \cdot 8\,748}{1-3} = \frac{4 - 26\,244}{-2} = 13\,120$$

Exercise 207. Rewrite each series in summation notation, then compute its sum.

(a) $7 + 14 + 28 + 56 + 112 + 224 + 448 + 896$.

(b) $20 + 10 + 5 + 5/2 + 5/4 + 5/8$.

(c) $1 + 1/3 + 1/9 + 1/27 + 1/81 + 1/243$. (Answers on p. 1836.)

50.2. Infinite Geometric Sequences and Series

Perhaps surprisingly, an infinite geometric series converges if $|r| < 1$. Moreover, there's a nice and simple formula for the limit:

$$\frac{\text{First Term}}{1 - \text{Common Ratio}}.$$

A bit more formally,

Fact 114. *Let $a \in \mathbb{R}$. If $|r| < 1$, then*

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}.$$

Proof. See p. 1614 (Appendices). □

If instead $|r| \geq 1$, then the infinite geometric series diverges (for $a \neq 0$):

Fact 115. *Let $a \neq 0$. If $|r| \geq 1$, then $a + ar + ar^2 + ar^3 + \ldots$ diverges.*

Proof. See p. 1615 (Appendices). □

You've probably seen this "trick":

Example 737. Suppose $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. Then

1. $2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$
2. So, $S - 2S = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\right) - \left(2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$ or $-S = -2$.
3. Hence, $S = 2$.

Following this last example, we can write down this "proof" of Fact 114:

"*Proof*". Suppose $S = a + ar + ar^2 + ar^3 + \ldots$. Then

1. $rS = ar + ar^2 + ar^3 + ar^4 + \ldots$
2. So, $S - rS = \left(a + ar + ar^2 + ar^3 + \ldots\right) - \left(ar + ar^2 + ar^3 + ar^4 + \ldots\right)$ or $(1-r)S = a$.
3. Hence, $S = \frac{a}{1-r}$. □

Unfortunately, the above "proof" is flawed. To see why, try it for $r = 2$:

Example 738. Suppose $S = 1 + 2 + 4 + 8 + 16 + \ldots$. Then

1. $2S = 2 + 4 + 8 + 16 + 32 + \ldots$
2. So, $S - 2S = (1 + 2 + 4 + 8 + 16 + \ldots) - (2 + 4 + 8 + 16 + 32 + \ldots)$ or $-S = -1$.
3. Hence, $S = 1$.

In the above example, "$S = 1$" is clearly wrong. But we used the exact same three-step procedure as in the previous example. So, what went wrong? Why did the procedure work in the previous example but not in this example?

The quick answer is that here S is divergent, so that the algebraic manipulations in the first two steps are illegitimate (in contrast, they were legitimate in the previous example where S was convergent). For a more detailed answer, see Ch. 146.7 (Appendices).

Exercise 208. Rewrite each series in summation notation, then compute its sum.

(a) $6 + 9/2 + 27/8 + \ldots.$

(b) $20 + 10 + 5 + \ldots$

(c) $1 + 1/3 + 1/9 + \ldots$ (Answers on p. 1836.)

51. Rules of Summation Notation

Fact 116. *Suppose $a_n, b_n \in \mathbb{R}$ for all n, $c \in \mathbb{R}$, and $k, l \in \mathbb{Z}^+$. Then*[313]

(a) Constant Rule: $$\sum_{n=1}^{k} c = ck;$$

(b) Constant Factor Rule: $$\sum_{n=1}^{k} (ca_n) = c\sum_{n=1}^{k} a_n;$$

(c) Sum Rule: $$\sum_{n=1}^{k} (a_n + b_n) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n;$$

(d) Difference Rule: $$\sum_{n=1}^{k} (a_n - b_n) = \sum_{n=1}^{k} a_n - \sum_{n=1}^{k} b_n;$$

(e) Breakup Rule: $$\sum_{n=1}^{k+l} a_n = \sum_{n=1}^{k} a_n + \sum_{n=k+1}^{k+l} a_n.;$$

(f) Change Start & Stop: $$\sum_{n=k+1}^{k+l} a_n = \sum_{n=1}^{l} a_{k+n}.$$

Proof. **(a)** $$\sum_{n=1}^{k} c = \overbrace{c + c + \cdots + c}^{k \text{ times}} = ck.$$

(b) $$\sum_{n=1}^{k} (ca_n) = ca_1 + ca_2 + \cdots + ca_k = c(a_1 + a_2 + \cdots + a_k) = c\sum_{n=1}^{k} a_n.$$

(c) $$\begin{aligned}\sum_{n=1}^{k} (a_n + b_n) &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_k + b_k) \\ &= (a_1 + a_2 + \cdots + a_k) + (b_1 + b_2 + \cdots + b_k) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n.\end{aligned}$$

(d) $$\begin{aligned}\sum_{n=1}^{k} (a_n - b_n) &= (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_k - b_k) \\ &= (a_1 + a_2 + \cdots + a_k) - (b_1 + b_2 + \cdots + b_k) = \sum_{n=1}^{k} a_n - \sum_{n=1}^{k} b_n.\end{aligned}$$

(e) $$\begin{aligned}\sum_{n=1}^{k+l} a_n &= a_1 + a_2 + \cdots + a_k + a_{k+1} + a_{k+2} + \cdots + a_{k+l} \\ &= (a_1 + a_2 + \cdots + a_k) + (a_{k+1} + a_{k+2} + \cdots + a_{k+l}) = \sum_{n=1}^{k} a_n + \sum_{n=k+1}^{k+l} a_n.\end{aligned}$$

(f) $$\sum_{n=k+1}^{k+l} a_n = a_{k+1} + a_{k+2} + a_{k+3} + \cdots + +a_{k+l} = \sum_{n=1}^{l} a_{k+n}.$$ □

[313] More of such identities here: ✿.

Exercise 209. Evaluate each of the following ($x \in \mathbb{R}$). (Answer on p. 1837.)

(a) $\sum_{n=5}^{100} 1$. **(b)** $\sum_{n=5}^{100} n$. **(c)** $\sum_{n=5}^{100} (n+1)$. **(d)** $\sum_{n=5}^{100} (3n+2)$. **(e)** $\sum_{n=5}^{100} nx$.

Exercise 210. Let $x \neq 1$ and $S_k = \sum_{n=1}^{k} nx^{n-1}$. Prove the following.

(a) $S_5 = \dfrac{1-6x^5+5x^6}{(1-x)^2}$. **(b)** $S_k = \dfrac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}$.

(Hint: Consider $S_k - xS_k$.) (Answer on p. 1837.)

Example 739. Consider $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{1\,000\times 1\,001}$.

Finding the sum of this series is easy using **partial fractions**. First, rewrite into summation notation:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{1\,000\times 1\,001}=\sum_{n=1}^{1\,000}\frac{1}{n(n+1)}.$$

Next, decompose the nth term into partial fractions:

$$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}=\frac{A(n+1)+Bn}{n(n+1)}=\frac{(A+B)n+A}{n(n+1)}.$$

Comparing coefficients, $A=1$ and $A+B=0$. So, $B=-1$.

Hence,
$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

And,
$$\begin{aligned}\sum_{n=1}^{1\,000}\frac{1}{n(n+1)}&=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{1\,000\times 1\,001}\\&=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}\cdots-\frac{1}{1\,000}+\frac{1}{1\,000}-\frac{1}{1\,001}\\&=1-\frac{1}{1\,001}=\frac{1\,000}{1\,001}.\end{aligned}$$

In the second line, *every* term with denominator 2 through 1 000 is happily cancelled out. Your syllabus calls this process of mass cancellation the **method of differences**. (Some writers instead call this **telescoping**.)[314]

More generally,

$$\sum_{n=1}^{k}\frac{1}{n(n+1)}=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{k(k+1)}=1-\frac{1}{k+1}$$

We can also show that the corresponding infinite series converges:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots=\lim_{k\to\infty}\sum_{n=1}^{k}\frac{1}{n(n+1)}=\lim_{k\to\infty}\left(1-\frac{1}{k+1}\right)=1.$$

[314] ProofWiki says that this "arises from the obvious physical analogy with the folding up of a telescope".

Example 740. Consider $\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\cdots+\frac{1}{1\,000\times1\,001\times1\,002}$.

First rewrite into summation notation: $\sum_{n=1}^{1\,000}\frac{1}{n(n+1)(n+2)}$.

Next, decompose the nth term into partial fractions:

$$\frac{1}{n(n+1)(n+2)}=\frac{A}{n}+\frac{B}{n+1}+\frac{C}{n+2}=\frac{A(n+1)(n+2)+Bn(n+2)+Cn(n+1)}{n(n+1)(n+2)}$$

$$=\frac{(A+B+C)n^2+(3A+2B+C)n+2A}{n(n+1)(n+2)}.$$

Comparing coefficients, $A+B+C\overset{1}{=}0$, $3A+2B+C\overset{2}{=}0$, and $2A=1$ or $A=0.5$.

Take $\overset{2}{=}$ minus $\overset{1}{=}$ to get $2A+B=0$ or $B=-1$. And now $\overset{1}{=}$ yields $C=0.5$.

Altogether, $$\frac{1}{n(n+1)(n+2)}=\frac{0.5}{n}-\frac{1}{n+1}+\frac{0.5}{n+2}.$$

And so, $$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\cdots+\frac{1}{1\,000\times1\,001\times1\,002}$$

$$=\frac{0.5}{1}-\frac{1}{2}+\frac{0.5}{3}+\frac{0.5}{2}-\frac{1}{3}+\frac{0.5}{4}+\frac{0.5}{3}-\frac{1}{4}+\frac{0.5}{5}+\cdots+\frac{0.5}{1\,000}-\frac{1}{1\,001}+\frac{0.5}{1\,002}.$$

Observe that the three terms with denominator 3 cancel out. And the same will happen to all terms with denominator 4 through 1 000.

This leaves only terms with denominators 1, 2, 1 001, and 1 002:

$$\sum_{n=1}^{1\,000}\frac{1}{n(n+1)(n+2)}=\frac{0.5}{1}-\frac{1}{2}+\frac{0.5}{2}+\frac{0.5}{1\,001}-\frac{1}{1\,001}+\frac{0.5}{1\,002}$$

$$=\frac{1}{4}-\frac{0.5}{1\,001}+\frac{0.5}{1\,002}=\frac{1}{4}-\frac{1}{2\cdot1001\cdot1002}=0.249\ldots$$

More generally,

$$\sum_{n=1}^{k}\frac{1}{n(n+1)(n+2)}=\frac{0.5}{1}-\frac{1}{2}+\frac{0.5}{2}-\frac{0.5}{k+1}+\frac{0.5}{k+2}=\frac{1}{4}-\frac{1}{2(k+1)}+\frac{1}{2(k+2)}$$

We can also show that the corresponding infinite series converges:

$$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\cdots=\lim_{k\to\infty}\sum_{n=1}^{k}\frac{1}{n(n+1)(n+2)}=\lim_{k\to\infty}\left(\frac{1}{4}-\frac{1}{2(k+1)}+\frac{1}{2(k+2)}\right)=\frac{1}{4}.$$

Above we used **partial fractions**. We will next use **surd rationalisation**.

Example 741. Consider $\dfrac{1}{\sqrt{1}+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\dfrac{1}{\sqrt{9\,999}+\sqrt{10\,000}}$.

Again, first rewrite into summation notation:

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{9\,999}+\sqrt{10\,000}}=\sum_{n=1}^{9\,999}\frac{1}{\sqrt{n}+\sqrt{n+1}}.$$

Next, rationalise the surds in the denominator of the nth term:

$$\frac{1}{\sqrt{n}+\sqrt{n+1}}=\frac{1}{\sqrt{n}+\sqrt{n+1}}\frac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n}-\sqrt{n+1}}=\frac{\sqrt{n}-\sqrt{n+1}}{n-(n+1)}$$

$$=\frac{\sqrt{n}-\sqrt{n+1}}{-1}=\sqrt{n+1}-\sqrt{n}.$$

So,

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{9\,999}+\sqrt{10\,000}}$$

$$=\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\cdots+\sqrt{10\,000}-\sqrt{9\,999}.$$

The red terms cancel out. Likewise with the blue. This leaves

$$\sum_{n=1}^{9\,999}\frac{1}{\sqrt{n}+\sqrt{n+1}}=\sqrt{10\,000}-\sqrt{1}=100-1=99.$$

More generally,

$$\sum_{n=1}^{k}\frac{1}{\sqrt{n}+\sqrt{n+1}}=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{k}+\sqrt{k+1}}=\sqrt{k+1}-1$$

"Clearly", the corresponding infinite series diverges:

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}=\lim_{k\to\infty}\sum_{n=1}^{k}\frac{1}{\sqrt{n}+\sqrt{n+1}}=\lim_{k\to\infty}\sum_{n=1}^{k}\left(\sqrt{k+1}-1\right)=\infty.$$

The next example again uses partial fractions and now also the **formula for the sum of an arithmetic series**:

Example 742. Consider $\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+1\,000}$.

Again, first rewrite into summation notation:

$$\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+1\,000}=\sum_{n=1}^{1\,000}\frac{1}{1+\cdots+n}.$$

Next, use the formula for the sum of an arithmetic series to rewrite the nth term:

$$1+\cdots+n=\frac{n(n+1)}{2} \quad \text{or} \quad \frac{1}{1+\cdots+n}=\frac{2}{n(n+1)}.$$

Now, decompose into partial fractions:

$$\frac{2}{n(n+1)}=\frac{2}{n}-\frac{2}{n+1}.$$

Altogether,

$$\begin{aligned}\sum_{n=1}^{1\,000}\frac{1}{1+\cdots+n}&=\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+1\,000}\\&=\frac{2}{1}-\frac{2}{2}+\frac{2}{2}-\frac{2}{3}+\frac{2}{3}-\frac{2}{4}+\cdots+\frac{2}{1\,000}-\frac{2}{1\,001}=\frac{2}{1}-\frac{2}{1\,001}=\frac{2\,000}{1\,001}.\end{aligned}$$

More generally,

$$\sum_{n=1}^{k}\frac{1}{1+\cdots+n}=\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+k}=2-\frac{2}{k+1}$$

And hence, the sum of the corresponding infinite series converges to 2:

$$\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots=\lim_{k\to\infty}\sum_{n=1}^{k}\frac{1}{1+\cdots+n}=\lim_{k\to\infty}\left(2-\frac{2}{k+1}\right)=2.$$

Example 743. We can also use the method of differences to find the sum of squares.

We'll prove
$$1^2+2^2+3^2+\cdots+k^2=\sum_{i=1}^{k}n^2=\frac{k(k+1)(2k+1)}{6}.$$

First, observe that
$$(n+1)^3-n^3=3n^2+3n+1.$$

So,
$$\sum_{i=1}^{k}\left[(n+1)^3-n^3\right]=\sum_{i=1}^{k}\left(3n^2+3n+1\right)=3\sum_{i=1}^{k}n^2+3\sum_{i=1}^{k}n+\sum_{i=1}^{k}1$$
$$\stackrel{1}{=}3\sum_{i=1}^{k}n^2+3\frac{k(k+1)}{2}+k.$$

But,
$$\sum_{i=1}^{k}\left[(n+1)^3-n^3\right]=2^3-1^3+3^3-2^3+4^3-3^3+\cdots+(k+1)^3-k^3$$
$$\stackrel{2}{=}(k+1)^3-1^3=k^3+3k^2+3k.$$

Plug $\stackrel{1}{=}$ into $\stackrel{2}{=}$:
$$3\sum_{i=1}^{k}n^2+3\frac{k(k+1)}{2}+k=k^3+3k^2+3k.$$

Rearranging,
$$\sum_{i=1}^{k}n^2=\frac{k^3+3k^2+3k-3k(k+1)/2-k}{3}$$
$$=\frac{2k^3+3k^2+k}{6}=\frac{k(k+1)(2k+1)}{6}.$$

Exercise 211. Rewrite each series in summation notation and find its sum. Next, write down its sum in the case where the series has k terms instead. Finally, determine if the corresponding infinite series converges. If it does, find its limit. (Answer on p. 1838.)

(a) $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\cdots+\frac{1}{999\,999}$. (Hint in footnote.)[315]

(b) $\lg\frac{1}{2}+\lg\frac{2}{3}+\lg\frac{3}{4}+\cdots+\lg\frac{999}{1\,000}$. (lg is the base-10 log.)

(c) $\frac{1}{2\sqrt{1}+1\sqrt{2}}+\frac{1}{3\sqrt{2}+2\sqrt{3}}+\cdots+\frac{1}{100\sqrt{99}+99\sqrt{100}}$. (Hint in footnote.)[316]

(d) $1^3+2^3+3^3+\cdots+100^3$. (Hint in footnote.)[317]

[315] Hint: Think about the square numbers.

[316] Do the surd rationalisation. Then persevere with the algebra and things will work out nicely.

[317] Consider $(n+1)^4-n^4$ and mimic the last example (be warned that the algebra will be more painful).

Part III.
Vectors

The cultural problem is a self-perpetuating monster: students learn about math from their teachers, and teachers learn about it from their teachers, so this lack of understanding and appreciation for mathematics in our culture replicates itself indefinitely. Worse, the perpetuation of this "pseudo-mathematics," this emphasis on the accurate yet mindless manipulation of symbols, creates its own culture and its own set of values. Those who have become adept at it derive a great deal of self-esteem from their success. The last thing they want to hear is that math is really about raw creativity and aesthetic sensitivity. Many a graduate student has come to grief when they discover, after a decade of being told they were "good at math," that in fact they have no real mathematical talent and are just very good at following directions. Math is not about following directions, it's about making new directions.

— Paul Lockhart (2002, 2009).

a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

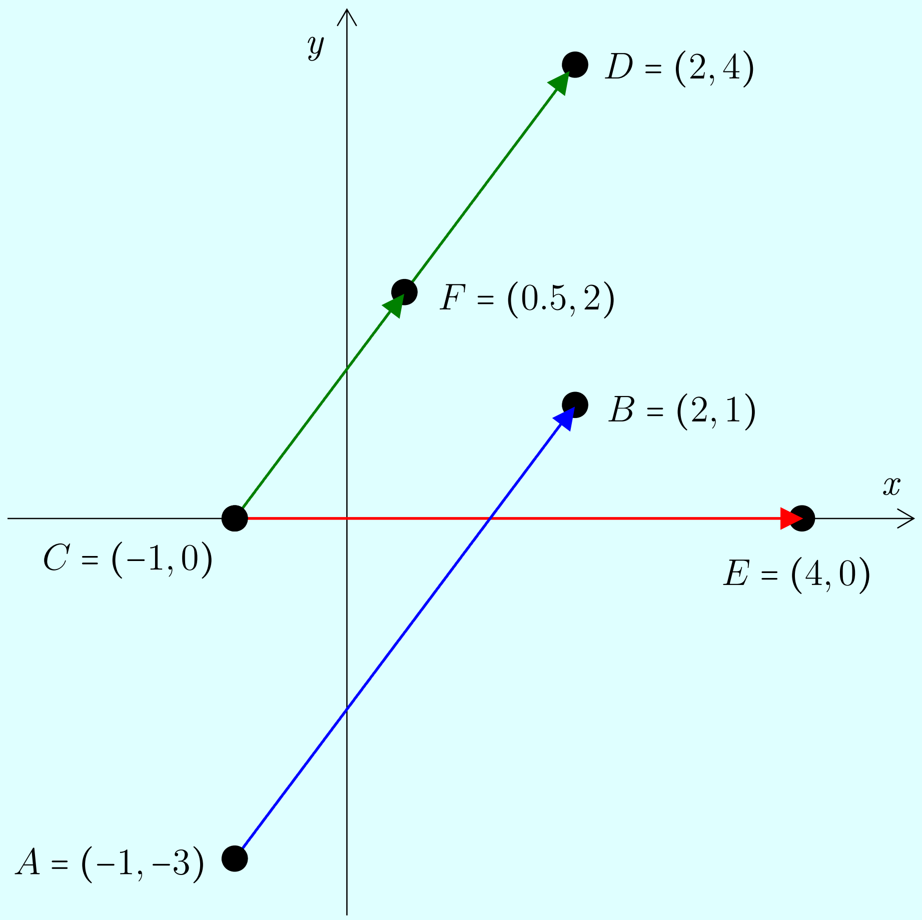
— George Pólya (1945).

53. Introduction to Vectors

The Latin word **vector** means **carrier**. You may recall from biology that mosquitoes are vectors, because they **carry** diseases to humans. Similarly, in mathematics, a vector "**carries**" us from one point to another.

Example 744. Let $A = (-1, -3)$ and $B = (2, 1)$ be points. Then $\overrightarrow{AB} = (3, 4)$ is the **vector** that "carries" us 3 units east and 4 units north from the point A to the point B.

The vector $\overrightarrow{AB}$ has tail A and head B—remember, a vector goes **from tail to head**. (Just remember: **arrowhead**—the arrowhead is where the vector's head is.)



Let $C = (-1, 0)$, $D = (2, 4)$, $E = (4, 0)$, and $F = (0.5, 2)$ be points.

The vector $\overrightarrow{CD} = (3, 4)$ "carries" us 3 units east and 4 units north from the tail C to the head D.

The vector $\overrightarrow{CE} = (5, 0)$ "carries" us 5 units east and 0 units north from the tail C to the head E.

The vector $\overrightarrow{CF} = (1.5, 2)$ "carries" us 1.5 units east and 2 units north from the tail C to the head F.

Formally, a vector in 2D space is an ordered pair of real numbers:

Definition 124. Given the points $A = (a_1, a_2)$ and $B = (b_1, b_2)$, the *vector from A to B*, denoted $\overrightarrow{AB}$, is this ordered pair of real numbers:

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2).$$

(Later on when we look at three-dimensional (3D) space, vectors will instead be ordered **triples** of real numbers.)

A vector is often contrasted with a **scalar**, which is simply any real number:

Definition 125. A *scalar* is any real number.

We now contrast a **vector**, a **scalar**, and a **point**:

A **vector** is a **two-dimensional (2D) mathematical object** with the properties of

magnitude (or **length**) and **direction**.

(A line is also a 2D object with magnitude or length, but not direction.)

In contrast, a **scalar** is a **one-dimensional object** with only the property of

magnitude.

And a **point** is a **zero-dimensional object** (with neither magnitude nor direction).

Example 745. You may recall from physics that **velocity** is a vector quantity, while **speed** is a scalar quantity. In particular, speed is the magnitude of velocity.

We'll have more to say about this in Ch. 58.

53.1. The Magnitude or Length of a Vector

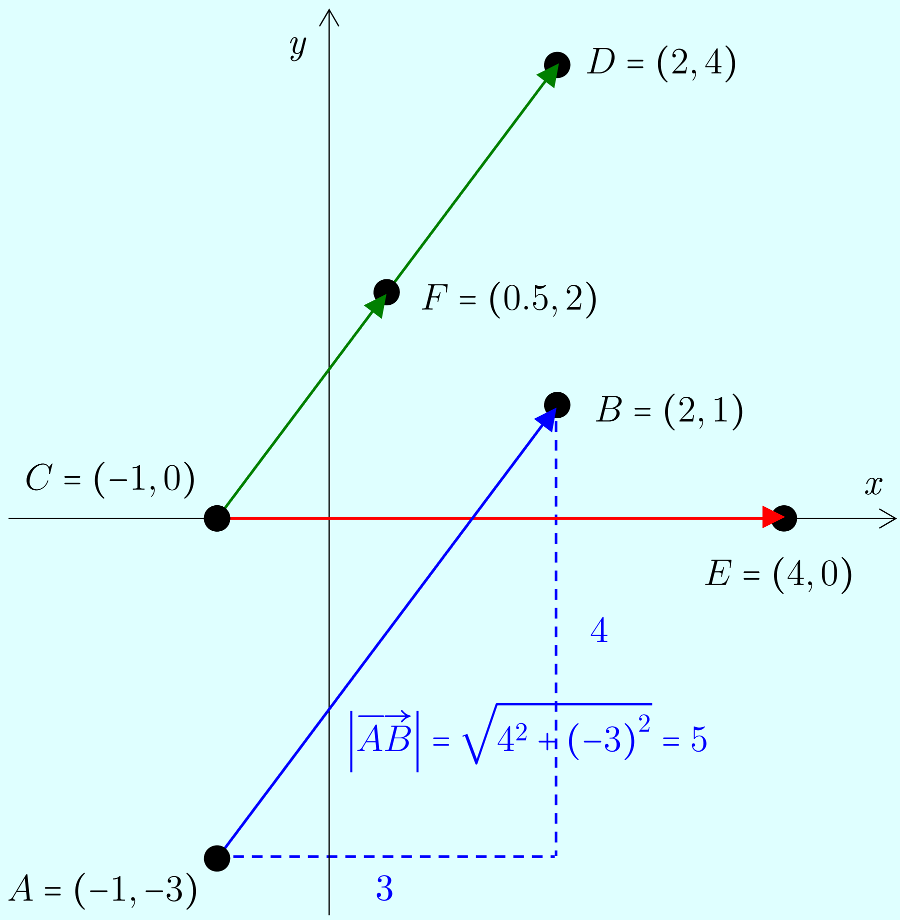
The **magnitude** or **length** of a vector is simply given by Pythagoras' Theorem:

Definition 126. Given the vector (p, q), its *magnitude* (or *length*), denoted $|(p, q)|$, is this number:

$$|(p, q)| = \sqrt{p^2 + q^2}.$$

Example 746. The magnitude or length of the vector $\overrightarrow{AB} = (3, 4)$ is

$$\left|\overrightarrow{AB}\right| = |(3, 4)| = \sqrt{4^2 + (-3)^2} = 5.$$



Similarly, the magnitudes of $\overrightarrow{CD} = (3, 4)$, $\overrightarrow{CE} = (5, 0)$, and $\overrightarrow{CF} = (1.5, 2)$ are

$$\begin{aligned}
\left|\overrightarrow{CD}\right| &= |(3, 4)| &&= \sqrt{4^2 + (-3)^2} = 5, \\
\left|\overrightarrow{CE}\right| &= |(5, 0)| &&= \sqrt{5^2 + 0^2} \quad = 5, \\
\left|\overrightarrow{CF}\right| &= |(1.5, 2)| &&= \sqrt{1.5^2 + 2^2} \quad = 2.5.
\end{aligned}$$

Remark 95. In more general contexts, **norm** is another synonym for **magnitude** (or **length**). But we won't use this term in this textbook.

53.2. When Are Two Vectors Identical?

Informally, two vectors (p,q) and (r,s) are identical or equal if and only if they have the same **direction** and **magnitude**. Formally,

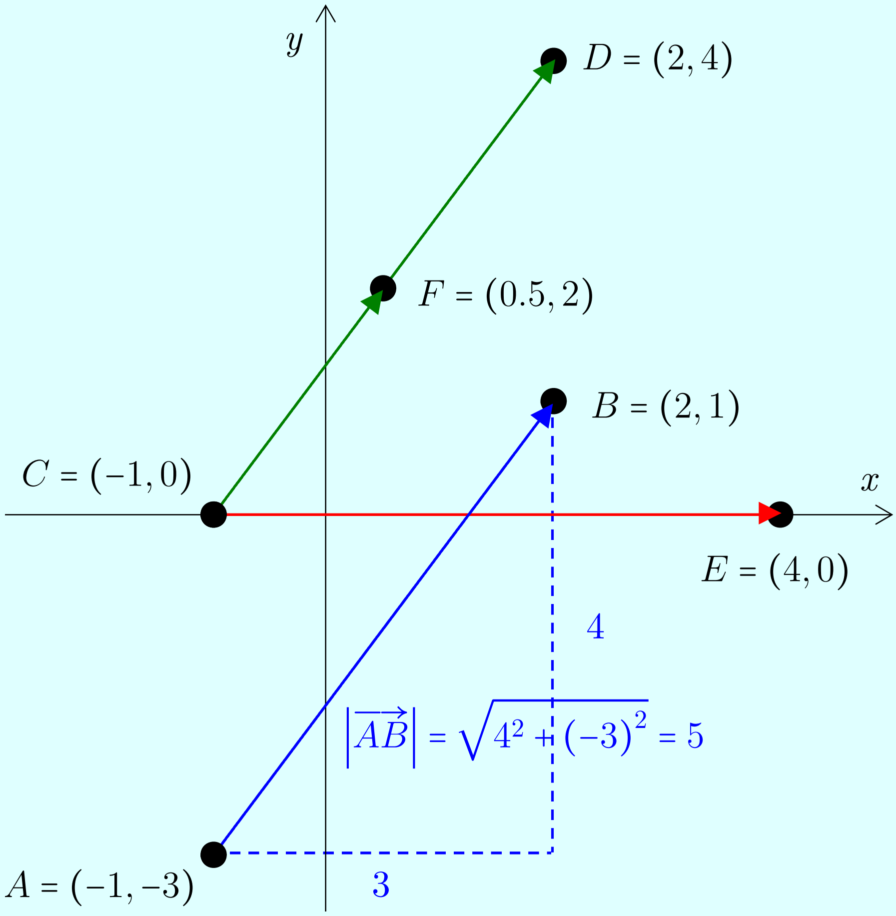
$$(p,q) = (r,s) \quad \Longleftrightarrow \quad p = r,\ q = s.$$

Example 747. Consider the vectors $\overrightarrow{AB} = (3,4)$ and $\overrightarrow{CD} = (3,4)$. Both have length 5. Also, both **point in the same direction**.[318]

Informally, $\overrightarrow{AB} = \overrightarrow{CD}$ because $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same length and direction.

A little more formally, $\overrightarrow{AB} = \overrightarrow{CD}$ because $3 = 3$ and $4 = 4$.

Note that when determining whether two vectors are identical, **their tail and head do not matter**. In the above example, $\overrightarrow{AB} = \overrightarrow{CD}$ even though they don't have the same tail or head.



Example 748. The vectors $\overrightarrow{CD} = (3,4)$ and $\overrightarrow{CF} = (1.5,2)$ point in the same direction.

However, $\overrightarrow{CD} \neq \overrightarrow{CF}$ because they have different lengths—$\left|\overrightarrow{CD}\right| = 5$, while $\left|\overrightarrow{CF}\right| = 2.5$.

More formally, $\overrightarrow{CD} \neq \overrightarrow{CF}$ because $(3,4) \neq (1.5,2)$.

Example 749. Each of $\overrightarrow{AB} = (3,4)$, $\overrightarrow{CD} = (3,4)$, and $\overrightarrow{CE} = (5,0)$ has length 5. However, $\overrightarrow{CE}$ "obviously" points in a different direction from $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

So,
$$\overrightarrow{CE} \neq \overrightarrow{AB} \quad \text{and} \quad \overrightarrow{CE} \neq \overrightarrow{CD}.$$

[318] Below, Definition 137 will formally define what it means for two vectors to "point in the same direction".

53.3. A Vector and a Point Are Different Things

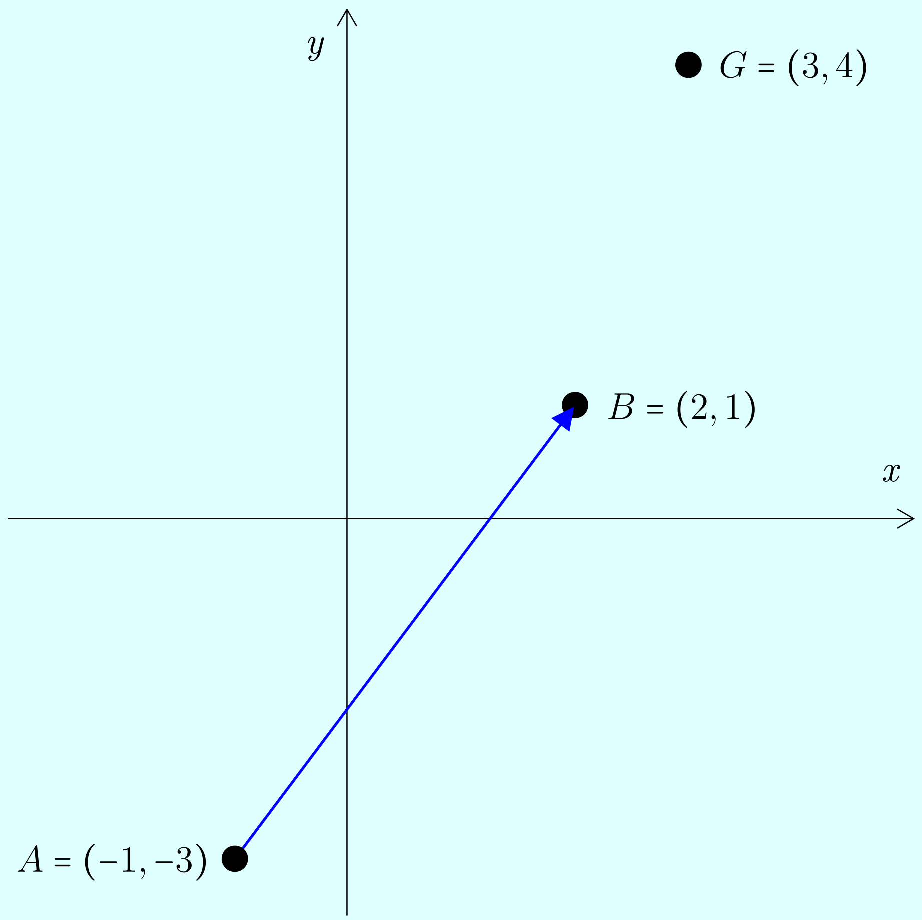
Here's another important point. Although a vector and a point can each be described by an ordered pair of real numbers, they are **entirely different mathematical objects**.

To repeat, a **vector** is a **two-dimensional object** possessing the properties of **length** and **direction**. In contrast, a **point** is a **zero-dimensional object**, with neither length nor direction. Example:

Example 750. Let $A = (-1, -3)$, $B = (2, 1)$, and $G = (3, 4)$ be points.

Then $\overrightarrow{AB} = (3, 4)$ is the **vector** that carries us 3 units east and 4 units north **from** A **to** B. It is a two-dimensional object endowed with the properties of **length** and **direction**.

In contrast, the **point** $G = (3, 4)$, although also described by an ordered pair of real numbers, is a zero-dimensional object, with neither length nor direction.



Again, $\overrightarrow{AB} = (3, 4)$ is a vector, while $G = (3, 4)$ is a point. They are *completely different mathematical objects*. Don't confuse them.

So far this textbook has used the notation (p, q) to mean three entirely different things:

(a) The **set of real numbers** between p and q (excluding p and q);

(b) The **point** with x-coordinate p and y-coordinate q; or

(c) The **vector** that "carries" us p units east and q units north.

As discussed in Remark 26, this could be confusing in theory, but shouldn't be in practice.

53.4. Two More Ways to Denote a Vector

The vector (p,q) can also be written in two other ways. First, we can write it "vertically":

$$(p,q) = \begin{pmatrix} p \\ q \end{pmatrix}.$$

Example 751. The vectors $\overrightarrow{AB} = (3,4)$, $\overrightarrow{CD} = (3,4)$, $\overrightarrow{CE} = (5,0)$, $\overrightarrow{CF} = (1.5,2)$ may also be written

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \overrightarrow{CD} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \overrightarrow{CE} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad \overrightarrow{CF} = \begin{pmatrix} 1.5 \\ 2 \end{pmatrix}.$$

As we'll see shortly, we'll be doing a lot of addition and multiplication with vectors. And so, this "vertical" notation for vectors is very useful, because it literally helps us see better. But in print, I'll often prefer using the (a,b) notation, simply because it takes up less space.

Second, we can denote a vector by a single, lower-case, bold-font letter:

$$(p,q) = \mathbf{u}.$$

Example 752. The vectors $\overrightarrow{AB} = (3,4)$, $\overrightarrow{CD} = (3,4)$, $\overrightarrow{CE} = (5,0)$, $\overrightarrow{CF} = (1.5,2)$ may also be written

$$\overrightarrow{AB} = \mathbf{a}, \quad \overrightarrow{CD} = \mathbf{b}, \quad \overrightarrow{CE} = \mathbf{c}, \quad \overrightarrow{CF} = \mathbf{d}.$$

(The choice of letters is somewhat arbitrary. For an obvious reason, $\mathbf{v}$ is a favourite.)

We'll often use the bold-font letter notation in print. However, it's hard to hand-write in bold font, so you can write $\overrightarrow{u}$ and $\overrightarrow{v}$ in place of $\mathbf{u}$ and $\mathbf{v}$.

Exercise 212. Let $A = (-1,-3)$, $B = (2,1)$, and $G = (3,4)$ be points.

Consider the five vectors $\overrightarrow{AG}$, $\overrightarrow{BA}$, $\overrightarrow{BG}$, $\overrightarrow{GA}$, $\overrightarrow{GB}$. Write down each in three different ways. What is each vector's tail, head, and length? How many units does each vector carry us in the x- and y-directions? (Answer on p. 1841.)

Exercise 213. Provide a counterexample to show that the following is not always true:

$$|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|.$$

(Answer on p. 1841.)

Remark 96. Just so you know, *yet another* way to denote the vector $\mathbf{v} = (v_1, v_2)$ is with brackets (and no comma): $\mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$.

This brackets notation for vectors does not appear in your A-Level Maths syllabus and exams. And so, we won't use it in this textbook.

53.5. Position Vectors

Given a point A, its **position vector** $\overrightarrow{OA}$ is simply the vector that carries us from the origin O to the point A:

Definition 127. Given a point A, its *position vector* is the vector $\overrightarrow{OA}$.

So, given a point $A = (a_1, a_2)$, its position vector is

$$\overrightarrow{OA} = \mathbf{a} = (a_1, a_2) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Example 753. The points $A = (-1, -3)$, $B = (2, 1)$, and $C = (-1, 0)$ have position vectors

$$\overrightarrow{OA} = \mathbf{a} = (-1, -3) = \begin{pmatrix} -1 \\ -3 \end{pmatrix}, \quad \overrightarrow{OB} = \mathbf{b} = (2, 1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \overrightarrow{OC} = \mathbf{c} = (-1, 0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Again, take care not to confuse a point with its position vector. Although A and $\overrightarrow{OA}$ may both be denoted by $(-1, -3)$, they are different mathematical objects—the former is a point while the latter is a vector.

53.6. The Zero Vector

Informally, the **zero vector** is the vector that carries us nowhere. Formally,

Definition 128. The *zero vector*, denoted $\mathbf{0}$, is the origin's position vector.

And so, the zero vector is $$\mathbf{0} = \overrightarrow{OO} = (0, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- Once again, do not confuse the point $O = (0, 0)$ with its position vector $\mathbf{0}$.
- Given any point P, the vector that carries us from P to P is the vector carries us precisely nowhere. Hence, $\overrightarrow{PP} = \mathbf{0}$.

The following result says that every vector has non-negative length. And moreover, the only vector with length 0 is the zero vector:

Fact 117. *Suppose* $\mathbf{v}$ *is a vector. Then* $|\mathbf{v}| \geq 0$. *Moreover,* $|\mathbf{v}| = 0 \iff \mathbf{v} = \mathbf{0}$.

Proof. Let $\mathbf{v} = (v_1, v_2)$. Then $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2} \geq 0$ and $|\mathbf{v}| = 0 \iff \mathbf{v} = (0, 0) = \mathbf{0}$.[319] □

[319]For a proof of this result in the general n-dimensional case, see p. 1618 (Appendices).

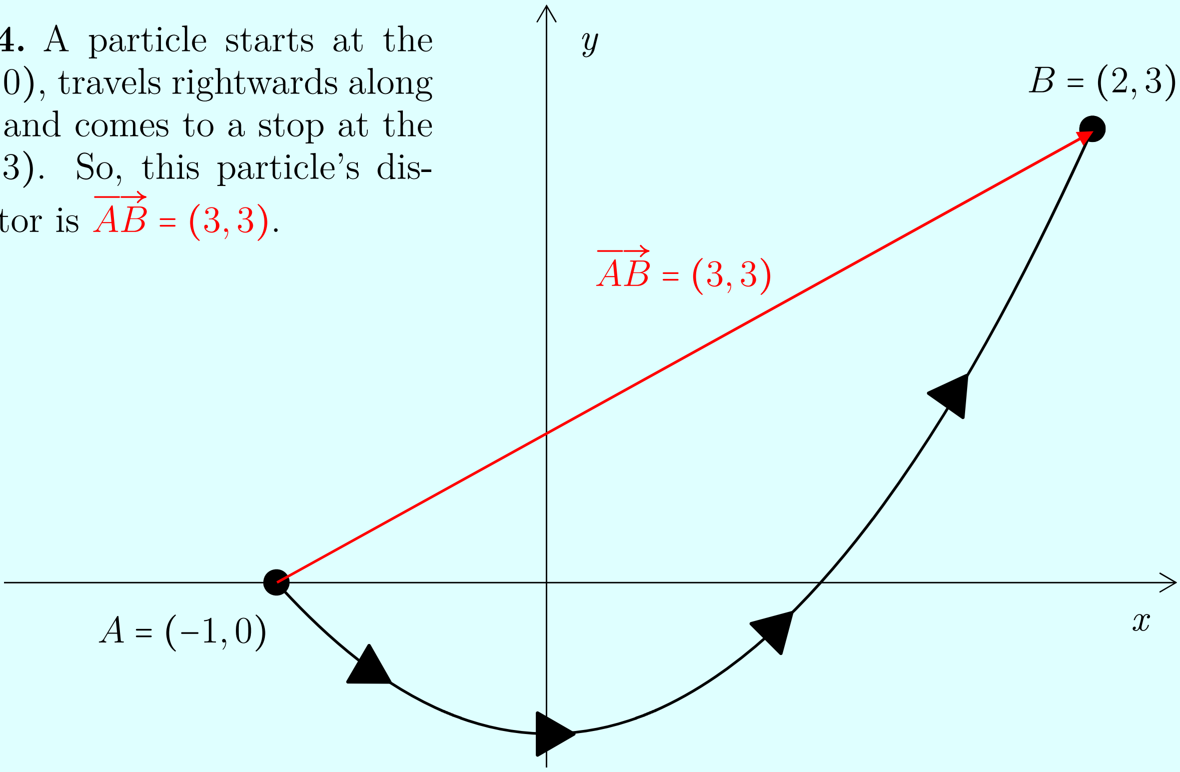
53.7. Displacement Vectors

Definition 129. Suppose a moving object starts at point A and ends at point B. Then we call $\overrightarrow{AB}$ its *displacement vector*.

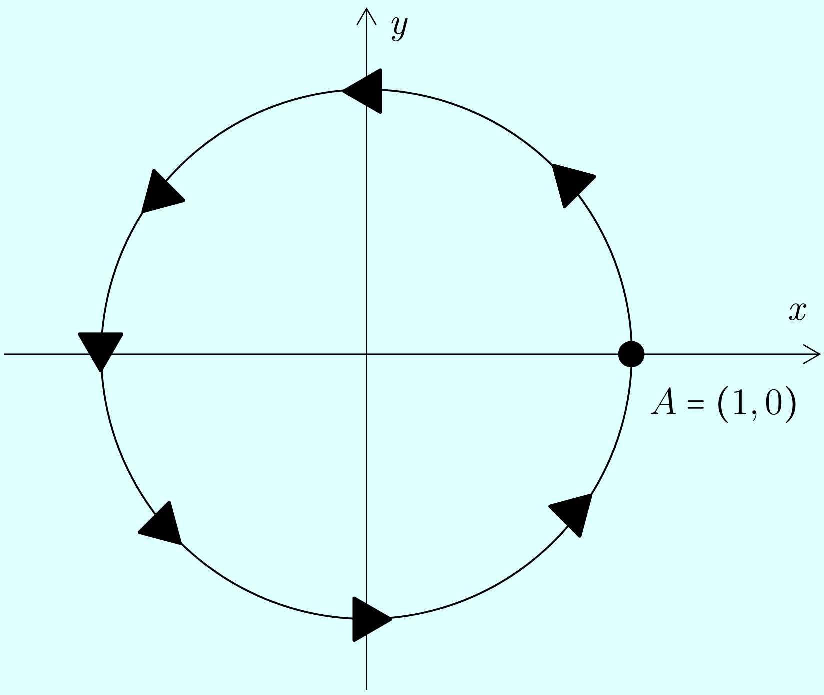
So, if a moving object starts at $A = (a_1, a_2)$ and ends at $B = (b_1, b_2)$, then regardless of the path taken by the object, we say that its displacement vector is

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2).$$

Example 754. A particle starts at the point $A = (-1, 0)$, travels rightwards along the black arc, and comes to a stop at the point $B = (2, 3)$. So, this particle's displacement vector is $\overrightarrow{AB} = (3, 3)$.



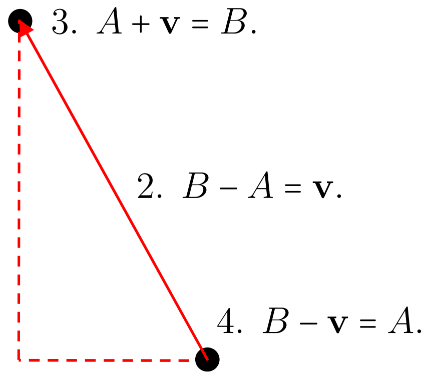
Example 755. A particle starts at the point $A = (1, 0)$, travels anti-clockwise around the unit circle centred on the origin, completes one full circle, and comes to a stop at the point A. So, this particle's displacement vector is $\overrightarrow{AA} = (0, 0) = \mathbf{0}$ (not depicted in figure).



53.8. Sum and Difference of Points and Vectors

In this subchapter, we'll learn these four things:

1. Point + Point = Undefined
2. Point − Point = Vector
3. Point + Vector = Point
4. Point − Vector = Point



1. Point + Point = Undefined

If A and B are points, then there is no such thing as $A + B$.

Example 756. Let $A = (1, 2)$ and $B = (5, 0)$ be points. Then the sum $A + B$ is undefined. It makes no sense to talk about the sum of two points.

The analogy is to points or locations in the real world:

Example 757. Consider Athens and Berlin, two points or locations. The sum Athens + Berlin is undefined. It makes no sense to talk about the sum of two points or locations.

2. Point − Point = Vector.

Definition 130. Given two points A and B, the *difference* $B - A$ is the vector $\overrightarrow{AB}$.

And so, given the points $A = (a_1, a_2)$ and $B = (b_1, b_2)$, their difference is

$$B - A = \overrightarrow{AB} = (b_1 - a_1, b_2 - a_2).$$

Example 758. Let $A = (1, 2)$ and $B = (5, 0)$. Then $B - A$ is the vector $\overrightarrow{AB}$:

$$B - A = \overrightarrow{AB} = (5, 0) - (1, 2) = (4, -2).$$

Similarly, the difference $A - B$ is defined to be the vector $\overrightarrow{BA}$:

$$A - B = \overrightarrow{BA} = (1, 2) - (5, 0) = (-4, 2).$$

We can continue with the same Athens-Berlin analogy:

Example 759. The vector "Berlin − Athens" is the journey from Athens to Berlin:

$$\text{Berlin} - \text{Athens} = (-500, 900).$$

That is, the journey from Athens to Berlin carries us 500 km west and 900 km north.[320]

Similarly, the vector "Athens − Berlin" is the reverse journey from Berlin to Athens:

$$\text{Athens} - \text{Berlin} = (500, -900).$$

That is, the journey from Athens to Berlin carries us 500 km east and 900 km south.

[320] I made up these numbers. The actual journey is longer (Google Maps).

3. Point + Vector = Point.

Definition 131. Given a point $A = (a_1, a_2)$ and a vector $\mathbf{v} = (v_1, v_2)$, their *sum* $A + \mathbf{v}$ is this point:

$$A + \mathbf{v} = (a_1 + v_1, a_2 + v_2).$$

Equivalently, if the vector $\mathbf{v}$'s tail is at the point A, then its head is at the point $A + \mathbf{v}$.

Example 760. Let $A = (1, 2)$ and $\mathbf{v} = (4, 4)$. Then their sum is the point $(5, 6)$:

$$A + \mathbf{v} = (1, 2) + (4, 4) = (5, 6).$$

Example 761. If $\mathbf{v} = (-500, 900)$, then

$$\text{Athens} + \mathbf{v} = \text{Athens} + (-500, 900) = \text{Berlin}.$$

Starting from Athens, travelling 500 km west and 900 km north brings us to Berlin.

4. Point − Vector = Point.

Definition 132. Given a point $B = (b_1, b_2)$ and a vector $\mathbf{v} = (v_1, v_2)$, their *difference* $B - \mathbf{v}$ is this point:

$$B - \mathbf{v} = (b_1 - v_1, b_2 - v_2).$$

Equivalently, if the vector $\mathbf{v}$'s head is at the point A, then its tail is at the point $A - \mathbf{v}$.

Example 762. Let $A = (1, 2)$ and $\mathbf{v} = (4, 4)$. Then their difference is this point:

$$A - \mathbf{v} = (1, 2) - (4, 4) = (-3, -2).$$

Example 763. If $\mathbf{v} = (-500, 900)$, then

$$\text{Berlin} - \mathbf{v} = \text{Berlin} - (-500, 900) = \text{Athens}.$$

If we end up in Berlin after travelling 500 km west and 900 km north, then we must have started in Athens.

Exercise 214. Consider the vector $(4, -3)$. (Answer on p. 1841.)

(a) If its tail is $(0, 0)$, then what is its head?
(b) If its head is $(0, 0)$, then what is its tail?
(c) If its tail is $(5, 2)$, then what is its head?
(d) If its head is $(5, 2)$, then what is its tail?

53.9. Sum, Additive Inverse, and Difference of Vectors

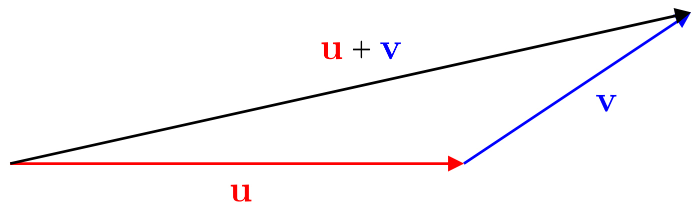
In this subchapter, we'll learn these three things:

1. Vector + Vector = Vector.
2. − Vector = Vector.
3. Vector − Vector = Vector.

1. Vector + Vector = Vector.

Definition 133. The *sum* of two vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is denoted $\mathbf{u} + \mathbf{v}$ and is the vector $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$.

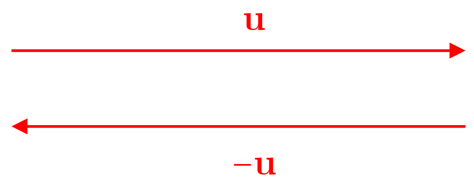
Place $\mathbf{v}$'s tail at $\mathbf{u}$'s head. Then $\mathbf{u} + \mathbf{v}$ is the vector from $\mathbf{u}$'s tail to $\mathbf{v}$'s head:



Example 764. The sum of $\mathbf{u} = (-1, 3)$ and $\mathbf{v} = (4, 4)$ is the vector $(3, 7)$:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}.$$

2. − Vector = Vector (Additive inverse).



Informally, the vector $-\mathbf{u}$ is the vector $\mathbf{u}$ flipped in the opposite direction. Formally,

Definition 134. The *additive inverse* of the vector $\mathbf{u} = (u_1, u_2)$ is this vector:

$$-\mathbf{u} = (-u_1, -u_2).$$

Example 765. The additive inverses of $\mathbf{u} = (-1, 3)$ and $\mathbf{v} = (4, 4)$ are

$$-\mathbf{u} = -\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \text{and} \quad -\mathbf{v} = -\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}.$$

3. Vector − Vector = Vector.

Definition 135. Given two vectors $\mathbf{u}$ and $\mathbf{v}$, their *difference* $\mathbf{u}-\mathbf{v}$ is the sum of $\mathbf{u}$ and the additive inverse of $\mathbf{v}$. That is,

$$\mathbf{u}-\mathbf{v}=\mathbf{u}+(-\mathbf{v}).$$

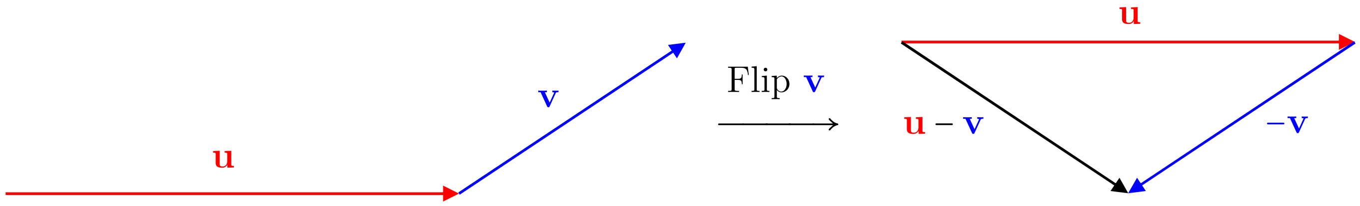
Fact 118. *If* $\mathbf{u}=(u_1,u_2)$ *and* $\mathbf{v}=(v_1,v_2)$ *are vectors, then*

$$\mathbf{u}-\mathbf{v}=(u_1-v_1,u_2-v_2).$$

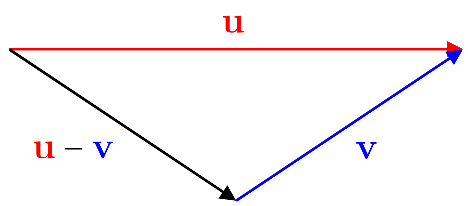
Proof. By Definition 134, $-\mathbf{v}=(-v_1,-v_2)$. And so by Definition 133,

$$\mathbf{u}-\mathbf{v}=\mathbf{u}+(-\mathbf{v})=(u_1-v_1,u_2-v_2).$$ □

To get $\mathbf{u}-\mathbf{v}$, first flip $\mathbf{v}$ to get $-\mathbf{v}$, then add $-\mathbf{v}$ to $\mathbf{u}$:



Or equivalently, place the heads of $\mathbf{u}$ and $\mathbf{v}$ at the same point. Then $\mathbf{u}-\mathbf{v}$ is the vector from the tail of $\mathbf{u}$ to the tail of $\mathbf{v}$:



Example 766. Suppose $\mathbf{u}=(-1,3)$ and $\mathbf{v}=(4,4)$. Then the difference $\mathbf{u}-\mathbf{v}$ is

$$\mathbf{u}-\mathbf{v}=\begin{pmatrix}-1\\3\end{pmatrix}-\begin{pmatrix}4\\4\end{pmatrix}=\begin{pmatrix}-5\\-1\end{pmatrix}.$$

Similarly, the difference $\mathbf{v}-\mathbf{u}$ is

$$\mathbf{v}-\mathbf{u}=\begin{pmatrix}4\\4\end{pmatrix}-\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}5\\1\end{pmatrix}.$$

In the previous subchapter, we *defined* $B - A = \overrightarrow{AB}$. We'll now *prove* that $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$:

Fact 119. *Suppose A and B are points. Then $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$.*

Proof. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. By Definition 124,

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2), \qquad \overrightarrow{OA} = (a_1, a_2), \quad \text{and} \quad \overrightarrow{OB} = (b_1, b_2).$$

By Fact 118, $\overrightarrow{OB} - \overrightarrow{OA} = (b_1 - a_1, b_2 - a_2)$. Thus, $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$. □

More generally,

Fact 120. *Suppose A, B, and C are points. Then $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$ and $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.*

Proof. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ be points. By Definition 130,

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2), \qquad \overrightarrow{AC} = (c_1 - a_1, c_2 - a_2), \quad \text{and} \quad \overrightarrow{CB} = (b_1 - c_1, b_2 - c_2).$$

And now by Fact 118,

$$\overrightarrow{AB} - \overrightarrow{AC} = (b_1 - a_1, b_2 - a_2) - (c_1 - a_1, c_2 - a_2) = (b_1 - c_1, b_2 - c_2) = \overrightarrow{CB}.$$

Observing that $-\overrightarrow{CB} = \overrightarrow{BC}$ and rearranging, we also have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. □

Example 767. Suppose $A = (-1, 2)$ and $B = (3, -1)$. Then

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Example 768. Suppose $C = (-1, 1)$ and $D = (3, -2)$. Then

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Exercise 215. Express each of these six vectors more simply: (Answer on p. 1841)

$$\overrightarrow{AC} + \overrightarrow{CB}, \qquad \overrightarrow{DC} + \overrightarrow{CA}, \qquad \overrightarrow{BD} + \overrightarrow{DA},$$

$$\overrightarrow{AD} - \overrightarrow{CD}, \qquad -\overrightarrow{DC} - \overrightarrow{BD}, \qquad \overrightarrow{BD} + \overrightarrow{DB}.$$

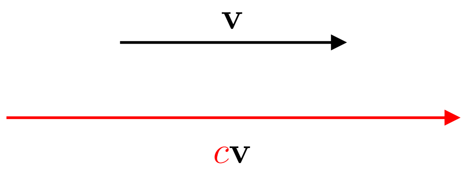
53.10. Scalar Multiplication of a Vector

Scalar multiplication of a vector works in the "obvious" fashion:

Definition 136. Given the vector $\mathbf{v} = (v_1, v_2)$ and the scalar $c \in \mathbb{R}$, the vector $c\mathbf{v}$ is

$$c\mathbf{v} = (cv_1, cv_2).$$

The vector $c\mathbf{v}$ is simply the vector that points in the same direction as $\mathbf{v}$, but has c times the length.



Fact 121. *If* $\mathbf{v}$ *is a vector and* $c \in \mathbb{R}$, *then* $|c\mathbf{v}| = |c|\,|\mathbf{v}|$.

Proof. See Exercise 216. □

Exercise 216. Let $c \in \mathbb{R}$ and $\mathbf{v} = (v_1, v_2)$. (Answer on p. 1841.)

(a) Write out $c\mathbf{v}$.

(b) Now prove Fact 121. (Hint: $\sqrt{x^2y} = |x|\sqrt{y}$.)

Exercise 217. Let $A = (1, -3)$, $B = (2, 0)$, and $C = (5, -1)$. (Answer on p. 1842.)

(a) Write down $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BC}$, $2\overrightarrow{AB}$, $3\overrightarrow{AC}$, and $4\overrightarrow{BC}$.

(b) Verify that $\left|2\overrightarrow{AB}\right| = 2\left|\overrightarrow{AB}\right|$, $\left|3\overrightarrow{AC}\right| = 3\left|\overrightarrow{AC}\right|$, and $\left|4\overrightarrow{BC}\right| = 4\left|\overrightarrow{BC}\right|$.

53.11. When Do Two Vectors Point in the Same Direction?

In words, we say that two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ *point in*

(a) The *same direction* if they are positive scalar multiples of each other;

(b) *Exact opposite directions* if they are negative scalar multiples of each other; and

(c) *Different directions* if they are not scalar multiples of each other.

In a bit more formal notation,

Definition 137. We say that two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ *point in*

(a) The *same direction* if $\mathbf{u} = k\mathbf{v}$ for some $k > 0$;

(b) *Exact opposite directions* if $\mathbf{u} = k\mathbf{v}$ for some $k < 0$; and

(c) *Different directions* if $\mathbf{u} \neq k\mathbf{v}$ for any $k \in \mathbb{R}$.

Example 769. Let $\mathbf{a} = (2, 0)$, $\mathbf{b} = (1, 0)$, $\mathbf{c} = (-3, 0)$, and $\mathbf{d} = (1, 1)$.

The vectors $\mathbf{a}$ and $\mathbf{b}$ point in the same direction because $\mathbf{a} = 2\mathbf{b}$.

The vectors $\mathbf{a}$ and $\mathbf{c}$ point in the exact opposite directions because $\mathbf{c} = -1.5\mathbf{a}$.

The vectors $\mathbf{a}$ and $\mathbf{d}$ point in different directions because $\mathbf{a} \neq k\mathbf{d}$ for any k.

Exercise 218. Continuing with the above example, explain if $\mathbf{b}$ points in the same, exact opposite, or different direction from each of $\mathbf{c}$ and $\mathbf{d}$. (Answer on p. 1842.)

Remark 97. The above definition specifies that $\mathbf{u}$ and $\mathbf{v}$ must be non-zero. So, the zero vector $\mathbf{0} = (0, 0)$ is a special case—it does **not** point in the same, exact opposite, or different direction as any other vector (not even itself).

53.12. When Are Two Vectors Parallel?

Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if they point in the same or the exact opposite directions and **non-parallel** otherwise:

Definition 138. Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are *parallel* if $\mathbf{u} = k\mathbf{v}$ for some $k \in \mathbb{R}$ and *non-parallel* otherwise.

So, *non-parallel* and *point in different directions* are synonyms.

As shorthand, we write $\mathbf{a} \parallel \mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ are parallel and $\mathbf{a} \nparallel \mathbf{b}$ if they are not.

Example 770. The vectors $\mathbf{a} = (2,0)$, $\mathbf{b} = (1,0)$, $\mathbf{c} = (-3,0)$ are parallel. And so as shorthand, we may write $\mathbf{a} \parallel \mathbf{b}$, $\mathbf{a} \parallel \mathbf{c}$, and $\mathbf{b} \parallel \mathbf{c}$.

The vector $\mathbf{d} = (1,1)$ is not parallel to $\mathbf{a}$, $\mathbf{b}$, or $\mathbf{c}$. Equivalently, $\mathbf{d} = (1,1)$ points in a different direction from $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. And so as shorthand, we may write $\mathbf{d} \nparallel \mathbf{a}$, $\mathbf{d} \nparallel \mathbf{b}$, and $\mathbf{d} \nparallel \mathbf{c}$.

Remark 98. Again, the zero vector $\mathbf{0} = (0,0)$ is the special case—it is neither parallel nor non-parallel to any other vector (not even itself).

Remark 99. Some other writers call two vectors that point in exact opposite directions **anti-parallel**. In contrast, we simply call them parallel. We will **not** use the term anti-parallel in this textbook.

53.13. Unit Vectors

Definition 139. A *unit vector* is any vector of length 1.

Example 771. The vectors $(1,0)$, $(0,1)$, and $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ are unit vectors:

$$|(1,0)| = \sqrt{1^2+0^2} = 1,$$
$$|(0,1)| = \sqrt{0^2+1^2} = 1,$$
$$\left|\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4}+\frac{2}{4}} = 1.$$

Example 772. The vectors $(1,1)$ and $(-1,-1)$ are **not** unit vectors:

$$|(1,1)| = \sqrt{1^2+1^2} = \sqrt{2} \neq 1,$$
$$|(-1,-1)| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2} \neq 1.$$

Given a vector $\mathbf{v}$, **its unit vector**, denoted $\hat{\mathbf{v}}$, is the vector that points in the same direction, but has length 1. Formally,

Definition 140. Given a non-zero vector $\mathbf{v}$, *its unit vector* (or the *unit vector in its direction*) is

$$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}.$$

It is easy to verify that thus defined, any vector's unit vector has length 1:

Fact 122. *Given any non-zero vector, its unit vector has length* 1.

Proof. By Fact 121, $|\hat{\mathbf{v}}| = \left|\frac{1}{|\mathbf{v}|}\mathbf{v}\right| = \left|\frac{1}{|\mathbf{v}|}\right| |\mathbf{v}| = \frac{1}{|\mathbf{v}|}|\mathbf{v}| = 1.$ □

Example 773. XXX

Fact 123. *Let* $\hat{\mathbf{v}}$ *be a unit vector. If* $c \in \mathbb{R}$, *then the vector* $c\hat{\mathbf{v}}$ *has length* $|c|$.

Proof. See Exercise 219. □

Example 774. XXX

Fact 124. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *be non-zero vectors. Then*

(a) $\hat{\mathbf{a}} = \hat{\mathbf{b}} \iff \mathbf{a}$ *and* $\mathbf{b}$ *point in the same direction;*

(b) $\hat{\mathbf{a}} = -\hat{\mathbf{b}} \iff \mathbf{a}$ *and* $\mathbf{b}$ *point in exact opposite directions;*

(c) $\hat{\mathbf{a}} = \pm\hat{\mathbf{b}} \iff \mathbf{a} \parallel \mathbf{b}$*;*

(d) $\hat{\mathbf{a}} \neq \pm\hat{\mathbf{b}} \iff \mathbf{a} \nparallel \mathbf{b}$.

Proof. See p. 1618 (Appendices). □

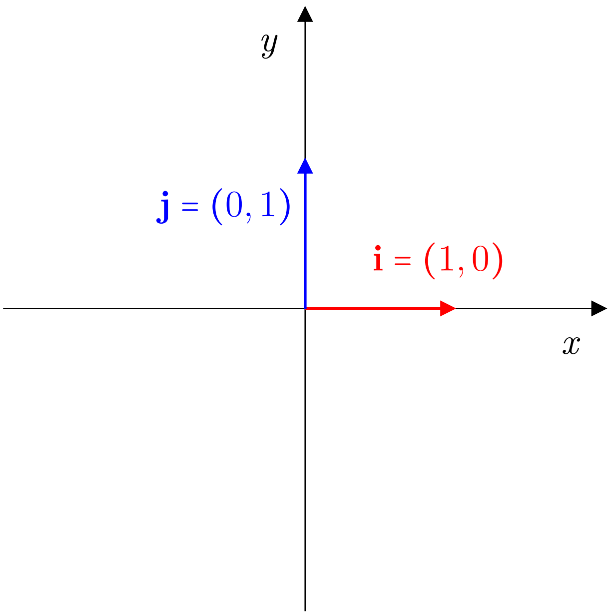
Exercise 219. Prove Fact 123. (Answer on p. 1842.)

Exercise 220. Let $A = (1, -3)$, $B = (2, 0)$, and $C = (5, -1)$ be points. Find the unit vectors of the six vectors below. (Answer on p. 1842.)

$$\overrightarrow{AB}, \qquad \overrightarrow{AC}, \qquad \overrightarrow{BC}, \qquad 2\overrightarrow{AB}, \qquad 3\overrightarrow{AC}, \qquad 4\overrightarrow{BC}.$$

Remark 100. Note that some writers also call $\hat{\mathbf{u}}$ the **normalised vector** of $\mathbf{u}$, but we shall not do so.

53.14. The Standard Basis Vectors

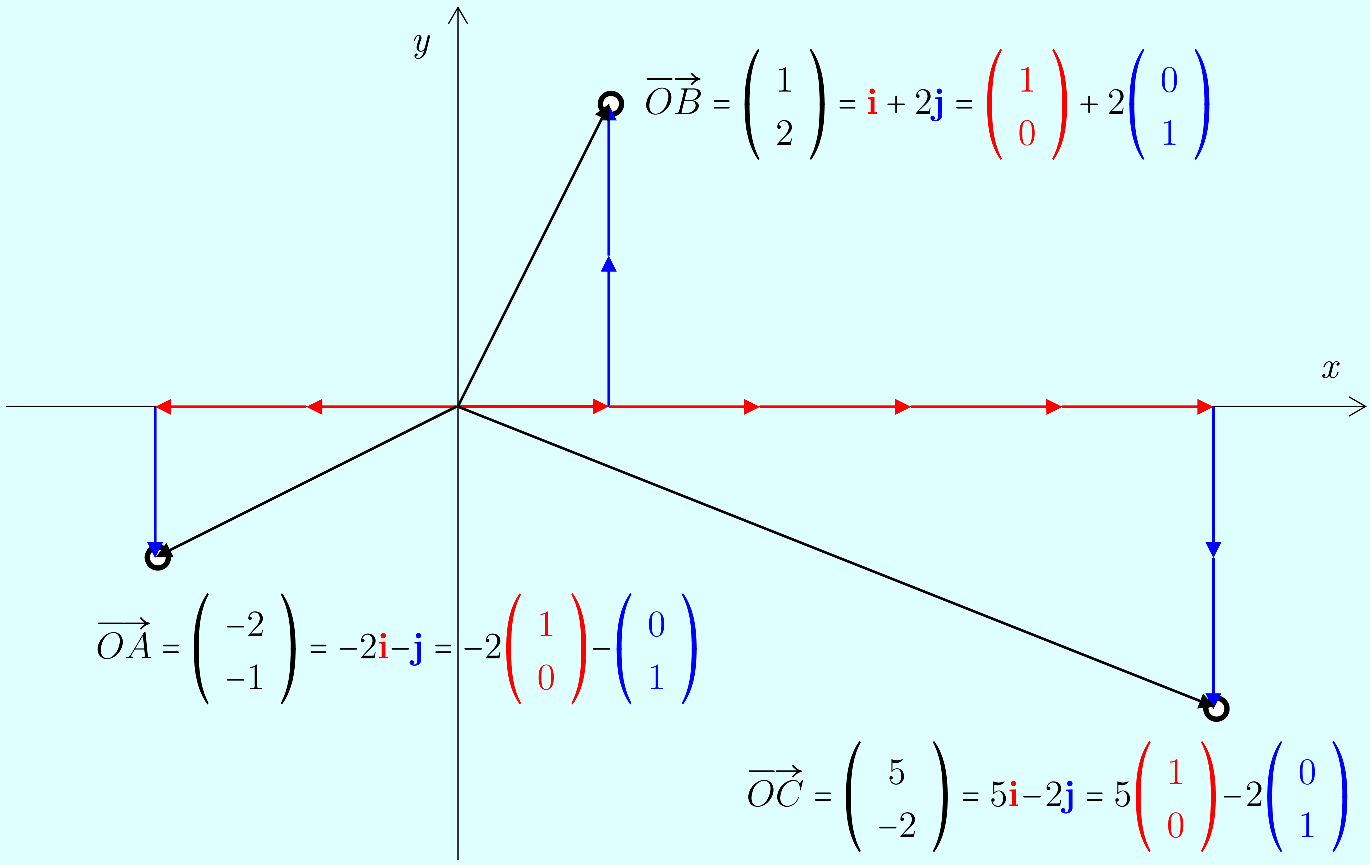


The **standard basis vectors** $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ are simply the unit vectors that point in the directions of the positive x- and y-axes. Formally,

Definition 141. The *standard basis vectors* (in 2D space) are $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

It turns out that *any* vector can be written as the **linear combination** (i.e. weighted sum) of $\mathbf{i}$'s and $\mathbf{j}$'s:

Example 775. Let $A = (-2, -1)$, $B = (1, 2)$, and $C = (5, -2)$ be points. Their position vectors can be written as linear combinations (i.e. weighted sums) of $\mathbf{i}$'s and $\mathbf{j}$'s:



$$\overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\overrightarrow{OA} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -2\mathbf{i} - \mathbf{j} = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\overrightarrow{OC} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} = 5\mathbf{i} - 2\mathbf{j} = 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

53.15. Any Vector Is A Linear Combination of Two Other Vectors

We just learnt that any vector can be written as the **linear combination** (i.e. weighted sum) of the standard basis vectors **i** and **j**. It turns out that more generally, we can do the same using *any* two vectors, so long as they are non-parallel:

Fact 125. *Let* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *be vectors. If* $\mathbf{a} \nparallel \mathbf{b}$, *then there exist* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}.$$

Proof. For a heuristic proof, see the next page. For a formal proof, see p. 1619 (Appendices). □

Example 776. Consider the vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (3, 4)$. Since $\mathbf{a} \nparallel \mathbf{b}$, by Fact 125, *any* vector can be expressed as the linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Consider for example the vector $\mathbf{u} = (2, 2)$. We will find $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b}$. To do so, first write

$$\mathbf{u} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \alpha\mathbf{a} + \beta\mathbf{b} = \alpha\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Write out the above vector equation as the following two cartesian equations:

$$2 \overset{1}{=} 1\alpha + 3\beta \quad \text{and} \quad 2 \overset{2}{=} 2\alpha + 4\beta.$$

Now solve this system of (two) equations: $\overset{2}{=}$ minus $2\times \overset{1}{=}$ yields $-2\beta = -2$ or $\beta = 1$, so that $\alpha = -1$. So,

$$\mathbf{u} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = -\mathbf{a} + \mathbf{b}.$$

Exercise 221. Let $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (3, 4)$. Express each of the vectors $\mathbf{v} = (3, 2)$ and $\mathbf{w} = (-1, 0)$ as the linear combination of $\mathbf{a}$ and $\mathbf{b}$. (Answer on p. 1842.)

Exercise 222. Explain why any vector can be written as a linear combination of the vectors $\mathbf{a} = (1, 3)$ and $\mathbf{b} = (7, 5)$. Then express each of the vectors $\mathbf{i} = (1, 0)$, $\mathbf{j} = (0, 1)$, and $\mathbf{d} = (1, 1)$ as the linear combination of $\mathbf{a}$ and $\mathbf{b}$. (Answer on p. 1843.)

Remark 101. For a somewhat recent application of Fact 125 in the A-Level exams, see N2013/I/6(i) (Exercise 617).

The condition that $\mathbf{a} \nparallel \mathbf{b}$ is important. If $\mathbf{a} \parallel \mathbf{b}$, then Fact 125 does not apply:

Example 777. Consider the vectors $\mathbf{a} = (1, 1)$ and $\mathbf{b} = (2, 2)$. Since $\mathbf{a} \parallel \mathbf{b}$, Fact 125 does not apply.

For example, we cannot express $\mathbf{v} = (1, 2)$ written as the linear combination of $\mathbf{a}$ and $\mathbf{b}$.

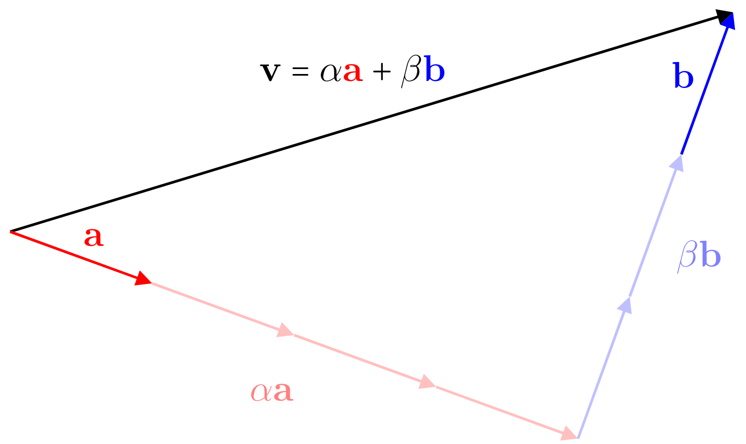
Example 778. The vectors $\mathbf{c} = (3, 1)$ and $\mathbf{d} = (-3, -1)$ point in exact opposite directions. Since $\mathbf{c} \parallel \mathbf{d}$, Fact 125 does not apply.

For example, we cannot express $\mathbf{v} = (1, 2)$ written as the linear combination of $\mathbf{c}$ and $\mathbf{d}$.

Here is a heuristic proof-by-picture of Fact 125.

Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{v}$ be non-zero vectors, with $\mathbf{a} \nparallel \mathbf{b}$.

Place $\mathbf{a}$'s tail at $\mathbf{v}$'s tail. Then also place $\mathbf{b}$'s head at $\mathbf{v}$'s head.

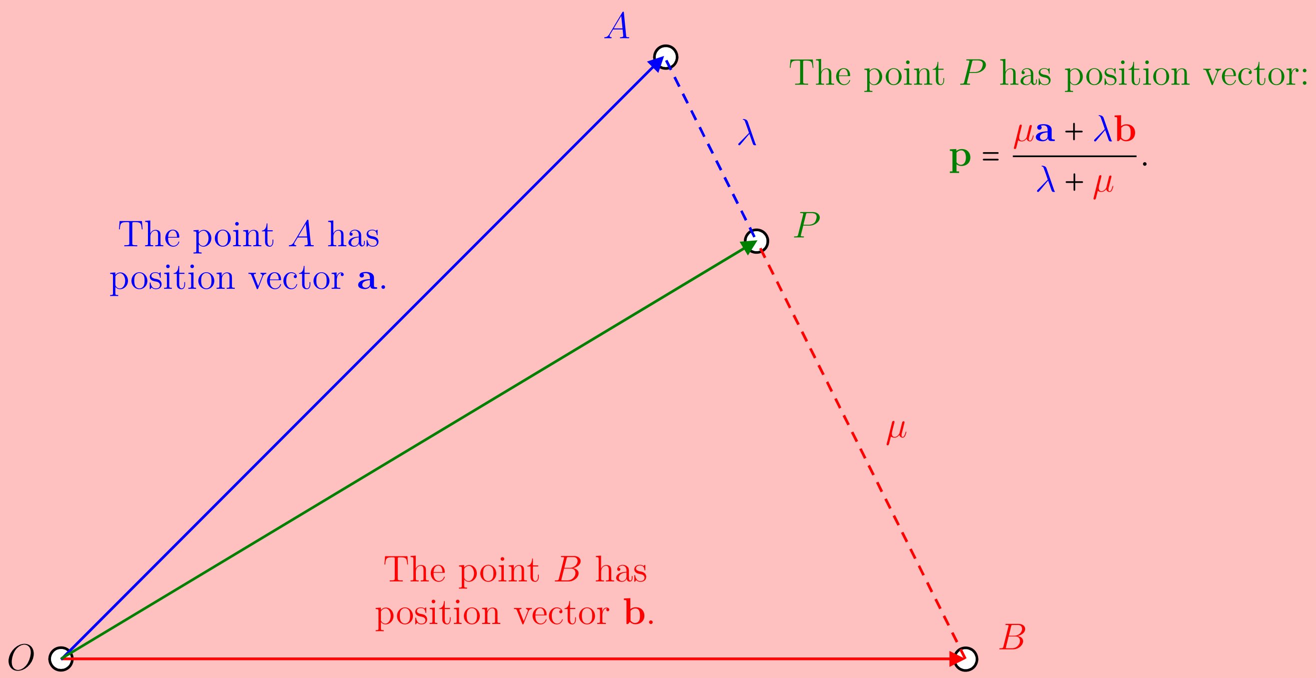


As the above figure suggests, "obviously", we can always find real numbers α and β so that the head of $\alpha\mathbf{a}$ and the tail of $\beta\mathbf{b}$ coincide. In other words, there are real numbers α and β such that $\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b}$.

53.16. The Ratio Theorem

Theorem 16. *Let A and B be points with position vectors $\mathbf{a}$ and $\mathbf{b}$. Suppose the point P is on the line segment AB and divides it in the ratio $\lambda : \mu$. Then P's position vector is*

$$\mathbf{p} = \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}.$$



Proof. By Fact 119, $\overrightarrow{AP} \stackrel{1}{=} \mathbf{p} - \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Now observe that $\overrightarrow{AP}$ points in the same direction as $\overrightarrow{AB}$, but has $\lambda / (\lambda + \mu)$ times the length. So,

$$\overrightarrow{AP} \stackrel{2}{=} \frac{\lambda}{\lambda + \mu}\overrightarrow{AB} = \frac{\lambda}{\lambda + \mu}(\mathbf{b} - \mathbf{a}).$$

Putting $\stackrel{1}{=}$ and $\stackrel{2}{=}$ together, we have

$$\mathbf{p} = \mathbf{a} + \overrightarrow{AP} = \mathbf{a} + \frac{\lambda}{\lambda + \mu}(\mathbf{b} - \mathbf{a}) = \frac{(\lambda + \mu)\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})}{\lambda + \mu} = \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}. \qquad \square$$

No need to *mug* the Ratio Theorem because List MF26 (p. 4) has this:

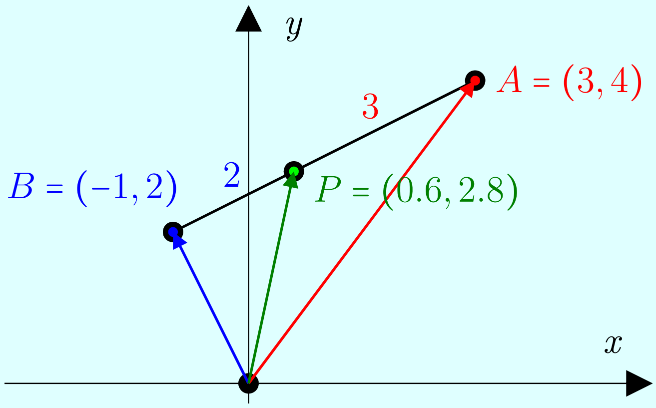
The point dividing AB in the ratio $\lambda : \mu$ has position vector $\frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$

Example 779. Let $A = (3, 4)$ and $B = (-1, 2)$ be points. Let P be the point that divides the line segment AB in the ratio $3 : 2$.

By the Ratio Theorem,

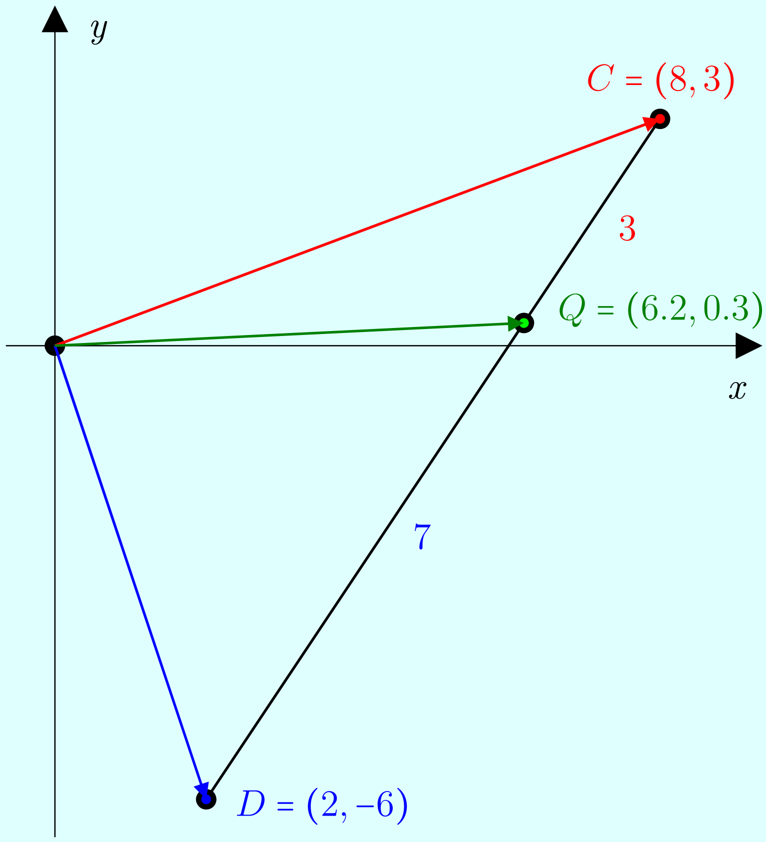
$$\mathbf{p} = \frac{2\mathbf{a} + 3\mathbf{b}}{3 + 2} = \frac{2}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 14 \end{pmatrix}.$$

So, $\quad P = (0.6, 2.8)$.



Example 780. Let $C = (8, 3)$ and $D = (2, -6)$ be points. Let Q be the point that divides the line segment CD in the ratio $3 : 7$. By the Ratio Theorem,

$$\mathbf{q} = \frac{7\mathbf{c} + 3\mathbf{d}}{3 + 7} = \frac{7}{10}\begin{pmatrix} 8 \\ 3 \end{pmatrix} + \frac{3}{10}\begin{pmatrix} 2 \\ -6 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 62 \\ 3 \end{pmatrix}, \qquad \text{and so,} \qquad Q = (6.2, 0.3).$$



Exercise 223. Let $A = (1, 2)$, $B = (3, 4)$, $C = (1, 4)$, $D = (2, 3)$, $E = (-1, 2)$, $F = (3, -4)$ be points. Find the points P, Q, and R which divide the line segments AB, CD, and EF in the ratios $5 : 6$, $5 : 1$, and $2 : 3$, respectively. (Answer on p. 1843.)

54. Lines

Recall[321] that any line l may be written as

$$l = \{(x, y) : ax + by + c = 0\},$$

where at least one of a or b is non-zero.

More simply, we say that the line l is *described* by this **cartesian equation**:[322]

$$ax + by + c = 0.$$

In this chapter, we'll learn a second method for describing lines, namely **vector equations**.

We'll start by introducing the concept of a line's **direction vector**:

54.1. Direction Vector

Informally, **a direction vector of a line** is *any* vector that's parallel to the line. Formally,

Definition 142. Given any two distinct points A and B on a line, we call the vector $\overrightarrow{AB}$ a *direction vector* of the line.

Example 781. Consider the line l described by the cartesian equation $y = 2x + 3$.

It contains the points

$$A = (0, 3) \quad \text{and} \quad B = (-1.5, 0).$$

Hence, a direction vector of l is

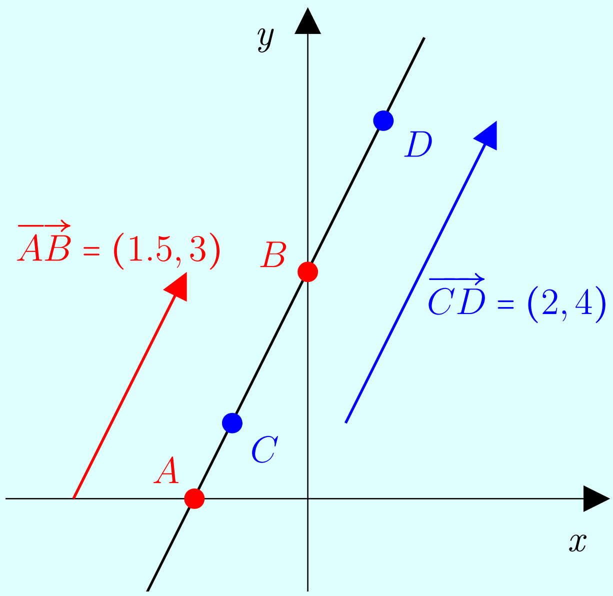
$$\overrightarrow{AB} = B - A = (0, 3) - (-1.5, 0) = (1.5, 3).$$

The line l also contains the points

$$C = (-1, 1) \quad \text{and} \quad D = (1, 5).$$

Hence, another direction vector of l is

$$\overrightarrow{CD} = D - C = (-1, 1) - (1, 5) = (2, 4).$$



[321] Ch. 8.

[322] Some writers also call this a **scalar equation**. We won't.

The above example shows that direction vectors are **not unique**. If $\mathbf{v}$ is a direction vector of a line, then so too is *any* vector that's parallel to $\mathbf{v}$.

But, no other vector is a direction vector of the line. That is, if $\mathbf{u} \nparallel \mathbf{v}$, then $\mathbf{u}$ is **not** a direction vector of the line. (And so, although the direction vector $\mathbf{v}$ isn't unique, we can say that it is **unique up to non-zero scalar multiplication**.)

Altogether, if a line has direction vector $\mathbf{v}$, then its direction vectors are exactly those that are parallel to $\mathbf{v}$. Formally,

Fact 126. *Suppose* $\mathbf{v}$ *is a line's direction vector. Then*

$$\mathbf{u} \text{ is also that line's direction vector} \quad \iff \quad \mathbf{u} \parallel \mathbf{v}.$$

Proof. See p. 1619 (Appendices). □

Example 782. The line[323] $y = 2x + 3$ has direction vectors

$$\overrightarrow{AB} = \begin{pmatrix} 1.5 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{CD} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

We can easily verify that $\overrightarrow{CD} \parallel \overrightarrow{AB}$:

$$\overrightarrow{CD} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \frac{4}{3}\begin{pmatrix} 1.5 \\ 3 \end{pmatrix} = \frac{4}{3}\overrightarrow{AB}.$$

The following are parallel to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and are thus also direction vectors of l:

$$-5\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -10 \\ -20 \end{pmatrix}, \quad \pi\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\pi \\ 4\pi \end{pmatrix}, \quad \text{and} \quad 17\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 34 \\ 68 \end{pmatrix}.$$

In contrast, these are not:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

[323] Strictly and pedantically speaking, "$y = 2x + 3$" is an equation and not a line—so, we should say instead, "The line described by $y = 2x + 3$," or, "The graph of $y = 2x + 3$," or, "The line $\{(x, y) : y = 2x + 3\}$. But this gets cumbersome. So we shan't be so strict and pedantic. We'll often simply say things like, "The line $y = 2x + 3$," as is done here.

Fact 127. *The line $ax + by + c = 0$ has direction vector $(b, -a)$.*

Proof. Let $D = (p, q)$ be any point on the line. Since D is on the line, it satisfies the line's cartesian equation—that is,

$$ap + bq + c = 0.$$

Now consider the point $E = (p + b, q - a)$. We now show that E also satisfies the line's cartesian equation and is thus is also on the line:

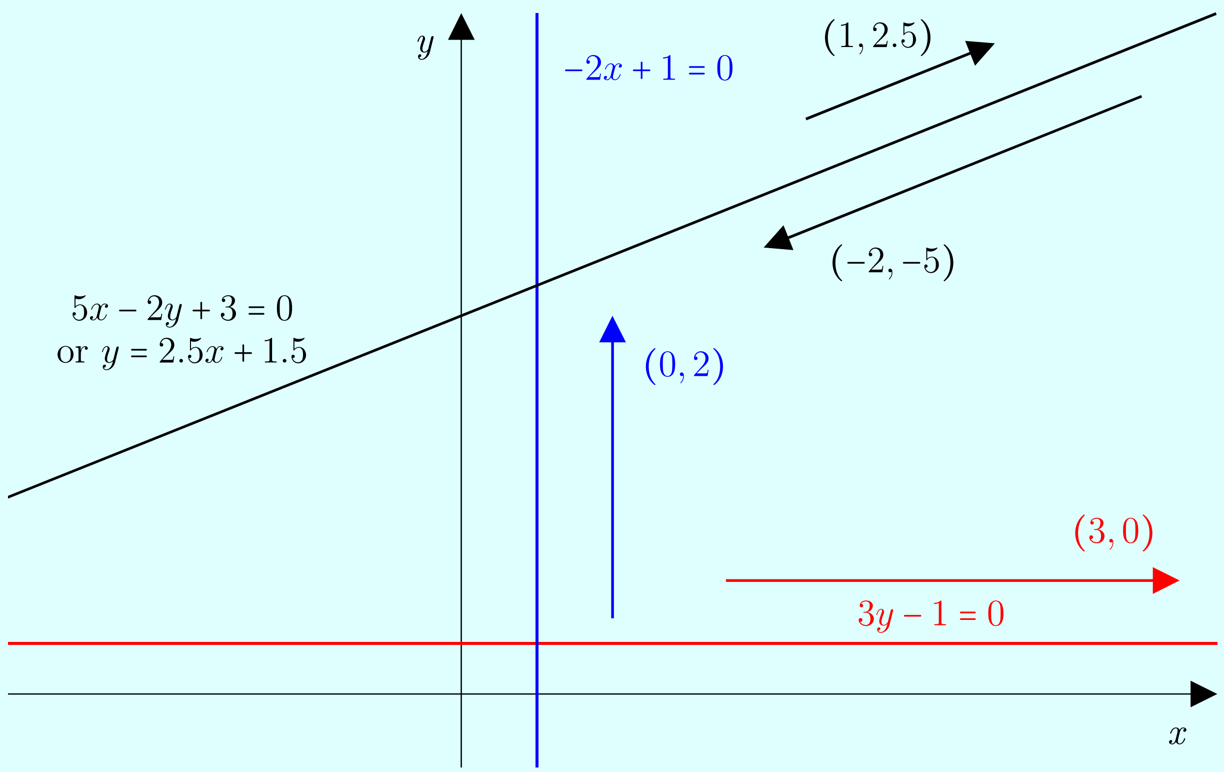
$$a(p + b) + b(q - a) + c = ap + ab + bq - ab + c = ap + bq + c = 0.$$

Since D and E are both points on the line, by Definition 142, the line has this direction vector:

$$\overrightarrow{DE} = E - D = (p + b, q - a) - (p, q) = (b, -a).$$ □

Example 783. By Fact 127, the line $5x - 2y + 3 = 0$ or $y = 2.5x + 1.5$ has direction vector $(-2, -5)$.

Since $(1, 2.5) \parallel (-2, -5)$, by Fact 126, it also has direction vector $(1, 2.5)$. (As we'll see on the next page, not coincidentally, this line's gradient is also 2.5.)



Next, the line $3y - 1 = 0$ or $y = 1/3$ has direction vector $(3, 0)$.

And the line $-2x + 1 = 0$ or $x = 0.5$ has direction vector $(0, 2)$.

The following result follows readily from Fact 127:

Corollary 21. **(a)** *A horizontal line has direction vector* $(1,0)$.
(b) *A vertical line has direction vector* $(0,1)$.
(c) *A line with gradient* m *has direction vector* $(1,m)$.

Proof. By Fact 127, the line $ax+by+c=0$ has direction vector $(b,-a)$.

(a) If the line is horizontal, then by Fact 25, $a=0$. And so, the line has direction vector $(b,0)$. Since $(1,0)\parallel(b,0)$, by Fact 126, the line also has direction vector $(1,0)$.

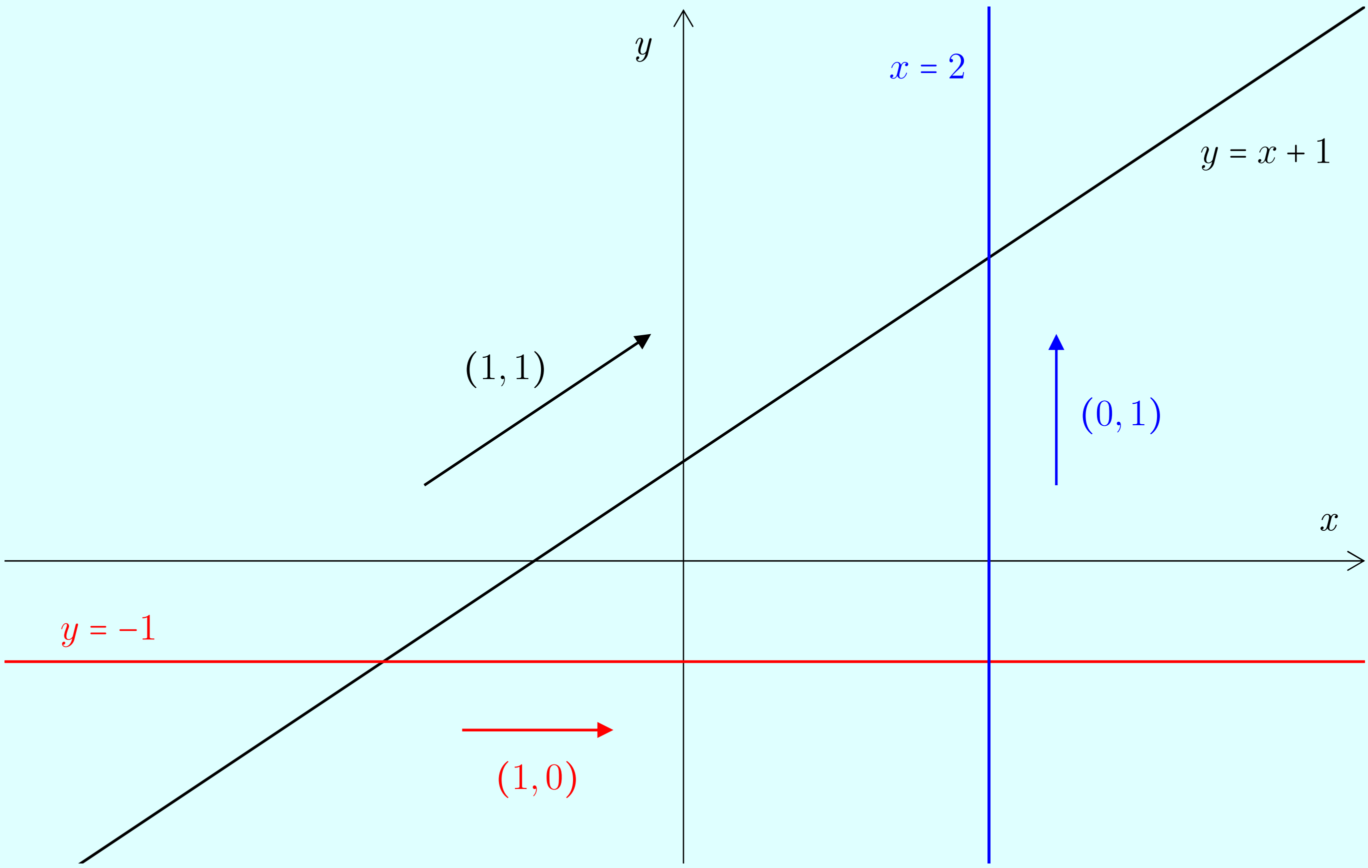
(b) Similarly, if the line is vertical, then by Fact 25, $b=0$. And so, the line has direction vector $(0,-a)$. Since $(0,1)\parallel(0,-a)$, by Fact 126, the line also has direction vector $(0,1)$.

(c) The line's gradient is $-b/a=m$. But $(b,-a)\parallel(-b/a,1)$. And so by Fact 126, the line also has direction vector $(1,m)$. □

Example 784. The horizontal line $y=-1$ has direction vector $(1,0)$.

The vertical line $x=2$ has direction vector $(0,1)$.

The oblique line $y=x+1$ has gradient 1 and thus direction vector $(1,1)$.



Exercise 224. For each line, write down a direction vector. (Answer on p. 1844.)

(a) $3x-y+2=0$. **(b)** $7y=-2x-3$. **(c)** $y=\pi$. **(d)** $x=-5$.

54.2. Cartesian to Vector Equations

Example 785. Let $l = \{(x, y) : 3x - y + 2 = 0\}$ be the set containing exactly those points (x, y) that satisfy the cartesian equation $3x - y + 2 = 0$ or $y = 3x + 2$. We say that l is the line that may be *described* by the cartesian equation $3x - y + 2 = 0$ or $y = 3x + 2$.

It turns out that we can also describe l using a **vector equation**.

To do so, first observe that l contains the point $P = (0, 2)$. Also, it has gradient 3 and thus direction vector $\mathbf{v} = (1, 3)$. Since l is a straight line, it must also contain these points:

$$P + 1\mathbf{v} = (0, 2) + 1(1, 3) = (1, 5) \quad \text{and} \quad P - 1\mathbf{v} = (0, 2) - 1(1, 3) = (-1, -1).$$

Indeed, l contains exactly those points R that can be expressed as $P + \lambda\mathbf{v} = (0, 2) + \lambda(1, 3)$ for some real number λ. That is,

$$l = \left\{R : R \overset{1}{=} (0, 2) + \lambda(1, 3) \quad (\lambda \in \mathbb{R})\right\}.$$

More simply, we may say that the **vector equation** $\overset{1}{=}$ *describes* l.

Equivalently, l contains exactly those points R whose position vector $\mathbf{r}$ may be expressed as $\mathbf{p} + \lambda\mathbf{v} = (0, 2) + \lambda(1, 3)$ for some real number λ. That is,

$$l = \left\{R : \mathbf{r} \overset{2}{=} (0, 2) + \lambda(1, 3) \quad (\lambda \in \mathbb{R})\right\}.$$

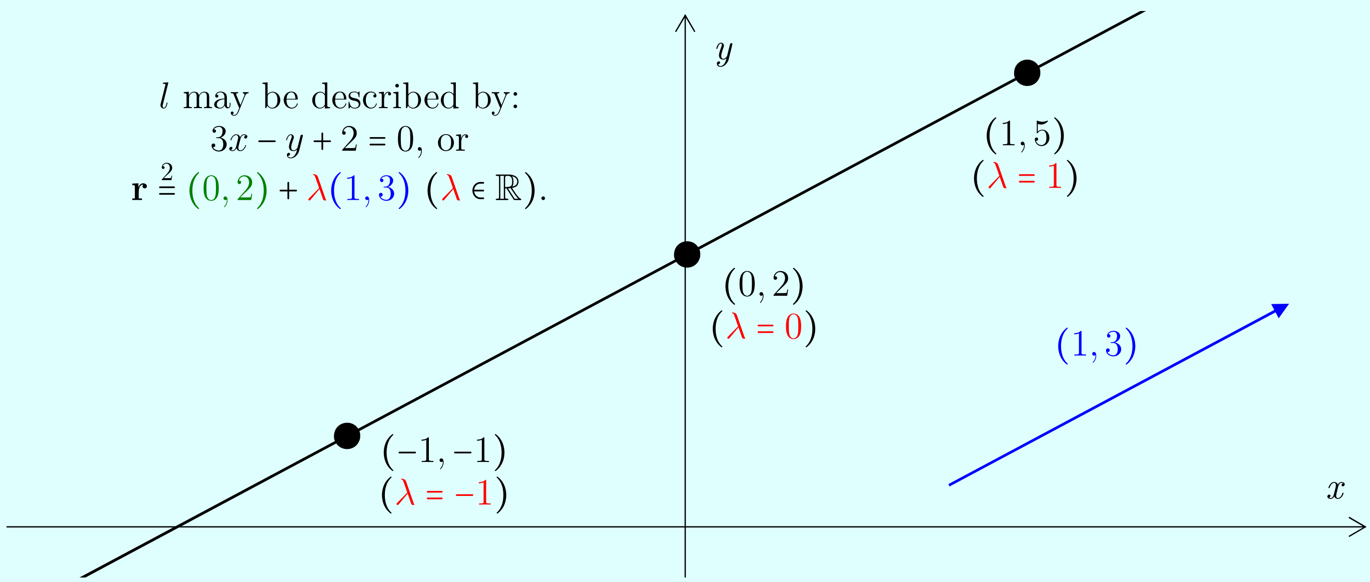
Again, we may more simply say that the **vector equation** $\overset{2}{=}$ *describes* l.

(By the way, $\overset{1}{=}$ and $\overset{2}{=}$ are subtly different—more on this in Ch. 54.3.)

As in Ch. 45 (Simple Parametric Equations), λ is a **parameter**. Here "$\lambda \in \mathbb{R}$" says that λ takes on every value in $\mathbb{R}$. And as λ varies, we get different points of the line.

For example, $\lambda = -1$, $\lambda = 0$, and $\lambda = 1$ produce these points:

$$(-1, -1) = (0, 2) - 1(1, 3), \quad (0, 2) = (0, 2) + 0(1, 3), \quad \text{and} \quad (1, 5) = (0, 2) + 1(1, 3).$$



(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Of course, l also contains infinitely many other points—each distinct value of $\lambda \in \mathbb{R}$ produces a distinct point on l.

We noted in Fact 126 that a line's direction vector is **unique up to non-zero scalar multiplication**. A direction vector of l is $\mathbf{v} = (1, 3)$, but so too is any non-zero scalar multiple of $\mathbf{v}$. And so, here are three more ways to write l:

$$\begin{aligned} l &= \left\{ (x, y) : \mathbf{r} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 100 \\ 300 \end{pmatrix} \quad (\lambda \in \mathbb{R}) \right\} \\ &= \left\{ (x, y) : \mathbf{r} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -100 \\ -300 \end{pmatrix} \quad (\lambda \in \mathbb{R}) \right\} \\ &= \left\{ (x, y) : \mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1.5 \\ 4.5 \end{pmatrix} \quad (\lambda \in \mathbb{R}) \right\}. \end{aligned}$$

More generally, for any $k \neq 0$ and $\mathbf{u} = k(1, 3)$, we may write

$$l = \{(x, y) : \mathbf{r} = (0, 2) + \lambda \mathbf{u} \quad (\lambda \in \mathbb{R})\}.$$

The foregoing discussion suggests the following general Definition of a line.

Definition 143. A *line* is any set of points that can be written as

$$\left\{ R : \overrightarrow{OR} = \mathbf{p} + \lambda \mathbf{v} \quad (\lambda \in \mathbb{R}) \right\},$$

where $\mathbf{p}$ and $\mathbf{v} \neq \mathbf{0}$ are vectors.

The above Definition says that a line contains exactly those points R whose position vector $\overrightarrow{OR} = \mathbf{r}$ may be expressed as

$$\overrightarrow{OR} = \mathbf{r} \overset{2}{=} \mathbf{p} + \lambda \mathbf{v} = (p_1, p_2) + \lambda (v_1, v_2) \qquad \text{(for some } \lambda \in \mathbb{R}\text{)}.$$

Equivalently, a line contains exactly those points R that may be expressed as

$$R \overset{1}{=} (p_1, p_2) + \lambda (v_1, v_2) \qquad \text{(for some } \lambda \in \mathbb{R}\text{)}$$

To repeat, here are what the vectors $\mathbf{p}$ and $\mathbf{v}$ and the number λ mean:

- $\mathbf{p} = (p_1, p_2)$ is the position vector of some point on the line;
- $\mathbf{v} = (v_1, v_2)$ is a direction vector of the line; and
- The **parameter** λ takes on every value in $\mathbb{R}$; each distinct value produces a distinct point on the line.

Definition 143 subsumes and our earlier definition of a line, Definition 41. The difference is that Definition 41 "works" only in 2D space. In contrast, Definition 143 is more general—it "works" in 2D space and, as we'll see, also in 3D space.[324]

[324]Indeed, it also "works" in any n-dimensional space.

And so, to write down a line's vector equation, we need simply find any point on the line and any direction vector of the line. More examples to illustrate how this works:

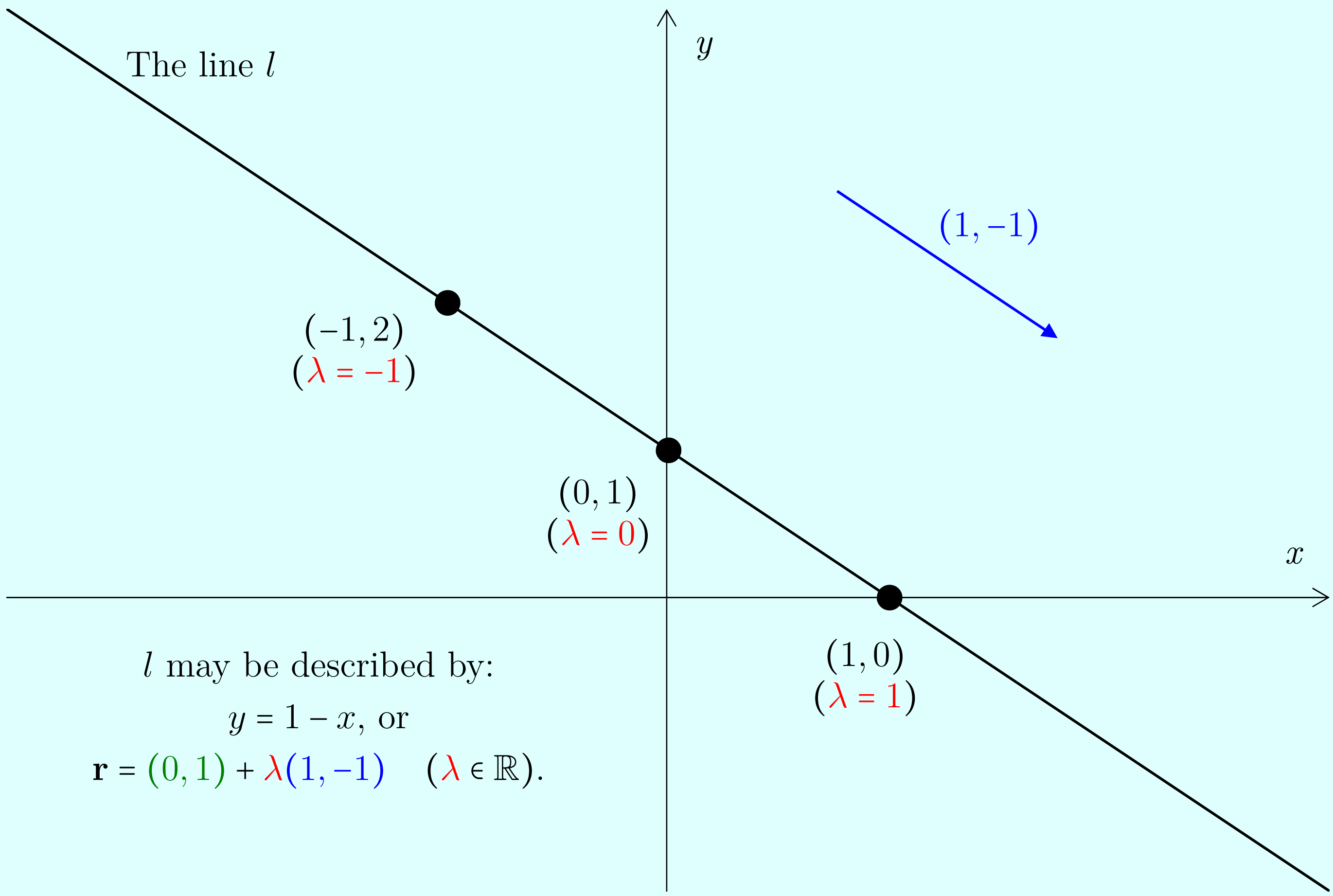
Example 786. Let l be the line $y = 1 - x$.

It contains the point $(0, 1)$. It has gradient -1 and so, by Corollary 21, the direction vector $(1, -1)$. Hence, we can also describe the line l by

$$R \stackrel{1}{=} (0,1) + \lambda(1,-1) \qquad \text{or} \qquad \mathbf{r} \stackrel{2}{=} (0,1) + \lambda(1,-1) \qquad (\lambda \in \mathbb{R}).$$

Both $\stackrel{1}{=}$ and $\stackrel{2}{=}$ say the same thing:

- $\stackrel{1}{=}$ says that l contains exactly those *points* R that may be written as $(0,1) + \lambda(1,-1)$, for some real number λ.
- $\stackrel{2}{=}$ says that l contains exactly those points R whose *position vector* $\mathbf{r} = \overrightarrow{OR}$ may be written as $(0,1) + \lambda(1,-1)$, for some real number λ.



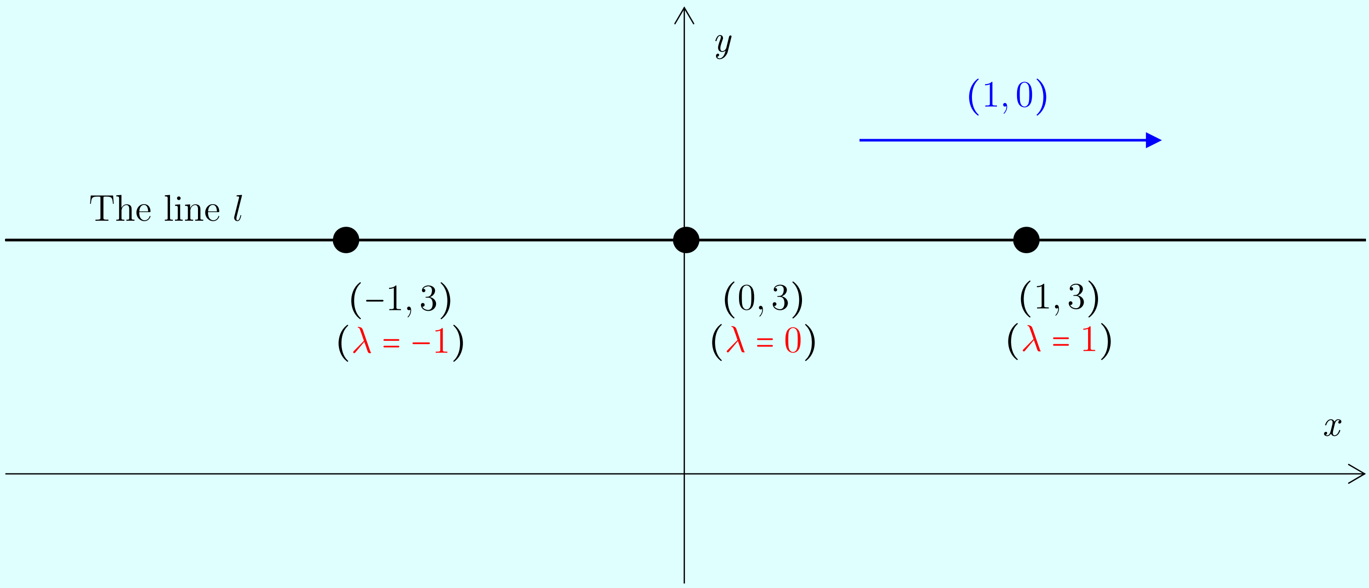
As the parameter λ takes on different values in $\mathbb{R}$, we get different points of l. So for example, $\lambda = -1$, $\lambda = 0$, and $\lambda = 1$ produce these points:

$$\begin{aligned} (-1,2) &= (0,1) - 1(1,-1), \\ (0,1) &= (0,1) + 0(1,-1), \\ (1,0) &= (0,1) + 1(1,-1). \end{aligned}$$

Example 787. Let l be the line $y = 3$.

It contains the point $(0,3)$. Since it's horizontal, by Corollary 21, it has direction vector $(1,0)$. So, we can also describe the line l by

$$R \overset{1}{=} (0,3) + \lambda(1,0) \qquad \text{or} \qquad \mathbf{r} \overset{2}{=} (0,3) + \lambda(1,0) \qquad (\lambda \in \mathbb{R}).$$



As λ varies, we get different points of l. So for example, $\lambda = -1$, $\lambda = 0$, and $\lambda = 1$ produce

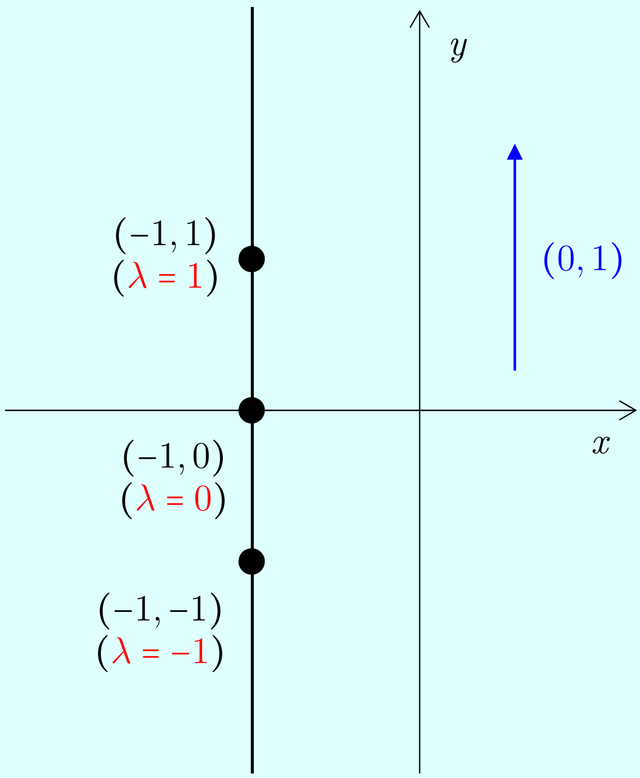
$$\begin{aligned}
(-1,3) &= (0,3) - 1(1,0), \\
(0,3) &= (0,3) + 0(1,0), \\
(1,3) &= (0,3) + 1(1,0).
\end{aligned}$$

Example 788. The line $x = 1$. contains the point $(-1,0)$. Since it's vertical, by Corollary 21, it has direction vector $(0,1)$. So, we can also describe the line by

$$R \overset{1}{=} (0,1) + \lambda(0,1) \quad \text{or}$$
$$\mathbf{r} \overset{2}{=} (0,1) + \lambda(0,1) \quad (\lambda \in \mathbb{R}).$$

As λ varies, we get different points of the line. For example, $\lambda = -1$, $\lambda = 0$, and $\lambda = 1$ produce

$$\begin{aligned}
(-1,2) &= (0,1) - 1(0,1), \\
(0,1) &= (0,1) + 0(0,1), \\
(1,0) &= (0,1) + 1(0,1).
\end{aligned}$$



Exercise 225. Each of the following (cartesian) equations describes a line. Rewrite each into a vector equation. Also, what points are produced when your parameter takes on the values -1, 0, and 1? (Answer on p. 1844.)

(a) $-5x + y + 1 = 0$. **(b)** $x - 2y - 1 = 0$. **(c)** $y - 4 = 0$. **(d)** $x - 4 = 0$.

Exercise 226. In Definition 143 (of a line), we impose the restriction that a line's direction vector $\mathbf{v}$ must be non-zero. By considering what the line becomes if $\mathbf{v}$ is the zero vector, explain why we impose this restriction. (Answer on p. 1844.)

Remark 102. Here we repeat our earlier warning. A line $\{(x, y) : ax + by + c = 0\}$ *is* a set of points. But for the sake of convenience, we often simply say that the line may be *described* by the **cartesian equation** $ax + by + c = 0$. And if we're especially lazy or sloppy, we might even say that the line *is* the equation $ax + by + c = 0$ (even though strictly speaking, this is wrong because a line is **not** an equation—it is a set).

Here likewise, a line $\{R = (x, y) : \mathbf{r} = \mathbf{p} + \lambda\mathbf{v}, \lambda \in \mathbb{R}\}$ *is* a set of points. But for the sake of convenience, we will often simply say that the line may be *described* by the **vector equation** $\mathbf{r} = \mathbf{p} + \lambda\mathbf{v}$ ($\lambda \in \mathbb{R}$). And if we're especially lazy or sloppy, we might even say that the line *is* the equation $ax + by + c = 0$ (even though again, strictly speaking, this is wrong because a line is **not** an equation—it is a set).

54.3. Three Pedantic Points to Test/Reinforce Your Understanding

As we've seen above, a line l may be described using either of these vector equations:

$$\overbrace{R}^{\text{Point}} \stackrel{1}{=} \overbrace{P}^{\text{Point}} + \overbrace{\lambda \mathbf{v}}^{\text{Vector}} \qquad (\lambda \in \mathbb{R}).$$

Or,
$$\overbrace{\mathbf{r}}^{\text{Vector}} \stackrel{2}{=} \overbrace{\mathbf{p}}^{\text{Vector}} + \overbrace{\lambda \mathbf{v}}^{\text{Vector}} \qquad (\lambda \in \mathbb{R}).$$

Pedantic Point 1. $\stackrel{1}{=}$ is consistent with what we learnt earlier (in Ch. 53.9):

$$\text{Point} = \text{Point} + \text{Vector}.$$

Likewise, $\stackrel{2}{=}$ is consistent with what we learnt earlier:

$$\text{Vector} = \text{Vector} + \text{Vector}.$$

So, both vector equations $\stackrel{1}{=}$ and $\stackrel{2}{=}$ are perfectly correct ways to describe the exact same line.

The difference is that $\stackrel{1}{=}$ does so "more directly" than $\stackrel{2}{=}$. Because, to repeat,

- $\stackrel{1}{=}$ says that l contains exactly those *points* R that may be written as $P + \lambda \mathbf{v}$, for some real number λ.
- $\stackrel{2}{=}$ says that l contains exactly those points R whose *position vector* $\mathbf{r} = \overrightarrow{OR}$ may be written as $\mathbf{p} + \lambda \mathbf{v}$, for some real number λ.

Pedantic Point 2. What would be **wrong** and unacceptable is this:

$$\overbrace{R}^{\text{Point}} \stackrel{3}{=} \overbrace{\mathbf{p}}^{\text{Vector}} + \lambda \overbrace{\mathbf{v}}^{\text{Vector}} \qquad (\lambda \in \mathbb{R}), \qquad ✗$$

As we learnt earlier (Ch. 53.9), Vector + Vector = Vector. But the LHS of $\stackrel{3}{=}$ is a **Point** while its RHS is a **Vector**. So, $\stackrel{3}{=}$ is false.

Similarly, the following is also **wrong** and unacceptable:

$$\overbrace{\mathbf{r}}^{\text{Vector}} \stackrel{4}{=} \overbrace{P}^{\text{Point}} + \lambda \overbrace{\mathbf{v}}^{\text{Vector}} \qquad (\lambda \in \mathbb{R}). \qquad ✗$$

As we also learnt earlier, Point + Vector = Point. But the LHS of $\stackrel{4}{=}$ is a **Vector** while its RHS is a **Point**. So, $\stackrel{4}{=}$ is false.

Pedantic Point 3. A line is a set of points and **not** a set of vectors. So, take care to note that the line l contains the **points** $R = (x, y)$ and $P = (p_1, p_2)$—it does **not** contain the vectors $\mathbf{r} = (x, y)$ and $\mathbf{p} = (p_1, p_2)$.

54.4. Vector to Cartesian Equations

Suppose a line is described by this **vector equation**:

$$\mathbf{r} = \mathbf{p} + \lambda \mathbf{v} \quad (\lambda \in \mathbb{R}).$$

Or equivalently,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (\lambda \in \mathbb{R}).$$

Then given any point (x, y) on this line, there must be some real number λ such that

$$x = p_1 + \lambda v_1 \qquad \text{and} \qquad y = p_2 + \lambda v_2.$$

We say that the line may be described by the above pair of **cartesian equations**.

Hm ... but aren't we supposed to be able to describe a line with just one cartesian equation? Well, if we'd like, we can do some easy algebra to eliminate the parameter λ:

Example 789. The line l is described by

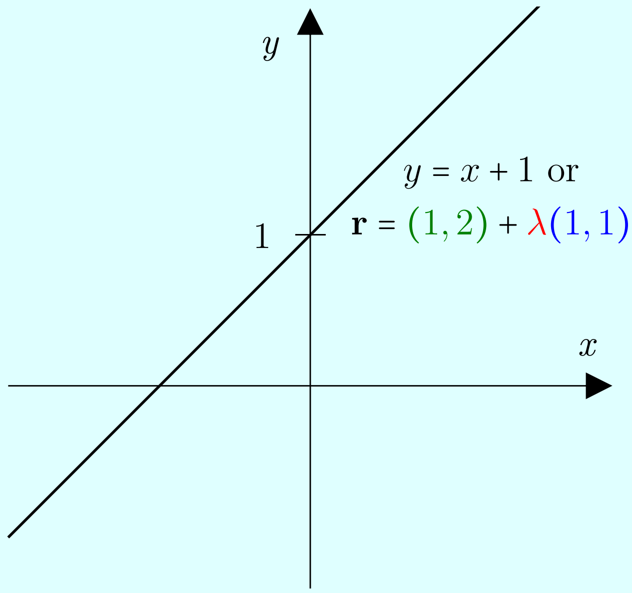
$$\mathbf{r} = (1, 2) + \lambda(1, 1) \quad (\lambda \in \mathbb{R}).$$

We can also describe l by this pair of cartesian equations:

$$x \overset{1}{=} 1 + \lambda \cdot 1 \quad \text{and} \quad y \overset{2}{=} 2 + \lambda \cdot 1.$$

We can eliminate the parameter λ through simple algebra. Take $\overset{2}{=}$ minus $\overset{1}{=}$:

$$y - x = 1 \quad \text{or} \quad y = x + 1.$$



Example 790. The line l is described by

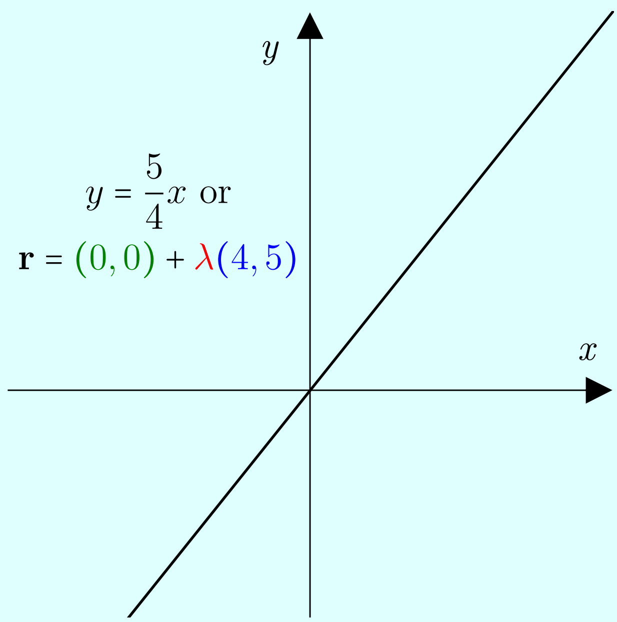
$$\mathbf{r} = (0, 0) + \lambda(4, 5) \quad (\lambda \in \mathbb{R}).$$

We can also describe l by this pair of cartesian equations:

$$x \overset{1}{=} 0 + \lambda \cdot 4 \quad \text{and} \quad y \overset{2}{=} 0 + \lambda \cdot 5.$$

Take $\overset{2}{=}$ minus $\frac{5}{4} \times \overset{1}{=}$:

$$y - \frac{5}{4}x = 0 \quad \text{or} \quad y = \frac{5}{4}x.$$



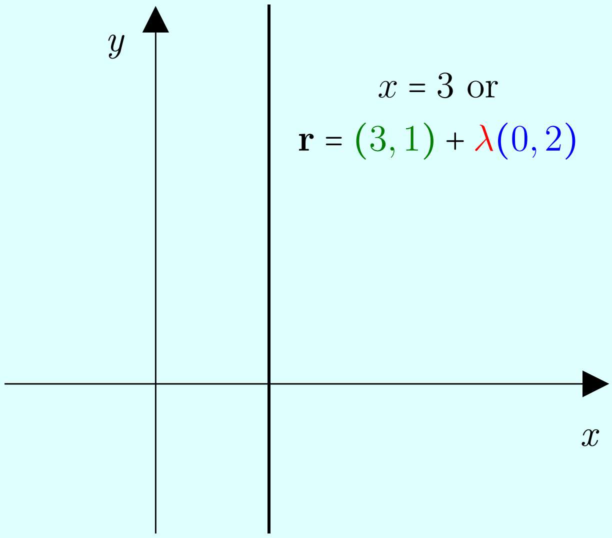
Example 791. The line l is described by

$$\mathbf{r} = (3,1) + \lambda(0,2) \quad (\lambda \in \mathbb{R}).$$

We can also describe l by this pair of cartesian equations:

$$x \overset{1}{=} 3 + \lambda \cdot 0 = 3 \quad \text{and} \quad y \overset{2}{=} 1 + \lambda \cdot 2.$$

Observe that in this example, we *cannot* use algebra to eliminate λ. It turns out that this is actually a **vertical line**.



As λ varies, the value of x is fixed at $x \overset{1}{=} 3$, while y varies along with λ. And so, instead of doing any algebra, we'll simply **discard** $\overset{2}{=}$. The above pair of cartesian equations then reduces to the single equation:

$$x \overset{1}{=} 3.$$

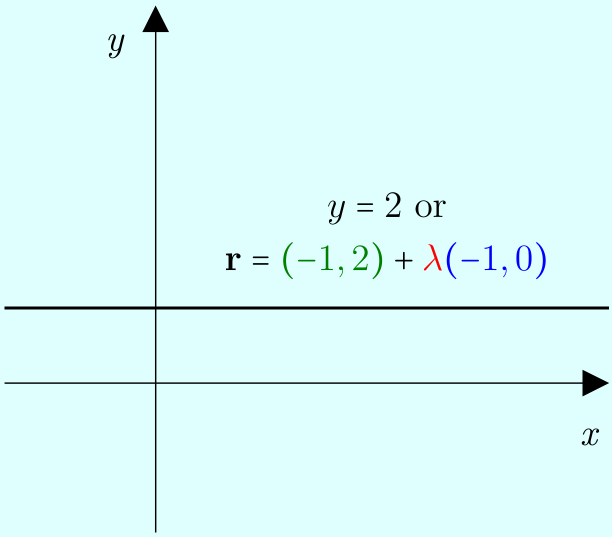
Example 792. The line l is described by

$$\mathbf{r} = (-1,2) + \lambda(-1,0) \quad (\lambda \in \mathbb{R}).$$

We can also describe l by this pair of cartesian equations:

$$x \overset{1}{=} -1 + \lambda \cdot (-1) \quad \text{and} \quad y \overset{2}{=} 2 + \lambda \cdot 0 = 2.$$

Again, in this example, we *cannot* use algebra to eliminate λ. It turns out that this is actually a **horizontal line**.



As λ varies, the value of y is fixed at $y \overset{2}{=} 2$, while x varies along with λ. And so, instead of doing any algebra, we'll simply **discard** $\overset{1}{=}$. The above pair of cartesian equations then reduces to the single equation:

$$y \overset{2}{=} 2.$$

Exercise 227. Each of the following vector equations describes a line. Rewrite each into cartesian equation form. (Answer on p. 1844.)

(a) $\mathbf{r} = (-1,3) + \lambda(1,-2) \quad (\lambda \in \mathbb{R})$. **(b)** $\mathbf{r} = (5,6) + \lambda(7,8) \quad (\lambda \in \mathbb{R})$.

(c) $\mathbf{r} = (0,-3) + \lambda(3,0) \quad (\lambda \in \mathbb{R})$. **(d)** $\mathbf{r} = (1,1) + \lambda(0,2) \quad (\lambda \in \mathbb{R})$.

In general,

Fact 128. *Let l be the line described by* $\mathbf{r} = (p_1, p_2) + \lambda(v_1, v_2) \quad (\lambda \in \mathbb{R})$.

(a) *If* $v_1, v_2 \neq 0$*, then l can be described by*

$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} \qquad \textit{or} \qquad y = \frac{v_2}{v_1} x + p_2 - \frac{v_2}{v_1} p_1.$$

(b) *If* $v_1 = 0$*, then l is vertical and can be described by* $x = p_1$.

(c) *If* $v_2 = 0$*, then l is horizontal and can be described by* $y = p_2$.

Proof. First, write $\qquad x \stackrel{1}{=} p_1 + \lambda v_1 \qquad$ and $\qquad y \stackrel{2}{=} p_2 + \lambda v_2$.

Take $v_1 \times \stackrel{2}{=}$ minus $v_2 \times \stackrel{1}{=}$:

$$v_1 y - v_2 x = v_1 p_2 + \lambda v_1 v_2 - v_2 p_1 - \lambda v_1 v_2 = v_1 p_2 - v_2 p_1.$$

Or, $$v_2 (x - p_1) \stackrel{3}{=} v_1 (y - p_2).$$

(a) If $v_1, v_2 \neq 0$, then $\stackrel{3}{=}$ divided by $v_1 v_2$ yields $\dfrac{x - p_1}{v_1} = \dfrac{y - p_2}{v_2}$.

(b) If $v_1 = 0$, then $\stackrel{3}{=}$ becomes $x = p_1$.

(c) If $v_2 = 0$, then $\stackrel{3}{=}$ becomes $y = p_2$. □

Armed with Fact 128, we now revisit the last four examples.

Example 793. The line $\mathbf{r} = (1, 2) + \lambda(1, 1)$ $(\lambda \in \mathbb{R})$ may also be described by

$$\frac{x - 1}{1} = \frac{y - 2}{1} \qquad \text{or} \qquad y = x + 1.$$

Example 794. The line $\mathbf{r} = (0, 0) + \lambda(4, 5)$ $(\lambda \in \mathbb{R})$ may also be described by

$$\frac{x - 0}{4} = \frac{y - 0}{5} \qquad \text{or} \qquad y = \frac{5}{4} x.$$

Example 795. The line $\mathbf{r} = (3, 1) + \lambda(0, 2)$ $(\lambda \in \mathbb{R})$ may also be described by $x = 3$.

Example 796. The line $\mathbf{r} = (-1, 2) + \lambda(-1, 0)$ $(\lambda \in \mathbb{R})$ may also be described by $y = 2$.

Exercise 228. Use Fact 128 to redo Exercise 227. (Answer on p. 1844.)

55. The Scalar Product

Definition 144. Given vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, their *scalar product*, denoted $\mathbf{u} \cdot \mathbf{v}$, is this number:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Example 797. Suppose $\mathbf{u} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$. Then

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5 \cdot 2 + (-3) \cdot 1 = 10 - 3 = 7,$$

$$\mathbf{u} \cdot \mathbf{w} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 0 \end{pmatrix} = 5 \cdot (-4) + (-3) \cdot 0 = -20 + 0 = -20,$$

$$\mathbf{u} \cdot \mathbf{x} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 7 \end{pmatrix} = 5 \cdot 8 + (-3) \cdot 7 = 40 - 21 = 19,$$

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 0 \end{pmatrix} = 2 \cdot (-4) + 1 \cdot 0 = -8 + 0 = -8.$$

The **scalar product** is itself simply a scalar (i.e. a real number). Hence the name.

Remark 103. The **scalar product** is also called the **dot product** or the **inner product**. But your A-Level exams and syllabus don't seem to use these terms. And so, neither shall we. We'll use only the term **scalar product**.

Right now, the scalar product may seem like a totally random and useless thing, but as we'll soon learn, it is plenty useful. Let us first learn about a few of its properties.

Recall from primary school that multiplication is **commutative**:

Example 798. $3 \times 5 = 15$ and $5 \times 3 = 15$.

Moreover, multiplication is **distributive** (over addition):

Example 799. $3 \times (5 + 11) = 3 \times 5 + 3 \times 11$ and $18 \times (7 - 31) = 18 \times 7 + 18 \times (-31)$.

It turns out that the scalar product is likewise **commutative** and **distributive**:

Fact 129. *Suppose* **a**, **b**, *and* **c** *are vectors. Then*

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. **(Commutative)**

(b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$. **(Distributive over Addition)**

The fact that the scalar product is both commutative and distributive is a simple consequence of the fact that multiplication is itself commutative and distributive. (The latter is, in turn, a fact we'll simply take for granted in this textbook.)

Proof. Let[325] $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, and $\mathbf{c} = (c_1, c_2)$. Then

(a)
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 = b_1 a_1 + b_2 a_2 = \mathbf{b} \cdot \mathbf{a}.$$

(b)
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = a_1 (b_1 + c_1) + a_2 (b_2 + c_2) = a_1 b_1 + a_2 b_2 + a_1 c_1 + a_2 c_2 = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$ □

Example 800. Continue to let $\mathbf{u} = (5, -3)$, $\mathbf{v} = (2, 1)$, $\mathbf{w} = (-4, 0)$, and $\mathbf{x} = (8, 7)$.

The scalar product is commutative:

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} = 7, \qquad \mathbf{w} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{w} = -20, \qquad \mathbf{x} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{x} = 19.$$

It is also distributive over addition:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 - 4 \\ 1 + 0 \end{pmatrix} = -10 - 3 = -13 = 7 - 20 = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w},$$

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \begin{pmatrix} 5 + 2 \\ -3 + 1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 0 \end{pmatrix} = -28 + 0 = -28 = -20 - 8 = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}.$$

Here's another "obvious" property of the scalar product:

Fact 130. *Suppose* **a** *and* **b** *be vectors and* $c \in \mathbb{R}$ *be a scalar. Then*

$$(c\mathbf{a}) \cdot \mathbf{b} = c (\mathbf{a} \cdot \mathbf{b}).$$

Proof. Let[326] $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. So, $c\mathbf{a} = (ca_1, ca_2)$. Hence,

$$(c\mathbf{a}) \cdot \mathbf{b} = (ca_1) b_1 + (ca_2) b_2 = c (a_1 b_1 + a_2 b_2) = c (\mathbf{a} \cdot \mathbf{b}).$$ □

Exercise 229. Let $\mathbf{v} = (2, 1)$, $\mathbf{w} = (-4, 0)$, and $\mathbf{x} = (8, 7)$. Above we already computed $\mathbf{v} \cdot \mathbf{w} = -8$. Now also compute these: (Answer on p. 1845.)

$$\mathbf{v} \cdot \mathbf{x}, \quad \mathbf{w} \cdot \mathbf{x}, \quad \mathbf{w} \cdot \mathbf{v}, \quad \mathbf{x} \cdot \mathbf{v}, \quad \mathbf{x} \cdot \mathbf{w}, \quad \mathbf{w} \cdot (\mathbf{x} + \mathbf{v}), \quad (2\mathbf{v}) \cdot \mathbf{x}, \text{ and } \mathbf{w} \cdot (2\mathbf{x}).$$

[325] The proof covers only the two-dimensional case. For a more general proof, see p. 1620 (Appendices).
[326] This proof covers only the two-dimensional case. For a more general proof, see p. 1620 (Appendices).

55.1. A Vector's Scalar Product with Itself

A vector's **length** is the square root of the scalar product with itself:

Fact 131. *Suppose* $\mathbf{v}$ *be a vector. Then* $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$ *and* $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$.

Proof. By Definition 126, $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$. By Definition 144, $\mathbf{v}\cdot\mathbf{v} = v_1v_1 + v_2v_2 = v_1^2 + v_2^2$. Hence, $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$ and $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$. □

Exercise 230. The lengths of $\mathbf{u} = (5, -3)$, $\mathbf{v} = (2, 1)$, $\mathbf{w} = (-4, 0)$, and $\mathbf{x} = (8, 7)$ are

$$|\mathbf{u}| = \sqrt{5^2 + (-3)^2} = \sqrt{34}$$

$$|\mathbf{v}| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$|\mathbf{w}| = \sqrt{(-4)^2 + 0^2} = 4$$

$$|\mathbf{x}| = \sqrt{8^2 + 7^2} = \sqrt{113}$$

And the square roots of the scalar product of each vector with itself are

$$\sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{(5, -3)\cdot(5, -3)} = \sqrt{25 + 9} = \sqrt{34}$$

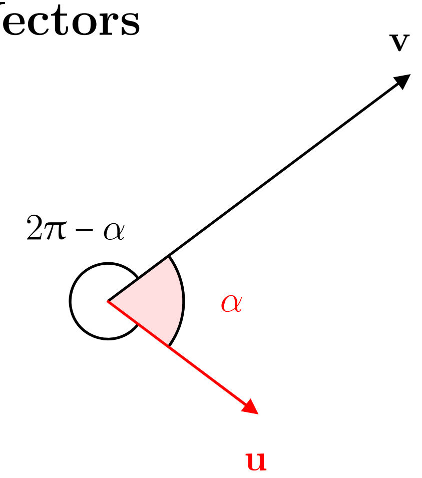
$$\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{(2, 1)\cdot(2, 1)} = \sqrt{4 + 1} = \sqrt{5}$$

$$\sqrt{\mathbf{w}\cdot\mathbf{w}} = \sqrt{(-4, 0)\cdot(-4, 0)} = \sqrt{16 + 0} = 4$$

$$\sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{(8, 7)\cdot(8, 7)} = \sqrt{64 + 49} = \sqrt{113}$$

Exercise 231. Let $\mathbf{a} = (-2, 3)$, $\mathbf{b} = (7, 1)$, and $\mathbf{c} = (5, -4)$. Verify that for each vector, its length equals the square root of its scalar product with itself. (Answer on p. 1845.)

56. The Angle Between Two Vectors



Place the tails of **u** and **v** at the same point.

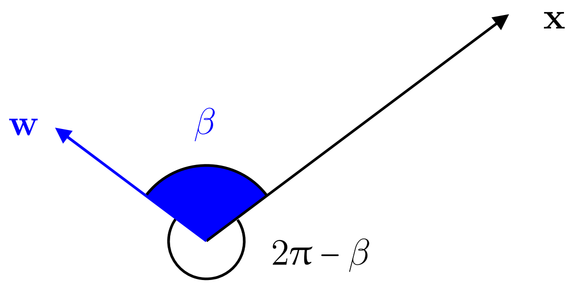
Informally, we might define the **angle** between **u** and **v** as the "amount" we must rotate **u** so that it points in the same direction as **v**.

The thing is, we can either rotate **u** **anticlockwise** by α or **clockwise** by $2\pi - \alpha$. So, we have an ambiguity here—which is *the* angle between **u** and **v**? Is it α or $2\pi - \alpha$?

To resolve this ambiguity, we will simply *define* the angle between two vectors as the **smaller of these two angles**. In other words, we'll define the angle between two vectors so that it's always between 0 and π (inclusive).

And so, in the above figure, the angle between **u** and **v** is α (and **not** $2\pi - \alpha$).

In contrast, in the figure below, the angle between **w** and **x** is β (and **not** $2\pi - \beta$).



Previously in Ch. 30, we gave an informal definition of angle (Definitions 88). As promised then, we now give our formal Definition of the **angle between two vectors**.

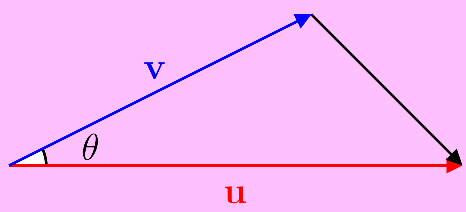
Warning: This definition comes seemingly outta nowhere. But don't worry, Exercise 232 (next page) will help you understand where this Definition comes from.

Definition 145. The *angle between two non-zero vectors* **u** *and* **v** is this number:

$$\cos^{-1}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Recall (Definition 99) that Range $\cos^{-1} = [0, \pi]$. So, by the above Definition, the angle between two vectors is indeed always between 0 and π.

Exercise 232. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors and θ be the angle between them.



This Exercise will help you understand why we define $\theta = \cos^{-1} \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$.

(a) Write down the vector that corresponds to the third side of the above triangle.

(b) Write down the lengths of the triangle's three sides in terms of $\mathbf{u}$ and $\mathbf{v}$.

(c) The Law of Cosines (Proposition 7) states that if a triangle has sides of lengths a, b, and c and has angle C opposite the side of length c, then

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Use the Law of Cosines to write down an equation involving θ, $\mathbf{u}$, and $\mathbf{v}$.

(d) Use distributivity (Fact 129) to prove that $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v}$.

(e) Now take the equation you wrote down in **(c)**, do the algebra (hint: use Fact 131), and hence show that

$$\theta = \cos^{-1} \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

(Answer on p. 1846.)

A simple rearrangement of Definition 145 produces this result:

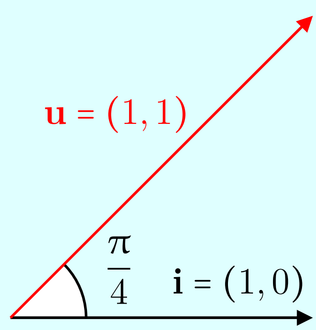
Fact 132. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are two non-zero vectors and* θ *is the angle between them, then*

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta.$$

Example 801. Let θ be the angle between the vectors $\mathbf{i} = (1,0)$ and $\mathbf{u} = (1,1)$. Since $\mathbf{i}$ points east, while $\mathbf{u}$ points north-east, primary-school trigonometry says $\theta = \pi/4$.

Let's verify that this is consistent with Definition 145:

$$\begin{aligned}
\cos^{-1}\frac{\mathbf{i}\cdot\mathbf{u}}{|\mathbf{i}|\,|\mathbf{u}|} &= \cos^{-1}\frac{(1,0)\cdot(1,1)}{|(1,0)|\,|(1,1)|} \\
&= \cos^{-1}\frac{1\cdot 1 + 0\cdot 1}{\sqrt{1^2+0^2}\sqrt{1^2+1^2}} \\
&= \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}. \qquad ✓
\end{aligned}$$



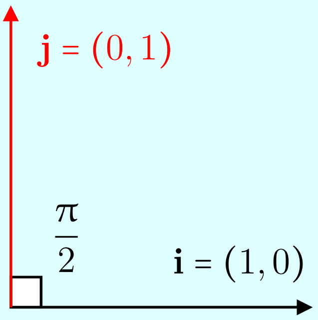
Example 802. Let θ be the angle between the vectors $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$. Since $\mathbf{i}$ points east, while $\mathbf{j}$ points north, primary-school trigonometry says $\theta = \pi/2$.

Let's verify that this is consistent with Definition 145:

$$\cos^{-1}\frac{\mathbf{i}\cdot\mathbf{j}}{|\mathbf{i}|\,|\mathbf{j}|} = \cos^{-1}\frac{(1,0)\cdot(0,1)}{|(1,0)|\,|(0,1)|}$$

$$= \cos^{-1}\frac{1\cdot 0 + 0\cdot 1}{\sqrt{1^2+0^2}\sqrt{0^2+1^2}}$$

$$= \cos^{-1} 0 = \frac{\pi}{2}. \qquad ✓$$

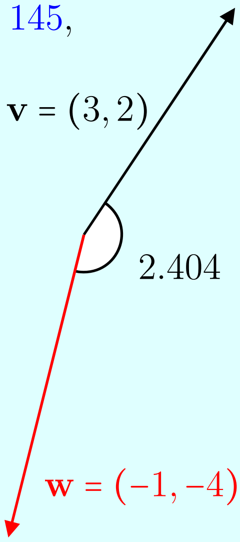


Example 803. Let $\mathbf{v} = (3,2)$ and $\mathbf{w} = (-1,-4)$ be vectors. By Definition 145,

$$\cos^{-1}\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} = \cos^{-1}\frac{(3,2)\cdot(-1,-4)}{|(3,2)|\,|(-1,-4)|}$$

$$= \cos^{-1}\frac{3\times(-1)+2\times(-4)}{\sqrt{3^2+2^2}\sqrt{(-1)^2+(-4)^2}}$$

$$= \cos^{-1}\left(\frac{-3-8}{\sqrt{13}\sqrt{17}}\right) = \cos^{-1}\left(\frac{-11}{\sqrt{221}}\right) \approx 2.404.$$



By the way, here's a possible concern. We've defined the angle between two vectors as

$$\cos^{-1}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

But recall (Definition 99) that Domain $\cos^{-1} = [-1,1]$. So, how can we be sure that the above expression is always well-defined? That is, how can we be sure that

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \in [-1,1] \qquad \text{or} \qquad -1 \le \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \le 1?$$

Fortunately, we can, thanks to Cauchy's Inequality:[327]

Fact 133. (Cauchy's Inequality.) *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are non-zero vectors. Then*

$$-1 \le \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \le 1.$$

Equivalently, $\qquad -|\mathbf{u}|\,|\mathbf{v}| \le \mathbf{u}\cdot\mathbf{v} \le |\mathbf{u}|\,|\mathbf{v}| \quad$ *or* $\quad (\mathbf{u}\cdot\mathbf{v})^2 \le |\mathbf{u}|^2\,|\mathbf{v}|^2.$

Proof. See p. 1620 (Appendices). □

[327] Also known as the Cauchy-Schwarz Inequality in its more general form.

Fact 134. *Suppose θ is the angle between two non-zero vectors* $\mathbf{u}$ *and* $\mathbf{v}$*. Then*

(a) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}=1 \iff \theta=0$;

(b) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\in(0,1) \iff \theta\in\left(0,\dfrac{\pi}{2}\right)$;

(c) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}=0 \iff \theta=\dfrac{\pi}{2}$;

(d) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\in(-1,0) \iff \theta\in\left(\dfrac{\pi}{2},\pi\right)$;

(e) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}=-1 \iff \theta=\pi$.

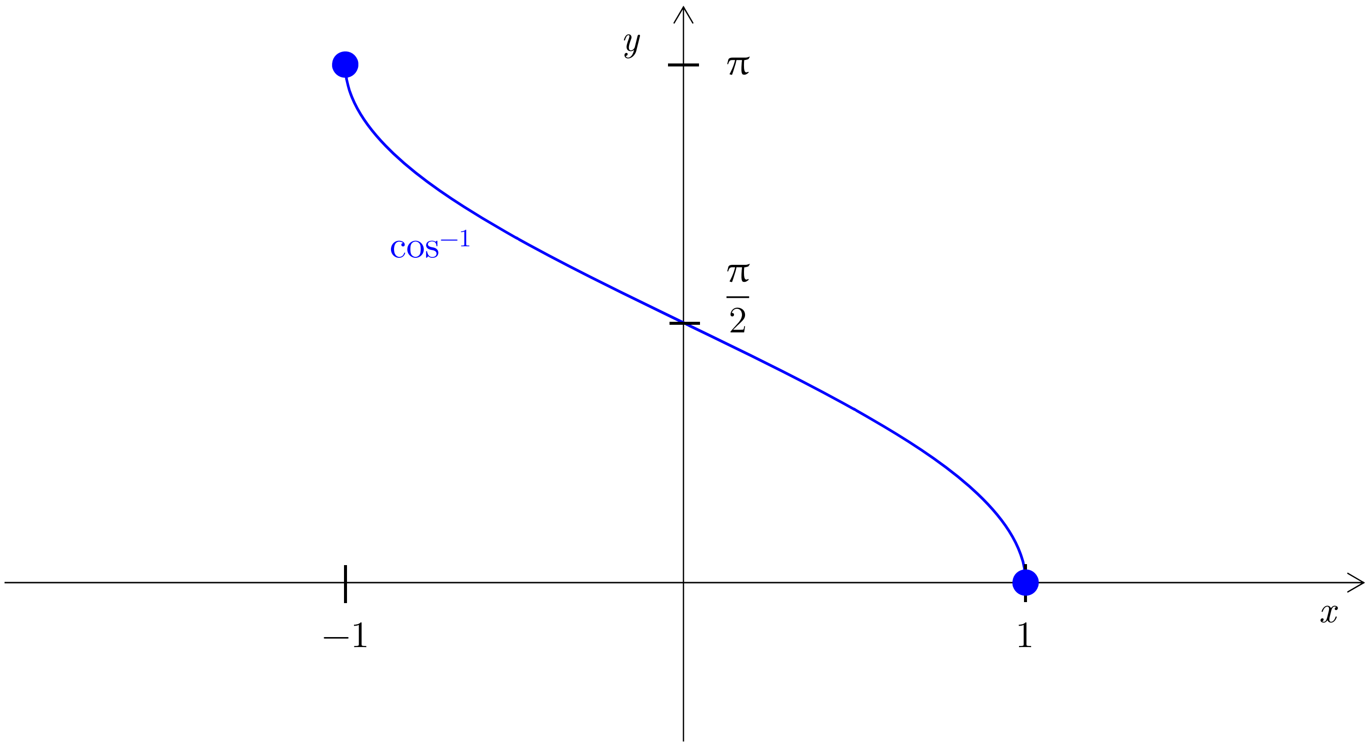
And thus,

(i) $\mathbf{u}\cdot\mathbf{v}>0 \iff \theta$ *is acute or zero;*

(ii) $\mathbf{u}\cdot\mathbf{v}=0 \iff \theta$ *is right;*

(iii) $\mathbf{u}\cdot\mathbf{v}<0 \iff \theta$ *is obtuse or straight.*

Proof. To prove this, simply apply Definition 145. □



Fact 134(ii) motivates the following Definition:

Definition 146. Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are *perpendicular* (or *normal* or *orthogonal*) if $\mathbf{u}\cdot\mathbf{v}=0$ and non-perpendicular if $\mathbf{u}\cdot\mathbf{v}\neq 0$.

As shorthand, we write $\mathbf{u}\perp\mathbf{v}$ if two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular; and $\mathbf{u}\not\perp\mathbf{v}$ if they aren't.

Remark 104. Again, note the special case of the zero vector $\mathbf{0}=(0,0)$—it is neither perpendicular nor non-perpendicular to any other vector (not even itself).

Fact 135. *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are non-zero vectors. Then*

(a) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = 1 \iff$ $\mathbf{u}$ *and* $\mathbf{v}$ *point in the same direction;*

(b) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = -1 \iff$ $\mathbf{u}$ *and* $\mathbf{v}$ *point in exact opposite directions;*

(c) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \pm 1 \iff \mathbf{u} \parallel \mathbf{v}$*;*

(d) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \in (-1,1) \iff$ $\mathbf{u}$ *and* $\mathbf{v}$ *point in different directions.*

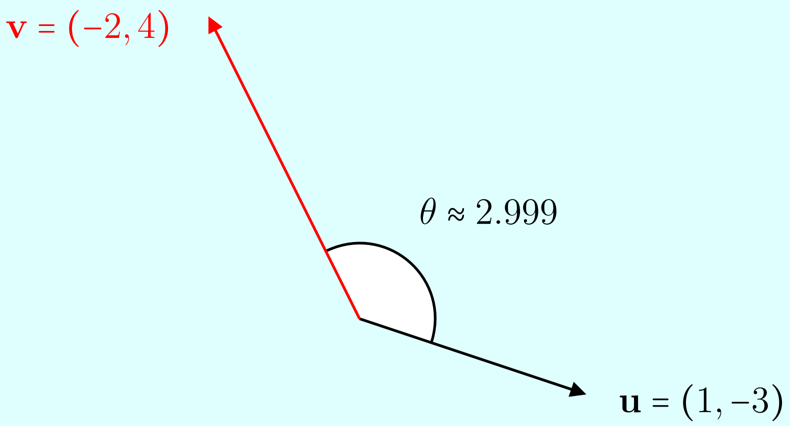
Proof. "Obviously", **(c)** and **(d)** simply follow from **(a)** and **(b)**. For the proof of **(a)** and **(b)**, see p. 1621 (Appendices). □

More examples:

Example 804. Let θ be the angle between the vectors $\mathbf{u} = (1,-3)$ and $\mathbf{v} = (-2,4)$. Then

$$\theta = \cos^{-1}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \cos^{-1}\frac{-2-12}{\sqrt{1^2+(-3)^2}\sqrt{(-2)^2+4^2}} = \cos^{-1}\frac{-14}{\sqrt{10}\sqrt{20}} = \cos^{-1}\frac{-14}{10\sqrt{2}} \approx 2.999.$$

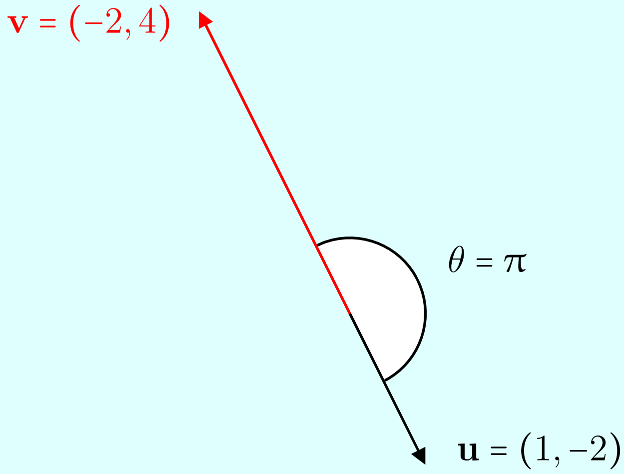
So, $\mathbf{u}$ and $\mathbf{v}$ are neither perpendicular nor parallel; instead, they point in different directions. Moreover, the angle between them is obtuse.



Example 805. Let θ be the angle between the vectors $\mathbf{u} = (1, -2)$ and $\mathbf{v} = (-2, 4)$. Then

$$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \cos^{-1} \frac{-2-8}{\sqrt{1^2 + (-2)^2}\sqrt{(-2)^2 + 4^2}} = \cos^{-1} \frac{-10}{\sqrt{5}\sqrt{20}} = \cos^{-1} -1 = \pi.$$

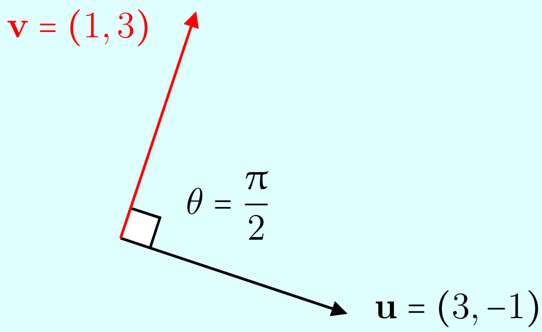
So, $\mathbf{u}$ and $\mathbf{v}$ are parallel; more specifically, they point in exact opposite directions. Moreover, the angle between them is straight.



Example 806. Let θ be the angle between the vectors $\mathbf{u} = (3, -1)$ and $\mathbf{v} = (1, 3)$. Then

$$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \cos^{-1} \frac{3-3}{\sqrt{3^2 + (-1)^2}\sqrt{1^2 + 3^2}} = \cos^{-1} 0 = \frac{\pi}{2}.$$

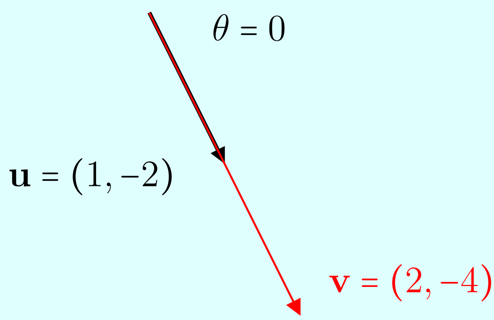
So, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular and the angle between them is right.



Example 807. Let θ be the angle between the vectors $\mathbf{u} = (1, -2)$ and $\mathbf{v} = (2, -4)$. Then

$$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \cos^{-1} \frac{2+8}{\sqrt{1^2 + (-2)^2}\sqrt{2^2 + (-4)^2}} = \cos^{-1} \frac{10}{\sqrt{5}\sqrt{20}} = \cos^{-1} 1 = 0.$$

So, $\mathbf{u}$ and $\mathbf{v}$ are parallel; more specifically, they point in the same direction. Moreover, the angle between them is zero.



We can use Fact 134(i)–(iii) to quickly check if the angle between two vectors is acute, zero, or obtuse:

Example 808. Even without any graphs or exact calculations, we can quickly see that

- The angle between the vectors $(817, -2)$ and $(39, -55)$ is acute, because

$$817 \cdot 39 + (-2) \cdot (-55) > 0.$$

- The vectors $\left(\sqrt{79300}, -470\right)$ and $\left(47, \sqrt{793}\right)$ are perpendicular, because

$$\sqrt{79300} \cdot 47 + (-470) \cdot \sqrt{793} = 0$$

- If $k < 0$, then the angle between $(67, k)$ and $(-485, 32)$ is obtuse, because

$$67 \cdot (-485) + 32k < 0.$$

Exercise 233. For each, find the angle between $\mathbf{u}$ and $\mathbf{v}$. Is it zero, acute, right, obtuse, or straight? Are $\mathbf{u}$ and $\mathbf{v}$ perpendicular or parallel? Do they point in the same, exact opposite, or different directions? (Answers on p. 1846.)

(a) $\mathbf{u} = (2, 0)$ and $\mathbf{v} = (0, 17)$.

(b) $\mathbf{u} = (5, 0)$ and $\mathbf{v} = (-3, 0)$.

(c) $\mathbf{u} = (1, 0)$ and $\mathbf{v} = \left(1, \sqrt{3}\right)$.

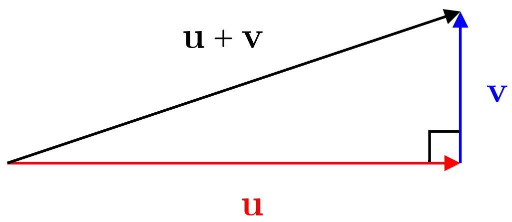
(d) $\mathbf{u} = (2, -3)$ and $\mathbf{v} = (1, 2)$.

56.1. Pythagoras' Theorem and Triangle Inequality

Recall **Pythagoras' Theorem** (Theorem 2)? Here it is again, but now in the language of vectors:

Theorem 17. (Pythagoras' Theorem.) *If* $\mathbf{u} \perp \mathbf{v}$, *then* $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$.

Proof. See Exercise 234. □



Exercise 234. Use Facts 129 and 131 to show that $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u}\cdot\mathbf{v} + |\mathbf{v}|^2$. Then use $\mathbf{u} \perp \mathbf{v}$ to complete the proof of the above Theorem. (Answer on p. 1847.)

Recall the **Triangle Inequality** (Fact 106)? Here it is again, but now in the language of vectors:

Fact 136. (Triangle Inequality.) *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors, then* $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

Proof. See Exercise 235. □

Exercise 235. Exercise 234 already showed that $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u}\cdot\mathbf{v} + |\mathbf{v}|^2$.

Prove Fact 136 using these steps: First apply Cauchy's Inequality (Fact 133) to the above equation. Then complete the square and take square roots. (Answer on p. 1847.)

56.2. Direction Cosines

Definition 147. The *x- and y-direction cosines* of the vector $\mathbf{v} = (v_1, v_2)$ are these numbers:

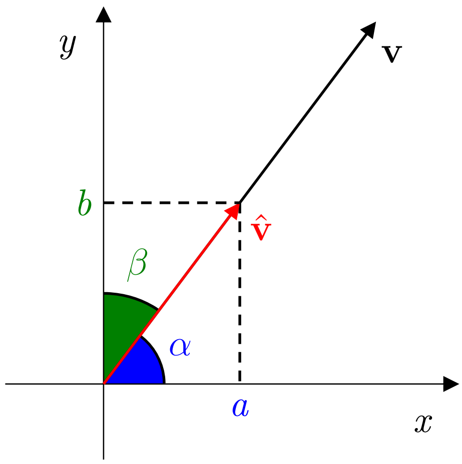
$$\frac{v_1}{|\mathbf{v}|} \quad \text{and} \quad \frac{v_2}{|\mathbf{v}|}.$$

Observe that the unit vector of $\mathbf{v} = (v_1, v_2)$ is

$$\hat{\mathbf{v}} = \left(\frac{v_1}{|\mathbf{v}|}, \frac{v_2}{|\mathbf{v}|}\right).$$

And so, equivalently, $\mathbf{v}$'s x- and y-direction cosines are the x- and y-coordinates of its unit vector.

We now explain why the x- and y-direction cosines are so named. Place the tail of $\mathbf{v} = (v_1, v_2)$ at the origin. Let α be the angle between $\mathbf{v}$ and the positive x-axis. Similarly, let β be the angle between $\mathbf{v}$ and the positive y-axis.[328]



Let $\hat{\mathbf{v}} = (a, b)$ be $\mathbf{v}$'s unit vector. It has length 1 and forms the hypotenuse of two right triangles. From the lower-right triangle, we have $a = \cos\alpha$.

Similarly, from the upper-left triangle, we have $b = \cos\beta$.

This explains why $\mathbf{v}$'s unit vector's x- and y-coordinates are also its x- and y-direction cosines.

[328] More formally, α is the angle between $\mathbf{v}$ and $\mathbf{i} = (1, 0)$, while β is the angle between $\mathbf{v}$ and $\mathbf{j} = (0, 1)$.

We can state and prove what was just said a bit more formally:

Fact 137. *Let* $\mathbf{v} = (v_1, v_2)$ *be a non-zero vector. Suppose* α *and* β *are the angles between* $\mathbf{v}$ *and each of* $\mathbf{i}$ *and* $\mathbf{j}$*. Then*

$$\cos\alpha = \frac{v_1}{|\mathbf{v}|} \qquad \text{and} \qquad \cos\beta = \frac{v_2}{|\mathbf{v}|}.$$

Proof. Since α is the angle between $\mathbf{v}$ and $\mathbf{i}$, by Definition 145, we have

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|\,|\mathbf{i}|} = \frac{v_1 \cdot 1 + v_2 \cdot 0}{|\mathbf{v}|\cdot 1} = \frac{v_1}{|\mathbf{v}|}.$$

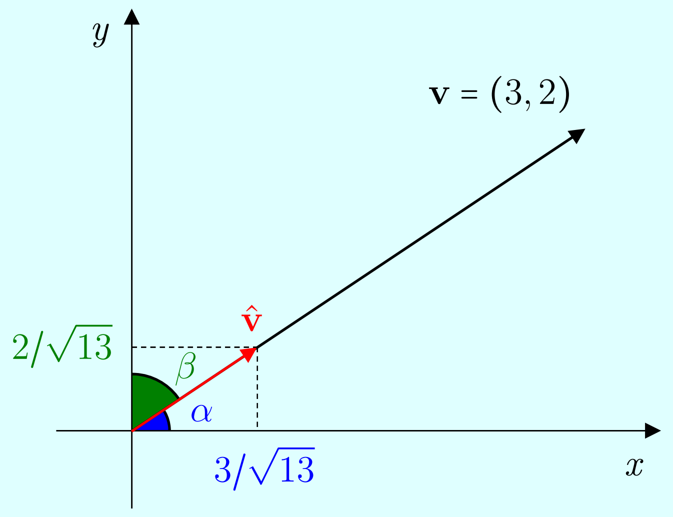
Similarly, since β is the angle between $\mathbf{v}$ and $\mathbf{j}$, we have

$$\cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{|\mathbf{v}|\,|\mathbf{j}|} = \frac{v_1 \cdot 0 + v_2 \cdot 1}{|\mathbf{v}|\cdot 1} = \frac{v_2}{|\mathbf{v}|}.$$ □

Example 809. Consider the vector $\mathbf{v} = (3, 2)$. Its x- and y-direction cosines are

$$\frac{v_1}{|\mathbf{v}|} = \frac{3}{\sqrt{3^2+2^2}} = \frac{3}{\sqrt{13}} \qquad \text{and} \qquad \frac{v_2}{|\mathbf{v}|} = \frac{2}{\sqrt{3^2+2^2}} = \frac{2}{\sqrt{13}}.$$

Which means, of course, that its unit vector is $\hat{\mathbf{v}} = \left(\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right)$.



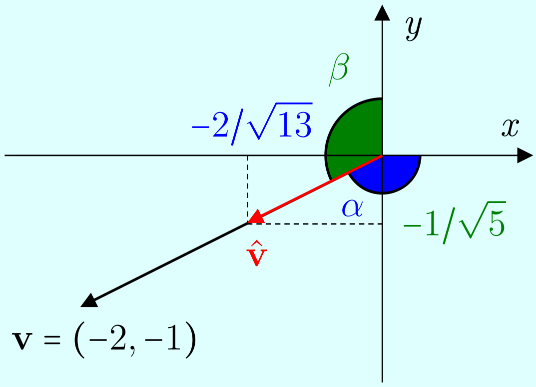
Let α and β be the angles it makes with the positive x- and y-axes. Then

$$\alpha = \cos^{-1}\frac{3}{\sqrt{13}} \approx 0.588 \quad \text{and} \quad \beta = \cos^{-1}\frac{2}{\sqrt{13}} \approx 0.983.$$

Example 810. Consider the vector $\mathbf{v} = (-2, -1)$. Its x- and y-direction cosines are

$$\frac{v_1}{|\mathbf{v}|} = \frac{-2}{\sqrt{(-2)^2 + (-1)^2}} = \frac{-2}{\sqrt{5}} \qquad \text{and} \qquad \frac{v_2}{|\mathbf{v}|} = \frac{-1}{\sqrt{(-2)^2 + (-1)^2}} = \frac{-1}{\sqrt{5}}.$$

Which means, of course, that its unit vector is $\hat{\mathbf{v}} = \left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$.



Let α and β be the angles it makes with the positive x- and y-axes. Then

$$\alpha = \cos^{-1}\frac{-2}{\sqrt{5}} \approx 2.678 \qquad \text{and} \qquad \beta = \cos^{-1}\frac{-1}{\sqrt{5}} \approx 2.034.$$

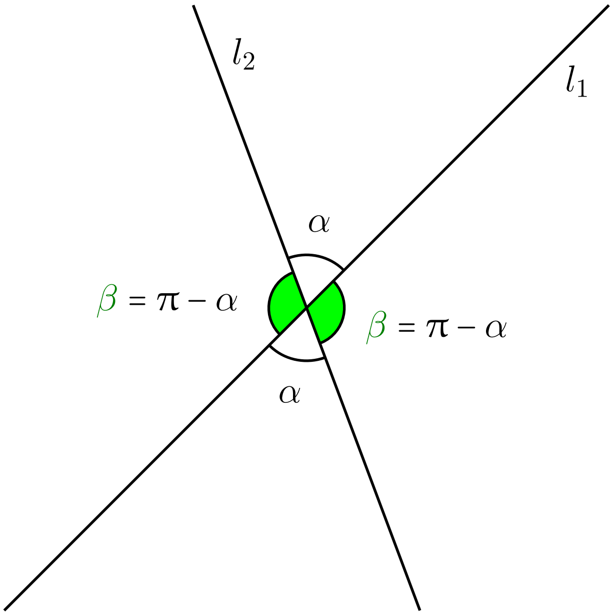
Exercise 236. For each vector, find its x- and y-direction cosines, and its unit vector.

(a) $(1, 3)$. **(b)** $(4, 2)$. **(c)** $(-1, 2)$. (Answer on p. 1847.)

57. The Angle Between Two Lines

(Below, Corollary 22 will give the "formula" for the angle between two lines. If you're a well-trained, mindless Singaporean monkey who cares only about "knowing" the "formula" without understanding where it comes from, you can skip straight to Corollary 22.)

"Obviously", any two non-parallel lines l_1 and l_2 in 2D space must intersect at exactly one point. And at this intersection point, two angles are formed. Let's call the smaller angle α, so that the larger angle is $\beta = \pi - \alpha$.



Now, we have a potential ambiguity: When we talk about *the* angle between l_1 and l_2, are we talking about the smaller angle α or the larger angle β?

To resolve this ambiguity, this textbook will adopt the convention that *the* angle between two lines is the smaller one. So, by this convention, *the* angle between l_1 and l_2 shall be α (and **not** β).

Note that by our adopted convention, the angle between two lines will always be acute (i.e. between 0 and $\pi/2$, inclusive) and **never** obtuse.

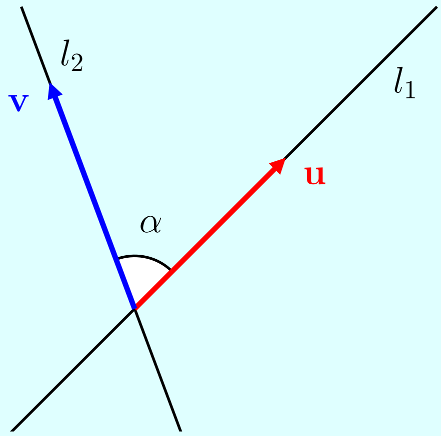
We now work towards a formal definition of the angle between two lines.

Example 811. Given the lines l_1 and l_2, we pick for each the direction vectors $\mathbf{u}$ and $\mathbf{v}$.

Observe that

$$\alpha = \begin{matrix}\text{Angle between}\\ l_1 \text{ and } l_2\end{matrix} = \begin{matrix}\text{Angle between}\\ \mathbf{u} \text{ and } \mathbf{v}.\end{matrix}$$

That is, α is the angle between the two lines; moreover, it is also the angle between the two vectors.



The above example suggests the following "Definition" for the angle between two lines. Given two lines l_1 and l_2, pick for each any direction vectors $\mathbf{u}$ and $\mathbf{v}$. Now define

$$\begin{matrix}\text{Angle between}\\ l_1 \text{ and } l_2\end{matrix} = \begin{matrix}\text{Angle between}\\ \mathbf{u} \text{ and } \mathbf{v}.\end{matrix}$$

This "Definition" works well in the above example, but only because the angle between $\mathbf{u}$ and $\mathbf{v}$ happens to be acute.

Unfortunately and as the next example illustrates, this "Definition" doesn't work so well if the angle between the two chosen direction vectors is instead obtuse:

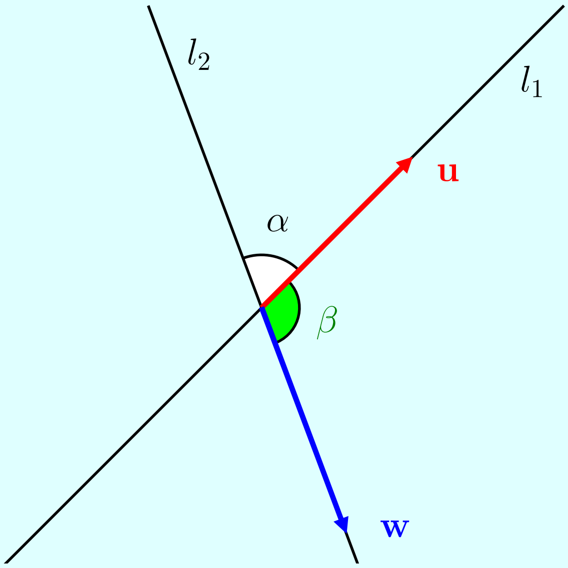
Example 812. We continue with the same two lines as before. We continue to pick the direction vector $\mathbf{u}$ for the line l_1. But this time, we pick the direction vector $\mathbf{w}$ for the line l_2.

The angle between the two lines remains the acute angle α.

However, the angle between the two chosen direction vectors is now the obtuse angle β.

And so, the above "Definition" fails because

$$\alpha = \begin{array}{c}\text{Angle between}\\ l_1 \text{ and } l_2\end{array} \neq \begin{array}{c}\text{Angle between}\\ \mathbf{u} \text{ and } \mathbf{w}\end{array} = \beta.$$



In this case, the angle between the two lines, α, is actually the **supplement** of the angle between the chosen two direction vectors, β. That is,

$$\alpha = \pi - \beta.$$

Let's introduce a new term, the **non-obtuse angle between two vectors**:

Example 813. The angle between the vectors $\mathbf{c}$ and $\mathbf{d}$ is β, which is obtuse. We say the **non-obtuse angle between them** is $\alpha = \pi - \beta$.

The angle between the vectors $\mathbf{e}$ and $\mathbf{f}$ is γ, which is acute. We say the non-obtuse angle between them is also γ.

Formally,

Definition 148. Let β be the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Then the *non-obtuse angle* between $\mathbf{u}$ and $\mathbf{v}$ is denoted α and is defined by

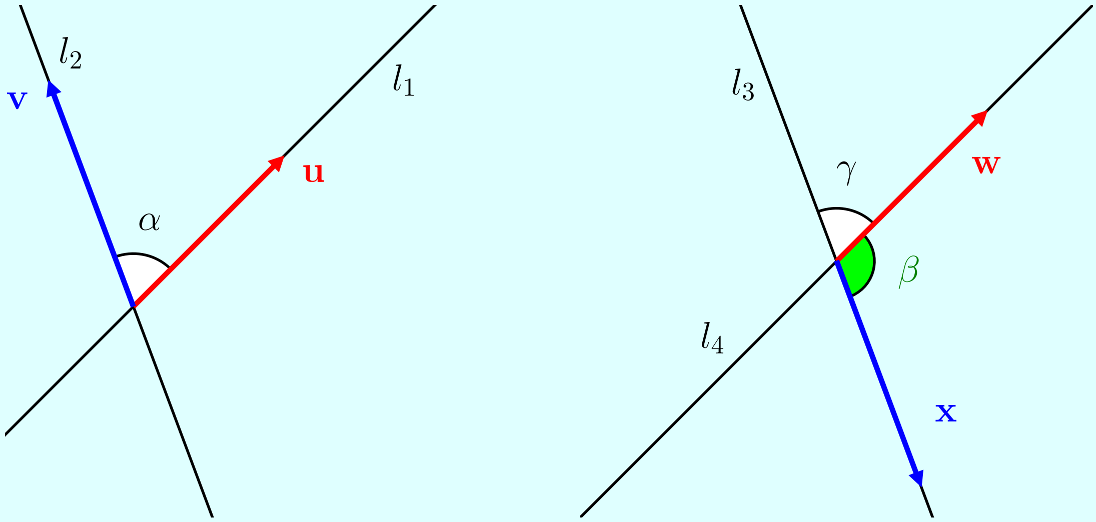
$$\alpha = \begin{cases} \beta, & \text{for } \beta \text{ not obtuse,} \\ \pi - \beta, & \text{for } \beta \text{ obtuse.} \end{cases}$$

We are now ready to write down our formal Definition of **the angle between two lines**:

Definition 149. Given two lines, pick for each any direction vector. We call the non-obtuse angle between these two vectors the *angle between the two lines*.

Example 814. Given the lines l_1 and l_2, we pick the direction vectors $\mathbf{u}$ and $\mathbf{v}$.

The angle between $\mathbf{u}$ and $\mathbf{v}$ is α, which is acute. And so, the non-obtuse angle between $\mathbf{u}$ and $\mathbf{v}$ is *also* α. So, by Definition 149, the angle between the two lines is α.



Given the lines l_3 and l_4, we pick the direction vectors $\mathbf{w}$ and $\mathbf{x}$.

The angle between $\mathbf{w}$ and $\mathbf{x}$ is β, which is obtuse. And so, the non-obtuse angle between $\mathbf{w}$ and $\mathbf{x}$ is $\gamma = \pi - \beta$. So, by Definition 149, the angle between the two lines is γ.

We just wrote down the Definition of the angle between two lines. We now work towards Corollary 22, which will give us our "formula" for the angle between two lines.

By Definition 145, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\cos^{-1} \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$.

It turns out that we can get the non-obtuse angle between $\mathbf{u}$ and $\mathbf{v}$ simply by slapping $|\cdot|$ (the absolute value function) onto the numerator:

Fact 138. *The non-obtuse angle between two non-zero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\cos^{-1} \frac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Proof. Suppose $0 \le \theta \le \pi/2$. Then $\mathbf{u}\cdot\mathbf{v} \ge 0$. And so, by Definition 148, the non-obtuse angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\cos^{-1} \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \cos^{-1} \frac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Suppose instead $\theta > \pi/2$. Then $\mathbf{u}\cdot\mathbf{v} < 0$. And so, by Definition 148 and the identity $\pi - \cos^{-1} x = \cos^{-1}(-x)$ (Fact 101), the non-obtuse angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\pi - \cos^{-1} \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \cos^{-1} \frac{-\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \cos^{-1} \frac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|}.$$ □

From Fact 138, this "formula" for the angle between two lines is immediate:

Corollary 22. *The angle between two lines with direction vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\cos^{-1}\frac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Example 815. Two lines l_1 and l_2 are described by

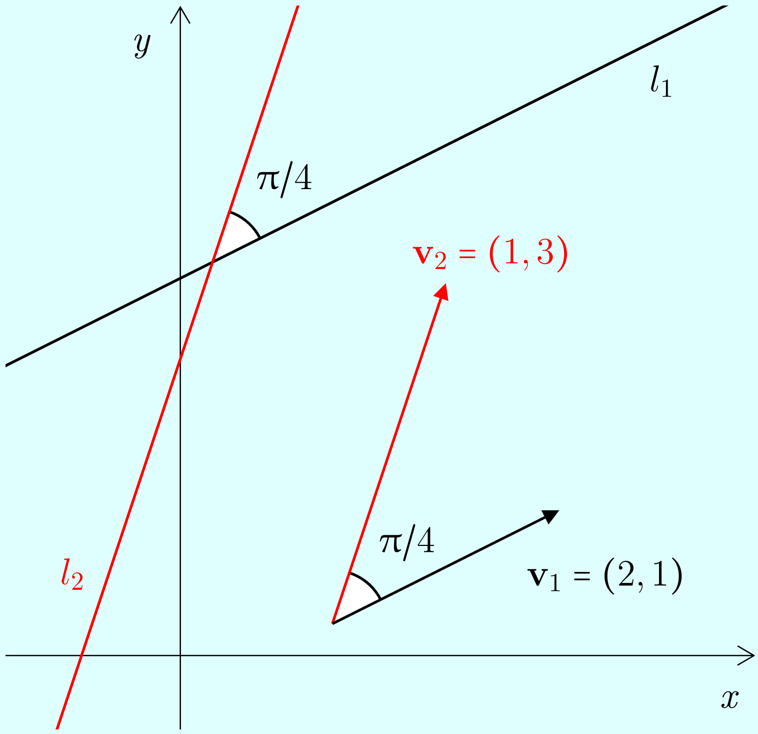
$$\mathbf{r} = (1,3) + \lambda\overbrace{(2,1)}^{\mathbf{v}_1} \qquad (\lambda\in\mathbb{R}),$$

$$\mathbf{r} = (-1,-1) + \lambda\overbrace{(1,3)}^{\mathbf{v}_2} \qquad (\lambda\in\mathbb{R}).$$

By Corollary 22, the angle between the two lines is

$$\cos^{-1}\frac{|\mathbf{v}_1\cdot\mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|} = \cos^{-1}\frac{|(2,1)\cdot(1,3)|}{|(2,1)|\,|(1,3)|}$$

$$= = \cos^{-1}\frac{|5|}{\sqrt{5}\sqrt{10}} = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

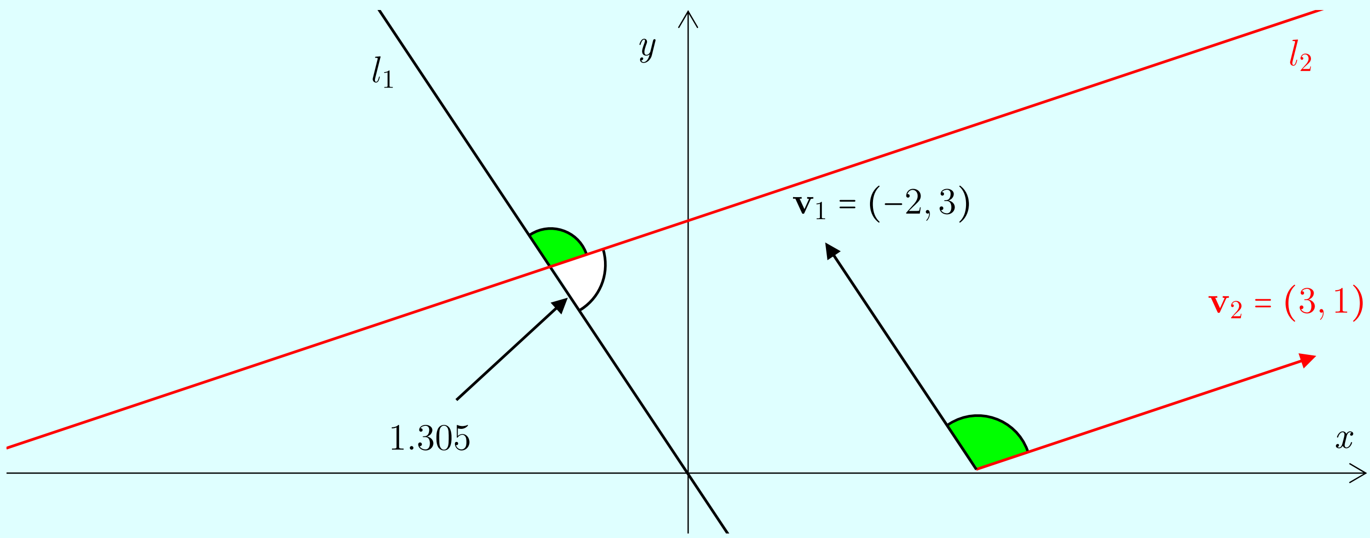


Example 816. Two lines l_1 and l_2 are described by

$$\mathbf{r} = (0,0) + \lambda\overbrace{(-2,3)}^{\mathbf{v}_1} \qquad \text{and} \qquad \mathbf{r} = (1,0) + \lambda\overbrace{(3,1)}^{\mathbf{v}_2} \qquad (\lambda\in\mathbb{R}).$$

By Corollary 22, the angle between the two lines is

$$\cos^{-1}\frac{|\mathbf{v}_1\cdot\mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|} = \cos^{-1}\frac{|(-2,3)\cdot(3,1)|}{|(-2,3)|\,|(3,1)|} = \cos^{-1}\frac{|-3|}{\sqrt{13}\sqrt{10}} \approx 1.305.$$



The following formal definition of when two lines are parallel or perpendicular will supersede the definitions given in Ch. 8.5.

Definition 150. Two lines are **(a)** *parallel* if they have parallel direction vectors; and **(b)** *perpendicular* if they have perpendicular direction vectors.

Corollary 23. *Suppose θ is the angle between two lines l_1 and l_2. Then* ***(a)*** $\theta = 0 \iff l_1 \parallel l_2$; ***(b)*** $\theta = \pi/2 \iff l_1 \perp l_2$.

Proof. See p. 1622 (Appendices). □

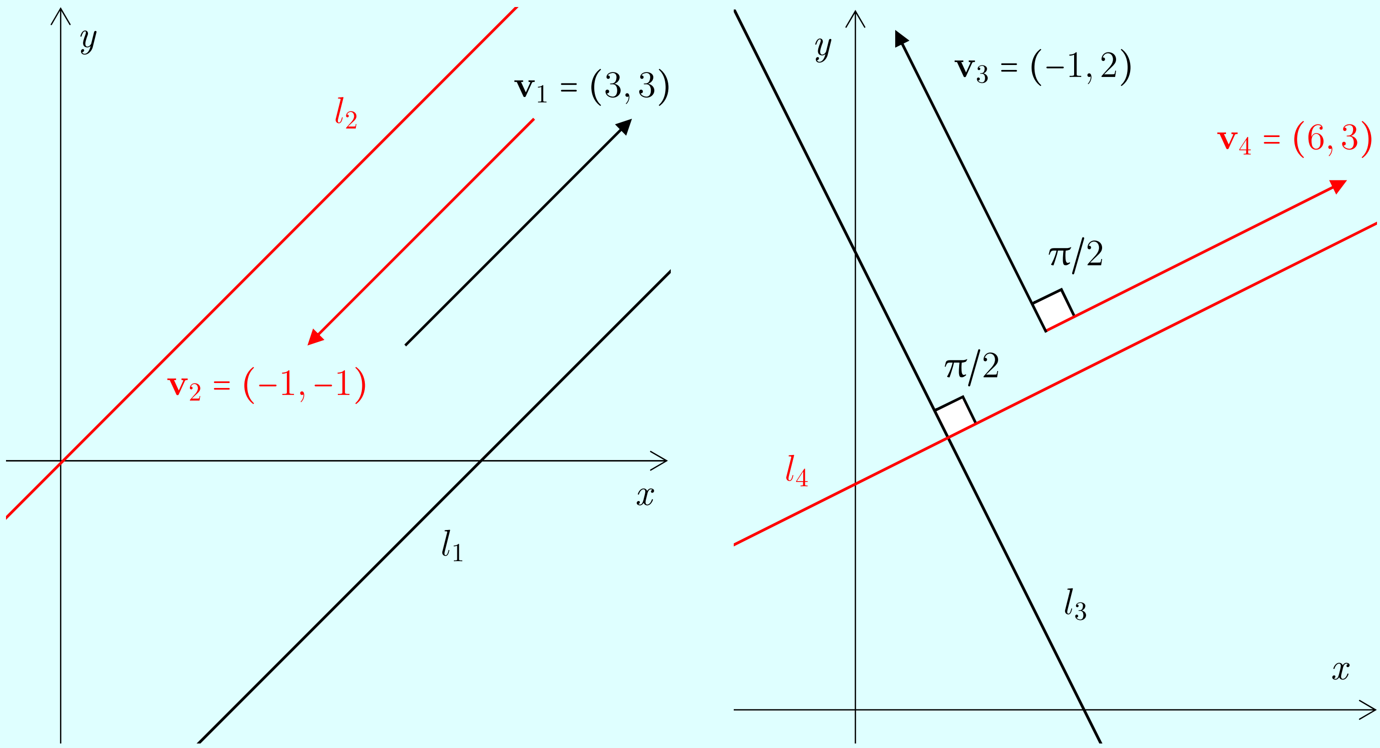
Example 817. Two lines l_1 and l_2 are described by

$$\mathbf{r} = (2,-2) + \lambda\overbrace{(3,3)}^{\mathbf{v}_1} \qquad \text{and} \qquad \mathbf{r} = (1,1) + \lambda\overbrace{(-1,-1)}^{\mathbf{v}_2} \qquad (\lambda \in \mathbb{R}).$$

By Corollary 22, the angle between l_1 and l_2 is

$$\cos^{-1}\frac{|\mathbf{v}_1 \cdot \mathbf{v}_2|}{|\mathbf{v}_1||\mathbf{v}_2|} = \cos^{-1}\frac{|(3,3)\cdot(-1,-1)|}{|(3,3)||(-1,-1)|} = \cos^{-1}\frac{|-6|}{\sqrt{18}\sqrt{2}} = \cos^{-1} 1 = 0.$$

By Corollary 23, the lines l_1 and l_2 are **parallel**.



Two lines l_3 and l_4 are described by

$$\mathbf{r} = (0,3) + \lambda\overbrace{(-1,2)}^{\mathbf{v}_3} \qquad \text{and} \qquad \mathbf{r} = (-1,1) + \lambda\overbrace{(6,3)}^{\mathbf{v}_4} \qquad (\lambda \in \mathbb{R}).$$

By Corollary 22, the angle between l_1 and l_2 is

$$\cos^{-1}\frac{|\mathbf{v}_3 \cdot \mathbf{v}_4|}{|\mathbf{v}_3||\mathbf{v}_4|} = \cos^{-1}\frac{|(-1,2)\cdot(6,3)|}{|(-1,2)||(6,3)|} = \cos^{-1}\frac{|0|}{\sqrt{5}\sqrt{45}} = \cos^{-1} 0 = \frac{\pi}{2}.$$

By Corollary 23, the lines l_3 and l_4 are **perpendicular**.

We can also define when a line and a vector are parallel or perpendicular:

Definition 151. A line and a vector are **(a)** *parallel* if the line has a direction vector that's parallel to the given vector; and **(b)** *perpendicular* if the line has a direction vector that's perpendicular to the given vector.

Here are two "obvious" Facts you may recall from primary school:

Fact 139. *Suppose two lines are ...*

(a) *Identical. Then they are also parallel.*

(b) *Distinct and parallel. Then they do not intersect.*

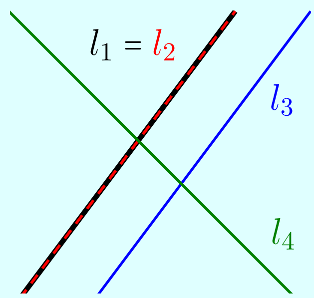
(c) *Distinct. Then they share at most one intersection point.*

Proof. See p. 1623 (Appendices). □

Fact 140. *If two lines (in the cartesian plane) are distinct and non-parallel, then they must share exactly one intersection point.*

Proof. See p. 1623 (Appendices). □

Example 818. The lines l_1 and l_2 are identical. And indeed, they are also parallel.



The lines l_1 and l_3 are distinct and parallel. And indeed they do not intersect.

The lines l_1 and l_4 are distinct and non-parallel. And indeed they share exactly one intersection point.

Exercise 237. Find the angle between each given pair of lines. State if they are parallel or perpendicular. (Answer on p. 1848.)

(a) $\mathbf{r} = (-1, 2) + \lambda\,(-1, 1)$ and $\mathbf{r} = (0, 0) + \lambda\,(2, -3)$ $(\lambda \in \mathbb{R})$.

(b) $\mathbf{r} = (-1, 2) + \lambda\,(1, 5)$ and $\mathbf{r} = (0, 0) + \lambda\,(8, 1)$ "

(c) $\mathbf{r} = (-1, 2) + \lambda\,(2, 6)$ and $\mathbf{r} = (0, 0) + \lambda\,(3, 2)$ "

Remark 105. Fact 140 applies only to 2D space. As we'll learn later, in 3D space, two lines can be distinct, non-parallel, and yet do not intersect. (We'll call such lines **skew lines**.)

In contrast, Fact 139 applies more generally to higher dimensions, including in 3D space.

58. Vectors vs Scalars

We now contrast vectors with scalars. We'll reuse Example 666 from Ch. 45 (Simple Parametric Equations):

Example 819. A moving particle's **position vector s** is a function of time t:

$$\mathbf{s}(t) = (s_x(t), s_y(t)) = (\cos t, \sin t), \qquad t \geq 0.$$

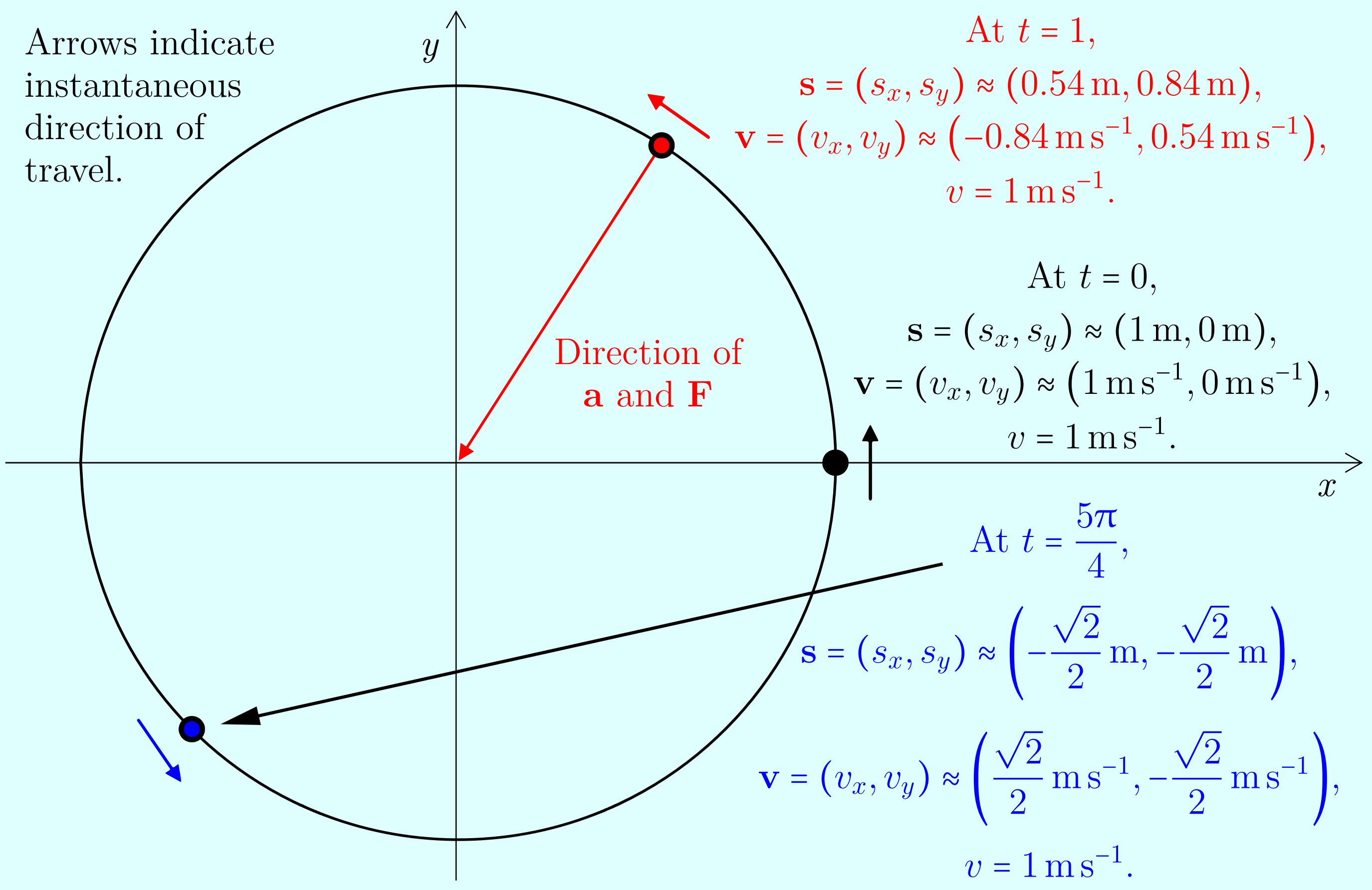
At time t (seconds, s), the particle is $\cos t$ metres (m) east (or to the right) of the origin $(0,0)$ and $\sin t$ m metres (m) north of (or above) the origin $(0,0)$.

For brevity, we'll often omit "(t)". That is, instead of the above, we'll often write

$$\mathbf{s} = (s_x, s_y) = (\cos t, \sin t), \qquad t \geq 0.$$

As time t progresses from 0 to 2π seconds, the particle traces out, anti-clockwise, the unit circle:

$$\begin{aligned}
t = 0 &\implies (s_x, s_y) = (1, 0), & t = \pi &\implies (s_x, s_y) = (-1, 0), \\
t = \pi/4 &\implies (s_x, s_y) = \left(\sqrt{2}/2, \sqrt{2}/2\right), & t = 5\pi/4 &\implies (s_x, s_y) = \left(-\sqrt{2}/2, -\sqrt{2}/2\right), \\
t = \pi/2 &\implies (s_x, s_y) = (0, 1), & t = 3\pi/2 &\implies (s_x, s_y) = (0, -1), \\
t = 3\pi/4 &\implies (s_x, s_y) = \left(-\sqrt{2}/2, \sqrt{2}/2\right), & t = 7\pi/4 &\implies (s_x, s_y) = \left(\sqrt{2}/2, -\sqrt{2}/2\right).
\end{aligned}$$



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The particle's **velocity vector** $\mathbf{v}(t)$ is the first derivative of $\mathbf{s}$ with respect to time t:

$$\mathbf{v}(t) = (v_x(t), v_y(t)) = \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \left(\frac{\mathrm{d}s_x}{\mathrm{d}t}, \frac{\mathrm{d}s_y}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}\cos t}{\mathrm{d}t}, \frac{\mathrm{d}\sin t}{\mathrm{d}t}\right) = (-\sin t, \cos t).$$

So, at time t, the particle is travelling in the eastern direction at $v_x = -\sin t \,\mathrm{m\,s^{-1}}$ (or equivalently, in the western direction at $\sin t \,\mathrm{m\,s^{-1}}$) and in the northern direction at $v_y = \cos t \,\mathrm{m\,s^{-1}}$.

The **magnitude** of the particle's velocity vector is denoted v and is called the particle's **speed**:

$$v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2}.$$

Since $\sin^2 t + \cos^2 t = 1$, for all t, we have

$$v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1.$$

Aha! So, interestingly, the particle travels at the constant speed of $1\,\mathrm{m\,s^{-1}}$. That is, at every instant in time t, it is moving $1\,\mathrm{m\,s^{-1}}$ in its direction of travel.

We now prove that the particle **always moves in a direction tangent to the circle**. In other words, its direction of travel is always perpendicular to its position vector.

To do so, we need simply prove that $\mathbf{v} \cdot \mathbf{s} = 0$ for all t:

$$\mathbf{v} \cdot \mathbf{s} = (-\sin t, \cos t) \cdot (\cos t, \sin t) = -\sin t \cos t + \cos t \sin t = 0. \quad ✓$$

Velocity is a **vector**—it has both **magnitude** and **direction**. In contrast, **speed** is a **scalar**—it has only **magnitude**.

(*Example continues on the next page ...*)

(*... Example continued from the previous page.*)

Similarly, the particle's **acceleration vector** is the first derivative of the velocity vector (or, equivalently, the second derivative of the position vector):

$$\mathbf{a}(t) = (a_x(t), a_y(t)) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{s}}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}v_x}{\mathrm{d}t}, \frac{\mathrm{d}v_y}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}(-\sin t)}{\mathrm{d}t}, \frac{\mathrm{d}\cos t}{\mathrm{d}t}\right) = (-\cos t, -\sin t).$$

So, at time t, the particle is accelerating eastwards at $a_x = -\cos t\,\mathrm{m\,s^{-2}}$ and northwards at $a_y = -\sin t\,\mathrm{m\,s^{-2}}$. Or equivalently, it is accelerating westwards at $\cos t\,\mathrm{m\,s^{-2}}$ and southwards at $\sin t\,\mathrm{m\,s^{-2}}$. (Note that $\mathrm{m\,s^{-2}}$ stands for **metre per second per second**.)

The **magnitude** of the particle's acceleration vector, denoted a, is

$$a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{(-\cos t)^2 + (-\sin t)^2} = \sqrt{1} = 1.$$

Aha! So, interestingly, the particle accelerates at the constant rate of $1\,\mathrm{m\,s^{-2}}$. That is, at every instant in time t, it is accelerating $1\,\mathrm{m\,s^{-2}}$ in its direction of acceleration.

(Note that for velocity, we gave its magnitude the special name of **speed**. But in contrast, the magnitude of acceleration has no special name. We simply call it the *magnitude of acceleration*.)

Above we proved that the particle's direction of movement is always tangent to the circle. Here we can similarly prove that its direction of acceleration is **always towards the centre of the circle**. (Or equivalently, the acceleration vector points in the exactly opposite direction as the position vector.) To do so, we need merely observe that the acceleration vector $\mathbf{a} = (-\cos t, -\sin t)$ and the position vector $\mathbf{s} = (\cos t, \sin t)$ point in exact opposite directions.[329]

Next, suppose the particle's **mass** is $m = 1\,\mathrm{kg}$. Recall (physics) Newton's Second Law:[330]

$$\overbrace{\text{Force}}^{\text{Vector}} = \overbrace{\text{Mass}}^{\text{Scalar}} \times \overbrace{\text{Acceleration}}^{\text{Vector}} \qquad \text{or} \qquad \mathbf{F} = m\mathbf{a}.$$

Note that mass is a scalar quantity. So, **force**, being a product of a scalar and a vector, is itself a vector quantity.

The force vector points in the same direction as the acceleration vector (i.e. towards the centre of the circle). Moreover, it has constant magnitude:

$$|\mathbf{F}| = |m\mathbf{a}| = |m|\,|\mathbf{a}| = (1\,\mathrm{kg}) \times \left(1\,\mathrm{m\,s^{-2}}\right) = 1\,\mathrm{kg\,m\,s^{-2}} = 1\,\mathrm{N},$$

where N stands for **newton** (the SI unit for force and which is equal to $\mathrm{kg\,m\,s^{-2}}$).

Physicists call such a force (which results in circular movement) a **centripetal force**.

[329]Note though that it would be wrong to write $\mathbf{a} = -\mathbf{s}$. This is because acceleration is measured in $\mathrm{m\,s^{-2}}$, while position is measured in m.

[330]See e.g. Exercise 188.

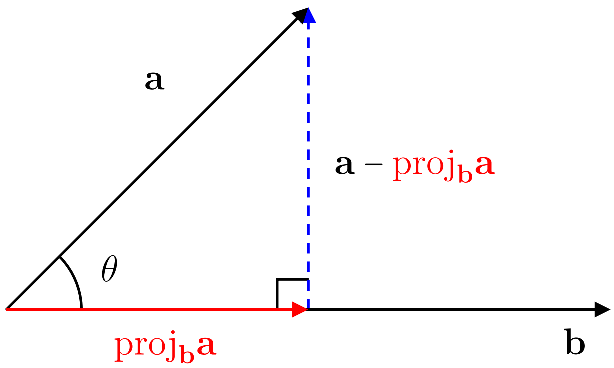
59. The Projection and Rejection Vectors

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors.

Motivation: We'd like the **projection of a on b**, denoted $\text{proj}_{\mathbf{b}}\mathbf{a}$, to be the vector that is[331]

- Parallel to $\mathbf{b}$; and
- Perpendicular to $\mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}$ (this vector is depicted in blue and will be called the **rejection vector**).

Let's work towards a formal definition of $\text{proj}_{\mathbf{b}}\mathbf{a}$.



First, since $\text{proj}_{\mathbf{b}}\mathbf{a} \parallel \mathbf{b}$, by Definition 138, we must have $\text{proj}_{\mathbf{b}}\mathbf{a} = k\hat{\mathbf{b}}$ for some $k \neq 0$.

Next, the length of $\text{proj}_{\mathbf{b}}\mathbf{a}$ is $|k|$. But what is k?

In the right triangle above, the hypotenuse corresponds to the vector $\mathbf{a}$, while the base corresponds to the vector $\text{proj}_{\mathbf{b}}\mathbf{a}$. If θ is the angle between these two vectors, then by our right-triangle definition of cosine,

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{|\text{proj}_{\mathbf{b}}\mathbf{a}|}{|\mathbf{a}|} \overset{1}{=} \frac{|k|}{|\mathbf{a}|}.$$

But by Definition 145, we also have $\qquad \cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \overset{2}{=} \dfrac{\mathbf{a}\cdot\hat{\mathbf{b}}}{|\mathbf{a}|}.$

Plug $\overset{1}{=}$ into $\overset{2}{=}$ to get $\qquad \dfrac{|k|}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\hat{\mathbf{b}}}{|\mathbf{a}|} \qquad$ or $\qquad |k| = \mathbf{a}\cdot\hat{\mathbf{b}}.$

The above discussion motivates this definition of the **projection vector**:[332]

Definition 152. Suppose $\mathbf{a}$ and $\mathbf{b} \neq \mathbf{0}$ are vectors. Then the *projection of* $\mathbf{a}$ *on* $\mathbf{b}$, denoted $\text{proj}_{\mathbf{b}}\mathbf{a}$, is this vector:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}, \qquad \text{or equivalently,} \qquad \text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}.$$

We'll also call the blue vector the **rejection vector**:

Definition 153. Suppose $\mathbf{a}$ and $\mathbf{b} \neq \mathbf{0}$ are vectors. Then the *rejection of* $\mathbf{a}$ *on* $\mathbf{b}$, denoted $\text{rej}_{\mathbf{b}}\mathbf{a}$, is this vector:

$$\text{rej}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}.$$

[331] These two properties are to hold so long as $\text{proj}_{\mathbf{b}}\mathbf{a}$ and $\mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}$ are non-zero.

[332] This definition imposes only the condition $\mathbf{b} \neq \mathbf{0}$ to ensure $\text{proj}_{\mathbf{b}}\mathbf{a}$ is well-defined. It doesn't impose the condition $\mathbf{a} \neq \mathbf{0}$ (even though the foregoing motivating discussion implicitly assumed this). So, we are allowed to speak of the projection of $\mathbf{0}$ on any non-zero vector $\mathbf{b}$—but of course, such a projection is always simply $\mathbf{0}$.

From the above two Definitions, Fact 141 is "obvious":

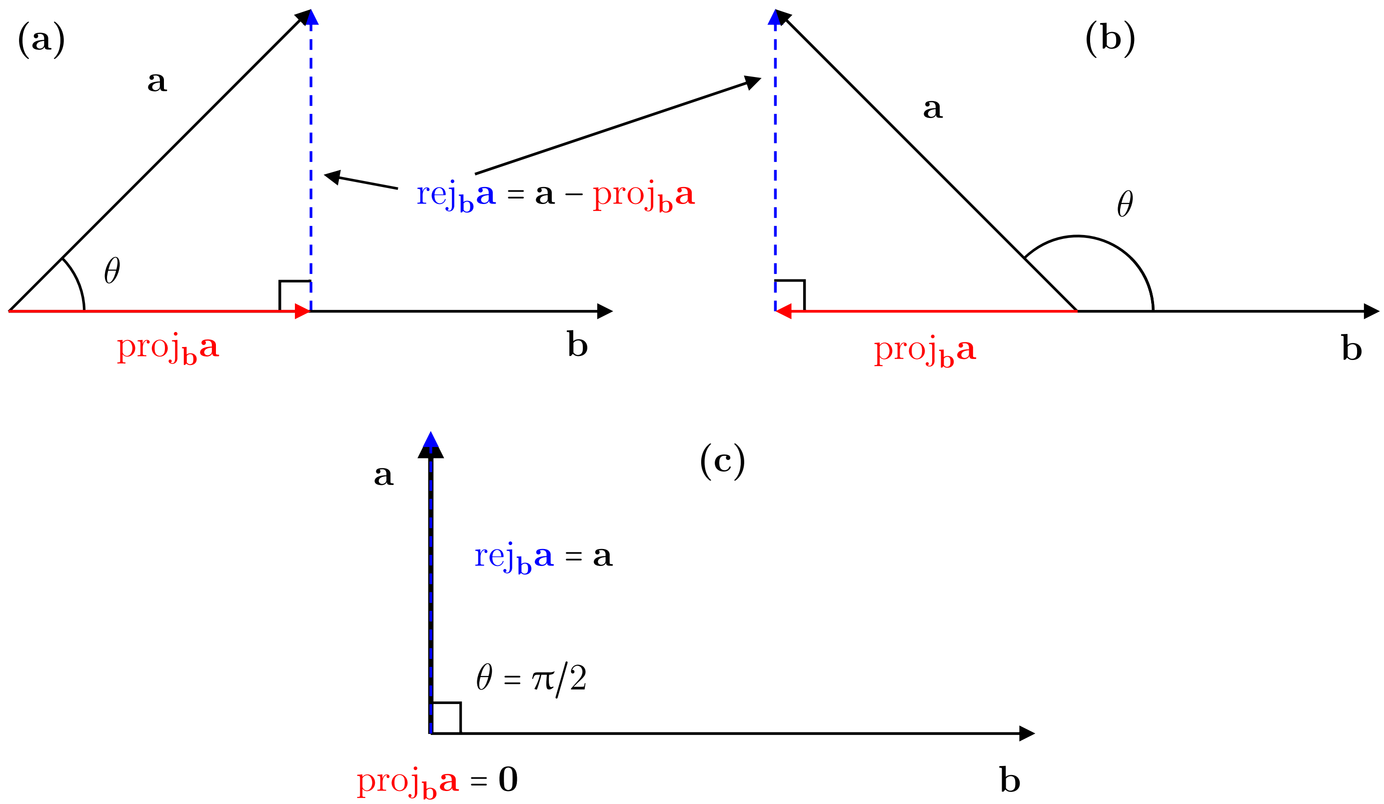
Fact 141. *Let* $\mathbf{a}$ *and* $\mathbf{b} \neq \mathbf{0}$ *be vectors and* $\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\mathbf{a} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}$.

(a) *If* $\mathbf{a} \cdot \hat{\mathbf{b}} > 0$, *then* $\text{proj}_{\mathbf{b}}\mathbf{a}$ *is a positive scalar multiple of* $\mathbf{b}$.

(b) *If* $\mathbf{a} \cdot \hat{\mathbf{b}} < 0$, *then* $\text{proj}_{\mathbf{b}}\mathbf{a}$ *is a negative scalar multiple of* $\mathbf{b}$.

(c) *If* $\mathbf{a} \cdot \hat{\mathbf{b}} = 0$, *then* $\text{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{0}$ *and* $\text{rej}_{\mathbf{b}}\mathbf{a} = \mathbf{a}$.

Geometric interpretation of Fact 141: Suppose θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Then
(a) If θ is acute, then $\text{proj}_{\mathbf{b}}\mathbf{a}$ points in the same direction as $\mathbf{b}$.
(b) If θ is obtuse, then $\text{proj}_{\mathbf{b}}\mathbf{a}$ points in the exact opposite direction as $\mathbf{b}$.
(c) If θ is right (i.e. if $\mathbf{a} \perp \mathbf{b}$), then $\text{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{0}$ and $\text{rej}_{\mathbf{b}}\mathbf{a} = \mathbf{a}$.



Above we already argued that $\text{proj}_{\mathbf{b}}\mathbf{a}$ has length $\left|\mathbf{a} \cdot \hat{\mathbf{b}}\right|$. We now formally prove this:

Fact 142. *Suppose* $\mathbf{a}$ *and* $\mathbf{b} \neq \mathbf{0}$ *are vectors. Then*

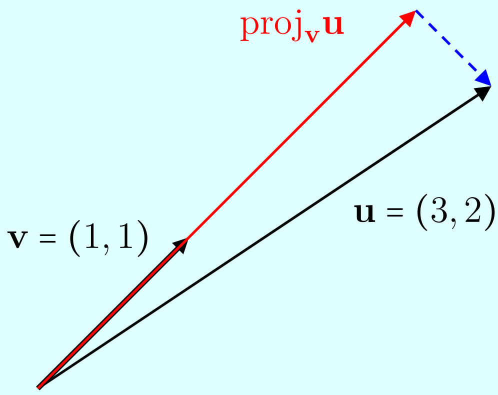
$$|\text{proj}_{\mathbf{b}}\mathbf{a}| = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right|.$$

Proof. By Definition 152 and Fact 121,

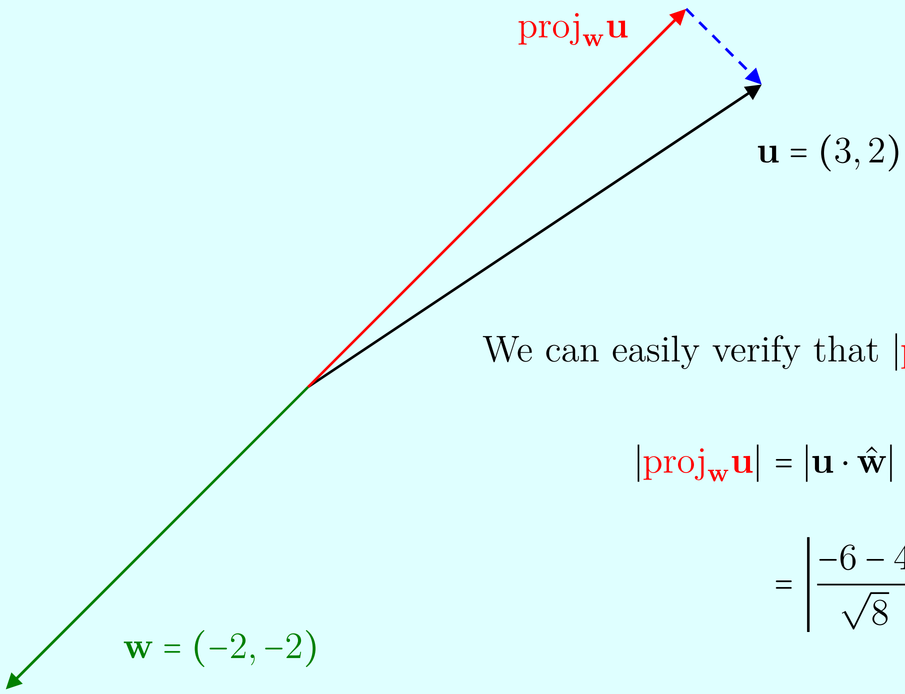
$$|\text{proj}_{\mathbf{b}}\mathbf{a}| = \left|\left(\mathbf{a} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}\right| = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right|\left|\hat{\mathbf{b}}\right| = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right| \cdot 1 = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right|.$$ □

Example 820. Let $\mathbf{u} = (3, 2)$ and $\mathbf{v} = (1, 1)$ be vectors and $\text{proj}_{\mathbf{v}}\mathbf{u}$ be the projection of $\mathbf{u}$ on $\mathbf{v}$. Then

$$|\text{proj}_{\mathbf{v}}\mathbf{u}| = |\mathbf{u} \cdot \hat{\mathbf{v}}| = \left| \frac{(3,2) \cdot (1,1)}{|(1,1)|} \right| = \frac{3+2}{\sqrt{2}} = \frac{5}{\sqrt{2}}.$$



Now consider $\mathbf{w} = (-2, -2)$—it points in the exact opposite direction from $\mathbf{v}$ and has twice the length. It turns out that the projection of $\mathbf{u}$ on $\mathbf{w}$, $\text{proj}_{\mathbf{w}}\mathbf{u}$, is identical to $\text{proj}_{\mathbf{v}}\mathbf{u}$.



We can easily verify that $|\text{proj}_{\mathbf{w}}\mathbf{u}| = |\text{proj}_{\mathbf{v}}\mathbf{u}|$:

$$|\text{proj}_{\mathbf{w}}\mathbf{u}| = |\mathbf{u} \cdot \hat{\mathbf{w}}| = \left| \frac{(3,2) \cdot (-2,-2)}{|(-2,-2)|} \right|$$

$$= \left| \frac{-6-4}{\sqrt{8}} \right| = \left| \frac{-10}{\sqrt{8}} \right| = \frac{5}{\sqrt{2}}. \qquad \checkmark$$

As the above example suggests, if $\mathbf{v}$ and $\mathbf{w}$ are parallel, then the projections of *any* vector $\mathbf{u}$ on $\mathbf{v}$ and $\mathbf{w}$ are identical. Formally,

Fact 143. *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors. If* $\mathbf{v} \parallel \mathbf{w}$, *then*

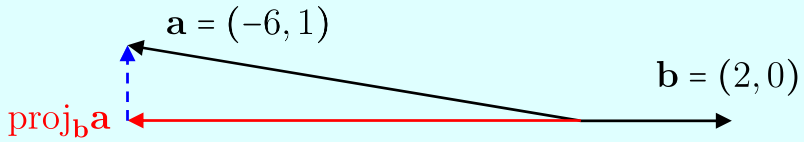
$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{proj}_{\mathbf{w}}\mathbf{u}.$$

Proof. If $\mathbf{v} \parallel \mathbf{w}$, then by Fact 124, $\hat{\mathbf{v}} = \pm\hat{\mathbf{w}}$. And so,

$$\text{proj}_{\mathbf{v}}\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{v}})\,\hat{\mathbf{v}} = [\mathbf{u} \cdot (\pm\hat{\mathbf{w}})]\,(\pm\hat{\mathbf{w}}) = (\mathbf{u} \cdot \hat{\mathbf{w}})\,\hat{\mathbf{w}} = \text{proj}_{\mathbf{w}}\mathbf{u}. \qquad \square$$

Example 821. Let $\mathbf{a} = (-6, 1)$ and $\mathbf{b} = (2, 0)$ be vectors and $\text{proj}_{\mathbf{b}}\mathbf{a}$ be the projection of $\mathbf{a}$ on $\mathbf{b}$. Then

$$|\text{proj}_{\mathbf{b}}\mathbf{a}| = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right| = \left|\frac{(-6,1)\cdot(2,0)}{|(2,0)|}\right| = \left|\frac{-12+0}{2}\right| = \left|\frac{-12}{2}\right| = 6/$$

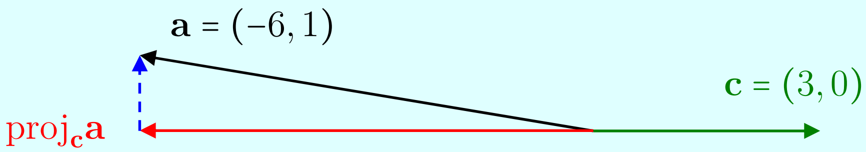


Now consider $\mathbf{c} = (3, 0)$—it points in the same direction as $\mathbf{b}$, but is half again as long. By Fact 143, the projection of $\mathbf{a}$ on $\mathbf{c}$ is the same as that of $\mathbf{a}$ on $\mathbf{b}$. That is,

$$\text{proj}_{\mathbf{c}}\mathbf{a} = \text{proj}_{\mathbf{b}}\mathbf{a}.$$

We can easily verify that $|\text{proj}_{\mathbf{c}}\mathbf{a}| = |\text{proj}_{\mathbf{b}}\mathbf{a}| = 6$:

$$|\text{proj}_{\mathbf{c}}\mathbf{a}| = |\mathbf{a} \cdot \hat{\mathbf{c}}| = \left|\frac{(-6,1)\cdot(3,0)}{|(3,0)|}\right| = \left|\frac{-18+0}{3}\right| = \left|\frac{-18}{3}\right| = 6.$$



Fact 143 can help us simplify some calculations:

Example 822. Let $\mathbf{u} = (5, -7)$ and $\mathbf{v} = \left(51\sqrt{347}, 68\sqrt{347}\right)$. What is $|\text{proj}_{\mathbf{v}}\mathbf{u}|$ (i.e. the length of the projection of $\mathbf{u}$ onto $\mathbf{v}$)?

Here it seems that the calculations will be pretty tedious. But observe that $\mathbf{v}$ is a multiple of $\mathbf{w} = (3, 4)$. So, $\mathbf{v} \parallel \mathbf{w}$. Hence, by Fact 143,

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{proj}_{\mathbf{w}}\mathbf{u}.$$

Thus, instead of computing $|\text{proj}_{\mathbf{v}}\mathbf{u}|$, we can simply compute $|\text{proj}_{\mathbf{w}}\mathbf{u}|$:

$$|\text{proj}_{\mathbf{v}}\mathbf{u}| = |\text{proj}_{\mathbf{w}}\mathbf{u}| = |\mathbf{u} \cdot \hat{\mathbf{w}}| = \left|\frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{w}|}\right| = \left|\frac{(5,-7)\cdot(3,4)}{|(3,4)|}\right| = \left|\frac{15 \cdot -28}{\sqrt{3^2+4^2}}\right| = \left|\frac{-13}{5}\right| = 2.6.$$

Exercise 238. Find the lengths of the projections of (Answer on p. 1848.)

(a) $(1, 0)$ on $(33, 33)$; and **(b)** $(33, 33)$ on $(1, 0)$.

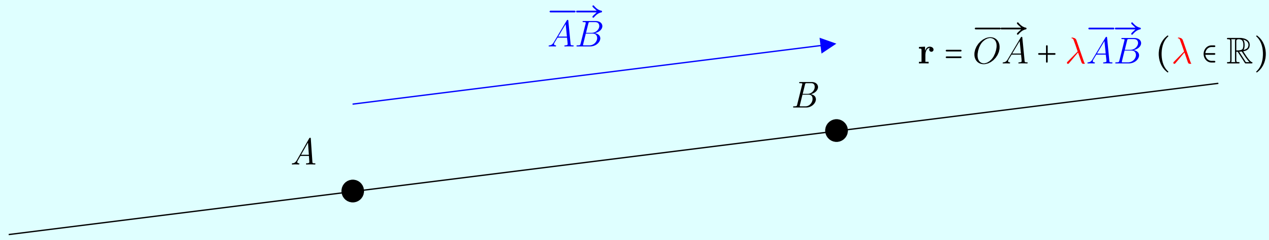
(c) Explain if this statement is true or false:

"Given any vectors $\mathbf{a}$ and $\mathbf{b}$, $|\text{proj}_{\mathbf{b}}\mathbf{a}| = |\text{proj}_{\mathbf{a}}\mathbf{b}|$."

60. Collinearity

Definition 154. Two or more points are *collinear* if some line contains all of them.

Example 823. The points A and B are collinear.



"Obviously", any two points must be collinear. Indeed, given any two points, there is a **unique line** that contains both of them:

Fact 144. *Suppose A and B are distinct points. Then the unique line that contains both A and B is described by*

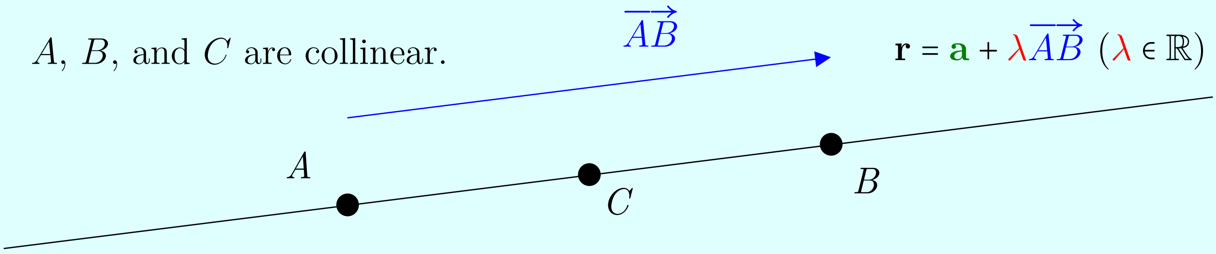
$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}, \qquad (\lambda \in \mathbb{R}).$$

Proof. First, plug in $\lambda = 0$ and $\lambda = 1$ to verify that the given line contains A and B.

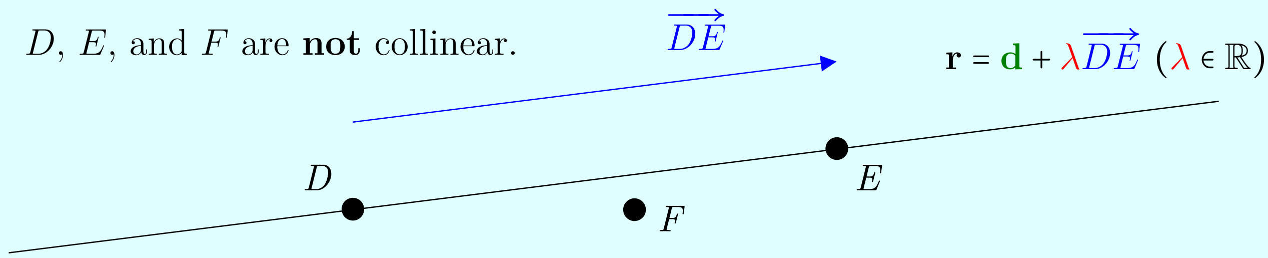
Next, this line is unique because any line that contains both A and B must have direction vector $\overrightarrow{AB}$ and so must be described by the given vector equation. $\square$

In contrast, three distinct points could be collinear but won't always be:

Example 824. The points A, B, and C are collinear:



In contrast, the points D, E, and F are **not**:



Here's one possible procedure for checking whether three points are collinear:

1. First use Fact 144 to write down the unique line that contains two of the three points.
2. Then check whether this line also contains the third point.

Example 825. To check if the points $A = (1,2)$, $B = (4,5)$, and $C = (7,8)$ are collinear,

1. First write down the unique line that contains both A and B:

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} = (1,2) + \lambda(3,3), \qquad (\lambda \in \mathbb{R}).$$

2. This line also contains C if and only if there exists $\hat{\lambda} \in \mathbb{R}$ such that

$$C = \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \hat{\lambda}\begin{pmatrix} 3 \\ 3 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} 7 &\overset{1}{=} 1 + 3\hat{\lambda}, \\ 8 &\overset{2}{=} 2 + 3\hat{\lambda}. \end{aligned}$$

As you can verify, $\hat{\lambda} = 2$ solves the above vector equation (or system of two equations).

So, our line also contains C. Hence, A, B, and C are collinear.

Example 826. To check if the points $D = (1,0)$, $E = (0,1)$, and $F = (0,0)$ are collinear,

1. First write down a line that contains both D and E:

$$\mathbf{r} = \overrightarrow{OD} + \lambda\overrightarrow{DE} = (1,0) + \lambda(-1,1), \qquad (\lambda \in \mathbb{R}).$$

2. This line also contains F if and only if there exists $\hat{\lambda} \in \mathbb{R}$ such that

$$F = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \hat{\lambda}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{or} \qquad \begin{aligned} 0 &\overset{1}{=} 1 - 1\hat{\lambda}, \\ 0 &\overset{2}{=} 0 + 1\hat{\lambda}. \end{aligned}$$

From $\overset{1}{=}$, we have $\hat{\lambda} = 1$, which contradicts $\overset{2}{=}$. So, there is no solution to the above vector equation (or system of two equations).

Hence, our line does **not** contain F. Thus, D, E, and F are **not** collinear.

Exercise 239. In each, determine if the points A, B, and C are collinear.

(a) $A = (3,1)$, $B = (1,6)$, and $C = (0,-1)$.

(b) $A = (1,2)$, $B = (0,0)$, and $C = (3,6)$. (Answer on p. 1849.)

61. The Vector Product

Definition 155. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors. Their *vector product*, denoted $\mathbf{u} \times \mathbf{v}$, is this number:

$$\mathbf{u} \times \mathbf{v} = u_1 v_2 - u_2 v_1.$$

Remark 106. The **vector product** is also called the **cross product**. But your A-Level exams and syllabus don't seem to use this term and so neither shall we. We'll use only the term **vector product**.

Example 827. Let $\mathbf{u} = (5, -3)$, $\mathbf{v} = (2, 1)$, $\mathbf{w} = (-4, 0)$, and $\mathbf{x} = (8, 7)$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5 \cdot 1 - (-3) \cdot 2 = 5 + 6 = 11,$$

$$\mathbf{u} \times \mathbf{w} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \times \begin{pmatrix} -4 \\ 0 \end{pmatrix} = 5 \cdot 0 - (-3) \cdot (-4) = 0 - 12 = -12,$$

$$\mathbf{u} \times \mathbf{x} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \times \begin{pmatrix} 8 \\ 7 \end{pmatrix} = 5 \cdot 7 - (-3) \cdot 8 = 35 + 24 = 59,$$

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 0 \end{pmatrix} = 2 \cdot 0 - 1 \cdot (-4) = 0 + 4 = 4.$$

We now discuss three properties of the vector product.

Recall that multiplication and the scalar product are both **distributive (over addition)** and **commutative**. It turns out that the vector product is also **distributive**:

Example 828. Continuing with the above example:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 - 4 \\ 1 + 0 \end{pmatrix} = 5 \cdot 1 - (-3) \cdot (-2)$$

$$= -1 = 11 + (-12) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}.$$

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \begin{pmatrix} 5 + 2 \\ -3 + 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 0 \end{pmatrix} = 7 \cdot 0 - (-2) \cdot (-4)$$

$$= -8 = -12 + 4 = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}.$$

However, the vector product is **not** commutative. Instead, it is ***anti*-commutative**:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

Example 829. Let $\mathbf{u} = (5, -3)$, $\mathbf{v} = (2, 1)$, and $\mathbf{w} = (-4, 0)$. We already showed that

$$\mathbf{u} \times \mathbf{v} = 11 \qquad \text{and} \qquad \mathbf{w} \times \mathbf{u} = -12.$$

We now show that $\mathbf{v} \times \mathbf{u} = -11$ and $\mathbf{w} \times \mathbf{u} = 12$:

$$\mathbf{v} \times \mathbf{u} = (2, 1) \times (5, -3) = 2 \cdot (-3) - 1 \cdot 5 = -6 - 5 = -11.$$

$$\mathbf{w} \times \mathbf{u} = (-4, 0) \times (5, -3) \ (-4) \cdot (-3) - 0 \cdot 5 = 12 - 0 = 12.$$

The third property is that a vector's vector product with itself is 0:

Example 830. Continuing with the above example, we have

$$\mathbf{u} \times \mathbf{u} = (5, -3) \times (5, -3) = 5 \cdot (-3) - (-3) \cdot 5 = -15 + 15 = 0.$$

$$\mathbf{v} \times \mathbf{v} = (2, 1) \times (2, 1) \ 2 \cdot 1 - 1 \cdot 2 = 2 - 2 = 0.$$

$$\mathbf{w} \times \mathbf{w} = (-4, 0) \times (-4, 0) \ (-4) \cdot 0 - 0 \cdot (-4) = 0 - 0 = 0.$$

In summary,

Fact 145. *Suppose* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors. Then*

(a) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ **(Distributive over Addition)**

(b) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ **(Anti-Commutative)**

(c) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ **(Self Vector Product Is Zero)**

Proof. See Exercise 240. □

Exercise 240. Let $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, and $\mathbf{c} = (c_1, c_2)$ be vectors. Prove the following (Fact 145): (Answer on p. 1850.)

(a) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (b) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (c) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

Fact 145(c) says that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and yields this result:

Corollary 24. *If* $\mathbf{a} \parallel \mathbf{b}$*, then* $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Proof. If $\mathbf{a} \parallel \mathbf{b}$, then there exists $c \neq 0$ such that $c\mathbf{a} = c\mathbf{b}$. So,

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (c\mathbf{a}) = c(\mathbf{a} \times \mathbf{a}) = c \cdot \mathbf{0} = \mathbf{0}. \qquad \square$$

Example 831. Let $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (-2, -4)$. Since $\mathbf{a} \parallel \mathbf{b}$, by Corollary 24, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

It turns out that the converse of Corollary 24 is also true:

Fact 146. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be non-zero vectors. If* $\mathbf{a} \times \mathbf{b} = \mathbf{0}$*, then* $\mathbf{a} \parallel \mathbf{b}$.

Proof. See p. 1628 (Appendices). $\square$

Example 832. Let $\mathbf{a} = (3, -1)$ and $\mathbf{b} = (2, k)$, where k is some unknown constant. We are now told that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. What is k?

By Fact 146, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ implies $\mathbf{a} \parallel \mathbf{b}$. So, $\mathbf{b}$ is a scalar multiple of $\mathbf{a}$.

Hence, $2/3 = k/(-1)$. And so, $k = -2/3$.

For future reference, let's combine Corollary 24 and Fact 146:

Corollary 25. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are non-zero vectors. Then*

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{a} \parallel \mathbf{b}.$$

Here's another "obvious" property of the vector product:

Fact 147. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are vectors and* $c \in \mathbb{R}$*. Then*

$$(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}).$$

Proof. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. Then $c\mathbf{a} = (ca_1, ca_2)$ and

$$(c\mathbf{a}) \times \mathbf{b} = (ca_1) \cdot b_2 - (ca_2) b_1 = c(a_1 b_2 - a_2 b_1) = c(\mathbf{a} \times \mathbf{b}). \qquad \square$$

Exercise 241. Let $\mathbf{a} = (1, -2)$, $\mathbf{b} = (3, 0)$, and $\mathbf{c} = (4, 1)$. Compute $\mathbf{a} \times \mathbf{b}$, $\mathbf{a} \times \mathbf{c}$, $\mathbf{b} \times \mathbf{c}$, $\mathbf{b} \times \mathbf{a}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{c} \times \mathbf{b}$, and $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$. (Answer on p. 1850.)

61.1. The Angle between Two Vectors Using the Vector Product

Recall (Fact 132) that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos\theta$.

With the vector product, we have a very similar result:

Fact 148. *Suppose θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Then*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin\theta.$$

Proof. See Exercise 242. □

Exercise 242. Let θ be the angle between the vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$.

(a) Express $|\mathbf{a}|$, $|\mathbf{b}|$, $|\mathbf{a} \times \mathbf{b}|$, and $\cos\theta$ in terms of a_1, a_2, b_1, and b_2. (You do not need to expand the squared terms.)

(b) Since $\theta \in [0, \pi]$, what can you say about the sign of $\sin\theta$? (That is, is $\sin\theta$ positive, negative, non-positive, or non-negative?)

(c) Now use a trigonometric identity to express $\sin\theta$ in terms of $\cos\theta$. (Hint: You should find that there are two possibilities. Use what you found in **(b)** to explain why you can discard one of these possibilities.)

(d) Plug the expression you wrote down for $\cos\theta$ in **(a)** into what you found in **(c)**.

(e) Prove[333] that $\left(a_1^2 + a_2^2\right)\left(b_1^2 + b_2^2\right) - \left(a_1 b_1 + a_2 b_2\right)^2 = \left(a_1 b_2 - a_2 b_1\right)^2$. (Hint: Simply expand the terms and do the algebra.)

(f) Use **(a)** and **(d)** to express $|\mathbf{a}| |\mathbf{b}| \sin\theta$ in terms of a_1, a_2, b_1, and b_2. Then use **(e)** to prove that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin\theta$. (Answer on p. 1850.)

Take $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin\theta$ and rearrange:

$$\sin\theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|}.$$

Now apply $\sin^{-1}$ to get this result:[334]

[333] By the way, this is simply an instance of Lagrange's Identity.

[334] Corollary 26 adds the condition that $\theta \in [0, \pi/2]$ (whereas the angle θ between two vectors can more generally be between 0 and π, inclusive). The reason for this addition is that Range $\sin^{-1} = [-\pi/2, \pi/2]$, so that if $\theta \in (\pi/2, \pi]$, then clearly the conclusion of Corollary 26 can't be true.

In fact, if $\theta \in (\pi/2, \pi]$, then we instead have $\theta = \pi - \sin^{-1} \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|}$, as we now show:

Proof. Suppose $\theta \in (\pi/2, \pi]$. Let $\beta = \pi - \theta \in [0, \pi/2)$. Then $\sin\theta = \sin(\pi - \beta) = \sin\beta$.
But $\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|} = \sin\beta$. Since $\beta \in [0, \pi/2)$, $\beta = \sin^{-1} \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|}$. Hence, $\theta = \pi - \beta = \pi - \sin^{-1} \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|}$. □

Corollary 26. *Suppose $\theta \in [0, \pi/2]$ is the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$*. Then*

$$\theta = \sin^{-1} \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|}.$$

Corollary 26 is thus the sine or vector product analogue of Definition 145.

Note though that we won't be using Corollary 26 to compute the angle between two vectors. This is because, as we'll see shortly, it's usually easier to compute the scalar product than the vector product.[335] And so, it's easier to just use Definition 145.

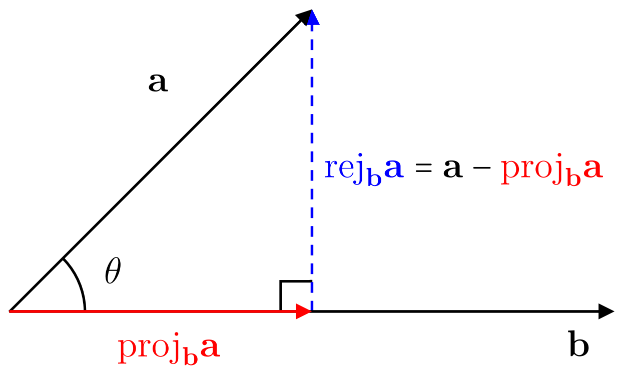
[335]Plus, as noted in the previous footnote, Corollary 26 only covers the case where $\theta \in [0, \pi/2]$.

61.2. The Length of the Rejection Vector

The vector product will be more useful only when we look at 3D space. Nonetheless, even in 2D space, it has this use:

Fact 149. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are vectors. Then*

$$|\text{rej}_{\mathbf{b}}\mathbf{a}| = \left|\mathbf{a} \times \hat{\mathbf{b}}\right|.$$



So, Fact 149 is the vector product analogue of Fact 142.

Proof. By the right-triangle definition of sine,[336]

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{|\text{rej}_{\mathbf{b}}\mathbf{a}|}{|\mathbf{a}|}.$$

Rearrange, then use $\left|\hat{\mathbf{b}}\right| = 1$ and Fact 148:

$$|\text{rej}_{\mathbf{b}}\mathbf{a}| = |\mathbf{a}|\sin\theta = |\mathbf{a}|\left|\hat{\mathbf{b}}\right|\sin\theta = \left|\mathbf{a} \times \hat{\mathbf{b}}\right|. \qquad \square$$

As we'll see next, Fact 149 will help us compute the distance between a point and the line.

[336] For a proof that makes no mention of the sine function, see p. 1627 (Appendices).

62. The Foot of the Perpendicular (from a Point to a Line)

We reproduce from Ch. 15.1 this result:

Corollary 4. *Suppose A is a point not on the line l. Then there exists a point B that is both* **(a)** *the unique point on l that's closest to A; and* **(b)** *the unique point on l such that $l \perp AB$.*

Recall that we also call the point B the **foot of the perpendicular** from A to l. Definition reproduced (also from Ch. 15.1):

Definition 156. Let A be a point that isn't on the line l. The *foot of the perpendicular from A to l* is the (unique) point B on l such that $AB \perp l$.

Earlier in Part I (p. 181), we didn't fully prove Corollary 4.[337] Now, armed with the language of vectors, we can do so quite easily. The proof below

- relies on several previously hard-fought results involving the scalar product;
- may seem a bit abstract and difficult, but only because it involves a bit of vector algebra (if you've had as much practice with vector algebra as you've had with "usual" algebra since primary school, this proof wouldn't seem difficult at all);
- is general in that it proves the claim not only in the case of 2D space, but also for 3D and all higher dimensional spaces.

Proof. Describe l by $R = P + \lambda_r \mathbf{v}$ ($\lambda \in \mathbb{R}$).

So, the vector from A to a point R on l is $\overrightarrow{AR} = \overrightarrow{AO} + \overrightarrow{OR} = \overrightarrow{OP} + \lambda_r \mathbf{v} - \overrightarrow{OA} = \overrightarrow{AP} + \lambda_r \mathbf{v}$.

Hence, $\left|\overrightarrow{AR}\right| = \left|\overrightarrow{AP} + \lambda_r \mathbf{v}\right| = \sqrt{\left(\overrightarrow{AP} + \lambda_r \mathbf{v}\right) \cdot \left(\overrightarrow{AP} + \lambda_r \mathbf{v}\right)} = \sqrt{\left|\overrightarrow{AP}\right|^2 + 2\lambda_r \overrightarrow{AP} \cdot \mathbf{v} + \lambda_r^2 |\mathbf{v}|^2}$.

Since $\sqrt{\cdot}$ is increasing, the above surd expression is minimised when the quadratic inside is minimised. And by Fact 34(d), the quadratic is minimised at

$$\lambda_s = -\frac{\text{“}b\text{”}}{2\text{“}a\text{”}} = -\frac{2\overrightarrow{AP} \cdot \mathbf{v}}{2|\mathbf{v}|^2} = -\frac{\overrightarrow{AP} \cdot \mathbf{v}}{|\mathbf{v}|^2}.$$

Thus, there exists a unique point $S = P + \lambda_s \mathbf{v} \stackrel{\star}{=} P - \dfrac{\overrightarrow{AP} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$ on l that minimises the distance between A and l. In other words, S is the point on l that's closest to A.

Next, $\overrightarrow{AR} \cdot \mathbf{v} = \left(\overrightarrow{AP} + \lambda_r \mathbf{v}\right) \cdot \mathbf{v} = \overrightarrow{AP} \cdot \mathbf{v} + \lambda_r |\mathbf{v}|^2$.

So, $\overrightarrow{AR} \perp l \iff \overrightarrow{AR} \perp \mathbf{v} \iff \overrightarrow{AR} \cdot \mathbf{v} = 0 \iff \overrightarrow{AP} \cdot \mathbf{v} + \lambda_t |\mathbf{v}|^2 = 0 \iff \lambda_t = -\overrightarrow{AP} \cdot \mathbf{v} / |\mathbf{v}|^2$.
Hence, there exists a unique point $T = P + \lambda_t \mathbf{v}$ on l such that $AT \perp l$.

Now, observe $\lambda_s = \lambda_t$. So, $S = T$ is the point B with the claimed properties **(a)** and **(b)**. □

[337] We did in the Appendices but not in the main text.

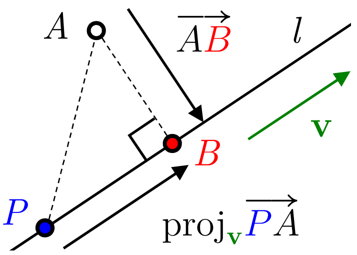
We reproduce from Ch. 15.1 this result:

Corollary 5. *Let l be the line $ax + by + c = 0$. Suppose A is a point not on l. Then the point B that is both* **(a)** *the unique point on l that's closest to A; and* **(b)** *the unique point on l such that $l \perp AB$ is*

$$B = \left(p - a\frac{ap + bq + c}{a^2 + b^2}, q - b\frac{ap + bq + c}{a^2 + b^2} \right).$$

(To repeat our warnings from Ch. 15.1, do *not* try mugging the above result. Instead, understand and remember the methods by which we can derive the above result—to be reviewed in the examples below.)

We can now restate the above result in the language of vectors:



Fact 150. *Let l be the line $R = P + \lambda \mathbf{v}$ $(\lambda \in \mathbb{R})$. Suppose A is a point not on l. Then the point B that is* **(a)** *the unique point on l that's closest to A; and* **(b)** *the unique point on l such that $l \perp AB$ is*

$$B = P + \text{proj}_{\mathbf{v}} \overrightarrow{PA}.$$

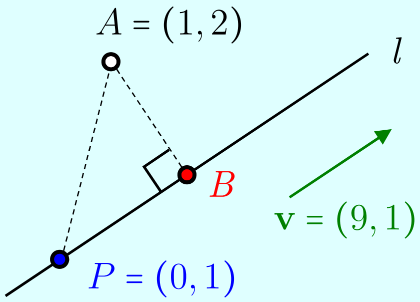
Proof. By Definition 152, $\text{proj}_{\mathbf{v}} \overrightarrow{PA} \stackrel{1}{=} \dfrac{\overrightarrow{PA} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = -\dfrac{\overrightarrow{AP} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$

Taking $\stackrel{\star}{=}$ from the previous proof, we have $B \stackrel{\star}{=} P - \dfrac{\overrightarrow{AP} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} \stackrel{1}{=} P + \text{proj}_{\mathbf{v}} \overrightarrow{PA}$. □

Fact 150 gives us an intuitive and easy-to-remember method for finding the point B:

Example 833. Let $A = (1,2)$ be a point and l be the line $\mathbf{r} = \overrightarrow{OP} + \lambda\mathbf{v} = (0,1) + \lambda(9,1)$ $(\lambda \in \mathbb{R})$. Let B be the point on l that's closest to A (B is also the foot of the perpendicular from A to l).

We'll find B using four methods:



Method 1 (Projection Vector). Compute $\overrightarrow{PA} = (1,2) - (0,1) = (1,1)$ and

$$\text{proj}_{\mathbf{v}}\overrightarrow{PA} = \text{proj}_{(9,1)}(1,1) = \frac{(1,1)\cdot(9,1)}{9^2+1^2}\begin{pmatrix} 9 \\ 1 \end{pmatrix} = \frac{9+1}{82}(9,1) = \frac{5}{41}(9,1).$$

Now simply apply Fact 150:

$$B = P + \text{proj}_{\mathbf{v}}\overrightarrow{PA} = (0,1) + \frac{5}{41}(9,1) = \frac{1}{41}(45,46).$$

Method 2 (Quadratic-Cartesian). In Part I, we learnt to find B this way:

The line l has cartesian equations $x = 9\lambda$ and $y = 1 + \lambda$—or, $y = x/9 + 1$.

Pick any arbitrary point $R = (r, r/9 + 1)$ on l.

The distance between A and R is $\sqrt{(1-r)^2 + (2 - r/9 - 1)^2} = \sqrt{82r^2/81 - 20r/9 + 2}$.

This last surd expression is minimised when the quadratic expression inside is minimised. And by Fact 34(d), this occurs when

$$r = -\frac{\text{“}b\text{”}}{2\text{“}a\text{”}} = -\frac{-20/9}{2(82/81)} = \frac{90}{82} = \frac{45}{41}.$$

Hence, $$B = \left(\frac{45}{41}, \frac{5}{41} + 1\right) = \left(\frac{45}{41}, \frac{46}{41}\right).$$

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Method 3 (Quadratic-Vector). Pick any arbitrary point $R = (0,1) + \lambda(9,1)$ on l. Then the distance between A and R is

$$\left|\overrightarrow{AR}\right| = \left|\overrightarrow{OR} - \overrightarrow{OA}\right| = |(0,1) + \lambda(9,1) - (1,2)| = |(9\lambda - 1, \lambda - 1)|$$

$$= \sqrt{(9\lambda - 1)^2 + (\lambda - 1)^2} = \sqrt{82\lambda^2 - 20\lambda + 2}.$$

As usual, this last surd expression is minimised when

$$\lambda = -\frac{\text{“}b\text{”}}{2\text{“}a\text{”}} = -\frac{-20}{2 \times 82} = \frac{5}{41}.$$

Hence, $$B = (0,1) + \frac{5}{41}(9,1) = \frac{1}{41}(45,46).$$

Method 4 (Scalar Product). Let $B = (0,1) + \lambda_b(9,1)$. Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (9\lambda_b - 1, \lambda_b - 1)$.

We have $\overrightarrow{AB} \perp l$ or $\overrightarrow{AB} \perp (9,1)$ or $\overrightarrow{AB} \cdot (9,1) = 0$ or

$$0 = (9\lambda_b - 1, \lambda_b - 1) \cdot (9,1) = 81\lambda_b - 9 + \lambda_b - 1 = 82\lambda_b - 10 \qquad \text{or} \qquad \lambda_b = 10/82 = 5/41.$$

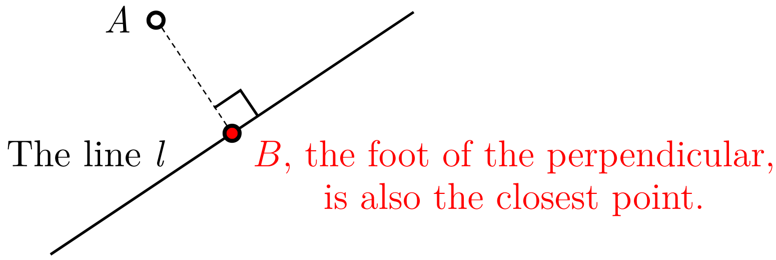
Hence, $$B = (0,1) + \frac{5}{41}(9,1) = \frac{1}{41}(45,46).$$

Exercise 243. Find the feet of the perpendiculars from the points $A = (-1,0)$ and $B = (3,2)$ to the line $\mathbf{r} = \overrightarrow{OP} + \lambda\mathbf{v} = (2,-3) + \lambda(5,1)$ $(\lambda \in \mathbb{R})$. (Answer on p. 1851.)

62.1. The Distance Between a Point and a Line

We reproduce from Ch. 15 this definition:

Definition 157. The *distance between a point A and a graph G* is the distance between A and B, where B is the point on G that's closest to A.



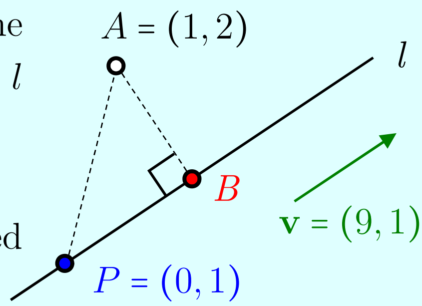
Let B be the point on the line l that's closest to the point A (and is also the foot of the perpendicular from A to l).

Then by the above definition, the distance between A and l is simply the distance between A and B—or equivalently, the length of the vector $\overrightarrow{AB}$. Let's jot this down as a formal result:

Corollary 27. *Suppose B is the point on the line l that's closest to the point A. Then the distance between A and l is $\left|\overrightarrow{AB}\right|$.*

We revisit Example 833:

Example 834. Let $A = (1,2)$ be a point and l be the line $\mathbf{r} = \overrightarrow{OP} + \lambda\mathbf{v} = (0,1) + \lambda(9,1)$ $(\lambda \in \mathbb{R})$. Let B be the point on l that's closest to A.



We'll find the distance between A and l using *five* methods.

Methods 1–4 will be the same as the four methods we used earlier for finding B.

Method 5 (new) uses the vector product.

Method 1 (Projection Vector). Compute $\overrightarrow{PA} = (1,2) - (0,1) = (1,1)$ and

$$\text{proj}_{\mathbf{v}}\overrightarrow{PA} = \text{proj}_{(9,1)}(1,1) = \frac{(1,1)\cdot(9,1)}{9^2+1^2}\begin{pmatrix} 9 \\ 1 \end{pmatrix} = \frac{9+1}{82}(9,1) = \frac{5}{41}(9,1).$$

Now by Fact 150, $B = P + \text{proj}_{\mathbf{v}}\overrightarrow{PA} = (0,1) + \frac{5}{41}(9,1) = \frac{1}{41}(45,46)$.

So, the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \left|\frac{1}{41}(4,-36)\right| = \frac{4}{41}|(1,-9)| = \frac{4}{41}\sqrt{1^2+(-9)^2} = \frac{4}{41}\sqrt{82} \approx 0.883.$$

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

With Method 1, we first explicitly find B, then compute $\left|\overrightarrow{AB}\right|$.

In contrast, with Methods 2–5, we'll find $\left|\overrightarrow{AB}\right|$ without explicitly finding B (but of course, if we wanted to, we could also easily find B):

Method 2 (Quadratic-Cartesian). The line l has cartesian equations $x = 9\lambda$ and $y = 1 + \lambda$—or, $y = x/9 + 1$.

Now, pick any arbitrary point $R = (r, r/9 + 1)$ on l.

The distance between A and R is $\sqrt{(1-r)^2 + (2 - r/9 - 1)^2} = \sqrt{82r^2/81 - 20r/9 + 2}$.

As usual, this last surd expression is minimised when

$$r = -\frac{\text{“}b\text{”}}{2\text{“}a\text{”}} = -\frac{-20/9}{2\left(82/81\right)} = \frac{90}{82} = \frac{45}{41}.$$

So, the minimised distance or more simply distance between A and l is

$$\sqrt{\frac{82}{81}\left(\frac{45}{41}\right)^2 - \frac{20}{9}\left(\frac{45}{41}\right) + 2} = \sqrt{2 \times \frac{25}{41} - \frac{100}{41} + 2} = \sqrt{-\frac{50}{41} + 2} = \sqrt{-\frac{50}{41} + 2} = \sqrt{\frac{32}{41}} \approx 0.883.$$

Method 3 (Quadratic-Vector). Pick any arbitrary point $R = (0,1) + \lambda(9,1)$ on l. The distance between A and R is

$$\left|\overrightarrow{AR}\right| = \left|\overrightarrow{OR} - \overrightarrow{OA}\right| = |(0,1) + \lambda(9,1) - (1,2)| = |(9\lambda - 1, \lambda - 1)|$$
$$= \sqrt{(9\lambda - 1)^2 + (\lambda - 1)^2} = \sqrt{82\lambda^2 - 20\lambda + 2}.$$

As usual, this last surd expression is minimised when

$$\lambda = -\frac{\text{“}b\text{”}}{2\text{“}a\text{”}} = -\frac{-20}{2 \times 82} = \frac{5}{41}.$$

So, the minimised distance or more simply distance between A and l is

$$\sqrt{82\left(\frac{5}{41}\right)^2 - 20\left(\frac{5}{41}\right) + 2} = \sqrt{\frac{2 \times 25}{41} - \frac{100}{41} + 2} = \sqrt{-\frac{50}{41} + 2} = \sqrt{\frac{32}{41}}.$$

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Method 4 (Scalar Product). Let $B = (0,1) + \lambda_b(9,1)$. Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (9\lambda_b - 1, \lambda_b - 1)$.

We have $\overrightarrow{AB} \perp l$ or $\overrightarrow{AB} \perp (9,1)$ or $\overrightarrow{AB} \cdot (9,1) = 0$ or

$$0 = (9\lambda_b - 1, \lambda_b - 1) \cdot (9,1) = 81\lambda_b - 9 + \lambda_b - 1 = 82\lambda_b - 10 \qquad \text{or} \qquad \lambda_b = 10/82 = 5/41.$$

So, the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \sqrt{(9\lambda_b - 1)^2 + (\lambda_b - 1)^2} = \sqrt{82\lambda_b^2 - 20\lambda_b + 2}$$
$$= \sqrt{82\left(\frac{5}{41}\right)^2 - 20\left(\frac{5}{41}\right) + 2} = \sqrt{2 \times \frac{25}{41} - \frac{100}{41} + 2} = \sqrt{-\frac{50}{41} + 2} = \sqrt{\frac{32}{41}}.$$

Method 5 (Vector Product). Recall that $\overrightarrow{AB}$ is $\text{rej}_{\mathbf{v}}\overrightarrow{PA}$—the rejection of $\overrightarrow{PA}$ on $\mathbf{v}$. Recall also Fact 149: $\left|\text{rej}_{\mathbf{v}}\overrightarrow{PA}\right| = \left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right|$. So,

$$\left|\overrightarrow{AB}\right| = \left|\text{rej}_{\mathbf{v}}\overrightarrow{PA}\right| = \left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right| = \left|(1,1) \times (9,1)/\sqrt{9^2 + 1^2}\right|$$
$$= |1 \times 1 - 1 \times 9| / \sqrt{82} = 8/\sqrt{82} \approx 0.883.$$

Methods 1–4 don't yield any neat and easy-to-remember general formulae, so we won't bother writing down any general formal result for them.

But Method 5 does, so let's jot down this formal result:

Corollary 28. *Let A be a point. Suppose l is the line $R = P + \lambda\mathbf{v}$ $(\lambda \in \mathbb{R})$. Then the distance between A and l is* $\left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right|$.

Another example illustrating all five methods:

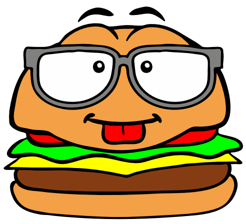
Example 835. XXX

Exercise 244. In each of the following, a point A and line l are given. Let B be the foot of the perpendicular from A to l. Find B and also the distance between A and l.

	The point A	**The line** l	**Answer on p.**
(a)	$(7,3)$	$\mathbf{r} = (8,3) + \lambda(9,3)$	1852.
(b)	$(8,0)$	Contains the points $(4,4)$ and $(6,11)$	1853.
(c)	$(8,5)$	$\mathbf{r} = (8,4) + \lambda(5,6)$	1854.

So far, we've looked only at **two-dimensional (2D) space** (or the **cartesian plane**). In the remainder of Part III, we'll look instead at **three-dimensional (3D) space**.

I will also often make use of Paul Seeburger's CalcPlot3D.[338] Whenever you see a tiny version of this icon:



click/touch it[339] and you'll be brought to the relevant 3D graph, where you can pan, zoom, rotate, etc., so as to get a better sense of what the 3D graph looks like.

(2021-11-06: Unfortunately Seeburger moved his website and all of these links are now broken. I'll fix them when I get the time.)

[338] I've examined dozens of 3D graphing software and all things considered (user-friendliness, accessibility, features, etc.), this is the best 3D graphing web app I've found so far. Please let me know if you know of any other better software/app. (I was gonna use GeoGebra, but it had too many critical flaws.)

[339] Note that you'll be routed through TinyURL.com first. The reason is that the CalcPlot3D links are often thousands of characters long and were confusing my computer.

63. Three-Dimensional (3D) Space

In 2D space, we had ordered pairs. In 3D space, we'll instead have **ordered triples**:[340]

Definition 158. Given an ordered triple (a, b, c), we call a its *first* or *x-coordinate*, b its *second* or *y-coordinate*, and c its third or *z-coordinate*.

Example 836. The ordered triple (Cow, Chicken, Dog) has x-coordinate Cow, y-coordinate Chicken, and z-coordinate Dog. As with ordered pairs, **the order of the coordinates matters**. So for example,

$$(\text{Cow, Chicken, Dog}) \neq (\text{Cow, Dog, Chicken}) \neq (\text{Chicken, Cow, Dog}).$$

In contrast, with a set of three elements, order doesn't matter:

$$\{\text{Cow, Chicken, Dog}\} = \{\text{Cow, Dog, Chicken}\} = \{\text{Chicken, Cow, Dog}\}.$$

Example 837. The ordered triple $(2, 5, -\pi)$ has x-coordinate 2, y-coordinate 5, and z-coordinate $-\pi$. Again, order matters, so that for example,

$$(2, 5, -\pi) \neq (2, -\pi, 5) \neq (5, 2, -\pi).$$

Again, in contrast, with a set of three elements, order doesn't matter:

$$\{2, 5, -\pi\} = \{2, -\pi, 5\} = \{5, 2, -\pi\}.$$

In 2D space, a **point** was simply any ordered pair of real numbers. Now in 3D space,

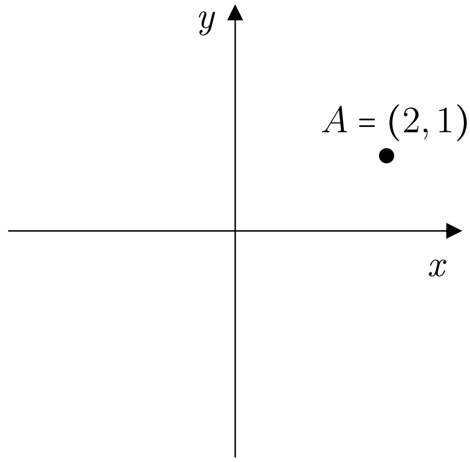
Definition 159. A *point* is any ordered triple of real numbers.

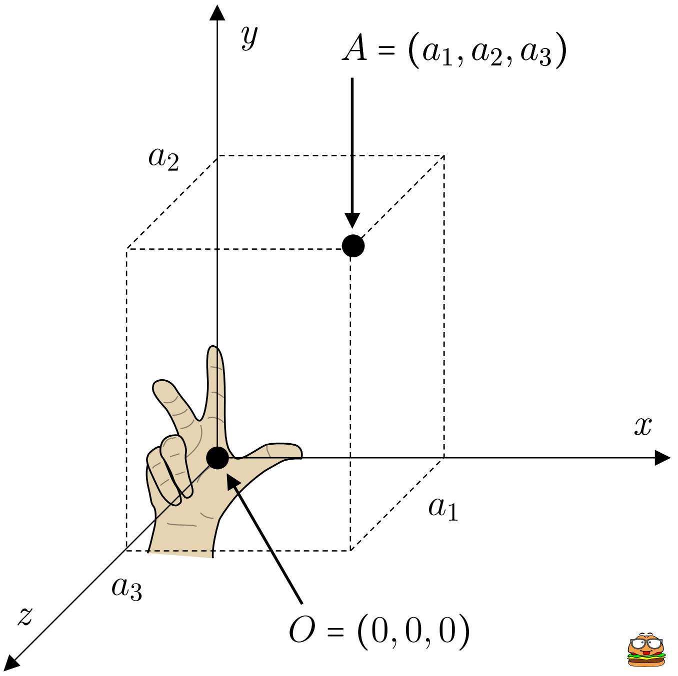
Example 838. The ordered triple (Cow, Chicken, Dog) is not a point because at least one of its coordinates is not a real number. (Indeed, all three aren't.)

Example 839. The ordered triple $(2, 5, -\pi)$ is a point because all three of its coordinates are real numbers.

[340]For the formal definition of an ordered triple (and n-tuple), see Definition 267 (Appendices).

In 2D space (the cartesian plane), we could depict points (ordered pairs of real numbers) by drawing on a piece of paper. The x**-axis** went right and the y**-axis** up.





In 3D space, we can again depict points (now ordered *triples* of real numbers) by drawing on a piece of paper. Again, the x-axis goes right and the y-axis up. But now, we also have the z-axis, which "comes out of the paper towards your face" and is perpendicular to both the x- and y-axes.

We say that this coordinate system follows the **right-hand rule**. To see why, have the palm of your right hand face you. Fold your ring and pinky fingers. Have your thumb point right, your index finger up, and your middle finger towards your face. Then these three fingers correspond to the x-, y-, and z-axes. (Try it!)

(If instead the z-axis "goes into the paper away from your face", then our coordinate system would instead follow the left-hand rule. Can you explain why?)

In 2D space, the **origin** was the point $O = (0, 0)$ (Definition 37) and was where the x- and y-axes intersected. And the generic point $A = (a_1, a_2)$ was a_1 units to the right and a_2 units above the origin.

Analogously, in 3D space,

Definition 160. The *origin* is the point $O = (0, 0, 0)$.

In 3D space, the origin is where the x-, y-, and z-axes intersect. And relative to the origin, the generic point $A = (a_1, a_2, a_3)$ is a_1 units right, a_2 units up, and a_3 units "out (towards your face)".

63.1. Graphs (in 3D)

In 2D space, a **graph** was any set of points, where points were ordered pairs of real numbers (Definition 38).

This remains true in 3D space, where a graph is *any* set of points (the only difference being that points are now ordered triples of real numbers). We reproduce from Ch. 7.3 this definition:

Definition 38. A *graph* (or *curve*) is any set of points.

Example 840. XXX

Example 841. XXX

In 2D space, the graph of an equation was the set of points for which the equation was true (Definition 39).

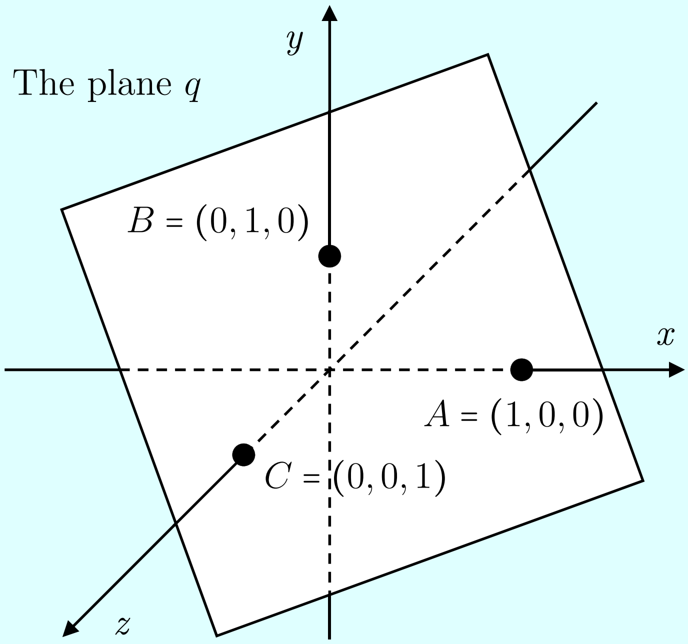
This remains true in 3D space (again, the only difference between that points are now ordered triples of real numbers). We reproduce from Ch. 7.4 this definition:

Definition 39. The *graph of an equation* is the set of points for which the equation is true.

Shortly, we'll learn about the equations (and systems of equations) used to describe planes (and lines). For now, here are two quick examples:

Example 842. Consider the equation:

$$x + y + z = 1.$$



It turns out that this equation describes a plane q in 3D space. (We'll learn more about this in Ch. 70.)

Specifically, q contains exactly those points (x, y, z) that satisfy

$$x + y + z = 1.$$

So for example, it contains the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

A little more formally, the plane q is a set:

$$q = \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}, x + y + z = 1\}.$$

In words, q is the set of ordered triples (x, y, z) such that x, y, and z are real numbers satisfy $x + y + z = 1$.

As with ordered pairs, we'll usually look only at ordered triples of real numbers, i.e. points. And so, we'll be a little lazy/sloppy and not bother mentioning that x, y, and z are real numbers. That is, we'll usually more simply write

$$q = \{(x, y, z) : x + y + z = 1\}.$$

In words, q is the set of points (x, y, z) such that x, y, and z satisfy $x + y + z = 1$.

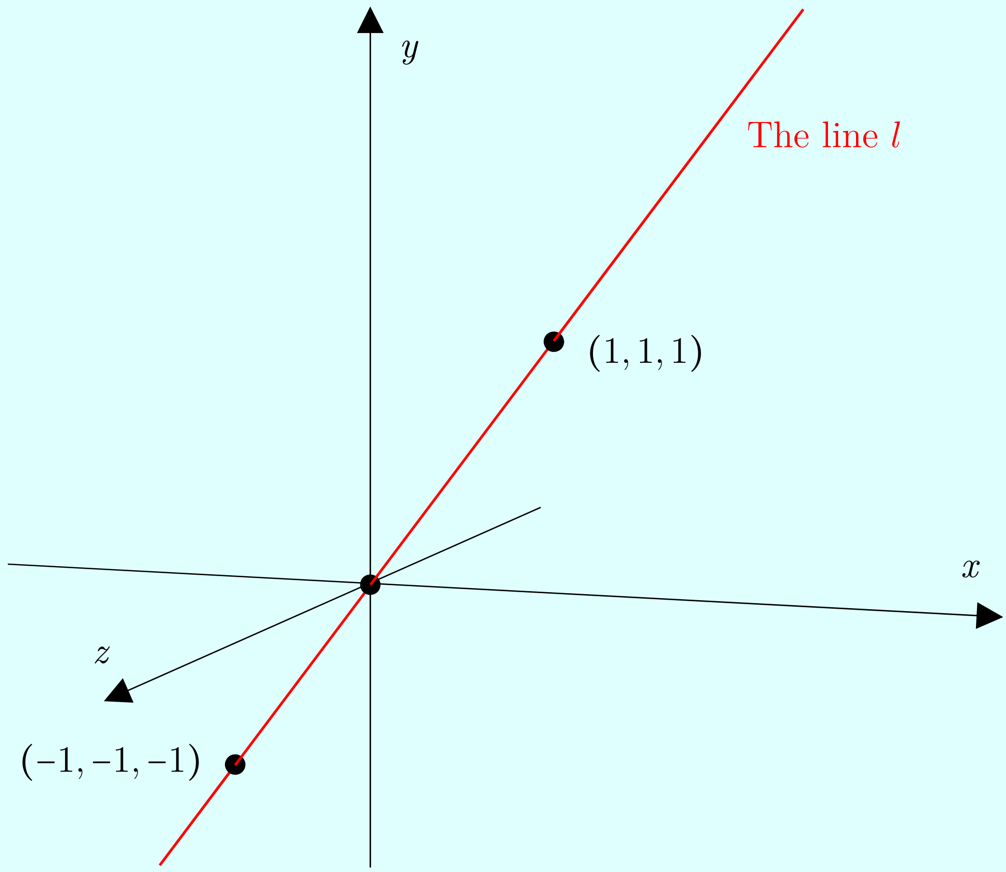
Example 843. Consider this system of (two) equations (in three variables):

$$x = y \qquad \text{and} \qquad y = z.$$

Or equivalently and more simply, $\qquad x = y = z.$

It turns out that this system of (two) equations describes a line l in 3D space. (We'll learn more about this in Ch. 67.) The line l contains exactly those points that can be written as $(\lambda, \lambda, \lambda)$, for some real number λ. ☺

So, for example, it contains the points $(1, 1, 1)$, $O = (0, 0, 0)$, and $(-1, -1, -1)$.



A little more formally, the line l is a set:

$$l = \{(x, y, z) : x = y = z\}.$$

In words, l is the set of ordered triples (x, y, z) such that x, y, and z are real numbers that satisfy $x = y = z$.

As per ☺ above, we can also write

$$l = \{(\lambda, \lambda, \lambda) : \lambda \in \mathbb{R}\} = \{\lambda (1, 1, 1) : \lambda \in \mathbb{R}\}.$$

In words, l is the set of points that can be written as $(\lambda, \lambda, \lambda)$ or $\lambda (1, 1, 1)$ for some real number λ.

64. Vectors (in 3D)

We now give the basic definitions and results concerning vectors in 3D space. Everything we learnt about vectors in 2D space finds its analogy in 3D space. (Indeed, we'll simply reproduce verbatim many of the definitions and results from before).

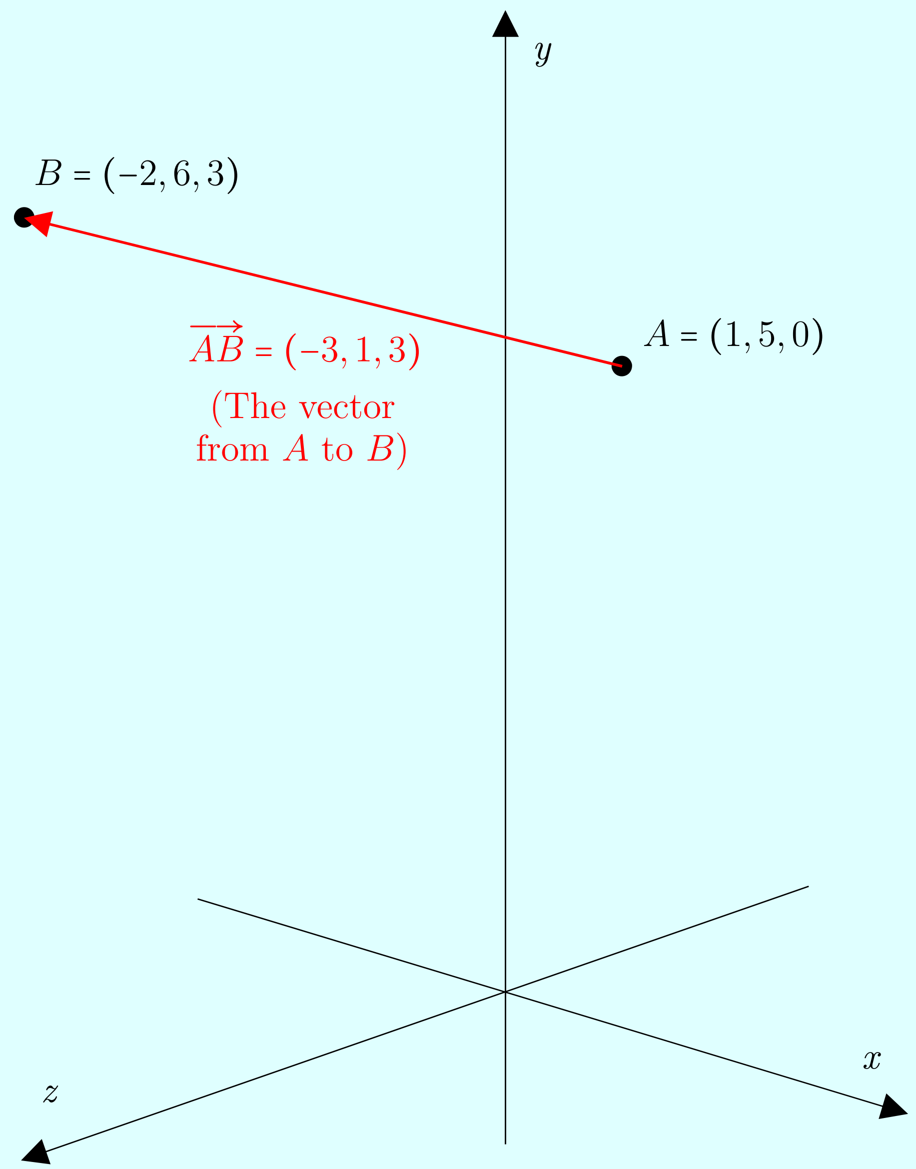
Most of the time, the analogy is obvious. We will therefore go fairly briskly in this chapter.

Definition 161. Given the points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, the *vector from A to B* is $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$.

Example 844. The vector from the point $A = (1, 5, 0)$ to the point $B = (-2, 6, 3)$ is

$$\overrightarrow{AB} = (-3, 1, 3) = \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} = \mathbf{u}.$$

(Observe that there are, again, at least four ways to denote a single vector.)



Definition 125. A *scalar* is any real number.

We may again contrast vectors with scalars: In 3D space, vectors are three-dimensional objects, while scalars are one-dimensional.

Definition 127. Given a point A, its *position vector* is the vector $\overrightarrow{OA}$.

And so, the point $A = (a_1, a_2, a_3)$ has position vector $\overrightarrow{OA} = \mathbf{a} = (a_1, a_2, a_3)$.

Example 845. The point $A = (1, 5, 0)$ has position vector $\overrightarrow{OA} = \mathbf{a} = (1, 5, 0)$.

Once again, do not confuse a point (a zero-dimensional object) with a vector (a three-dimensional object).

Definition 128. The *zero vector*, denoted $\mathbf{0}$, is the origin's position vector.

And so, in 3D space, the zero vector is $\mathbf{0} = \overrightarrow{OO} = (0, 0, 0)$.

Definition 129. Suppose a moving object starts at point A and ends at point B. Then we call $\overrightarrow{AB}$ its *displacement vector*.

And so, if a moving object starts at $A = (a_1, a_2, a_3)$ and ends at $B = (b_1, b_2, b_3)$, then its displacement vector is $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$.

Exercise 245. Let $A = (2, 5, 8)$ and $B = (0, 1, 1)$ be points.

(a) What is the vector from A to B?

(b) What are the position vectors of A and B?

(c) If a particle starts at A, travels to B, then travels back to A and stops there, then what is its displacement vector? (Answer on p. 1855.)

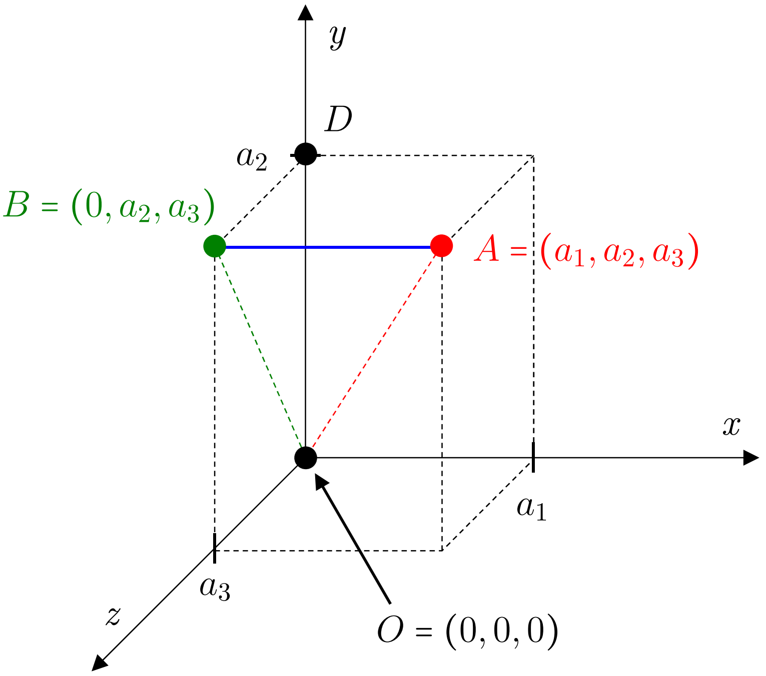
64.1. The Magnitude or Length of a Vector

In 2D space, the length of a vector $\mathbf{u} = (u_1, u_2)$ was defined as $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$ (Def. 126). Our Definition of a vector's length in 3D space is very much analogous:

Definition 162. Given the vector $\mathbf{u} = (u_1, u_2, u_3)$, its *magnitude* or *length*, denoted $|\mathbf{u}|$, is this number:

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Example 846. If $\mathbf{u} = (1, 2, 3)$, then the length of $\mathbf{u}$ is $|\mathbf{u}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.



To see why the above definition makes sense, pick any point $A = (a_1, a_2, a_3)$.

It is natural to define the length of the vector $\overrightarrow{OA}$ to be the length of the line segment OA, i.e. $|OA|$.

Our goal then is to find $|OA|$. To do so, we first consider the point $B = (0, a_2, a_3)$.

Observe that the line segment OB is the hypotenuse of the right triangle ODB. And so, by Pythagoras' Theorem,

$$|OB| = \sqrt{a_2^2 + a_3^2}.$$

Next, the line segment OA is the hypotenuse of the right triangle OBA. Moreover, $|BA| = a_1$. And so, again by Pythagoras' Theorem,

$$|OA| = \sqrt{|BA|^2 + |OB|^2} = \sqrt{a_1^2 + \left(\sqrt{a_2^2 + a_3^2}\right)^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

This completes our explanation of why the above Definition makes sense.

As before, the length of every vector must be non-negative. Moreover, a vector has zero length if and only if it is the zero vector:

Fact 117. *Suppose* $\mathbf{v}$ *is a vector. Then* $|\mathbf{v}| \geq 0$. *Moreover,* $|\mathbf{v}| = 0 \iff \mathbf{v} = \mathbf{0}$.

Exercise 246. Let $A = (2, 5, 8)$ and $B = (0, 1, 1)$ be points. What is the length of the vector from A to B? (Answer on p. 1855.)

64.2. Sums and Differences of Points and Vectors

As before,

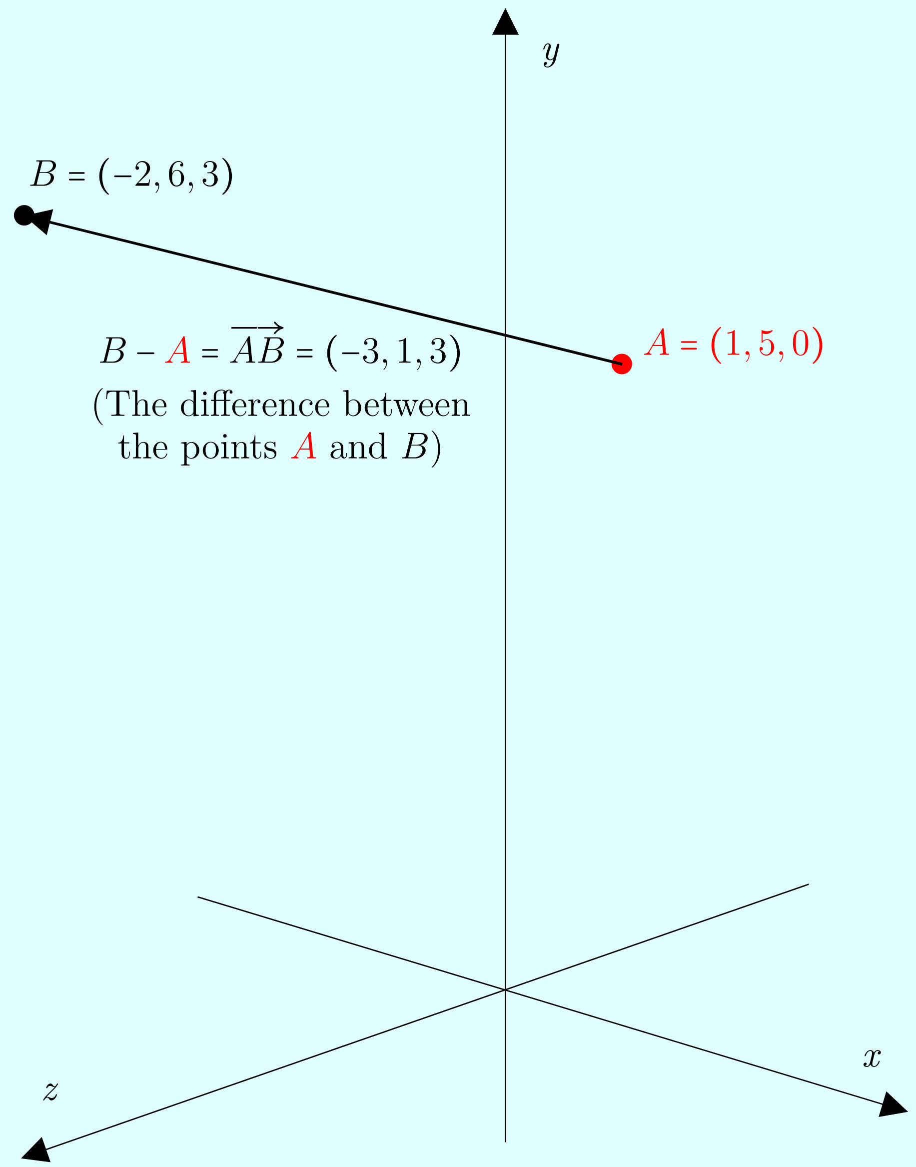
1. **Point + Point = Undefined.**
2. **Point − Point = Vector.**
3. **Point + Vector = Point.**
4. **Point − Vector = Point.**

Definition 130. Given two points A and B, the *difference* $B - A$ is the vector $\overrightarrow{AB}$.

And so, given the points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, their difference is

$$B - A = \overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3).$$

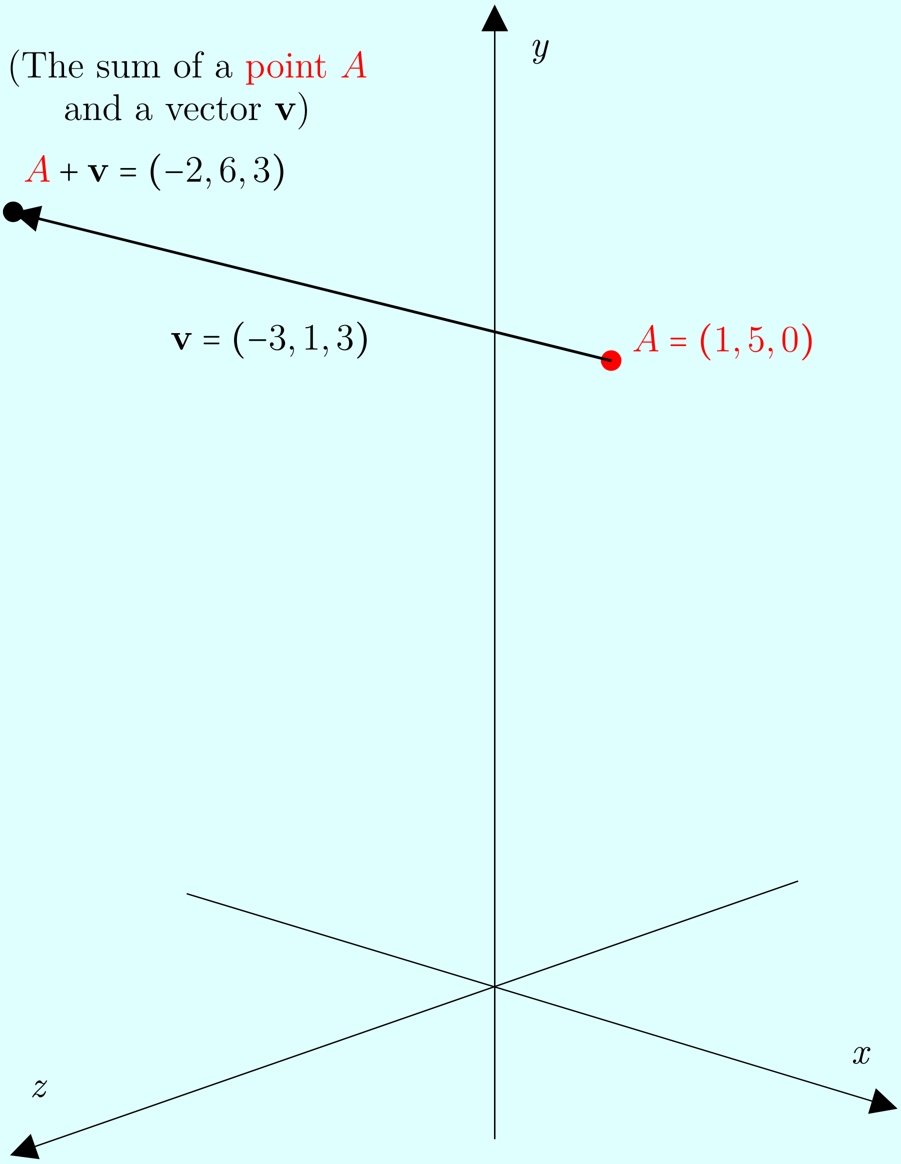
Example 847. Given the points $A = (1, 5, 0)$ and $B = (-2, 6, 3)$, their difference is the vector $B - A = \overrightarrow{AB} = (-2 - 1, 6 - 5, 3 - 0) = (-3, 1, 3)$.



Definition 163. Given a point $A = (a_1, a_2, a_3)$ and a vector $\mathbf{v} = (v_1, v_2, v_3)$, their *sum* $A + \mathbf{v}$ is this point:

$$A + \mathbf{v} = (a_1 + v_1, a_2 + v_2, a_3 + v_3).$$

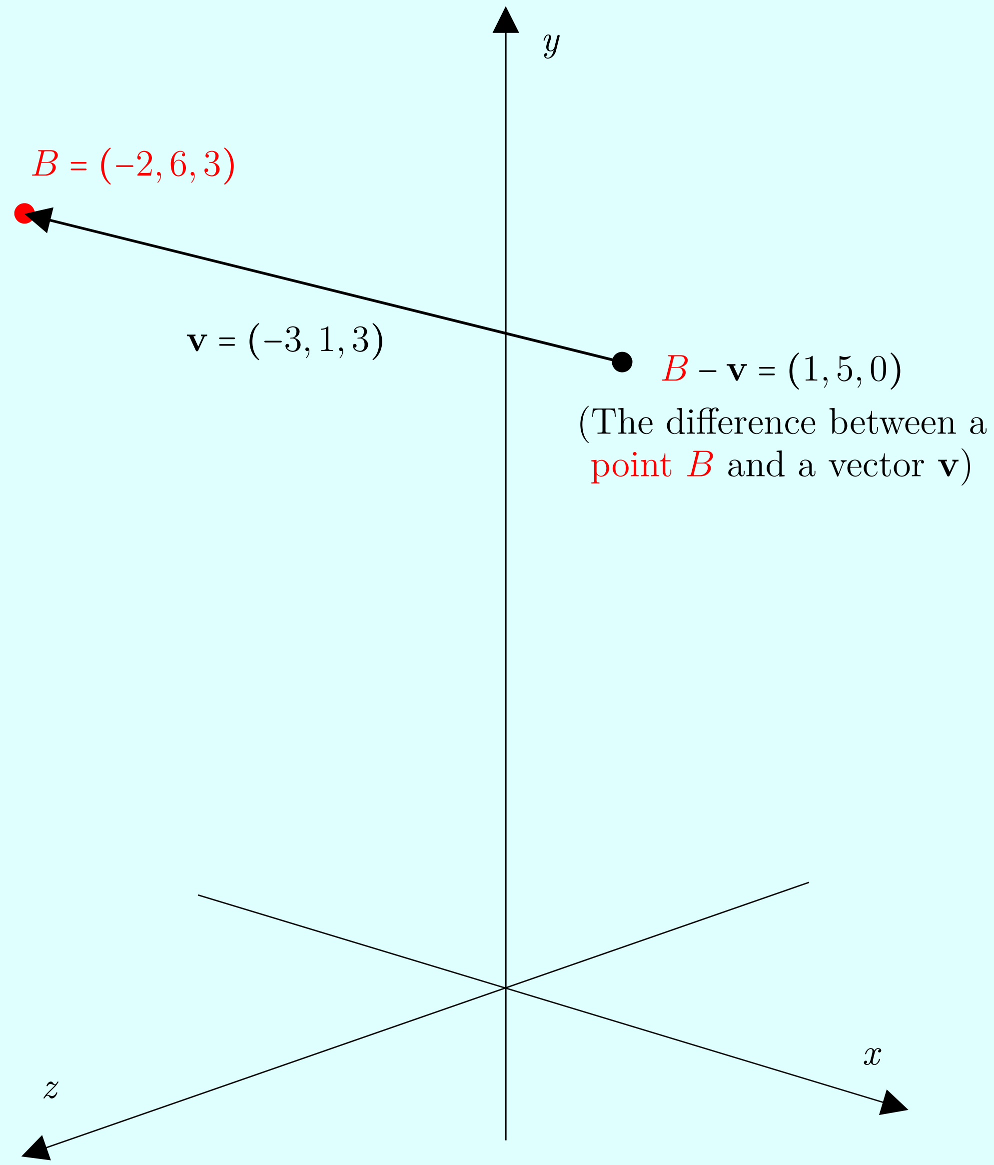
Example 848. Given the point $A = (1, 5, 0)$ and the vector $\mathbf{v} = (-3, 1, 3)$, their *sum* is the point $A + \mathbf{v} = (1 - 3, 5 + 1, 0 + 3) = (-2, 6, 3)$.



Definition 164. Given a point $B = (b_1, b_2, b_3)$ and a vector $\mathbf{v} = (v_1, v_2, v_3)$, their *difference* $B - \mathbf{v}$ is this point:

$$B - \mathbf{v} = (b_1 - v_1, b_2 - v_2, b_3 - v_3).$$

Example 849. Given the point $B = (-2, 6, 3)$ and the vector $\mathbf{v} = (-3, 1, 3)$, their *difference* is the point $B - \mathbf{v} = (-2 - (-3), 6 - 1, 3 - 3) = (1, 5, 0)$.



Exercise 247. Let $A = (1, 2, 3)$, $B = (-1, 0, 7)$, and $C = (5, -2, 3)$ be points. What are **(a)** $A + B$; **(b)** $A - B$; **(c)** $A + (B + C)$; and **(d)** $A + (B - C)$? (Answer on p. 1855.)

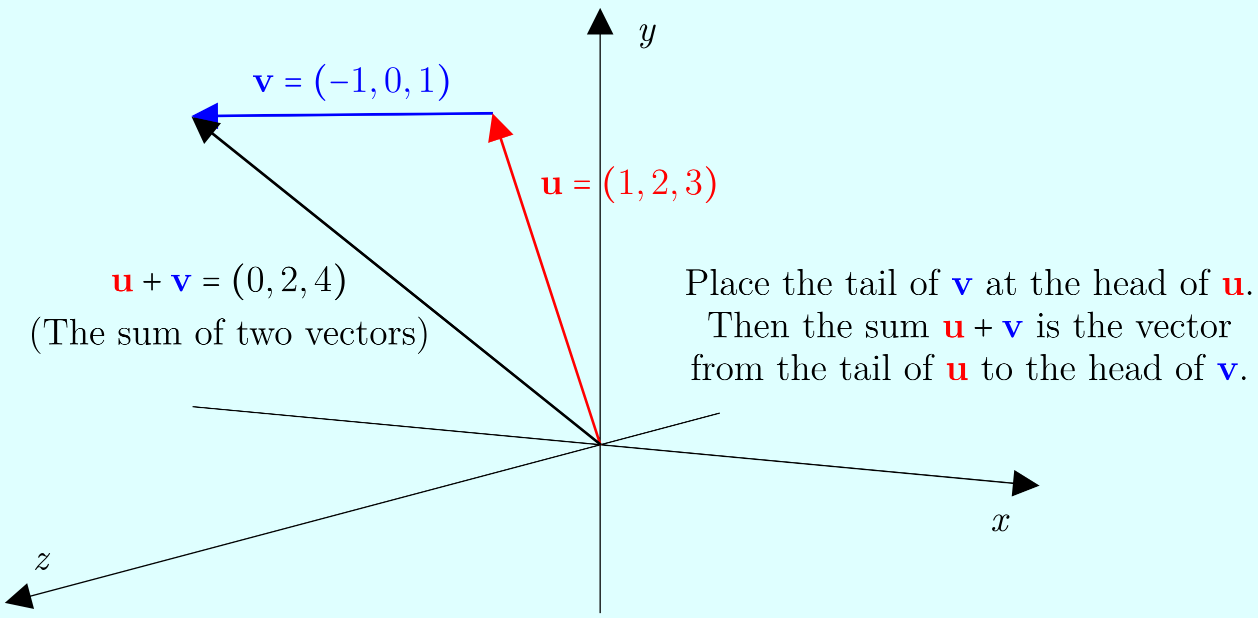
64.3. Sum, Additive Inverse, and Difference of Vectors

As before,

1. **Vector + Vector = Vector.**
2. **− Vector = Vector.**
3. **Vector − Vector = Vector.**

Definition 165. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors. Then their sum, denoted $\mathbf{u} + \mathbf{v}$, is the vector $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$.

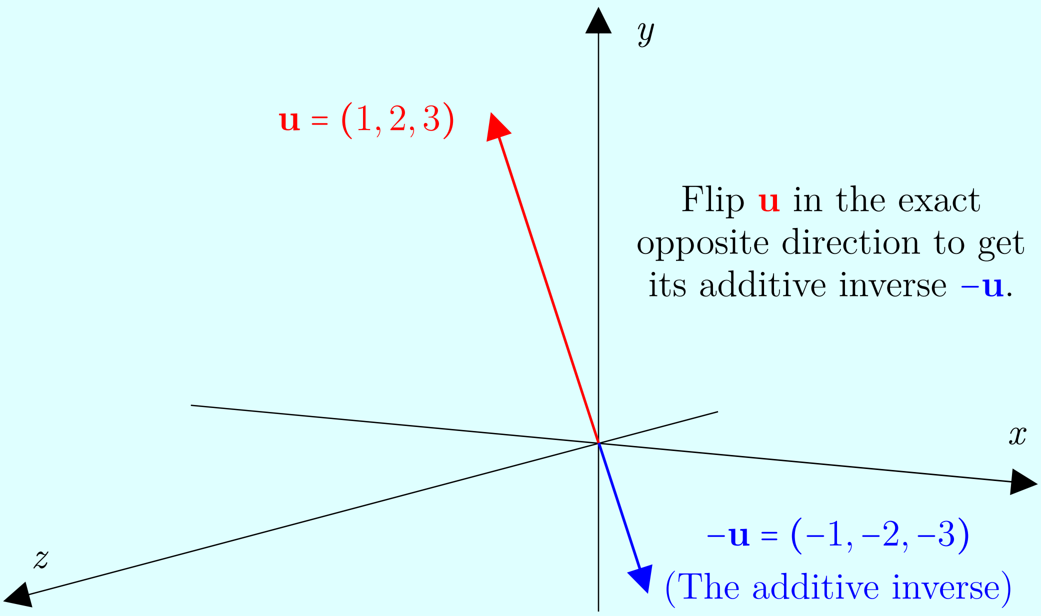
Example 850. Given the vectors $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (-1, 0, 1)$, their sum is the vector $\mathbf{u} + \mathbf{v} = (1 - 1, 2 + 0, 3 + 1) = (0, 2, 4)$.



Definition 166. The *additive inverse of* $\mathbf{u} = (u_1, u_2, u_3)$ is this vector:

$$-\mathbf{u} = (-u_1, -u_2, -u_3).$$

Example 851. The additive inverse of $\mathbf{u} = (1, 2, 3)$ is the vector $-\mathbf{u} = (-1, -2, -3)$.



Definition 135. Given two vectors $\mathbf{u}$ and $\mathbf{v}$, their *difference* $\mathbf{u}-\mathbf{v}$ is the sum of $\mathbf{u}$ and the additive inverse of $\mathbf{v}$. That is,

$$\mathbf{u}-\mathbf{v}=\mathbf{u}+(-\mathbf{v}).$$

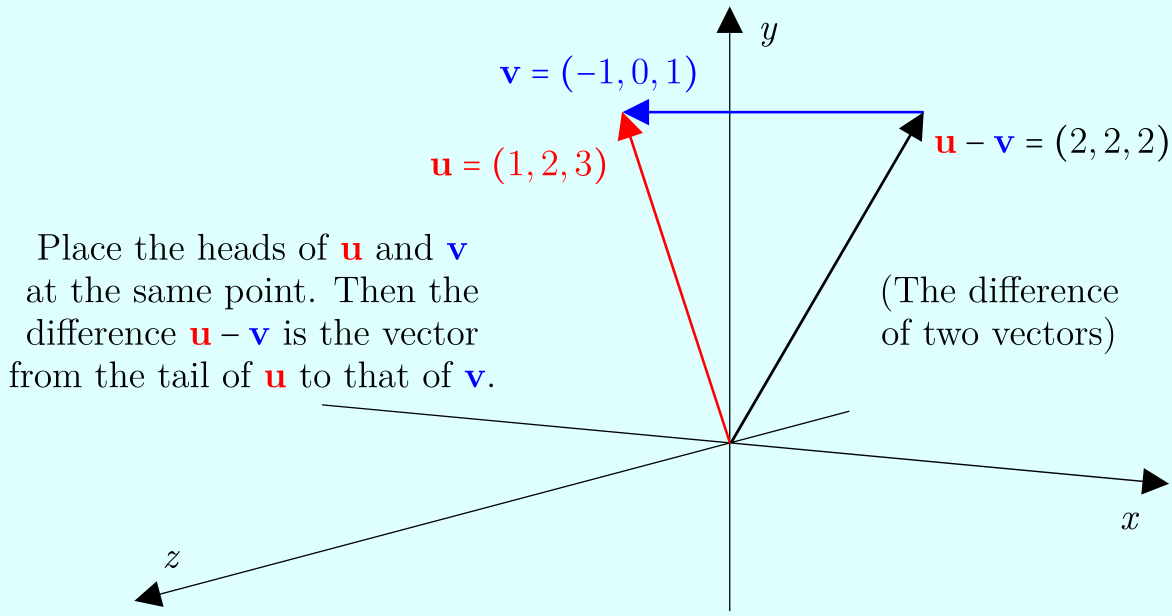
Fact 151. *If* $\mathbf{u}=(u_1,u_2,u_3)$ *and* $\mathbf{v}=(v_1,v_2,v_3)$ *are vectors, then*

$$\mathbf{u}-\mathbf{v}=(u_1-v_1,u_2-v_2,u_3-v_3).$$

Proof. By Definition 166, $-\mathbf{v}=(-v_1,-v_2,-v_3)$. And so by Definition 135,

$$\mathbf{u}-\mathbf{v}=\mathbf{u}+(-\mathbf{v})=(u_1-v_1,u_2-v_2,u_3-v_3). \qquad \square$$

Example 852. Given the vectors $\mathbf{u}=(1,2,3)$ and $\mathbf{v}=(-1,0,1)$, their difference is the vector $\mathbf{u}-\mathbf{v}=(1-(-1),2-0,3-1)=(2,2,2)$.



Exercise 248. Let $\mathbf{u}=(1,2,3)$, $\mathbf{v}=(-1,0,7)$, and $\mathbf{w}=(5,-2,3)$ be vectors. What are **(a)** $\mathbf{u}+\mathbf{v}$; **(b)** $\mathbf{u}-\mathbf{v}$; **(c)** $\mathbf{u}+(\mathbf{v}+\mathbf{w})$; and **(d)** $\mathbf{u}+(\mathbf{v}-\mathbf{w})$? (Answer on p. 1855.)

Fact 119. *Suppose A and B are points. Then $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$.*

Proof. Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Then

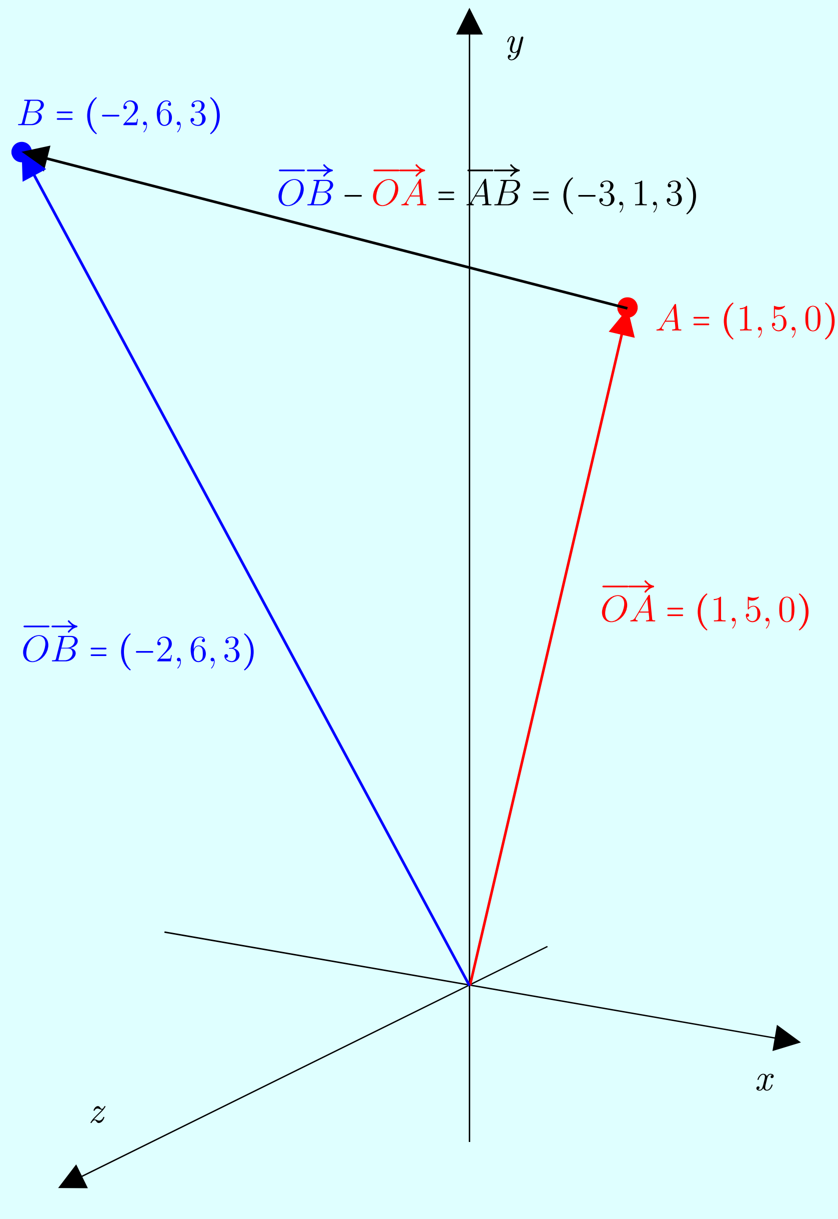
$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3), \quad \overrightarrow{OA} = (a_1, a_2, a_3), \quad \text{and} \quad \overrightarrow{OB} = (b_1, b_2, b_3).$$

By Fact 151, $\overrightarrow{OB} - \overrightarrow{OA} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$. Thus, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. □

Example 853. Let $A = (1, 5, 0)$ and $B = (-2, 6, 3)$ be points.

Then $\overrightarrow{AB} = B - A = (-2, 6, 3) - (1, 5, 0) = (-3, 1, 3)$, $\overrightarrow{OA} = (1, 5, 0)$, and $\overrightarrow{OB} = (-2, 6, 3)$.

And indeed, $\overrightarrow{OB} - \overrightarrow{OA} = (-2, 6, 3) - (1, 5, 0) = (-3, 1, 3) = \overrightarrow{AB}$. ✓



Fact 120. *Suppose A, B, and C are points. Then $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$ and $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.*

Proof. Let $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$. Then

$$\begin{aligned} \overrightarrow{AB} &= \left(b_1 - a_1,\ b_2 - a_2,\ b_3 - a_3\right), \\ \overrightarrow{AC} &= \left(c_1 - a_1,\ c_2 - a_2,\ c_3 - a_3\right), \\ \overrightarrow{CB} &= \left(b_1 - c_1,\ b_2 - c_2,\ b_3 - c_3\right), \end{aligned}$$

□

Proof. And,

$$\begin{aligned} \overrightarrow{AB} - \overrightarrow{AC} &= (b_1 - a_1, b_2 - a_2, b_3 - a_3) - (c_1 - a_1, c_2 - a_2, c_3 - a_3) \\ &= (b_1 - c_1, b_2 - c_2, b_3 - c_3) = \overrightarrow{CB}. \end{aligned}$$

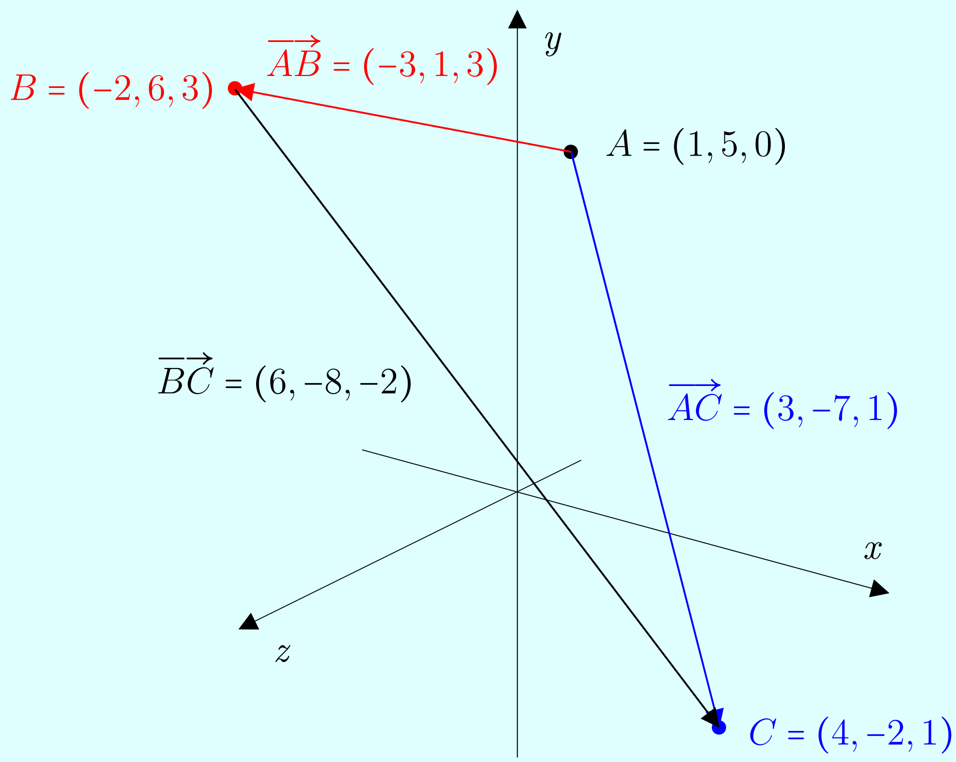
Observing that $-\overrightarrow{CB} = \overrightarrow{BC}$ and rearranging, we also have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. □

Example 854. Let $A = (1, 5, 0)$, $B = (-2, 6, 3)$, and $C = (4, -2, 1)$ be points.

Then $\overrightarrow{AB} = B - A = (-2, 6, 3) - (1, 5, 0) = (-3, 1, 3)$, $\overrightarrow{AC} = C - A = (4, -2, 1) - (1, 5, 0) = (3, -7, 1)$, and $\overrightarrow{BC} = C - B = (4, -2, 1) - (-2, 6, 3) = (6, -8, -2)$.

And indeed, $\overrightarrow{AB} - \overrightarrow{AC} = (-3, 1, 3) - (3, -7, 1) = (-6, 8, 2) = -\overrightarrow{BC} = \overrightarrow{CB}$. ✓

Also, $\overrightarrow{AB} + \overrightarrow{BC} = (-3, 1, 3) + (6, -8, -2) = (3, -7, 1) = \overrightarrow{AC}$. ✓



Exercise 249. Let $A = (5, -1, 0)$, $B = (3, 6, -5)$, and $C = (2, 2, 3)$ be points. Find $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$; and show that $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$ and $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. (Answer on p. 1855.)

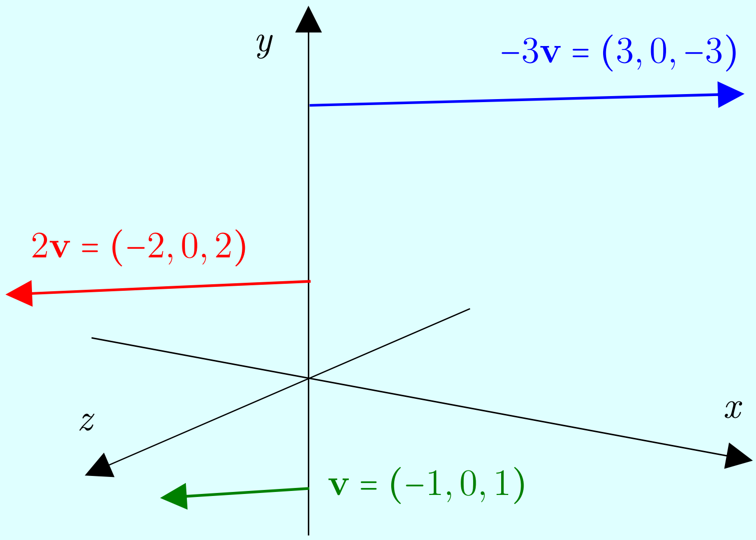
64.4. Scalar Multiplication and When Two Vectors Are Parallel

Definition 167. Given the vector $\mathbf{v} = (v_1, v_2, v_3)$ and the scalar $c \in \mathbb{R}$, the vector $c\mathbf{v}$ is

$$c\mathbf{v} = (cv_1, cv_2, cv_3).$$

Fact 121. *If* $\mathbf{v}$ *is a vector and* $c \in \mathbb{R}$*, then* $|c\mathbf{v}| = |c||\mathbf{v}|$.

Example 855. Let $\mathbf{v} = (-1, 0, 1)$ be a vector. Then $2\mathbf{v} = (-2, 0, 2)$ and $-3\mathbf{v} = (3, 0, -3)$.



Now,
$$|\mathbf{v}| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

And so by Fact 121, we have

$$|2\mathbf{v}| = 2\sqrt{2} \qquad \text{and} \qquad |-3\mathbf{v}| = 3\sqrt{2}.$$

Definition 137. We say that two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ *point in*

(a) The *same direction* if $\mathbf{u} = k\mathbf{v}$ for some $k > 0$;

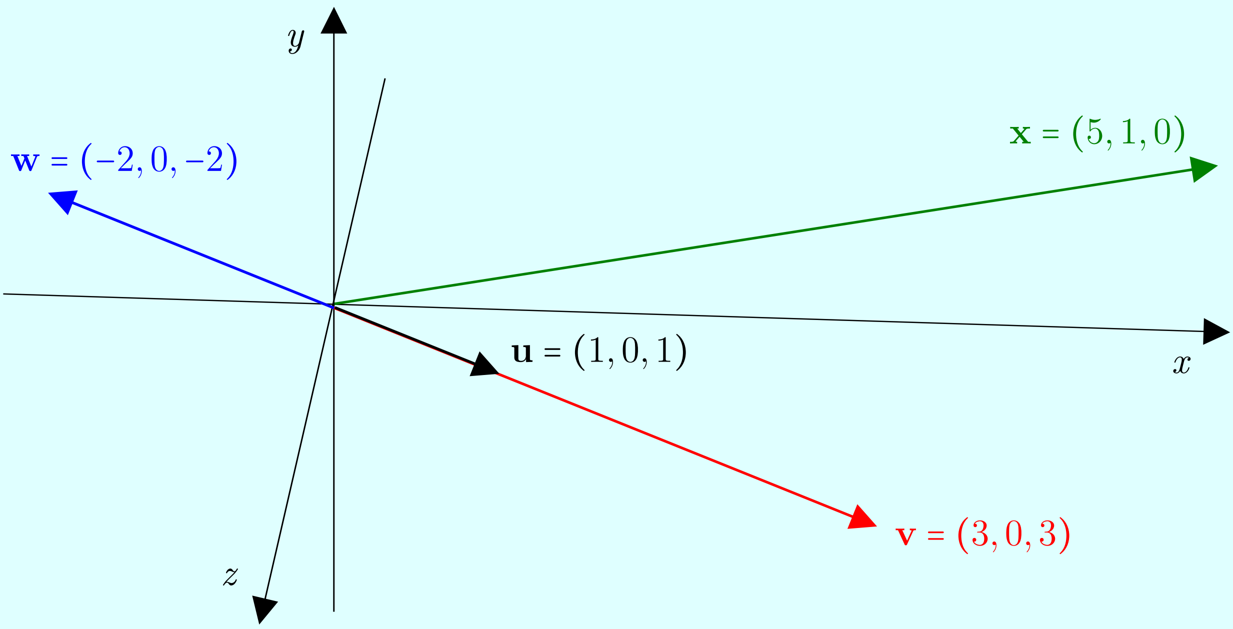
(b) *Exact opposite directions* if $\mathbf{u} = k\mathbf{v}$ for some $k < 0$; and

(c) *Different directions* if $\mathbf{u} \neq k\mathbf{v}$ for any $k \in \mathbb{R}$.

Definition 138. Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are *parallel* if $\mathbf{u} = k\mathbf{v}$ for some $k \in \mathbb{R}$ and *non-parallel* otherwise.

Example 856. Let $\mathbf{u} = (1, 0, 1)$. Then $\mathbf{u}$ points in

- The same direction as $\mathbf{v} = (3, 0, 3)$ because $\mathbf{v} = 3\mathbf{u}$.
- The exact opposite direction as $\mathbf{w} = (-2, 0, -2)$ because $\mathbf{w} = -2\mathbf{u}$.
- A different direction from $\mathbf{x} = (5, 1, 0)$ because $\mathbf{x} \neq k\mathbf{u}$ for any k.



So, $\mathbf{u}$ is parallel to both $\mathbf{v}$ and $\mathbf{w}$, but not to $\mathbf{x}$. As shorthand, we may write

$$\mathbf{u} \parallel \mathbf{v}, \mathbf{w} \qquad \text{and} \qquad \mathbf{u} \nparallel \mathbf{x}.$$

Remark 107. Again, note the special case of the zero vector $\mathbf{0} = (0, 0, 0)$. It does not point in the same, exact opposite, or different direction as any other vector. Also, it is neither parallel nor non-parallel to any other vector.

Exercise 250. Continue to let $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (3, 0, 3)$, $\mathbf{w} = (-2, 0, -2)$, and $\mathbf{x} = (5, 1, 0)$. State if each of the following pairs of vectors point in the same, exact opposite, or different directions; and also if they are parallel. (Answer on p. 1855.)

(a) $\mathbf{v}$ and $\mathbf{w}$. **(b)** $\mathbf{v}$ and $\mathbf{x}$. **(c)** $\mathbf{w}$ and $\mathbf{x}$. **(d)** $\mathbf{u}$ and $\mathbf{0}$.

64.5. Unit Vectors

The following definitions and results about unit vectors are exactly the same as before:

Definition 139. A *unit vector* is any vector of length 1.

Definition 140. Given a non-zero vector $\mathbf{v}$, *its unit vector* (or the *unit vector in its direction*) is

$$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}.$$

Fact 122. *Given any non-zero vector, its unit vector has length* 1.

Fact 123. *Let* $\hat{\mathbf{v}}$ *be a unit vector. If* $c \in \mathbb{R}$, *then the vector* $c\hat{\mathbf{v}}$ *has length* $|c|$.

Fact 124. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *be non-zero vectors. Then*

(a) $\hat{\mathbf{a}} = \hat{\mathbf{b}} \iff \mathbf{a}$ *and* $\mathbf{b}$ *point in the same direction;*
(b) $\hat{\mathbf{a}} = -\hat{\mathbf{b}} \iff \mathbf{a}$ *and* $\mathbf{b}$ *point in exact opposite directions;*
(c) $\hat{\mathbf{a}} = \pm\hat{\mathbf{b}} \iff \mathbf{a} \parallel \mathbf{b}$*;*
(d) $\hat{\mathbf{a}} \neq \pm\hat{\mathbf{b}} \iff \mathbf{a} \nparallel \mathbf{b}$.

Exercise 251. Find the length and unit vector of each vector. (Answer on p. 1856.)

(a) $\mathbf{a} = (1, 2, 3)$. **(b)** $\mathbf{b} = (4, 5, 6)$. **(c)** $\mathbf{a} - \mathbf{b}$.
(d) $2\mathbf{a}$. **(e)** $3\mathbf{b}$. **(f)** $-4(\mathbf{a} - \mathbf{b})$.

64.6. The Standard Basis Vectors

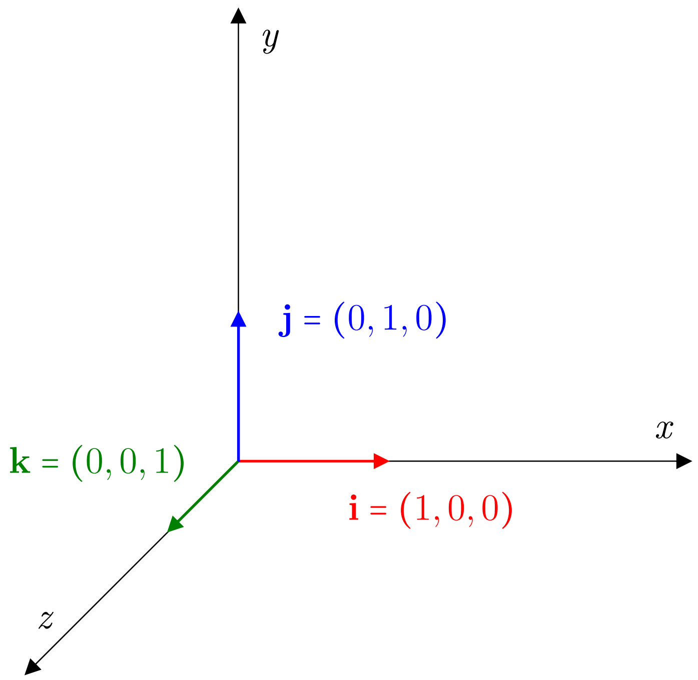
Recall[341] that in 2D space, the (two) **standard basis vectors** were the unit vectors that point in the directions of the positive x- and y-axes:

$$\mathbf{i} = (1, 0) \qquad \text{and} \qquad \mathbf{j} = (0, 1).$$

Analogously, in 3D space, the (three) **standard basis vectors** are the three unit vectors that point in the directions of the positive x-, y-, and z-axes:

Definition 168. The *standard basis vectors* are

$$\mathbf{i} = (1, 0, 0), \qquad \mathbf{j} = (0, 1, 0), \quad \text{and} \quad \mathbf{k} = (0, 0, 1).$$



Not surprisingly, every vector can be written as the linear combination of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$:[342]

Example 857. Let $\mathbf{u} = (7, 5, 3)$. Then $\mathbf{u} = 7\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$.

Exercise 252. Write each of the vectors $\mathbf{v} = (9, 0, -1)$ and $\mathbf{w} = (-7, 3, 5)$ as a linear combination of the standard basis vectors. (Answer on p. 1856.)

[341] Ch. 53.14.

[342] You may also recall (Fact 125) that in 2D space, *every* vector can be written as the linear combination of two non-parallel vectors. It turns out that there is an analogous result in 3D space.

For this result, we first define what it means for three (or more) vectors to be **linearly independent**:

Definition 169. Three (or more) non-zero vectors are *linearly independent* if the first vector cannot be written as a linear combination of the other two vectors.

We then have the following Fact (proof omitted). This Fact is beyond H2 Maths and isn't something you need worry about.

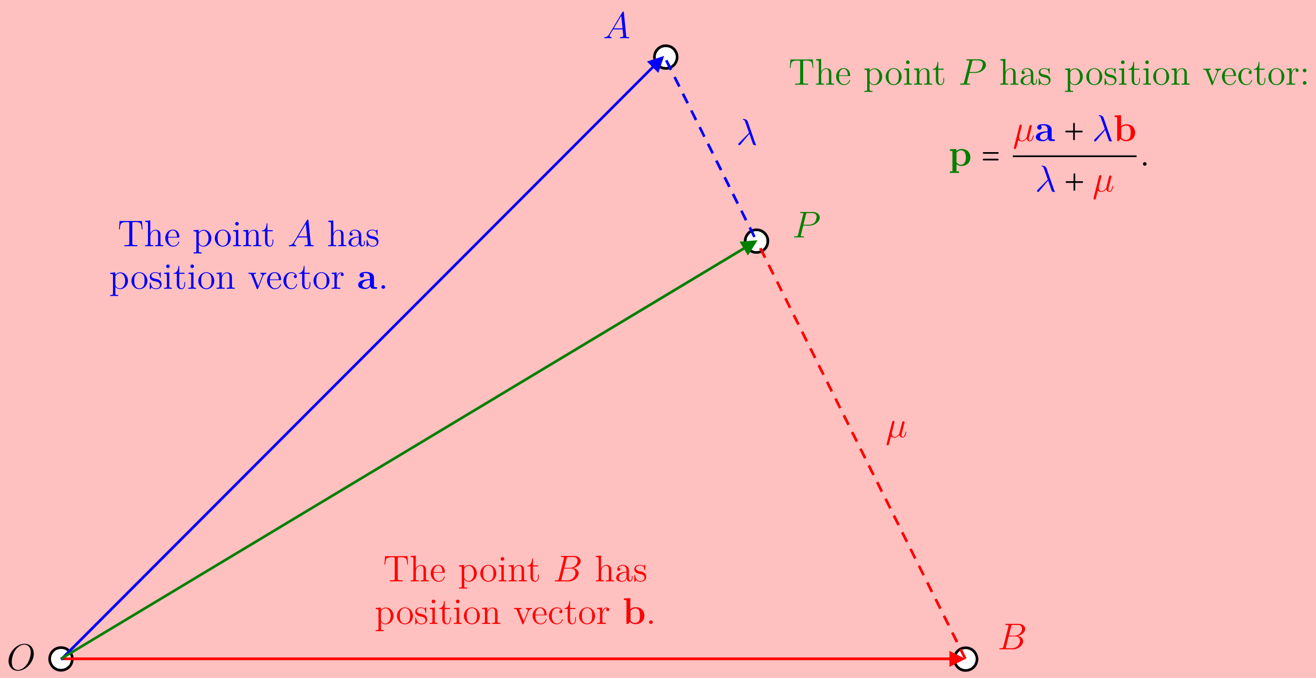
Fact 152. *Every vector can be written as the linear combination of three linearly independent vectors.*

64.7. The Ratio Theorem

The **Ratio Theorem** is exactly the same as before and now reproduced:

Theorem 16. *Let A and B be points with position vectors $\mathbf{a}$ and $\mathbf{b}$. Suppose the point P is on the line segment AB and divides it in the ratio $\lambda : \mu$. Then P's position vector is*

$$\mathbf{p} = \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}.$$



Exercise 253. Let $A = (1, 2, 3)$ on $B = (4, 5, 6)$ be points. Find the point that divides the line segment AB in the ratio $2 : 3$. (Answer on p. 1856.)

Let's end this chapter with three more exercises:

Exercise 254. *Fill in the blanks.* (Answers on p. 1856.)

(a) Informally, a vector is an "arrow" with two properties: ______ and ______.

(b) A point and a vector are entirely different objects and should not be confused. Nonetheless, each can be described by ______.

(c) Let $A = (a_1, a_2, a_3)$ be a point and $\mathbf{a} = (a_1, a_2, a_3)$ be a vector. We say that $\mathbf{a}$ is A's ________.

(d) The vector $\mathbf{a} = (a_1, a_2, a_3)$ carries us from the ______ to the point $A = (a_1, a_2, a_3)$.

Exercise 255. Let $A = (a_1, a_2, a_3)$ be a point. Write down the vector from the origin to A in every possible way. (Answers on p. 1856.)

Exercise 256. Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be points. What are $A+B$, $A+\overrightarrow{OB}$, $\overrightarrow{OA} + \overrightarrow{OB}$, $\overrightarrow{OA} - \overrightarrow{OB}$, $\overrightarrow{OA} - \overrightarrow{BA}$? (Answers on p. 1856.)

65. The Scalar Product (in 3D)

In 2D space, we defined the scalar product of $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ to be

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

We define the **scalar product** in 3D space analogously:

Definition 170. Given vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, their *scalar product*, denoted $\mathbf{u} \cdot \mathbf{v}$, is this number:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example 858. Let $\mathbf{u} = (5, -3, 1)$, $\mathbf{v} = (2, 1, -2)$, and $\mathbf{w} = (0, -4, 3)$. Then

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 10 - 3 - 2 = 5.$$

$$\mathbf{u} \cdot \mathbf{w} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = 0 + 12 + 3 = 15.$$

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = 0 - 4 - 6 = -10.$$

Recall[343] that in 2D space, the scalar product was both **commutative** and **distributive over addition**. The same remains true of the scalar product in 3D space:

Fact 153. *Suppose* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors. Then*

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. **(Commutative)**

(b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$. **(Distributive over Addition)**

Proof. Let[344] $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_3)$. Then

(a)
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = b_1 a_1 + b_2 a_2 + b_3 a_3 = \mathbf{b} \cdot \mathbf{a}.$$

(b)
$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= a_1 (b_1 + c_1) + a_2 (b_2 + c_2) + a_3 (b_3 + c_3) \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 + a_1 c_1 + a_2 c_2 + a_3 c_3 = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}. \end{aligned}$$
□

[343] Fact 129.

[344] Our proof here covers only the 3D case. For a more general proof, see p. 1620 (Appendices).

Example 859. Continue to let $\mathbf{u} = (5, -3, 1)$, $\mathbf{v} = (2, 1, -2)$, and $\mathbf{w} = (0, -4, 3)$.

To illustrate commutativity, we can easily verify that

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} = 5 \qquad \text{and} \qquad \mathbf{w} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{w} = 5.$$

And to illustrate distributivity, we compute the following:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2+0 \\ 1-4 \\ -2+3 \end{pmatrix} = 10 + 9 + 1 = 20.$$

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \begin{pmatrix} 5+2 \\ -3+1 \\ 1-2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = 0 + 8 - 3 = 5.$$

And again, a vector's length is the square root of its scalar product with itself:

Fact 131. *Suppose* $\mathbf{v}$ *be a vector. Then* $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ *and* $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$.

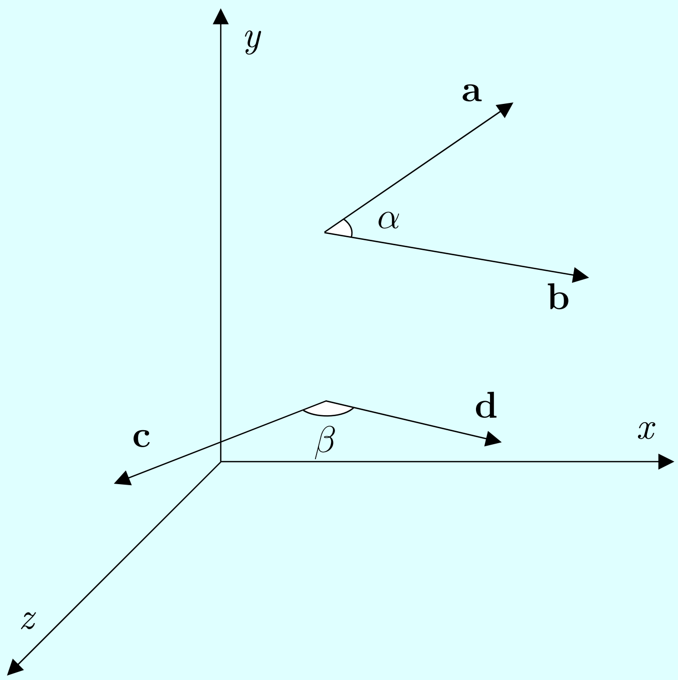
Proof. By Definition 162, $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. By Definition 144, $\mathbf{v} \cdot \mathbf{v} = v_1 v_1 + v_2 v_2 + v_3 v_3 = v_1^2 + v_2^2 + v_3^2$. Hence, $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ and $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$. □

Exercise 257. Compute $(1, 2, 3) \cdot (4, 5, 6)$ and $(-2, 4, -6) \cdot (1, -2, 3)$. (Answer on p. 1857.)

65.1. The Angle between Two Vectors

Place the tails of two vectors at the same point. Then as before, **the angle between these two vectors** is, informally, simply the (smaller) "amount" by which we must rotate one of the two vectors so that both point in the same direction.

Example 860. As shown in the figure below, **a**, **b**, **c**, and **d** are vectors. The angle between **a** and **b** is α, while that between **c** and **d** is β.



(Observe that here it so happens that α is acute, while β is obtuse.)

And formally, we'll use the exact same Definition as before:

Definition 145. The *angle between two non-zero vectors* **u** *and* **v** is this number:

$$\cos^{-1}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

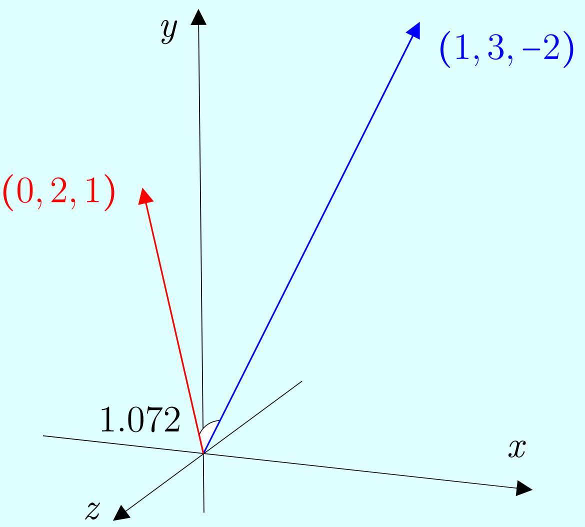
And as before, a simple rearrangement of Definition 145 yields:

Fact 132. *If* **u** *and* **v** *are two non-zero vectors and* θ *is the angle between them, then*

$$\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}|\,|\mathbf{v}|\cos\theta.$$

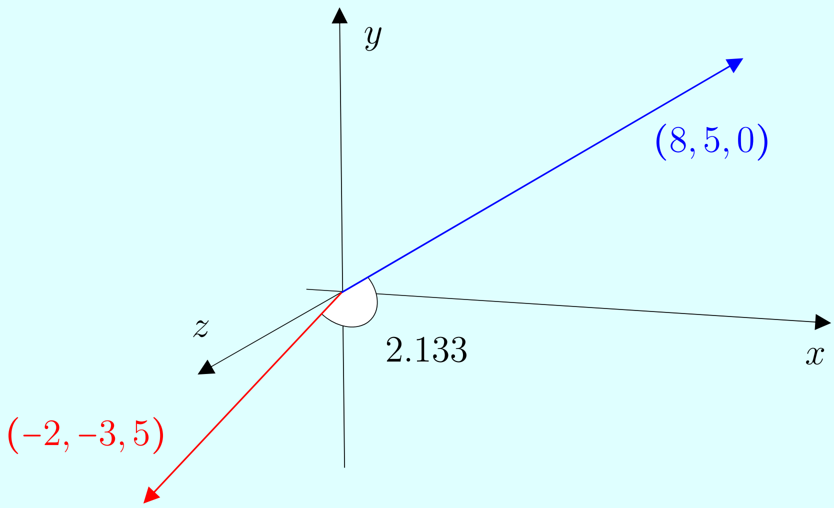
Example 861. The angle between the vectors $(1, 3, -2)$ and $(0, 2, 1)$ is

$$\cos^{-1} \frac{(1,3,-2)\cdot(0,2,1)}{|(1,3,-2)|\,|(0,2,1)|} = \cos^{-1} \frac{4}{\sqrt{14}\sqrt{5}} \approx 1.072.$$



Example 862. The angle between $(8, 5, 0)$ and $(-2, -3, 5)$ is

$$\cos^{-1} \frac{(8,5,0)\cdot(-2,-3,5)}{|(8,5,0)|\,|(-2,-3,5)|} = \cos^{-1} \frac{-31}{\sqrt{89}\sqrt{38}} \approx 2.133.$$



The following results and Definition are reproduced verbatim from before:

Fact 133. (Cauchy's Inequality.) *Suppose* **u** *and* **v** *are non-zero vectors. Then*

$$-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \leq 1.$$

Equivalently, $\quad -|\mathbf{u}|\,|\mathbf{v}| \leq \mathbf{u}\cdot\mathbf{v} \leq |\mathbf{u}|\,|\mathbf{v}| \quad$ *or* $\quad (\mathbf{u}\cdot\mathbf{v})^2 \leq |\mathbf{u}|^2\,|\mathbf{v}|^2.$

Fact 134. *Suppose θ is the angle between two non-zero vectors* $\mathbf{u}$ *and* $\mathbf{v}$*. Then*

(a) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = 1 \iff \theta = 0;$

(b) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \in (0,1) \iff \theta \in \left(0, \dfrac{\pi}{2}\right);$

(c) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = 0 \iff \theta = \dfrac{\pi}{2};$

(d) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \in (-1,0) \iff \theta \in \left(\dfrac{\pi}{2}, \pi\right);$

(e) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = -1 \iff \theta = \pi.$

And thus,

(i) $\mathbf{u}\cdot\mathbf{v} > 0 \iff \theta$ *is acute or zero;*

(ii) $\mathbf{u}\cdot\mathbf{v} = 0 \iff \theta$ *is right;*

(iii) $\mathbf{u}\cdot\mathbf{v} < 0 \iff \theta$ *is obtuse or straight.*

Definition 146. Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are *perpendicular* (or *normal* or *orthogonal*) if $\mathbf{u}\cdot\mathbf{v} = 0$ and non-perpendicular if $\mathbf{u}\cdot\mathbf{v} \neq 0$.

Fact 135. *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are non-zero vectors. Then*

(a) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = 1 \iff$ $\mathbf{u}$ *and* $\mathbf{v}$ *point in the same direction;*

(b) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = -1 \iff$ $\mathbf{u}$ *and* $\mathbf{v}$ *point in exact opposite directions;*

(c) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \pm 1 \iff \mathbf{u} \parallel \mathbf{v};$

(d) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \in (-1,1) \iff$ $\mathbf{u}$ *and* $\mathbf{v}$ *point in different directions.*

Theorem 17. (Pythagoras' Theorem.) *If* $\mathbf{u} \perp \mathbf{v}$*, then* $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$.

Fact 136. (Triangle Inequality.) *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors, then* $|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$.

Exercise 258. Find the angle between each pair of vectors. Also, state whether each pair of vectors is parallel or perpendicular. (Answer on p. 1857.)

(a) $\mathbf{a} = (1,2,3)$ and $\mathbf{b} = (4,5,6)$ **(b)** $\mathbf{u} = (-2,4,-6)$ and $\mathbf{v} = (1,-2,3)$

65.2. Direction Cosines

In 2D space, we defined the x- and y-direction cosines of the vector $\mathbf{u} = (u_1, u_2)$ to be $u_1/|\mathbf{u}|$ and $u_2/|\mathbf{u}|$. We shall define **direction cosines** in 3D space analogously:

Definition 171. The x-, y-, and z-*direction cosines* of the vector $\mathbf{v} = (v_1, v_2, v_3)$ are

$$\frac{v_1}{|\mathbf{v}|}, \qquad \frac{v_2}{|\mathbf{v}|}, \qquad \text{and} \qquad \frac{v_3}{|\mathbf{v}|}.$$

Again, the direction cosines are so named because each direction cosine is equal to the cosine of the angle the given vector makes with each (positive) axis:

Fact 154. *Let $\mathbf{v} = (v_1, v_2, v_3)$ be a non-zero vector. Suppose α, β, and γ are the angles between $\mathbf{v}$ and each of* **i**, **j**, *and* **k**. *Then*

$$\cos\alpha = \frac{v_1}{|\mathbf{v}|}, \qquad \cos\beta = \frac{v_2}{|\mathbf{v}|}, \qquad \text{and} \qquad \cos\gamma = \frac{v_3}{|\mathbf{v}|}.$$

Proof. Since α is the angle between $\mathbf{v}$ and $\mathbf{i}$, we have

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|\,|\mathbf{i}|} = \frac{v_1\cdot 1 + v_2\cdot 0 + v_3\cdot 0}{|\mathbf{v}|\cdot 1} = \frac{v_1}{|\mathbf{v}|}.$$

Similarly, since β is the angle between $\mathbf{v}$ and $\mathbf{j}$, we have

$$\cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{|\mathbf{v}|\,|\mathbf{j}|} = \frac{v_1\cdot 0 + v_2\cdot 1 + v_3\cdot 0}{|\mathbf{v}|\cdot 1} = \frac{v_2}{|\mathbf{v}|}.$$

And since γ is the angle between $\mathbf{v}$ and $\mathbf{k}$, we have

$$\cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{|\mathbf{v}|\,|\mathbf{k}|} = \frac{v_1\cdot 0 + v_2\cdot 0 + v_3\cdot 1}{|\mathbf{v}|\cdot 1} = \frac{v_3}{|\mathbf{v}|}.$$

□

Example 863. Let $\mathbf{v} = (2, 3, 2)$. Compute $|\mathbf{v}| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{17}$.

So $\mathbf{v}$'s x-, y-, and z-direction cosines are $2/\sqrt{17}$, $3/\sqrt{17}$, and $2/\sqrt{17}$.

And the angles it makes with the positive x-, y-, and z-axes are

$$\cos^{-1}\frac{2}{\sqrt{17}} \approx 1.064, \qquad \cos^{-1}\frac{3}{\sqrt{17}} \approx 0.756, \qquad \text{and} \qquad \cos^{-1}\frac{2}{\sqrt{17}} \approx 1.064.$$

Exercise 259. For each vector, write down its unit vector and x-, y-, and z-direction cosines. Then compute also the angles it makes with the positive x-, y-, and z-axes.

(a) $(1, 3, -2)$. **(b)** $(4, 2, -3)$. **(c)** $(-1, 2, -4)$. (Answer on p. 1857.)

66. The Projection and Rejection Vectors (in 3D)

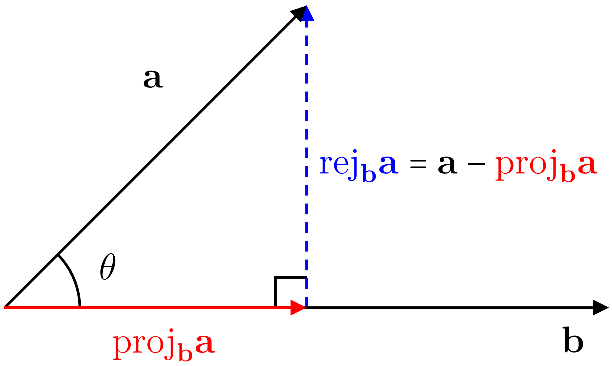
Our definitions and results about the **projection** and **rejection vectors** carry over from 2D space in the "obvious" fashion. Here are the same Definitions reproduced:

Definition 152. Suppose $\mathbf{a}$ and $\mathbf{b} \neq \mathbf{0}$ are vectors. Then the *projection of* $\mathbf{a}$ *on* $\mathbf{b}$, denoted $\text{proj}_{\mathbf{b}}\mathbf{a}$, is this vector:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\mathbf{a} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}, \qquad \text{or equivalently,} \qquad \text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}.$$

Definition 153. Suppose $\mathbf{a}$ and $\mathbf{b} \neq \mathbf{0}$ are vectors. Then the *rejection of* $\mathbf{a}$ *on* $\mathbf{b}$, denoted $\text{rej}_{\mathbf{b}}\mathbf{a}$, is this vector:

$$\text{rej}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}.$$



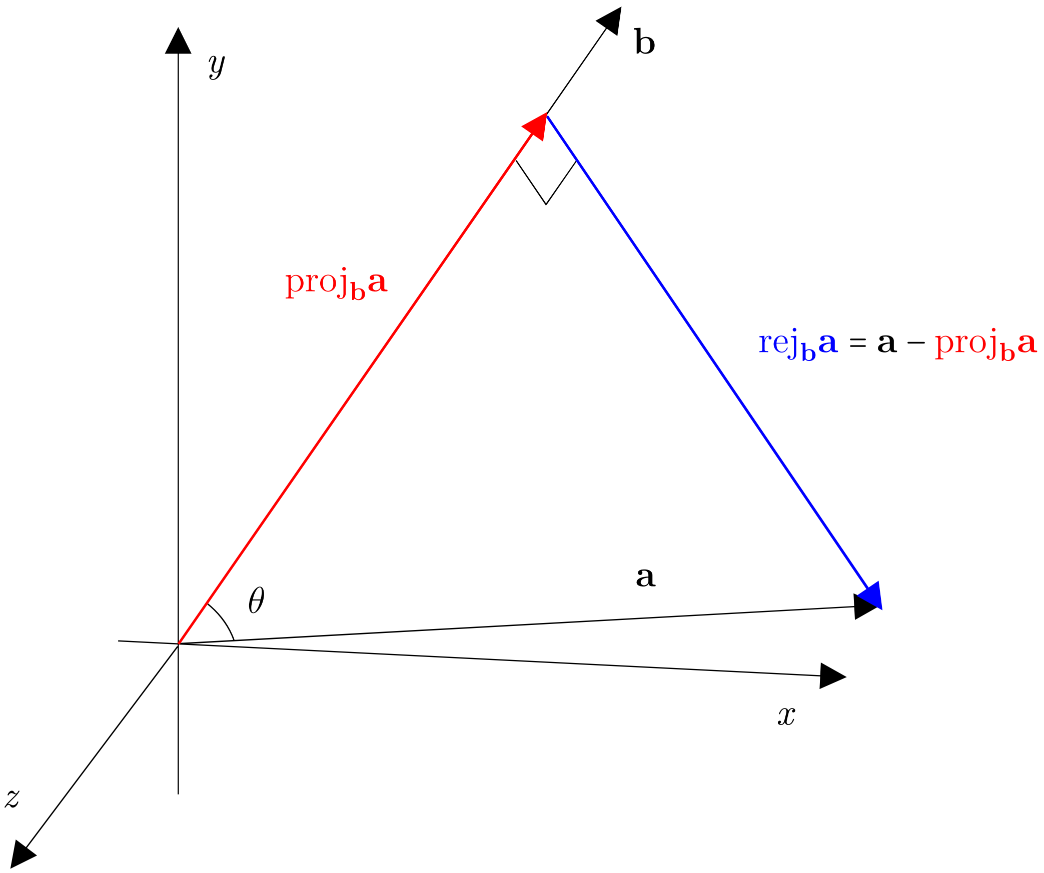
As before, the two key properties[345] are that

$$\text{proj}_{\mathbf{b}}\mathbf{a} \parallel \mathbf{b}$$

and

$$\text{rej}_{\mathbf{b}}\mathbf{a} \perp \text{proj}_{\mathbf{b}}\mathbf{a}, \mathbf{b}.$$

The 2D figure above is simply reproduced from before. Here's a figure depicting the projection and rejection vectors in 3D:

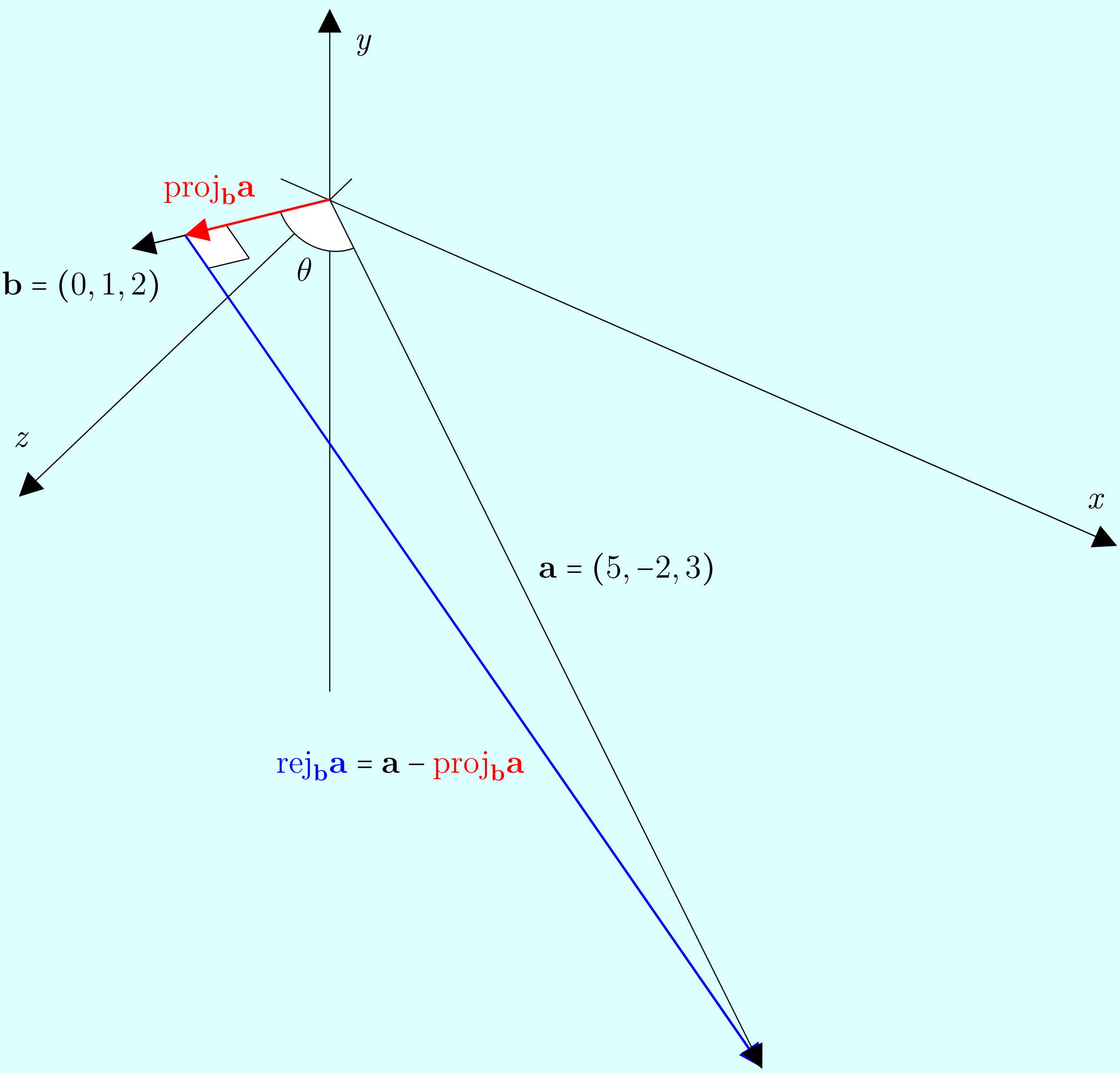


[345] Again, these two properties must hold provided $\text{proj}_{\mathbf{b}}\mathbf{a}$ and $\text{rej}_{\mathbf{b}}\mathbf{a}$ are both non-zero.

Example 864. Let $\mathbf{a} = (5, -2, 3)$ and $\mathbf{b} = (0, 1, 2)$. Then $\hat{\mathbf{b}} = (0, 1, 2) / \sqrt{5}$ and

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\mathbf{a} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}} = \frac{(5, -2, 3) \cdot (0, 1, 2)}{5}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \frac{0 - 2 + 6}{5}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \frac{4}{5}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.8 \\ 1.6 \end{pmatrix},$$

$$\text{rej}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a} = (5, -2, 3) - (0, 0.8, 1.6) = (5, -2.8, 1.4)$$



We can easily verify that $\text{proj}_{\mathbf{b}}\mathbf{a} = k\mathbf{b}$ for some k and hence that $\text{proj}_{\mathbf{b}}\mathbf{a} \parallel \mathbf{b}$:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = (0, 0.8, 1.6) = 0.8\,(0, 1, 2) = 0.8\mathbf{b}. \qquad ✓$$

We can also verify that $\text{rej}_{\mathbf{b}}\mathbf{a} \cdot \mathbf{b} = 0$ and hence that $\text{rej}_{\mathbf{b}}\mathbf{a} \perp \mathbf{b}$ (and also $\text{rej}_{\mathbf{b}}\mathbf{a} \perp \text{proj}_{\mathbf{b}}\mathbf{a}$):

$$\text{rej}_{\mathbf{b}}\mathbf{a} \cdot \mathbf{b} = (5, -2.8, 1.4) \cdot (0, 1, 2) = 0 - 2.8 + 2.8 = 0. \qquad ✓$$

As before, the length of the projection vector is given by the scalar product:

Fact 142. *Suppose* $\mathbf{a}$ *and* $\mathbf{b} \neq \mathbf{0}$ *are vectors. Then*

$$|\text{proj}_{\mathbf{b}}\mathbf{a}| = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right|.$$

Example 865. Continue to let $\mathbf{a} = (5, -2, 3)$ and $\mathbf{b} = (0, 1, 2)$. We already found

$$\hat{\mathbf{b}} = (0, 1, 2) / \sqrt{5} \qquad \text{and} \qquad \text{proj}_{\mathbf{b}}\mathbf{a} = (0, 0.8, 1.6).$$

Now, $$|\text{proj}_{\mathbf{b}}\mathbf{a}| = |0.8 (0, 1, 2)| = 0.8\sqrt{0^2 + 1^2 + 2^2} = 0.8\sqrt{5}.$$

Also, $$\left|\mathbf{a} \cdot \hat{\mathbf{b}}\right| = (5, -2, 3) \cdot (0, 1, 2) / \sqrt{5} = 4/\sqrt{5} = 4\sqrt{5}/5 = 0.8\sqrt{5}.$$

And so, it is indeed true that $$|\text{proj}_{\mathbf{b}}\mathbf{a}| = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right|.$$

As before, the sign of $\mathbf{a} \cdot \mathbf{b}$ tells us whether $\text{proj}_{\mathbf{b}}\mathbf{a}$ points in the same or exact opposite direction as $\mathbf{b}$:

Fact 141. *Let* $\mathbf{a}$ *and* $\mathbf{b} \neq \mathbf{0}$ *be vectors and* $\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\mathbf{a} \cdot \hat{\mathbf{b}}\right) \hat{\mathbf{b}}$.

(a) *If* $\mathbf{a} \cdot \hat{\mathbf{b}} > 0$, *then* $\text{proj}_{\mathbf{b}}\mathbf{a}$ *is a positive scalar multiple of* $\mathbf{b}$.

(b) *If* $\mathbf{a} \cdot \hat{\mathbf{b}} < 0$, *then* $\text{proj}_{\mathbf{b}}\mathbf{a}$ *is a negative scalar multiple of* $\mathbf{b}$.

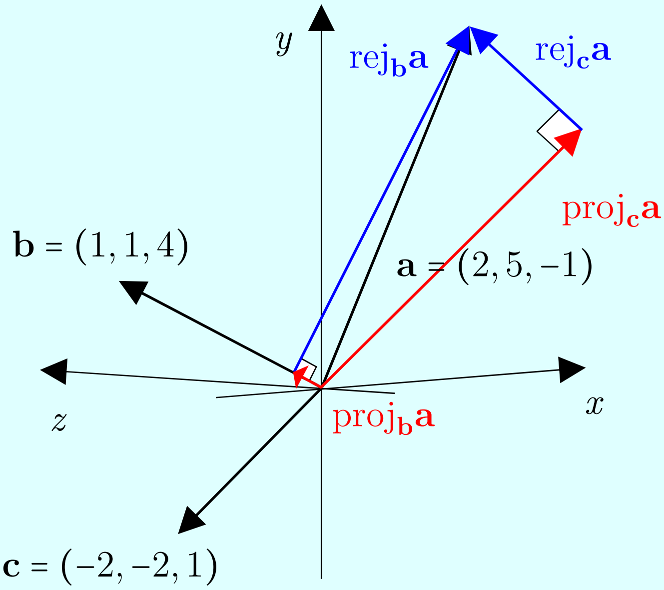
(c) *If* $\mathbf{a} \cdot \hat{\mathbf{b}} = 0$, *then* $\text{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{0}$ *and* $\text{rej}_{\mathbf{b}}\mathbf{a} = \mathbf{a}$.

Example 866. Let $\mathbf{a} = (2, 5, -1)$, $\mathbf{b} = (1, 1, 4)$, and $\mathbf{c} = (-2, -2, 1)$. Then

(a) $\mathbf{a} \cdot \mathbf{b} = 2 + 5 - 4 > 0$, so that the angle between $\mathbf{a}$ and $\mathbf{b}$ is acute and $\text{proj}_{\mathbf{b}}\mathbf{a}$ points in the same direction as $\mathbf{b}$.

(b) $\mathbf{a} \cdot \mathbf{c} = -4 - 10 - 1 < 5$, so that the angle between $\mathbf{a}$ and $\mathbf{c}$ is obtuse and $\text{proj}_{\mathbf{c}}\mathbf{a}$ points in the exact opposite direction as $\mathbf{b}$.

(c) $\mathbf{b} \cdot \mathbf{c} = -2 - 2 + 4 = 0$, so that the angle between $\mathbf{a}$ and $\mathbf{c}$ is right and $\text{proj}_{\mathbf{c}}\mathbf{b} = \mathbf{0}$. Moreover, $\text{rej}_{\mathbf{c}}\mathbf{b} = \mathbf{b}$.



Exercise 260. Continuing with the above example, find $\text{proj}_{\mathbf{b}}\mathbf{a}$, $\text{rej}_{\mathbf{b}}\mathbf{a}$, $\text{proj}_{\mathbf{c}}\mathbf{a}$, and $\text{rej}_{\mathbf{c}}\mathbf{a}$. Then verify that $\text{rej}_{\mathbf{b}}\mathbf{a} \perp \mathbf{b}$ and $\text{rej}_{\mathbf{c}}\mathbf{a} \perp \mathbf{c}$. (Answer on p. 1858.)

As before, if $\mathbf{v} \parallel \mathbf{w}$, then the projections of any vector on each of $\mathbf{v}$ and $\mathbf{w}$ are identical:

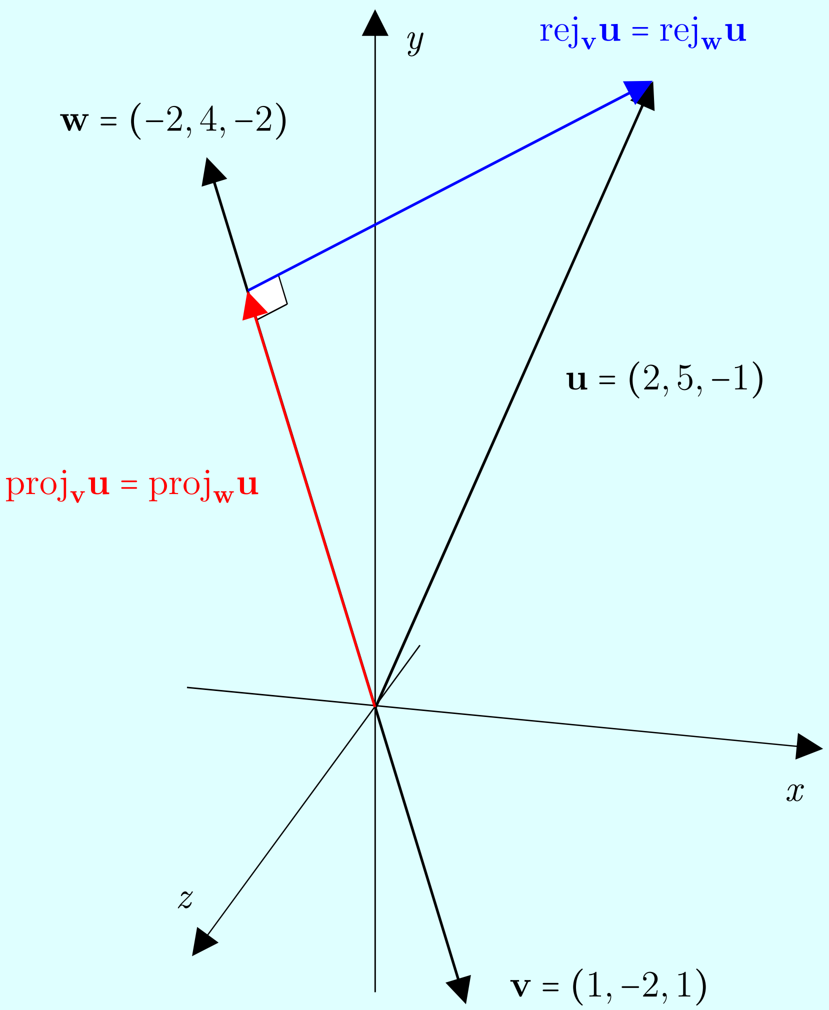
Fact 143. *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors. If* $\mathbf{v} \parallel \mathbf{w}$, *then*

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{proj}_{\mathbf{w}}\mathbf{u}.$$

Example 867. Let $\mathbf{u} = (2, 5, -1)$, $\mathbf{v} = (1, -2, 1)$, and $\mathbf{w} = (-2, 4, -2)$. Since $\mathbf{v} \parallel \mathbf{w}$, by the above Fact, it should be that $\text{proj}_{\mathbf{v}}\mathbf{u} = \text{proj}_{\mathbf{w}}\mathbf{u}$, as we now verify:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{v}})\,\hat{\mathbf{v}} = \frac{(2,5,-1)\cdot(1,-2,1)}{1^2 + (-2)^2 + 1^2}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{2-10+1}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

$$\text{proj}_{\mathbf{w}}\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{w}})\,\hat{\mathbf{w}} = \frac{(2,5,-1)\cdot(-2,4,-2)}{(-2)^2 + 4^2 + (-2)^2}\begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = \frac{-4+20+2}{24}\begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$



Exercise 261. Given $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (4, 5, 6)$, find $\text{proj}_{\mathbf{b}}\mathbf{a}$, $\text{rej}_{\mathbf{b}}\mathbf{a}$, $|\text{proj}_{\mathbf{b}}\mathbf{a}|$, and $|\text{rej}_{\mathbf{b}}\mathbf{a}|$. Verify that $\text{proj}_{\mathbf{b}}\mathbf{a} \parallel \mathbf{b}$ and $\text{rej}_{\mathbf{b}}\mathbf{a} \perp \mathbf{b}$. Does $\text{proj}_{\mathbf{b}}\mathbf{a}$ point in the same or exact opposite direction as $\mathbf{b}$? (Answer on p. 1858.)

67. Lines (in 3D)

Our definition of a **line** in 3D space is the general one given earlier and now reproduced:

Definition 143. A *line* is any set of points that can be written as

$$\left\{R : \overrightarrow{OR} = \mathbf{p} + \lambda\mathbf{v} \quad (\lambda \in \mathbb{R})\right\},$$

where $\mathbf{p}$ and $\mathbf{v} \neq \mathbf{0}$ are vectors.

As before, the above Definition says that a line contains exactly those points R whose position vector $\overrightarrow{OR} = \mathbf{r}$ may be expressed as

$$\overrightarrow{OR} = \mathbf{r} = \mathbf{p} + \lambda\mathbf{v} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{for some real number } \lambda.$$

Equivalently, a line contains exactly those points R that may be expressed as

$$R = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{for some real number } \lambda.$$

As before, here are what the vectors $\mathbf{p}$ and $\mathbf{v}$ and the number λ mean:

- $\mathbf{p} = (p_1, p_2, p_3)$ is the position vector of some point on the line;
- $\mathbf{v} = (v_1, v_2, v_3)$ is a direction vector of the line; and
- The **parameter** λ takes on every value in $\mathbb{R}$; each distinct value produces a distinct point on the line.

Direction vectors are defined exactly as before:

Definition 142. Given any two distinct points A and B on a line, we call the vector $\overrightarrow{AB}$ a *direction vector* of the line.

And as before, direction vectors are **unique up to non-zero scalar multiplication**. In other words, if a line has direction vector $\mathbf{v}$, then that line's direction vectors are exactly those that are parallel to $\mathbf{v}$. Formally,

Fact 126. *Suppose* $\mathbf{v}$ *is a line's direction vector. Then*

$$\mathbf{u} \textit{ is also that line's direction vector} \quad \iff \quad \mathbf{u} \parallel \mathbf{v}.$$

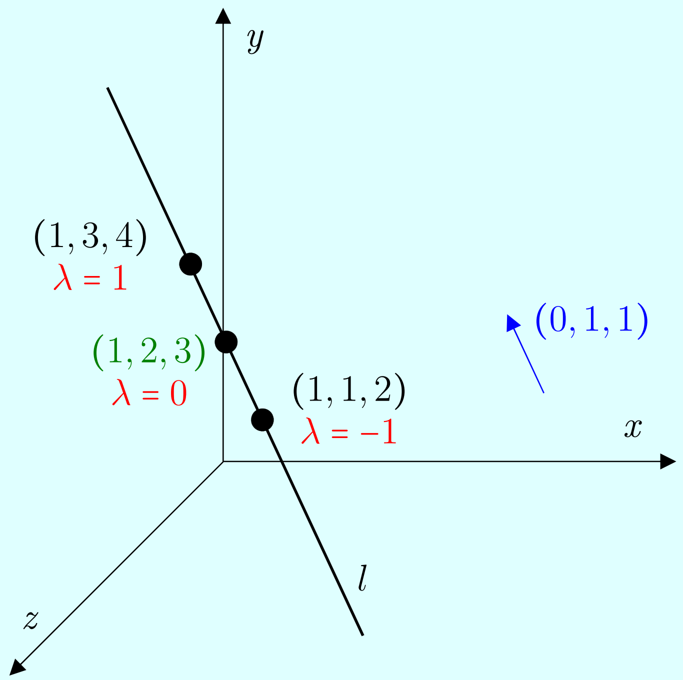
Example 868. Let l be the line described by this vector equation:

$$\mathbf{r} = \overrightarrow{OP} + \lambda \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad (\lambda \in \mathbb{R}).$$

The line l contains the point $P = (1, 2, 3)$ and has direction vector $\mathbf{v} = (0, 1, 1)$.

As the parameter λ varies, we get different points of l. So for example, when λ takes on the values 0, 1, and -1, we get these three position vectors (and corresponding points):

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$



Note that the direction vector $\mathbf{v} = (0, 1, 1)$ has x-coordinate 0. Informally, one implication of this is that the line doesn't "move" in the direction of the x-axis.

A little more formally, the line is perpendicular to the x-axis. Indeed, we can easily verify that $\mathbf{v}$ is perpendicular to the first standard basis vector $\mathbf{i} = (1, 0, 0)$:

$$\mathbf{v} \cdot \mathbf{i} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \mathbf{i} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 = 0.$$

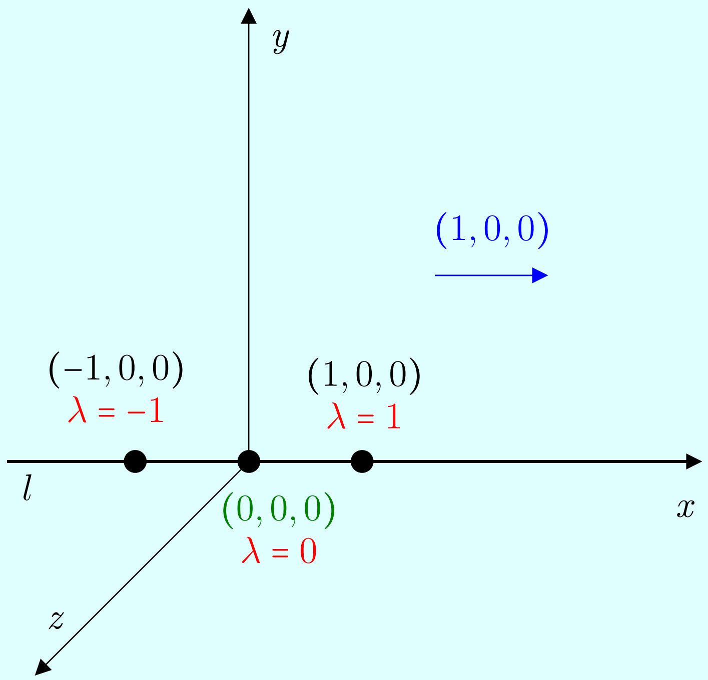
Example 869. Let l be the line described by this vector equation:

$$\mathbf{r} = \overrightarrow{OP} + \lambda\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (\lambda \in \mathbb{R}).$$

The line l contains the point $P = (0, 0, 0)$ and has direction vector $\mathbf{v} = (1, 0, 0)$.

As the parameter λ varies, we get different points of l. So for example, when λ takes on the values 0, 1, and -1, we get these three position vectors (and corresponding points):

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$



Note that the direction vector $\mathbf{v} = (1, 0, 0)$ has y- and z-coordinates 0. Again, this means that the line is perpendicular to the y- and z-axes—we can easily verify that $\mathbf{v}$ is perpendicular to both $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

Indeed, this line actually coincides with the x-axis—it passes through the origin $(0, 0, 0)$ and its direction vector is parallel to $\mathbf{i}$.

67.1. Vector to Cartesian Equations

Suppose a line l is described by this **vector equation**:

$$\mathbf{r} = (p_1, p_2, p_3) + \lambda(v_1, v_2, v_3) \qquad (\lambda \in \mathbb{R}).$$

Then for each point (x, y, z) on l, there is some $\lambda \in \mathbb{R}$ such that

$$x = p_1 + \lambda v_1, \qquad y = p_2 + \lambda v_2, \quad \text{and} \quad z = p_3 + \lambda v_3.$$

These three **cartesian equations** also describe l.

In 2D space, we could eliminate the parameter λ and thus reduce the two cartesian equations to one. Here in 3D space, we can also eliminate the parameter λ, but this time we'll merely reduce the three cartesian equations to **two**.

We start with three examples where **none** of the direction vector's coordinates are zero:

Example 870. Let l be the line described by this **vector equation**:

$$\mathbf{r} \stackrel{\star}{=} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad (\lambda \in \mathbb{R}).$$

That is, let $$l = \left\{ R : \mathbf{r} \stackrel{\star}{=} (1, 2, 3) + \lambda(4, 5, 6) \right\}.$$

In words, l is the set of points R whose position vector can be written as $(1, 2, 3) + \lambda(4, 5, 6)$, for some real number λ.

Write out $\stackrel{\star}{=}$ as the following three cartesian equations:

$$x \stackrel{1}{=} 1 + 4\lambda, \qquad y \stackrel{2}{=} 2 + 5\lambda, \qquad z \stackrel{3}{=} 3 + 6\lambda.$$

Rearrange each equation so that λ is on one side:

$$\lambda \stackrel{1}{=} \frac{x-1}{4}, \qquad \lambda \stackrel{2}{=} \frac{y-2}{5}, \qquad \lambda \stackrel{3}{=} \frac{z-3}{6}.$$

Eliminating λ leaves these two **cartesian equations** that describe the line l:

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}.$$

And so, we may also write

$$l = \left\{ (x, y, z) : \frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6} \right\}.$$

In words, l is the set of points (x, y, z) that satisfy the equations $\dfrac{x-1}{4} = \dfrac{y-2}{5} = \dfrac{z-3}{6}$.

Example 871. Let l_0 be the line described by this vector equation:

$$\mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \qquad (\lambda \in \mathbb{R}).$$

Write out the three cartesian equations:

$$x \stackrel{1}{=} -2 + \lambda, \qquad y \stackrel{2}{=} 5 + 5\lambda, \qquad z \stackrel{3}{=} 0 - 2\lambda.$$

Rearrange each equation so that λ is on one side:

$$\lambda \stackrel{1}{=} \frac{x+2}{1}, \qquad \lambda \stackrel{2}{=} \frac{y-5}{5}, \qquad \lambda \stackrel{3}{=} \frac{z}{-2}.$$

So, l_0 may also be described by these two cartesian equations:

$$\frac{x+2}{1} = \frac{y-5}{5} = \frac{z}{-2}.$$

Example 872. Let l_1 be the line described by this vector equation:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \qquad (\lambda \in \mathbb{R}).$$

Write out the three cartesian equations:

$$x \stackrel{1}{=} 0 + 2\lambda, \qquad y \stackrel{2}{=} 0 + 3\lambda, \qquad z \stackrel{3}{=} 0 + 5\lambda.$$

Rearrange each equation so that λ is on one side:

$$\lambda \stackrel{1}{=} \frac{x}{2}, \qquad \lambda \stackrel{2}{=} \frac{y}{3}, \qquad \lambda \stackrel{3}{=} \frac{z}{5}.$$

So, l_1 may also be described by these two cartesian equations:

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{5}.$$

We now look at examples where **exactly one** of the direction vector's coordinates is zero:

Example 873. The line l_2 is described by $\mathbf{r} = (1, 2, 3) + \lambda(0, 5, 6) \qquad (\lambda \in \mathbb{R})$.

Write out the three cartesian equations: $x \stackrel{1}{=} 1 + 0\lambda = 1$, $y \stackrel{2}{=} 2 + 5\lambda$, $z \stackrel{3}{=} 3 + 6\lambda$.

Observe that the direction vector $(0, 5, 6)$ has x-coordinate 0. And so, l_2 must be perpendicular to the x-axis. Indeed, the x-coordinate of *every* point on l_2 is fixed as $x \stackrel{1}{=} 1$.

Leave $\stackrel{1}{=}$ alone. But as before, rearrange $\stackrel{2}{=}$ and $\stackrel{3}{=}$ so that λ is on one side:

$$\lambda \stackrel{2}{=} \frac{y-2}{5} \qquad \text{and} \qquad \lambda \stackrel{3}{=} \frac{z-3}{6}.$$

Altogether then, l_2 may also be described by these two cartesian equations:

$$x = 1 \qquad \text{and} \qquad \frac{y-2}{5} = \frac{z-3}{6}.$$

Example 874. The line l_3 is described by $\mathbf{r} = (1, 2, 3) + \lambda(4, 0, 6) \qquad (\lambda \in \mathbb{R})$.

Write out the three cartesian equations: $x \stackrel{1}{=} 1 + 4\lambda$, $y \stackrel{2}{=} 2 + 0\lambda = 2$, $z \stackrel{3}{=} 3 + 6\lambda$.

Observe that the direction vector $(4, 0, 6)$ has y-coordinate 0. And so, l_3 must be perpendicular to the y-axis. Indeed, the y-coordinate of *every* point on l_3 is fixed as $y \stackrel{2}{=} 2$.

Leave $y \stackrel{2}{=} 2$ alone. But as before, rearrange $\stackrel{1}{=}$ and $\stackrel{3}{=}$ so that λ is on one side:

$$\lambda \stackrel{1}{=} \frac{x-1}{4} \qquad \text{and} \qquad \lambda \stackrel{3}{=} \frac{z-3}{6}.$$

Thus, l_3 may also be described by these two cartesian equations:

$$y = 2 \qquad \text{and} \qquad \frac{x-1}{4} = \frac{z-3}{6}.$$

Example 875. The line l_4 is described by $\mathbf{r} = (1, 2, 3) + \lambda(4, 5, 0) \qquad (\lambda \in \mathbb{R})$.

Write out the three cartesian equations: $x \stackrel{1}{=} 1 + 4\lambda$, $y \stackrel{2}{=} 2 + 5\lambda$, $z \stackrel{3}{=} 3 + 0\lambda = 3$.

Observe that the direction vector $(4, 5, 0)$ has z-coordinate 0. And so, l_4 must be perpendicular to the z-axis. Indeed, the z-coordinate of *every* point on l_4 is fixed as $z \stackrel{3}{=} 3$.

Leave $z \stackrel{3}{=} 3$ alone. But as before, rearrange $\stackrel{1}{=}$ and $\stackrel{2}{=}$ so that λ is on one side:

$$\lambda \stackrel{2}{=} \frac{x-1}{4} \qquad \text{and} \qquad \lambda \stackrel{3}{=} \frac{y-2}{5}.$$

Thus, l_4 may also be described by these two cartesian equations:

$$z = 3 \qquad \text{and} \qquad \frac{x-1}{4} = \frac{y-2}{5}.$$

We now look at examples where **exactly two** of the direction vector's coordinates are zero:

Example 876. The line l_5 is described by $\mathbf{r} = (1,2,3) + \lambda(0,0,6) \qquad (\lambda \in \mathbb{R})$.

Write out the three cartesian equations:

$$x \stackrel{1}{=} 1 + 0\lambda = 1, \qquad y \stackrel{2}{=} 2 + 0\lambda = 2, \qquad z \stackrel{3}{=} 3 + 6\lambda.$$

Observe that the direction vector $(0,0,6)$ has x- and y-coordinates 0. And so, l_5 must be perpendicular to both the x- and y-axes.

Indeed, the x-and y-coordinates of *every* point of l_5 are fixed as $x \stackrel{1}{=} 1$ and $y \stackrel{2}{=} 2$.

On the other hand, z is free to vary along with λ. Unlike in any of our previous examples, there is no restriction on what z can be. And so we call z the **free variable**.

Hence, in this example, there's actually no algebra to be done. We simply discard $\stackrel{3}{=}$ and say that l_5 may be described by these two cartesian equations:

$$x \stackrel{1}{=} 1 \qquad \text{and} \qquad y \stackrel{2}{=} 2.$$

Example 877. The line l_6 is described by $\mathbf{r} = (1,2,3) + \lambda(0,5,0) \qquad (\lambda \in \mathbb{R})$.

Write out the three cartesian equations:

$$x \stackrel{1}{=} 1 + 0\lambda = 1, \qquad y \stackrel{2}{=} 2 + 5\lambda, \qquad z \stackrel{3}{=} 3 + 0\lambda = 3.$$

Observe that the direction vector $(0,5,0)$ has x- and z-coordinates 0. And so, l_6 must be perpendicular to both the x- and z-axes.

Indeed, the x-and z-coordinates of *every* point of l_6 are fixed as $x \stackrel{1}{=} 1$ and $z \stackrel{3}{=} 3$.

Hence, the free variable is y. We simply discard $\stackrel{2}{=}$ and say that l_6 may be described by these two cartesian equations:

$$x \stackrel{1}{=} 1 \qquad \text{and} \qquad z \stackrel{3}{=} 3.$$

Example 878. The line l_7 is described by $\mathbf{r} = (1,2,3) + \lambda(4,0,0) \qquad (\lambda \in \mathbb{R})$.

Write out the three cartesian equations:

$$x \stackrel{1}{=} 1 + 4\lambda, \qquad y \stackrel{2}{=} 2 + 0\lambda = 2, \qquad z \stackrel{3}{=} 3 + 0\lambda = 3.$$

Observe that the direction vector $(4,0,0)$ has y- and z-coordinates 0. And so, l_7 must be perpendicular to both the y- and z-axes.

Indeed, the y-and z-coordinates of *every* point of l_7 are fixed as $y \stackrel{2}{=} 2$ and $z \stackrel{3}{=} 3$.

Hence, the free variable is x. We simply discard $\stackrel{1}{=}$ and say that l_7 may be described by these two cartesian equations:

$$y \stackrel{2}{=} 2 \qquad \text{and} \qquad z \stackrel{3}{=} 3.$$

In total, we have **seven possible cases**, depending on which of the direction vector's coordinates are zero. These seven cases are summarised in the following Fact (and were illustrated by the above examples).

Fact 155. *Suppose the line* l *is described by* $\mathbf{r} = (p_1, p_2, p_3) + \lambda(v_1, v_2, v_3)$ $(\lambda \in \mathbb{R})$.

(1) *If* $v_1, v_2, v_3 \neq 0$, *then* l *can be described by*

$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3}.$$

(2) *If* $v_1 = 0$ *and* $v_2, v_3 \neq 0$, *then* l *is perpendicular to the* x*-axis and can be described by*

$$x = p_1 \qquad \textit{and} \qquad \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3}.$$

(3) *If* $v_2 = 0$ *and* $v_1, v_3 \neq 0$, *then* l *is perpendicular to the* y*-axis and can be described by*

$$y = p_2 \qquad \textit{and} \qquad \frac{x - p_1}{v_1} = \frac{z - p_3}{v_3}.$$

(4) *If* $v_3 = 0$ *and* $v_1, v_2 \neq 0$, *then* l *is perpendicular to the* z*-axis and can be described by*

$$z = p_3 \qquad \textit{and} \qquad \frac{x - p_1}{v_1} = \frac{y - p_2}{v_2}.$$

(5) *If* $v_1, v_2 = 0$, *then* l *is perpendicular to the* x*- and* y*-axes and can be described by*

$$x = p_1 \qquad \textit{and} \qquad y = p_2.$$

(6) *If* $v_1, v_3 = 0$, *then* l *is perpendicular to the* x*- and* z*-axes and can be described by*

$$x = p_1 \qquad \textit{and} \qquad z = p_3.$$

(7) *If* $v_2, v_3 = 0$, *then* l *is perpendicular to the* y*- and* z*-axes and can be described by*

$$y = p_2 \qquad \textit{and} \qquad y = p_2.$$

Proof. See p. 1624 (Appendices). □

Exercise 262. Each vector equation below describes a line. Rewrite each into cartesian form. Also, state if each line is perpendicular to any axes. (Answer on p. 1859.)

(a) $\mathbf{r} = (-1, 1, 1) + \lambda(3, -2, 1)$ $(\lambda \in \mathbb{R})$. **(b)** $\mathbf{r} = (5, 6, 1) + \lambda(7, 8, 1)$ $(\lambda \in \mathbb{R})$.
(c) $\mathbf{r} = (0, -3, 1) + \lambda(3, 0, 1)$ $(\lambda \in \mathbb{R})$. **(d)** $\mathbf{r} = (9, 9, 9) + \lambda(1, 0, 0)$ $(\lambda \in \mathbb{R})$.
(e) $\mathbf{r} = (0, 0, 0) + \lambda(4, 8, 5)$ $(\lambda \in \mathbb{R})$. **(f)** $\mathbf{r} = (1, 3, 5) + \lambda(0, -4, 0)$ $(\lambda \in \mathbb{R})$.

67.2. Cartesian to Vector Equations

In the last subchapter, we started with a line's vector equation, then wrote down its cartesian equations. We'll now go the other way round.

To write down a line's vector equation, all we need are a point that's on the line and a direction vector of the line.

Example 879. Let l be the line described by these cartesian equations:

$$\frac{3x-9}{6}=\frac{2y-8}{2}=\frac{z-1}{3}.$$

First, rewrite the above cartesian equations so that the coefficients on x, y, and z are all 1. This is easily done by dividing the numerator and denominator of each fraction by the variable's coefficient:

$$\frac{x-3}{2}=\frac{y-4}{1}=\frac{z-1}{3}.$$

Reading off, the line l contains the point $(3,4,1)$ and has direction vector $(2,1,3)$. So, it can also be described by this vector equation:

$$\mathbf{r}=(3,4,1)+\lambda(2,1,3) \qquad (\lambda\in\mathbb{R}).$$

Example 880. Let l_1 be the line described by $\dfrac{-x+7}{-5}=\dfrac{0.5y+1}{0.3}=z-2.$

Rewrite as $$\frac{x-7}{5}=\frac{y+2}{0.6}=\frac{z-2}{1}.$$

Reading off, l_1 contains the point $(7,-2,2)$ and has direction vector $(5,0.6,1)$. So, it can also be described by

$$\mathbf{r}=(7,-2,2)+\lambda(5,0.6,1) \qquad (\lambda\in\mathbb{R}).$$

Example 881. Let l_2 be the line described by $\dfrac{5x}{2}=\dfrac{y-12}{6}=\dfrac{3z-15}{9}.$

Rewrite as $$\frac{x-0}{0.4}=\frac{y-12}{6}=\frac{z-5}{3}.$$

Reading off, l_2 contains the point $(0,12,5)$ and has direction vector $(0.4,6,3)$. So, it can also be described by

$$\mathbf{r}=(0,12,5)+\lambda(0.4,6,3) \qquad (\lambda\in\mathbb{R})$$

Three examples where **exactly one** of the direction vector's coordinates is zero:

Example 882. Let l_3 be the line described by

$$x \stackrel{1}{=} 17 \qquad \text{and} \qquad \frac{5y-12}{100} \stackrel{2}{=} \frac{3z-15}{9}.$$

By $\stackrel{1}{=}$, *every* point on l_3 has x-coordinate 17. Thus, l_3 is perpendicular to the x-axis.

Rewrite $\stackrel{2}{=}$ as
$$\frac{y-2.4}{20} = \frac{z-5}{3}.$$

Reading off, l_3 contains the point $(17, 2.4, 5)$ and has direction vector $(0, 20, 3)$. So, it can also be described by

$$\mathbf{r} = (17, 2.4, 5) + \lambda(0, 20, 3) \qquad (\lambda \in \mathbb{R}).$$

Example 883. Let l_4 be the line described by

$$y \stackrel{1}{=} -2 \qquad \text{and} \qquad \frac{-x}{3} \stackrel{2}{=} \frac{z+10}{-5}.$$

By $\stackrel{1}{=}$, *every* point on l_4 has y-coordinate -2. Thus, l_4 is perpendicular to the y-axis.

Rewrite $\stackrel{2}{=}$ as
$$\frac{x}{-3} = \frac{z-(-10)}{-5}.$$

Reading off, l_4 contains the point $(0, -2, -10)$ and has direction vector $(-3, 0, -5)$. So, it can also be described by

$$\mathbf{r} = (0, -2, -10) + \lambda(-3, 0, -5) \qquad (\lambda \in \mathbb{R}).$$

Example 884. Let l_5 be the line described by

$$4z \stackrel{1}{=} 3 \qquad \text{and} \qquad \frac{7x-6}{35} \stackrel{2}{=} \frac{2y+10}{18}.$$

By $\stackrel{1}{=}$, *every* point on l_5 has z-coordinate $3/4$. Thus, l_5 is perpendicular to the z-axis.

Rewrite $\stackrel{2}{=}$ as
$$\frac{x-6/7}{5} = \frac{y-(-5)}{9}.$$

Reading off, l_5 contains the point $(6/7, -5, 3/4)$ and has direction vector $(5, 9, 0)$. So, it can also be described by

$$\mathbf{r} = (6/7, -5, 3/4) + \lambda(5, 9, 0) \qquad (\lambda \in \mathbb{R}).$$

Three examples where **exactly two** of the direction vector's coordinates are zero:

Example 885. Let l_6 be the line described by $x = 5$ and $z = 9$.

Every point on l_6 has x- and z-coordinates 5 and 9. Hence, the direction vector must have 0 as its x- and z-coordinates. (Equivalently, this line must be perpendicular to both the x- and z-axes.)

The free variable is y. Altogether then, l_6 contains exactly those points $(5, y, 9)$, for all real numbers y. For example, it contains the points $(5, 0, 9)$ and $(5, -100, 9)$. Hence, for any non-zero k, $(0, k, 0)$ is a direction vector of l_6.

For simplicity, we pick $(0, 1, 0)$ as our direction vector and describe l_6 by

$$\mathbf{r} = (5, 0, 9) + \lambda(0, 1, 0) \qquad (\lambda \in \mathbb{R}).$$

Example 886. Let l_7 be the line described by $3x = -12$ and $y = 0$.

Every point on l_7 has x- and y-coordinates -4 and 0. Hence, the direction vector must have 0 as its x- and y-coordinates. (Equivalently, this line must be perpendicular to both the x- and y-axes.)

The free variable is z. Altogether then, l_7 contains exactly those points $(-4, 0, z)$, for all real numbers z. For example, it contains the points $(-4, 0, 0)$ and $(-4, 0, \pi)$. Hence, for any non-zero k, $(0, 0, k)$ is a direction vector of l_7.

For simplicity, we pick $(0, 0, 1)$ as our direction vector and describe l_7 by

$$\mathbf{r} = (-4, 0, 0) + \lambda(0, 0, 1) \qquad (\lambda \in \mathbb{R}).$$

Example 887. Let l_8 be the line described by $y = -11$ and $-4z = 52$.

Every point on l_8 has y- and z-coordinates -11 and -13. Hence, the direction vector must have 0 as its y- and z-coordinates. (Equivalently, this line must be perpendicular to both the y- and z-axes.)

The free variable is x. Altogether then, l_8 contains exactly those points $(x, -11, -13)$, for all real numbers x. For example, it contains the points $(0, -11, -13)$ and $\left(\sqrt{2}, -11, -13\right)$. Hence, for any non-zero k, $(k, 0, 0)$ is a direction vector of l_8.

For simplicity, we pick $(1, 0, 0)$ as our direction vector and describe l_8 by

$$\mathbf{r} = (0, -11, -13) + \lambda(1, 0, 0) \qquad (\lambda \in \mathbb{R}).$$

Exercise 263. Each pair of cartesian equations below describes a line. Rewrite each into vector form. State if each is perpendicular to any axes. (Answer on p. 1859.)

(a) $\dfrac{7x-2}{5} = \dfrac{0.3y-5}{7} = \dfrac{8z}{7}$.

(b) $2x = 3y = 5z$.

(c) $17x - 4 = \dfrac{3y-1}{2} = 3z$.

(d) $3y = 11$ and $\dfrac{x-3}{2} = \dfrac{5z-2}{7}$.

(e) $\dfrac{x}{5} = 13$ and $2z = 1$.

(f) $13x + 5 = 0$ and $y = 5z - 2$.

67.3. Parallel and Perpendicular Lines

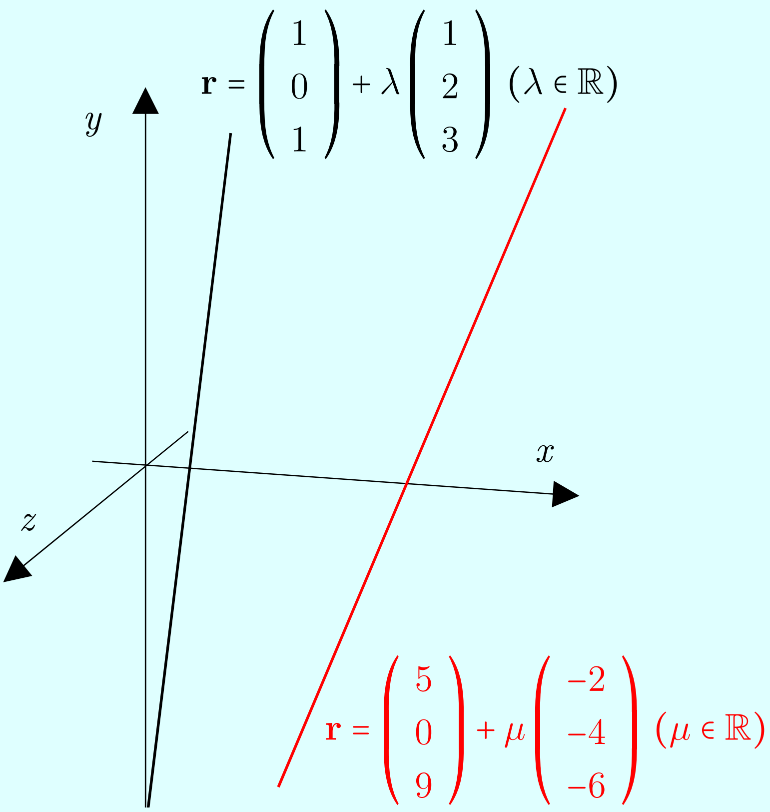
Our definition of when two lines in 3D space are **parallel** or **perpendicular** is exactly the same as before and now reproduced:

Definition 150. Two lines are **(a)** *parallel* if they have parallel direction vectors; and **(b)** *perpendicular* if they have perpendicular direction vectors.

Example 888. Suppose two lines are described by

$$\mathbf{r} = (1,0,1) + \lambda(1,2,3) \qquad \text{and} \qquad \mathbf{r} = (5,0,9) + \mu(-2,-4,-6) \qquad (\lambda, \mu \in \mathbb{R}).$$

Since $(1,2,3) \parallel (-2,-4,-6)$, by the above Definition, the two lines are parallel.



Example 889. Suppose two lines are described by

$$\mathbf{r} = (5,-1,4) + \lambda(8,2,-1) \qquad \text{and} \qquad \mathbf{r} = (3,1,6) + \mu(1,-2,4) \qquad (\lambda, \mu \in \mathbb{R}).$$

We have $(8,2,-1) \cdot (1,-2,4) = 8 - 4 - 4 = 0$. So, $(8,2,-1) \perp (1,-2,4)$ and by the above Definition, the two lines are perpendicular.

Example 890. Suppose two lines are described by

$$\mathbf{r} = (3,2,-1) + \lambda(1,0,0) \qquad \text{and} \qquad \mathbf{r} = (0,0,0) + \mu(1,1,0) \qquad (\lambda, \mu \in \mathbb{R}).$$

Since $(1,0,0) \nparallel (1,1,0)$ and $(1,0,0) \not\perp (1,1,0)$, the two lines are neither parallel nor perpendicular.

The following Fact remains true in 3D space:

Fact 139. *Suppose two lines are ...*

(a) *Identical. Then they are also parallel.*

(b) *Distinct and parallel. Then they do not intersect.*

(c) *Distinct. Then they share at most one intersection point.*

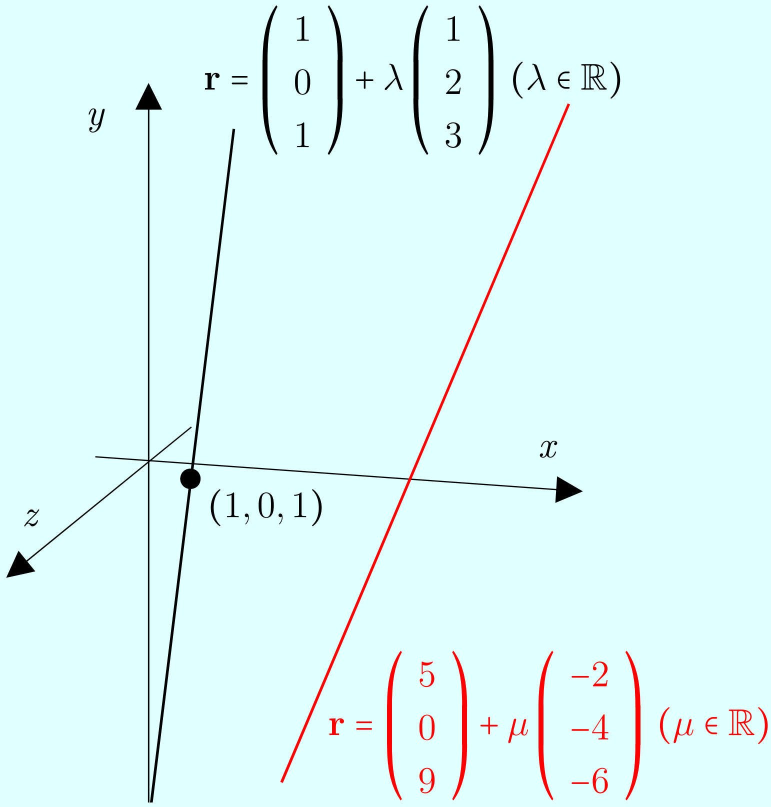
Example 891. As in Example 888, two lines are described by

$$\mathbf{r} = (1,0,1) + \lambda\,(1,2,3) \text{ and}$$
$$\mathbf{r} = (5,0,9) + \mu\,(-2,-4,-6) \quad (\lambda,\mu \in \mathbb{R}).$$

Since $(1,2,3) \parallel (-2,-4,-6)$, the two lines are parallel.

Observe that the point $(1,0,1)$ is on the first line (plug in $\lambda = 0$), but not on the second (to see this, observe that the only point on the second line with x-coordinate 1 corresponds to $\mu = 2$). Thus, the two lines are distinct.

Since they are parallel and distinct, by Fact 139(b), they do not intersect.



Example 892. Two lines are described by

$$\mathbf{r} = (3,6,9) + \lambda\,(1,2,3) \qquad \text{and} \qquad \mathbf{r} = (0,0,0) + \mu\,(-2,-4,-6) \qquad (\lambda,\mu \in \mathbb{R}).$$

Since $(1,2,3) \parallel (-2,-4,-6)$, by the above Definition, the two lines are parallel.

Observe that the point $(3,6,9)$ is on both lines (plug in $\lambda = 0$ and $\mu = -1.5$).

Since the two lines are parallel and do intersect, by Fact 139(b), they cannot be distinct. Equivalently, they must be identical.

We will next learn how to determine whether two lines in 3D space intersect and if they do, how to find their intersection point.

67.4. Intersecting Lines

Fact 139(c) says that two distinct lines share at most one intersection point. Here are two examples of when two distinct lines in 3D space intersect and how we go about finding their (only) intersection point.

Example 893. Two lines are described by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

These two lines are not parallel and hence distinct. And so, by Fact 139(c), they share at most one intersection point.

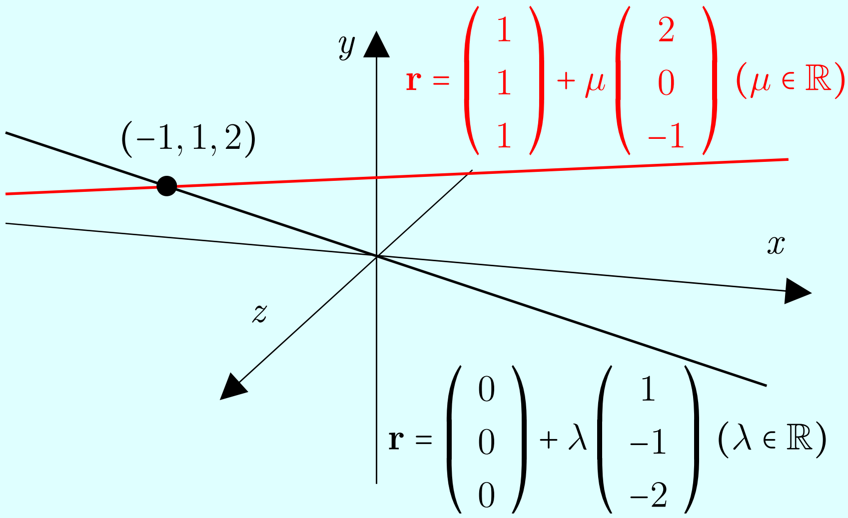
Suppose they intersect. Then there must be real numbers $\hat{\lambda}$ and $\hat{\mu}$ such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \hat{\mu} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} \hat{\lambda} &\overset{1}{=} 1 + 2\hat{\mu}, \\ -\hat{\lambda} &\overset{2}{=} 1, \\ -2\hat{\lambda} &\overset{3}{=} 1 - \hat{\mu}. \end{aligned}$$

From $\overset{2}{=}$, $\hat{\lambda} = -1$. Plug this into $\overset{1}{=}$ to get $\hat{\mu} = -1$. You can verify that these values of $\hat{\lambda}$ and $\hat{\mu}$ also satisfy $\overset{3}{=}$. Hence, the two lines do indeed intersect.

To find their intersection point, plug $\hat{\lambda} = -1$ or $\hat{\mu} = -1$ into either line's vector equation:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \underbrace{\hat{\lambda}}_{-1} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \underbrace{\hat{\mu}}_{-1} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$



Example 894. Two lines are described by

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

These two lines are not parallel and hence distinct. And so, by Fact 139(c), they share at most one intersection point.

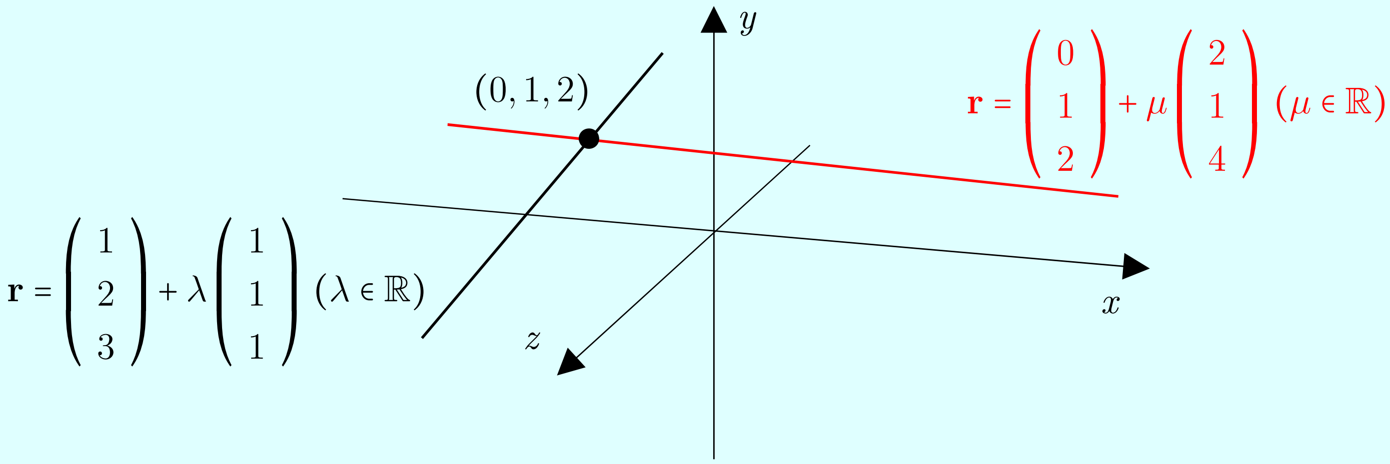
Suppose they intersect. Then there must be real numbers $\hat{\lambda}$ and $\hat{\mu}$ such that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \hat{\mu} \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} 1 + \hat{\lambda} &\overset{1}{=} 2\hat{\mu}, \\ 2 + \hat{\lambda} &\overset{2}{=} 1 + \hat{\mu}, \\ 3 + \hat{\lambda} &\overset{3}{=} 2 + 4\hat{\mu}. \end{aligned}$$

$\overset{1}{=}$ minus $\overset{2}{=}$ yields $-1 = \hat{\mu} - 1$ or $\hat{\mu} = 0$. Plug this into $\overset{1}{=}$ to get $\hat{\lambda} = -1$. You can verify that these values of $\hat{\lambda}$ and $\hat{\mu}$ also satisfy $\overset{3}{=}$. Hence, the two lines intersect.

To find their intersection point, plug $\hat{\lambda} = -1$ or $\hat{\mu} = 0$ into either line's vector equation:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \underbrace{\hat{\lambda}}_{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \underbrace{\hat{\mu}}_{0} \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$



67.5. Skew Lines

Recall[346] that in 2D space, two non-parallel lines *must* intersect. In contrast, in 3D space, two non-parallel lines need not intersect. We have a special name for such lines:

Definition 172. Two lines are said to be *skew* if they are not parallel and do not intersect.

Example 895. Two lines are described by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

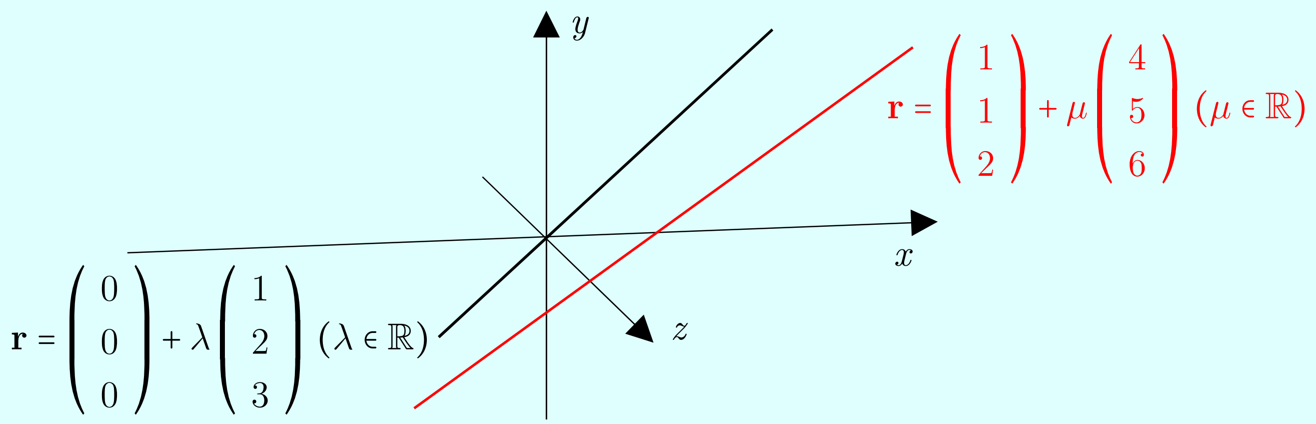
These two lines are not parallel and hence distinct. And so by Fact 139(c), they share at most one intersection point.

To check if they intersect, suppose there are real numbers $\hat{\lambda}$ and $\hat{\mu}$ such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \stackrel{\star}{=} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \hat{\mu} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} \hat{\lambda} &\stackrel{1}{=} 1 + 4\hat{\mu}, \\ 2\hat{\lambda} &\stackrel{2}{=} 1 + 5\hat{\mu}, \\ 3\hat{\lambda} &\stackrel{3}{=} 2 + 6\hat{\mu}. \end{aligned}$$

Now, $2 \times \stackrel{1}{=}$ minus $\stackrel{2}{=}$ yields $0 = 1 + 3\hat{\mu}$ or $\hat{\mu} = -1/3$. Plug this back into $\stackrel{1}{=}$ to get $\hat{\lambda} = -1/3$. But these values of $\hat{\lambda}$ and $\hat{\mu}$ contradict $\stackrel{3}{=}$.

This contradiction means that there are no real numbers $\hat{\lambda}$ and $\hat{\mu}$ such that $\stackrel{\star}{=}$ holds. In other words, the two lines **do not intersect**. And since they are not parallel either, by the above Definition, they are **skew**.



[346] Fact 140.

Example 896. Two lines are described by

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \quad (\lambda, \mu \in \mathbb{R}).$$

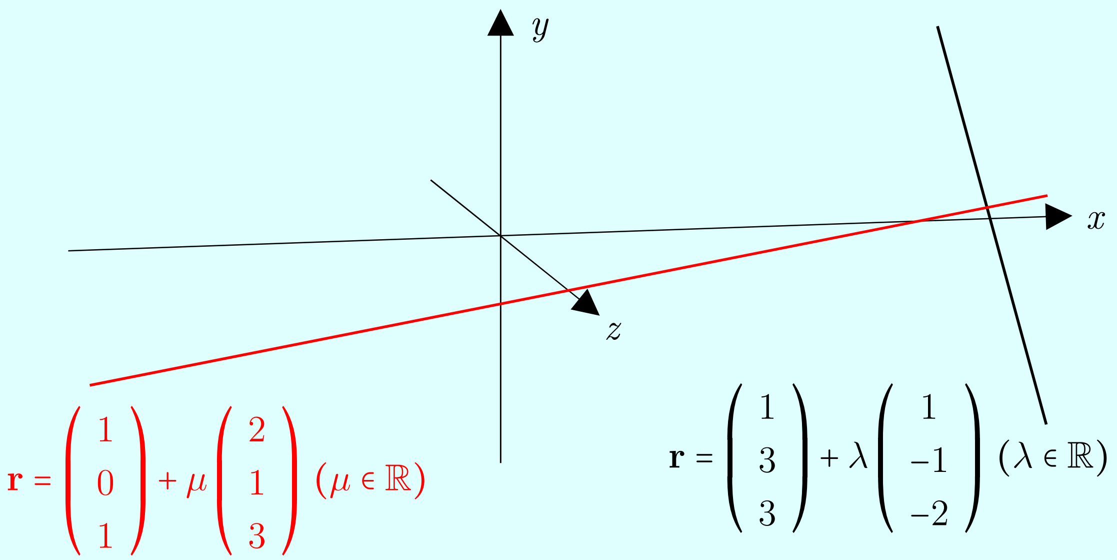
These two lines are not parallel and hence distinct. And so by Fact 139(c), they share at most one intersection point.

To check if they intersect, suppose there are real numbers $\hat{\lambda}$ and $\hat{\mu}$ such that

$$\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \stackrel{\star}{=} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \hat{\mu} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \text{or} \quad \begin{aligned} 1 + \hat{\lambda} &\stackrel{1}{=} 1 + 2\hat{\mu}, \\ 3 - \hat{\lambda} &\stackrel{2}{=} \hat{\mu}, \\ 3 - 2\hat{\lambda} &\stackrel{3}{=} 1 + 3\hat{\mu}. \end{aligned}$$

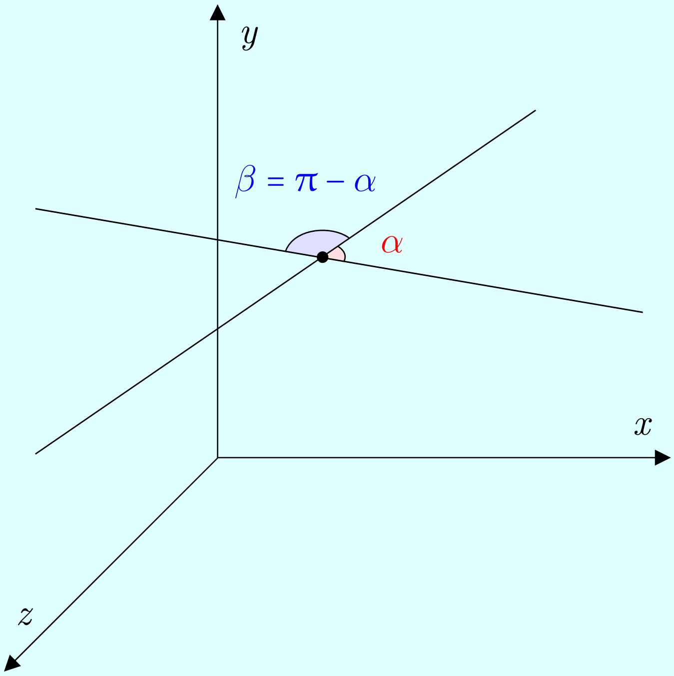
Now, $\stackrel{1}{=}$ minus $2\times \stackrel{2}{=}$ minus yields $3\hat{\lambda} - 5 = 1$ or $\hat{\lambda} = 2$. Plug this back into $\stackrel{1}{=}$ to get $\hat{\mu} = 1$. But these values of $\hat{\lambda}$ and $\hat{\mu}$ contradict $\stackrel{3}{=}$.

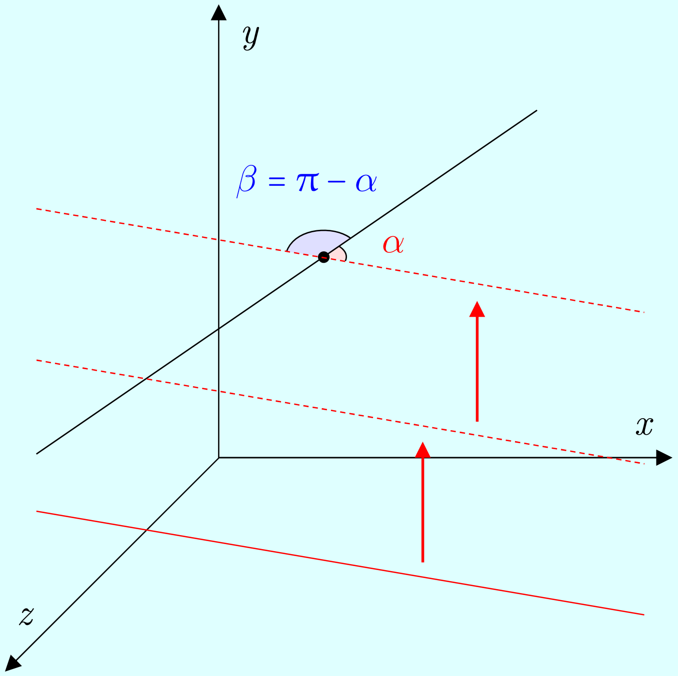
And so again, the two lines **do not intersect**. And since they are not parallel either, they are **skew**.



Example 897. As in 2D space, two intersecting lines in 3D space form two angles α and $\beta = \pi - \alpha$ at their intersection point. We define the smaller of these two angles to be the angle between the two lines.

So in the figure on the right, the angle between the two lines is α and not β.





In the figure on the left, the black and solid red lines **do not intersect**. But even so, we will still find it useful to talk about the angle between them.

To do so, translate the red line upwards so that it intersects the black line. As usual, two angles α and $\beta = \pi - \alpha$ are formed at the intersection point. We then define the angle between the black and solid red lines to be the smaller of these two angles, namely α.

Our formal definition of the **angle between two lines** is reproduced from before:

Definition 149. Given two lines, pick for each any direction vector. We call the non-obtuse angle between these two vectors the *angle between the two lines*.

And so, we have the same results as before:

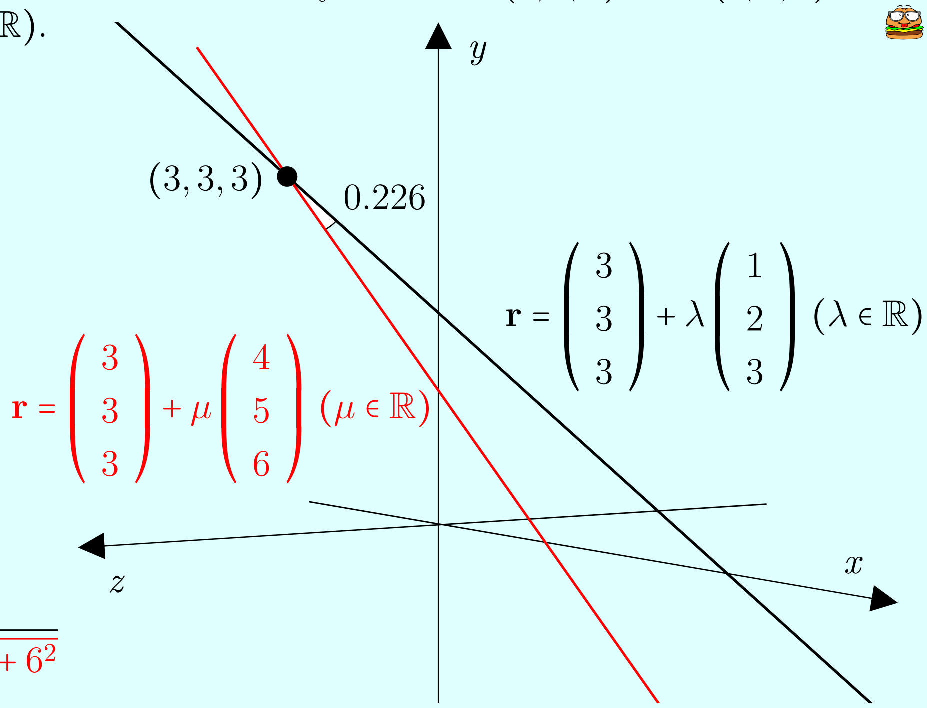
Corollary 22. *The angle between two lines with direction vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\cos^{-1} \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Corollary 23. *Suppose θ is the angle between two lines l_1 and l_2. Then **(a)** $\theta = 0 \iff l_1 \parallel l_2$; **(b)** $\theta = \pi/2 \iff l_1 \perp l_2$.*

First, two examples where we find the angle between two intersecting lines:

Example 898. Two lines are described by $\mathbf{r} = (3,3,3) + \lambda(1,2,3)$ and $\mathbf{r} = (3,3,3) + \mu(4,5,6)$ $(\lambda, \mu \in \mathbb{R})$.



Observe that these two lines intersect at $(3,3,3)$. (And so, they aren't skew.)

The angle between these two lines is the non-obtuse angle between their direction vectors and is given by Corollary 22:

$$\cos^{-1} \frac{|(1,2,3) \cdot (4,5,6)|}{|(1,2,3)|\,|(4,5,6)|}$$

$$= \cos^{-1} \frac{|4+10+18|}{\sqrt{1^2+2^2+3^2}\sqrt{4^2+5^2+6^2}}$$

$$= \cos^{-1} \frac{32}{\sqrt{14}\sqrt{77}} \approx 0.226.$$

This angle is neither zero nor right. And so by Corollary **23**, the two lines are neither parallel nor perpendicular.

Example 899. Two lines are described by

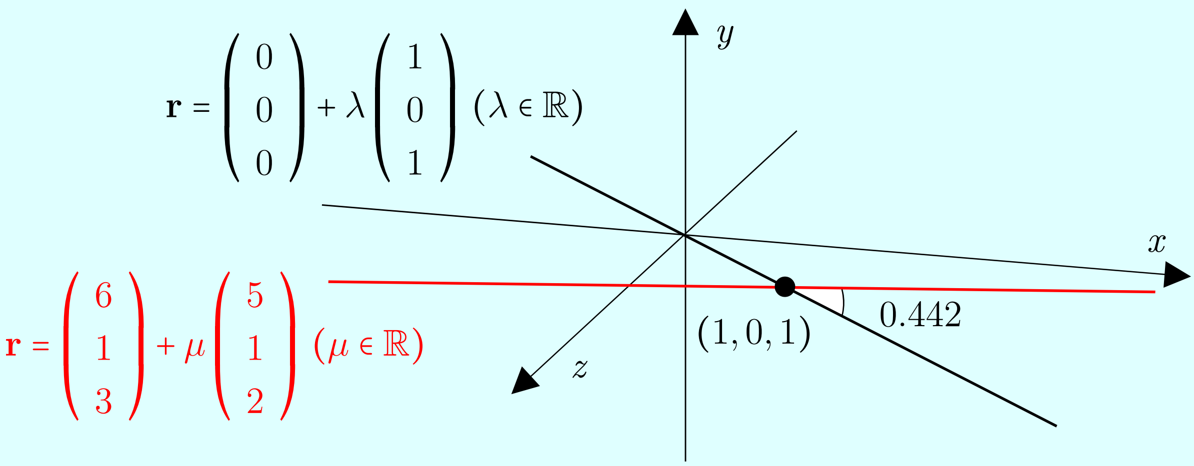
$$\mathbf{r} = (0,0,0) + \lambda(1,0,1) \qquad \text{and} \qquad \mathbf{r} = (6,1,3) + \mu(5,1,2) \qquad (\lambda, \mu \in \mathbb{R}).$$

Observe that these two lines intersect at $(1,0,1)$. (And so, they are not skew.)

Again, the angle between these two lines is given by Corollary 22:

$$\cos^{-1} \frac{|(1,0,1) \cdot (5,1,2)|}{|(1,0,1)|\,|(5,1,2)|} = \cos^{-1} \frac{|5+0+2|}{\sqrt{1^2+0^2+1^2}\sqrt{5^2+1^2+2^2}} = \cos^{-1} \frac{7}{\sqrt{2}\sqrt{30}} \approx 0.442.$$

This angle is neither zero nor right. And so by Corollary **23**, the two lines are neither parallel nor perpendicular.



And now, two examples where we find the angle between two **non-intersecting lines**.

Example 900. Two lines are described by

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \hat{\mu} \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

If they intersect, then there are real numbers $\hat{\lambda}$ and $\hat{\mu}$ such that

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \overset{\star}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \hat{\mu} \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} 1 + \hat{\lambda} &\overset{1}{=} 0, \\ 2 + 2\hat{\lambda} &\overset{2}{=} 3\hat{\mu}, \\ 2 + \hat{\lambda} &\overset{3}{=} -2\hat{\mu}. \end{aligned}$$

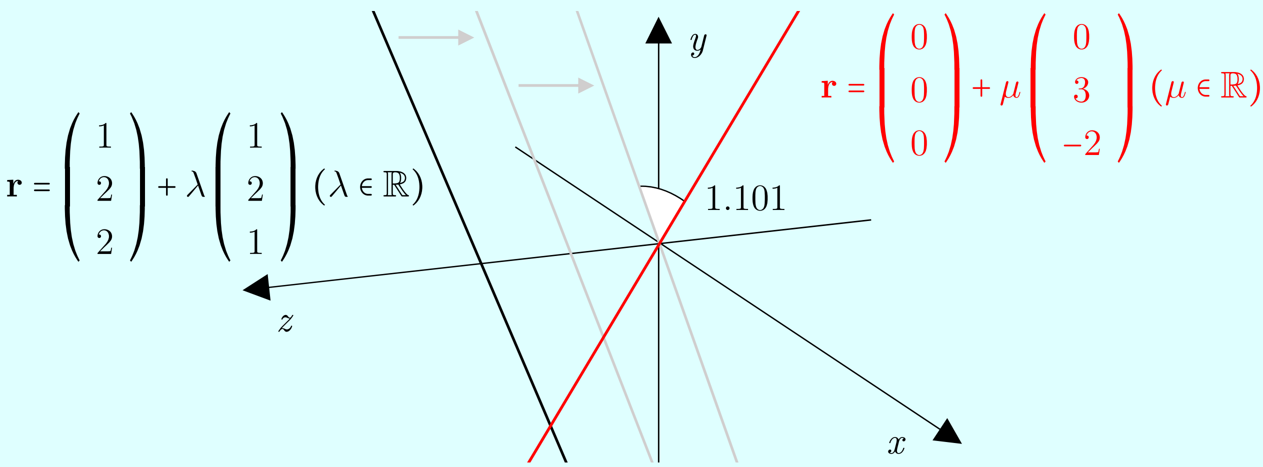
From $\overset{1}{=}$, $\hat{\lambda} = -1$. Plug this into $\overset{2}{=}$ to get $\hat{\mu} = 0$. But now, these values of $\hat{\lambda}$ and $\hat{\mu}$ contradict $\overset{3}{=}$. Hence, the two lines do not intersect.

Even though the two lines do not intersect, we will still find it useful to talk about the angle between them. This we can compute as usual:

$$\cos^{-1} \frac{|(1,2,1) \cdot (0,3,-2)|}{|(1,2,1)||(0,3,-2)|} = \cos^{-1} \frac{|0+6-2|}{\sqrt{1^2+2^2+1^2}\sqrt{0^2+3^2+(-2)^2}} = \cos^{-1} \frac{4}{\sqrt{6}\sqrt{13}} \approx 1.101.$$

This angle is neither zero nor right. And so by Corollary **23**, the two lines are neither parallel nor perpendicular.

Since the two lines do not intersect and are not parallel, they are skew.



The two lines do not intersect. Nonetheless, we can always translate one of the two lines so that they intersect. In the above figure, we've translated the black line so that it intersects the red line at the origin.

Example 901. Two lines are described by

$$\mathbf{r} = (0,1,2) + \lambda(9,1,3) \qquad \text{and} \qquad \mathbf{r} = (4,5,6) + \mu(3,2,1) \qquad (\lambda, \mu \in \mathbb{R}).$$

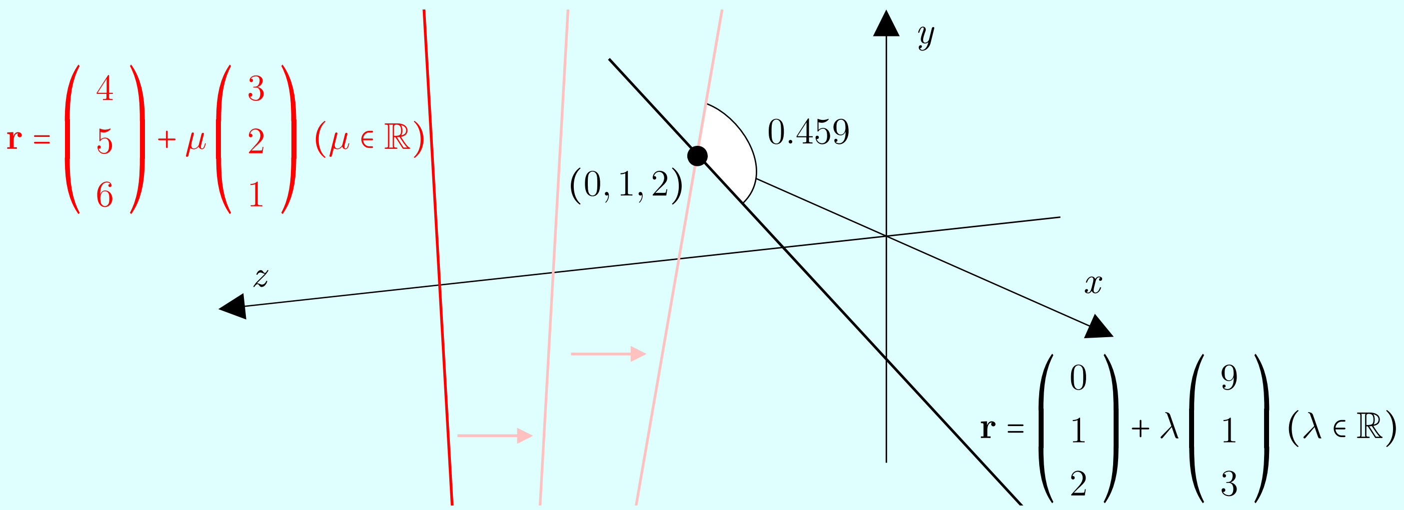
If they intersect, then there are real numbers $\hat{\lambda}$ and $\hat{\mu}$ such that

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} 9 \\ 1 \\ 3 \end{pmatrix} \overset{\star}{=} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \hat{\mu} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} 9\hat{\lambda} &\overset{1}{=} 4 + 3\hat{\mu}, \\ 1 + \hat{\lambda} &\overset{2}{=} 5 + 2\hat{\mu}, \\ 2 + 3\hat{\lambda} &\overset{3}{=} 6 + \hat{\mu}. \end{aligned}$$

$\overset{1}{=}$ minus $3\times$ $\overset{3}{=}$ yields $-6 = -14$, which is a contradiction. Hence, the two lines do not intersect. Nonetheless, we can as usual compute the angle between them:

$$\cos^{-1} \frac{|(9,1,3) \cdot (3,2,1)|}{|(9,1,3)||(3,2,1)|} = \cos^{-1} \frac{|27+2+3|}{\sqrt{9^2+1^2+3^2}\sqrt{3^2+2^2+1^2}} = \cos^{-1} \frac{32}{\sqrt{91}\sqrt{14}} \approx 0.459.$$

This angle is neither zero nor right. And so by Corollary **23**, the two lines are neither parallel nor perpendicular. Since they do not intersect either, they are skew.



The two lines do not intersect. Nonetheless, we can always translate one of the two lines so that they intersect. In the above figure, we've translated the red line so that it intersects the black line at the point $(0,1,2)$.

Exercise 264. Each of **(a)**–**(d)** gives a pair of lines in vector form. Find any intersection points and the angle between the two lines. State if the two lines are parallel, perpendicular, identical, or skew. (Answer on p. 1860.)

(a) $\mathbf{r} = (0,1,1) + \lambda(1,-1,1)$ and $\mathbf{r} = (1,3,3) + \mu(0,0,2)$.

(b) $\mathbf{r} = (-1,2,3) + \lambda(0,1,0)$ and $\mathbf{r} = (0,0,0) + \mu(8,-3,5)$.

(c) $\mathbf{r} = (7,3,4) + \lambda(8,3,4)$ and $\mathbf{r} = (9,3,7) + \mu(3,-4,-3)$.

(d) $\mathbf{r} = (0,0,1) + \lambda(1,2,1)$ and $\mathbf{r} = (1,0,0) + \mu(-3,-6,-3)$.

67.7. Collinearity (in 3D)

Our Definition of **collinearity** is the same as before:

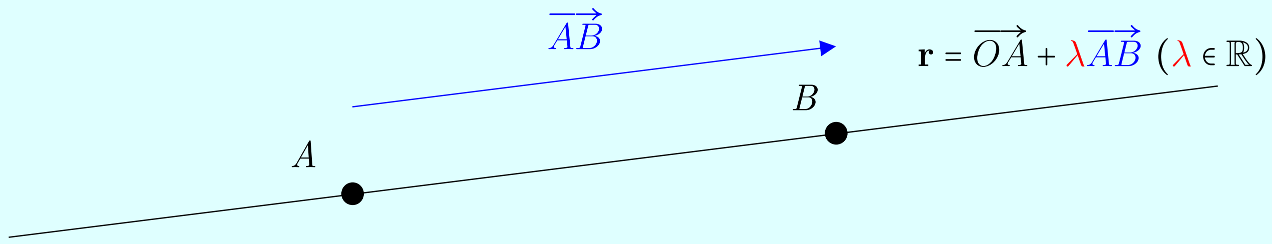
Definition 154. Two or more points are *collinear* if some line contains all of them.

Fact 144 is reproduced from before and says that any two points are always collinear:

Fact 144. *Suppose A and B are distinct points. Then the unique line that contains both A and B is described by*

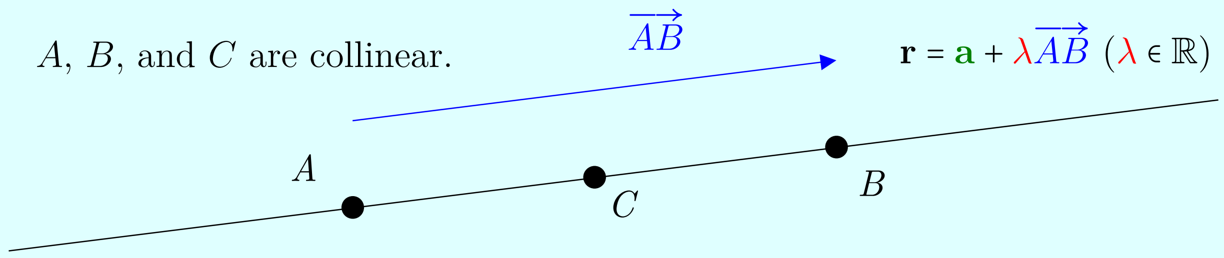
$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}, \qquad (\lambda \in \mathbb{R}).$$

Example 902. Any two points A and B are collinear.

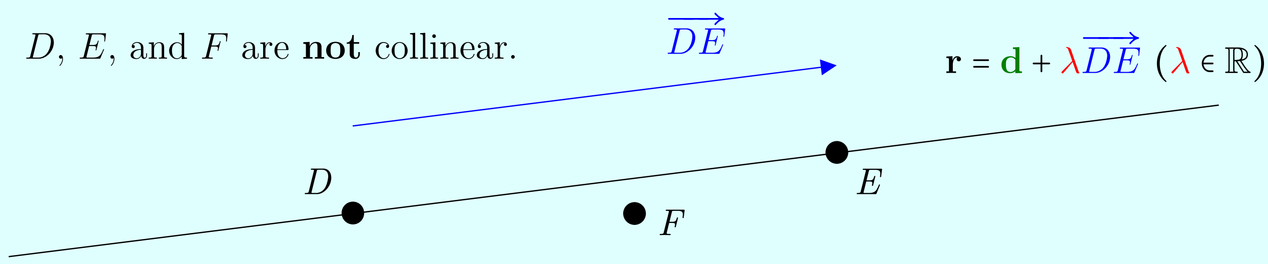


And as before, three distinct points could be collinear but won't always generally be:

Example 903. The points A, B, and C are collinear:



In contrast, the points D, E, and F are not collinear.



We'll use the exact same procedure to check whether three points are collinear:

1. First use Fact 144 to write down the unique line that contains two of the three points.
2. Then check whether this line also contains the third point.

Two examples:

Example 904. Let $A = (1, 2, 3)$, $B = (4, 5, 6)$, and $C = (7, 8, 9)$ be points.

To check if they are collinear,

1. First write down the unique line that contains both A and B:

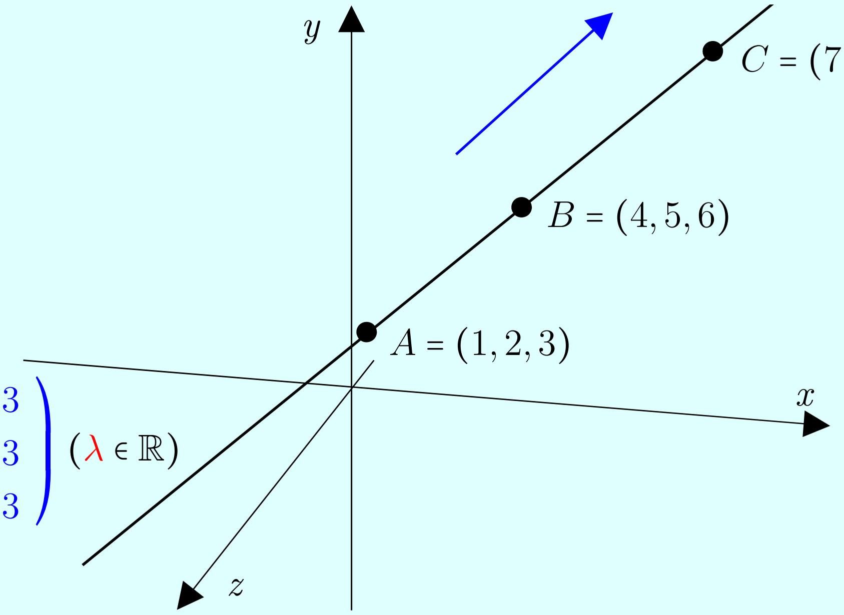
$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \qquad (\lambda \in \mathbb{R}).$$

2. This line also contains C if and only if there exists $\hat{\lambda} \in \mathbb{R}$ such that

$$C = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} 7 &\stackrel{1}{=} 1 + 3\hat{\lambda}, \\ 8 &\stackrel{2}{=} 2 + 3\hat{\lambda}, \\ 9 &\stackrel{3}{=} 3 + 3\hat{\lambda}. \end{aligned}$$

As you can verify, $\hat{\lambda} = 2$ solves the above vector equation (or system of three equations). So, our line also contains C.

We conclude that A, B, and C are collinear.



$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \quad (\lambda \in \mathbb{R})$$

Example 905. Let $D = (1,0,0)$, $E = (0,1,0)$, and $F = (0,0,1)$ be points. To check if they are collinear,

1. First write down a line that contains both D and E:

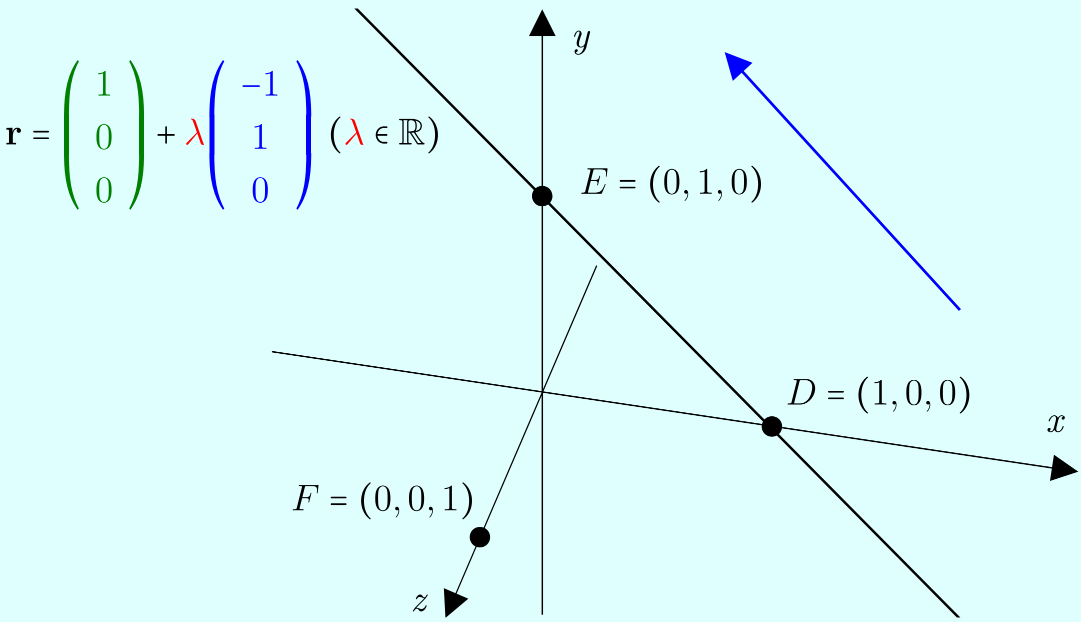
$$\mathbf{r} = \overrightarrow{OD} + \lambda\overrightarrow{DE} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad (\lambda \in \mathbb{R}).$$

2. This line also contains F if and only if there exists $\hat{\lambda} \in \mathbb{R}$ such that

$$F = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} 0 &\overset{1}{=} 1 - 1\hat{\lambda}, \\ 0 &\overset{2}{=} 0 + 1\hat{\lambda}, \\ 0 &\overset{3}{=} 0 + 0\hat{\lambda}. \end{aligned}$$

From $\overset{1}{=}$, we have $\hat{\lambda} = 1$. But this contradicts $\overset{2}{=}$. This contradiction means that there is no solution to the above vector equation (or system of three equations). So, the line we wrote down above does **not** contain F.

We conclude that D, E, and F are **not** collinear.



Exercise 265. Determine if A, B, and C are collinear. (Answer on p. 1861.)

(a) $A = (3,1,2)$, $B = (1,6,5)$, and $C = (0,-1,0)$.

(b) $A = (1,2,4)$, $B = (0,0,1)$, and $C = (3,6,10)$.

68. The Vector Product (in 3D)

In 2D space, the vector product was simply a scalar (real number):

Example 906. Given $\mathbf{u} = (1,2)$ and $\mathbf{v} = (3,4)$, their vector product $\mathbf{u} \times \mathbf{v}$ is the number $1 \times 4 - 2 \times 3 = -2$.

In contrast, in 3D space, the vector product will be a **vector** (hence the name)!

As we'll see later, we'll often have the need to find a vector that's perpendicular to two other vectors. It is this need that motivates the concept of the vector product (in 3D).

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors. Can we find some vector $\mathbf{c} = (c_1, c_2, c_3)$ that's perpendicular to both $\mathbf{a}$ and $\mathbf{b}$?

Well, if $\mathbf{c} \perp \mathbf{a}, \mathbf{b}$, then $\mathbf{a} \cdot \mathbf{c} \stackrel{1}{=} 0$ and $\mathbf{b} \cdot \mathbf{c} \stackrel{2}{=} 0$. Or,

$$
\begin{aligned}
(a_1, a_2, a_3) \cdot (c_1, c_2, c_3) &= a_1 c_1 + a_2 c_2 + a_3 c_3 \stackrel{1}{=} 0, \\
(b_1, b_2, b_3) \cdot (c_1, c_2, c_3) &= b_1 c_1 + b_2 c_2 + b_3 c_3 \stackrel{2}{=} 0.
\end{aligned}
$$

Our goal is to find $\mathbf{c}$ that solves $\stackrel{1}{=}$ and $\stackrel{2}{=}$. Observe that $b_3 \times \stackrel{1}{=}$ minus $a_3 \times \stackrel{2}{=}$ yields

$$
\begin{aligned}
0 &\stackrel{3}{=} a_1 b_3 c_1 + a_2 b_3 c_2 \cancel{+ a_3 b_3 c_3} - a_3 b_1 c_1 - a_3 b_2 c_2 \cancel{- a_3 b_3 c_3} \\
&= c_2 \left(a_2 b_3 - a_3 b_2 \right) - c_1 \left(a_3 b_1 - a_1 b_3 \right).
\end{aligned}
$$

Now, notice that $c_1 \stackrel{4}{=} a_2 b_3 - a_3 b_2$ and $c_2 \stackrel{5}{=} a_3 b_1 - a_1 b_3$ solves $\stackrel{3}{=}$.

Next, get the corresponding value of c_3 by plugging $\stackrel{4}{=}$ and $\stackrel{5}{=}$ into $\stackrel{1}{=}$:

$$
0 = a_1 \overbrace{(\cancel{a_2 b_3} - a_3 b_2)}^{c_1} + a_2 \overbrace{(a_3 b_1 \cancel{- a_1 b_3})}^{c_2} + a_3 c_3 = -a_1 \cancel{a_3} b_2 + a_2 \cancel{a_3} b_1 + \cancel{a_3} c_3 \quad \text{or} \quad c_3 = a_1 b_2 - a_2 b_1.
$$

Hence, a vector that solves $\stackrel{1}{=}$ and $\stackrel{2}{=}$ (i.e. is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$) is

$$
\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.
$$

We will simply use the above as our Definition of the **vector product**:

Definition 173. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors. Their *vector product*, denoted $\mathbf{a} \times \mathbf{b}$, is this vector:

$$
\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.
$$

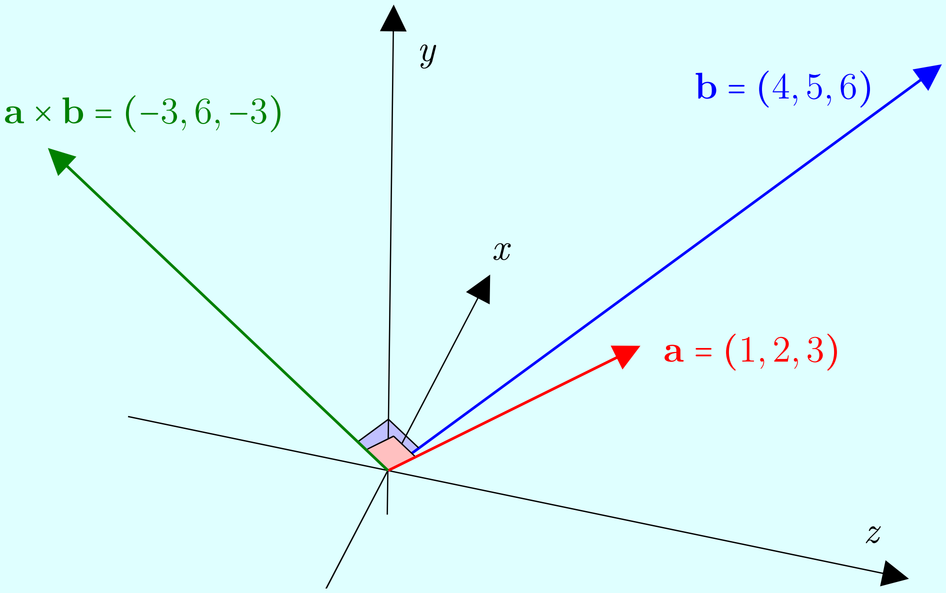
By the way, no need to *mug* Definition 173, because it's already on List MF26 (p. 4).

Example 907. The vector product of $\mathbf{a} = (1,2,3)$ and $\mathbf{b} = (4,5,6)$ is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \cdot 6 - 3 \cdot 5 \\ 3 \cdot 4 - 1 \cdot 6 \\ 1 \cdot 5 - 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$$

From our above discussion, we already know that $\mathbf{a} \times \mathbf{b} \perp \mathbf{a}, \mathbf{b}$. Indeed, this was *the* geometric property that motivated our definition of the vector product. Nonetheless, as an exercise, let's go ahead and verify that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (-3,6,-3) \cdot (1,2,3) = -3 + 12 - 9 = 0, \quad ✓$$
$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = (-3,6,-3) \cdot (4,5,6) = -12 + 30 - 18 = 0. \quad ✓$$

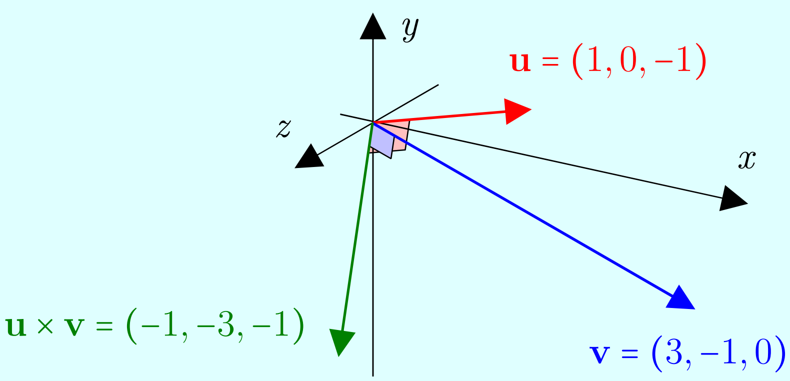


Example 908. The vector product of $\mathbf{u} = (1,0,-1)$ and $\mathbf{v} = (3,-1,0)$ is

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 - (-1) \cdot (-1) \\ -1 \cdot 3 - 1 \cdot 0 \\ 1 \cdot (-1) - 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}.$$

Again, let's verify that $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}, \mathbf{v}$:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (-1,-3,-1) \cdot (1,0,-1) = -1 + 0 + 1 = 0, \quad ✓$$
$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (-1,-3,-1) \cdot (3,-1,0) = -3 + 3 + 0 = 0. \quad ✓$$



Here's the formal statement of the vector product's key geometric property:

Fact 156. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are vectors with* $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$*. Then* $\mathbf{a} \times \mathbf{b} \perp \mathbf{a}, \mathbf{b}$*.*

Proof. See Exercise 266(c). □

Exercise 266. Let $\mathbf{u} = (0, 1, 2)$, $\mathbf{v} = (3, 4, 5)$, $\mathbf{w} = (-1, -2, -3)$, and $\mathbf{x} = (1, 0, 5)$.

(a) Find $\mathbf{u} \times \mathbf{v}$ and verify that $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}, \mathbf{v}$.

(b) Find $\mathbf{w} \times \mathbf{x}$ and verify that $\mathbf{w} \times \mathbf{x} \perp \mathbf{w}, \mathbf{x}$. (Answer on p. 1862.)

(c) Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Prove that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.

All of our results about the vector product in 2D space continue to hold in 3D space and are now reproduced. First, it remains true that the vector product is **distributive** and **anti-commutative**. Moreover, the **vector product of a vector with itself is zero**:

Fact 145. *Suppose* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors. Then*

(a)	$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$	**(Distributive over Addition)**
(b)	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	**(Anti-Commutative)**
(c)	$\mathbf{a} \times \mathbf{a} = \mathbf{0}$	**(Self Vector Product Is Zero)**

Proof. See Exercise 267(a), (b), and (c). □

Next, we have the following "obvious" property reproduced from before:

Fact 147. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are vectors and* $c \in \mathbb{R}$*. Then*

$$(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}).$$

Proof. See Exercise 267(d). □

Exercise 267. Suppose $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$, and $d \in \mathbb{R}$.

(a) Prove that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

(b) First verify that $(4, 5, 6) \times (1, 2, 3) = -(1, 2, 3) \times (4, 5, 6)$. Then prove that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

(c) Prove that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

(d) Let $d \in \mathbb{R}$. Prove that $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b})$. (Answer on p. 1863.)

Also, from Fact 145(c), we again have the following result. The proof is exactly the same as before and is simply reproduced:

Corollary 24. *If* $\mathbf{a} \parallel \mathbf{b}$*, then* $\mathbf{a} \times \mathbf{b} = \mathbf{0}$*.*

Proof. If $\mathbf{a} \parallel \mathbf{b}$, then there exists $c \neq 0$ such that $c\mathbf{a} = c\mathbf{b}$. So,

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (c\mathbf{a}) = c(\mathbf{a} \times \mathbf{a}) = c \cdot \mathbf{0} = \mathbf{0}.$$ □

And again, we have the converse of Corollary 24:

Fact 146. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be non-zero vectors. If* $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, *then* $\mathbf{a} \parallel \mathbf{b}$.

Proof. See p. 1628 (Appendices). □

So again, together, Corollary 24 and Fact 146 yield

Corollary 25. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are non-zero vectors. Then*

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \quad \iff \quad \mathbf{a} \parallel \mathbf{b}.$$

Example 909. Let $\mathbf{s} = (1, 2, 3)$ and $\mathbf{t} = (2, 4, 6)$ be vectors. Since $\mathbf{s} \parallel \mathbf{t}$, by Corollary 25, we must have $\mathbf{s} \times \mathbf{t} = \mathbf{0}$, as we now verify:

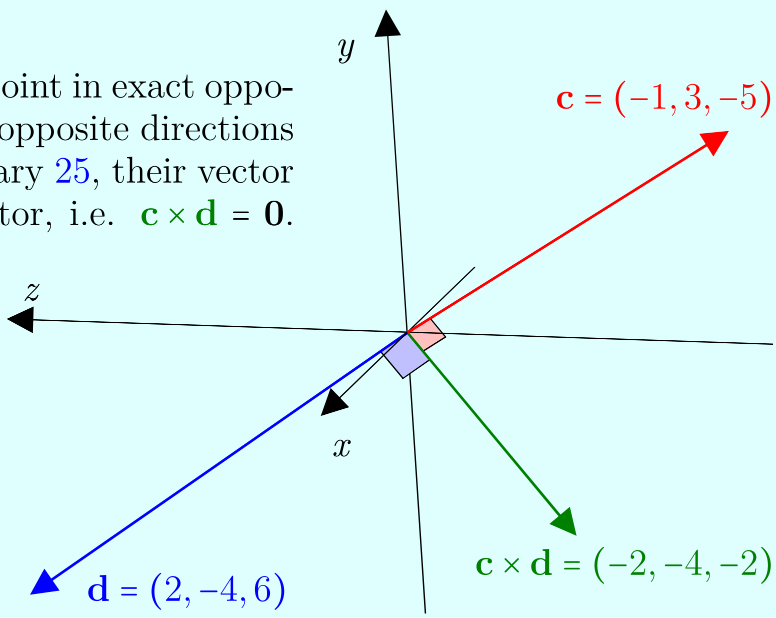
$$\mathbf{s} \times \mathbf{t} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \cdot 6 - 3 \cdot 4 \\ 3 \cdot 2 - 1 \cdot 6 \\ 1 \cdot 4 - 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Example 910. The vector product of $\mathbf{c} = (-1, 3, -5)$ and $\mathbf{d} = (2, -4, 6)$ is

$$\mathbf{c} \times \mathbf{d} = \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} \times \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix}.$$

You can verify that $\mathbf{c} \times \mathbf{d} \perp \mathbf{c}, \mathbf{d}$.

Note that $\mathbf{c}$ and $\mathbf{d}$ "nearly" but do **not** point in exact opposite directions. If they pointed in exact opposite directions (and were thus parallel), then by Corollary 25, their vector product would have to be the zero vector, i.e. $\mathbf{c} \times \mathbf{d} = \mathbf{0}$. Which isn't the case here.



The next result says that a vector is parallel to $\mathbf{a} \times \mathbf{b}$ if and only if it's perpendicular to both $\mathbf{a}$ and $\mathbf{b}$:

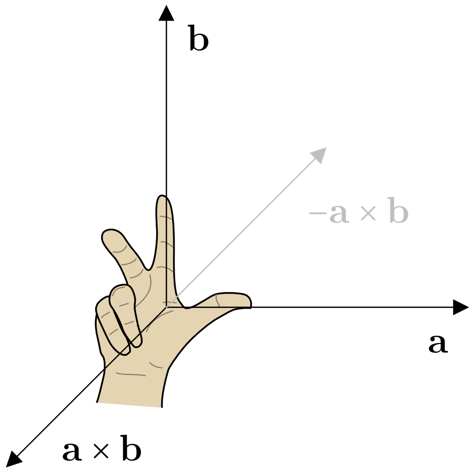
Fact 157. *Suppose* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors, with* $\mathbf{a} \nparallel \mathbf{b}$. *Then*

$$\mathbf{c} \parallel \mathbf{a} \times \mathbf{b} \quad \iff \quad \mathbf{c} \perp \mathbf{a}, \mathbf{b}.$$

Proof. See p. 1628 (Appendices). □

68.1. The Right-Hand Rule

Given two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$, there are exactly *two* (unit) vectors that are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. One is (the unit vector of) our vector product $\mathbf{a} \times \mathbf{b}$. The other is (the unit vector of) the vector that points in the exact opposite direction—this, of course, is simply $-\mathbf{a} \times \mathbf{b}$.



The vector product $\mathbf{a} \times \mathbf{b}$ is defined so that it satisfies the **right-hand rule**. To see why, have the palm of your right hand face you. Fold your ring and pinky fingers. Have your thumb point right, your index finger up, and your middle finger towards your face. Then these three fingers correspond to the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. (Try it!)

In contrast, $-\mathbf{a} \times \mathbf{b}$, the other vector that's perpendicular to $\mathbf{a}$ and $\mathbf{b}$, satisfies the left-hand rule. (Can you explain why?)

Remark 108. We have

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \qquad \text{and} \qquad -\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 \\ a_2b_1 - a_1b_2 \end{pmatrix}.$$

Why is it that one of these two arbitrary-looking vectors satisfies the right-hand rule, while the other satisfies the left-hand rule? That this is so is not at all obvious and is beyond the scope of this textbook.[347]

Fun Fact

Why do we use the **right-hand rule** rather than the left-hand rule? One possible explanation might be that the right-handed majority is, as usual, being tyrannical.

But more likely, this is simply an arbitrary convention, similar to how most of the world drives on the right (Remark 23).

Indeed, according to Mitiguy (2009),[348]

> *Until 1965, the Soviet Union used the left-hand rule, logically reasoning that the left-hand rule is more convenient because a right-handed person can simultaneously write while performing cross products.*

[347] The short answer is that **(i)** we earlier adopted the convention that our coordinate system obeys the right-hand rule; and **(ii)** $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ is **positively oriented** with respect to $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ (what exactly *positively oriented* means is the bit that's beyond the scope of this textbook). Had we instead adopted the convention that our coordinate system obeys the left-hand rule, then as currently defined, our vector product $\mathbf{a} \times \mathbf{b}$ would also obey the left-hand rule.

[348] I can't find any other sources for this story. So I'd treat it as apocryphal rather than fact.

68.2. The Length of the Vector Product

As before, the vector product $\mathbf{a} \times \mathbf{b}$ has length $|\mathbf{a}||\mathbf{b}|\sin\theta$. Formally,

Fact 148. *Suppose θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Then*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta.$$

Proof. Exercise 268 guides you through a proof of this Fact. □

Exercise 268. Let θ be the angle between the vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

(a) Express $|\mathbf{a}|$, $|\mathbf{b}|$, $|\mathbf{a} \times \mathbf{b}|$, and $\cos\theta$ in terms of a_1, a_2, a_3, b_1, b_2, and b_3. (You need not expand the squared terms.)

(b) Since $\theta \in [0, \pi]$, what can you say about the sign of $\sin\theta$? (That is, is $\sin\theta$ positive, negative, non-positive, or non-negative?)

(c) Now use a trigonometric identity to express $\sin\theta$ in terms of $\cos\theta$. (Hint: You should find that there are two possibilities. Use what you found in **(b)** why you can discard one of these possibilities.)

(d) Plug the expression you wrote down for $\cos\theta$ in **(a)** into what you found in **(c)**.

(e) Prove the following algebraic identity.[349] (Hint: Fully expand each of LHS and RHS. Then conclude that LHS = RHS.)

$$\begin{aligned} &\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - \left(a_1b_1 + a_2b_2 + a_3b_3\right)^2 \\ = \;& \left(a_2b_3 - a_3b_2\right)^2 + \left(a_3b_1 - a_1b_3\right)^2 + \left(a_1b_2 - a_2b_1\right)^2. \end{aligned}$$

(f) Use **(a)** and **(d)** to express $|\mathbf{a}||\mathbf{b}|\sin\theta$ in terms of a_1, a_2, a_3, b_1, b_2, and b_3. Then use **(e)** to prove that

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta.$$

(Answer on p. 1864.)

[349] By the way, this is again simply an instance of Lagrange's Identity.

68.3. The Length of the Rejection Vector

In 2D space, the vector product was a scalar (real number). In contrast, in 3D space, it is a vector (hence the name).

Nonetheless and perhaps surprisingly, Fact 149—which says the rejection vector's length is given by the vector product—remains true and is now reproduced:

Fact 149. *Suppose* **a** *and* **b** *are vectors. Then*

$$|\text{rej}_{\mathbf{b}}\mathbf{a}| = \left|\mathbf{a} \times \hat{\mathbf{b}}\right|.$$

Proof. See p. 1627 (Appendices). □

Example 911. The points $A = (1, 5, -2)$, $B = (2, 3, 1)$, and $C = (2, 7, -1)$ form a right triangle, with $\overrightarrow{AB} = (1, -2, 3)$, $\overrightarrow{AC} = (1, 2, 1)$, and $\overrightarrow{BC} = (0, 4, -2)$.

The lengths of the line segments AB and AC are simply

$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14} \qquad \text{and} \qquad \left|\overrightarrow{AC}\right| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$$

As an exercise, let's verify that Fact 149 "works". Observe that

$$\overrightarrow{AB} = -\text{rej}_{\overrightarrow{AC}}\overrightarrow{BC} \quad \text{and} \quad \overrightarrow{AC} = \text{rej}_{\overrightarrow{AB}}\overrightarrow{BC}.$$

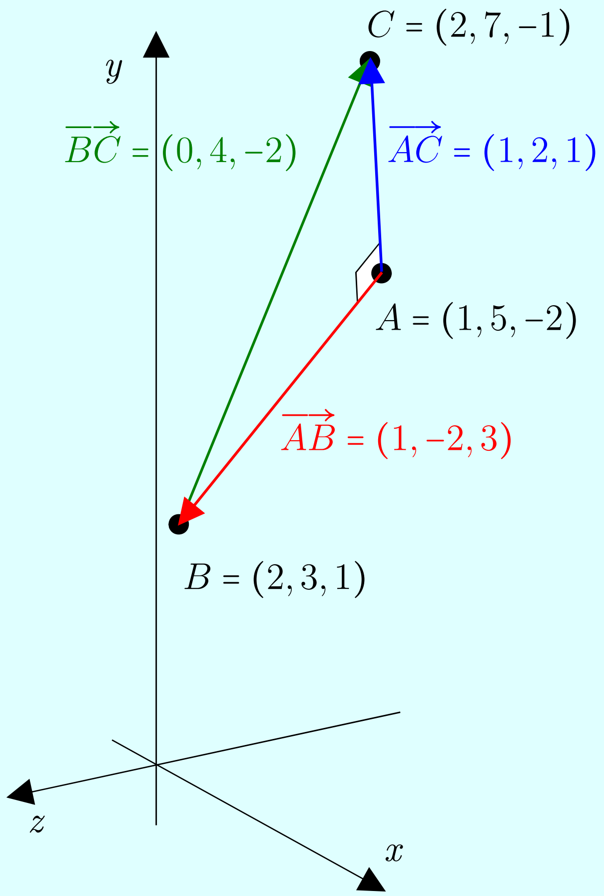
And so, by Fact 149,

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC} \times \widehat{\overrightarrow{AC}}\right| = \frac{\left|\overrightarrow{BC} \times \overrightarrow{AC}\right|}{\left|\overrightarrow{AC}\right|} = \frac{|(8, -2, -4)|}{|(1, 2, 1)|}$$

$$= \sqrt{\frac{8^2 + (-2)^2 + (-4)^2}{1^2 + 2^2 + 1^2}} = \sqrt{\frac{84}{6}} = \sqrt{14}. \checkmark$$

$$\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BC} \times \widehat{\overrightarrow{AB}}\right| = \frac{\left|\overrightarrow{BC} \times \overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|} = \frac{|(8, -2, -4)|}{|(1, -2, 3)|}$$

$$= \sqrt{\frac{(-8)^2 + 2^2 + 4^2}{1^2 + (-2)^2 + 3^2}} = \sqrt{\frac{84}{14}} = \sqrt{6}. \checkmark$$



69. The Foot of the Perpendicular (3D)

As in Ch. 62, we reproduce from Ch. 15.1 the following result and definition:

Corollary 4. *Suppose A is a point not on the line l. Then there exists a point B that is both* **(a)** *the unique point on l that's closest to A; and* **(b)** *the unique point on l such that $l \perp AB$.*

Definition 174. Let A be a point that isn't on the line l. The *foot of the perpendicular from A to l* is the (unique) point B on l such that $AB \perp l$.

In 3D space, this result from Ch. 62 still holds:

Fact 150. *Let l be the line $R = P + \lambda \mathbf{v}$ ($\lambda \in \mathbb{R}$). Suppose A is a point not on l. Then the point B that is* ***(a)*** *the unique point on l that's closest to A; and* ***(b)*** *the unique point on l such that $l \perp AB$ is*

$$B = P + \text{proj}_{\mathbf{v}} \overrightarrow{PA}.$$

In Ch. 62, we already proved both of the above results generally (i.e. in 2D, 3D, and also higher dimensional spaces).

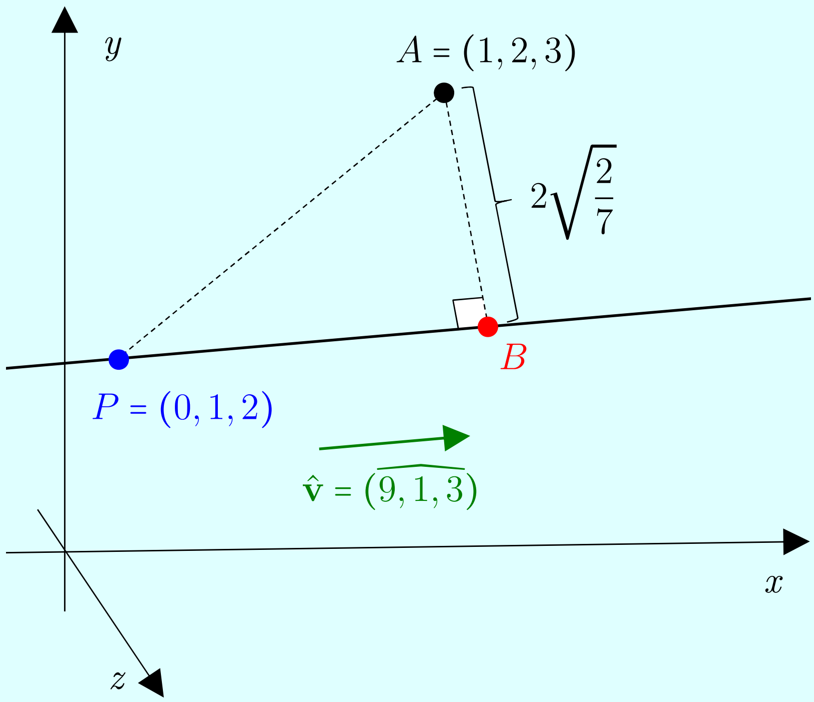
Again, the distance between a point and a graph is the minimum distance between them:

Definition 61. The *distance between a point A and a graph G* is the distance between A and B, where B is the point on G that's closest to A.

Corollary 27. *Suppose B is the point on the line l that's closest to the point A. Then the distance between A and l is $\left|\overrightarrow{AB}\right|$.*

Example 912. Let $A = (1,2,3)$ be a point and l be the line $\mathbf{r} = \overrightarrow{OP} + \lambda\mathbf{v} = (0,1,2) + \lambda(9,1,3)$ $(\lambda \in \mathbb{R})$. Let B be the point on l that's closest to A (B is also the foot of the perpendicular from A to l).

As in Ch. 62, we'll find B or the distance between A and l using four or five methods.



Method 1 (Projection Vector). Compute $\overrightarrow{PA} = (0,1,2) - (1,2,3) = (1,1,1)$ and

$$\text{proj}_{\mathbf{v}}\overrightarrow{PA} = \text{proj}_{(9,1,3)}(1,1,1) = \frac{(1,1,1)\cdot(9,1,3)}{9^2+1^2+3^2}(9,1,3) = \frac{9+1+3}{91}(9,1,3) = \frac{1}{7}(9,1,3).$$

Now simply apply Fact 150:

$$B = P + \text{proj}_{\mathbf{v}}\overrightarrow{PA} = (0,1,2) + \frac{1}{7}(9,1,3) = \frac{1}{7}(9,8,17).$$

And the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \left|\frac{1}{7}(9,8,17) - (1,2,3)\right| = \left|\frac{1}{7}(2,-6,-4)\right|$$
$$= \frac{1}{7}\sqrt{2^2+(-6)^2+(-4)^2} = \frac{1}{7}\sqrt{56} = \frac{1}{7}\sqrt{4\times 7} = 2\sqrt{\frac{2}{7}}.$$

(Example continues on the next page ...)

(... *Example continued from the previous page.*)

Method 2 (Quadratic-Cartesian). The line l has cartesian equations $\dfrac{x}{9} = y-1 = \dfrac{z-2}{3}$.
The distance between A and any arbitrary point $R = (9r, r+1, 3r+2)$ on l is

$$\sqrt{(9r-1)^2 + (r+1-2)^2 + (3r+2-3)^2} = \sqrt{91r^2 - 26r + 3}.$$

As usual, this last surd expression is minimised when

$$r = -\frac{\text{“}b\text{”}}{2\text{“}a\text{”}} = -\frac{-26}{2 \times 91} = \frac{1}{7}.$$

Hence, $$B = \frac{1}{7}(9, 8, 17).$$

And the distance between A and l is

$$\sqrt{91 \times \left(\frac{1}{7}\right)^2 - 26 \times \frac{1}{7} + 3} = \sqrt{\frac{13}{7} - \frac{26}{7} + 3} = \sqrt{\frac{8}{7}} = 2\sqrt{\frac{2}{7}}.$$

Method 3 (Quadratic-Vector). Pick any arbitrary point $R = (0, 1, 2) + \lambda(9, 1, 3)$ on l.
The distance between A and R is

$$\left|\overrightarrow{AR}\right| = \left|\overrightarrow{OR} - \overrightarrow{OA}\right| = |(0,1,2) + \lambda(9,1,3) - (1,2,3)| = |(9\lambda - 1, \lambda - 1, 3\lambda - 1)|$$

$$= \sqrt{(9\lambda - 1)^2 + (\lambda - 1)^2 + (3\lambda - 1)^2} = \sqrt{91\lambda^2 - 26\lambda + 3}.$$

As usual, this last surd expression is minimised when

$$\lambda = -\frac{\text{“}b\text{”}}{2\text{“}a\text{”}} = -\frac{-26}{2 \times 91} = \frac{1}{7}.$$

Hence, $$B = (0, 1, 2) + \frac{1}{7}(9, 1, 3) = \frac{1}{7}(9, 8, 17).$$

And the distance between A and l is

$$\sqrt{91 \times \left(\frac{1}{7}\right)^2 - 26 \times \frac{1}{7} + 3} = \sqrt{\frac{13}{7} - \frac{26}{7} + 3} = \sqrt{\frac{8}{7}} = 2\sqrt{\frac{2}{7}}.$$

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Method 4 (Scalar Product). Let $B = (0,1,2) + \lambda_b(9,1,3)$. Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (9\lambda_b - 1, \lambda_b - 1, 3\lambda_b - 1)$.

We have $\quad \overrightarrow{AB} \perp l, \quad$ or $\overrightarrow{AB} \perp (9,1,3), \quad$ or $\overrightarrow{AB} \cdot (9,1,3) = 0$, or

$$0 = (9\lambda_b - 1, \lambda_b - 1, 3\lambda_b - 1) \cdot (9,1,3) = 81\lambda_b - 9 + \lambda_b - 1 + 9\lambda_b - 3 = 91\lambda_b - 13,$$

or $\lambda_b = 1/7$. Hence, $B = (0,1,2) + \dfrac{1}{7}(9,1,3) = \dfrac{1}{7}(9,8,17)$.

And the distance between A and l is

$$\begin{aligned}\left|\overrightarrow{AB}\right| &= |(9\lambda_b - 1, \lambda_b - 1, 3\lambda_b - 1)| = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(\frac{1}{7} - 1\right)^2 + \left(\frac{3}{7} - 1\right)^2} \\ &= \sqrt{\frac{4}{49} + \frac{36}{49} + \frac{16}{49}} = \frac{1}{7}\sqrt{56}.\end{aligned}$$

Method 5 (Vector Product). Recall that $\overrightarrow{AB}$ is $\text{rej}_{\mathbf{v}}\overrightarrow{PA}$—the rejection of $\overrightarrow{PA}$ on $\mathbf{v}$. Recall also Fact 149: $\left|\text{rej}_{\mathbf{v}}\overrightarrow{PA}\right| = \left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right|$. So,

$$\begin{aligned}\left|\overrightarrow{AB}\right| = \left|\text{rej}_{\mathbf{v}}\overrightarrow{PA}\right| = \left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right| &= \left|(1,1,1) \times (9,1,3)/\sqrt{9^2 + 1^2 + 3^2}\right| \\ &= |(3-1, 9-3, 1-9)|/\sqrt{91} = \sqrt{2^2 + 6^2 + (-8)^2}/\sqrt{91} = \sqrt{104/91} = \sqrt{8/7}.\end{aligned}$$

(*Example continues on the next page ...*)

Formal statement of Method 5, reproduced from Ch. 62:

Corollary 28. *Let A be a point. Suppose l is the line $R = P + \lambda\mathbf{v}$ ($\lambda \in \mathbb{R}$). Then the distance between A and l is* $\left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right|$.

Another example illustrating the four or five methods:

Example 913. Let $A = (-1, 0, 1)$ be a point and l be the line described by

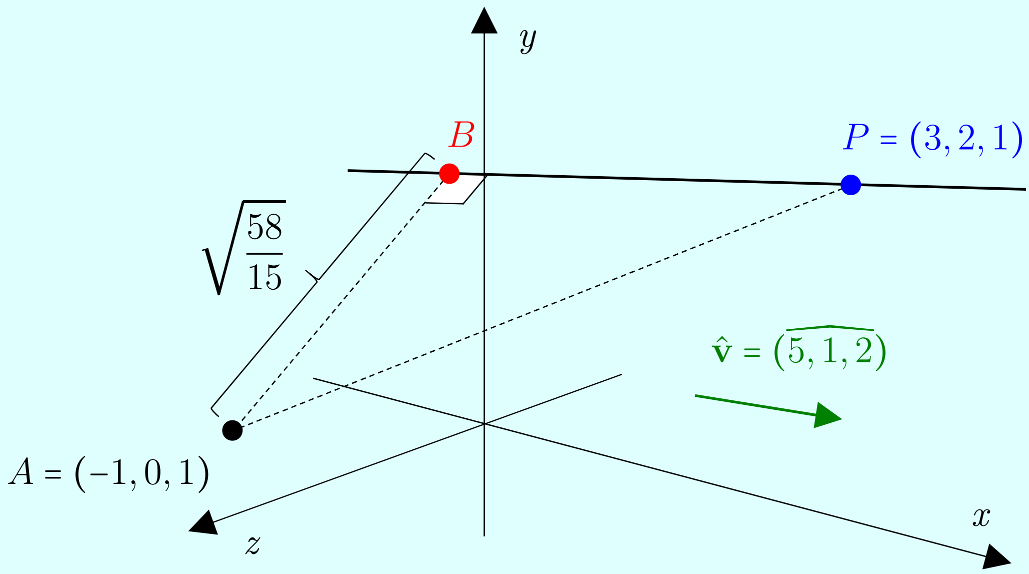
$$\mathbf{r} = \overrightarrow{OP} + \lambda \mathbf{v} = (3, 2, 1) + \lambda (5, 1, 2) \qquad (\lambda \in \mathbb{R}).$$

Method 1 (Formula Method). First, $\overrightarrow{PA} = (-1, 0, 1) - (3, 2, 1) = (-4, -2, 0)$. So,

$$\begin{aligned} \overrightarrow{PB} = \text{proj}_{\mathbf{v}} \overrightarrow{PA} = \text{proj}_{(5,1,2)} (-4, -2, 0) &= \frac{(-4, -2, 0) \cdot (5, 1, 2)}{5^2 + 1^2 + 2^2} (5, 1, 2) \\ &= \frac{-20 - 2 + 0}{30} (5, 1, 2) = -\frac{22}{30} (5, 1, 2) = -\frac{11}{15} (5, 1, 2). \end{aligned}$$

And so by Fact 150, the foot of the perpendicular from A to l is

$$B = P + \text{proj}_{\mathbf{v}} \overrightarrow{PA} = (3, 2, 1) - \frac{11}{15} (5, 1, 2) = \frac{1}{15} (-10, 19, -7).$$



By Corollary 28, the distance between A and l is

$$\left|\overrightarrow{BA}\right| = \left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right| = \left|(-4, -2, 0) \times \frac{(5, 1, 2)}{\sqrt{5^2 + 1^2 + 2^2}}\right| = \left|\frac{(-4, 8, 6)}{\sqrt{30}}\right| = \sqrt{\frac{58}{15}}.$$

Method 2 (Perpendicular Method). Let $B = (3, 2, 1) + \tilde{\lambda} (5, 1, 2)$. Write down $\overrightarrow{AB}$:

$$\overrightarrow{AB} = B - A = (3, 2, 1) + \tilde{\lambda} (5, 1, 2) - (-1, 0, 1) = \left(5\tilde{\lambda} + 4, \tilde{\lambda} + 2, 2\tilde{\lambda}\right).$$

Since $\overrightarrow{AB} \perp l$, we have $\overrightarrow{AB} \perp \mathbf{v}$ or,

$$0 = \left(5\tilde{\lambda} + 4, \tilde{\lambda} + 2, 2\tilde{\lambda}\right) \cdot (5, 1, 2) = 5\left(5\tilde{\lambda} + 4\right) + \left(\tilde{\lambda} + 2\right) + 2\left(2\tilde{\lambda}\right) = 30\tilde{\lambda} + 22.$$

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Rearranging, $\tilde{\lambda} = -22/30 = -11/15$ and so,

$$B = (3,2,1) + \frac{11}{15}(5,1,2) = \frac{1}{15}(-10,19,-7).$$

Happily, this is the same as what we found in Method 1. And now,

$$\overrightarrow{AB} = B - A = \frac{1}{15}(-10,19,-7) - (-1,0,1) = \frac{1}{15}(5,19,-22).$$

Thus, the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \frac{1}{15}\left|(5,19,-22)\right| = \frac{1}{15}\sqrt{5^2+19^2+(-22)^2} = \frac{\sqrt{870}}{15} = \sqrt{\frac{58}{15}}.$$

Method 3 (or the **Calculus Method**). Let R be a generic point on l, so that $\overrightarrow{AR} = (5\lambda+4, \lambda+2, 2\lambda)$ and the distance between A and R is

$$\left|\overrightarrow{AR}\right| = \sqrt{(5\lambda+4)^2 + (\lambda+2)^2 + (2\lambda)^2} = \sqrt{30\lambda^2 + 44\lambda + 20}.$$

Again, first differentiate the expression $30\lambda^2 + 44\lambda + 20$ with respect to λ:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(30\lambda^2 + 44\lambda + 20\right) = 60\lambda + 44.$$

Then by the First Order Condition (FOC), we have

$$(60\lambda + 44)\big|_{\lambda=\tilde{\lambda}} = 0 \qquad \text{or} \qquad \tilde{\lambda} = -\frac{44}{60} = -\frac{11}{15}.$$

Happily, this is the same as what we found in Method 2. And now, as before, we can find B and $\left|\overrightarrow{AB}\right|$. Alternatively, we could simply have found $\tilde{\lambda}$ by using "$-b/2a$":

$$\tilde{\lambda} = \text{“}-b/2a\text{”} = -\frac{44}{2\cdot 30} = -\frac{11}{15}.$$

Exercise 269. For each of the following, use all three methods you just learnt to find the foot of the perpendicular from A to l; and the distance between A and l.

	The point A	**The line l**	**Answer on p.**
(a)	$(7,3,4)$	$\mathbf{r} = (8,3,4) + \lambda(9,3,7)$	1865.
(b)	$(8,0,2)$	Contains the points $(4,4,3)$ and $(6,11,5)$	1866.
(c)	$(8,5,9)$	$\mathbf{r} = (8,4,5) + \lambda(5,6,0)$	1867.

70. Planes: Introduction

In the remainder of Part III (Vectors), we'll study **planes**.

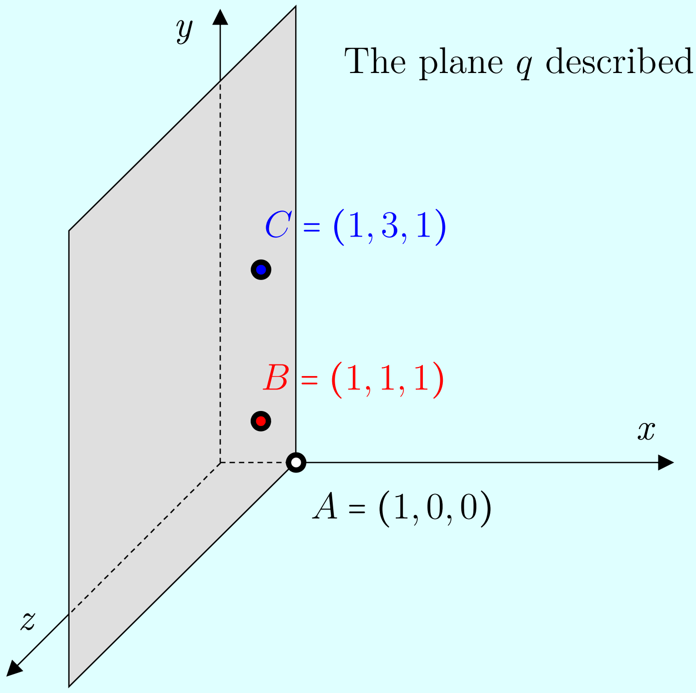
Informally, a **plane** is a "flat 2D surface" (like a piece of paper). A bit more formally, a plane is, like a line, simply a set of points.

Let's start by taking a quick look at some examples of planes.

Example 914. Consider the plane q described by the cartesian equation $x = 1$.

It contains exactly those points whose x-coordinate is 1. So, it contains $A = (1, 0, 0)$, $B = (1, 1, 1)$, $C = (1, 3, 1)$, and every other point with x-coordinate 1.

In contrast, it does not contain $(2, 0, 0)$, $(\pi, 1, 1)$, $\left(\sqrt{2}, 3, 1\right)$, or any other point whose x-coordinate isn't 1.



Formally, the plane q is a **set of points**:

$$q = \{(x, y, z) : x = 1\}.$$

In words, q is the set containing exactly those points (x, y, z) whose x-coordinate is 1.

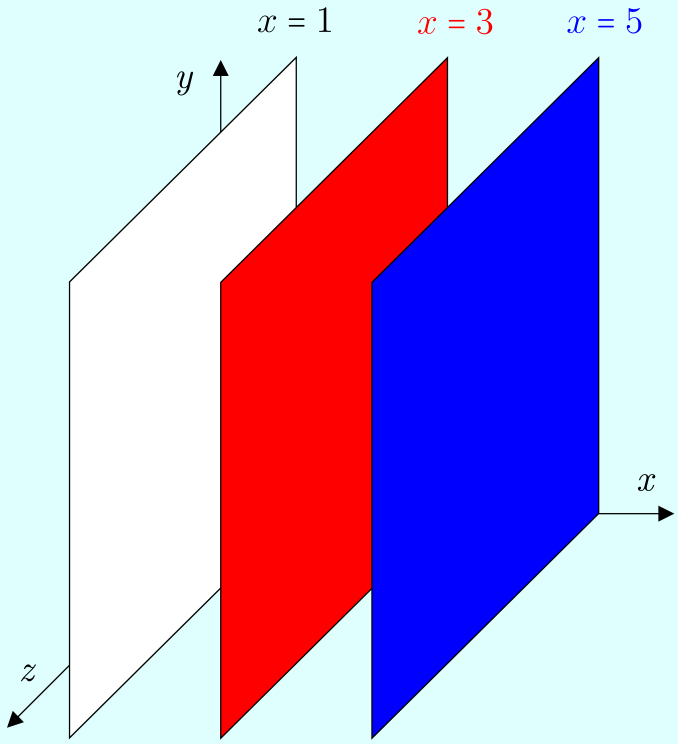
You should take a moment to convince yourself that the plane q, which is the set of points whose x-coordinates are 1, does indeed form a "flat 2D surface".

Example 915. Three planes are described by these cartesian equations:

$$x = 1, \qquad x = 3, \quad \text{and} \quad x = 5.$$

Later on, we will learn what it means for two planes to be **parallel** and how to calculate the **distance between two planes**. But for now, we merely assert that "obviously",

- The three planes are parallel.
- The distance between the first and second planes is 2.
- The distance between the second and third planes is also 2.



Exercise 270. The planes q_1 and q_2 are described by $y = 2$ and $z = 3$.

(a) Sketch the graphs of both planes in a single figure.

Then find two points that are on

(b) q_1 but not q_2;

(c) q_2 but not q_1; and

(d) Both q_1 and q_2. (Answer on p. 1868.)

More examples:

Example 916. Consider the plane q described by the cartesian equation $y = 2x$.

It is the set of points (x, y, z) that satisfies the equation $y = 2x$. Formally,

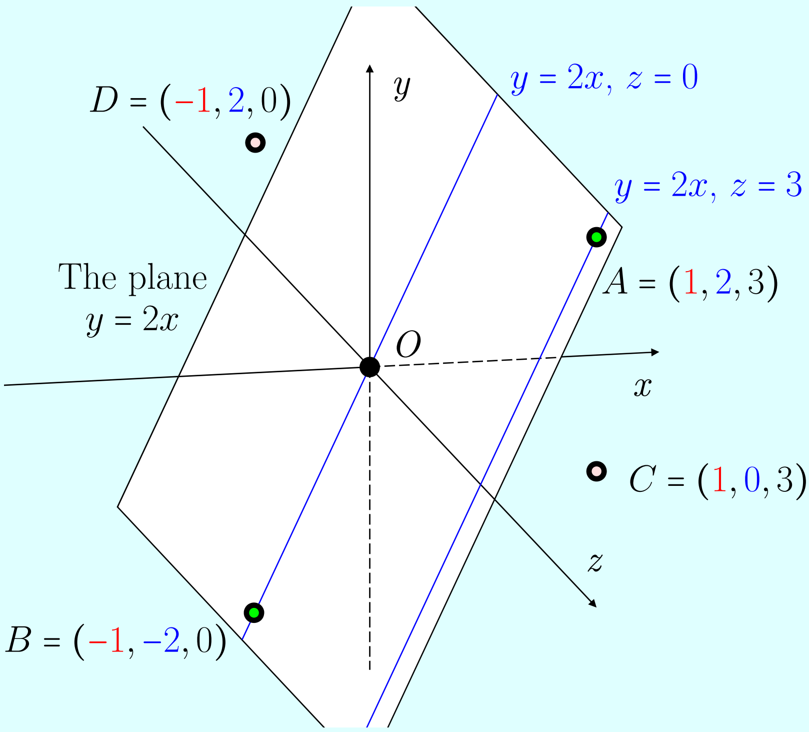
$$q = \{(x, y, z) : y = 2x\}.$$

So for example, it contains

- The origin $O = (0, 0, 0)$ because $0 = 2 \cdot 0$;
- The point $A = (1, 2, 3)$ because $2 = 2 \cdot 1$; and
- The point $B = (-1, -2, 0)$ because $-2 = 2 \cdot (-1)$.

In contrast, it does **not** contain

- The point $C = (1, 0, 3)$ because $0 \neq 2 \cdot 1$; or
- The point $D = (-1, 2, 0)$ because $2 \neq 2 \cdot (-1)$.



Also, the plane q contains the lines $y = 2x,\ z = 0$ and $y = 2x,\ z = 3$.

Indeed, for every $k \in \mathbb{R}$, the line $y = 2x,\ z = k$ is contained in the plane q.[350]

[350]Here's a proof of this assertion. Consider the line $y = 2x$, $z = k$. Let P be any point on the line. Observe that P obviously satisfies the plane's equation $y = 2x$. So, $P \in q$. We have just shown that any arbitrary point P on the line is also on q. Hence, q contains the line.

Example 917. Consider the plane q described by the cartesian equation $x+y=z$.

It is the set of points (x,y,z) that satisfies the equation $x+y=z$. Formally,

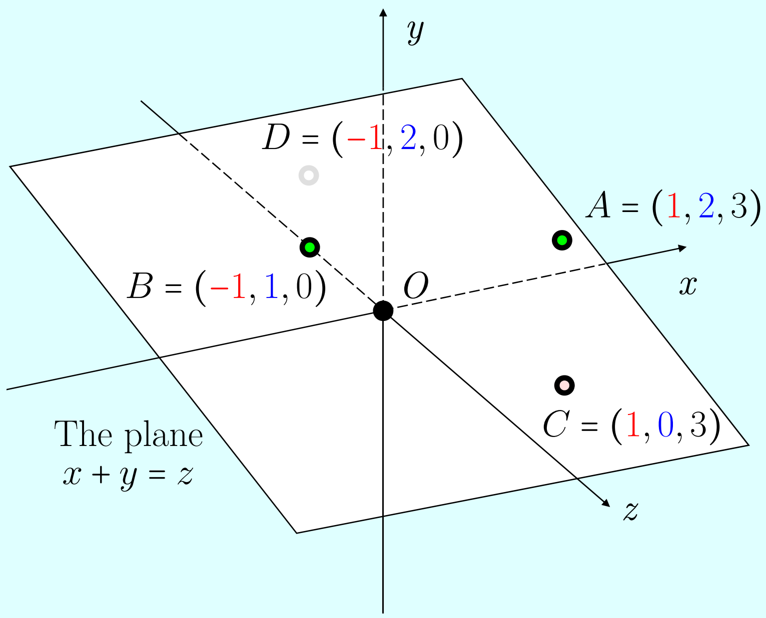
$$q=\{(x,y,z):x+y=z\}.$$

So for example, it contains

- The origin $O=(0,0,0)$ because $0+0=0$;
- The point $A=(1,2,3)$ because $1+2=3$; and
- The point $B=(-1,1,0)$ because $-1+1=0$.

In contrast, it does **not** contain

- The point $C=(1,0,3)$ because $1+0\neq 3$; or
- The point $D=(-1,2,0)$ because $-1+2\neq 0$.



Note that as depicted, D is "behind" the plane.

As the above examples suggest, it turns out that in general, any **plane** q is simply the graph of this cartesian equation:

$$ax+by+cz=d,$$

where $a,b,c,d\in\mathbb{R}$ (and at least one of a, b, or c is non-zero).

In other words, the plane q is the set of points (x,y,z) that satisfy $ax+by+cz=d$.

Formally, the plane q is a set of points:

$$q=\{(x,y,z):ax+by+cz=d\}.$$

In the coming chapters, we will explain *why* a plane may be described by the above cartesian equation. We will also learn what the vector (a,b,c) and the number d mean geometrically.

70.1. The Analogy Between a Plane, a Line, and a Point

In 3D space,

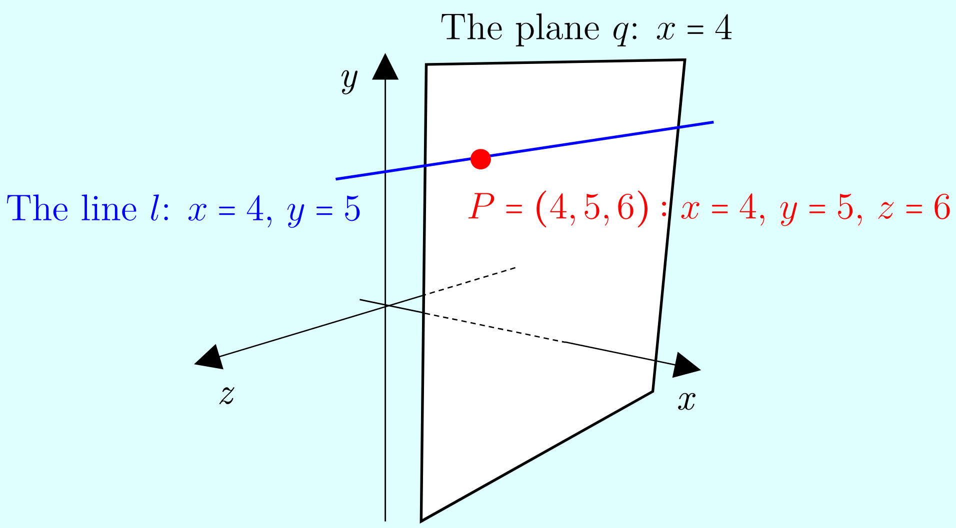
A **plane**	is a	**two-dimensional**	object and can be described by		**one**		equation(s).
A **line**	"	**one-dimensional**	"	and	"	**two**	"
A **point**	"	**zero-dimensional**	"	and	"	**three**	"

Example 918. We are given 3D space, the set of all points.

We first impose the equation or constraint $x = 4$. That is, we keep only those points (in 3D space) that satisfy the equation $x = 4$ and "throw away" all other points. This leaves us with the plane q. We say that the plane q is described by **one** equation:

$$x = 4.$$

"$3 - 1 = 2$": By imposing **1** constraint on **3**D space, we end up with a **2**D object.



Now suppose we *also* impose the constraint $y = 5$. That is, we take the plane q, but keep only those points on q that satisfy the equation $y = 5$ (and "throw away" all other points). This gives us the line l. We say that the line l is described by **two** equations:

$$x = 4 \qquad \text{and} \qquad y = 5.$$

"$3 - 2 = 1$": By imposing **2** constraints on **3**D space, we end up with a **1**D object.

Finally, we impose a third constraint $z = 6$. That is, we take the line l, but keep only those points on l that satisfy the equation $z = 6$ (and "throw away" all other points). This gives us the point P. Hence, the point P is described by **three** equations:

$$x = 4, \qquad y = 5, \qquad \text{and} \qquad z = 6.$$

"$3 - 3 = 0$": By imposing **3** constraints on **3**D space, we end up with a **0**D object.

We observe that each additional equation (or constraints) "chops off" a dimension. This observation also generalises to spaces of other dimensions. For example, in 2D space,

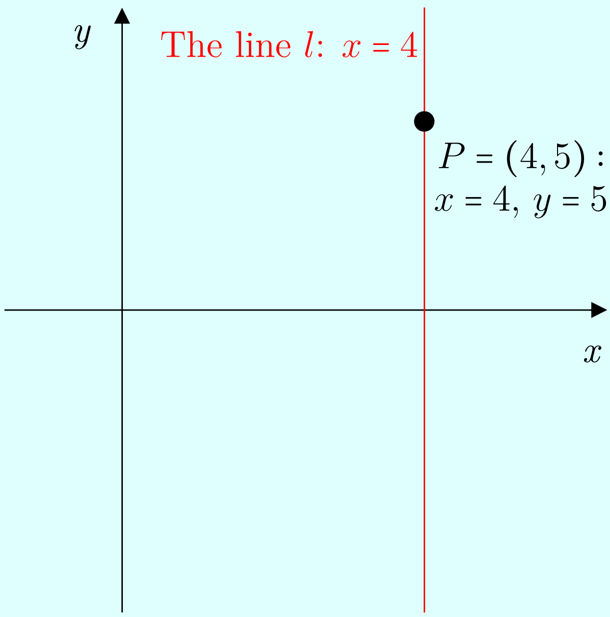
"$2-1=1$":	A **line**	is a **1D**	object that can be described by	**one** equation(s).
"$2-2=0$":	A **point**	" **0D**	"	**two** "

Example 919. In this example, we return to 2D space.

We first impose the equation or constraint $x = 4$. That is, we keep only those points (in 2D space) that satisfy the equation $x = 4$ and "throw away" all other points. This leaves us with the line l. We say that the line l is described by **one** equation:

$$x = 4.$$

"$2-1=1$": By imposing **1** constraint on **2**D space, we end up with a **1**D object.



Now suppose we also impose the constraint $y = 5$. That is, we take the line l, but keep only those points on l that satisfy the equation $y = 5$ (and "throw away" all other points). This gives us the point $P = (4,5)$. We say that the point P is described by **two** equations:

$$x = 4 \qquad \text{and} \qquad y = 5.$$

"$2-2=0$": By imposing **2** constraints on **2**D space, we end up with a **0**D object.

Similarly, in 4D space (**not** in H2 Maths):

"$4-1=3$":	**One** equation describes a 3D space.			
"$4-2=2$":	A **plane**	is a **2D**	object that can be described by	**one** equations.
"$4-3=1$":	A **line**	" **1D**	"	**two** "
"$4-4=0$":	A **point**	" **0D**	"	**three** "

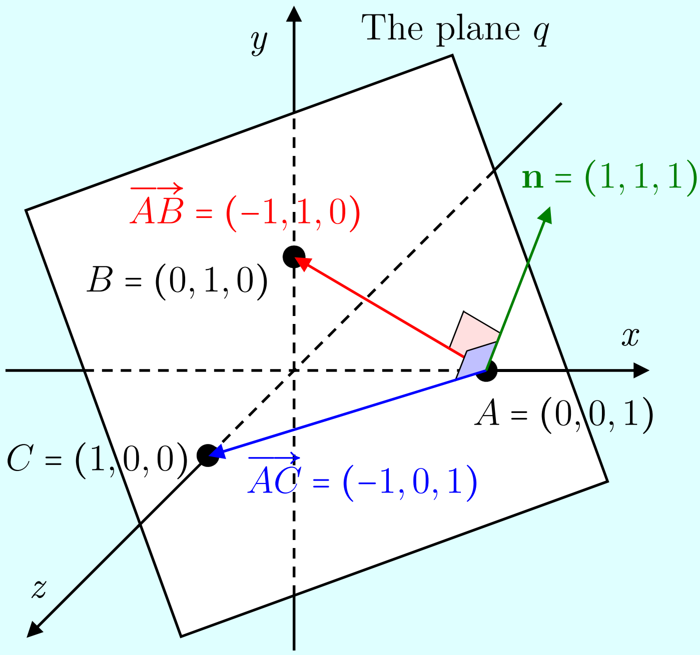
And so on and so forth in all higher-dimensional spaces as well.

Example 920. Let q be the plane that contains the points $A = (1,0,0)$, $B = (0,1,0)$, and $C = (0,0,1)$. Informally and intuitively, a plane is a "flat surface".

Since it's a "flat surface", there must be some vector that is perpendicular to it. We will call any such vector a **normal vector** of the plane.[351]

To find a normal vector of q, all we need do is pick any two vectors **on** q and compute their vector product.

Let's pick, say, $\overrightarrow{AB} = (-1,1,0)$ and $\overrightarrow{AC} = (-1,0,1)$. Their vector product (which we'll call $\mathbf{n}$) is



$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{n}.$$

As we learnt in Ch. 68, the vector product $\mathbf{n} = (1,1,1)$ must be perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$. It turns out that $\mathbf{n}$ is *also* perpendicular to *every* vector on q (we'll formally state and prove this as Fact 164 below). And so, we call $\mathbf{n}$ a **normal vector** of q.

Now, let R denote a generic point on q. Then the vector $\overrightarrow{AR}$ is on q. Which means

$$\overrightarrow{AR} \perp \mathbf{n} = (1,1,1), \qquad \text{or equivalently,} \qquad \overrightarrow{AR} \cdot (1,1,1) = 0.$$

Let's manipulate this last equation a little:

$$\begin{aligned}
& & \overrightarrow{AR} \cdot (1,1,1) &= 0 & \\
\iff & & \left(\overrightarrow{OR} - \overrightarrow{OA}\right) \cdot (1,1,1) &= 0 & \left(\because \overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA}\right) \\
\iff & & \overrightarrow{OR} \cdot (1,1,1) - \overrightarrow{OA} \cdot (1,1,1) &= 0 & \text{(Distributivity)} \\
\iff & & \overrightarrow{OR} \cdot (1,1,1) &= \overrightarrow{OA} \cdot (1,1,1) & \\
\iff & & \overrightarrow{OR} \cdot (1,1,1) &= (1,0,0) \cdot (1,1,1) = 1. &
\end{aligned}$$

We say that this last equation $\overrightarrow{OR} \cdot (1,1,1) = 1$ is a **vector equation** that *describes* the plane q. To be a bit more formal and precise, we'd write q as this set (of points):

$$q = \left\{ R : \overrightarrow{OR} \cdot (1,1,1) = 1 \right\}.$$

In words, q is the set containing exactly those points R that satisfy $\overrightarrow{OR} \cdot (1,1,1) = 1$.

(*Example continues on the next page ...*)

[351] We'll formally define what a normal vector of a plane is in Definition 177.

(... *Example continued from the previous page.*)

If we let $\mathbf{r}$ denote the position vector of the generic point R, then here is another **vector equation** that also describes q:

$$\mathbf{r} \cdot (1, 1, 1) = 1.$$

Again, to be a bit more formal and precise, we'd write

$$q = \{R : \mathbf{r} \cdot (1, 1, 1) = 1\}.$$

In words, q is the set containing exactly those points R that satisfy $\mathbf{r} \cdot (1, 1, 1) = 1$.

The above discussion motivates the following formal definition of a **plane**:

Definition 175. A *plane* is any set of points that can be written as

$$\left\{R : \overrightarrow{OR} \cdot \mathbf{n} = d\right\} \qquad \text{or} \qquad \{R : \mathbf{r} \cdot \mathbf{n} = d\},$$

where $\mathbf{n}$ is some non-zero vector and $d \in \mathbb{R}$.

In words, a plane is the set containing exactly those points R that satisfy

$$\overrightarrow{OR} \cdot \mathbf{n} = d \qquad \text{or} \qquad \mathbf{r} \cdot \mathbf{n} = d.$$

We can also say that a plane is *described* by either of the above **vector equations**.

By the way, in the above example, we spoke of vectors being **on** a plane. It's probably a good idea to formally and precisely define what this means:

Definition 176. A vector $\mathbf{v}$ is *on* a plane q if there are points $S, T \in q$ such that $\mathbf{v} = \overrightarrow{ST}$.

Remark 109. If $\mathbf{v}$ is on q, then for the sake of convenience, we will sometimes be sloppy and say that q **contains** $\mathbf{v}$.

This is sloppy because strictly speaking, it's wrong to say that a plane q *contains* a vector $\mathbf{v}$. A plane contains *points* and **not** *vectors*. Nonetheless, for the sake of convenience, we will often simply (and incorrectly) say that a plane contains a vector.

Example 921. The plane $q = \left\{R : \overrightarrow{OR} \cdot (-3,0,2) = -5\right\}$ contains

- the point $A = (1,0,-1)$ because A satisfies the plane's vector equation:

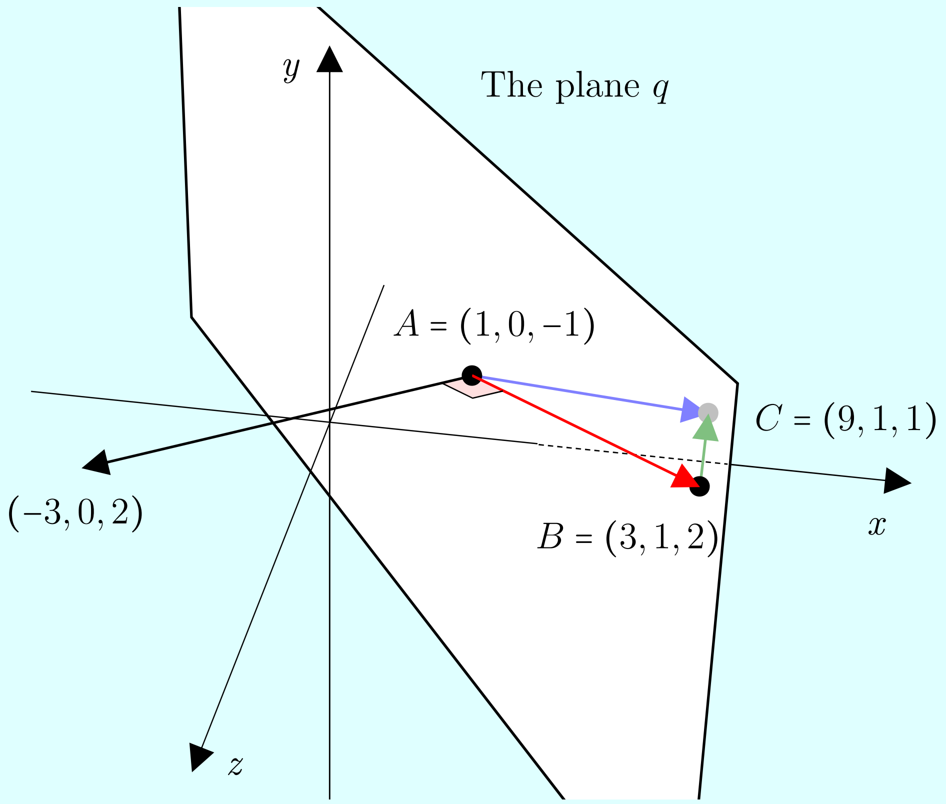
$$\overrightarrow{OA} \cdot (-3,0,2) = (1,0,-1) \cdot (-3,0,2) = -3 + 0 - 2 = -5; \qquad ✓$$

- the point $B = (3,1,2)$:

$$\overrightarrow{OB} \cdot (-3,0,2) = (3,1,2) \cdot (-3,0,2) = -9 + 0 + 4 = -5; \qquad ✓$$

- but not the point $C = (9,1,1)$:

$$\overrightarrow{OC} \cdot (-3,0,2) = (9,1,1) \cdot (-3,0,2) = -27 + 0 + 2 = -25. \qquad ✗$$



Since q contains the points A and B, the vector $\overrightarrow{AB}$ is on q. (We could also say that "q contains the vector $\overrightarrow{AB}$," but this is a bit sloppy—n. 109.)

Now, are the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ on q? As you may have guessed, the answer is, "No, they are not." But to prove this, we'll have to wait until Fact 163 below.[352]

[352] Here's an incorrect "proof": "C is not on q. Therefore, the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ not on q." This proof is incorrect because in order to prove that $\overrightarrow{AC}$ is not on q, we need to prove that $\overrightarrow{AC} \neq \overrightarrow{PQ}$ for *any* two points P and Q on q. The mere observation that C is not on q does not suffice.

Remark 110. Strictly speaking, this statement is wrong:

1. "Let q be the plane $\overrightarrow{OR} \cdot (-3, 0, 2) = 5$," ✗

This is because q is not an equation but a set of points. So, strictly speaking, we should write

2. "Let q be the plane $\left\{R : \overrightarrow{OR} \cdot (-3, 0, 2) = 5\right\}$." ✓

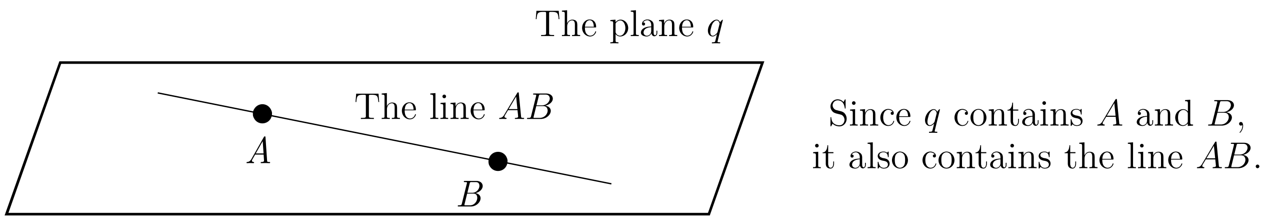
Nonetheless, having to always write Statement 2 gets tiresome. So, we'll be lazy and sloppy and often write the (incorrect) Statement 1, with the understanding that what we really mean is the (correct) Statement 2.

(We already made this same point earlier with lines—n. 323.)

Exercise 271. Does the plane $q = \left\{R : \overrightarrow{OR} \cdot (-5, 7, 3) = -1\right\}$ contain any of these points: $A = (5, -3, 1)$, $B = (1, -2, 6)$, $C = (-2, 2, -3)$? (Answer on p. 1869.)

Our first result about planes is simple and intuitively "obvious":

Fact 158. *Let A and B be distinct points and q be a plane. If q contains A and B, then it also contains all the points in the line AB.*



Proof. See p. 1630 (Appendices).[353] □

Example 922. In the last example, we verified that the plane $q = \left\{ R : \overrightarrow{OR} \cdot (-3, 0, 2) = 5 \right\}$ contains the points $A = (1, 0, -1)$ and $B = (3, 1, 2)$. By the above Fact then, q also contains every point in the line AB.

Remark 111. A couple of pages ago, we made the point that strictly speaking, a plane cannot contain a vector (Remark 109).

Here we can make a similar remark: Strictly speaking, a plane *cannot* contain a line (equivalently, a line cannot be *in* or *an element of* a plane).

This is because a line is a set of points (where a point is an ordered triple of real numbers). A plane is *also* a set of points. But one set of points cannot be *in* or *an element of* another set of points.

Instead, what's possible is for a set of points to be a *subset* of another set of points. In particular, a line can be a subset of a plane.

Nonetheless, as with planes and vectors, we shall be sloppy with planes and lines:

We will often go ahead and say that "a line is *on* a plane" or "a plane *contains* a line", even though strictly speaking, either statement is incorrect. We'll make such statements with the understanding that what we really mean is that "the line is a *subset* of the plane" or "the plane contains all the points in the line".

And so, continuing with the last example, we'll often be sloppy and say that "the line AB is *on* q" or "q *contains* the line AB" (even though strictly speaking, either statement is incorrect).

[353] I have relegated many of the proofs in this chapter to the Appendices even though they are not difficult and would usually have been in the main text. However, this chapter contains an inordinate number of results and I decided to do so to avoid overwhelming the student.

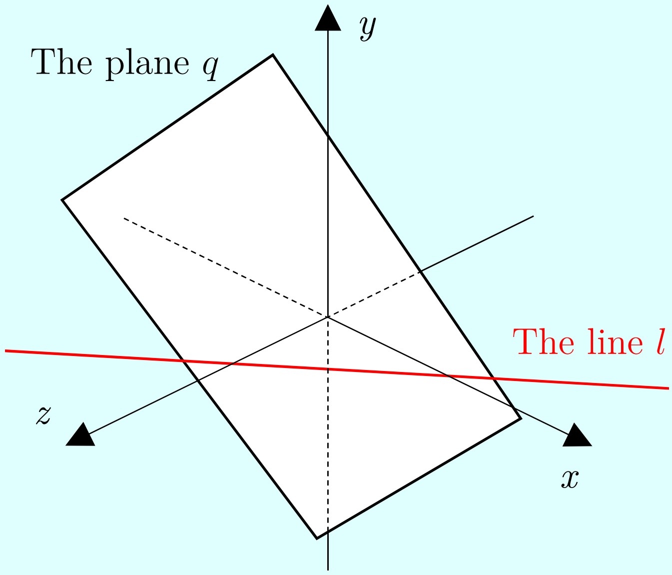
Example 923. Let q be the plane $\mathbf{r} \cdot (4, 1, 5) = 0$ and l be the line $\mathbf{r} = \overrightarrow{OA} + \lambda \mathbf{v} = (-1, -1, 1) + \lambda (5, 0, -4)$ $(\lambda \in \mathbb{R})$.

The plane q contains the line l. Here are two ways to show this:

Method 1. Observe l contains the points $A = (-1, -1, 1)$ and $B = A + \mathbf{v} = (-1, -1, 1) + (5, 0, -4) = (4, -1, -3)$.

But q also contains A and B (as you should be able to verify). And so, by the above Fact, q contains the line AB, which is also the line l.

Method 2. Let $R = (-1, -1, 1) + \lambda (5, 0, -4)$ be an arbitrary point on the line l. We show that R satisfies q's vector equation:



$$\begin{aligned}\overrightarrow{OR} \cdot (4, 1, 5) &= [(-1, -1, 1) + \lambda (5, 0, -4)] \cdot (4, 1, 5) \\ &= 4(-1 + 5\lambda) + (-1) + 5(1 - 4\lambda) \\ &= -4 - 20\lambda - 1 + 5 + 20\lambda = 0.\end{aligned}$$

We've just shown that q contains the arbitrary point R on l. So, q contains every point on l. (Sloppily, we may say that q contains l.)

Exercise 272. Suppose the plane q is described by $\mathbf{r} \cdot (4, -3, 2) = -10$, while the line l is described by $\mathbf{r} = (7, 3, 1) + \lambda (3, 6, -2)$. Determine if q contains l. (Answer on p. 1869.)

We next examine the normal vector in greater detail.

71.1. The Normal Vector

A plane's **normal vector** is perpendicular to every vector on that plane. Formally,

Definition 177. A *normal vector* of a plane is a vector that's perpendicular to every vector on that plane.

More simply, instead of saying, "$\mathbf{n}$ is a normal vector of the plane q," we'll also say, "$\mathbf{n}$ is normal to q". And as shorthand, we'll write $\mathbf{n} \perp q$.

Not surprisingly, the vector $\mathbf{n}$ used in Definition 175 of the plane is a normal vector:

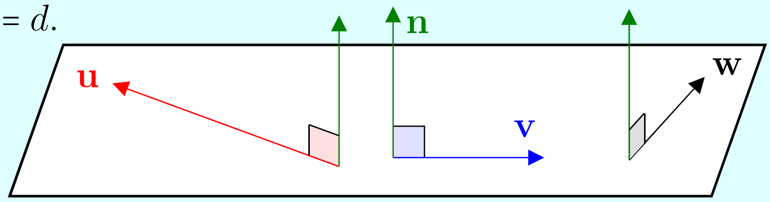
Fact 159. *If* $q = \left\{R : \overrightarrow{OR} \cdot \mathbf{n} = d\right\}$ *is a plane, then* $\mathbf{n} \perp q$.

Proof. See p. 1630 (Appendices). □

Example 924. Let q be the plane $\mathbf{r} \cdot \mathbf{n} = d$.

By the above Fact, $\mathbf{n} \perp q$—i.e., $\mathbf{n}$ is perpendicular to *every* vector on q.

So if the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are on q, then $\mathbf{n} \perp \mathbf{u}, \mathbf{v}, \mathbf{w}$.

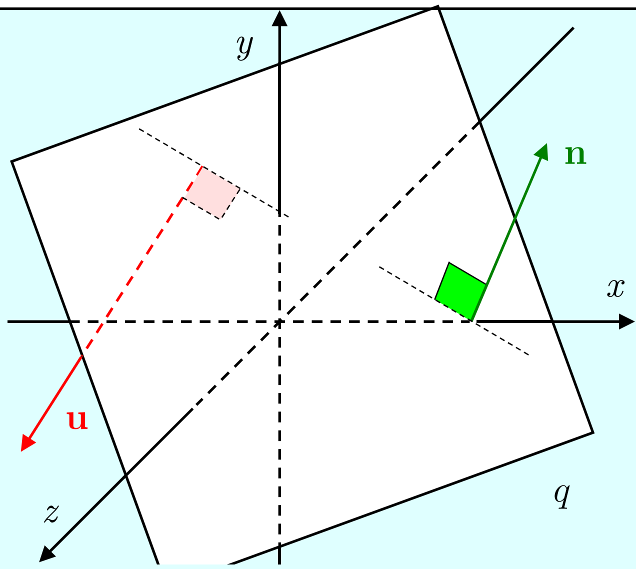


A plane's normal vector is **not unique**:

Example 925. Let q be the plane $\mathbf{r} \cdot (1, 1, 1) = 1$, so that $\mathbf{n} = (1, 1, 1)$ is a normal vector of q. Let

$$\mathbf{m} = (2, 2, 2)$$
$$\mathbf{u} = (-1.5, -1.5, -1.5)$$
$$\mathbf{v} = \left(\sqrt{5}, \sqrt{5}, \sqrt{5}\right)$$

The vectors $\mathbf{m} = 2\mathbf{n}$, $\mathbf{u} = -1.5\mathbf{n}$, and $\mathbf{v} = \sqrt{5}\mathbf{n}$ are parallel to $\mathbf{n}$. So, they are "obviously" also normal vectors of q.



As the above example suggests, if $\mathbf{n}$ is a normal vector of the plane q, then so too is any vector $\mathbf{m}$ that's parallel to $\mathbf{n}$:

Fact 160. *Let q be a plane and $\mathbf{n}$ and $\mathbf{m}$ be vectors. Suppose $\mathbf{n} \perp q$. Then*

$$\mathbf{m} \parallel \mathbf{n} \implies \mathbf{m} \perp q.$$

Proof. See p. 1630 (Appendices). □

It turns out that the converse of Fact 160 is also true. That is, suppose $\mathbf{n}$ is a normal vector of the plane q. If $\mathbf{m}$ is also a normal vector of q, then $\mathbf{m} \parallel \mathbf{n}$:

Theorem 18. *Let q be a plane and $\mathbf{n}$ and $\mathbf{m}$ be vectors. Suppose $\mathbf{n} \perp q$. Then*

$$\mathbf{m} \perp q \quad \Longrightarrow \quad \mathbf{m} \parallel \mathbf{n}.$$

Proof. See p. 1633 (Appendices). □

Fact 160 says that a normal vector $\mathbf{n}$ of a plane q is **not unique**: *Every* vector parallel to $\mathbf{n}$ is also a normal vector of q.

Theorem 18 then says the converse: *Only* vectors that are parallel to $\mathbf{n}$ are normal vectors of q.

Combine these two results together:

Corollary 29. *Let q be a plane and $\mathbf{n}$ and $\mathbf{m}$ be vectors. Suppose $\mathbf{n} \perp q$. Then*

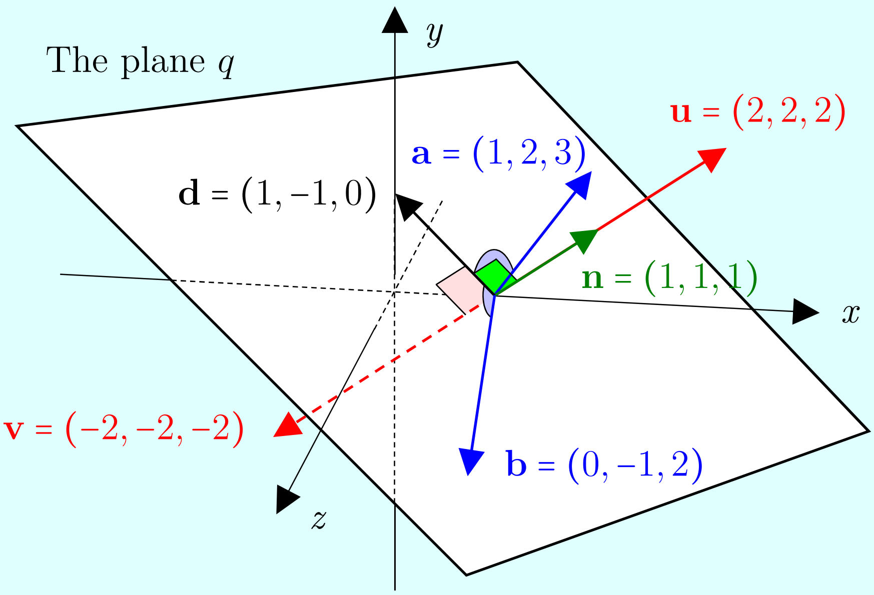
$$\mathbf{m} \perp q \quad \Longleftrightarrow \quad \mathbf{m} \parallel \mathbf{n}.$$

In other words, given a plane's normal vector $\mathbf{n}$, that plane's normal vectors are *exactly* those which are parallel to $\mathbf{n}$.

Example 926. Let q again be the plane $\mathbf{r} \cdot (1,1,1) = 1$.

By Corollary 29 ($\Longleftarrow$), every vector that's parallel to $\mathbf{n} = (1,1,1)$ is also normal to q. For example, the following vectors are are parallel to $(1,1,1)$ and so also normal to q:

$$\mathbf{u} = (2,2,2),$$
$$\mathbf{v} = (-2,-2,-2),$$
$$\mathbf{w} = \left(\sqrt{5}, \sqrt{5}, \sqrt{5}\right).$$



Conversely, by Corollary 29 ($\Longrightarrow$), every normal vector of q **must** be parallel to $(1,1,1)$. For example, the vectors $\mathbf{a} = (1,2,3)$, $\mathbf{b} = (0,-1,2)$, $\mathbf{c} = (1,1,0.9)$, and $\mathbf{0}$ are **not** parallel to $(1,1,1)$ and so are **not** normal to q.

The vector $\mathbf{d} = (1,-1,0)$ is on the plane q and, as depicted, $\mathbf{d} \perp \mathbf{n}, \mathbf{u}, \mathbf{v}$, but $\mathbf{d} \not\perp \mathbf{a}, \mathbf{b}$.

Exercise 273. The plane q is described by $\mathbf{r} \cdot (1,-1,1) = -2$. Determine if $\mathbf{a} = (2,-2,2)$, $\mathbf{b} = (2,2,-2)$, and $\mathbf{c} = \left(-\sqrt{2}, \sqrt{2}, -\sqrt{2}\right)$ are normal vectors of q. (Answer on p. 1869.)

Suppose the plane q can be described by $\mathbf{r} \cdot \mathbf{n} = d$. If $\mathbf{m} = k\mathbf{n}$ for some $k \neq 0$, then "obviously", q can also be described by

$$\mathbf{r} \cdot \overbrace{\mathbf{m}}^{k\mathbf{n}} = kd.$$

For future reference, let's jot this down as a formal result:

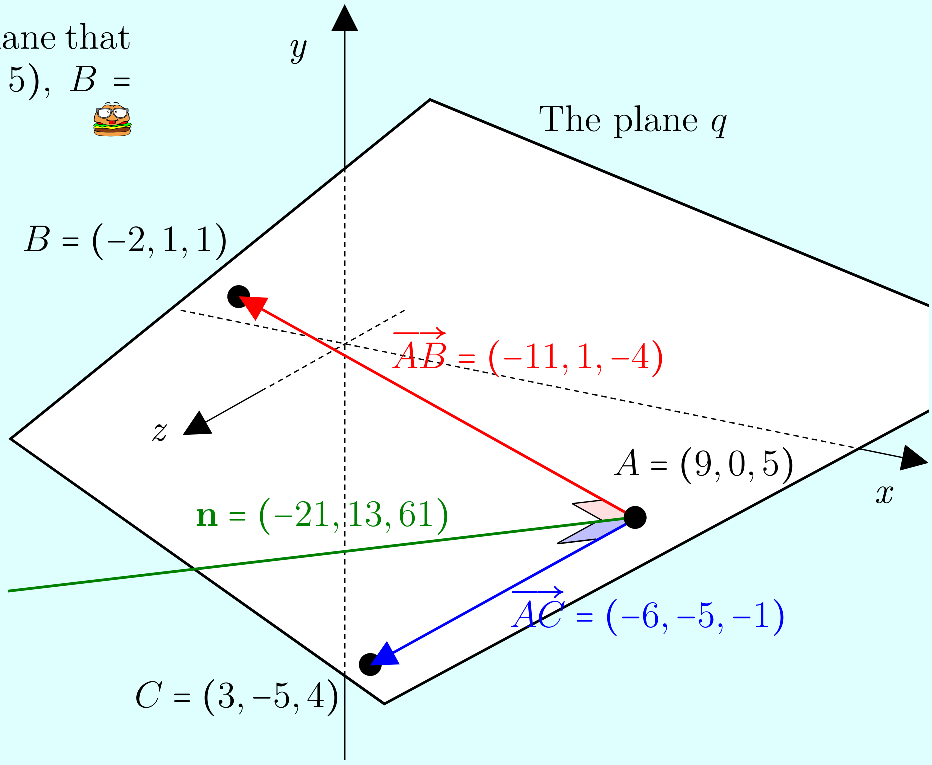
Fact 161. *Suppose q is a plane with $q = \{R : \mathbf{r} \cdot \mathbf{n} = d\}$.*

If $\mathbf{m} = k\mathbf{n}$ for some $k \neq 0$, then we also have $q = \{R : \mathbf{r} \cdot \mathbf{m} = kd\}$.

Example 927. The plane q described by $\mathbf{r} \cdot (1, 2, 3) = 4$ can also be described by

$$\mathbf{r} \cdot (2, 4, 6) = 8, \qquad \mathbf{r} \cdot (-1, -2, -3) = -4, \quad \text{or} \quad \mathbf{r} \cdot \left(\sqrt{5}, 2\sqrt{5}, 3\sqrt{5}\right) = 4\sqrt{5}.$$

Example 928. Let q be the plane that contains the points $A = (9, 0, 5)$, $B = (-2, 1, 1)$, and $C = (3, -5, 4)$.



Let's first write down two vectors on q:

$$\overrightarrow{AB} = (-11, 1, -4),$$
$$\overrightarrow{AC} = (-6, -5, -1).$$

So, a normal vector of q is

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -21 \\ 13 \\ 61 \end{pmatrix}.$$

Compute $d = \overrightarrow{OA} \cdot \mathbf{n} = (9, 0, 5) \cdot (-21, 13, 61) = -189 + 0 + 305 = 116$.

Hence, q may be described by $\mathbf{r} \cdot (-21, 13, 61) = 116$.

Another normal vector of q is $2(-21, 13, 61) = (-42, 26, 122)$. Thus, q may also be described by $\mathbf{r} \cdot (-42, 26, 122) = 2 \cdot 116 = 232$.

Exercise 274. The plane q contains the points $A = (1, -1, 2)$, $B = (-2, 3, 0)$, and $C = (0, -1, 1)$. (Answer on p. 1869.)

(a) Find a normal vector of q.

(b) Hence write down a vector equation that describes the plane q.

(c) Write down another normal vector of q.

(d) Hence write down another vector equation that describes the plane q.

Exercise 275. Let q, l, $\mathbf{v}$, and P be, respectively, a plane, a line, a vector, and a point (all in 3D space). Explain whether each statement **(i)** could be perfectly correct; or **(ii)** is only ever strictly speaking incorrect, but nonetheless written by us anyway (because we're sloppy and lazy): (Answer on p. 1869.)

(a) q contains l; **(b)** q contains P; **(c)** l contains P; **(d)** q contains $\mathbf{v}$;
(e) l is on or in q; **(f)** P is on or in q; **(g)** P is on or in l; **(h)** $\mathbf{v}$ is on or in q.

Next, let q be a plane with normal vector $\mathbf{n}$. Definition 177 says that if a vector $\mathbf{v}$ is on q, then it must also be perpendicular to $\mathbf{n}$.

It turns out that the converse is also true. That is, if $\mathbf{v}$ is perpendicular to $\mathbf{n}$, then it must be on q. Formally,

Fact 162. *Suppose q is a plane with normal vector $\mathbf{n}$. Then*

$$\mathbf{v} \perp \mathbf{n} \qquad \Longrightarrow \qquad \mathbf{v} \textit{ is on } q.$$

Proof. See p. 1632 (Appendices). □

Putting Definition 177 and Fact 162 together, a plane's vectors are exactly those that are perpendicular to its normal vector:

Corollary 30. *Suppose q is a plane with normal vector $\mathbf{n}$. Then*

$$\mathbf{v} \perp \mathbf{n} \quad \Longleftrightarrow \quad \mathbf{v} \textit{ is on } q.$$

Example 929. Let $\mathbf{n} = (5,1,6)$ be a vector and q be the plane $\mathbf{r} \cdot \mathbf{n} = -3$. As you can verify, q contains the points $A = (-1,2,0)$, $B = (-2,1,1)$, and $C = (3,0,-3)$.

Since A, B, and C are on q, so too are the vectors $\overrightarrow{AB} = (-1,-1,1)$ and $\overrightarrow{AC} = (4,-2,-3)$. Hence, we should have $\overrightarrow{AB}, \overrightarrow{AC} \perp \mathbf{n}$, as we now verify:

$$\overrightarrow{AB} \cdot \mathbf{n} = (-1,-1,1) \cdot (5,1,6) = -5 - 1 + 6 = 0. \qquad ✓$$
$$\overrightarrow{AC} \cdot \mathbf{n} = (4,-2,-3) \cdot (5,1,6) = 20 - 2 - 18 = 0. \qquad ✓$$

Now, is the vector $\mathbf{u} = (-3,3,2)$ on the plane q?

Well, it's not obvious. We could try to find two points S and T on q such that $\mathbf{u} = \overrightarrow{ST}$. But this could be slow and laborious.

A quicker method would be to use Corollary 30. That is, simply check if $\mathbf{u} \perp \mathbf{n}$:

$$\mathbf{u} \cdot \mathbf{n} = (-3,3,2) \cdot (5,1,6) = -15 + 3 + 12 = 0. \qquad ✓$$

So yup, $\mathbf{u}$ is on q.[354]

Let's now consider the vector $\mathbf{v} = (-7,2,3)$. Is it on q? Again, simply check if $\mathbf{v} \perp \mathbf{n}$.

$$\mathbf{v} \cdot \mathbf{n} = (-7,2,3) \cdot (5,1,6) = -35 + 2 + 18 = -15. \qquad ✗$$

So nope, $\mathbf{v}$ is not on q.

Exercise 276. Let q be the plane described by $\mathbf{r} \cdot (8,-2,1) = 5$. Are the vectors $\mathbf{a} = (3,7,-5)$, $\mathbf{b} = (1,6,4)$, and $\mathbf{c} = (3,10,1)$ are on q? (Answer on p. 1869.)

[354] For anyone not convinced, let $D = A + \mathbf{u} = (-1,2,0) + (-3,3,2) = (-4,5,2)$. We can verify that $D \in q$. And now, since $A, D \in q$, by Definition 176, the vector $\overrightarrow{AD} = \mathbf{u}$ is on q.

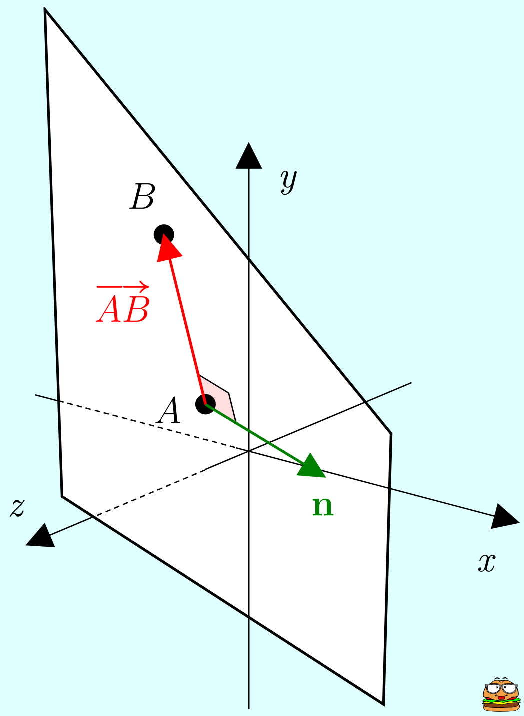
Suppose the plane q contains the point P. Then q contains exactly those points R for which the vector $\overrightarrow{PR}$ is on q. Formally,

Fact 163. *Let q be a plane and P and R be points. Suppose $P \in q$. Then*

$$R \in q \qquad \Longleftrightarrow \qquad \text{The vector } \overrightarrow{PR} \text{ is on } q.$$

Proof. See p. 1631 (Appendices). □

Example 930. The plane $q = \left\{R : \overrightarrow{OR} \cdot (0, -2, 3) = 1\right\}$ contains the point $A = (0, 1, 1)$. The point B is such that $\overrightarrow{AB} = (1, 3, 2)$.



Here are two methods for showing that $B \in q$:

Method 1. First find $B = A + \overrightarrow{AB} = (0, 1, 1) + (1, 3, 2) = (1, 4, 3)$. Then show that B satisfies the plane's vector equation:

$$\overrightarrow{OB} \cdot \mathbf{n} = (1, 4, 3) \cdot (0, -2, 3) = 0 - 8 + 9 = 1. \qquad \checkmark$$

Method 2 (quicker). Simply check if $\overrightarrow{AB} \perp \mathbf{n}$:

$$\overrightarrow{AB} \cdot \mathbf{n} = (1, 3, 2) \cdot (0, -2, 3) = 0 - 6 + 6 = 0. \qquad \checkmark$$

Yup, $\overrightarrow{AB} \perp \mathbf{n}$. And so by Fact 163, $B \in q$.

We now revisit Example 921:

Example 921. We already showed that the plane $q = \left\{R : \overrightarrow{OR} \cdot (-3, 0, 2) = -5\right\}$ contains the points $A = (1, 0, -1)$ and $B = (3, 1, 2)$, but not the point $C = (9, 1, 1)$. We also concluded that the vector $\overrightarrow{AB}$ is on q (because q contains A and B).

However, we were unable to say if the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are on q. But now, with Fact 163, we know that they are not.

Exercise 277. The plane q is described by $\mathbf{r} \cdot (7, -1, 3) = 19$ and $A = (1, 4, -1)$ is a point. The point B is such that $\overrightarrow{AB} = (7, 3, -2)$. Is the point B on q? (Answer on p. 1869.)

Given two (non-parallel) vectors on a plane, their vector product is normal to the plane:

Fact 164. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are non-parallel vectors on a plane* q*, then* $\mathbf{a}\times\mathbf{b}\perp q$*.*

Proof. See p. 1636 (Appendices). □

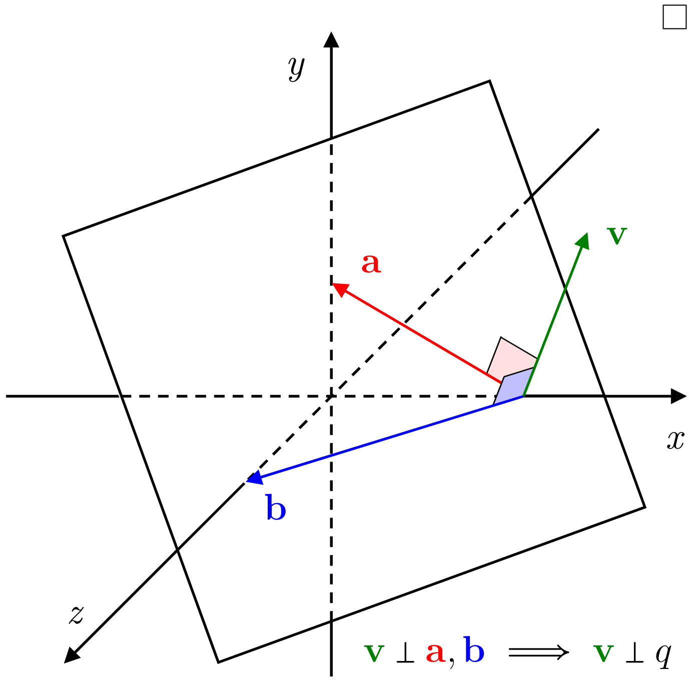
Now, suppose $\mathbf{a}$ and $\mathbf{b}$ are (non-parallel) vectors on a plane q. If $\mathbf{v}\perp q$, then by Definition 177, $\mathbf{v}\perp\mathbf{a},\mathbf{b}$.

It turns out that the converse is also true. That is, if $\mathbf{v}\perp\mathbf{a},\mathbf{b}$, then $\mathbf{v}\perp q$. This is because if $\mathbf{v}\perp\mathbf{a},\mathbf{b}$, then by Fact 157, $\mathbf{v}\parallel\mathbf{a}\times\mathbf{b}$. But by Fact 164, $\mathbf{a}\times\mathbf{b}\perp q$. And so by Corollary 29, $\mathbf{v}\perp q$.

Let's jot this down as a formal result:

Corollary 31. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are non-parallel vectors on a plane* q*. Then*

$$\mathbf{v}\perp q \iff \mathbf{v}\perp\mathbf{a},\mathbf{b}.$$



Corollary 32. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are non-parallel vectors on the plane* q*. Then*

$$\mathbf{c}\perp\mathbf{a}\times\mathbf{b} \iff \mathbf{c} \text{ is on } q.$$

Proof. By Fact 164, $\mathbf{a}\times\mathbf{b}\perp q$. And so by Corollary 30, $\mathbf{c}\perp\mathbf{a}\times\mathbf{b}\iff\mathbf{c}$ is on q. □

Example 931. The plane q contains the points $A=(0,0,1)$, $B=(4,2,0)$, and $C=(-5,0,4)$.

Let $\mathbf{v}=(6,-7,10)$. Is $\mathbf{v}\perp q$?

First, write down two non-parallel vectors on q. Two obvious candidates are

$$\overrightarrow{AB}=(4,2,-1) \qquad \text{and} \qquad \overrightarrow{AC}=(-5,0,3).$$

Then check if $\mathbf{v}\perp\overrightarrow{AB},\overrightarrow{AC}$:

$$\mathbf{v}\cdot\overrightarrow{AB}=(6,-7,10)\cdot(4,2,-1)=24-14-10=0, \qquad ✓$$
$$\mathbf{v}\cdot\overrightarrow{AC}=(6,-7,10)\cdot(-5,0,3)=-30+0+30=0. \qquad ✓$$

Since $\overrightarrow{AB}\nparallel\overrightarrow{AC}$ and $\mathbf{v}\perp\overrightarrow{AB},\overrightarrow{AC}$, by Corollary 31, $\mathbf{v}\perp q$.

Exercise 278. The vectors $\mathbf{a}=(1,-1,1)$ and $\mathbf{b}=(-2,2,-2)$ are on the plane q. Is $\mathbf{n}=(0,1,2)$ a normal vector of q? What about $\mathbf{m}=(1,3,2)$? (Answer on p. 1869.)

72. Planes in Cartesian Form

Example 932. The plane q is described by the **vector equation** $\mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot(1,2,3) \overset{v}{=} 4$.

The plane q contains those points $R = (x,y,z)$ whose position vector satisfies $\overset{v}{=}$.

We can also rewrite $\overset{v}{=}$ as $$\overbrace{(x,y,z)}^{\mathbf{r}}\cdot\overbrace{(1,2,3)}^{\mathbf{n}} \overset{v}{=} 4.$$

But, $$(x,y,z)\cdot(1,2,3) = x + 2y + 3z.$$

So, q is also described by the **cartesian equation** $x + 2y + 3z \overset{c}{=} 4$.

In general, suppose q is the plane described by this **vector equation**:

$$\mathbf{r}\cdot\mathbf{n} = (x,y,z)\cdot(a,b,c) \overset{v}{=} d.$$

Then q can also be described by this **cartesian equation**:

$$ax + by + cz \overset{c}{=} d.$$

Formally,

Fact 165. *Suppose (a,b,c) is a non-zero vector and $d \in \mathbb{R}$. Then*

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \overset{v}{=} d \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : ax + by + cz \overset{c}{=} d \right\}.$$

Example 933. The plane described by $\mathbf{r}\cdot(5,0,-1) = 3$ may also be described by $5x - z = 3$.

Example 934. The plane described by $\mathbf{r}\cdot(-1,7,2) = 0$ may also be described by $-x + 7y + 2z = 0$.

Example 935. The plane described by $5x + 6y + 7z = 8$ may also be described by $\mathbf{r}\cdot(5,6,7) = 8$.

Example 936. The plane described by $y = 5$ may also be described by $\mathbf{r}\cdot(0,1,0) = 5$.

Fact 166. *The plane described by* $\mathbf{r} \cdot (a, b, c) = d$ *contains the origin if and only if* $d = 0$.

Proof. The origin is the point $(x, y, z) = (0, 0, 0)$ and satisfies the equation $\mathbf{r} \cdot (a, b, c) = d$ or $ax + by + cz = d$ if and only if $d = 0$. So, the plane described by $\mathbf{r} \cdot (a, b, c) = d$ contains the origin if and only if $d = 0$. □

Example 937. Consider the plane q described by

$$ax + by + cz = 0 \qquad \text{or} \qquad \mathbf{r} \cdot (a, b, c) = 0.$$

Even without knowing a, b, and c, we know q contains the origin.

Example 938. Consider the plane q described by

$$ax + by + cz = 8 \qquad \text{or} \qquad \mathbf{r} \cdot (a, b, c) = 8.$$

Even without knowing a, b, and c, we know q does **not** contain the origin.

Exercise 279. Each of the following is a plane given in vector form. Rewrite each in cartesian form and state if each contains the origin. (Answer on p. 1870.)

(a) $\mathbf{r} \cdot (1, 2, 3) = 17$. **(b)** $\mathbf{r} \cdot (-1, 0, -2) = 0$. **(c)** $\mathbf{r} \cdot (0, -2, 5) = -3$.

Exercise 280. Each of the following is a plane given in cartesian form. Rewrite each in vector form and state if each contains the origin. (Answer on p. 1870.)

(a) $x + 5 = 17y + z$. **(b)** $y + 1 = 0$. **(c)** $x + z = y - 2$.

72.1. Finding Points on a Plane

Example 939. The plane q is described in vector or cartesian form by

$$\mathbf{r} \cdot (1,2,3) = 4 \qquad \text{or} \qquad x + 2y + 3z = 4.$$

Given a plane's cartesian equation, we can use trial-and-error to find points on that plane: Simply try out values of x, y, and z that satisfy the cartesian equation. (Tip: As always, zero is our friend.)

So for example, these points are on q (as you should verify yourself):

$$A = (4,0,0), \qquad B = (0,2,0), \quad \text{and} \quad C = (1,0,1).$$

In contrast, these are **not**:

$$D = (1,0,0), \qquad E = (0,1,1), \quad \text{and} \quad F = (-3,4,5).$$

Example 940. The plane q is described in vector or cartesian form by

$$\mathbf{r} \cdot (3,1,1) = -4 \qquad \text{or} \qquad 3x + y + z = -4.$$

These points are on q:

$$A = (0,-4,0), \qquad B = (0,0,-4), \quad \text{and} \quad C = (-1,-1,0).$$

These are **not**:

$$D = (1,0,0), \qquad E = (0,1,1), \quad \text{and} \quad F = (-3,4,5).$$

Example 941. The plane q is described in vector or cartesian form by

$$\mathbf{r} \cdot (-5,1,0) = 1 \qquad \text{or} \qquad -5x + y = 1.$$

These points are on q:

$$A = (0,1,0), \qquad B = (0,1,1), \quad \text{and} \quad C = (-1,-4,0).$$

These are **not**:

$$D = (1,0,0), \qquad E = (0,2,1), \quad \text{and} \quad F = (-3,4,5).$$

Exercise 281. For each plane (given in vector form), rewrite into cartesian form. Then find three points it contains and another three it doesn't. (Answer on p. 1870.)

(a) $\mathbf{r} \cdot (0,0,1) = 32.$ **(b)** $\mathbf{r} \cdot (5,3,1) = -2.$ **(c)** $\mathbf{r} \cdot (1,-2,3) = 0.$

72.2. Finding Vectors on a Plane

Example 942. The plane q is described by $\mathbf{r}\cdot(1,2,3)=4$ or $x+2y+3z=4$.

It has normal vector $\mathbf{n}=(a,b,c)=(1,2,3)$.

Recall (Corollary 30) that a vector $\mathbf{v}$ is on q if and only if $\mathbf{v}\perp\mathbf{n}$. So, to find vectors on q, we need simply find vectors that are perpendicular to $\mathbf{n}$—that is, vectors whose scalar product with $\mathbf{n}$ is zero.

We will now *construct* one such vector $\mathbf{u}=(u_1,u_2,u_3)$. That is, we'll pick values of u_1, u_2, and u_3 so that $\mathbf{u}\cdot\mathbf{n}=0$.

As always, zero is our friend. Let's start by picking $u_3=0$: So, $\mathbf{u}=(u_1,u_2,0)$.

We'll now play a simple little trick. Suppose we set $u_2=-a$ and $u_1=b$:

$$\mathbf{u}=(b,-a,0)=(2,-1,0).$$

Then things nicely cancel out: $\mathbf{u}\cdot\mathbf{n}=(b,-a,0)\cdot(a,b,c)=ba-ab+0=0.$

Et voilà! By construction, $\mathbf{u}$ is perpendicular to $\mathbf{n}$ and is thus on q.

Using the same method, we can easily construct two more vectors that are also on q:

$$\mathbf{v}=(c,0,-a)=(3,0,-1) \qquad \text{and} \qquad \mathbf{w}=(0,c,-b)=(0,3,-2).$$

(As you can easily verify, $\mathbf{v}\cdot\mathbf{n}=0$ and $\mathbf{w}\cdot\mathbf{n}=0$.)

And of course, the additive inverses of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are also on q:

$$-\mathbf{u}=(-2,1,0) \qquad -\mathbf{v}=(-3,0,1) \quad \text{and} \quad -\mathbf{w}=(0,-3,2).$$

That q contains the six vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $-\mathbf{u}$, $-\mathbf{v}$, and $-\mathbf{w}$ is formally stated as Fact 167 below.

Even without the above method, we can easily find vectors that are on q. For example, it is not difficult to see that $\mathbf{d}=(1,1,-1)$ is also on q because

$$\mathbf{d}\cdot\mathbf{n}=(1,1,-1)\cdot(1,2,3)=1+2-3=0.$$

It is equally easy to show that a vector is **not** on q. For example, $\mathbf{e}=(3,2,1)$ and $\mathbf{f}=(1,-1,1)$ are not on q because $\mathbf{e}\cdot\mathbf{n}\neq 0$ and $\mathbf{f}\cdot\mathbf{n}\neq 0$ (as you can verify).

Fact 167. *The plane with normal vector (a,b,c) contains points whose differences are these vectors:*

$$(b,-a,0),\ (c,0,-a),\ (0,c,-b),\ (-b,a,0),\ (-c,0,a),\ \textit{and}\ (0,-c,b).$$

Example 943. The plane q is described by $\mathbf{r} \cdot (3,1,1) = -4$ or $3x + y + z = -4$. It has normal vector $\mathbf{n} = (a,b,c) = (3,1,1)$. And so by Fact 167, q contains these vectors:

$$\begin{array}{lll} (b,-a,0) = (1,-3,0), & (c,0,-a) = (1,0,-3), & (0,c,-b) = (0,1,-1), \\ (-b,a,0) = (-1,3,0), & (-c,0,a) = (-1,0,3), & (0,-c,b) = (0,-1,1). \end{array}$$

We can also easily find other vectors on q. For example, $\mathbf{d} = (-1,1,2)$ is on q because

$$\mathbf{d} \cdot \mathbf{n} = (-1,1,2) \cdot (3,1,1) = -3 + 1 + 2 = 0.$$

In contrast, $\mathbf{e} = (3,2,1)$ and $\mathbf{f} = (1,-1,1)$ are not on q because $\mathbf{e} \cdot \mathbf{n} \neq 0$ and $\mathbf{f} \cdot \mathbf{n} \neq 0$.

Example 944. The plane q is described by $\mathbf{r} \cdot (-5,1,0) = -4$ or $-5x + y = 1$. It has normal vector $\mathbf{n} = (a,b,c) = (-5,1,0)$. And so by Fact 167, q contains these vectors:

$$\begin{array}{lll} (b,-a,0) = (1,5,0), & (c,0,-a) = (0,0,5), & (0,c,-b) = (0,0,-1), \\ (-b,a,0) = (-1,-5,0), & (-c,0,a) = (0,0,-5), & (0,-c,b) = (0,0,1). \end{array}$$

Actually, here we can make another useful and important observation. Notice that $\mathbf{n}$'s z-coordinate is $\mathbf{0}$. And so, if a vector $\mathbf{u} = (u_1, u_2, u_3)$ is perpendicular to $\mathbf{n}$ (and is hence on q), then so too is the vector (u_1, u_2, λ) for *any* $\lambda \in \mathbb{R}$.

So for example, since $(1,5,0)$ is on q, so too are these vectors:

$$(1,5,0), \qquad (1,5,1), \qquad \left(1,5,-\sqrt{2}\right), \qquad (1,5,999), \qquad (1,5,\pi), \qquad \text{etc.}$$

Moreover, for *any* $\lambda \in \mathbb{R}$, the vector $(0,0,\lambda)$ must be perpendicular to $\mathbf{n}$. In particular, the standard basis vector $\mathbf{k} = (0,0,1)$ is perpendicular to $\mathbf{n}$ and is thus also on q.

Example 945. The plane q is described by $\mathbf{r} \cdot (0,0,1) = 5$ or $z = 5$. It has normal vector $\mathbf{n} = (a,b,c) = (0,0,1)$. Observe that $\mathbf{n}$'s x- and y-coordinates are both 0.

Following the observation made in the previous example, if a vector $\mathbf{u} = (u_1, u_2, u_3)$ is perpendicular to $\mathbf{n}$ (and so is on q), then so too is the vector (λ, μ, u_3) for *any* $\lambda, \mu \in \mathbb{R}$.

Here we also make a new observatio: Since only the z-coordinate of $\mathbf{n}$ is non-zero, if $\mathbf{u} = (u_1, u_2, u_3) \perp \mathbf{n}$ or $\mathbf{u} \cdot \mathbf{n} = 0$, it must be that $u_3 = 0$.

Altogether then, the vectors that are perpendicular to $\mathbf{n}$ (and so are on q) are exactly those vectors that can be written as $(\lambda, \mu, 0)$ for $\lambda, \mu \in \mathbb{R}$. So for example, the vectors $(9,0,0)$, $(0,5,0)$, and $\left(-\pi, \sqrt{2}, 0\right)$ are perpendicular to $\mathbf{n}$ (and so are on q).

In contrast, the vectors $(9,0,1)$, $(0,5,2)$, and $\left(-\pi, \sqrt{2}, 3\right)$ are not.

Exercise 282. Find three non-parallel vectors on each plane: **(a)** $\mathbf{r} \cdot (1,-2,3) = 0$; **(b)** $\mathbf{r} \cdot (5,3,1) = -2$; **(c)** $\mathbf{r} \cdot (1,0,4) = 5$; **(d)** $\mathbf{r} \cdot (0,7,0) = 32$. (Answer on p. 1870.)

73. Planes in Parametric Form

In the last two chapters, we learnt to describe planes in **vector** and **cartesian form**. In this chapter, we'll learn to describe planes in **parametric form**.

To do so, we first introduce an important result.

Recall (Fact 125) that in 2D space, if $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors, then any vector $\mathbf{c}$ can be written as the **linear combination (LC)** of $\mathbf{a}$ and $\mathbf{b}$. That is, given any vector $\mathbf{c}$ (in 2D space), there exist real numbers λ and μ such that

$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}.$$

It turns out that the same is true of vectors **on a plane** in 3D space. That is, in 3D space, a vector is on a plane if and only if it can be written as a LC of two non-parallel vectors on that plane. Or equivalently, the vectors on a plane q are **exactly** those that can be written as a LC of *any* two non-parallel vectors on q. Formally,

Theorem 19. *Let q be a plane and $\mathbf{a}$ and $\mathbf{b}$ be non-parallel vectors on q. Suppose $\mathbf{c}$ is a non-zero vector. Then*

$$\mathbf{c} \text{ is a vector on } q \iff \text{There exist } \lambda, \mu \in \mathbb{R} \text{ such that } \mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}.$$

Remark 112. Take care to note that Theorem 19 is an if and only if ($\iff$) statement which says two things. First, $\Longleftarrow$ says,

"If a vector can be written as a LC of $\mathbf{a}$ and $\mathbf{b}$, then it is on q."

Or equivalently, "Every vector that's a LC of $\mathbf{a}$ and $\mathbf{b}$ is on q."

Second, the converse $\Longrightarrow$ says,

"If a vector is on q, then it can be written as a LC of $\mathbf{a}$ and $\mathbf{b}$."

Or equivalently, "Every vector on q is as a LC of $\mathbf{a}$ and $\mathbf{b}$."

Proof. First note that since $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors on q, by Fact 164, $\mathbf{a}\times\mathbf{b} \perp q$.

We first prove $\Longleftarrow$. Suppose there exist $\lambda, \mu \in \mathbb{R}$ such that $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$. We show that $\mathbf{a}\times\mathbf{b} \perp \mathbf{c}$, so that by Fact 162, $\mathbf{c}$ is also a vector on the plane:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= (\mathbf{a}\times\mathbf{b})\cdot(\lambda\mathbf{a}+\mu\mathbf{b}) = (\mathbf{a}\times\mathbf{b})\cdot(\lambda\mathbf{a}) + (\mathbf{a}\times\mathbf{b})\cdot(\mu\mathbf{b}) \\ &= \lambda(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} + \mu(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0 + 0 = 0. \checkmark\end{aligned}$$

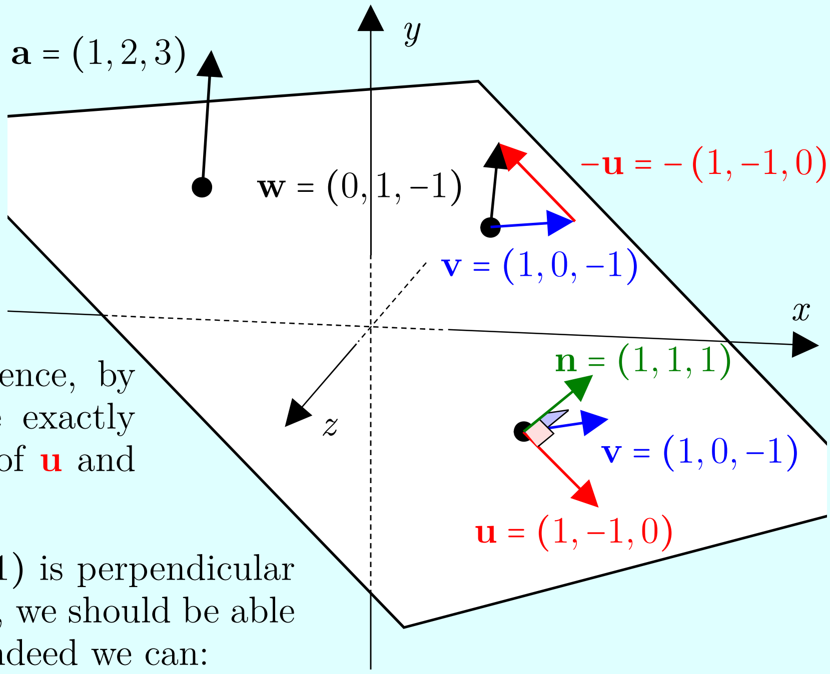
The proof of $\Longrightarrow$ is harder and relegated to p. 1636 (Appendices). □

As we'll see in Ch. 73.1, Theorem 19 will allow us to describe planes in **parametric form**.

But first, let's better acquaint ourselves with Theorem 19 with some examples:

Example 946. The plane q described by $x+y+z=1$ has normal vector $\mathbf{n} = (1,1,1)$.

The vectors $\mathbf{u} = (1,-1,0)$ and $\mathbf{v} = (1,0,-1)$ are both perpendicular to $\mathbf{n}$ (verify this!). So, by Corollary 30, both are on q.



Observe moreover that $\mathbf{u} \nparallel \mathbf{v}$. Hence, by Theorem 19, the vectors on q are exactly those that can be written as a LC of $\mathbf{u}$ and $\mathbf{v}$.

For example, the vector $\mathbf{w} = (0,1,-1)$ is perpendicular to $\mathbf{n}$ and is on q. So, by Theorem 19, we should be able to write $\mathbf{w}$ as a LC of $\mathbf{u}$ and $\mathbf{v}$, as indeed we can:

$$\mathbf{w} = (0,1,-1) = (1,0,-1) - (1,-1,0) = \mathbf{v} - \mathbf{u}.$$

In contrast, the vector $\mathbf{a} = (1,2,3)$ is not perpendicular to $\mathbf{n}$ and so is **not** on q. So, by Theorem 19, we cannot write $\mathbf{a}$ as a LC of $\mathbf{u}$ and $\mathbf{v}$. To verify this, write

$$\mathbf{a} = \lambda\mathbf{u} + \mu\mathbf{v} \qquad \text{or} \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \text{or} \qquad \begin{aligned} 1 &\overset{1}{=} \lambda + \mu \\ 2 &\overset{2}{=} -\lambda \\ 3 &\overset{3}{=} -\mu. \end{aligned}$$

Take $\overset{1}{=}$ plus $\overset{2}{=}$ to get $3 = \mu$, which contradicts $\overset{3}{=}$. So, there are no $\lambda, \mu \in \mathbb{R}$ such that $\mathbf{a} = \lambda\mathbf{u} + \mu\mathbf{v}$. Or equivalently, $\mathbf{a}$ **cannot** be written as a LC of $\mathbf{u}$ and $\mathbf{v}$.

Example 947. The plane q described by $x - y = 5$ has normal vector $\mathbf{n} = (1, -1, 0)$.

The vectors $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (1, 1, 0)$ are perpendicular to $\mathbf{n}$ and are so on q. Moreover, $\mathbf{u} \nparallel \mathbf{v}$. Hence, by Theorem 19, the vectors on q are exactly those that can be written as a LC of $\mathbf{u}$ and $\mathbf{v}$.

For example, the vector $\mathbf{w} = (1, 1, 1)$ is perpendicular to $\mathbf{n}$ and is on q. So, by Theorem 19, we should be able to write $\mathbf{w}$ as a LC of $\mathbf{u}$ and $\mathbf{v}$, as indeed we can:

$$\mathbf{w} = (1, 1, 1) = (0, 0, 1) + (1, 1, 0) = \mathbf{u} + \mathbf{v}.$$

In contrast, the vector $\mathbf{a} = (0, 1, 0)$ is not perpendicular to $\mathbf{n}$ and is not on q. So, by Theorem 19, we cannot write $\mathbf{a}$ as a LC of $\mathbf{u}$ and $\mathbf{v}$. To verify this, write

$$\mathbf{a} = \lambda\mathbf{u} + \mu\mathbf{v} \qquad \text{or} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \text{or} \qquad \begin{array}{l} 0 \stackrel{1}{=} \mu \\ 1 \stackrel{2}{=} \mu \\ 0 \stackrel{3}{=} \lambda. \end{array}$$

Clearly, $\stackrel{1}{=}$ contradicts $\stackrel{2}{=}$. So, there are no $\lambda, \mu \in \mathbb{R}$ such that $\mathbf{a} = \lambda\mathbf{u} + \mu\mathbf{v}$. Or equivalently, **a cannot** be written as a LC of $\mathbf{u}$ and $\mathbf{v}$.

Example 948. The plane q described by $z = 7$ has normal vector $\mathbf{n} = (0, 0, 1)$.

The vectors $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 1, 0)$ are perpendicular to $\mathbf{n}$ and so are on q. Moreover, $\mathbf{u} \nparallel \mathbf{v}$. Hence, by Theorem 19, the vectors on q are exactly those that can be written as a LC of $\mathbf{u}$ and $\mathbf{v}$.

For example, the vector $\mathbf{w} = (1, 2, 0)$ is perpendicular to $\mathbf{n}$ and is on q. So, by Theorem 19, we should be able to write $\mathbf{w}$ as a LC of $\mathbf{u}$ and $\mathbf{v}$, as indeed we can:

$$\mathbf{w} = (1, 2, 1) = (1, 0, 0) + 2\,(0, 1, 0) = \mathbf{u} + 2\mathbf{v}.$$

In contrast, the vector $\mathbf{a} = (0, 1, 1)$ is not perpendicular to $\mathbf{n}$ and is not on q. So, by Theorem 19, we cannot write $\mathbf{a}$ as a LC of $\mathbf{u}$ and $\mathbf{v}$. To verify this, write

$$\mathbf{a} = \lambda\mathbf{u} + \mu\mathbf{v} \qquad \text{or} \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{or} \qquad \begin{array}{l} 0 \stackrel{1}{=} \lambda \\ 1 \stackrel{2}{=} \mu \\ 1 \stackrel{3}{=} 0. \end{array}$$

Clearly, $\stackrel{3}{=}$ is a contradiction. So, there are no $\lambda, \mu \in \mathbb{R}$ such that $\mathbf{a} = \lambda\mathbf{u} + \mu\mathbf{v}$. Or equivalently, **a cannot** be written as a LC of $\mathbf{u}$ and $\mathbf{v}$.

Exercise 283. For each plane **(a)**–**(c)**, (Answer on p. 1871.)

(i) Write down the corresponding cartesian equation.

(ii) Write down two non-parallel vectors on the plane (call them $\mathbf{u}$ and $\mathbf{v}$). (You should explain why $\mathbf{u}$ and $\mathbf{v}$ are non-parallel.)

(iii) Write down another vector $\mathbf{w}$ on the plane. Explain why we can express $\mathbf{w}$ as a LC of $\mathbf{u}$ and $\mathbf{v}$. Then do so.

(iv) Explain whether it is possible to express the vector $(1,1,1)$ as a LC of $\mathbf{u}$ and $\mathbf{v}$. And if it is possible, do so.

(a) $\mathbf{r} \cdot (1,2,3) = 4$. **(b)** $\mathbf{r} \cdot (1,0,-1) = 0$. **(c)** $\mathbf{r} \cdot (9,1,1) = -5$.

73.1. Planes in Parametric Form

Suppose q is a plane and P is a point on q. Then Fact 163 says that a point R is on the plane q if and only if the vector $\overrightarrow{PR}$ is on q. In formal notation,

$$R \in q \quad \overset{1}{\iff} \quad \overrightarrow{PR} \text{ is on } q.$$

Next, suppose also that **a** and **b** are non-parallel vectors on q. Then Theorem 19 says that a vector **v** is on the plane q if and only if it is a LC of **a** and **b**. In formal notation,

$$\mathbf{v} \text{ is on } q \quad \overset{2}{\iff} \quad \text{There exist } \lambda, \mu \in \mathbb{R} \text{ such that } \mathbf{v} = \lambda \mathbf{a} + \mu \mathbf{b}.$$

Now combine $\overset{1}{\iff}$ and $\overset{2}{\iff}$:

$$R \in q \quad \overset{3}{\iff} \quad \text{There exist } \lambda, \mu \in \mathbb{R} \text{ such that } \overrightarrow{PR} = \lambda \mathbf{a} + \mu \mathbf{b}.$$

In words, a point R is on the plane q if and only if the vector $\overrightarrow{PR}$ is a LC of **a** and **b**.

Equivalently, the plane q contains exactly those points R for which the vector $\overrightarrow{PR}$ is a LC of **a** and **b**. That is,

$$q \overset{4}{=} \left\{ R : \overrightarrow{PR} = \lambda \mathbf{a} + \mu \mathbf{b} \quad (\lambda, \mu \in \mathbb{R}) \right\}.$$

We previously learnt how to describe planes in **vector** and **cartesian forms**.

With $\overset{4}{=}$, we now have us a third way to describe planes and is called the **parametric form** of a plane.

Let's summarise the above discussion as a formal result:

Fact 168. *Let q be a plane, P be a point, and* **a** *and* **b** *be non-parallel vectors. Suppose P,* **a***, and* **b** *are on q. Then*

$$R \in q \quad \iff \quad \text{there exist } \lambda, \mu \in \mathbb{R} \text{ such that } \overrightarrow{PR} = \lambda \mathbf{a} + \mu \mathbf{b}.$$

Or equivalently, $$q \overset{4}{=} \left\{ R : \overrightarrow{PR} = \lambda \mathbf{a} + \mu \mathbf{b} \quad (\lambda, \mu \in \mathbb{R}) \right\}.$$

Since $\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$, we have

$$\overrightarrow{PR} = \lambda \mathbf{a} + \mu \mathbf{b} \quad \iff \quad \overrightarrow{OR} = \overrightarrow{OP} + \lambda \mathbf{a} + \mu \mathbf{b}.$$

So, the above Fact can also be rewritten as

Corollary 33. *Let q be a plane, P be a point, and* **a** *and* **b** *be non-parallel vectors. Suppose P,* **a***, and* **b** *are on q. Then*

$$R \in q \quad \iff \quad \text{there exist } \lambda, \mu \in \mathbb{R} \text{ such that } \overrightarrow{OR} = \overrightarrow{OP} + \lambda \mathbf{a} + \mu \mathbf{b}.$$

Or equivalently, $$q \overset{5}{=} \left\{ R : \overrightarrow{OR} = \overrightarrow{OP} + \lambda \mathbf{a} + \mu \mathbf{b} \quad (\lambda, \mu \in \mathbb{R}) \right\}.$$

As before, instead of fully writing out the plane q in set notation (as was done in $\stackrel{4}{=}$ and $\stackrel{5}{=}$), we can simply say that q is *described* by the **parametric equation**

$$\overbrace{\overrightarrow{OR}}^{\text{Vec.}} \stackrel{6}{=} \overbrace{\overrightarrow{OP}}^{\text{Vec.}} + \lambda\mathbf{a} + \mu\mathbf{b} \qquad (\lambda, \mu \in \mathbb{R});$$

or,

$$\overbrace{R}^{\text{Pt}} \stackrel{7}{=} \overbrace{P}^{\text{Pt}} + \lambda\mathbf{a} + \mu\mathbf{b} \qquad (\lambda, \mu \in \mathbb{R}).$$

With $\stackrel{7}{=}$, we have a nice, informal geometric interpretation: A point R is on the plane q if and only if

R can be reached from P through a LC of "steps" in the directions of $\mathbf{a}$ and $\mathbf{b}$.

Example 949. Let q be the plane described by $\mathbf{r}\cdot(1,1,1)=1$ or $x+y+z=1$.

It contains the point $P=(1,0,0)$ and the non-parallel vectors $\mathbf{a}=(-1,1,0)$ and $\mathbf{b}=(-1,0,1)$. So, by Corollary 33,

$$\begin{aligned} q &= \left\{R:\overrightarrow{OR}=\overrightarrow{OP}+\lambda\mathbf{a}+\mu\mathbf{b} \qquad (\lambda,\mu\in\mathbb{R})\right\} \\ &= \left\{R:\overrightarrow{OR}\overset{6}{=}\begin{pmatrix}1\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}1-\lambda-\mu\\ \lambda\\ \mu\end{pmatrix} \qquad (\lambda,\mu\in\mathbb{R})\right\}. \end{aligned}$$

More simply, we say that q is *described* by the **parametric equation** $\overset{6}{=}$:

$$\overrightarrow{OR}\overset{6}{=}\begin{pmatrix}1\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}1-\lambda-\mu\\ \lambda\\ \mu\end{pmatrix} \qquad (\lambda,\mu\in\mathbb{R}).$$

As the parameters λ and μ vary, we get different points on q. For example, $(\lambda,\mu)=(3,5)$ produces the point S:

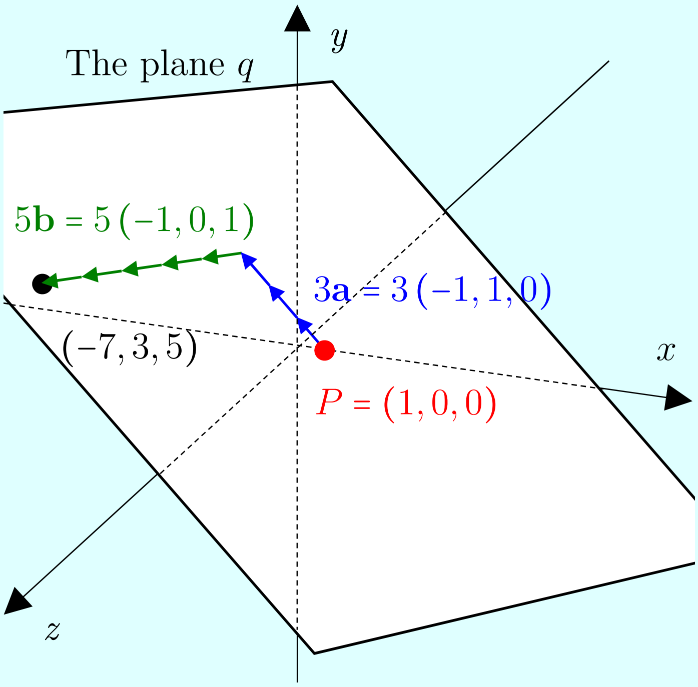
$$S=(1,0,0)+3(-1,1,0)+5(-1,0,1)=(-7,3,5).$$

Starting from the point P, we can reach S by taking 3 "steps" in the direction of $\mathbf{a}$, then 5 "steps" in the direction of $\mathbf{b}$.

Similarly, $(\lambda,\mu)=(0,0)$ produces the point T:

$$T=(1,0,0)+0(-1,1,0)+0(-1,0,1)=(1,0,0).$$

Starting from the point P, we were already at the point $T=P$.



Example 950. Let q be the plane described by $z = 3$.

It contains the point $P = (0, 0, 3)$ and the non-parallel vectors $\mathbf{a} = (1, 0, 0)$ and $\mathbf{b} = (0, 1, 0)$. So, by Corollary 33,

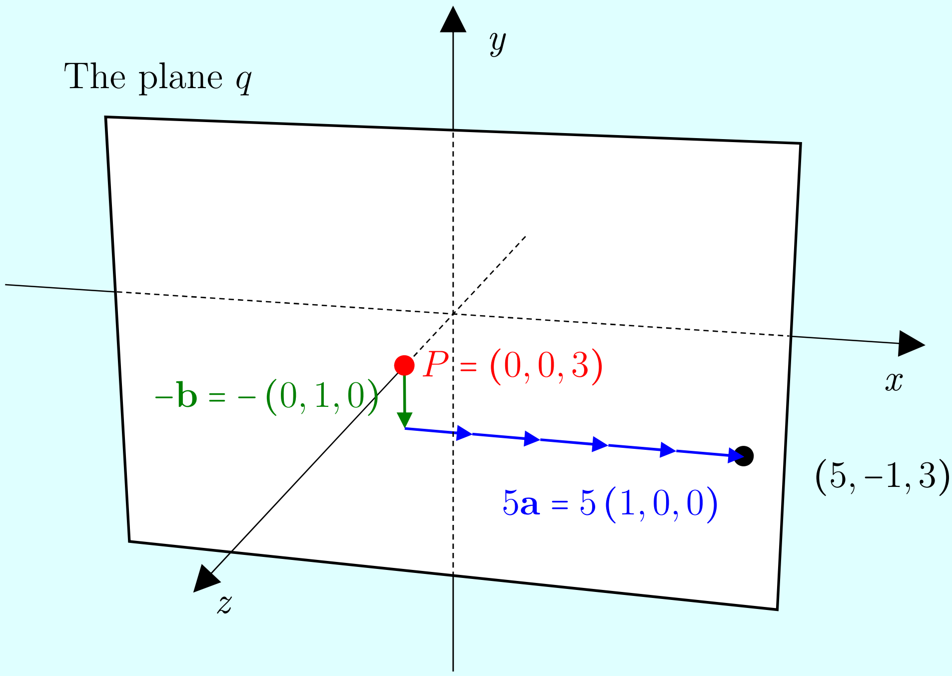
$$q = \left\{R : \overrightarrow{OR} = \overrightarrow{OP} + \lambda\mathbf{a} + \mu\mathbf{b} \qquad (\lambda, \mu \in \mathbb{R})\right\}$$
$$= \left\{R : \overrightarrow{OR} \overset{6}{=} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ \mu \\ 3 \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R})\right\}.$$

More simply, we say that q is *described* by the **parametric equation** $\overset{6}{=}$.

As the parameters λ and μ vary, we get different points on the plane q. For example, $(\lambda, \mu) = (5, -1)$ produces the point S:

$$S = (0, 0, 3) + 5(1, 0, 0) + (-1)(0, 1, 0) = (5, -1, 3).$$

Starting from the point P, we can reach the point S by taking 1 "step" in the direction **opposite** to $\mathbf{b}$, then 5 "steps" in the direction of $\mathbf{a}$.



As another example, $(\lambda, \mu) = (0, 1)$ produces the point T (not depicted):

$$T = (0, 0, 3) + 0(1, 0, 0) + 1(0, 1, 0) = (0, 1, 3).$$

Starting from the point P, we can reach T by taking 1 "step" in the direction of $\mathbf{b}$.

Example 951. Let q be the plane described by $-x+3y-5z=7$.

It contains the point $P=(-7,0,0)$ and the non-parallel vectors $\mathbf{a}=(3,1,0)$ and $\mathbf{b}=(5,0,-1)$. So, by Corollary 33,

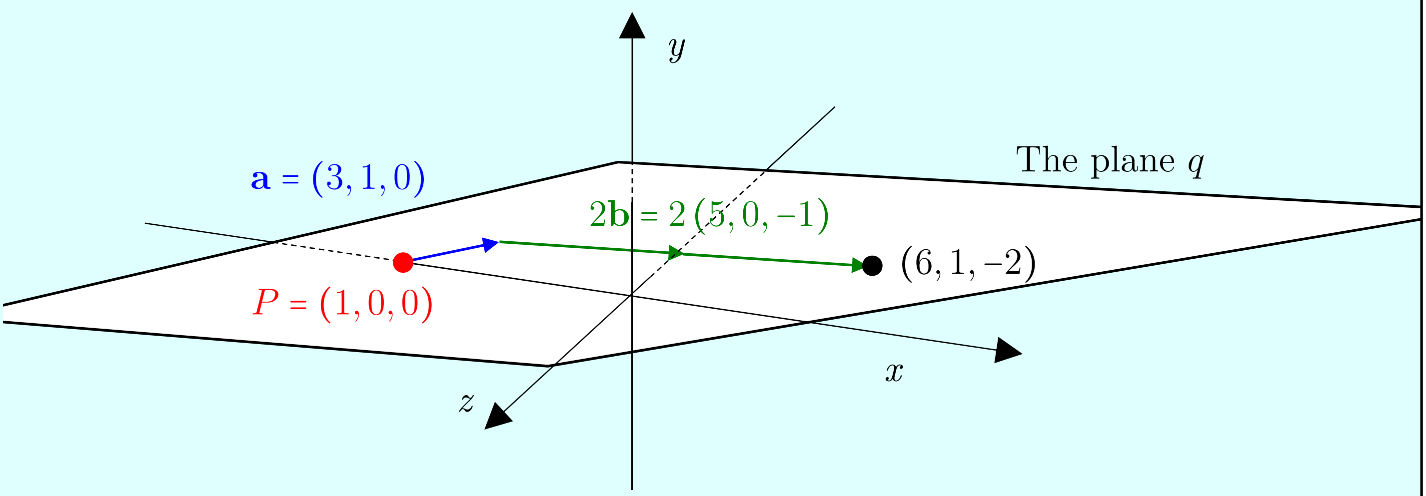
$$\begin{aligned} q &= \left\{R : \overrightarrow{OR} = \overrightarrow{OP} + \lambda\mathbf{a} + \mu\mathbf{b} \qquad (\lambda,\mu\in\mathbb{R})\right\} \\ &= \left\{R : \overrightarrow{OR} \overset{6}{=} \begin{pmatrix} -7 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -7+3\lambda+5\mu \\ \lambda \\ -\mu \end{pmatrix} \qquad (\lambda,\mu\in\mathbb{R})\right\}. \end{aligned}$$

More simply, we may say that q is *described* by the **parametric equation** $\overset{6}{=}$.

As the parameters λ and μ vary, we get different points on the plane q. For example, $(\lambda,\mu)=(1,2)$ produces the point S:

$$S=(-7,0,0)+1(3,1,0)+2(5,0,-1)=(6,1,-2).$$

Starting from the point P, we can reach S by taking 1 "step" in the direction of $\mathbf{a}$, then 2 "steps" in the direction of $\mathbf{b}$.



As another example, $(\lambda,\mu)=(-1,3)$ produces the point T (not depicted):

$$T=(-7,0,0)+(-1)(3,1,0)+3(5,0,-1)=(5,-1,-3).$$

Starting from the point A, we can reach T by taking 1 "step" in the direction **opposite** to $\mathbf{a}$ and 3 "steps" in the direction of $\mathbf{b}$.

Exercise 284. Rewrite each plane into **both** cartesian and parametric forms. (Answer on p. 1872.)

(a) $\mathbf{r}\cdot(-1,2,5)=5$. **(b)** $\mathbf{r}\cdot(0,0,1)=0$. **(c)** $\mathbf{r}\cdot(1,-3,5)=-2$.

73.2. Parametric to Vector or Cartesian Form

Given a plane in **parametric form**, rewriting it into **vector** or **cartesian form** is easy:

Example 952. We are given a plane in parametric form:

$$\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 7+8\lambda+9\mu \\ 3+3\lambda+3\mu \\ 4+4\lambda+7\mu \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

This plane contains the vectors $(8,3,4)$ and $(9,3,7)$. So, a normal vector is

$$(8,3,4) \times (9,3,7) = (9,-20,-3).$$

The plane contains the point $(7,3,4)$. Since $(7,3,4) \cdot (9,-20,-3) = 63 - 60 - 12 = -9$, the plane may be described by

$$\mathbf{r} \cdot (9,-20,-3) = -9 \qquad \text{or} \qquad 9x - 20y - 3z = -9.$$

Example 953. We are given a plane in parametric form:

$$\mathbf{r} = \begin{pmatrix} 17+3\lambda-2\mu \\ 2\mu-2 \\ 5\lambda \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

First, rewrite the above as

$$\mathbf{r} = \begin{pmatrix} 17 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

This plane contains the vectors $(3,0,5)$ and $(-2,2,0)$. So, a normal vector is

$$(3,0,5) \times (-2,2,0) = (-10,-10,6).$$

Observe that $(-10,-10,6) \parallel (5,5,-3)$. Hence, by Fact 160, $(5,5,-3)$ is also a normal vector of the plane. (It's nice to "simplify" the normal vector as much as possible—this will usually make our subsequent calculations slightly easier.)

The plane contains the point $(17,-2,0)$. Since $(17,-2,0) \cdot (5,5,-3) = 85 - 10 + 0 = -75$, the plane may be described by

$$\mathbf{r} \cdot (5,5,-3) = -75 \qquad \text{or} \qquad 5x + 5y - 3z = -75.$$

Example 954. We are given a plane in parametric form:

$$\mathbf{r} = \begin{pmatrix} \lambda - \mu - 2 \\ 14 + 5\lambda + 3\mu \\ 5 + \mu + 7\lambda \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

First, rewrite the above as

$$\mathbf{r} = \begin{pmatrix} -2 \\ 14 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

This plane contains the vectors $(1, 5, 7)$ and $(-1, 3, 1)$. So, a normal vector is

$$(1, 5, 7) \times (-1, 3, 1) = (-16, -8, 8).$$

Observe that $(-16, -8, 8) \parallel (2, 1, -1)$. So, $(2, 1, -1)$ is also a normal vector.

The plane contains the point $(-2, 14, 5)$. Since $(-2, 14, 5) \cdot (2, 1, -1) = -4 + 14 - 5 = 5$, the plane may be described by

$$\mathbf{r} \cdot (2, 1, -1) = 5 \qquad \text{or} \qquad 2x + y - z = 5.$$

Exercise 285. Rewrite each plane into **both** vector and cartesian form.

(a) $\mathbf{r} = (1, 2, 3) + \lambda (4, 5, 6) + \mu (7, 8, 9) \qquad (\lambda, \mu \in \mathbb{R})$.

(b) $\mathbf{r} = (\lambda - \mu, 4\lambda + 5, 0) \qquad (\lambda, \mu \in \mathbb{R})$.

(c) $\mathbf{r} = (1 + \mu, 1 + \lambda, \lambda + \mu) \qquad (\lambda, \mu \in \mathbb{R})$. (Answer on p. 1872.)

74. Four Ways to Uniquely Determine a Plane

Recall that there are two ways to uniquely determine a line. A line can be uniquely determined by **(a) two distinct points**; or **(b) a point and a vector**.

Similarly, there are **Four Ways** to uniquely determine a plane:[355]

1. **A point** and **a normal vector**;
2. **A point** and **two vectors** (that aren't parallel);
3. **Two points** and **a vector** (that isn't parallel to the vector between the two points); or
4. **Three points** (that aren't collinear).

More formally and precisely,

Fact 169. *Let A be a point and* $\mathbf{u}$ *be a vector.*

(a) *The unique plane that contains A and has* $\mathbf{u}$ *as its normal vector is* $\left\{R : \overrightarrow{OR} \cdot \mathbf{u} = \overrightarrow{OA} \cdot \mathbf{u}\right\}$.

Let $\mathbf{v}$ *be a vector that isn't parallel to* $\mathbf{u}$*.*

2. The unique plane that contains A, $\mathbf{u}$*, and* $\mathbf{v}$ *is* $\{R : R = A + \lambda\mathbf{u} + \mu\mathbf{v} \quad (\lambda, \mu \in \mathbb{R})\}$.

Let $B \neq A$ be a point such that $\overrightarrow{AB} \nparallel \mathbf{u}$.

3. The unique plane that contains A, B, and $\mathbf{u}$ *is* $\left\{R : R = A + \lambda\mathbf{u} + \mu\overrightarrow{AB} \quad (\lambda, \mu \in \mathbb{R})\right\}$.

Let C be a point that isn't collinear with A and B.

4. The unique plane that contains A, B, and C is $\left\{R : R = A + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC} \quad (\lambda, \mu \in \mathbb{R})\right\}$.

Proof. **(a)** See Fact 279 (Appendices).

(b) Apply Corollary 33.

(c) Since $\overrightarrow{AB} \nparallel \mathbf{u}$, we can again apply Corollary 33.

(d) Since A, B, and C aren't collinear, $\overrightarrow{AB} \nparallel \overrightarrow{AC}$. And now, we can again apply Corollary 33. □

We'll give two examples of each of the Four Ways. These examples will also serve as a summary of what we've learnt so far about planes.

First, two examples where we're given **a point** and **a normal vector** of the plane:

[355] This is an assertion that we formally prove only in Ch. **??** (Appendices).

Example 955. A plane contains the point $(1,2,3)$ and has normal vector $(1,1,0)$.

Compute $(1,2,3)\cdot(1,1,0) = 1+2+0 = 3$. So, this plane may be described in vector or cartesian form by

$$\mathbf{r}\cdot(1,1,0) = 3 \qquad \text{or} \qquad x+y=3.$$

This plane also contains the non-parallel vectors $(1,-1,0)$ and $(0,0,1)$. So, it may be described in parametric form by

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ 2-\lambda \\ 3+\mu \end{pmatrix}, \qquad (\lambda,\mu\in\mathbb{R}).$$

Example 956. A plane contains the point $(0,0,1)$ and has normal vector $(2,-1,1)$.

Compute $(0,0,1)\cdot(2,-1,1) = 0+0+1 = 1$. So, this plane may be described in vector or cartesian form by

$$\mathbf{r}\cdot(2,-1,1) = 1 \qquad \text{or} \qquad 2x-y+z=1.$$

This plane also contains the non-parallel vectors $(1,2,0)$ and $(0,1,1)$. So, it may be described in parametric form by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda+\mu \\ 1+\mu \end{pmatrix}, \qquad (\lambda,\mu\in\mathbb{R}).$$

Second, two examples where we're given **one point** and **two vectors** (that aren't parallel):

Example 957. A plane contains the point $(1,2,3)$ and the non-parallel vectors $(5,4,3)$ and $(1,-1,2)$. So, it may be described in parametric form by

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+5\lambda+\mu \\ 2+4\lambda-\mu \\ 3+3\lambda+2\mu \end{pmatrix}, \qquad (\lambda,\mu\in\mathbb{R}).$$

This plane has normal vector $(5,4,3)\times(1,-1,2) = (11,-7,-9)$.

Compute $(1,2,3)\cdot(11,-7,-9) = 11-14-27 = -30$. So, this plane may be described in vector or cartesian form by

$$\mathbf{r}\cdot(11,-7,-9) = -30 \qquad \text{or} \qquad 11x-7y-9z=-30.$$

Example 958. A plane contains the point $(5,0,1)$ and the non-parallel vectors $(1,1,8)$ and $(1,0,1)$. So, it may be described in parametric form by

$$\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5+\lambda+\mu \\ \lambda \\ 1+8\lambda+\mu \end{pmatrix}, \qquad (\lambda, \mu \in \mathbb{R}).$$

This plane has normal vector $(1,1,8) \times (1,0,1) = (1,7,-1)$.

Compute $(5,0,1) \cdot (1,7,-1) = 5+0-1 = 4$. So, this plane may be described in vector or cartesian form by

$$\mathbf{r} \cdot (1,7,-1) = 4 \qquad \text{or} \qquad x + 7y - z = 4.$$

Third, two examples where we're given **two points** and **a vector** (that isn't parallel to the vector between the two points):

Example 959. A plane contains the points $(0,0,3)$ and $(1,4,5)$, and the vector $(3,2,1)$.

The vector between the two points is $(1,4,5) - (0,0,3) = (1,4,2)$ and isn't parallel to the vector $(3,2,1)$. So, this plane may be described in parametric form by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\lambda+\mu \\ 2\lambda+4\mu \\ 3+\lambda+2\mu \end{pmatrix}, \qquad (\lambda, \mu \in \mathbb{R}).$$

This plane has normal vectors $(3,2,1) \times (1,4,2) = (0,-5,10)$ and hence also $(0,-1,2)$.

Compute $(0,0,3) \cdot (0,-1,2) = 0+0+6 = 6$. So, this plane may be described in vector or cartesian form by

$$\mathbf{r} \cdot (0,-1,2) = 6 \qquad \text{or} \qquad -y + 2z = 6.$$

Example 960. A plane contains the points $(8,-2,0)$ and $(3,6,9)$, and the vector $(0,1,1)$.

The vector between the two points is $(3,6,9) - (8,-2,0) = (-5,8,9)$ and isn't parallel to the vector $(0,1,1)$. So, this plane may be described in parametric form by

$$\mathbf{r} = \begin{pmatrix} 8 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 8-5\mu \\ -2+\lambda+8\mu \\ \lambda+9\mu \end{pmatrix}, \qquad (\lambda, \mu \in \mathbb{R}).$$

This plane has normal vector $(0,1,1) \times (-5,8,9) = (1,-5,5)$.

Compute $(8,-2,0) \cdot (1,-5,5) = 8+10+0 = 18$. So, this plane may be described in vector and cartesian forms by $\mathbf{r} \cdot (1,-5,5) = 18$ and $x - 5y + 5z = 18$.

Fourth and last, two examples where we're given **three points** (that aren't collinear):

Example 961. A plane contains the points $(1,2,3)$, $(4,5,8)$, and $(2,3,5)$.

The vector between the first two points is $(4,5,8)-(1,2,3)=(3,3,5)$, while that between the first and last is $(2,3,5)-(1,2,3)=(1,1,2)$. Since $(3,3,5)\nparallel(1,1,2)$, this plane may be described in parametric form by

$$\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\3\\5\end{pmatrix}+\mu\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}1+3\lambda+\mu\\2+3\lambda+\mu\\3+5\lambda+2\mu\end{pmatrix},\qquad(\lambda,\mu\in\mathbb{R}).$$

This plane has normal vector $(3,3,5)\times(1,1,2)=(1,-1,0)$.

Compute $(1,2,3)\cdot(1,-1,0)=1-2+0=-1$. So, this plane may be described in vector and cartesian forms by $\mathbf{r}\cdot(1,-1,0)=-1$ and $x-y=-1$.

Example 962. A plane contains the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

The vector between the first two points is $(0,1,0)-(1,0,0)=(-1,1,0)$, while that between the first and last is $(0,0,1)-(1,0,0)=(-1,0,1)$. Since $(-1,1,0)\nparallel(-1,0,1)$, this plane may be described in parametric form by

$$\mathbf{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}1-\lambda-\mu\\\lambda\\\mu\end{pmatrix},\qquad(\lambda,\mu\in\mathbb{R}).$$

This plane has normal vector $(-1,1,0)\times(-1,0,1)=(1,1,1)$.

Compute $(1,0,0)\cdot(1,1,1)=1+0+0=1$. So, this plane may be described in vector and cartesian forms by $\mathbf{r}\cdot(1,1,1)=1$ and $x+y+z=1$.

As the above examples suggest, we can easily "go "from any one of to any other of the Four Ways.

That is, starting from **(a)** a point and a normal vector (hence uniquely determining some plane q), we can easily find

(b) two other non-parallel vectors on q;

(c) another point and another vector on q (where the vector is not parallel to the vector connecting the two known points); and

(d) another two points on q (where the three points are not collinear). (How?)[356]

Similarly, starting from **(b)**, we can easily "go to" **(a)**, **(c)**, or **(d)**. (How?)[357]

[356] Say we're given **(a)** a point $A=(3,2,1)$ and a normal vector $\mathbf{n}=(1,2,3)$ of a plane q. Then **(b)** two non-parallel vectors on q are $\mathbf{u}=(0,3,-2)$ and $\mathbf{v}=(3,0,-1)$; and **(c)**+**(d)** another two points on q are $B=A+\mathbf{u}$ and $C=A+\mathbf{v}$, where A, B, and C are not collinear.

[357] Say we're given **(b)** a point $A=(3,2,1)$ and two non-parallel vectors $\mathbf{u}=(0,3,-2)$ and $\mathbf{v}=(3,0,-1)$ on q. Then **(a)** a normal vector of q is $\mathbf{u}\times\mathbf{v}$; and **(c)**+**(d)** another two points on q are $B=A+\mathbf{u}$ and $C=A+\mathbf{v}$, where A, B, and C are not collinear.

Similarly, starting from **(c)**, we can easily "go to" **(a)**, **(b)**, or **(d)**. (How?)[358]

Similarly, starting from **(d)**, we can easily "go to" **(a)**, **(b)**, or **(c)**. (How?)[359]

Exercise 286. In each of **(a)**–**(c)**, three points are given. Describe the plane that contains all three points in vector, cartesian, **and** parametric form.

(a) $(7,3,4)$, $(8,3,4)$, and $(9,3,7)$.

(b) $(8,0,2)$, $(4,4,3)$, and $(2,7,2)$.

(c) $(8,5,9)$, $(8,4,5)$, and $(5,6,0)$.

(Answer on p. 1873.)

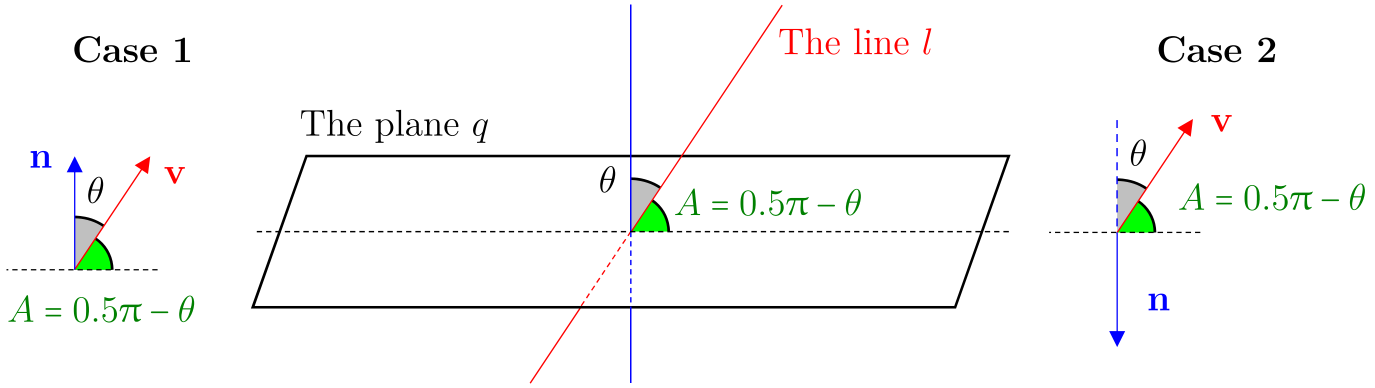
[358] Say we're given **(c)** two points $A = (3,2,1)$ and $B = (4,4,4)$ and a vector $\mathbf{v} = (3,0,-1)$ on q, where $\mathbf{v} \nparallel \overrightarrow{AB}$. Then **(a)** $\overrightarrow{AB} \times \mathbf{v}$ is a normal vector of q; **(b)** $\overrightarrow{AB}$ is another vector on q that isn't parallel to $\mathbf{v}$; and **(d)** $C = A + \mathbf{v}$ is another point on q such that A,B, and C are not collinear.

[359] Say we're given **(d)** three non-collinear points $A = (3,2,1)$, $B = (4,4,4)$, and $C = (6,2,0)$ on q. Then **(a)** $\overrightarrow{AB} \times \overrightarrow{AC}$ is a normal vector of q; **and** **(b)**+**(c)** $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two non-parallel vectors on q.

75. The Angle between a Line and a Plane

Suppose the line l has direction vector $\mathbf{v}$ and the plane q has normal vector $\mathbf{n}$.

Let θ be the **non-obtuse angle** between $\mathbf{v}$ and $\mathbf{n}$.



We want A in the above figure to be the angle between the line l and the plane q.

There are two possible cases: The angle between $\mathbf{v}$ and $\mathbf{n}$ is **(1)** θ; or **(2)** $\pi - \theta$.

But in either case, A is the complement of θ. That is, $A = \dfrac{\pi}{2} - \theta$.

By Fact 138,
$$\theta = \cos^{-1} \frac{|\mathbf{v} \cdot \mathbf{n}|}{|\mathbf{v}|\,|\mathbf{n}|}.$$

By Fact 101,
$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

Altogether,
$$A = \frac{\pi}{2} - \cos^{-1} \frac{|\mathbf{v} \cdot \mathbf{n}|}{|\mathbf{v}|\,|\mathbf{n}|} = \sin^{-1} \frac{|\mathbf{v} \cdot \mathbf{n}|}{|\mathbf{v}|\,|\mathbf{n}|}.$$

The above discussion motivates this definition:

Definition 178. The *angle between a line* with direction vector $\mathbf{v}$ and *a plane* with normal vector $\mathbf{n}$ is this number:

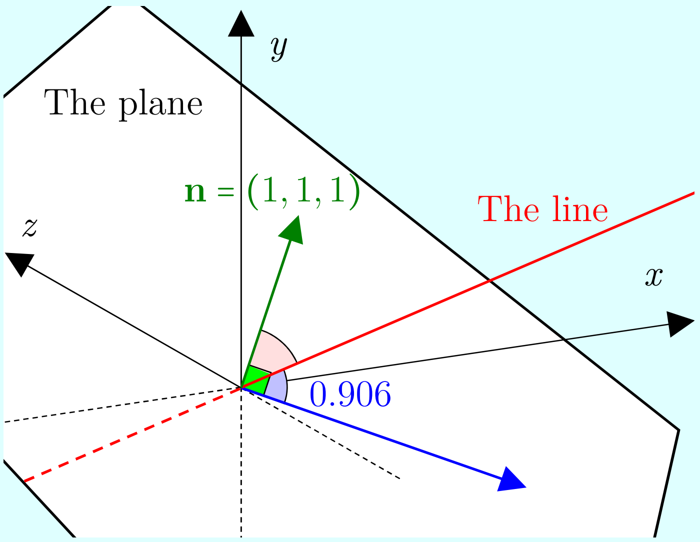
$$\sin^{-1} \frac{|\mathbf{v} \cdot \mathbf{n}|}{|\mathbf{v}|\,|\mathbf{n}|}.$$

Example 963. The angle between a line with direction vector $(9, 1, 3)$ and a plane with normal vector $(1, 1, 1)$ is

$$\sin^{-1} \frac{|(9,1,3) \cdot (1,1,1)|}{|(9,1,3)|\,|(1,1,1)|}$$

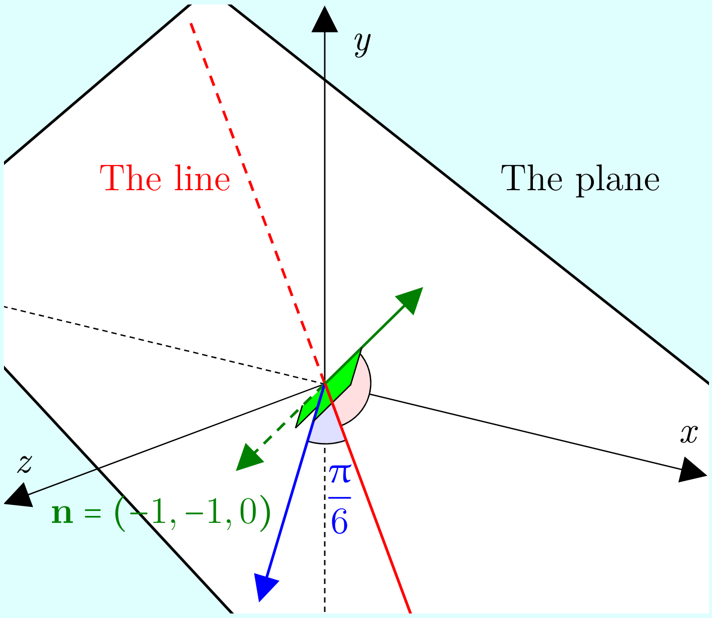
$$= \sin^{-1} \frac{|9+1+3|}{\sqrt{9^2+1^2+3^2}\sqrt{1^2+1^2+1^2}}$$

$$= \sin^{-1} \frac{|13|}{\sqrt{91}\sqrt{3}} \approx 0.906.$$



Example 964. The angle between a line with direction vector $(1,0,1)$ and a plane with normal vector $(-1,-1,0)$ is

$$\sin^{-1}\frac{|(1,0,1)\cdot(-1,-1,0)|}{|(1,0,1)||(-1,-1,0)|}$$
$$=\sin^{-1}\frac{|-1+0+0|}{\sqrt{1^2+0^2+1^2}\sqrt{(-1)^2+(-1)^2+0^2}}$$
$$=\sin^{-1}\frac{|-1|}{\sqrt{2}\sqrt{2}}=\sin^{-1}\frac{1}{2}\approx\frac{\pi}{6}.$$



Example 965. The angle between a line with direction vector $(1,0,1)$ and a plane with normal vector $(0,1,0)$ is

$$\sin^{-1}\frac{|(1,0,1)\cdot(0,1,0)|}{|(1,0,1)||(0,1,0)|}=\sin^{-1}\frac{|0+0+0|}{\sqrt{1^2+0^2+1^2}\sqrt{0^2+1^2+0^2}}=\sin^{-1}\frac{|0|}{\sqrt{2}\sqrt{1}}=\sin^{-1}0=0.$$

Exercise 287. Find the angle between the given line and plane. (Answer on p. 1874.)

(a) Line: $\mathbf{r}=(-1,2,3)+\lambda(-1,1,0)\quad(\lambda\in\mathbb{R})$.
Plane: $\mathbf{r}\cdot(3,4,5)=0$.

(b) Line: Contains the points $(-1,2,3)$ and $(-1,4,9)$.
Plane: $\mathbf{r}=(2,0,0)+\lambda(-3,1,0)+\mu(0,5,-3)\ (\lambda\in\mathbb{R})$.

(c) Line: Contains the points $(-1,2,3)$ and $(0,11,11)$.
Plane: Contains the points $(1.5,0,0)$ and $(0,0,1.5)$ and the vector $(4,-1,0)$.

75.1. When a Line and a Plane Are Parallel, Perp., or Intersect

Definition 179. Let θ be the angle between a line and a plane. The line and plane are said to be **(a)** *parallel* if $\theta = 0$; and **(b)** *perpendicular* if $\theta = \pi/2$.

As usual, if a line l and plane q are parallel, write $l \parallel q$. If not, write $l \nparallel q$.

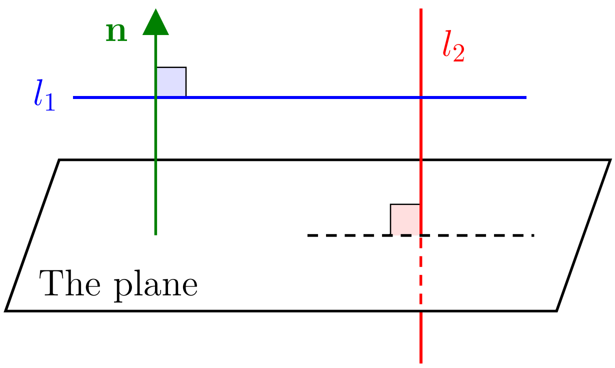
And if they're perpendicular, write write $l \perp q$.
If not, write $l \not\perp q$.

The plane q has normal vector $\mathbf{n}$, while the lines l_1 and l_2 have direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

"Obviously", $l_1 \perp q \iff \mathbf{v}_1 \parallel \mathbf{n}$.

And $l_2 \parallel q \iff \mathbf{v}_2 \perp \mathbf{n}$.

Let's state and prove these obviosities formally:



Fact 170. *Suppose the line l has direction vector* $\mathbf{v}$ *and the plane q has normal vector* $\mathbf{n}$*. Then* **(a)** $l \parallel q \iff \mathbf{v} \perp \mathbf{n}$*; and* **(b)** $l \perp q \iff \mathbf{v} \parallel \mathbf{n}$.

Proof. Let θ be the angle between l and q. By Definitions 179, 178, and 146, and Fact 135:

(a) $$l \parallel q \iff \theta = 0 \iff \sin^{-1}\frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{v}|\,|\mathbf{n}|} = 0 \iff \mathbf{v}\cdot\mathbf{n} = 0 \iff \mathbf{v} \perp \mathbf{n}.$$

(b) $$l \perp q \iff \theta = \frac{\pi}{2} \iff \sin^{-1}\frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{v}|\,|\mathbf{n}|} = \frac{\pi}{2} \iff \frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{v}|\,|\mathbf{n}|} = 1 \iff \mathbf{v} \parallel \mathbf{n}.$$ □

The next result is intuitively "obvious":

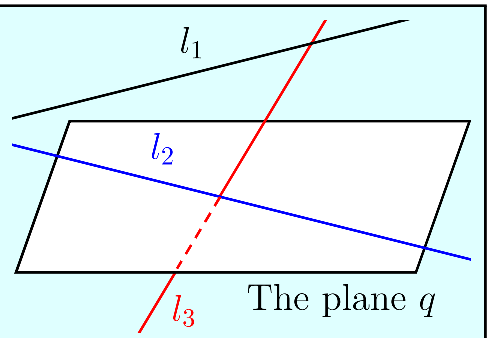
Fact 171. *Given a line and a plane, exactly one of the three following possibilities holds: The line and plane are*

(a) *Parallel and do not intersect at all; or*
(b) *Parallel and the line lies entirely on the plane; or*
(c) *Non-parallel and intersect at exactly one point.*

Proof. See p. 1638 (Appendices). □

Example 966. The three lines l_1, l_2, and l_3, and plane q illustrate all three possibilities given in Fact 171:

1. $l_1 \parallel q$ and does not intersect q.
2. $l_2 \parallel q$ and lies entirely on q.
3. $l_3 \nparallel q$ and intersects q at exactly one point.



By Fact 171, the line l and plane q intersect at exactly one point $\iff l \nparallel q$.

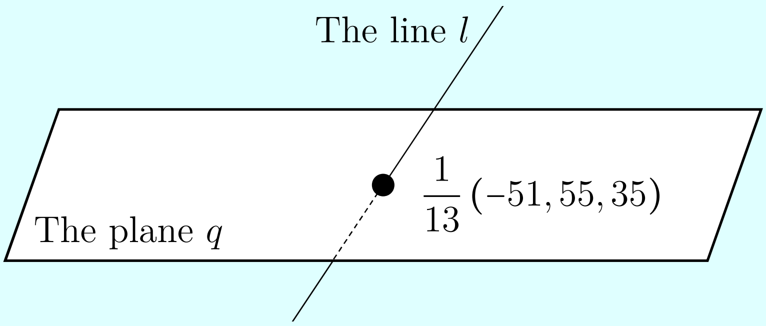
Example 967. The line l and plane q are described by

$$\mathbf{r} = (3,5,5) + \lambda\overbrace{(9,1,3)}^{\mathbf{v}} \ (\lambda \in \mathbb{R}) \qquad \text{and} \qquad \mathbf{r} \cdot \overbrace{(1,1,1)}^{\mathbf{n}} = 3.$$

Observe that $\mathbf{v} \cdot \mathbf{n} = (9,1,3) \cdot (1,1,1) \neq 0$, so that $\mathbf{v} \not\perp \mathbf{n}$ and hence, $l \nparallel q$.

Thus, l and q share exactly one intersection point.

To find this intersection point, simply plug in a generic point of l into the equation for q:



$$\left[(3,5,5) + \hat{\lambda}(9,1,3)\right] \cdot (1,1,1) = 3 \qquad \iff \qquad 13 + 13\hat{\lambda} = 3 \qquad \iff \qquad \hat{\lambda} = -\frac{10}{13}.$$

So, l and q intersect at $(3,5,5) + \hat{\lambda}(9,1,3) = (3,5,5) - \frac{10}{13}(9,1,3) = \frac{1}{13}(-51,55,35)$.

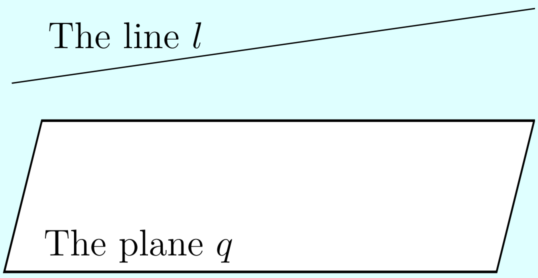
Example 968. The line l and the plane q are described by

$$\mathbf{r} = (3,5,5) + \lambda\overbrace{(9,1,3)}^{\mathbf{v}} \ (\lambda \in \mathbb{R}) \qquad \text{and} \qquad \mathbf{r} \cdot \overbrace{(1,0,-3)}^{\mathbf{n}} = -6.$$

Observe that $\mathbf{v} \cdot \mathbf{n} = (9,1,3) \cdot (1,0,-3) = 0$, so that $\mathbf{v} \perp \mathbf{n}$ and hence $l \parallel q$. Thus, either

l lies entirely on q; or

l and q don't intersect at all.



It's easy to figure out which of these two possibilities holds: We already know that l contains the point $(3,5,5)$; so, simply check if this point is also on the plane q:

$$(3,5,5) \cdot (1,0,-3) = 3 + 0 - 15 = -12 \neq -6.$$

Since this point does not satisfy q's vector equation, it is not on q. So, it cannot be that l lies entirely on q. Hence, it must be that l and q do not intersect at all.

Example 969. The line l and the plane q are described by

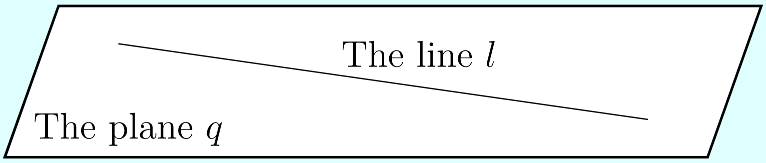
$$\mathbf{r} = (3,5,3) + \lambda\overbrace{(9,1,3)}^{\mathbf{v}} \ (\lambda \in \mathbb{R}) \qquad \text{and} \qquad \mathbf{r} \cdot \overbrace{(1,0,-3)}^{\mathbf{n}} = -6.$$

Again, $l \parallel q$. So again, two possibilities: Either l lies entirely on q or they don't intersect at all.

To find out which, same trick as before: Check if the point $(3,5,3)$ on l is also on q:

$$(3,5,3) \cdot (1,0,-3) = 3 + 0 - 9 = -6 \qquad \checkmark$$

Yup, this time it is. Since l and q are parallel and share an intersection point, it must be that l lies entirely on q.



Exercise 288. In each of **(a)**–**(c)**, a line l and a plane q are given. For each, determine which of the three possibilities given in Fact 171 holds. And if the line and plane intersect, find their intersection point. (Answer on p. 1874.)

(a) l: $\mathbf{r} = (4,5,6) + \lambda(2,3,5) \quad (\lambda \in \mathbb{R})$.
q: $\mathbf{r} \cdot (-10,0,4) = -26$.

(b) l: Contains the points $(5,5,6)$ and $(3,2,1)$.
q: $\mathbf{r} = (3,0,1) + \lambda(2,0,5) + \mu(2,1,5) \ (\lambda \in \mathbb{R})$.

(c) l: Contains the points $(4,5,6)$ and $(6,8,11)$.
q: Contains the points $(2,0,-2)$ and $(2,1,-2)$ and the vector $(3,0,10)$.

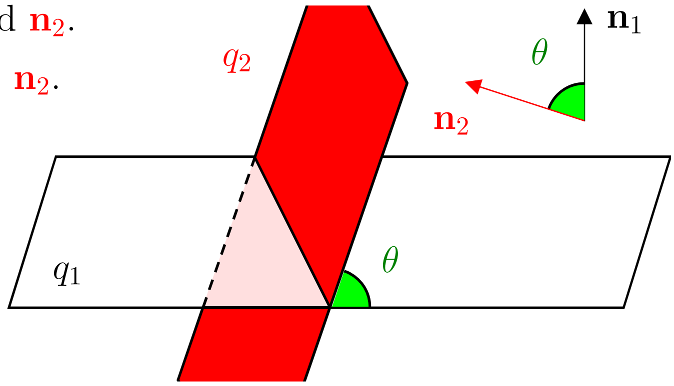
76. The Angle between Two Planes

The planes q_1 and q_2 have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$.

Let θ be the (non-obtuse) angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

Then θ is also the angle between q_1 and q_2.

The above discussion motivates this definition:



Definition 180. The *angle between two planes* is the non-obtuse angle between their normal vectors.

Now apply Fact 138:

Fact 172. *The angle between two planes with normal vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *is*

$$\cos^{-1}\frac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Example 970. The angle between the planes $\mathbf{r}\cdot(2,1,3)=26$ and $\mathbf{r}\cdot(-3,0,5)=-25$ is

$$\theta=\cos^{-1}\frac{|(2,1,3)\cdot(-3,0,5)|}{|(2,1,3)|\,|(-3,0,5)|}=\cos^{-1}\frac{|9|}{\sqrt{14}\sqrt{34}}\approx 1.146.$$

Example 971. The angle between the planes $\mathbf{r}\cdot(1,1,1)=12$ and $\mathbf{r}\cdot(-1,-1,0)=-1$ is

$$\theta=\cos^{-1}\frac{|(1,1,1)\cdot(-1,-1,0)|}{|(1,1,1)|\,|(-1,-1,0)|}=\cos^{-1}\frac{|-2|}{\sqrt{3}\sqrt{2}}=\cos^{-1}\sqrt{\frac{2}{3}}\approx 0.615$$

Exercise 289. Find the angle between the given planes. (Answers on p. 1875.)

(a) $\mathbf{r}\cdot(-1,-2,-3)=1$ and $\mathbf{r}\cdot(3,4,5)=2$.

(b) One plane contains the vectors $(1,-1,0)$ and $(3,5,-1)$. The other contains the vectors $(0,1,0)$ and $(10,2,3)$.

(c) One plane contains the points $(1,1,0)$, $(3,0,0)$, and $(0,0,1)$. The other contains the points $(1,-1,0)$, $(1,0,-1)$, and $(0,3,1)$.

76.1. When Two Planes Are Parallel, Perp., or Intersect

Definition 181. If θ is the angle between two planes, then the two planes are

(a) *Parallel* if $\theta = 0$; and **(b)** *Perpendicular* if $\theta = \pi/2$.

As usual, if two planes q and r, write $q \parallel r$. If not, write $q \nparallel r$.

And if they're perpendicular, write write $q \perp r$. If not, write $q \not\perp r$.

The next result is "obvious":

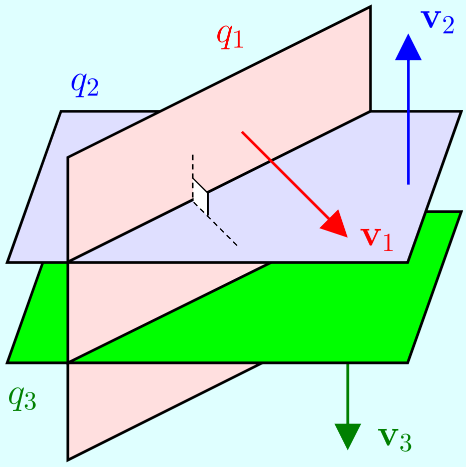
Fact 173. *Suppose q and r are planes with normal vectors $\mathbf{u}$ and $\mathbf{v}$. Then*

(a) $q \parallel r \iff \mathbf{u} \parallel \mathbf{v}$; *and* **(b)** $q \perp r \iff \mathbf{u} \perp \mathbf{v}$.

Proof. See p. 1639 (Appendices). □

Example 972. The planes q_1, q_2, and q_3 have normal vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. We have

- $\mathbf{v}_1 \perp \mathbf{v}_2$ and $q_1 \perp q_2$.
- $\mathbf{v}_1 \perp \mathbf{v}_3$ and $q_1 \perp q_3$.
- $\mathbf{v}_2 \parallel \mathbf{v}_3$ and $q_2 \parallel q_3$.



Another intuitively "obvious" result:

Fact 174. *If two planes are parallel, then they are either identical or do not intersect.*

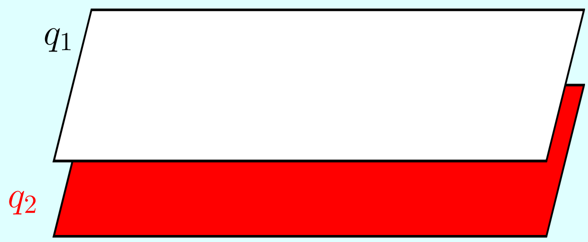
Proof. See p. 1639 (Appendices). □

Example 973. The planes q_1 and q_2 are described by $\mathbf{r}\cdot(3,-3,-1) = 1$ and $\mathbf{r}\cdot(-6,6,2) = 5$.

Since $(3,-3,-1) \parallel (-6,6,2)$, $q_1 \parallel q_2$. 🍔

Now pick any point on q_1—for example, $(0,0,-1)$. Check if this point is on q_2:

$$(0,0,-1)\cdot(-6,6,2) = 0 + 0 - 2 = -2 \neq 5.$$



It isn't. Since the two planes are not identical, by Fact 174, they do not intersect at all.

Example 974. The planes q_1 and q_2 are described by $\mathbf{r}\cdot(1,0,2) = -3$ and $\mathbf{r}\cdot(-2,0,-4) = 6$.

Since $(1,0,2) \parallel (-2,0,-4)$, $q_1 \parallel q_2$.

Now pick any point on q_1—for example, $(-1,0,-1)$. Check if this point is on q_2:

$$(-1,0,-1)\cdot(-2,0,-4) = 2 + 0 + 4 = 6.$$

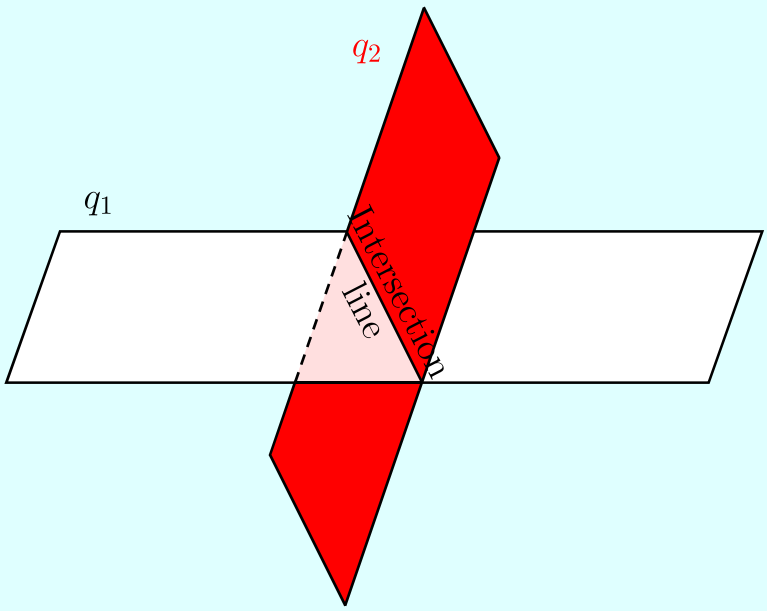
Yup, it is. Since the two planes share at least one intersection point, by Fact 174, they are identical.

Here's another "obvious" result:

Fact 175. *If two planes are not parallel, then they must intersect.*

Proof. See p. 1640 (Appendices). □

Example 975. Since $q_1 \nparallel q_2$, by Fact 175, they must intersect.



In fact, and as should be intuitively "obvious", q_1 and q_2 must intersect along a line.

As the above example suggests, **two non-parallel planes must intersect along a line**.

Now, what do we know about this intersection line? Well, "obviously", its direction vector is parallel to both planes and is thus also perpendicular to both planes' normal vectors.

Let $\mathbf{n}$ and $\mathbf{m}$ be two planes' normal vectors. By Fact 157, the only vectors perpendicular to both $\mathbf{n}$ and $\mathbf{m}$ are those parallel to $\mathbf{n}\times\mathbf{m}$.

Hence, their intersection line must have direction vector $\mathbf{n}\times\mathbf{m}$. Altogether,

Fact 176. *Suppose two non-parallel planes have normal vectors* $\mathbf{n}$ *and* $\mathbf{m}$*. Then their intersection is a line with direction vector* $\mathbf{n}\times\mathbf{m}$*.*

Proof. See p. 1640 (Appendices). □

Example 976. The planes q_1 and q_2 are described by

$$\mathbf{r}\cdot\mathbf{n}_1 = \mathbf{r}\cdot(-1,2,-3) = 4 \qquad \text{and} \qquad \mathbf{r}\cdot\mathbf{n}_2 = \mathbf{r}\cdot(5,-6,7) = 0.$$

Since $\mathbf{n}_1 \nparallel \mathbf{n}_2$, by Fact 176, q_1 and q_2 must intersect along a line with direction vector $\mathbf{n}_1 \times \mathbf{n}_2 = (-1,2,-3)\times(5,-6,7) = (-4,-8,-4)$ or $(1,2,1)$.

Recall that a direction vector and a point fully describe a line. We already have a direction vector. So, let's find some point P that is on the intersection line.

To do so, first write out the two planes' cartesian equations:

$$-x+2y-3z = 4 \qquad \text{and} \qquad 5x-6y+7z = 0.$$

The solutions to the above system of (two) equations gives us the two planes' intersection points. Note that with three variables and two equations, this system of equations has infinitely many solutions and hence **infinitely many intersection points**. And of course, the set of all these intersection points *is* the intersection line.

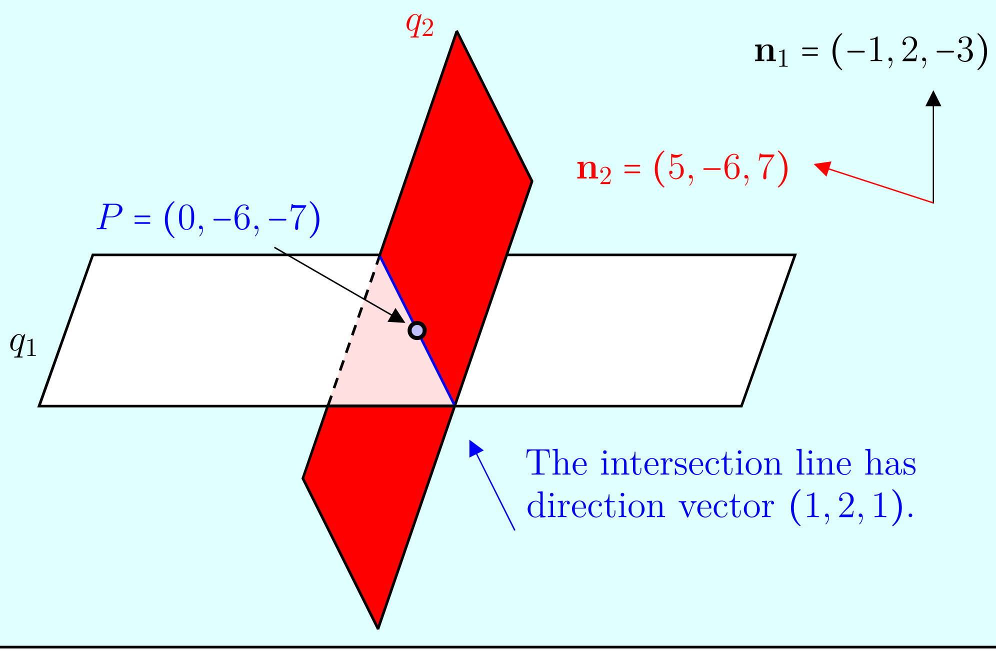
To find any one such intersection point, zero is as usual our friend. Let's look for an intersection point whose x-coordinate is zero. In other words, let's simply plug $x=0$ into the above equations to get

$$2y-3z \stackrel{1}{=} 4 \qquad \text{and} \qquad -6y+7z \stackrel{2}{=} 0.$$

And now, we can easily solve this system of (two) equations (with two variables): $\stackrel{2}{=}$ plus $3\times \stackrel{1}{=}$ yields $-2z = 12$ or $z=-6$ and hence, $y=-7$.

So, an intersection point shared by q_1 and q_2 is $P=(0,-6,-7)$. Hence, their intersection line is

$$\mathbf{r} = (0,-6,-7)+\lambda(1,2,1) \qquad (\lambda\in\mathbb{R}).$$



Together, Facts 174 and 176 say the following:

Corollary 34. *Given two planes, exactly one of these three possibilities holds: They are*

(a) *Identical and thus also parallel;*

(b) *Parallel and do not intersect at all; or*

(c) *Non-parallel and intersect along a line.*

Proof. If two distinct planes are parallel, then by Fact 174, they do not intersect at all. And if they aren't parallel, then by Fact 176, they intersect along a line. □

The above result implies that two distinct planes intersect if and only if they are not parallel.

Example 977. The planes q_1 and q_2 are described by

$$\mathbf{r} \cdot \mathbf{n}_1 = \mathbf{r} \cdot (-3, 7, 1) = 2 \qquad \text{and} \qquad \mathbf{r} \cdot \mathbf{n}_2 = \mathbf{r} \cdot (1, 2, 1) = 0.$$

Since $\mathbf{n}_1 \nparallel \mathbf{n}_2$, by Fact 176, they must intersect along a line with direction vector $(-3, 7, 1) \times (1, 2, 1) = (5, 4, -13)$.

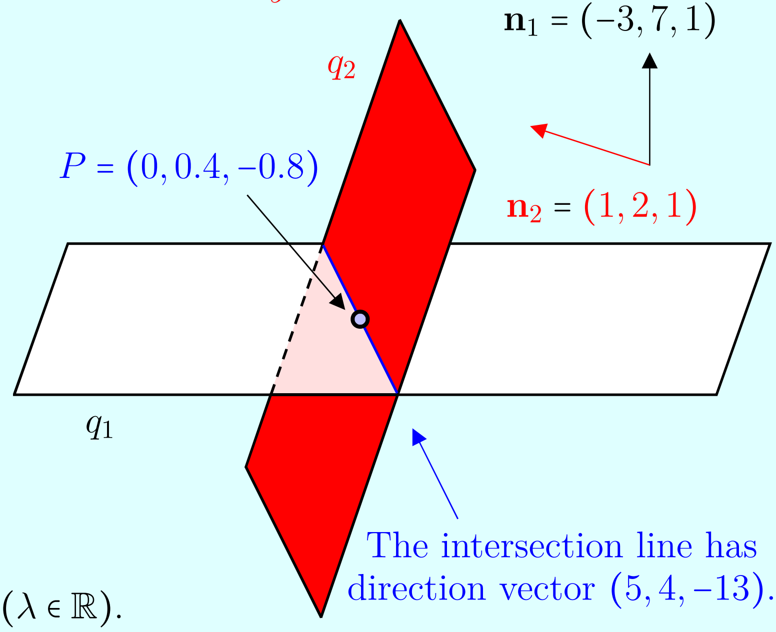
To find an intersection point, write out the two planes' cartesian equations:

$$-3x + 7y + z = 2 \qquad \text{and} \qquad x + 2y + z = 0.$$

Again, there'll be infinitely many intersection points. To find one, we again plug $x = 0$ into the above equations to get

$$7y + z \overset{1}{=} 2 \qquad \text{and} \qquad 2y + z \overset{2}{=} 0.$$

Solving, $y = 0.4$ and $z = -0.8$. So, an intersection point shared by q_1 and q_2 is $P = (0, 0.4, -0.8)$. Hence, their intersection line is



$$\mathbf{r} = (0, 0.4, -0.8) + \lambda (5, 4, -13) \qquad (\lambda \in \mathbb{R}).$$

Example 978. The planes q_1 and q_2 are described by

$$\mathbf{r} \cdot \mathbf{n}_1 = \mathbf{r} \cdot (0,4,5) = 0 \qquad \text{and} \qquad \mathbf{r} \cdot \mathbf{n}_2 = \mathbf{r} \cdot (3,4,5) = 1.$$

Since $\mathbf{n}_1 \nparallel \mathbf{n}_2$, by Fact 176, they must intersect along a line with direction vector $(0,4,5) \times (3,4,5) = (0,15,-12)$ or $(0,5,-4)$.

To find an intersection point, write out the two planes' cartesian equations:

$$4y + 5z = 0 \qquad \text{and} \qquad 3x + 4y + 5z = 1.$$

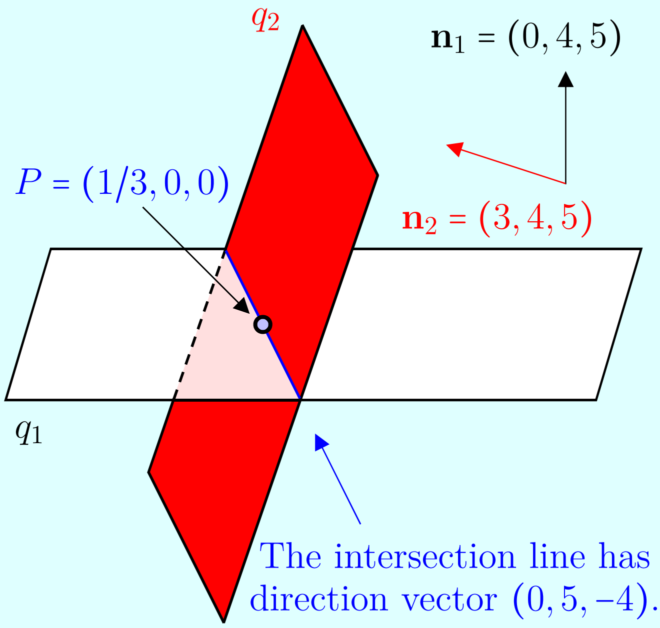
Again, there'll be infinitely many intersection points. It turns out that this time, our "plug in $x = 0$ trick" won't work so nicely. Let's try it anyway and see what happens:

$$4y + 5z \stackrel{1}{=} 0 \qquad \text{and} \qquad 4y + 5z \stackrel{2}{=} 1,$$

which are clearly contradictory. What this contradiction means is that the two planes do **not** share any intersection point whose x-coordinate is 0.

No big deal. Instead of plugging in $x = 0$, let's try $y = 0$ instead:

$$5z \stackrel{3}{=} 0 \qquad \text{and} \qquad 3x + 5z \stackrel{4}{=} 1.$$



Solving, we have $z = 0$ and $x = 1/3$. So, an intersection point shared by q_1 and q_2 is $P = (1/3,0,0)$. Hence, their intersection line is

$$\mathbf{r} = (1/3,0,0) + \lambda (0,5,-4) \qquad (\lambda \in \mathbb{R}).$$

Exercise 290. In each of **(a)**–**(g)**, a pair of planes q_1 and q_2 is given. For each pair, determine which of the three possibilities in Corollary 34 holds. If the two planes intersect, find the set of intersection points. (Answer on p. 1875.)

(a) $\mathbf{r} \cdot (4,9,3) = 61$ and $\mathbf{r} \cdot (1,1,2) = 19$.

(b) q_1 contains the point $(1,3,-2)$ and the vectors $(1,-1,0)$ and $(1,-1,1)$; while q_2 contains the point $(2,3,5)$ and the vectors $(6,-1,0)$ and $(8,0,-1)$.

(c) q_1 contains the points $(1,1,6)$, $(7,7,0)$, and $(5,3,3)$; while q_2 contains the points $(7,3,1)$, $(5,5,1)$, and $(3,5,2)$.

(d) q_1 contains the points $(5,3,2)$, $(1,5,3)$, and $(10,0,1)$; while q_2 contains the points $(5,-1,4)$, $(8,8,-2)$, and $(3,5,2)$.

(e) $\mathbf{r} \cdot (7,1,1) = 42$ and $\mathbf{r} \cdot (1,1,2) = 6$.

(f) $\mathbf{r} \cdot (0,1,3) = 0$ and $\mathbf{r} \cdot (-1,1,3) = 2$.

77. The Distance between a Point and a Plane

Earlier, from Ch. 15.1, we had these result and definition:

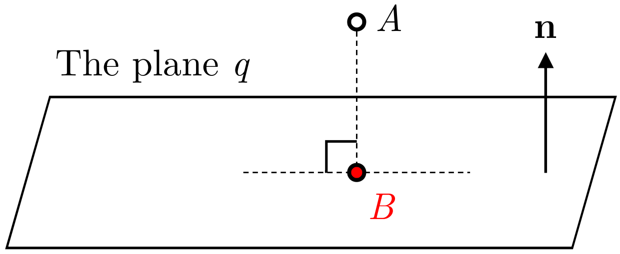
Corollary 4. *Suppose A is a point not on the line l. Then there exists a point B that is both* **(a)** *the unique point on l that's closest to A; and* **(b)** *the unique point on l such that $l \perp AB$.*

Definition 182. Let A be a point that isn't on the line l. The *foot of the perpendicular from A to l* is the (unique) point B on l such that $AB \perp l$.

In this chapter, we have, analogously, these result and definition:

Fact 177. *Suppose A is a point not on the plane q. Then there exists a point B that is both* **(a)** *the unique point on Q that's closest to A; and* **(b)** *the unique point on q such that $q \perp AB$.*

Proof. See p. 764 (Appendices). □



Fact 177. *Suppose A is a point not on the plane q. Then there exists a point B that is both* **(a)** *the unique point on Q that's closest to A; and* **(b)** *the unique point on q such that $q \perp AB$.*

Proof. Let q be the plane $\mathbf{r} \cdot \mathbf{n} = d$. Let l be the line $R = A + \lambda \mathbf{n}$ ($\lambda \in \mathbb{R}$).

Since $l \nparallel q$, by Fact 171, l intersects q at exactly one point—call it B. So, $B = A + \lambda_b \mathbf{n}$ for some $\lambda_b \in \mathbb{R}$.

Since $\overrightarrow{AB} = \lambda_b \mathbf{n}$, we have $q \perp AB$ (and $\overrightarrow{AB}$ is perpendicular to every vector on q).

Let $C \neq B$ be any other point on q. Then $\overrightarrow{BC}$ is a vector on q. So, $\overrightarrow{AB} \perp \overrightarrow{BC}$ or $\overrightarrow{AB} \cdot \overrightarrow{BC} \overset{1}{=} 0$. Also, $\left|\overrightarrow{BC}\right|^2 \overset{2}{\neq} 0$

Now, $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB} + \overrightarrow{BC}\right| = \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{BC} \overset{1}{=} \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 \overset{2}{>} \left|\overrightarrow{AB}\right|^2$.

We've just shown that **(a)** B is closer to A than any other point on q.

Now, $\overrightarrow{AC} \cdot \overrightarrow{BC} = \left(\overrightarrow{AB} + \overrightarrow{BC}\right) \cdot \overrightarrow{BC} = \overrightarrow{AB} \cdot \overrightarrow{BC} + \left|\overrightarrow{BC}\right|^2 \overset{1}{=} \left|\overrightarrow{BC}\right|^2 \overset{2}{\neq} 0$. Hence, $\overrightarrow{AC} \cdot \overrightarrow{BC} \neq 0$ or $\overrightarrow{AC} \not\perp \overrightarrow{BC}$. Thus, $q \not\perp AC$.

We've just shown that **(b)** besides B, there is no point C on q such that $q \perp AC$. □

Definition 183. Let A be a point that isn't on the line q. The *foot of the perpendicular from A to q* is the (unique) point B on q such that $AB \perp q$.

In Chs. 62 and 69, we learnt to find

(a) The foot of the perpendicular from a point to a line; and

(b) The distance between a point and a line.

In this chapter, we'll analogously learn to find

(a) The foot of the perpendicular from a point to a plane; and

(b) The distance between a point and a plane.

We again reproduce from Ch. 15 this definition:

Definition 184. The *distance between a point A and a graph G* is the distance between A and B, where B is the point on G that's closest to A.

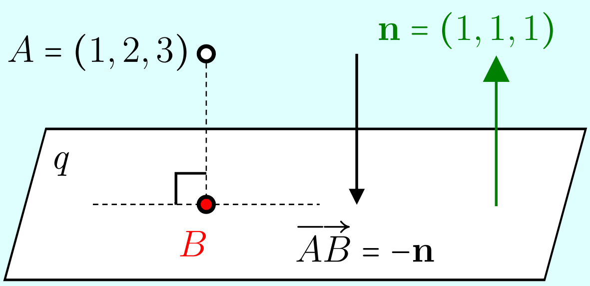
Let A be a point, q be a plane, and B be the foot of the perpendicular from A to q. By Fact 177, B is also the point on q that's closest to A. So, the distance between A and q is simply $\left|\overrightarrow{AB}\right|$.

Example 979. Let $A = (1,2,3)$ be a point, q be the plane $\mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot(1,1,1) = 3$, and B be the foot of the perpendicular from A to q.

Find B and $\left|\overrightarrow{AB}\right|$ (the distance between A and q).

Since B is the foot of the perpendicular, we have $AB \perp q$ or $\overrightarrow{AB} \parallel \mathbf{n}$ or for some $k \neq 0$, $k\mathbf{n} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

So, $\overrightarrow{OB} = \overrightarrow{OA} + k\mathbf{n} = (1,2,3) + k(1,1,1)$.



Since $B \in q$, $\overrightarrow{OB}\cdot(1,1,1) = 3$ or $[(1,2,3) + k(1,1,1)]\cdot(1,1,1) = 3$ or $6 + 3k = 3$.

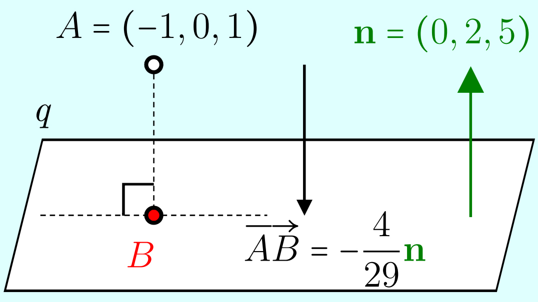
So, $k = -1$, $B = A + k\mathbf{n} = A - 1\mathbf{n} = (1,2,3) - 1(1,1,1) = (0,1,2)$, and the distance between A and q is

$$\left|\overrightarrow{AB}\right| = |k\mathbf{n}| = |k|\,|\mathbf{n}| = |-1|\,|(1,1,1)| = \sqrt{3}.$$

Example 980. Let $A = (-1,0,1)$ be a point, q be the plane described by $\mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot(0,2,5) = 1$, and B be the foot of the perpendicular from A to q.

Find B and $\left|\overrightarrow{AB}\right|$ (the distance between A and q).

For some $k \neq 0$, $\overrightarrow{OB} = \overrightarrow{OA} + k\mathbf{n} = (-1,0,1) + k(0,2,5)$.



Not to scale.

Since $B \in q$, $\overrightarrow{OB}\cdot(0,2,5) = 1$ or $[(-1,0,1)+k(0,2,5)]\cdot(0,2,5) = 1$ or $5 + 29k = 1$.

So, $k = -4/29$, $B = A + k\mathbf{n} = A - \frac{4}{29}\mathbf{n} = (-1,0,1) - \frac{4}{29}(0,2,5) = \left(-1, -\frac{8}{29}, \frac{9}{29}\right)$, and the distance between A and q is

$$\left|\overrightarrow{AB}\right| = |k\mathbf{n}| = |k|\,|\mathbf{n}| = \left|-\frac{4}{29}\right||(0,2,5)| = \frac{4}{29}|(0,2,5)| = \frac{4}{29}\sqrt{29} = \frac{4}{\sqrt{29}}.$$

Exercise 291. For each of the following, let B be the foot of the perpendicular from A to q. Use the Perpendicular Method to find B and the distance between A and q.

	The point A	**The plane q**
(a)	$(7,3,4)$	$\mathbf{r}\cdot(9,3,7) = 109$.
(b)	$(8,0,2)$	$\mathbf{r}\cdot(2,7,2) = 42$.
(c)	$(8,5,9)$	$\mathbf{r}\cdot(5,6,0) = 64$.

(Answer on p. 1877.)

By mimicking the above examples, we can derive general formulae for B and $\left|\overrightarrow{AB}\right|$:

Fact 178. *Let q be a plane described by $\mathbf{r}\cdot\mathbf{n} = d$, A be a point that isn't on the plane q, and B be the foot of the perpendicular from A to q.*

Suppose
$$k = \frac{d - \overrightarrow{OA}\cdot\mathbf{n}}{|\mathbf{n}|^2}.$$

Then **(a)** $B = A + k\mathbf{n}$; *and* **(b)** $\left|\overrightarrow{AB}\right| = |k|\,|\mathbf{n}|$.

Proof. See Exercise 292. □

Observe that if $\mathbf{n} = (a,b,c)$ and $A = (x,y,z)$, then in the above result, we also have

$$k = \frac{d - (ax + by + cz)}{a^2 + b^2 + c^2}.$$

You may find this second formula for k easier to remember.

Exercise 292. Prove Fact 178**(a)** and **(b)**. (Answer on p. 1878.)
(Hint: Carefully and exactly mimic what was done in the last two examples.)

Do not try *mugging* Fact 178. This is a waste of your brainpower. And disaster strikes if you're unable to remember the formulae.

It's far easier and better to understand the method used to derive Fact 178 and illustrated in the last two examples. By actually understanding what's going on, you also easily remember how you can use the method.

Nonetheless, as a quick illustration, let's redo the last two examples using Fact 178:

Example 981. Let $A = (1, 2, 3)$ be a point, q be the plane described by $\mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot(1, 1, 1) = 3$, and B be the foot of the perpendicular from A to q.

Compute $|\mathbf{n}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ and

$$k = \frac{d - \overrightarrow{OA}\cdot\mathbf{n}}{|\mathbf{n}|^2} = \frac{3 - (1, 2, 3)\cdot(1, 1, 1)}{3} = \frac{3 - (1 + 2 + 3)}{3} = \frac{-3}{3} = -1.$$

So, by Fact 178, $B = A + k\mathbf{n} = (1, 2, 3) - 1(1, 1, 1) = (0, 1, 2)$ and $\left|\overrightarrow{AB}\right| = |k||\mathbf{n}| = 1\cdot\sqrt{3} = \sqrt{3}$.

Example 982. Let $A = (-1, 0, 1)$ be a point, q be the plane described by $\mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot(0, 2, 5) = 1$, and B be the foot of the perpendicular from A to q.

Compute $|\mathbf{n}| = \sqrt{0^2 + 2^2 + 5^2} = \sqrt{29}$ and

$$k = \frac{d - \overrightarrow{OA}\cdot\mathbf{n}}{|\mathbf{n}|^2} = \frac{1 - (-1, 0, 1)\cdot(0, 2, 5)}{29} = \frac{3 - (0 + 0 + 5)}{3} = -\frac{4}{29}.$$

So, by Fact 178, $B = A + k\mathbf{n} = (-1, 0, 1) - \frac{4}{29}(0, 2, 5) = \left(-1, -\frac{8}{29}, \frac{4}{29}\right)$ and $\left|\overrightarrow{AB}\right| = |k||\mathbf{n}| = \frac{4}{29}\cdot\sqrt{29} = \frac{4}{\sqrt{29}}$.

The next result is not one that students could reasonably have been expected to know. Which means, of course, that it made a sudden appearance in 2017 (Exercise 608).

Corollary 35. *Suppose a plane is described by $\mathbf{r}\cdot\mathbf{n} = d$. Then the distance between the plane and the origin is*

$$\frac{|d|}{|\mathbf{n}|}.$$

Also, if $\mathbf{n}$ is a unit vector, then the distance between the plane and the origin is $|d|$.

Proof. See Exercise 293. □

Exercise 293. Prove Corollary 35. (Hint: Use Fact 178). (Answer on p. 1879.)

We again revisit the same two examples:

Example 983. Let q be the plane described by $\mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot(1,1,1) = 3$. Then by Corollary 35, the distance between q and the origin is

$$\frac{|d|}{|\mathbf{n}|} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

Example 984. Let q be the plane described by $\mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot(0,2,5) = 1$. Then by Corollary 35, the distance between q and the origin is

$$\frac{|d|}{|\mathbf{n}|} = \frac{1}{\sqrt{29}}.$$

Two new examples:

Example 985. Let q be the plane described by $\mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot(0,5,1) = 7$, $A = (-1,2,5)$ be a point, and B be the foot of the perpendicular from A to q.

For some $k \neq 0$, $\overrightarrow{OB} = \overrightarrow{OA} + k\mathbf{n} = (-1,2,5) + k(0,5,1)$.

Since $B \in q$, $\overrightarrow{OB}\cdot(0,5,1) = 7$ or $[(-1,2,5) + k(0,5,1)]\cdot(0,5,1) = 7$ or $15 + 26k = 7$.

So, $k = -8/26 = -4/13$, $B = A - \frac{4}{13}\mathbf{n} = (-1,2,5) - \frac{4}{13}(0,5,1) = \frac{1}{13}(-13,6,61)$, and the distance between A and q is

$$\left|\overrightarrow{AB}\right| = |k|\,|\mathbf{n}| = \frac{4}{13}\cdot\sqrt{26} = \frac{4\sqrt{2}}{\sqrt{13}}.$$

Again, you shouldn't bother *mugging* Fact 178. Nonetheless, let's use it to quickly check that our above results are correct:

Compute $|\mathbf{n}| = \sqrt{0^2 + 5^2 + 1^2} = \sqrt{26}$ and

$$k = \frac{d - \overrightarrow{OA}\cdot\mathbf{n}}{|\mathbf{n}|^2} = \frac{7 - (-1,2,5)\cdot(0,5,1)}{26} = \frac{7-15}{26} = -\frac{4}{13}.$$

So, by Fact 178, $B = A + k\mathbf{n} = (-1,2,5) - \frac{4}{13}(0,5,1) = \frac{1}{13}(-13,6,61)$ and

$$\left|\overrightarrow{AB}\right| = |k|\,|\mathbf{n}| = 4\sqrt{2/13}.$$

Example 986. Let q be the plane described by $\mathbf{r}\cdot\mathbf{n}=\mathbf{r}\cdot(1,2,3)=32$, $A=(0,0,0)$ be a point, and B be the foot of the perpendicular from A to q.

For some $k\neq 0$, $\overrightarrow{OB}=\overrightarrow{OA}+k\mathbf{n}=(0,0,0)+k(1,2,3)$.

Since $B\in q$, $\overrightarrow{OB}\cdot(1,2,3)=32$ or $[(0,0,0)+k(1,2,3)]\cdot(1,2,3)=32$ or $14k=32$.

So, $k=32/14=16/7$, $B=A+\dfrac{16}{7}\mathbf{n}=(0,0,0)+\dfrac{16}{7}(1,2,3)=\dfrac{16}{7}(1,2,3)$. and the distance between A and q is

$$\left|\overrightarrow{AB}\right|=|k|\,|\mathbf{n}|=\frac{16}{7}\cdot\sqrt{14}=\frac{16\sqrt{2}}{\sqrt{7}}.$$

Again, you shouldn't bother *mugging* Fact 178. Nonetheless, let's use it to quickly check that our above results are correct:

Compute $|\mathbf{n}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$ and

$$k=\frac{d-\overrightarrow{OA}\cdot\mathbf{n}}{|\mathbf{n}|^2}=\frac{32-(0,0,0)\cdot(1,2,3)}{14}=\frac{32-0}{14}=\frac{16}{7}.$$

So, by Fact 178, $B=A+k\mathbf{n}=(0,0,0)+\dfrac{16}{7}(1,2,3)=\dfrac{16}{7}(1,2,3)$ and

$$\left|\overrightarrow{AB}\right|=|k|\,|\mathbf{n}|=16\sqrt{2/7}.$$

Exercise 294. Let $S=(-1,0,7)$ and $T=(3,2,1)$ be points and q be the plane described by $\mathbf{r}\cdot(5,-3,1)=0$. Use both methods you've learnt in this chapter to find **(a)** the feet of the perpendiculars from S and T to q. Then find **(b)** the distances from q to the points S and T; and also **(c)** the distance between q and the origin. (Answer on p. 1879.)

Remark 113. Earlier when finding the distance between a point and a line, we also had what we called the Quadratic Methods.

It turns out there's actually also a similar method for finding the distance between a point and a plane. However, because this involves multivariate calculus and is not on your syllabus, I have placed a very brief this discussion of this method to Ch. 144.14(Appendices).

78. Coplanarity

Definition 185. Two or more points are *coplanar* if some plane contains all of them.

"Obviously", given any line, there is a plane that contains this line.

Recall (Fact 144): Any two points are collinear. So "obviously", they must also be coplanar.

Recall (Ch. 74): Three non-collinear points uniquely determine a plane. So, any three points must also be coplanar.

In contrast, four points need not be coplanar. Given four distinct points, we'll use these steps to check if they're coplanar:

1. Write down the plane that contains three of the points.
2. Then check whether this plane also contains the fourth point.

Example 987. Let $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, and $D = (1, 1, -1)$ be points. To check if they are coplanar, we'll write down the (unique) plane q that contains A, B, and C. We'll then check if $D \in q$.

As we learnt in Ch. 74, given three non-collinear points, there are several ways to find q.

Here we'll first compute q's normal vector $\overrightarrow{AB} \times \overrightarrow{AC} = (1, 1, 1)$.

We then also compute $\overrightarrow{OA} \cdot (1, 1, 1) = 1 + 0 + 0 = 1$. We conclude that q is $\mathbf{r} \cdot (1, 1, 1) = 1$.

We now check whether $D \in q$:

$$\overrightarrow{OD} \cdot (1, 1, 1) = (1, 1, -1) \cdot (1, 1, 1) = 1 + 1 - 1 = 1. \qquad ✓$$

So yes, the four points are coplanar.

Example 988. Let $A = (2, 3, 5)$, $B = (8, -1, 0)$, $C = (0, 1, 0)$, and $D = (-3, -2, -1)$ be points. To check if they are coplanar, we'll write down the plane q that contains A, B, and C. We'll then check if $D \in q$.

We can describe q in parametric form as

$$\mathbf{r} = \overrightarrow{OC} + \lambda\overrightarrow{AB} + \mu\overrightarrow{BC} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -4 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} -8 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6\lambda - 8\mu \\ 1 - 4\lambda + 2\mu \\ -5\lambda \end{pmatrix} (\lambda, \mu \in \mathbb{R}).$$

Now check if $D \in q$ by plugging D into the the above parametric equation:

$$\begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 6\lambda - 8\mu \\ 1 - 4\lambda + 2\mu \\ -5\lambda \end{pmatrix} \quad \text{or} \quad \begin{aligned} -3 &\stackrel{1}{=} 6\lambda - 8\mu, \\ -2 &\stackrel{2}{=} 1 - 4\lambda + 2\mu, \\ -1 &\stackrel{3}{=} -5\lambda. \end{aligned}$$

From $\stackrel{3}{=}$, $\lambda = 0.2$. So, from $\stackrel{1}{=}$, $\mu = 21/40$. These values of λ and μ don't satisfy $\stackrel{2}{=}$. Hence, $D \notin q$ and the four points are **not** coplanar.

Here's an occasionally useful shortcut:

Fact 179. *If three of four points are collinear, then the four points are coplanar.*

Proof. Given four points A, B, C, and D, suppose A, B, and C are collinear. Some plane q contains A, B, and D. By Fact 158, q contains the line AB and hence also the point C. Thus, q contains all four points. □

Example 989. Let $A = (1,2,3)$, $B = (4,5,6)$, $C = (10,11,12)$, and $D = (9,1,7)$ be points.

The line AB is $\mathbf{r} = (1,2,3) + \lambda(3,3,3)$ $(\lambda \in \mathbb{R})$. By setting $\lambda = 3$, we see that AB also contains the point C. So, A, B, and C are collinear.

Hence, by Fact 179, A, B, C, and D are coplanar. (Indeed, given *any* point E, it is similarly true that the four points A, B, C, and E must be coplanar.)

Note though that the converse of Fact 179 is false (see Exercise 295).

Exercise 295. In Example 987, we already showed that the points $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$, and $D = (1,1,-1)$ are coplanar. Now show that no three of these four points are collinear. (You will thus have produced a counterexample to the converse of Fact 179.) (Answer on p. 1880.)

Exercise 296. In each of the following, determine if the four points given are coplanar. If they are, write down the plane that contains all four points. (Answer on p. 1880.)

(a) $A = (0,1,5)$, $B = (-3,-1,1)$, $C = (2,7,5)$, and $D = (6,6,1)$.

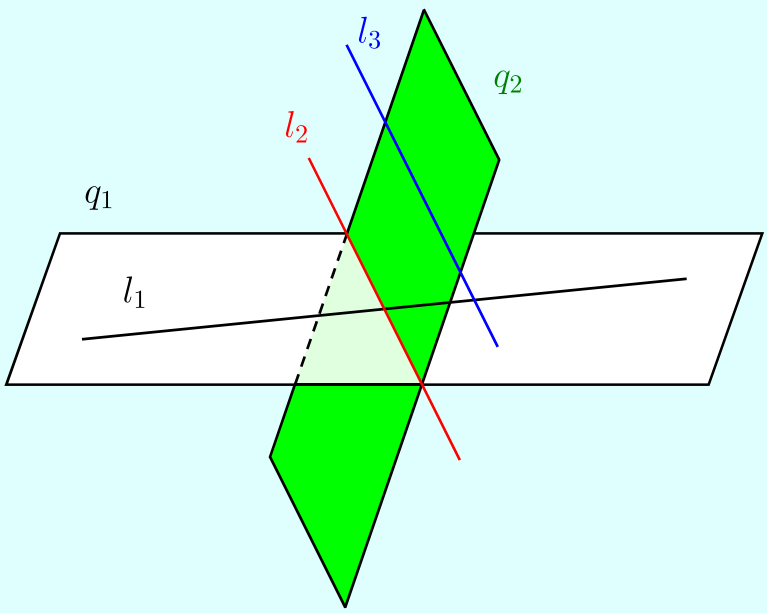
(b) $A = (-1,3,-5)$, $B = (0,0,0)$, $C = (6,1,-2)$, and $D = (4,7,-12)$.

(c) $A = (0,1,2)$, $B = (1,2,3)$, $C = (2,3,4)$, and $D = (19,0,-5)$.

78.1. Coplanarity of Lines

Definition 186. Two or more lines are *coplanar* if some plane contains all of them.

Example 990. The lines l_1 and l_2 are coplanar because the plane q_1 contains both of them. Similarly, l_2 and l_3 are coplanar because the plane q_2 contains both of them.



In contrast, l_1 and l_3 are not coplanar because no plane contains both of them.

Somewhat obviously and trivially,

Fact 180. *If two lines are identical, then they are also parallel and coplanar.*

Proof. Let the two (identical) lines be described by $\mathbf{r} = \overrightarrow{OP} + \lambda\mathbf{u}$ $(\lambda \in \mathbb{R})$.

They have parallel direction vectors. And so by Definition 150, they are parallel.

Let $\mathbf{v}$ be any vector that points in a different direction from $\mathbf{u}$. Then the plane $\mathbf{r} = \overrightarrow{OP} + \lambda\mathbf{u} + \mu\mathbf{v}$ $(\lambda, \mu \in \mathbb{R})$ contains both lines (to verify this, simply let $\mu = 0$). □

In the less trivial case of two distinct lines, there are three possibilities:

Fact 181. *Suppose l_1 and l_2 are distinct lines described by $R = P + \alpha\mathbf{u}$ and $R = Q + \beta\mathbf{v}$ $(\alpha, \beta \in \mathbb{R})$. Then exactly one of the following three possibilities holds: The two lines are*

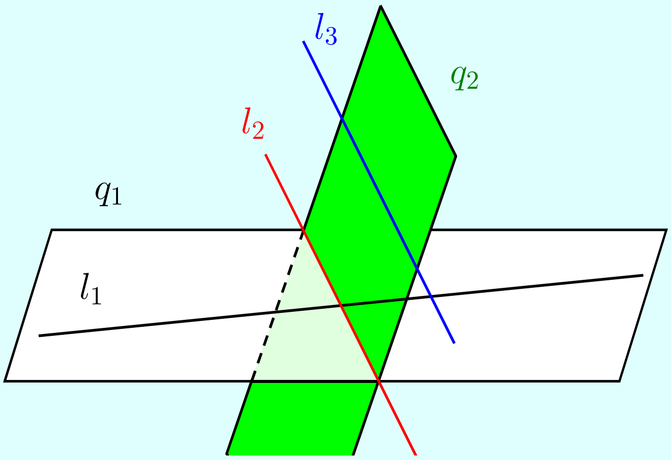
(a) *Parallel and do not intersect; moreover, the unique plane that contains both lines is $q_a = \left\{R : R = P + \lambda\mathbf{u} + \mu\overrightarrow{PQ} \quad (\lambda, \mu \in \mathbb{R})\right\}$; or*

(b) *Not parallel and share exactly one intersection point; moreover, the unique plane that contains both lines is described by $q_b = \{R : R = P + \lambda\mathbf{u} + \mu\mathbf{v} \quad (\lambda, \mu \in \mathbb{R})\}$; or*

(c) *Skew (i.e. neither parallel nor intersect) and are not coplanar.*

Proof. See p. 1643 (Appendices). □

Example 991. We illustrate Fact 181 using the last example:



(a) The lines l_2 and l_3 are parallel, coplanar, and do not intersect.

(b) The lines l_1 and l_2 are non-parallel, coplanar, and share exactly one intersection point.

(c) The lines l_1 and l_3 are skew (i.e. aren't parallel & don't intersect) and not coplanar.

It's probably easier to remember this Corollary, which is immediate from the last result:

Corollary 36. *Two lines are coplanar if and only if they are not skew.*

And now, remember that when two distinct lines are not skew (either parallel or intersect), there are only two possible Cases:

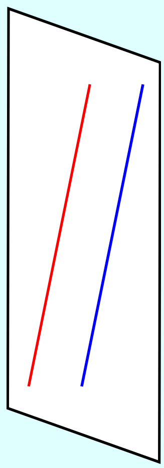
1. If they're parallel, they don't intersect; and
2. If they're not parallel, they intersect exactly once.

And in either Case, Fact 181 says there's a unique plane that contains both lines. This plane can be easily found using any of the Four Ways (Ch. 74):

Example 992. Two lines are described by

$$\mathbf{r} = (8,1,1) + \alpha\overbrace{(3,6,9)}^{\mathbf{u}} \qquad \text{and} \qquad \mathbf{r} = (4,5,6) + \beta\overbrace{(1,2,3)}^{\mathbf{v}} \quad (\alpha,\beta \in \mathbb{R}).$$

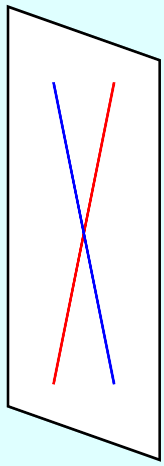
Since $\mathbf{u} \parallel \mathbf{v}$, the two lines are parallel. Since $(8,1,1) - (4,5,6) = (4,-4,-5) \nparallel \mathbf{u}$, the two lines are distinct and don't intersect.



Compute $\mathbf{v} - \mathbf{u} = (-4,4,5) = \mathbf{w}$. Note that $\mathbf{v} \nparallel \mathbf{w}$. So, the unique plane that contains $(8,1,1)$, $\mathbf{v}$, and $\mathbf{w}$ (and hence also the unique plane that contains both lines) is

$$\mathbf{r} = (8,1,1) + \lambda\overbrace{(1,2,3)}^{\mathbf{v}} + \mu\overbrace{(-4,4,5)}^{\mathbf{w}} \qquad (\lambda,\mu \in \mathbb{R}).$$

Example 993. Two lines are described by



$$\mathbf{r} = (0,0,0) + \alpha\overbrace{(0,1,0)}^{\mathbf{u}} \quad \text{and} \quad \mathbf{r} = (4,17,0) + \beta\overbrace{(1,0,0)}^{\mathbf{v}}, \quad (\alpha, \beta \in \mathbb{R}).$$

Since $\mathbf{u} \nparallel \mathbf{v}$, the two lines are **not parallel**. To check if they intersect, write

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \hat{\alpha}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \\ 0 \end{pmatrix} + \hat{\beta}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{aligned} 0 &\overset{1}{=} 4 + \hat{\beta}, \\ \hat{\alpha} &\overset{2}{=} 17, \\ 0 &\overset{3}{=} 0. \end{aligned}$$

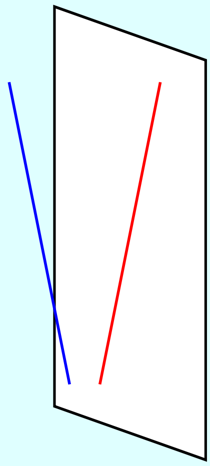
Solving, $\hat{\alpha} = 17$ and $\hat{\beta} = -4$. So, the two lines **intersect** at

$$(0,0,0) + 17(0,1,0) = (4,17,0) - 4(1,0,0) = (0,17,0).$$

Hence, the two lines aren't skew. The unique plane that contains both lines is also the unique plane that contains $(0,0,0)$, $\mathbf{u}$, and $\mathbf{v}$. And this plane is

$$\mathbf{r} = (0,0,0) + \lambda\mathbf{u} + \mu\mathbf{v}, \qquad (\lambda, \mu \in \mathbb{R}).$$

Example 994. Two lines are described by



$$\mathbf{r} = (0,1,2) + \alpha\overbrace{(9,1,3)}^{\mathbf{u}} \quad \text{and} \quad \mathbf{r} = (4,5,6) + \beta\overbrace{(3,2,1)}^{\mathbf{v}}, \quad (\alpha, \beta \in \mathbb{R}).$$

Since $\mathbf{u} \nparallel \mathbf{v}$, the two lines are **not parallel**. To check if they intersect, write

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \hat{\alpha}\begin{pmatrix} 9 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \hat{\beta}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{aligned} 9\hat{\alpha} &\overset{1}{=} 4 + 3\hat{\beta}, \\ 1 + \hat{\alpha} &\overset{2}{=} 5 + 2\hat{\beta}, \\ 2 + 3\hat{\alpha} &\overset{3}{=} 6 + \hat{\beta}. \end{aligned}$$

Take $\overset{1}{=}$ minus $3\times \overset{3}{=}$ to get $-6 = -14$, a contradiction. So, the two lines **do not intersect**. So, they're **skew** and **not coplanar**.

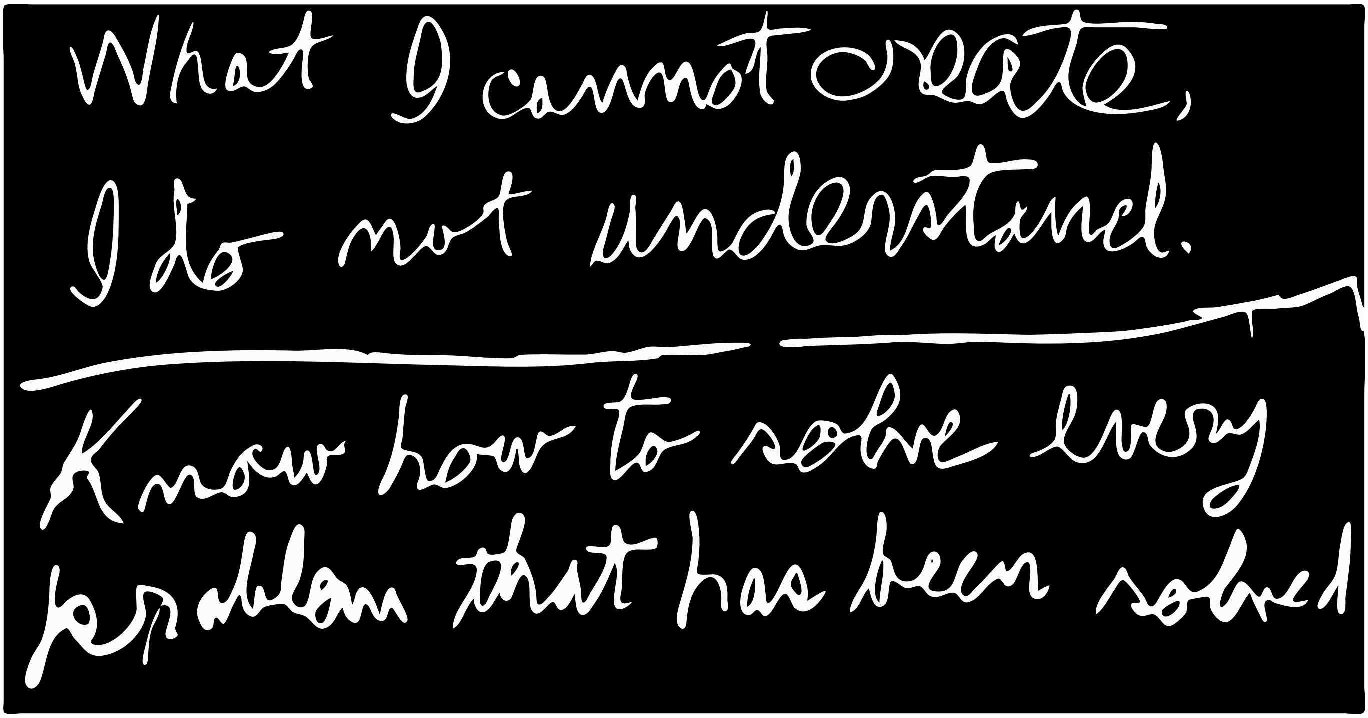
Remark 114. In this textbook, we define two lines to be *skew* if they are not parallel and do not intersect (Definition 172). Corollary 36 then follows as a result.

However, some writers take another route: They first define two lines to be *skew* if they are not coplanar. That is, they use Corollary 36 as their definition of skew lines. They *then* prove that our Definition 172 follows as a result.

Exercise 297. Determine if each pair of lines is parallel, coplanar, or intersect. Also, find any intersection points and any plane that contains both lines.(Answer on p. 1881.)

	l_1	l_2	
(a)	$\mathbf{r} = (8,1,5) + \lambda\ (3,2,1)$	$\mathbf{r} = (1,2,3) + \lambda\ (5,6,7)$	$(\lambda \in \mathbb{R})$
(b)	$\mathbf{r} = (0,0,6) + \lambda\ (3,9,0)$	$\mathbf{r} = (1,1,1) + \lambda\ (1,3,0)$	“
(c)	$\mathbf{r} = (6,5,5) + \lambda\ (1,0,1)$	$\mathbf{r} = (9,3,6) + \lambda\ (0,1,1)$	“
(d)	$\mathbf{r} = (-1,3,8) + \lambda\ (-5,0,1)$	$\mathbf{r} = (9,3,6) + \lambda\ (10,0,2)$	“

Part IV.
Complex Numbers



What I cannot create, I do not understand.

Know how to solve every problem that has been solved.

— Richard Feynman's blackboard at the time of his death (1988).

79. Complex Numbers: Introduction

the historical sequence of extensions of the number system, from natural numbers to integers to rationals to reals to complex numbers, can with hindsight be interpreted as a quest to make more and more equations have solutions.

— Ian Stewart (2015).

Here's a brief and simplistic motivation of complex numbers:

1. Solve $x - 1 = 0$. Easy; the solution is a **natural number**: $x = 1$.
2. To solve $x + 1 = 0$, we must invent **negative numbers**. The solution is $x = -1$.
3. To solve $x^2 = 2$, we must invent **irrational numbers**. The solution is $x = \pm\sqrt{2}$.
4. To solve $x^2 = -1$, we must invent the **imaginary unit**:

Definition 187. The *imaginary unit*, denoted i, is the number that satisfies $\mathrm{i}^2 = -1$.

Or equivalently: $\mathrm{i} = \sqrt{-1}$. And so, the solution to $x^2 = -1$ is $x = \pm\mathrm{i} = \pm\sqrt{-1}$.

In this textbook, we'll happily (and naïvely) assume that the "usual" rules of arithmetic also apply to the complex numbers.[360]

Any real-number multiple of the imaginary unit is called an **imaginary number**:

Definition 188. An *imaginary number* is any ib, where $b \in \mathbb{R}$.

Example 995. Imaginary numbers: $2\mathrm{i} = 2\sqrt{-1}$, $-\mathrm{i} = -\sqrt{-1}$, $5.2\mathrm{i} = 5.2\sqrt{-1}$, $-99\mathrm{i} = -99\sqrt{-1}$.

Example 996. The number i is *both* the imaginary unit and an imaginary number.

Example 997. Perhaps surprisingly, 0 is also an imaginary number! This is because we can write

$$0 = 0\mathrm{i}.$$

And so, by Definition 188, 0 is imaginary.

Indeed, 0 is both real and imaginary.

This may seem weird, but you can treat it as just another arbitrary convention (that turns out to be sometimes more convenient than the alternative convention of *not* treating 0 as an imaginary number).

We can now define a **complex number** as the sum of any real and any imaginary number:

[360] In this textbook, we've been neither clear nor explicit about what these rules are. We have simply assumed that everyone, including you the student, "knows" what they are.

Definition 189. A *complex number* is any $a + ib$, where $a, b \in \mathbb{R}$.

The *set of all complex numbers* is $\mathbb{C} = \{a + \mathrm{i}b : a, b \in \mathbb{R}\}$.

Example 998. *All* of these are complex numbers: 0, 1, −5, 4/3, −2.881, $\sqrt{2}$, π, i, 2i, $\sqrt{2}$i, $-\pi$i, 3 + 2i, −5 − i, −2 + 0.5i.

Indeed, *all* numbers we'll encounter in A-Level Maths (and this textbook) are complex numbers.[361]

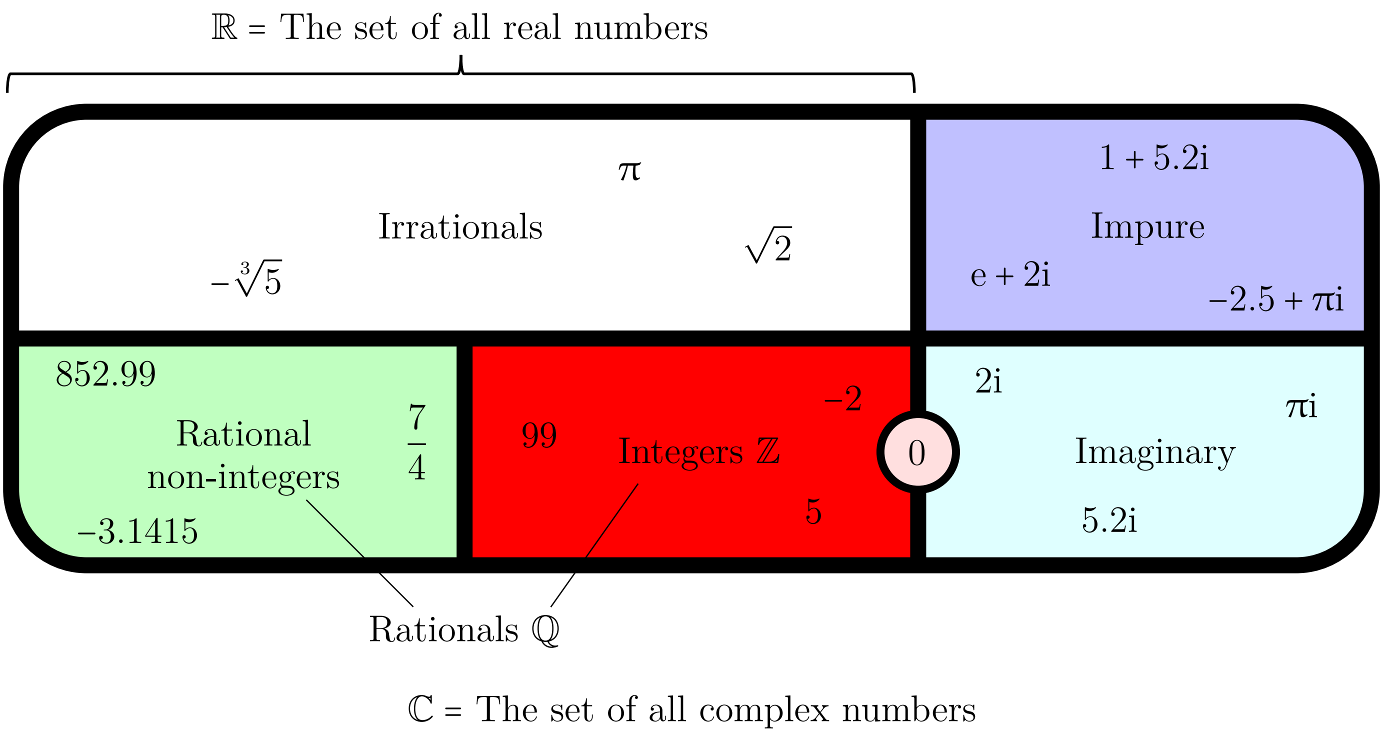
The last three of the above numbers (3 + 2i, −5 − i, −2 + 0.5i) are complex numbers that are neither real nor imaginary.[362] Such numbers are of the form $a + ib$, where $a, b \neq 0$.

Definition 190. A *impure number* is any complex number that is neither real nor imaginary, i.e. any $a + ib$, where $a, b \in \mathbb{R}$ and $a, b \neq 0$.

Example 999. Impure numbers: 3 + 2i, −5 − i, −2 + 0.5i.

Remark 115. The term *impure number* is not standard. But we'll use it anyway as it's convenient and allows us to avoid saying the mouthful *complex number that is neither real nor imaginary*.

Earlier Ch. 4.10 gave a Venn diagram of the real numbers. We now expand that diagram to produce this Venn diagram of the complex numbers:



With **one very special exception**, every complex number is either real, imaginary, or impure (and not any two of these). The one very special exception is the number 0, which is *both* real and imaginary.

We have $\mathbb{C} \supseteq \mathbb{R} \supseteq \mathbb{Q} \supseteq \mathbb{Z}$.

[362]Reminder: **Infinity is not a number.** In particular, infinity is not a complex number.
Examples of numbers that aren't complex: Quaternions, octonions, and hypercomplex numbers.

Remark 116. Only a real number can be rational or irrational.[363] There's no such thing as a rational or irrational imaginary or impure number—except for 0, which is real, rational, *and* imaginary.

Similarly, only a real number can be positive or negative. There's no such thing as a positive or negative imaginary or impure number—0 is both real and imaginary, but is neither positive nor negative.[364]

Similarly, only a real number can be an integer. There's no such thing as an integer imaginary or impure number—except for 0, which is real, an integer, *and* imaginary.

Example 1000. Each of the numbers $\mathrm{i} = \sqrt{-1}$, $-\mathrm{i} = -\sqrt{-1}$, $5\mathrm{i}$, $1+5\mathrm{i}$, $-\sqrt{2}\mathrm{i}$, $1+\sqrt{2}\mathrm{i}$ is either imaginary or impure. None is real, rational, irrational, positive, negative, or an integer.

Example 1001. Zero is real, rational, *and* imaginary, but *not* positive, negative, or impure.

Example 1002. Slowly work through this table:

	$9-2\mathrm{i}$	$\sqrt{3}\mathrm{i}$	0	4	$-4-2\mathrm{i}$	i	$\sqrt{3}$	-1	$-\mathrm{i}$	1.394
Complex	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Imaginary		✓	✓			✓			✓	
Impure	✓				✓					
Real			✓	✓			✓	✓		✓
An integer			✓	✓				✓		
Positive				✓			✓			✓
Negative								✓		
Rational			✓	✓				✓		✓
Irrational							✓			✓

[363]Recall (Ch. 4.9) that we defined $\mathbb{Q} = \{x \in \mathbb{R} : x = a/b \text{ for some } a, b \in \mathbb{Z}\}$ and $\{\text{Irrationals}\} = \mathbb{R} \setminus \mathbb{Q}$.
[364]See ✿ (especially ajotaxte's answer).

Exercise. Fill in the table. (Answer on p. 1882.)

	-2.5	$1+i\pi$	$1+\pi$	$1-\sqrt{6}$	$1-8i$	$\sqrt{5}+0i$	$-200i$
Complex							
Imaginary							
Impure							
Real							
An integer							
Positive							
Negative							
Rational							
Irrational							

Exercise 298. Explain if each statement is true or false: (Answer on p. 1882.)

(a) "$\mathbb{Z} \cup \{\text{Rational non-integers}\} = \mathbb{Q}$".

(b) "$\mathbb{Q} \cap \{\text{Irrationals}\} = \varnothing$."

(c) "$0 \in \mathbb{C} \setminus \mathbb{R}$".

(d) "$\{\text{Imaginary numbers}\} \cap \{\text{Impure numbers}\} = \varnothing$."

(e) "$\mathbb{Z} \subseteq \{\text{Imaginary numbers}\}$."

(f) "$\mathbb{Z} \cap \{\text{Imaginary numbers}\} = \varnothing$."

(g) "$(\{\text{Imaginary numbers}\} \cup \{\text{Impure numbers}\}) \cap \mathbb{R} = \varnothing$."

79.1. The Real and Imaginary Parts of Complex Numbers

Definition 191. Given a complex number $z = a + ib$, its *real part* is $\operatorname{Re} z = a$ and its *imaginary part* is $\operatorname{Im} z = b$.

Example 1003. Let $z = 3 + 2\mathrm{i}$. Then $\operatorname{Re} z = \operatorname{Re}(3 + 2\mathrm{i}) = 3$ and $\operatorname{Im} z = \operatorname{Im}(3 + 2\mathrm{i}) = 2$.

Example 1004. Let $w = 7$. Then $\operatorname{Re} w = \operatorname{Re} 7 = 7$ and $\operatorname{Im} w = \operatorname{Im} 7 = 0$.

Example 1005. Let $\omega = 19\mathrm{i}$. Then $\operatorname{Re} \omega = \operatorname{Re}(19\mathrm{i}) = 0$ and $\operatorname{Im} \omega = \operatorname{Im}(19\mathrm{i}) = 19$.

Remark 117. Your A-Level examiners like using the symbols z, w, and ω (lower-case Greek letter *omega*) to denote complex numbers and so that's what we'll try to do too.

"Obviously", two complex numbers z and w are equal if and only if

$$\operatorname{Re} z = \operatorname{Re} w \qquad \text{and} \qquad \operatorname{Im} z = \operatorname{Im} w.$$

Example 1006. Let $z = 3 + ib$ and $w = a - 17\mathrm{i}$. If $z = w$. then

$$a = 3 \qquad \text{and} \qquad b = -17.$$

Remark 118. Your A-Level examiners seem to follow the convention of writing $a + ib$ or $x + \mathrm{i}y$ rather than $a + b\mathrm{i}$ or $x + y\mathrm{i}$. That is, the imaginary unit i is written before a variable like b or y. And so that's also what we'll do too. (The logic seems to be that constants come before variables and the imaginary unit i is a constant.)

Note though that we will still write $1 + 2\mathrm{i}$ or $-3 - 4\mathrm{i}$ rather than $1 + \mathrm{i}2$ or $-3 - \mathrm{i}4$. That is, any constants like 1 or 4 will still be written before the imaginary unit i.

Exercise 299. Exactly two of a, b, c, and d are identical. Which? (Answer on p. 1882.)

$$a = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{2}i, \qquad b = \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \qquad c = \sin\frac{\pi}{3} - \sin\frac{\pi}{3}i, \qquad d = \frac{\sqrt{3}}{2} - \cos\left(-\frac{\pi}{4}\right)i.$$

79.2. Complex Numbers in Ordered Pair Notation

It is also often convenient to write complex numbers in **ordered pair notation**, with the first term being the real part and the second term being the imaginary.

Example 1007. $z = 3 + 2\mathrm{i} = (3, 2)$.

Example 1008. $w = 7 = (7, 0)$.

Example 1009. $\omega = 19\mathrm{i} = (0, 19)$.

In general, given a complex number z, we can also write

$$z = \operatorname{Re} z + \mathrm{i} \operatorname{Im} z = (\operatorname{Re} z, \operatorname{Im} z).$$

Exercise 300. Rewrite each number in ordered pair notation. (Answer on p. 1882.)

(a) $z = 33\,(1 + \mathrm{ei})$ **(b)** $w = (237 + \pi) - \left(\sqrt{2} - 3\right)\mathrm{i}$ **(c)** $\operatorname{Re}\omega = p$, $\operatorname{Im}\omega = q$.

79.3. The Usual Ordering Does Not Apply to Non-Real Numbers

The "usual ordering" refers to $<$, $\leq$, $>$, or $\geq$.

In this textbook, we haven't precisely defined what exactly any of these four symbols means. We've simply relied on your informal and intuitive understanding of what each means:

Example 1010. $5 > 4$, $3 \leq 8$, $2 > -6$, $1 \geq 0$.

The title of this brief subchapter says this—if a or b is not real, then the following statements are simply left undefined or meaningless:

$$a < b,\ a \leq b,\ a > b,\ a \geq b.$$

Example 1011. These statements are left undefined or meaningless:

$$3 + 5\mathrm{i} > 4,\ \mathrm{i} \leq 5\mathrm{i},\ 0 > -\mathrm{i},\ \mathrm{i} \geq -1 - \mathrm{i}.$$

That's all you need know for the A-Levels. For a deeper discussion, see e.g. ✿.

80. Some Arithmetic of Complex Numbers

To repeat, we'll happily (and naïvely) assume that the "usual" rules of arithmetic also apply to the complex numbers. In which case, **addition** and **subtraction** are especially simple:

80.1. Addition and Subtraction

Example 1012. Let $z = -2 + = (-2, 1)\,\mathrm{i}$ and $w = 3\mathrm{i} = (0, 3)$. Then

$$z + w = -2 + 4\mathrm{i} \qquad \text{and} \qquad z - w = -2 - 2\mathrm{i}.$$

Or, $\quad z + w = (-2 + 0, 1 + 3) = (-2, 4) \qquad$ and $\qquad z - w = (-2 - 0, 1 - 3) = (-2, -2)$.

Example 1013. Let $z = 7 - \mathrm{i} = (7, -1)$ and $w = 2 + 5\mathrm{i} = (2, 5)$. Then

$$z + w = 9 + 4\mathrm{i} \qquad \text{and} \qquad z - w = 5 - 6\mathrm{i}.$$

Or, $\quad z + w = (7 + 2, -1 + 5) = (9, 4) \qquad$ and $\qquad z - w = (7 - 2, -1 - 5) = (5, -6)$.

In general,

Fact 182. *Suppose* $z = a + \mathrm{i}b = (a, b)$ *and* $w = c + \mathrm{i}d = (c, d)$. *Then*

(a) $z + w = a + c + \mathrm{i}\,(b + d) = (a + c, b + d)$; *and*

(b) $z - w = a - c + \mathrm{i}\,(b - d) = (a - c, b - d)$.

Exercise 301. For each, compute $z + w$ and $z - w$. (Answer on p. 1883.)

(a) $z = -5 + 2\mathrm{i}$, $w = 7 + 3\mathrm{i}$. **(b)** $z = 3 - \mathrm{i}$, $w = 11 + 2\mathrm{i}$. **(c)** $z = 1 + 2\mathrm{i}$, $w = 3 - \sqrt{2}\mathrm{i}$.

80.2. Multiplication

Here are the powers of i:

$$\mathrm{i}=\mathrm{i},\qquad \mathrm{i}^2=\mathrm{i}\times\mathrm{i}=-1,\qquad \mathrm{i}^3=\mathrm{i}\times\mathrm{i}^2=-\mathrm{i},\qquad \mathrm{i}^4=\mathrm{i}\times\mathrm{i}^3=1,$$

$$\mathrm{i}^5=\mathrm{i}\times\mathrm{i}^4=\mathrm{i},\qquad \mathrm{i}^6=\mathrm{i}\times\mathrm{i}=-1,\qquad \mathrm{i}^7=\mathrm{i}\times\mathrm{i}^2=-\mathrm{i},\qquad \mathrm{i}^8=\mathrm{i}\times\mathrm{i}^3=1,$$

$$\mathrm{i}^9=\mathrm{i}\times\mathrm{i}^8=\mathrm{i},\qquad \mathrm{i}^{10}=\mathrm{i}\times\mathrm{i}=-1,\qquad \mathrm{i}^{11}=\mathrm{i}\times\mathrm{i}^2=-\mathrm{i},\qquad \mathrm{i}^{12}=\mathrm{i}\times\mathrm{i}^3=1,$$

etc.

Observe that $\mathrm{i}^4=1$. And so, the cycle repeats after every fourth power.

The "usual" rules of multiplication hold:

Example 1014. Let $z=\mathrm{i}$ and $w=1+\mathrm{i}$. Then $zw=\mathrm{i}(1+\mathrm{i})=\mathrm{i}(1)+\mathrm{i}^2=\mathrm{i}-1$.

Example 1015. Let $z=-2+\mathrm{i}$ and $w=3\mathrm{i}$. Then

$$zw=(-2+\mathrm{i})(3\mathrm{i})=(-2)(3\mathrm{i})+\mathrm{i}(3\mathrm{i})=-6\mathrm{i}+3\mathrm{i}^2=-3-6\mathrm{i}.$$

Google does the basic arithmetic of complex numbers as well as Wolfram Alpha, but much more quickly. So here in Part IV, whenever you see the G logo, click on it to go to the relevant computation done by Google.

Example 1016. Let $z=2-\mathrm{i}$ and $w=-1+\mathrm{i}$. Then

$$zw=(2-\mathrm{i})(-1+\mathrm{i})=-2+2\mathrm{i}+\mathrm{i}-\mathrm{i}^2=-1+3\mathrm{i}.$$

Example 1017. Let $z=3+2\mathrm{i}$ and $w=-7+4\mathrm{i}$. Then

$$zw=(3+2\mathrm{i})(-7+4\mathrm{i})=-21+12\mathrm{i}-14\mathrm{i}+8\mathrm{i}^2=-29-2\mathrm{i}.$$

In general,

Fact 183. *If $z=a+\mathrm{i}b=(a,b)$ and $w=c+\mathrm{i}d=(c,d)$, then*

$$zw=ac-bd+\mathrm{i}(ad+bc)=(ac-bd,ad+bc).$$

Proof. See Exercise 303. □

Recall that $(x+y)^2 = x^2 + 2xy + y^2$ and $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$. Hence,

$$(a+\mathrm{i}b)^2 = a^2 + 2a(\mathrm{i}b) + (\mathrm{i}b)^2 = a^2 + 2\mathrm{i}ab - b^2,$$
$$(a+\mathrm{i}b)^3 = a^3 + 3a^2(\mathrm{i}b) + 3a(\mathrm{i}b)^2 + (\mathrm{i}b)^3 = a^3 + 3\mathrm{i}a^2b - 3ab^2 - \mathrm{i}b^3.$$

Let's jot these down formally:

Fact 184. **(a)** $(a+ib)^2 = a^2 + 2\mathrm{i}ab - b^2$.
(b) $(a+ib)^3 = a^3 + 3\mathrm{i}a^2b - 3ab^2 - \mathrm{i}b^3$.

Example 1018. Let $z = 3 + 2\mathrm{i}$. To compute z^2, we can use Fact 184(a):

$$z^2 = (3+2\mathrm{i})^2 = 3^2 + 2\cdot 3\cdot 2\mathrm{i} - 2^2 = 9 + 12\mathrm{i} - 4 \stackrel{1}{=} 5 + 12\mathrm{i}.$$

Instead of using Fact 184(a), we could do the usual multiplication:

$$z^2 = (3+2\mathrm{i})(3+2\mathrm{i}) = 9 + 6\mathrm{i} + 6\mathrm{i} - 4 = 5 + 12\mathrm{i}.$$

And to compute z^3, we can use Fact 184(b):

$$z^3 = (3+2\mathrm{i})^3 = 3^3 + 3\cdot 3^2\cdot 2\mathrm{i} - 3\cdot 3\cdot 2^2 - 2^3\mathrm{i} = 27 + 54\mathrm{i} - 36 - 8\mathrm{i} = -9 + 46\mathrm{i}.$$

Again, instead of using Fact 184(b), we could do the usual multiplication:

$$z^3 = z^2 z = (5+12\mathrm{i})(3+2\mathrm{i}) = 15 + 10\mathrm{i} + 36\mathrm{i} - 24 = -9 + 46\mathrm{i}.$$

Exercise 302. For each, compute zw, z^2, and z^3. (Answer on p. 1883.)

(a) $z = -5 + 2\mathrm{i}$, $w = 7 + 3\mathrm{i}$. **(b)** $z = 3 - \mathrm{i}$, $w = 11 + 2\mathrm{i}$. **(c)** $z = 1 + 2\mathrm{i}$, $w = 3 - \sqrt{2}\mathrm{i}$.

Exercise 303. Prove Fact 183. (Answer on p. 1884.)

Exercise 304. If $2 + \mathrm{i}$ solves $az^3 + bz^2 + 3z - 1 = 0$, then what are a and b? (Answer on p. 1884.)

80.3. Conjugation

Example 1019. Let $z = 1 + \mathrm{i}$. Then the **(complex) conjugate** of z is $z^* = 1 - \mathrm{i}$. We call $z = 1 + \mathrm{i}$ and $z^* = 1 - \mathrm{i}$ a **(complex) conjugate pair**.

Definition 192. Given the complex number $z = a+ib$, its *(complex) conjugate* is $z^* = a-ib$. Also, $z = a + ib$ and $z^* = a - ib$ are called a *(complex) conjugate pair*.

Example 1020. The conjugate of $w = -5 - \left(17 + \sqrt{2}\right)\mathrm{i}$ is $w^* = -5 + \left(17 + \sqrt{2}\right)\mathrm{i}$; and w and w^* are a conjugate pair.

Example 1021. The conjugate of $\omega = 10$ is $\omega^* = 10$; and ω and ω^* are a conjugate pair.

Example 1022. The conjugate of $a = 2\mathrm{i}$ is $a^* = -2\mathrm{i}$; and a and a^* are a conjugate pair.

"Obviously", the conjugate of the conjugate of a complex number z is z itself:

Fact 185. $(z^*)^* = z$.

Example 1023. Let $z = 1 + \mathrm{i}$. Then $z^* = 1 - \mathrm{i}$ and $(z^*)^* = 1 + \mathrm{i} = z$.

Example 1024. Let $w = -5 - 17\mathrm{i}$. Then $w^* = -5 + 17\mathrm{i}$ and $(w^*)^* = -5 - 17\mathrm{i} = w$.

Example 1025. Let $\omega = 10$. Then $\omega^* = 10$ and $(\omega^*)^* = 10 = \omega$.

Example 1026. Let $a = 2\mathrm{i}$. Then $a^* = -2\mathrm{i}$ and $(a^*)^* = 2\mathrm{i} = a$.

Recall[365] that $a + b$ and $a - b$ were called a **conjugate pair** because

$$(a + b)(a - b) \stackrel{1}{=} a^2 - b^2.$$

Recall also that when a denominator contained a surd, we could often use $\stackrel{1}{=}$ to **rationalise** ("make rational") the denominator:

Example 1027. $\dfrac{3}{1+\sqrt{5}} = \dfrac{3}{1+\sqrt{5}}\dfrac{1-\sqrt{5}}{1-\sqrt{5}} = \dfrac{3\left(1-\sqrt{5}\right)}{1^2-5^2} = \dfrac{3\left(1-\sqrt{5}\right)}{-4} = \dfrac{3}{4}\left(\sqrt{5}-1\right)$.

Here we can play a similar trick. Observe that if $z = a + ib$, then

$$zz^* = (a + \mathrm{i}b)(a - \mathrm{i}b) \stackrel{2}{=} a^2 - (\mathrm{i}b)^2 = a^2 - \mathrm{i}^2b^2 \stackrel{2}{=} a^2 + b^2.$$

We can often use $\stackrel{2}{=}$ to **realise** ("make real") a denominator that contains a complex number:

[365] Ch. 5.6.

Example 1028. Let $z = 1 + \mathrm{i}$. Consider the reciprocal of z:

$$\frac{1}{z} = \frac{1}{1+\mathrm{i}}.$$

In general, it is easier to deal with "simpler" denominators. We might thus like to rid the above denominator of any complex numbers.

To do so, simply multiply by $z^*/z^* = 1$:

$$\frac{1}{z} = \frac{1}{z}\frac{z^*}{z^*} = \frac{1}{1+\mathrm{i}}\frac{\overbrace{1-\mathrm{i}}^{z^*}}{1-\mathrm{i}} \overset{2}{=} \frac{1-\mathrm{i}}{1^2+1^2} = \frac{1-\mathrm{i}}{2} = \frac{1}{2} - \frac{1}{2}\mathrm{i}.$$

Fact 186. *If $z = a + \mathrm{i}b = (a, b)$, then*

(a) $zz^* = a^2 + \mathrm{i}b^2 = \left(a^2, b^2\right)$; *and* **(b)** $\dfrac{1}{z} = \dfrac{z^*}{|z|^2} = \dfrac{1}{a^2+b^2}(a, -b)$.

Proof. See Exercise 306. □

A few more examples:

Example 1029. Let $z = -3 + 5\mathrm{i} = (-3, 5)$. Then $z^* = -3 - 5\mathrm{i} = (-3, -5)$ and

$$\frac{1}{z} = \frac{z^*}{3^2+5^2} = \frac{z^*}{34}^* = \frac{-3-5\mathrm{i}}{34} = -\frac{3}{34} - \frac{5}{34}\mathrm{i} = \frac{1}{34}(-3, -5).$$

Example 1030. Let $w = 1 - \mathrm{i} = (1, -1)$. Then $w^* = 1 + \mathrm{i} = (1, 1)$ and

$$\frac{1}{w} = \frac{w^*}{1^2+1^2} = \frac{w^*}{2} = \frac{1+\mathrm{i}}{2} = \frac{1}{2} + \frac{1}{2}\mathrm{i} = \frac{1}{2}(1, 1).$$

Example 1031. Let $\omega = 1 + \mathrm{i} = (1, 1)$. Then $\omega^* = 1 - \mathrm{i} = (1, -1)$ and

$$\frac{1}{\omega} = \frac{1}{1^2+1^2}\omega^* = \frac{1-\mathrm{i}}{2} = \frac{1}{2} - \frac{1}{2}\mathrm{i} = \frac{1}{2}(1, -1).$$

Exercise 305. In each, express z's conjugate and reciprocal in the form $a + \mathrm{i}b$.

(a) $z = -5 + 2\mathrm{i}$. **(b)** $z = 3 - \mathrm{i}$. **(c)** $z = 1 + 2\mathrm{i}$. (Answer on p. 1884.)

Exercise 306. Prove Fact 186. (Answer on p. 1884.)

Remark 119. Just so you know, most writers denote the conjugate of z by $\overline{z}$. However, your A-Level examiners use z^* and so that's what we'll do too.

80.4. Division

We can now divide one complex number by another:

Fact 187. *If* $z = a + \mathrm{i}b = (a, b)$ *and* $w = c + \mathrm{i}d = (c, d)$ *with* $w \neq 0$, *then*

$$\frac{z}{w} = \frac{zw^*}{|w|^2} = \frac{1}{c^2 + d^2}(ac + bd, bc - ad).$$

Proof. $\dfrac{z}{w} = \dfrac{z}{w}\dfrac{w^*}{w^*} = \dfrac{zw^*}{c^2 + d^2} = \dfrac{1}{c^2 + d^2}(ac + bd, bc - ad)$. □

Example 1032. Let $z = -2 + \mathrm{i}$ and $w = 3\mathrm{i}$. Then

$$\frac{z}{w} = \frac{-2 + \mathrm{i}}{3\mathrm{i}} = \frac{zw^*}{0^2 + 3^2} = \frac{(-2 + \mathrm{i})(-3\mathrm{i})}{9} = \frac{3 + 6\mathrm{i}}{9} = \frac{1}{3} + \frac{2}{3}\mathrm{i}.$$

Example 1033. Let $z = 3 + \mathrm{i}$ and $w = 1 - \mathrm{i}$. Then

$$\frac{z}{w} = \frac{3 + \mathrm{i}}{1 - \mathrm{i}} = \frac{zw^*}{1^2 + 1^2} = \frac{(3 + \mathrm{i})(1 + \mathrm{i})}{2} = \frac{2 + 4\mathrm{i}}{2} = 1 + 2\mathrm{i}.$$

Example 1034. Let $z = 1 + \mathrm{i}$ and $w = 3 - 2\mathrm{i}$. Then

$$\frac{z}{w} = \frac{1 + \mathrm{i}}{3 - 2\mathrm{i}} = \frac{zw^*}{3^2 + 2^2} = \frac{(1 + \mathrm{i})(3 + 2\mathrm{i})}{13} = \frac{1 + 5\mathrm{i}}{13} = \frac{1}{13} + \frac{5}{13}\mathrm{i}.$$

Example 1035. Let $z = 2 - \mathrm{i}$ and $w = -1 + \mathrm{i}$. Then

$$\frac{z}{w} = \frac{2 - \mathrm{i}}{-1 + \mathrm{i}} = \frac{zw^*}{1^2 + 1^2} = \frac{(2 - \mathrm{i})(-1 - \mathrm{i})}{2} = \frac{-3 - \mathrm{i}}{2} = -\frac{3}{2} - \frac{1}{2}\mathrm{i}.$$

Example 1036. Let $z = 3 + 2\mathrm{i}$ and $w = -7 + 4\mathrm{i}$. Then

$$\frac{z}{w} = \frac{3 + 2\mathrm{i}}{-7 + 4\mathrm{i}} = \frac{zw^*}{7^2 + 4^2} = \frac{(3 + 2\mathrm{i})(-7 - 4\mathrm{i})}{65} = \frac{-13 - 26\mathrm{i}}{65} = -\frac{1}{5} - \frac{2}{5}\mathrm{i}.$$

Example 1037. Let $z = -3 + 6\mathrm{i}$ and $w = 2 + \mathrm{i}\pi$. Then

$$\frac{z}{w} = \frac{-3 + 6\mathrm{i}}{2 + \mathrm{i}\pi} = \frac{zw^*}{2^2 + \pi^2} = \frac{(-3 + 6\mathrm{i})(2 - \mathrm{i}\pi)}{4 + \pi^2} = \frac{-6 + 3\mathrm{i}\pi + 12\mathrm{i} + 6\pi}{4 + \pi^2} = 6\frac{\pi - 1}{4 + \pi^2} + 3\frac{\pi + 4}{4 + \pi^2}\mathrm{i}.$$

Exercise 307. Express z/w in the form $a + \mathrm{i}b$. (Answer on p. 1884.)

(a) $z = 1 + 3\mathrm{i}$, $w = -\mathrm{i}$. **(b)** $z = 2 - 3\mathrm{i}$, $w = 1 + \mathrm{i}$. **(c)** $z = \sqrt{2} - \pi i$, $w = 3 - \sqrt{2}\mathrm{i}$.
(d) $z = 11 + 2\mathrm{i}$, $w = \mathrm{i}$. **(e)** $z = -3$, $w = 2 + \mathrm{i}$. **(d)** $z = 7 - 2i$, $w = 5 + \mathrm{i}$.

81. Solving Polynomial Equations

Recall (Ch. 14) that if the quadratic equation $ax^2 + bx + c = 0$ has non-negative discriminant (i.e. $b^2 - 4ac \geq 0$), then it has two **real** roots, which are given by

$$x \overset{1}{=} \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Now that we've learnt a little about complex numbers, we can more simply say that regardless of the sign of the discriminant:

Fact 188. *Every quadratic equation has two complex roots given by* $\overset{1}{=}$.

Proof. See Theorem 20 below (the Fundamental Theorem of Algebra). □

Example 1038. Consider the quadratic equation $x^2 - 2x + 2 = 0$.

Its discriminant is negative: $b^2 - 4ac = (-2)^2 - 4(1)(2) = -4 < 0$.

Nonetheless, like every quadratic equation, it has two complex roots:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm \frac{\sqrt{4} \times \sqrt{-1}}{2} = 1 \pm \frac{2\mathrm{i}}{2} = 1 \pm \mathrm{i}.$$

In this case, both roots are non-real.

Example 1039. The quadratic equation $x^2 - 3x + 2 = 0$ has positive discriminant: $b^2 - 4ac = (-3)^2 - 4(1)(2) = 1 > 0$. Thus, both of its complex roots are real:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{1}}{2} = 1, 2.$$

Example 1040. The quadratic equation $x^2 - 2x + 1 = 0$ has discriminant zero: $b^2 - 4ac = (-2)^2 - 4(1)(1) = 0$. As usual, its two roots are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{0}}{2} = 1.$$

Hmm ... this time there's only one root, namely 1. Doesn't this contradict Fact 188?

Well, here we'll cheat a little, by calling 1 a **repeated** or **double root** of the quadratic equation $x^2 - 2x + 1 = 0$. You can think of this as a sort of cheap accounting trick to ensure that Fact 188 (and later on also Theorem 20) are "correct".[366]

Exercise 308. Solve each equation. (Answer on p. 1885.)

(a) $x^2 + x + 1 = 0$. **(b)** $x^2 + 2x + 2 = 0$. **(c)** $3x^2 + 3x + 1 = 0$.

[366] This is a somewhat simplistic explanation. Repeated or multiple roots actually have greater significance than merely ensuring Fact 188 or Theorem 20 holds.

81.1. The Fundamental Theorem of Algebra

By Fact 188, every quadratic equation has two (possibly repeated) complex roots. It turns out this is more generally true: Every nth-degree polynomial equation in one variable has n (possibly repeated) roots. This is the **Fundamental Theorem of Algebra (FTA)**:

Theorem 20. (The Fundamental Theorem of Algebra) *Suppose $a_0 \neq 0$. Then the following equation has n (possibly repeated) roots:*

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0.$$

Proof. Omitted.[367] □

Example 1041. The 2nd-degree polynomial (or quadratic) equation $x^2 - 1 = 0$ has two roots, namely 1 and -1.

Example 1042. The 2nd-degree polynomial (or quadratic) equation $x^2 + 1 = 0$ has two roots, namely i and $-i$.

Example 1043. By the FTA, the 3rd-degree polynomial equation (or cubic equation) $x^3 - 8 = 0$ has three roots. Let's find them using what we learnt in Ch. 38.

Observe that $2^3 - 8 = 0$. So one root is 2 and $x - 2$ is a factor of $x^3 - 8$.

To find the other two factors, write

$$x^3 - 8 = (x-2)\left(ax^2 + bx + c\right) = ax^3 + (b - 2a)\,x^2 + ?x - 2c,$$

Comparing coefficients, we have $a = 1$, $b = 2$, and $c = 4$. (Note that ? stands for a coefficient we don't bother to calculate because it isn't necessary. We have three unknowns a, b, and c; and so, it is only necessary to compute three of these coefficients.) Thus,

$$x^3 - 8 = (x-2)\left(x^2 + 2x + 4\right).$$

We can then further factorise $x^2 + 2x + 4$ using the usual quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2 \cdot 1} = -1 \pm \sqrt{3}\mathrm{i}.$$

Altogether then, the three roots of the 3rd-degree polynomial equation $x^3 - 8 = 0$ are

$$2, \qquad -1 + \sqrt{3}\mathrm{i}, \qquad -1 - \sqrt{3}\mathrm{i}.$$

And here is the cubic polynomial $x^3 - 8$ factorised into its three linear factors:

$$x^3 - 8 = (x-2)\left(x + 1 - \sqrt{3}\mathrm{i}\right)\left(x + 1 + \sqrt{3}\mathrm{i}\right).$$

Exercise 309. Verify that $-1 \pm \sqrt{3}\mathrm{i}$ solve $x^3 - 8 = 0$. (Answer on p. 1885.)

[367] See e.g. Schilling, Lankham, & Nachtergaele (2016, Ch. 3).

As noted earlier, there may sometimes be repeated or multiple roots:

Example 1044. The 2nd-degree polynomial (or quadratic) equation

$$x^2 - 2x + 1 = (x-1)^2 = 0$$

has two repeated or multiple roots, namely 1 and 1.

Example 1045. The 3rd-degree polynomial (or cubic) equation

$$x^3 - 6x^2 + 12x - 8 = (x-2)^2 = 0$$

has three repeated or multiple roots, namely 2, 2, and 2.

The FTA can be useful even if we have no idea how to solve an equation.

Example 1046. We may have no idea how to solve the 17th-degree polynomial equation

$$x^{17} + 3x^4 - 2x + 1 = 0.$$

Nonetheless, the FTA gives us a useful piece of information, namely that this equation *must* have 17 roots or solutions (though some may possibly be repeated).

Exercise 310. Solve $x^3 + 64 = 0$. (Answer on p. 1885.)

Exercise 311. You're told that 1 solves both of the equations given below. Find the other roots of each equation. (Answer on p. 1885.)

(a) $x^3 + x^2 - 2 = 0$. **(b)** $x^4 - x^2 - 2x + 2 = 0$.

81.2. The Complex Conjugate Root Theorem

Example 1047. The equation $x^2 - 2x + 2 = 0$ has roots $1 + \mathrm{i}$ and $1 - \mathrm{i}$.

Example 1048. The equation $7x^2 + x + 1 = 0$ has roots $-\frac{1}{14} + \frac{3\sqrt{3}}{14}\mathrm{i}$ and $-\frac{1}{14} - \frac{3\sqrt{3}}{14}\mathrm{i}$.

The above examples suggest that if $z = p + \mathrm{i}q$ solves the quadratic equation $ax^2 + bx + c = 0$, then so too does its conjugate $z^* = c - \mathrm{i}d$. It turns out that this is generally true of any polynomial equation, *provided the coefficients are real*:

Theorem 21. (Complex Conjugate Root Theorem.) *Suppose $c_0, c_1, \ldots, c_n \in \mathbb{R}$. If $z = a + \mathrm{i}b$ solves $c_n x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0 = 0$, then so does $z^* = a - \mathrm{i}b$.*

Proof. See p. 1646 (Appendices). □

Example 1049. If $z = 2 - \mathrm{i}$ solves $x^3 - x^2 - 7x + 15 = 0$, then by Theorem 21, the conjugate $z^* = 2 + \mathrm{i}$ also solves the same equation.

Example 1050. If both i and $0.5\mathrm{i}$ solve $4x^4 + 5x^2 + 1 = 0$, then by Theorem 21, the conjugates $-\mathrm{i}$ and $-0.5\mathrm{i}$ also solve the same equation.

Example 1051. If $-1 + 2\mathrm{i}$ is a root of $x^3 - 3x^2 - 5x - 25 = 0$, then what are the other two?

Well, by Theorem 21, we know that $-1 - 2\mathrm{i}$ must be another root.

So, both $x - (-1 + 2\mathrm{i}) = x + 1 - 2\mathrm{i}$ and $x - (-1 - 2\mathrm{i}) = x + 1 + 2\mathrm{i}$ are factors of $x^3 - 3x^2 - 5x - 25$.

Compute $$(x + 1 - 2\mathrm{i})(x + 1 + 2\mathrm{i}) = (x + 1)^2 - (2\mathrm{i})^2 = x^2 + 2x + 5.$$

Now write, $$x^3 - 3x^2 - 5x - 25 = \left(x^2 + 2x + 5\right)(ax + b) = ax^3 + ?x^2 + ?x + 5b.$$

Comparing coefficients, we have $a = 1$ and $b = -5$. Thus,

$$x^3 - 3x^2 - 5x - 25 = \left(x^2 + 2x + 5\right)(x - 5) = (x + 1 - 2\mathrm{i})(x + 1 + 2\mathrm{i})(x - 5).$$

Altogether then, the three roots of the cubic equation $x^3 - 3x^2 - 5x - 25 = 0$ are

$$-1 + 2\mathrm{i}, \qquad -1 - 2\mathrm{i}, \qquad 5.$$

Example 1052. Let $p, q \in \mathbb{R}$. If $3 + 2\mathrm{i}$ solves $x^2 + px + q = 0$, then what are p and q?

Well, by Theorem 21, $3 - 2\mathrm{i}$ also solves this equation. Thus,

$$x^2 + px + q = [x - (3 + 2\mathrm{i})][x - (3 - 2\mathrm{i})] = (x - 3)^2 - (2\mathrm{i})^2 = x^2 - 6x + 13.$$

Comparing coefficients, we have $p = -6$ and $q = 13$.

Example 1053. We're given that i solves $x^4 + x^3 - 5x^2 + x - 6 = 0$. What are the other three roots?

Well, by Theorem 21, we know that $-\mathrm{i}$ must be another root.

So, both $x - \mathrm{i}$ and $x + \mathrm{i}$ are factors of $x^4 + x^3 - 5x^2 + x - 6$.

Compute $$(x - \mathrm{i})(x + \mathrm{i}) = x^2 - \mathrm{i}^2 = x^2 + 1.$$

Now, $$x^4 + x^3 - 5x^2 + x - 6 = \left(x^2 + 1\right)\left(ax^2 + bx + c\right) = ax^4 + bx^3 + ?x^2 + ?x + c.$$

Comparing coefficients, we have $a = 1$, $b = 1$, and $c = -6$. Thus,

$$x^4 + x^3 - 5x^2 + x - 6 = \left(x^2 + 1\right)\left(x^2 + x - 6\right).$$

By the quadratic formula or otherwise, we have

$$x^2 + x - 6 = (x + 3)(x - 2).$$

Altogether then, $$x^4 + x^3 - 5x^2 + x - 6 = (x - \mathrm{i})(x + \mathrm{i})(x + 3)(x - 2).$$

And the four roots of the quartic equation $x^4 + x^3 - 5x^2 + x - 6 = 0$ are

$$\mathrm{i}, \qquad -\mathrm{i}, \qquad -3, \qquad 2.$$

By the way, the condition that all coefficients $c_0, c_1, \ldots, c_n$ in the polynomial equation are real is important. If this condition is violated, then the Theorem's conclusion may not hold:

Example 1054. We are told that $-2 + \mathrm{i}$ solves $x^2 - (5 + 4\mathrm{i})x + (-17 + \mathrm{i}) \overset{1}{=} 0$.

Observe that not all of the coefficients in $\overset{1}{=}$ are real. And so, Theorem 21's conclusion may not hold.

And indeed, it does not. As you should verify yourself, the conjugate $-2 - \mathrm{i}$ does not solve $\overset{1}{=}$. Instead, the other solution to $\overset{1}{=}$ is $7 + 3\mathrm{i}$.

Exercise 312. You're told that $2 - 3\mathrm{i}$ solves both of the equations below. Find the other roots of each equation. (Answer on p. 1886.)

(a) $x^4 - 6x^3 + 18x^2 - 14x - 39 = 0$. **(b)** $-2x^4 + 21x^3 - 93x^2 + 229x - 195 = 0$.

Exercise 313. Let $p, q \in \mathbb{R}$. If $1 - \mathrm{i}$ solves $x^2 + px + q = 0$, then what are p and q? (Answer on p. 1886.)

82. The Argand Diagram

In Ch. 7.2 (and also earlier in secondary school), we learnt that ordered pairs of real numbers can be depicted geometrically as points on the **cartesian plane**, where the horizontal or x-axis corresponds to the first coordinate of an ordered pair, while the vertical or y-axis corresponds to the second.

We just learnt that the complex number $z = a + ib$ may also be written as an ordered pair:

$$z = (a, b) .$$

And so, we shall *also* depict complex numbers geometrically as **points** on the the **complex plane** or **Argand diagram**. The horizontal or x-axis is now our **real axis**, while the vertical axis is now our **imaginary axis**.

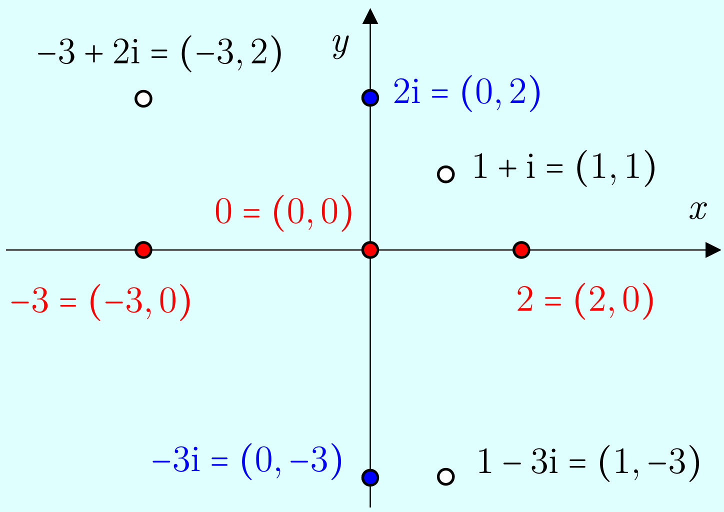
Example 1055. The Argand diagram below depicts seven complex numbers.

The real numbers $-3 = (-3, 0)$ and $2 = (2, 0)$ are on the horizontal axis.

The imaginary numbers $2\mathrm{i} = (0, 2)$ and $-3\mathrm{i} = (0, -3)$ are on the vertical axis.

The number $0 = (0, 0)$ is *both* real and imaginary. And sure enough, it is on *both* the horizontal and vertical axes.

The impure imaginary numbers $-3 + 2\mathrm{i} = (-3, 2)$, $1 + \mathrm{i} = (1, 1)$, and $1 - 3\mathrm{i} = (1, -3)$ are not on either axis.



Exercise 314. Draw 2, -1, $2\mathrm{i}$, $1+2\mathrm{i}$, $-1-3\mathrm{i}$ on an Argand diagram. (Answer on p. 1886.)

Remark 120. Both of the following mathematical objects can be depicted on a plane.

- The **set of complex numbers** or the **complex plane** or **Argand diagram** $\mathbb{C} = \{a + \mathrm{i}b : a, b \in \mathbb{R}\}$; and
- The **set of ordered pairs of real numbers** or the **cartesian plane** $\{(x, y) : x, y \in \mathbb{R}\}$.

However, you should be aware that the complex plane and cartesian are **different mathematical objects**. Don't worry, you need merely be aware that they are different; you needn't know what exactly the differences are.

83. Complex Numbers in Polar Form

So far, we've written complex numbers in **cartesian** (or **standard** or **rectangular**) **form**, i.e. as either

$$z = a + \mathrm{i}b \qquad \text{or} \qquad z = (a, b).$$

In this chapter, we'll learn to write down a complex number in **polar form**. (And in the next chapter, we'll learn how to do so in **exponential form**.)

To write down $z = a + \mathrm{i}b = (a, b)$ in cartesian form, we need two pieces of information:

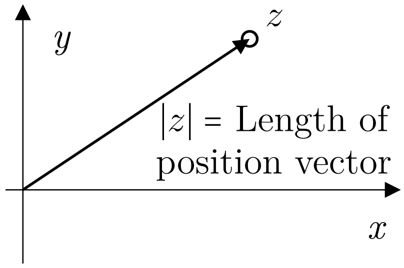
its **real part** $\operatorname{Re} z = a$; and its **imaginary part** $\operatorname{Im} z = b$.

To write down z in polar form, we likewise need two pieces of information:

its **modulus**, denoted $|z|$; and its **argument**, denoted $\arg z$.

83.1. The Modulus

We just learnt to depict a complex number z on an Argand diagram. Now consider z's position vector (i.e. the vector from 0 to the point that is z). The **modulus** of z, denoted $|z|$, is simply the **magnitude** or **length** of its position vector:



Definition 193. The *modulus* of the complex number $z = a + ib$ is denoted $|z|$ and defined as this number:

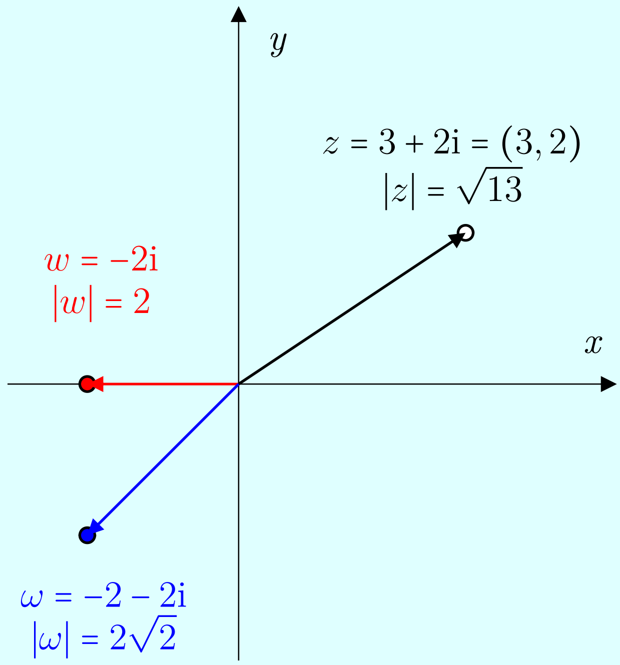
$$|z| = \sqrt{a^2 + b^2}.$$

Or equivalently,
$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}.$$

Example 1056. Let

$$z = 3 + 2\text{i} = (3, 2)$$
$$w = -2\text{i} = (0, -2),$$
$$\omega = -2 - 2\text{i} = (-2, -2).$$

Then
$$|z| = \sqrt{3^2 + 2^2} = \sqrt{13},$$
$$|w| = \sqrt{0^2 + (-2)^2} = 2,$$
$$|\omega| = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}.$$



Exercise 315. Compute the moduli of these numbers: (Answer on p. 1887.)

$$2, \qquad -1, \qquad 2\text{i}, \qquad 1 + 2\text{i}, \qquad -1 - 3\text{i}.$$

83.2. The Argument: An Informal Introduction

The **argument** of a complex number z is denoted $\arg z$.

Informally, it is the angle that z's position vector makes with the positive x-axis, with some qualifications:

- If z is on or above the x-axis (i.e. $\operatorname{Im} z \geq 0$), then $\arg z \in [0, \pi]$.
- If z is below the x-axis (i.e. $\operatorname{Im} z < 0$), then $\arg z \in (-\pi, 0)$.

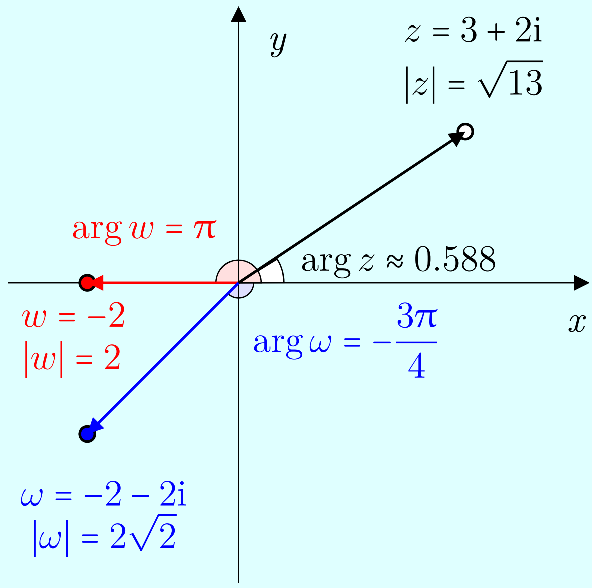
Example 1057. Let $z = 3 + 2\text{i} = (3, 2)$, $w = -2\text{i} = (0, -2)$, and $\omega = -2 - 2\text{i} = (-2, -2)$.

Then $\arg z = \tan^{-1} \dfrac{2}{3} \approx 0.588$, $\arg w = \pi$, and $\arg \omega = -\dfrac{3\pi}{4}$.

Notice $\arg z$ and $\arg w$ are measured **anti-clockwise** from the positive x-axis, while $\arg \omega$ is measured **clockwise** from the positive x-axis. We'll adopt these informal rules:

- If z is on or above the x-axis (i.e. $\operatorname{Im} z \geq 0$), then $\arg z \in [0, \pi]$.
- If z is below the x-axis (i.e. $\operatorname{Im} z < 0$), then $\arg z \in (-\pi, 0)$.

Altogether, for any $z \neq 0$, we have $\arg z \in (-\pi, \pi]$. (We'll leave $\arg 0$ undefined.)



Exercise 316. Find the arguments of 2, -1, 2i, $1 + 2\text{i}$, and $-1 - 3\text{i}$.(Answer on p. 1887.)

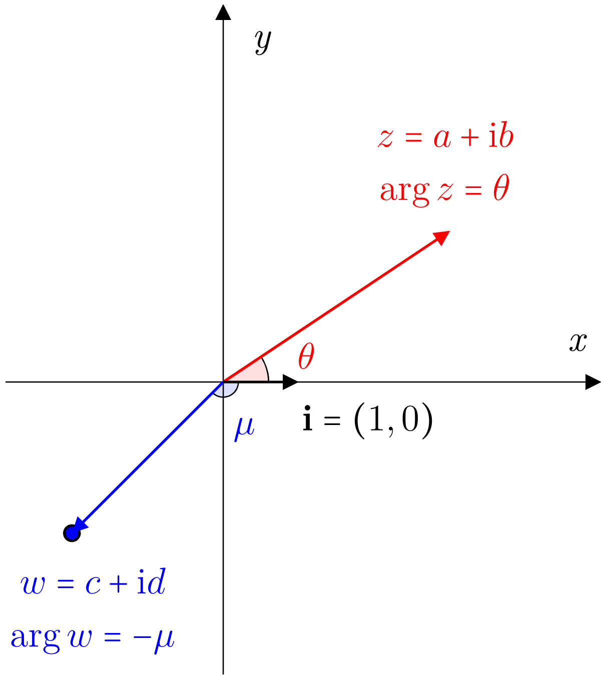
Exercise 317. Depict $z = 2 - \text{i}$ and $w = -3 + 2\text{i}$ on an Argand diagram. What are moduli and arguments? (Answer on p. 1887.)

83.3. The Argument: Formally Defined

We now work towards a formal definition of the argument. We'll do so using what we learnt about vectors.

Let $z = (a, b)$ and $w = (c, d)$ be non-zero complex numbers, where z is on or above the x-axis (i.e. $b \geq 0$), while w is below it (i.e. $d < 0$).

Let $\mathbf{z} = (a, b)$ and $\mathbf{w} = (c, d)$ be the corresponding position vectors. Let $\mathbf{i} = (1, 0)$ be the unit vector that points in the direction of the positive x-axis.



Let θ be the angle between $\mathbf{z}$ and $\mathbf{i}$; and μ be the angle between $\mathbf{w}$ and $\mathbf{i}$. Following our informal discussion on the previous page, we want

$$\arg z = \theta \qquad \text{and} \qquad \arg w = -\mu.$$

Recall (Definition 145 in Ch. 145) that the angle between $\mathbf{z}$ and $\mathbf{i}$ is

$$\theta = \cos^{-1}\frac{\mathbf{z}\cdot\mathbf{i}}{|\mathbf{z}|\,|\mathbf{i}|} = \cos^{-1}\frac{(a,b)\cdot(1,0)}{|(a,b)|\,|(1,0)|} = \cos^{-1}\frac{a+0}{\sqrt{a^2+b^2}\cdot 1} = \cos^{-1}\frac{a}{\sqrt{a^2+b^2}} = \cos^{-1}\frac{a}{|z|}.$$

Similarly, the angle between $\mathbf{w}$ and $\mathbf{i}$ is

$$\mu = \cos^{-1}\frac{\mathbf{w}\cdot\mathbf{i}}{|\mathbf{w}|\,|\mathbf{i}|} = \cos^{-1}\frac{(c,d)\cdot(1,0)}{|(c,d)|\,|(1,0)|} = \cos^{-1}\frac{c+0}{\sqrt{c^2+d^2}\cdot 1} = \cos^{-1}\frac{c}{\sqrt{c^2+d^2}} = \cos^{-1}\frac{c}{|w|}.$$

Thus, $$\arg z = \theta = \cos^{-1}\frac{a}{|z|} \qquad \text{and} \qquad \arg w = -\mu = -\cos^{-1}\frac{c}{|w|}.$$

The above discussion motivates this formal definition of a complex number's **argument**:

Definition 194. Let $z = a + ib \neq 0$. The *argument* of z is denoted $\arg z$ and is the number defined by[368]

$$\arg z \stackrel{1}{=} \begin{cases} \cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}, & \text{if } b \geq 0, \\[2ex] -\cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}, & \text{if } b < 0. \end{cases}$$

Or equivalently, $$\arg z = \begin{cases} \cos^{-1}\dfrac{\operatorname{Re} z}{|z|}, & \text{if } \operatorname{Im} z \geq 0, \\[2ex] -\cos^{-1}\dfrac{\operatorname{Re} z}{|z|}, & \text{if } \operatorname{Im} z < 0. \end{cases}$$

[368] We could also think of $\arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ as the function with the mapping rule $\stackrel{1}{=}$.

As previously noted, $\arg 0$ is undefined.

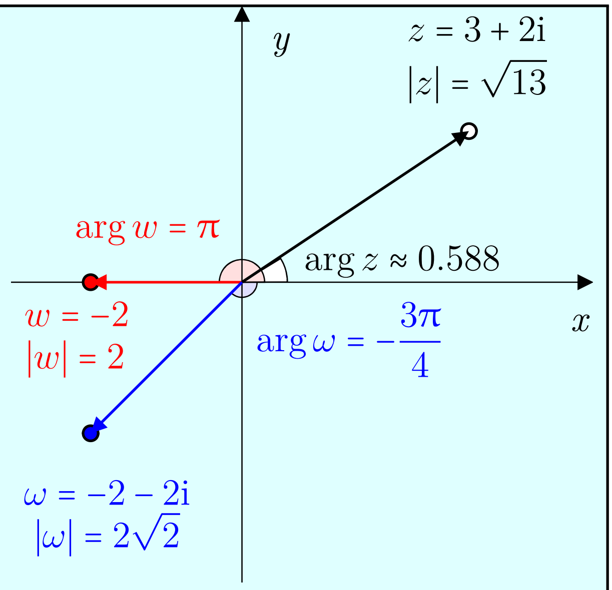
Example 1058. Let $z = 3 + 2\mathrm{i} = (3, 2)$, $w = -2 = (-2, 0)$, and $\omega = -2 - 2\mathrm{i} = (-2, -2)$.

We have $|z| = \sqrt{13}$, $|w| = 2$, and $|\omega| = 2\sqrt{2}$. And so by Definition 194:

$$\arg z = \cos^{-1}\frac{\operatorname{Re} z}{|z|} = \cos^{-1}\frac{3}{\sqrt{13}} \approx 0.588,$$

$$\arg w = \cos^{-1}\frac{\operatorname{Re} w}{|w|} = \cos^{-1}\frac{-2}{2} = \pi,$$

$$\arg \omega = -\cos^{-1}\frac{\operatorname{Re} \omega}{|\omega|} = -\cos^{-1}\frac{-2}{2\sqrt{2}} = -\frac{3\pi}{4}.$$



Note that there's a negative sign before arccosine for ω (because ω is below the x-axis). In contrast, there isn't one for either z or w (because they are on or above the x-axis).

The next result is immediate from the Argand diagram and Definition 194:

Fact 189. *Let z be a non-zero complex number. Then*

(a) *z is imaginary* $\iff$ *z is on the y-axis* $\iff$ $\arg z = \pm\frac{\pi}{2}$.

(b) *z is a positive real number* $\iff$ *z is on the positive x-axis* $\iff$ $\arg z = 0$.

(c) *z is a negative real number* $\iff$ *z is on the negative x-axis* $\iff$ $\arg z = \pi$.

Exercise 318. Find each number's argument, but this time using Definition 194. (Check that your answers are the same as before.) (Answer on p. 1887.)

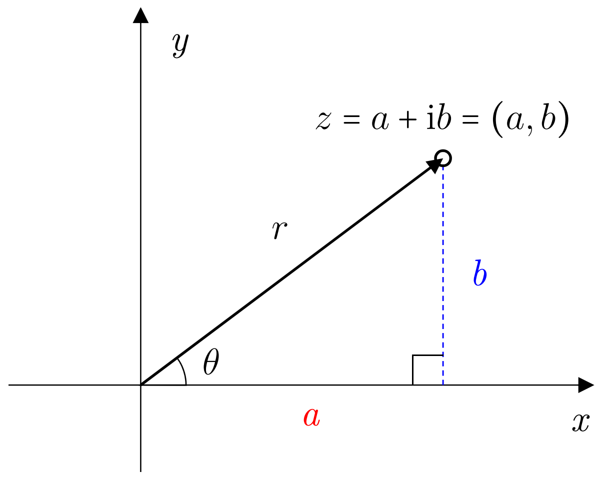
$$2, \quad -1, \quad 2\mathrm{i}, \quad 1 + 2\mathrm{i}, \quad -1 - 3\mathrm{i}, \quad z = 2 - \mathrm{i}, \quad w = -3 + 2\mathrm{i}.$$

Remark 121. We've learnt two methods for computing the argument of a complex number. The first may be called the "look at the graph and use arctangent". The second simply uses Definition 194. I personally do not find the second difficult to remember. And so, going forward in this textbook, that's what I'll use. But you should use whichever you find is easier for you.

83.4. Complex Numbers in Polar Form

Suppose $z = a + ib = (a, b)$ is a complex number with $a, b > 0$. Let $\theta = \arg z$ and $r = |z|$. Then

$$\cos\theta = \frac{\text{A}}{\text{H}} = \frac{a}{r} \qquad \text{and} \qquad \sin\theta = \frac{\text{O}}{\text{H}} = \frac{b}{r}.$$



Rearranging, $a = r\cos\theta$ and $b = r\sin\theta$.

Thus, we may also write z in **polar form**:

$$z = a + ib = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

Let's jot this down as a formal result:

Fact 190. *Suppose z is a non-zero complex number with $|z| = r$ and $\arg z = \theta$. Then*

$$z = r(\cos\theta + i\sin\theta).$$

Proof. We already proved this result above, but only in the case where both a and b are positive. For a complete proof, see p. 1646 (Appendices). □

Example 1059. Let $z = 5 - 2i = (5, -2)$. Then

$$r = |z| = \sqrt{5^2 + (-2)^2} = \sqrt{29} \qquad \text{and} \qquad \theta = \arg z = -\cos^{-1}\frac{5}{\sqrt{29}} \approx -0.381.$$

So, z may also be written in polar form as

$$z = r(\cos\theta + i\sin\theta) \approx \sqrt{29}\,(\cos -0.381 + i\sin -0.381).$$

Example 1060. Let $z = 1 + 3i = (1, 3)$. Then

$$r = |z| = \sqrt{1^2 + 3^2} = \sqrt{10} \qquad \text{and} \qquad \theta = \arg z = \cos^{-1}\frac{1}{\sqrt{10}} \approx 0.322.$$

Thus, $$z = r(\cos\theta + i\sin\theta) \approx \sqrt{10}\,(\cos 0.322 + i\sin 0.322).$$

Example 1061. Let $z = -4 + 7i = (-4, 7)$. Then

$$r = |z| = \sqrt{(-4) + 7^2} = \sqrt{65} \qquad \text{and} \qquad \theta = \arg z = \cos^{-1}\frac{-4}{\sqrt{65}} \approx 2.090.$$

Thus, $$z = r(\cos\theta + i\sin\theta) \approx \sqrt{65}\,(\cos 2.090 + i\sin 2.090).$$

Exercise 319. Rewrite each complex number in polar form. (Hint: We already computed their moduli and arguments in earlier exercises.) (Answer on p. 1887.)

$$2, \quad -1, \quad 2i, \quad 1 + 2i, \quad -1 - 3i, \quad z = 2 - i, \quad w = -3 + 2i.$$

84. Complex Numbers in Exponential Form

In this chapter, we introduce **Euler's Formula**, then use Euler's Formula to write down complex numbers in **exponential form**.

Theorem 22. (Euler's Formula) *Suppose* $\theta \in \mathbb{R}$. *Then* $e^{i\theta} = \cos\theta + i\sin\theta$.

Proof. See p. 1647 (Appendices). □

Richard Feynman called the above "the most remarkable formula in mathematics".[369]

Plug $\theta = \pi$ into Euler's Formula to get

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0 = -1.$$

Rearrange to get **Euler's Identity**:

Corollary 37. (Euler's Identity) $e^{i\pi} + 1 = 0$.

Euler's Identity is one of the most extraordinary and beautiful equations in all of mathematics. It links together five of the most fundamental mathematical constants:

$$e, \quad i, \quad \pi, \quad 1, \quad \text{and} \quad 0.$$

Fun Fact

Leonhard Euler (1707–83) was a stud. There are so many mathematical results and objects named after him that there is even a Wikipedia entry just to list the things that have been named after him!

This can sometimes result in confusion. For example, what we call **Euler's Formula** is called **Euler's Identity** by others and vice versa.

As another example, **Euler's number** $e = 2.718\ldots$ is different from **Euler's constant** $\gamma = 0.577\ldots$

Even if we ignored his output before he turned blind at around 60, he'd still figure amongst the greatest mathematicians in history.

[369] *The Feynman Lectures on Physics* (1964, p. 22–10).

84.1. Complex Numbers in Exponential Form

Let z be a non-zero complex number with $r = |z|$ and $\theta = \arg z$. Then by Fact 190,

$$z \stackrel{1}{=} r(\cos\theta + \mathrm{i}\sin\theta).$$

By Euler's Formula, $\cos\theta + \mathrm{i}\sin\theta \stackrel{2}{=} \mathrm{e}^{\mathrm{i}\theta}$. Now plug $\stackrel{2}{=}$ into $\stackrel{1}{=}$ to get z in **exponential form**:

$$z = r\mathrm{e}^{\mathrm{i}\theta}.$$

Let's jot this down as a formal result:

Fact 191. *Suppose z is a non-zero complex number with $r = |z|$ and $\theta = \arg z$. Then*

$$z = r\mathrm{e}^{\mathrm{i}\theta}.$$

Example 1062. Let $z = 5 - 2\mathrm{i} = (5, -2)$. Then

$$r = |z| = \sqrt{5^2 + (-2)^2} = \sqrt{29} \qquad \text{and} \qquad \theta = \arg z = -\cos^{-1}\frac{5}{\sqrt{29}} \approx -0.381.$$

So, we can write z in exponential form as

$$z = r\mathrm{e}^{\mathrm{i}\theta} = \sqrt{29}\mathrm{e}^{-0.381\mathrm{i}}.$$

Example 1063. Let $z = 1 + 3\mathrm{i} = (1, 3)$. Then

$$r = |z| = \sqrt{1^2 + 3^2} = \sqrt{10} \qquad \text{and} \qquad \theta = \arg z = \cos^{-1}\frac{1}{\sqrt{10}} \approx 0.322.$$

Thus,
$$z = r\mathrm{e}^{\mathrm{i}\theta} = \sqrt{10}\mathrm{e}^{0.322\mathrm{i}}.$$

Example 1064. Let $z = -4 + 7\mathrm{i} = (-4, 7)$. Then

$$r = |z| = \sqrt{(-4) + 7^2} = \sqrt{65} \qquad \text{and} \qquad \theta = \arg z = \cos^{-1}\frac{-4}{\sqrt{65}} \approx 2.090.$$

Thus,
$$z = r\mathrm{e}^{\mathrm{i}\theta} = \sqrt{65}\mathrm{e}^{2.090\mathrm{i}}.$$

Exercise 320. Rewrite each complex number in polar form. (Hint: We already computed their moduli and arguments in earlier exercises.) (Answer on p. 1887.)

$$2, \quad -1, \quad 2\mathrm{i}, \quad 1 + 2\mathrm{i}, \quad -1 - 3\mathrm{i}, \quad z = 2 - \mathrm{i}, \quad w = -3 + 2\mathrm{i}.$$

Altogether, we've learnt to write complex numbers in three ways:

Cartesian (or standard or rectangular) form

$$z = a + \mathrm{i}b = \operatorname{Re} z + \mathrm{i} \operatorname{Im} z$$

Polar form

$$z = r(\cos\theta + \mathrm{i}\sin\theta) = |z|(\cos\arg z + \mathrm{i}\sin\arg z)$$

Exponential form

$$z = r\mathrm{e}^{\mathrm{i}\theta} = |z|\,\mathrm{e}^{\mathrm{i}\arg z}$$

Remark 122. Your A-Level examiners do not seem to use the term *exponential form*. What we call *exponential form* is simply called *polar form* by them. (And what we call *polar form* is also called *polar form* by them.)

84.2. Some Useful Formulae for Sine and Cosine

Fact 192. *Suppose $\theta \in \mathbb{R}$. Then*

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Proof. By Theorem 22 (Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$),

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)}{2}$$

$$= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} = \frac{2\cos\theta}{2} = \cos\theta;$$

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{\cos\theta + i\sin\theta - [\cos(-\theta) + i\sin(-\theta)]}{2i}$$

$$= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2i} = \frac{2i\sin\theta}{2i} = \sin\theta. \quad \square$$

Remark 123. Some writers use Fact 192 as their *definitions* of sine and cosine.

85. More Arithmetic of Complex Numbers

Now that we know how to write complex numbers in polar and exponential forms, their arithmetic becomes easier. Consider **multiplication**:

Fact 193. *Let z and w be non-zero complex numbers. Then*

$$\textbf{(a)}\ |zw| = |z|\,|w|; \qquad \textit{and} \qquad \textbf{(b)}\ \arg(zw) = \arg z + \arg w + 2k\pi,$$

where in **(b)**,
$$k = \begin{cases} -1, & \textit{if } \arg z + \arg w > \pi, \\ 0, & \textit{if } \arg z + \arg w \in (-\pi, \pi], \\ 1, & \textit{if } \arg z + \arg w \le -\pi. \end{cases}$$

Proof. For **(a)**, see Exercise 322. For **(b)**, see p. 1648 (Appendices). □

The additional term $2k\pi$ in Fact 193**(b)** is to ensure that $\arg(zw) \in (-\pi, \pi]$, as is required by the definition of the argument. A few examples will make this clear:

Example 1065. Let

$$z = 5 - 2\mathrm{i} = (5, -2) \approx \sqrt{29}\,(\cos -0.381 + \mathrm{i}\sin -0.381) = \sqrt{29}\mathrm{e}^{-0.381\mathrm{i}},$$
$$w = 1 + 3\mathrm{i} = (1, 3) \approx \sqrt{10}\,(\cos 1.249 + \mathrm{i}\sin 1.249) = \sqrt{10}\mathrm{e}^{1.249\mathrm{i}}.$$

By Fact 193, **(a)** $|zw| = \sqrt{29}\sqrt{10} = \sqrt{290}$; and

(b) $\arg(zw) = \arg z + \arg w + 2k\pi \approx -0.381 + 1.249 + 0\pi \approx 0.869.$

Now, how did I know to choose $k = 0$ here? Well, by definition, the argument of any complex number is in the interval $(-\pi, \pi]$. And so, when applying Fact 193**(b)**, we always simply choose k to ensure that $\arg(zw) \in (-\pi, \pi]$. Here we already have $\arg z + \arg w \approx -0.381 + 1.249 \approx 0.869 \in (-\pi, \pi]$. And so, we simply choose $k = 0$.

With **(a)** and **(b)**, we can write zw down in both **polar** and **exponential forms**:

$$zw \approx \sqrt{290}\,(\cos 0.869 + \mathrm{i}\sin 0.869) = \sqrt{290}\mathrm{e}^{0.869\mathrm{i}}.$$

To write zw down in **cartesian form**, we can use $|zw|$ and $\arg(zw)$ to compute

$$\mathrm{Re}\,(zw) \approx \sqrt{290}\cos 0.869 = 10.994; \qquad \text{and} \qquad \mathrm{Im}\,(zw) \approx \sqrt{290}\sin 0.869 = 13.005.$$

Of course, since the real and imaginary parts of both z and w are all integers, so too must be the real and imaginary parts of zw. And so we have in fact $\mathrm{Re}\,(zw) = 11$ and $\mathrm{Im}\,(zw) = 13$. Thus, $zw = 11 + 13\mathrm{i}$.

Alternatively, we can do the usual multiplication, which yields us the exact value of zw:

$$zw = (5 - 2\mathrm{i})(1 + 3\mathrm{i}) = 5 + 15\mathrm{i} - 2\mathrm{i} + 6 = 11 + 13\mathrm{i}.$$

Example 1066. Let

$$\begin{aligned} z &= -4+7\mathrm{i} = (-4,7) \approx \sqrt{65}\,(\cos 2.090 + \mathrm{i}\sin 2.090) = \sqrt{65}\mathrm{e}^{2.090\mathrm{i}}, \\ w &= \ \ 1+9\mathrm{i} \ \ = \ \ (1,9) \ \ \approx \sqrt{82}\,(\cos 1.460 + \mathrm{i}\sin 1.460) = \sqrt{82}\mathrm{e}^{1.460\mathrm{i}}. \end{aligned}$$

By Fact 193, **(a)** $|zw| = \sqrt{65}\sqrt{82}$; and

(b) $\arg(zw) = \arg z + \arg w + 2k\pi \approx 2.090 + 1.460 - 2\pi \approx -2.733$.

Since $\arg z + \arg w \approx 2.090 + 1.460 > \pi$, we choose $k = -1$.

With **(a)** and **(b)**, we can write zw down in both **polar** and **exponential forms**:

$$zw \approx \sqrt{65}\sqrt{82}\,(\cos -2.733 + \mathrm{i}\sin -2.733) = \sqrt{65}\sqrt{82}\mathrm{e}^{-2.733\mathrm{i}}.$$

To write zw down in **cartesian form**, we can use $|zw|$ and $\arg(zw)$ to compute

$$\operatorname{Re}(zw) \approx \sqrt{65}\sqrt{82}\cos -2.733 \approx -67; \qquad \text{and} \qquad \operatorname{Im}(zw) \approx \sqrt{65}\sqrt{82}\sin -2.733 \approx -29.$$

Thus, $$zw = -67 - 29\mathrm{i}.$$

Alternatively, we can do the usual multiplication, which yields us the exact value of zw:

$$zw = (-4+7\mathrm{i})\,(1+9\mathrm{i}) = -4 - 36\mathrm{i} + 7\mathrm{i} - 63 = -67 - 29\mathrm{i}.$$

Example 1067. Let

$$\begin{aligned} z &= -2-\mathrm{i} = (-2,-1) \approx \ \sqrt{5}\,(\cos -2.678 + \mathrm{i}\sin -2.678) \ = \sqrt{5}\mathrm{e}^{-2.678\mathrm{i}}, \\ w &= 1-3\mathrm{i} = \ (1,-3) \ \approx \sqrt{10}\,(\cos -1.249 + \mathrm{i}\sin -1.249) = \sqrt{10}\mathrm{e}^{-1.249\mathrm{i}}. \end{aligned}$$

By Fact 193, **(a)** $|zw| = \sqrt{5}\sqrt{10} = 5\sqrt{2}$; and

(b) $\arg(zw) = \arg z + \arg w + 2k\pi \approx -2.678 - 1.249 + 2\pi \approx 2.356$.

Since $\arg z + \arg w \approx -2.678 - 1.249 \leq \pi$, we choose $k = 1$.

With **(a)** and **(b)**, we can write zw down in both **polar** and **exponential forms**:

$$zw \approx 5\sqrt{2}\,(\cos 2.356 + \mathrm{i}\sin 2.356) = 5\sqrt{2}\mathrm{e}^{2.356\mathrm{i}}.$$

To write zw down in **cartesian form**, we can use $|zw|$ and $\arg(zw)$ to compute

$$\operatorname{Re}(zw) \approx 5\sqrt{2}\cos 2.356 \approx -5; \qquad \text{and} \qquad \operatorname{Im}(zw) \approx 5\sqrt{2}\sin 2.356 \approx 5.$$

Thus, $$zw = -5 + 5\mathrm{i}.$$

Alternatively, we can do the usual multiplication, which yields us the exact value of zw:

$$zw = (-2-\mathrm{i})\,(1-3\mathrm{i}) = -2 + 6\mathrm{i} - \mathrm{i} - 3 = -5 + 5\mathrm{i}.$$

Multiplying a complex number by a positive real number leaves the argument unchanged:

Corollary 38. *Suppose z is a complex number and $a > 0$. Then* $\arg(az) = \arg z$.

Proof. By Fact 193, $\arg(az) = \arg a + \arg z + 2k\pi = \arg z + 2k\pi = \arg z$, where we choose $k = 0$ because $\arg z \in (-\pi, \pi]$. □

Example 1068. Let z=7-9i, so that $5z = 35 - 45\mathrm{i}$. Then $\arg z = \arg(5z) \approx -0.910$.

In contrast, multiplying z by -1 changes the argument by either $-\pi$ or $+\pi$:

Corollary 39. (a) *If* $\arg z > 0$, *then* $\arg(-z) = \arg z - \pi$.

(b) *If* $\arg z \le 0$, *then* $\arg(-z) = \arg z + \pi$.

Proof. By Fact 193(b), $\arg(-z) = \arg(-1 \cdot z) = \arg(-1) + \arg z + 2k\pi = \pi + \arg z + 2k\pi$.

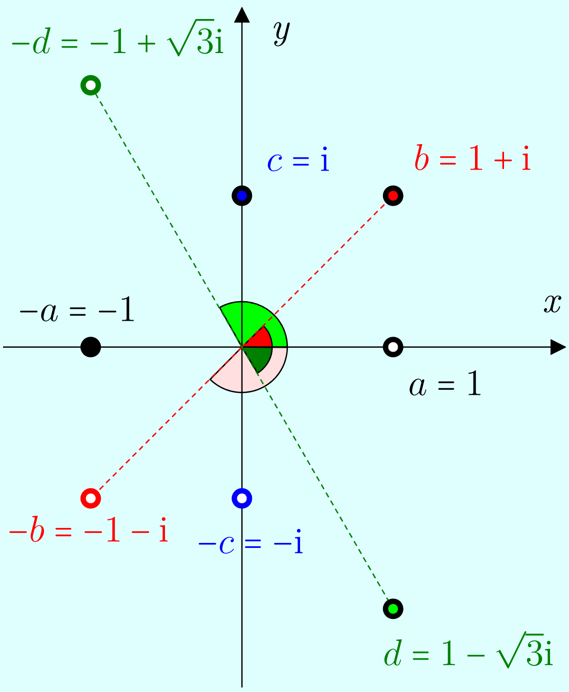
(a) If $\arg z > 0$, then $k = -1$ and thus, $\arg(-z) \stackrel{1}{=} \pi + \arg z - 2\pi = \arg z - 2\pi$.

(b) If $\arg z \le 0$, then $k = 0$ and thus, $\arg(-z) \stackrel{1}{=} \pi + \arg z - 0\pi = \arg z + \pi$ □

Example 1069. Let z=7-9i, so that $-z = -7 + 9\mathrm{i}$. Since $\arg z \approx -0.910 \le 0$, by Corollary 39, we have $\arg(-z) = \arg z + \pi \approx -0.910 + \pi \approx 2.232$.

Example 1070. Let $a = 1$, $b = 1 + \mathrm{i}$, $c = \mathrm{i}$, and $d = 1 - \sqrt{3}\mathrm{i}$. Then

$$\begin{aligned}
\arg a &= \arg 1 = 0 \le 0,\\
\arg b &= \arg(1+\mathrm{i}) = \frac{\pi}{4} > 0,\\
\arg c &= \arg \mathrm{i} = \frac{\pi}{2} > 0,\\
\arg d &= \arg\left(1 - \sqrt{3}\mathrm{i}\right) = \frac{-\pi}{3} \le 0.
\end{aligned}$$



And so by Corollary 39,

$$\begin{aligned}
\arg(-a) &= \arg(-11) = \arg a + \pi = \pi,\\
\arg(-b) &= \arg(-1-\mathrm{i}) = \arg b - \pi = \frac{-3\pi}{4},\\
\arg(-c) &= \arg(-\mathrm{i}) = \arg c - \pi = \frac{-\pi}{2},\\
\arg(-d) &= \arg\left(-1 + \sqrt{3}\mathrm{i}\right) = \arg d + \pi = \frac{2\pi}{3}.
\end{aligned}$$

Combining the last two Corollaries, we have that multiplying a complex number by a negative real number changes the argument by either $-\pi$ or $+\pi$:

Corollary 40. *Suppose z is a complex number and $a > 0$.*

(a) *If* $\arg z > 0$, *then* $\arg(-az) = \arg(-z) = \arg z - \pi$.

(b) *If* $\arg z \leq 0$, *then* $\arg(-az) = \arg(-z) = \arg z + \pi$.

Example 1071. Let z=7-9i, so that $-5z = -35 + 45\mathrm{i}$. Since $\arg z \approx -0.910 \leq 0$, by Corollary 40, we have $\arg(-5z) = \arg z + \pi \approx -0.910 + \pi \approx 2.232$.

Example 1072. Suppose $a = 1$, $b = 1 + \mathrm{i}$, $c = \mathrm{i}$, and $d = 1 - \sqrt{3}\mathrm{i}$. Then

$$\arg a = \arg 1 = 0 \leq 0, \qquad \arg c = \arg \mathrm{i} = \frac{\pi}{2} > 0,$$

$$\arg b = \arg(1+\mathrm{i}) = \frac{\pi}{4} > 0, \qquad \arg d = \arg\left(1 - \sqrt{3}\mathrm{i}\right) = \frac{-\pi}{3} \leq 0.$$

And so, by Corollary 40,

$$\arg(-3a) = \arg(-33) = \arg(-a) = \arg a + \pi = \pi,$$

$$\arg(-3b) = \arg(-3-3\mathrm{i}) = \arg(-b) = \arg b - \pi = \frac{-3\pi}{4},$$

$$\arg(-3c) = \arg(-3\mathrm{i}) = \arg(-c) = \arg c - \pi = \frac{-\pi}{2},$$

$$\arg(-3d) = \arg\left(-3 + 3\sqrt{3}\mathrm{i}\right) = \arg(-d) = \arg d + \pi = \frac{2\pi}{3}.$$

Exercise 321. For each, find $|zw|$, $\arg(zw)$, $|-2zw|$, and $\arg(-2zw)$. Then express both zw and $-2zw$ in polar, exponential, *and* cartesian forms. (Answer on p. 1888.)

(a) $z = 1$, $w = -3$. **(b)** $z = 2\mathrm{i}$, $w = 1 + 2\mathrm{i}$. **(c)** $z = -1 - 3\mathrm{i}$, $w = 3 + 4\mathrm{i}$.
(d) $z = -2 + 5\mathrm{i}$, $w = \mathrm{i}$. **(e)** $z = -1 - \mathrm{i}$, $w = -1 - 2\mathrm{i}$. **(f)** $z = -5 - 3\mathrm{i}$, $w = 5 - \mathrm{i}$.

Exercise 322. This Exercise guides you through a proof of Fact 193**(a)**. Let $r = |z|$, $\theta = \arg z$, $s = |w|$, and $\phi = \arg w$. (Answer on p. 1889.)

(a) Express z and w in polar form.

(b) Expand zw. Then use a trigonometric identity to show that

$$zw = rs\left[\cos(\theta + \phi) + \mathrm{i}\sin(\theta + \phi)\right].$$

(c) Now show that $|zw| = |z|\,|w|$.

85.1. The Reciprocal

Fact 194. *Suppose w is a non-zero complex number. Then*

(a)
$$\left|\frac{1}{w}\right| = \frac{1}{|w|}.$$

(b) *If w is not a negative real number, then* $\arg\dfrac{1}{w} = -\arg w$.

(c) *If w is a negative real number, then* $\arg\dfrac{1}{w} = \arg w = \pi$.

Proof. **(c)** is "obvious". For a proof of **(a)** and **(b)**, see p. 1649 (Appendices). □

Example 1073. Let

$$\begin{aligned} z &= 5 - 2\text{i} = (5, -2) \approx \sqrt{29}\,(\cos -0.381 + \text{i}\sin -0.381) = \sqrt{29}\text{e}^{-0.381\text{i}}, \\ w &= 1 + 3\text{i} = (1, 3) \approx \sqrt{10}\,(\cos 1.249 + \text{i}\sin 1.249) = \sqrt{10}\text{e}^{1.249\text{i}}. \end{aligned}$$

By Fact 194, **(a)** $\left|\dfrac{1}{z}\right| = \dfrac{1}{\sqrt{29}}$ and **(b)** $\arg\dfrac{1}{z} = -\arg z \approx 0.381$. Also, **(a)** $\left|\dfrac{1}{w}\right| = \dfrac{1}{\sqrt{10}}$ and **(b)** $\arg\dfrac{1}{w} = -\arg w \approx -1.249$. So,

$$\begin{aligned} \frac{1}{z} &\approx \frac{1}{\sqrt{29}}(\cos 0.381 + \text{i}\sin 0.381) \approx \frac{1}{\sqrt{29}}\text{e}^{0.381\text{i}}, \\ \frac{1}{w} &\approx \frac{1}{\sqrt{10}}(\cos -1.249 + \text{i}\sin -1.249) \approx \frac{1}{\sqrt{10}}\text{e}^{-1.249\text{i}}. \end{aligned}$$

Example 1074. Let $\omega = -5$, so that $1/\omega = -1/5$. Hence, $\arg\omega = \arg(1/\omega) = \pi$.

Exercise 323. Find the moduli and arguments of each number and its reciprocal. Then write down the latter in exponential, polar, *and* cartesian form. (Answer on p. 1889.)

(a) $z = 1$. **(b)** $w = 2\text{i}$. **(c)** $z = -17$. **(d)** $w = -8\text{i}$.
(e) $z = -2 + 5\text{i}$. **(f)** $w = -1 - \text{i}$. **(g)** $z = 1 - 3\text{i}$. **(h)** $w = 3 + 4\text{i}$.

Fact 195. *Suppose z and w are non-zero complex numbers. Then*

$$\textbf{(a)}\ \left|\frac{z}{w}\right| = \frac{|z|}{|w|}; \qquad \textit{and} \qquad \textbf{(b)}\ \arg\frac{z}{w} = \arg z - \arg w + 2k\pi,$$

where in **(b)**,
$$k = \begin{cases} -1, & \text{if } \arg z - \arg w > \pi, \\ 0, & \text{if } \arg z - \arg w \in (-\pi, \pi], \\ 1, & \text{if } \arg z - \arg w \leq -\pi. \end{cases}$$

Proof. For **(a)**, see Exercise 325. For **(b)**, see p. 1650 (Appendices). □

Example 1075. Let

$$z = 5 - 2\mathrm{i} = (5, -2) \approx \sqrt{29}\left(\cos -0.381 + \mathrm{i}\sin -0.381\right) = \sqrt{29}\mathrm{e}^{-0.381\mathrm{i}},$$
$$w = 1 + 3\mathrm{i} = (1, 3) \approx \sqrt{10}\left(\cos 1.249 + \mathrm{i}\sin 1.249\right) = \sqrt{10}\mathrm{e}^{1.249\mathrm{i}}.$$

By Fact 195, we have

$$\textbf{(a)} \qquad \left|\frac{z}{w}\right| = \frac{\sqrt{29}}{\sqrt{10}} = \sqrt{2.9},$$

$$\textbf{(b)} \quad \arg\frac{z}{w} = \arg z - \arg w + 2k\pi \approx -0.381 - 1.249 + 0 \approx -1.630,$$

where we chose $k = 0$ because $\arg z - \arg w \in (-\pi, \pi]$.

With **(a)** and **(b)**, we can write z/w down in both **polar** and **exponential forms**:

$$\frac{z}{w} \approx \sqrt{2.9}\left(\cos -1.630 + \mathrm{i}\sin -1.630\right) = \sqrt{2.9}\mathrm{e}^{-1.630\mathrm{i}}.$$

To write z/w down in **cartesian form**, we can use $|z/w|$ and $\arg(z/w)$ to compute

$$\operatorname{Re}\frac{z}{w} \approx \sqrt{2.9}\cos -1.630 \approx -0.101 \qquad \text{and} \qquad \operatorname{Im}\frac{z}{w} \approx \sqrt{2.9}\sin -1.630 \approx -1.700.$$

Thus, $$\frac{z}{w} = -0.1 - 1.7\mathrm{i}.$$

Alternatively, we can simply do the usual division, which yields us the exact value of zw:

$$\frac{z}{w} = \frac{5 - 2\mathrm{i}}{1 + 3\mathrm{i}} = \frac{5 - 2\mathrm{i}}{1 + 3\mathrm{i}} \cdot \frac{1 - 3\mathrm{i}}{1 - 3\mathrm{i}} = \frac{5 - 15\mathrm{i} - 2\mathrm{i} - 6}{1^2 + 3^2} = \frac{-1 - 17\mathrm{i}}{10} = -0.1 - 1.7\mathrm{i}.$$

Example 1076. Let

$$\begin{aligned} z &= -4+7\mathrm{i} = (-4,7) \approx \sqrt{65}\,(\cos 2.090 + \mathrm{i}\sin 2.090) = \sqrt{65}\mathrm{e}^{2.090\mathrm{i}}, \\ w &= \ \ 1+9\mathrm{i} \ \ = \ \ (1,9) \ \ \approx \sqrt{82}\,(\cos 1.460 + \mathrm{i}\sin 1.460) = \sqrt{82}\mathrm{e}^{1.460\mathrm{i}}. \end{aligned}$$

By Fact 195, **(a)** $\left|\dfrac{z}{w}\right| = \dfrac{\sqrt{65}}{\sqrt{82}}$,

(b) $\arg\dfrac{z}{w} = \arg z - \arg w + 2k\pi \approx 2.090 - 1.460 + 0 \approx 0.630$,

where we chose $k = 0$ because $\arg z - \arg w \in (-\pi, \pi]$.

With **(a)** and **(b)**, we can write z/w down in both **polar** and **exponential forms**:

$$\frac{z}{w} \approx \frac{\sqrt{65}}{\sqrt{82}}\,(\cos 0.630 + \mathrm{i}\sin 0.630) = \frac{\sqrt{65}}{\sqrt{82}}\mathrm{e}^{0.630\mathrm{i}}.$$

To write z/w down in **cartesian form**, we can use $|z/w|$ and $\arg(z/w)$ to compute

$$\operatorname{Re}\frac{z}{w} \approx \frac{\sqrt{65}}{\sqrt{82}}\cos 0.630 \approx 0.719 \qquad \text{and} \qquad \operatorname{Im}\frac{z}{w} \approx \frac{\sqrt{65}}{\sqrt{82}}\sin 0.630 \approx 0.525.$$

Thus, $$\frac{z}{w} \approx 0.719 + 0.525\mathrm{i}.$$

Alternatively, we can simply do the usual division, which yields us the exact value of z/w:

$$\frac{z}{w} = \frac{-4+7\mathrm{i}}{1+9\mathrm{i}} = \frac{-4+7\mathrm{i}}{1+9\mathrm{i}}\frac{1-9\mathrm{i}}{1-9\mathrm{i}} = \frac{-4+36\mathrm{i}+7\mathrm{i}+63}{1^2+9^2} = \frac{59+43\mathrm{i}}{82} = \frac{59}{82} + \frac{43}{82}\mathrm{i}.$$

Example 1077. Let

$$\begin{aligned} z &= -2-\mathrm{i} = (-2,-1) \approx \sqrt{5}\,(\cos -2.678 + \mathrm{i}\sin -2.678) = \sqrt{5}\mathrm{e}^{-2.678\mathrm{i}}, \\ w &= 1-3\mathrm{i} = (1,-3) \approx \sqrt{10}\,(\cos -1.249 + \mathrm{i}\sin -1.249) = \sqrt{10}\mathrm{e}^{-1.249\mathrm{i}}. \end{aligned}$$

By Fact 195: **(a)** $\left|\dfrac{z}{w}\right| = \dfrac{\sqrt{5}}{\sqrt{10}} = \dfrac{1}{\sqrt{2}}$,

(b) $\arg\dfrac{z}{w} = \arg z - \arg w + 2k\pi \approx -2.678 + 1.249 + 0 \approx -1.429$,

where we chose $k = 0$ because $\arg z - \arg w \in (-\pi, \pi]$.

With **(a)** and **(b)**, we can write z/w down in both **polar** and **exponential forms**:

$$\frac{z}{w} \approx \frac{1}{\sqrt{2}}(\cos -1.429 + \mathrm{i}\sin -1.429) = \frac{1}{\sqrt{2}}\mathrm{e}^{-1.429\mathrm{i}}.$$

To write z/w down in **cartesian form**, we can use $|z/w|$ and $\arg(z/w)$ to compute

$$\operatorname{Re}\frac{z}{w} \approx \frac{1}{\sqrt{2}}\cos -1.429 \approx 0.100 \qquad \text{and} \qquad \operatorname{Im}\frac{z}{w} \approx \frac{1}{\sqrt{2}}\sin -1.429 \approx -0.700.$$

Thus, $$\frac{z}{w} = 0.1 - 0.7\mathrm{i}.$$

Alternatively, we can simply do the usual division, which yields us the exact value of z/w:

$$\frac{z}{w} = \frac{-2-\mathrm{i}}{1-3\mathrm{i}} = \frac{-2-\mathrm{i}}{1-3\mathrm{i}}\frac{1+3\mathrm{i}}{1+3\mathrm{i}} = \frac{-2-6\mathrm{i}-\mathrm{i}+3}{1^2+3^2} = \frac{1-7\mathrm{i}}{10} = 0.1-0.7\mathrm{i}$$

Exercise 324. For each, find $|z/w|$ and $\arg(z/w)$. Then express z/w in polar, exponential, *and* cartesian forms. (Answer on p. 1890.)

(a) $z = 1$, $w = -3$. **(b)** $z = 2\mathrm{i}$, $w = 1+2\mathrm{i}$. **(c)** $z = -1-3\mathrm{i}$, $w = 3+4\mathrm{i}$.
(d) $z = -2+5\mathrm{i}$, $w = \mathrm{i}$. **(e)** $z = -1-\mathrm{i}$, $w = -1-2\mathrm{i}$. **(f)** $z = -5-3\mathrm{i}$, $w = 5-\mathrm{i}$.

Exercise 325. Use Facts 193 and 194 to prove Fact 195(a). (Answer on p. 1890.)

Part V.
Calculus

Revision in progress (November 2021).

And hence messy at the moment.
Appy polly loggies for any inconvenience caused.

The calculus was the first achievement of modern mathematics, and it is difficult to overestimate its importance.

— John von Neumann (1947).

if you forget about understanding what's going on and concentrate on mechanical manipulations, you'll forget how to do even the mechanical manipulations.

— Timothy Gowers (2012).

Zudem ist es ein Irrtum zu glauben, daß die Strenge in der Beweisführung die Feindin der Einfachheit wäre. An zahlreichen Beispielen finden wir im Gegenteil bestätigt, daß die strenge Methode auch zugleich die einfachere und leichter faßliche ist.

Besides it is an error to believe that rigor in the proof is the enemy of simplicity. On the contrary we find it confirmed by numerous examples that the rigorous method is at the same time the simpler and the more easily comprehended.

— David Hilbert (1900, 1902t).

86. Limits

The idea of **limits** isn't on your syllabus.[370] But it is fundamental to calculus. And it really isn't all that difficult, especially if presented in informal and intuitive terms (as I try to do here). It is therefore worthwhile investing a little time in it, just so things become that much clearer.[371]

Ch. 25 already briefly introduced you to limits (in the context of **asymptotes**—**vertical**, **horizontal**, and **oblique**). This chapter will look at limits again, but at greater depth.

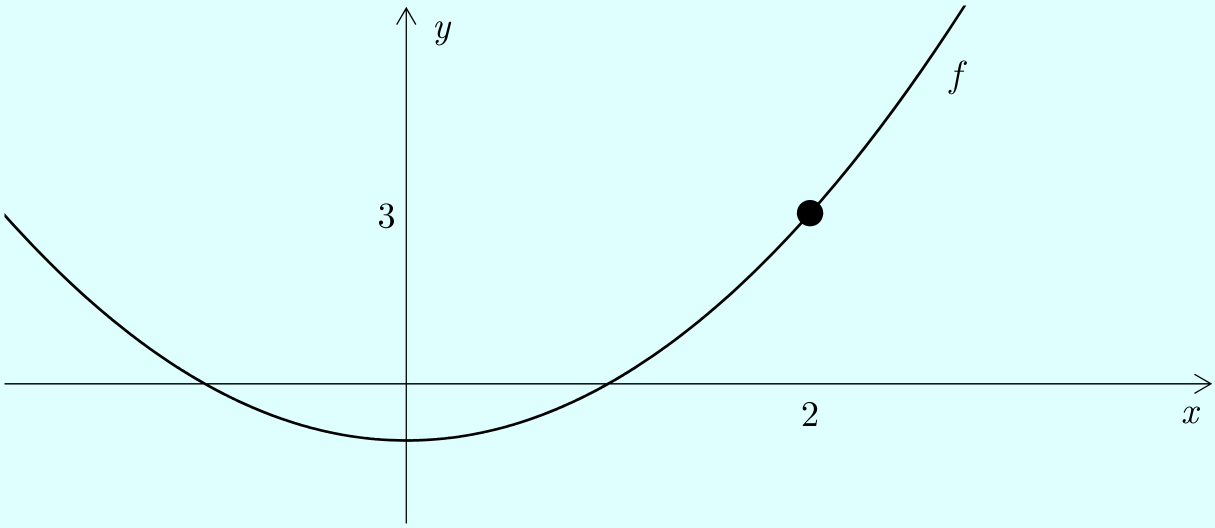
Remark 124. To the unconvinced Type 1 Pragmatist thinking of skipping this chapter: Think again. In recent years, your A-Level examiners have seen fit to screw students over with curveball, totally-out-of-the-syllabus questions[372] involving limits. See especially Exercise 575(c) (N2017/I/9). So yea, this chapter's probably worth a quick read.

Reminder

A **nice function** (Ch 17.4) is this textbook's (non-standard but convenient) term for a real-valued function of a real variable (or equivalently, a function whose domain and codomain are both sets of real numbers).

[370]Ignoring the Central Limit Theorem, the word *limit* appears on your syllabus only once (p. 9, "concept of definite integral as a limit of sum"), almost in passing, and solely in relation to the definite integral.
[371]To keep things simple, we discuss only **functional limits** (and not **sequential limits**).
[372]We already discussed this "phenomenon" in my Preface/Rant—see p. xlix.

Example 1078. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 1$.



Informal observations:

For **all** values of x that are **"close"** but **not equal** to 2,
$f(x)$ is "close" (or possibly even equal) to 3.

Or, By making x **"sufficiently close"** but **not equal** to 2,
$f(x)$ can be made as "close" as we like to 3.

Either of the above two informal statements[373] may be written formally as

$$\lim_{x \to 2} f(x) = 3, \quad \textbf{(1)}$$

which we read aloud as, The limit of f at 2 is 3. **(2)**

Or, As $x \to 2$, $f(x) \to 3$, **(3)**

which we read aloud as, As x approaches 2, $f(x)$ approaches 3. **(4)**

These last four statements are equivalent and formal. You need not know their precise meaning. An intuitive and informal understanding will suffice.

You may be thinking, "The above example is quite silly or useless. We already know that $f(2) = 3$. So what have we added by also writing $\lim_{x \to 2} f(x) = 3$?"

And you'd be right. The above is just our very first introductory example to limits. To see why limits aren't so silly or useless, we'll have to look at more examples:

Subtle Point 1: The condition **not equal to** is important. When considering the limit of a function g at a, we do **not** care about $g(a)$, the value of the function at a. We **only** care

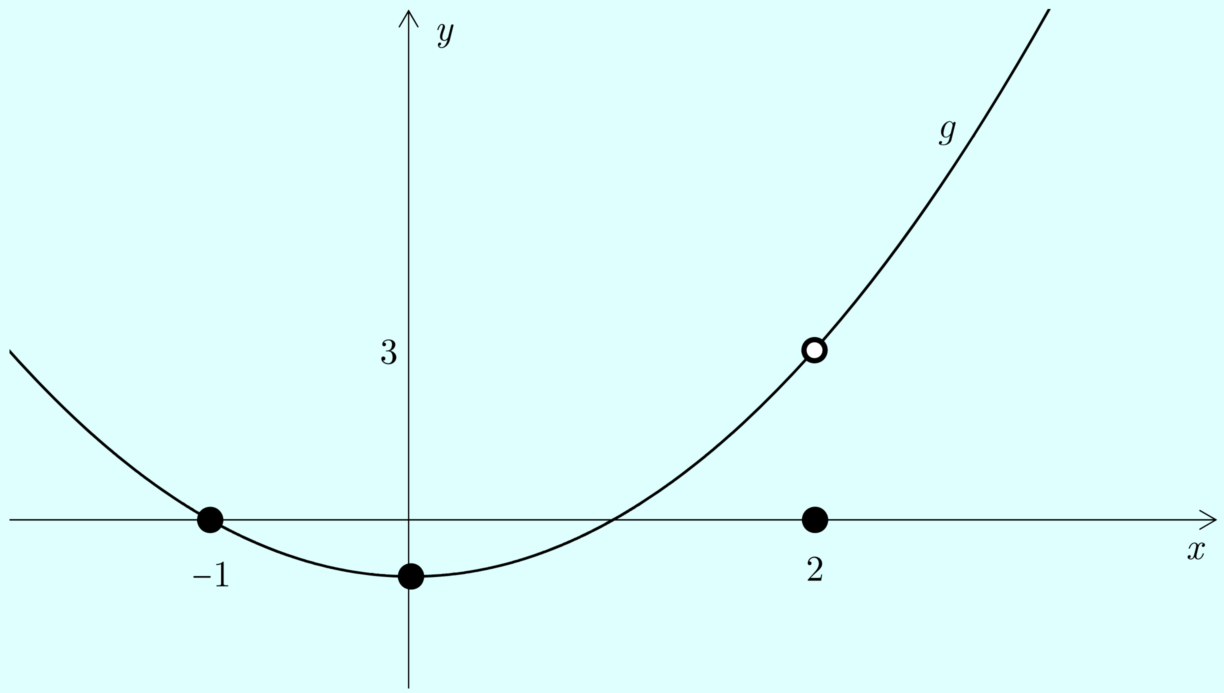
[373] In particular, the terms "close" and "sufficiently close" are informal and hence in scare quotes.

about the values of x that are "close" to a:

Example 1079. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^2 - 1 & \text{for } x \neq 2, \\ 0 & \text{for } x = 2. \end{cases}$$

The function g is very similar to the function f (last example), except that now $g(2) = 0$ and we have a "hole" in our curve. As we'll learn later, this is an example of a **removable discontinuity**.



Observe though that these two informal statements remain true:

For **all** values of x that are **"close"** but **not equal** to 2,
$g(x)$ is "close" (or possibly even equal) to 3.

By making x **"sufficiently close"** but **not equal** to 2,
$g(x)$ can be made as "close" as we like to 3.

So, each of these four equivalent and formal statements remains true:

(1) $\lim_{x \to 2} g(x) = 3$.

(2) The limit of g at 2 is 3.

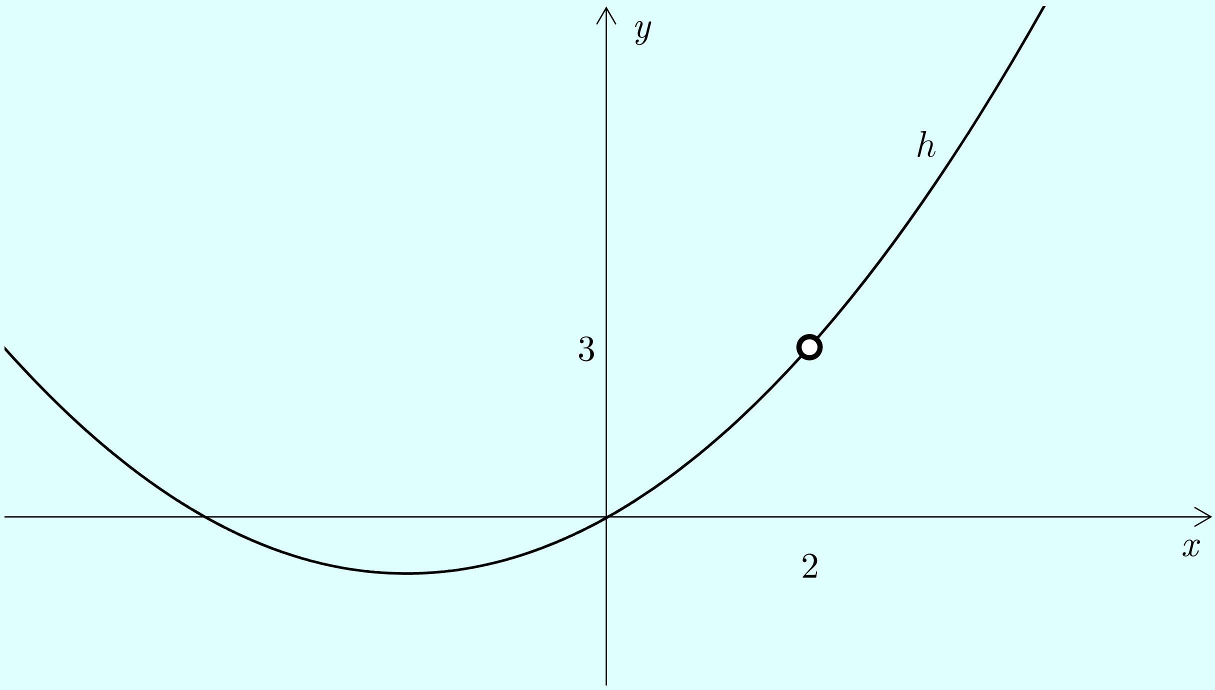
(3) As $x \to 2$, $g(x) \to 3$.

(4) As x approaches 2, $g(x)$ approaches 3.

Subtle Point 2: Actually, the condition "not equal to" goes even further—even if $h(a)$ is undefined, $\lim_{x \to a} h(x)$ may still be well-defined:

Example 1080. Define $h : \mathbb{R} \setminus \{2\} \to \mathbb{R}$ by $h(x) = x^2 - 1$.

The function h is very similar to g, except that $h(2)$ is now simply left **undefined**.



Observe that these two informal statements remain true:

For **all** values of x that are **"close"** but **not equal** to 2,
$h(x)$ is "close" (or possibly even equal) to 3.

By making x **"sufficiently close"** but **not equal** to 2,
$h(x)$ can be made as "close" as we like to 3.

So, each of the following four equivalent and formal statements remains true:

(1) $\lim_{x \to 2} h(x) = 3$.

(2) The limit of h at 2 is 3.

(3) As $x \to 2$, $h(x) \to 3$.

(4) As x approaches 2, $h(x)$ approaches 3.

Definition 195 (*informal*)**.** These four statements are equivalent and formal:

(1) $\lim_{x \to a} f(x) = L$.

(2) The limit of f at a is L.

(3) As $x \to a$, $f(x) \to L$.

(4) As x approaches a, $f(x)$ approaches L.

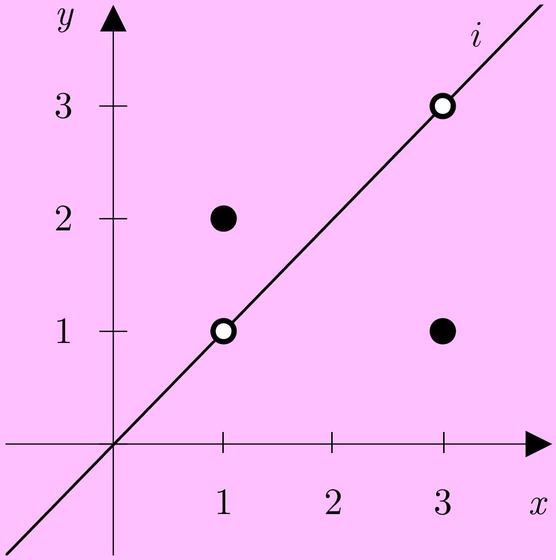
Informally, each of **(1)**–**(4)** says,[374]

For **all** values of x that are **"close"** but **not equal** to a,
$f(x)$ is "close" (or possibly even equal) to L.

Or, By making x **"sufficiently close"** but **not equal** to a,
$f(x)$ can be made as "close" as we like to L.

[374] For a formal definition of limits, see Ch. 146.2 (Appendices).

Exercise 326. The function $i : \mathbb{R} \to \mathbb{R}$ is graphed below. Find the limits of i at 0, 1, 2, and 3. (Answer on p. 1891.)

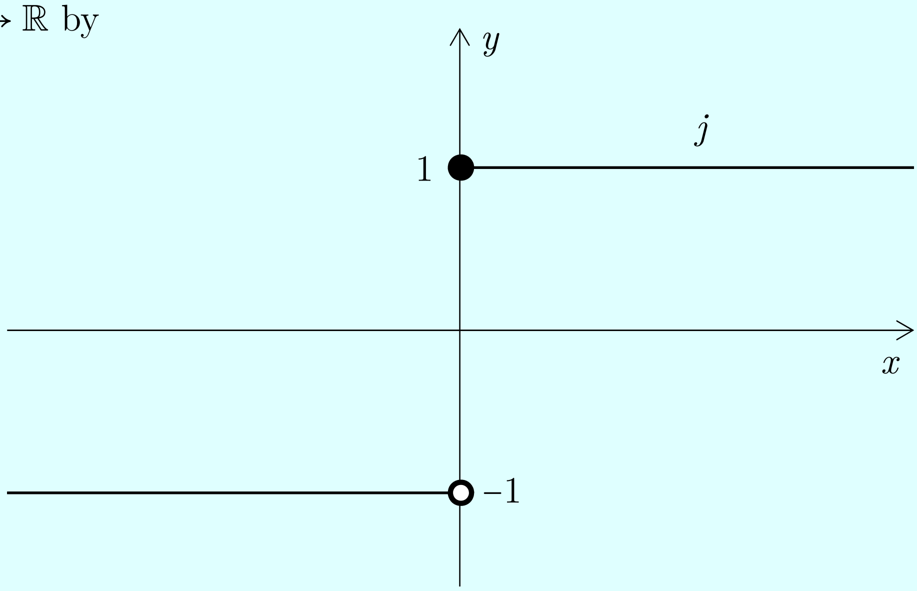


86.2. Examples Where The Limit Does Not Exist

To better understand limits, we now look at examples where the limit does **not** exist:

Example 1081. Define $j : \mathbb{R} \to \mathbb{R}$ by

$$j(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$



Consider $\lim_{x \to 0} j(x)$, the limit of j at 0.

Could it be that $\lim_{x \to 0} j(x) = -1$? No, because there are values of x that are "close" to 0 but for which $j(x) = 1$. Hence, it is not true that for **all** values of x that are **"close"** but **not equal** to 0, $j(x)$ is "close" to -1. So,

$$\lim_{x \to 0} j(x) \neq -1.$$

Next, could it be that $\lim_{x \to 0} j(x) = 1$? No, because there are values of x that are "close" to 0 but for which $j(x) = -1$. Hence, it is not true that for **all** values of x that are **"close"** but **not equal** to 0, $j(x)$ is "close" to 1. So,

$$\lim_{x \to 0} j(x) \neq 1.$$

We've just argued that $\lim_{x \to 0} j(x)$ does not equal either -1 or 1. Let's now also argue more generally that $\lim_{x \to 0} j(x)$ cannot be equal to *any* real number L. This is because for values of x that are **"close"** but **not equal** to 0, there is no single real number L that $j(x)$ is "close" to.

So, we conclude $\qquad \lim_{x \to 0} j(x)$ does not exist.

As we'll learn later, this is an example of a **jump discontinuity**.

(*Example continues on the next page ...*)

We now introduce also the concepts of a **left-hand limit** and a **right-hand limit**. These concepts are simple and will improve your understanding.

(... *Example continued from the previous page.*)

These four statements are equivalent, formal, and true:

1. $\lim_{x \to 0^-} j(x) = -1$.

2. The **left-hand limit** of j at 0 is -1.

3. As $x \to 0^-$, $j(x) \to -1$.

4. As x approaches 0 **from the left**, $j(x)$ approaches -1.

Informally, each of the last four statements says this:

For **all** values of x that are **"close"** to but ***less* than** 0,
$j(x)$ is "close" (or possibly even equal) to -1.

Or, By making x **"sufficiently close"** but ***less* than** 0,
$j(x)$ can be made as "close" as we like to -1.

Similarly, these four statements are equivalent, formal, and true:

1. $\lim_{x \to 0^+} j(x) = 1$.

2. The **right-hand limit** of j at 0 is 1.

3. As $x \to 0^+$, $j(x) \to 1$.

4. As x approaches 0 **from the right**, $j(x)$ approaches 1.

Informally, each of the last four statements says this:

For **all** values of x that are **"close"** to but ***more* than** 0,
$j(x)$ is **"close"** (or possibly even equal) to 1.

Or, By making x **"sufficiently close"** but ***more* than** 0,
$j(x)$ can be made as "close" as we like to 1.

Definition 196 (*informal*)**.** These four statements are equivalent and formal:

1. $\lim_{x \to a^-} f(x) = L$.

2. The left-hand limit of f at a is L.

3. As $x \to a^-$, $f(x) \to L$.

4. As x approaches a from the left, $f(x)$ approaches L.

Informally, each of the above four statements says,[375]

For **all** values of x that are **"close"** but **less than** a,
$f(x)$ is "close" (or possibly even equal) to L.

Or By making x **"sufficiently close"** but **less than** a,
$f(x)$ can be made as "close" as we like to L.

Definition 197 (*informal*)**.** These four statements are equivalent and formal:

1. $\lim_{x \to a^+} f(x) = L$.

2. The right-hand limit of f at a is L.

3. As $x \to a^+$, $f(x) \to L$.

4. As x approaches a from the right, $f(x)$ approaches L.

Informally, each of the above four statements says,[376]

For **all** values of x that are **"close"** but **more than** a,
$f(x)$ is "close" (or possibly even equal) to L.

Or By making x **"sufficiently close"** but **more than** a,
$f(x)$ can be made as "close" as we like to L.

Not surprisingly, the limit of f at a is L *if and only if* the left- and right-hand limits of f at a are both also L:

Fact 196. *Suppose f is a nice function. Then*

$$\lim_{x \to a} f(x) = L \qquad \Longleftrightarrow \qquad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

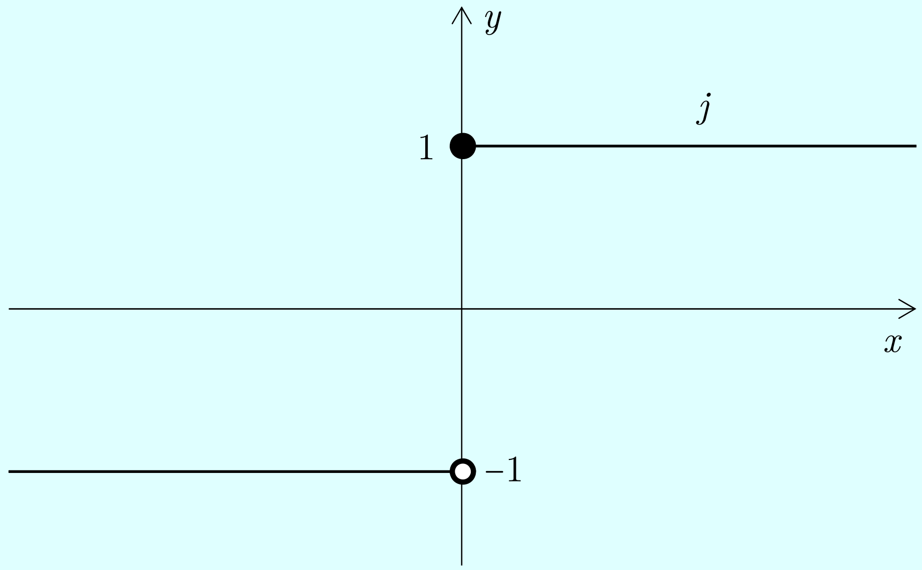
Proof. See p. 1655 (Appendices). □

Corollary 41. *If the left- and right-hand limits of f at a are not equal, then the limit of f at a does not exist.*

[375] For a formal definition of left-hand limits, see Definition 320 (Appendices).
[376] For a formal definition of right-hand limits, see Definition 320 (Appendices).

Example 1082. Continue to define $j : \mathbb{R} \to \mathbb{R}$ by

$$j(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$

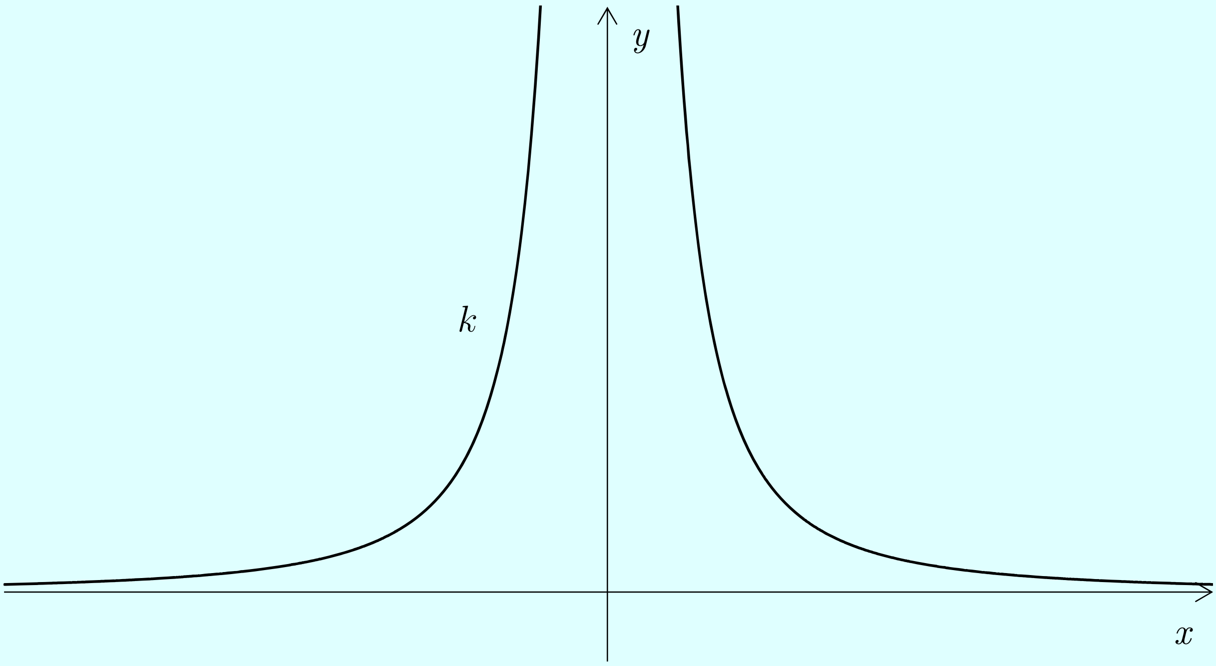


The left- and right-hand limits of j at 0 *do* exist:

$$\lim_{x \to 0^-} j(x) = -1 \qquad \text{and} \qquad \lim_{x \to 0^+} j(x) = 1.$$

However, they are not equal. And so by Fact 196, the limit of j at 0, $\lim_{x \to 0} j(x)$, does not exist.

Example 1083. Define $k : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $k(x) = \frac{1}{x^2}$.



The limit of k at 0—or $\lim_{x \to 0} k(x)$—**does not exist** because there is no number L such that

For **all** values of x that are **"close"** but **not equal** to 0,
$k(x)$ is "close" (or possibly even equal) to L.

Nonetheless, we are allowed to say that "the limit of k at 0 is infinity" and write

$$\lim_{x \to 0} k(x) = \infty.$$

(As we learnt in Ch. 25, we'll also say that the vertical line $x = 0$—also the y-axis—is a **vertical asymptote** for the function k.)

Here you may be confused—we've just made two seemingly contradictory statements:

- "The limit of k at 0 **does not exist**." (Or, "$\lim_{x \to 0} k(x)$ does not exist.")
- "The limit of k at 0 is infinity." (Or, "$\lim_{x \to 0} k(x) = \infty$.")

But strangely enough, both of the above statements are true. How can this be?

The key here is to recall the point made and emphasised on p. 52—∞ is **not a number**. Instead, it is merely an occasionally convenient symbol.

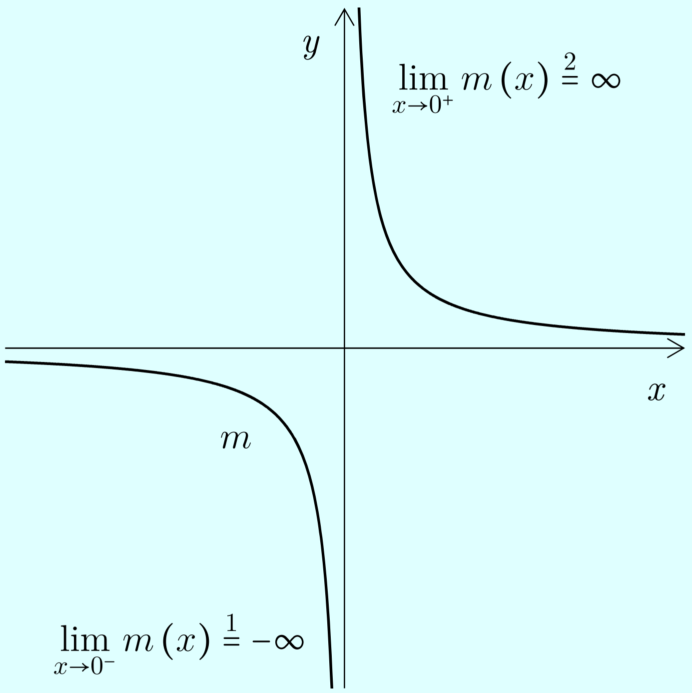
We've written, "The limit of k at 0 is infinity," or, "$\lim_{x \to 0} k(x) \stackrel{\star}{=} \infty$."

Importantly, $\stackrel{\star}{=}$ is simply convenient **shorthand** for this informal statement:

As x "approaches" 0, $k(x)$ "grows" without upper bound.[377]

Importantly, $\stackrel{\star}{=}$ does **not** say that $\lim_{x \to 0} k(x)$ is equal or identical to some object called ∞. Indeed, by writing $\stackrel{\star}{=}$, we do **not** even commit to the existence of an object called ∞.

Example 1084. Define $m : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $m(x) = 1/x$.



The limit of m at 0—or $\lim_{x \to 0} m(x)$—**does not exist**, because there is no number L such that

For **all** values of x that are **"close"** but **not equal** to 0,
$m(x)$ is "close" (or possibly even equal) to L.

It is likewise true that the **left-** and **right-hand limits** of m at 0—or $\lim_{x \to 0^-} m(x)$ and $\lim_{x \to 0^+} m(x)$—**do not exist**, because there is no number L such that

For **all** values of x that are **"close"** to but ***less* than** 0, $m(x)$ is "close" (or possibly even equal) to L;

or

For **all** values of x that are **"close"** to but ***more* than** 0, $m(x)$ is "close" (or possibly even equal) to L.

Yet at the same time, we may write

$$\lim_{x \to 0^-} m(x) \stackrel{1}{=} -\infty \qquad \text{and} \qquad \lim_{x \to 0^+} m(x) \stackrel{2}{=} \infty.$$

(So again, $x = 0$ or the y-axis is a vertical asymptote of m.)

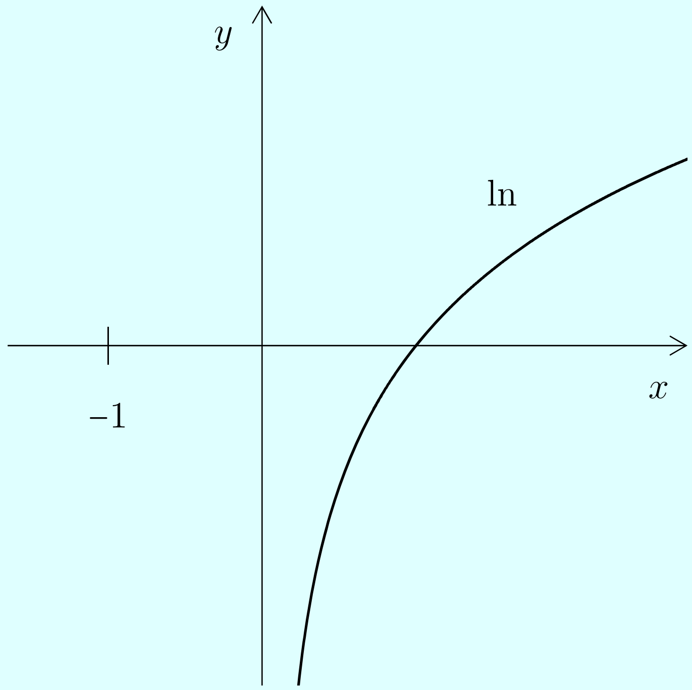
Again, neither $\stackrel{1}{=}$ nor $\stackrel{2}{=}$ acknowledges the existence of objects called $-\infty$ or ∞. Instead, informally, each simply says,

1. As x "approaches" 0 "from the left", $m(x)$ "keeps growing towards $-\infty$".
2. As x "approaches" 0 "from the right", $m(x)$ "keeps growing towards ∞".

By the way, in contrast with the previous example, these two statements are false:

$$\lim_{x \to 0} m(x) = \infty. \quad ✗ \qquad \lim_{x \to 0} m(x) = -\infty. \quad ✗$$

Example 1085. Recall that the natural logarithm function ln has domain $\mathbb{R}^+$.



Consider the limit of ln at -1 or $\lim_{x \to -1} \ln x$.

One key motivation for the concept of limits is to help us understand how a function behaves "near" a point.

If a function is undefined "near" a point, then there is nothing to understand. In which case, it makes sense to simply say that the limit does not exist at that point.[378]

So, here for example, since ln is undefined "near" -1, it makes sense to simply say that $\lim_{x \to -1} \ln x$ **does not exist**.

Indeed, given any negative number a, it is likewise true that $\lim_{x \to a} \ln x$ **does not exist**. Again, the reason is that ln is undefined "near" any negative number a.

By the way, what is the limit of ln at 0? Following our previous examples, we observe that the right-hand limit of ln at 0 is $-\infty$. We can write this as

$$\lim_{x \to 0^+} \ln x \stackrel{1}{=} -\infty$$

Or informally, as x "approaches" 0 "from the right", $\ln x$ "keeps growing towards $-\infty$". (By the way, does $\lim_{x \to 0^+} \ln x$ exist?)[379]

It is also true that for all values of x that are *in the domain* of ln, as x "approaches" 0, $\ln x$ "keeps growing towards $-\infty$". So—and this is a bit subtle—it is also true that

$$\lim_{x \to 0} \ln x \stackrel{2}{=} -\infty.$$

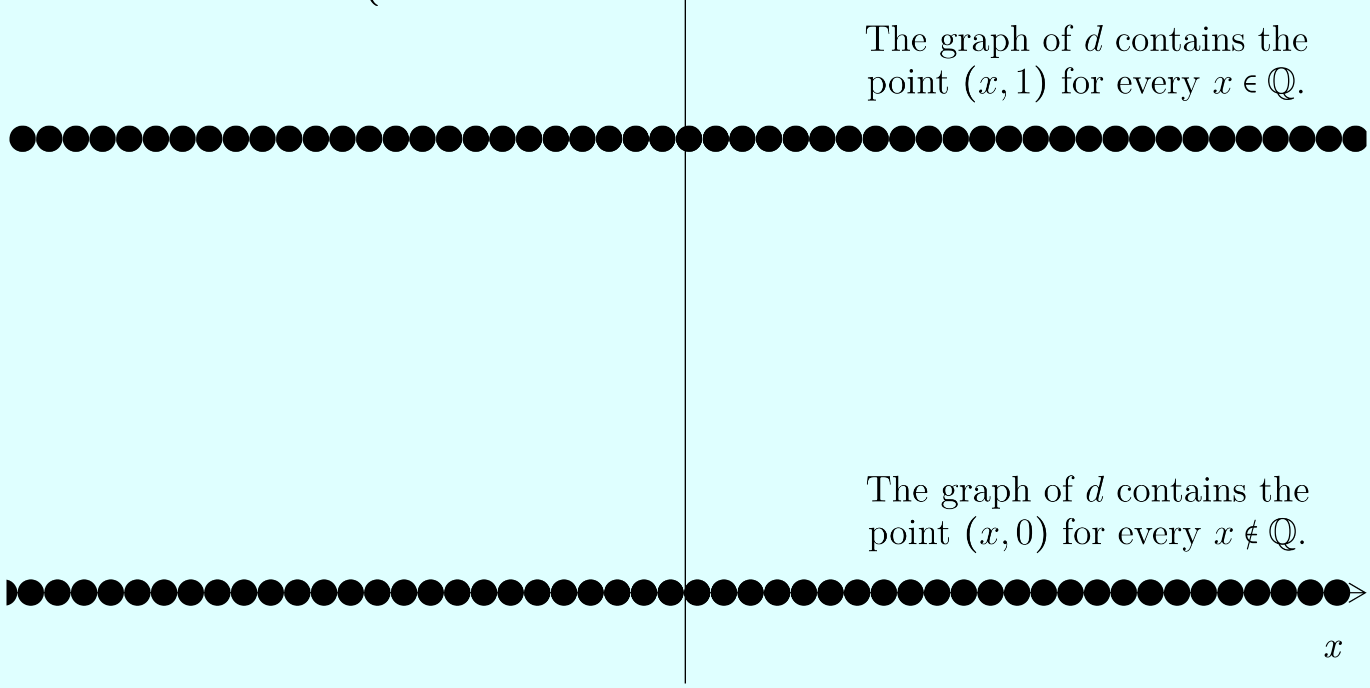
(Again, does $\lim_{x \to 0} \ln x$ exist?)[380]

[378]This sentence is a little informal, imprecise, and incorrect. More formally, precisely, and correctly, we

We now revisit the **Dirichlet function** (last examined on p. 236):

Example 1086. The Dirichlet function $d : \mathbb{R} \to \mathbb{R}$ is defined by:

$$d(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$



Observe that for every real number a, $\lim_{x \to a} d(x)$, does not exist.

To see why, consider any rational number, say 2. Since 2 is rational, we have $d(2) = 1$.

However, for values of x "near" 2, there is no number L that $d(x)$ stays "close" to. Instead, "near" 2, $d(x)$ takes on the values 0 and 1 "infinitely often". And so,

$$\lim_{x \to 2} d(x) \text{ does not exist.}$$

Next, consider any irrational number, say $\sqrt{2}$. Since $\sqrt{2}$ is irrational, we have $d\left(\sqrt{2}\right) = 0$.

However, for values of x "near" $\sqrt{2}$, there is no number L that $d(x)$ stays "close" to. Instead, "near" $\sqrt{2}$, $d(x)$ takes on the values 0 and 1 "infinitely often". And so,

$$\lim_{x \to \sqrt{2}} d(x) \text{ does not exist.}$$

---

should instead say that it makes sense to define limits only at points that are "near" other points in the function's domain (see Appendices, in particular Definitions 317 and 319).

379 No, $\lim_{x \to 0^+} \ln x$ does not exist. Again, this does not contradict $\overset{1}{=}$, which is merely shorthand for a more precise and formal statement whose meaning we've given informally as, "as x 'approaches' 0 'from the right', $\ln x$ 'keeps growing towards $-\infty$'".

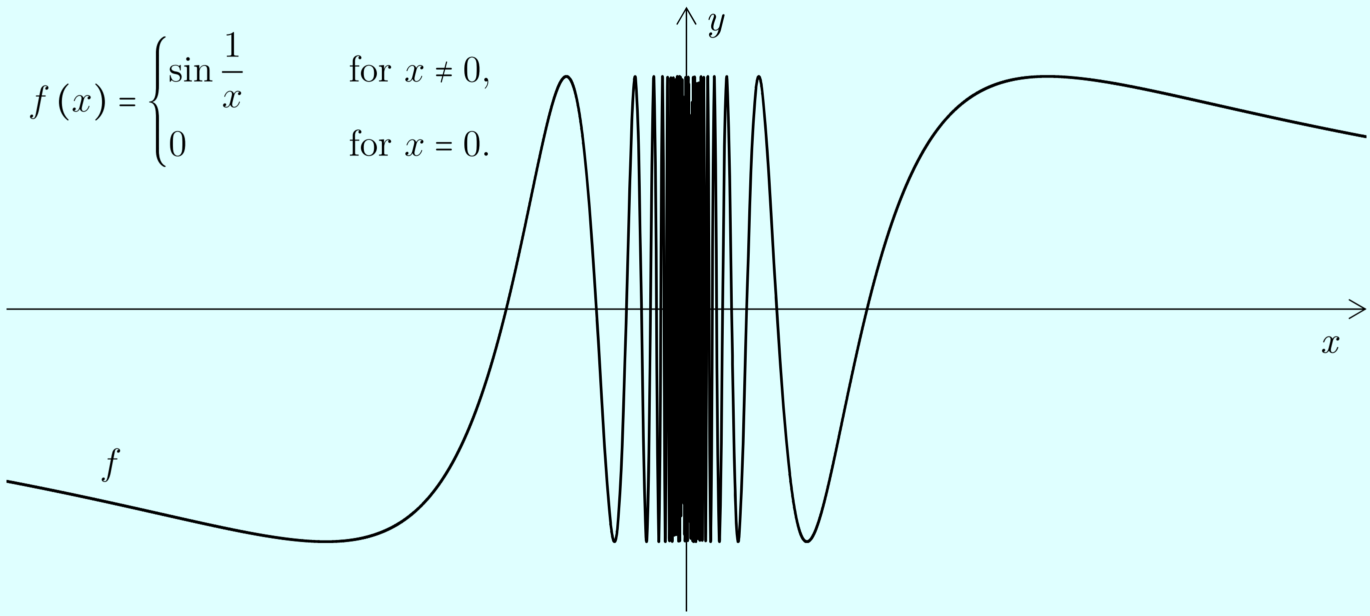
380 Again, no, $\lim_{x \to 0} \ln x$. And again, this does not contradict $\overset{2}{=}$.

Our next example is perhaps even stranger:

Example 1087. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

This is a very strange function indeed. Like sin, f takes on values between -1 and 1. But as x gets "closer" to 0, $f(x)$ fluctuates ever more rapidly between -1 and 1. (Why?) Indeed, when we're very "close" to 0, it's impossible to accurately depict the graph of f.



Observe that for all values of x that are "close to" but not equal to 0, there is no number L that $f(x)$ is "close to". Instead, when x is "close to" 0, $f(x)$ takes on every value in $[-1, 1]$ "infinitely often"! So, "near" 0, there is no number L that $f(x)$ stays "close to".

In other words, $\lim_{x \to 0} f(x)$ does not exist.

Exercise 327. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 2 & \text{for } x > 0. \end{cases}$$

What are $\lim_{x \to -5} f(x)$, $\lim_{x \to 0} f(x)$, and $\lim_{x \to 5} f(x)$? (Answer on p. **1891**.)

86.3. Rules for Limits

Happily, the usual arithmetic operations are preserved when we take limits, so that we have the following simple, "obvious", and predictable **Rules for Limits**:

Theorem 23. (Rules for Limits) *Let f and g be nice functions; and $k, L, M \in \mathbb{R}$. Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then*

(a) $\lim_{x\to a}[kf(x)] \overset{\text{F}}{=} kL$ **(Constant Factor Rule for Limits)**

(b) $\lim_{x\to a}[f(x) \pm g(x)] \overset{\pm}{=} L + M$ **(Sum and Difference Rules for Limits)**

(c) $\lim_{x\to a}[f(x)g(x)] \overset{\times}{=} LM$ **(Product Rule for Limits)**

(d) $\lim_{x\to a}\dfrac{1}{g(x)} \overset{\text{R}}{=} \dfrac{1}{M}$ *(for $M \neq 0$)* **(Reciprocal Rule for Limits)**

(e) $\lim_{x\to a}\dfrac{f(x)}{g(x)} \overset{\div}{=} \dfrac{L}{M}$ *(for $M \neq 0$)* **(Quotient Rule for Limits)**

(f) $\lim_{x\to a} k \overset{\text{C}}{=} k$ **(Constant Rule for Limits)**

(g) $\lim_{x\to a} x^k \overset{\text{P}}{=} a^k$ **(Power Rule for Limits)**

Proof. See p. 1660 (Appendices). □

Example 1088. Let $k = 5$. Define $f, g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2 + 1 \qquad \text{and} \qquad g(x) = \sqrt{x}.$$

Let's find the limits of f and g at 1 using the above Rules:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(x^2+1\right) \overset{\pm}{=} \lim_{x\to 1} x^2 + \lim_{x\to 1} 1 \overset{\text{C}}{=} \lim_{x\to 1} x^2 + 1 \overset{\text{P}}{=} 1^2 + 1 = 2.$$

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}\sqrt{x} \overset{\text{P}}{=} 1^{0.5} = 1.$$

Also by the above Rules, we have

(a) $\lim_{x\to 1}[kf(x)] \overset{\text{F}}{=} 5 \times 2 = 10$

(b) $\lim_{x\to 1}[f(x) + g(x)] \overset{\pm}{=} 2 + 1 = 3$

$\lim_{x\to 1}[f(x) - g(x)] \overset{\pm}{=} 2 - 1 = 1$

(c) $\lim_{x\to 1}[f(x)g(x)] \overset{\times}{=} 2 \times 1 = 2$

(d) $\lim_{x\to 1}\dfrac{1}{g(x)} \overset{\text{R}}{=} \dfrac{1}{1} = 1$ (since $\lim_{x\to 1} g(x) \neq 0$)

(e) $\lim_{x\to 1}\dfrac{f(x)}{g(x)} \overset{\div}{=} \dfrac{2}{1} = 2$ (since $\lim_{x\to 1} g(x) \neq 0$)

(f) $\lim_{x\to 1} k \overset{\text{C}}{=} 5$

(g) $\lim_{x\to 1} x^k \overset{\text{P}}{=} 1^5 = 1$

Remark 125. As stated, limits are **not** on your H2 Maths syllabus. And so, *a fortiori*, neither are the above Rules for Limits.

Nonetheless, I have decided to include the above Rules in the main text anyway, because they are so simple, "obvious", and easy to remember. Moreover, they'll be quite useful later when we learn to compute derivatives.

Exercise 328. Let $k = 7$. Define $h, i : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \ln(x+1) \qquad \text{and} \qquad i(x) = \sin x + 2.$$

Sketch the graphs of h and i. Based on your graphs, write down $\lim_{x\to 0} h(x)$ and $\lim_{x\to 0} i(x)$. Then also write down $\lim_{x\to 0}[kh(x)]$, $\lim_{x\to 0}[h(x)+i(x)]$, $\lim_{x\to 0}[h(x)-i(x)]$, $\lim_{x\to 0}[h(x)i(x)]$, $\lim_{x\to 0}\frac{1}{i(x)}$, $\lim_{x\to 0}\frac{h(x)}{i(x)}$, $\lim_{x\to 0} k$, and $\lim_{x\to 0} x^k$. (Answer on p. 1891.)

87. Continuity, Revisited

Ch. 18 briefly discussed the concept of **continuity** and stated that a function whose entire graph can be drawn without lifting your pencil is continuous.

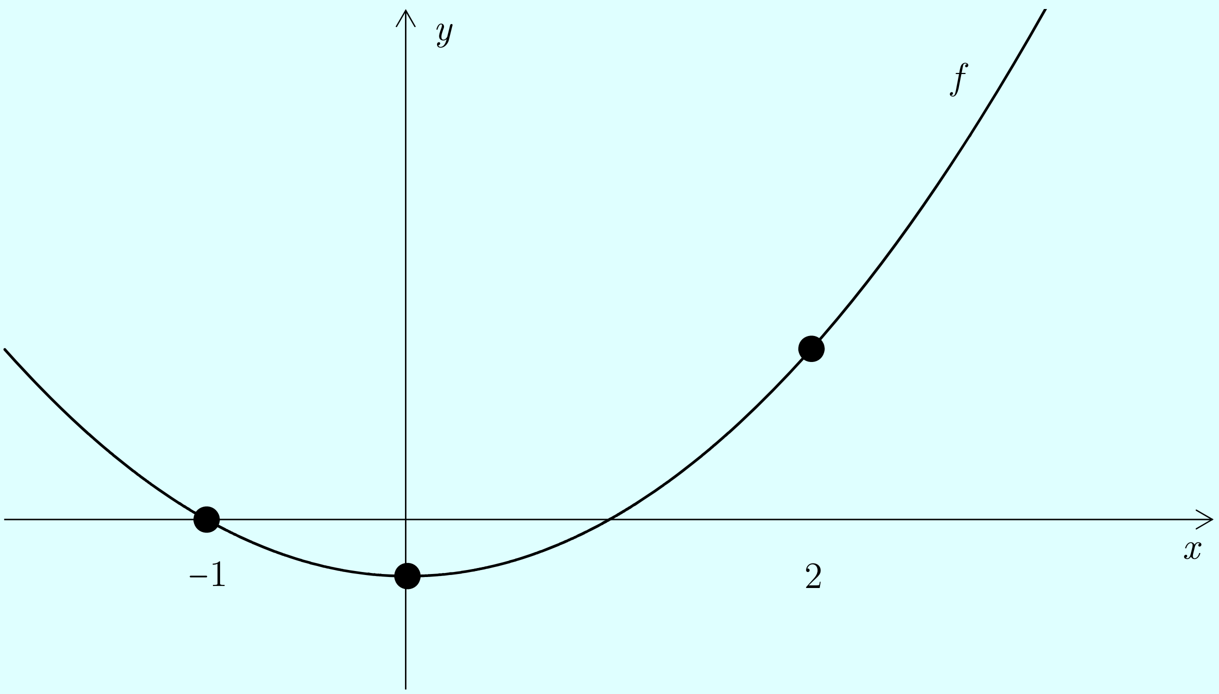
Now that we have an intuitive grasp of limits, we can formally and precisely define continuity:

Definition 198. Let f be a nice function whose domain contains the point a. We say that f is *continuous at a* if[381]

$$\lim_{x \to a} f(x) = f(a),$$

or if a is an isolated point of D.[382]

Example 1089. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 1$.



The limits of f at -1, 0, and 2 are

$$\lim_{x \to -1} f(x) = 0, \qquad \lim_{x \to 0} f(x) = -1, \quad \text{and} \quad \lim_{x \to 2} f(x) = 3.$$

The values of f at -1, 0, and 2 are

$$f(-1) = 0, \qquad f(0) = -1, \quad \text{and} \quad f(2) = 3.$$

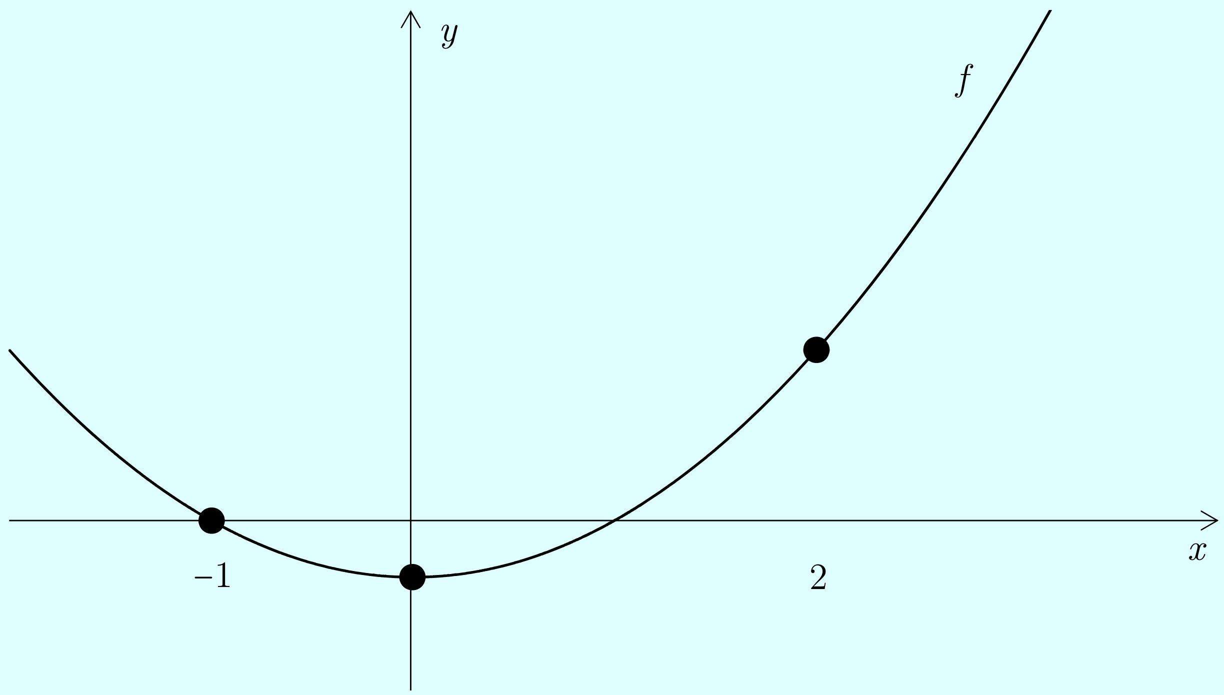
Conclude: f is continuous at -1, 0, and 2, because

$$\lim_{x \to -1} f(x) = f(-1), \qquad \lim_{x \to 0} f(x) = f(0), \quad \text{and} \quad \lim_{x \to 2} f(x) = f(2).$$

[382]You can treat the fine print here as an annoying technicality that doesn't matter for A-Level Maths. Ch. 87.7 (optional) briefly discusses this technicality.

Definition 199. A function is *continuous on a set* if it is continuous at every point in that set. A function is *continuous* if it is continuous on its domain.

Example 1090. Continue with the last example.



Above, we concluded that f is continuous at -1, 0, and 2, because

$$\lim_{x\to-1} f(x) = f(-1), \qquad \lim_{x\to 0} f(x) = f(0), \quad \text{and} \quad \lim_{x\to 2} f(x) = f(2).$$

It turns out that for every $a \in \text{Domain } f = \mathbb{R}$, it is also true that

$$\lim_{x\to a} f(x) = f(a).$$

So, f is continuous at every point a in its domain $\mathbb{R}$.

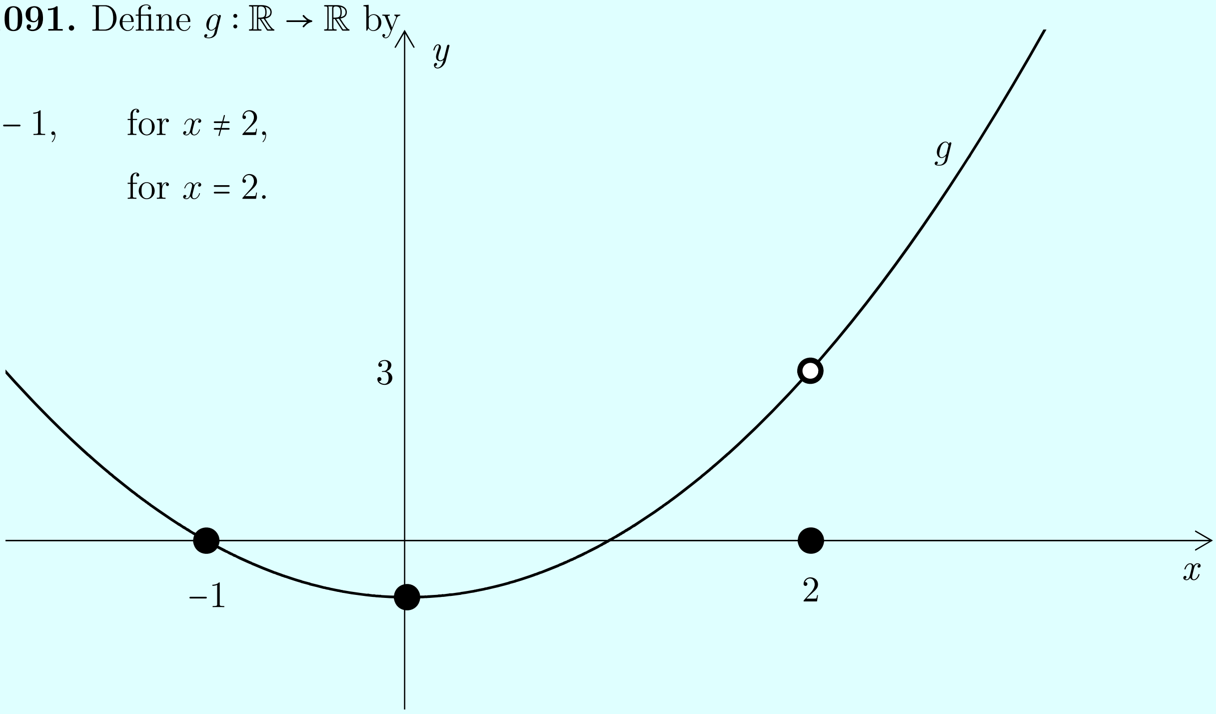
By Definition 199 then, f is **(a)** continuous on $\mathbb{R}$; and **(b)** a continuous function.

87.1. Functions with a Single Discontinuity

Definition 200. Let f be a nice function and $a \in \text{Domain } f$. If f is not continuous at a, then we say that f is *discontinuous at a* (or f *has a discontinuity at a*).

Example 1091. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^2 - 1, & \text{for } x \neq 2, \\ 0, & \text{for } x = 2. \end{cases}$$



The function g is almost identical to f in the last example. The only difference is that whereas $f(2) = 3$, we now have $g(2) = 0$ and we have a "hole" at $(2, 3)$.

It is again the case that g is continuous at -1 because

$$\lim_{x \to -1} g(x) = 0, \qquad g(-1) = 0, \quad \text{and hence} \quad \lim_{x \to -1} g(x) = g(-1).$$

Similarly, g is continuous at 0 because

$$\lim_{x \to 0} g(x) = -1, \qquad g(0) = -1, \quad \text{and hence} \quad \lim_{x \to 0} g(x) = g(0).$$

However, g is **not** continuous at 2 because

$$\lim_{x \to 2} g(x) = 3, \quad \text{but} \quad g(2) = 0, \quad \text{so that} \quad \lim_{x \to 2} g(x) \neq g(2).$$

We say that g is *discontinuous* (or has a *discontinuity*) at 2.

It turns out that g is continuous at every point in its domain $\mathbb{R}$ except at 2.

Since g has a single discontinuity (namely at 2), it is **not** a continuous function. This is because the existence of a single discontinuity disqualifies a function from being called continuous.

Nonetheless, we may say that g is continuous on $\mathbb{R} \setminus \{2\}$ or $(-\infty, 2) \cup (2, \infty)$. (Equivalently, g is continuous everywhere except at 2.)

It turns out that this discontinuity is an example of a *removable discontinuity*. Informally, a removable discontinuity is a "hole" at $(2, 3)$.

Informally, we could "patch" this "hole" by simply moving the "black dot" at $(2, 0)$ up to $(2, 3)$ (where the "hole" is)—if we did this, then the new function thus created would be continuous.

Two more examples of removable discontinuities:

Example 1092. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} x, & \text{for } x \neq 0, \\ 1, & \text{for } x = 0. \end{cases}$$

Figure to be
inserted here.

The function h has a removable discontinuity at 0.

Example 1093. Define $i : \mathbb{R} \to \mathbb{R}$ by

$$i(x) = \begin{cases} -x, & \text{for } x < 0, \\ 1 & \text{for } x = 0, \\ x, & \text{for } x > 0. \end{cases}$$

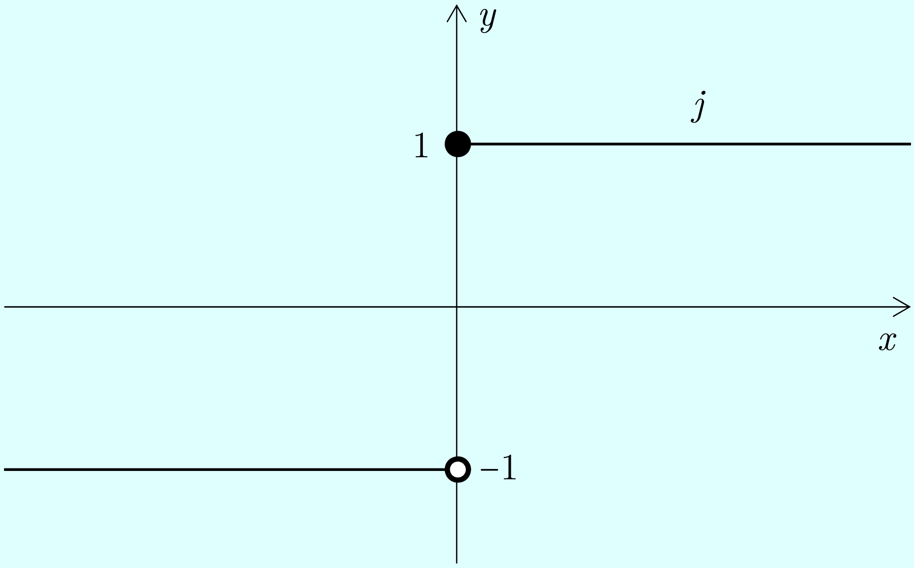
Figure to be
inserted here.

The function i has a removable discontinuity at 0.

We've just looked at three examples of removable discontinuities. We now look at examples of **jump discontinuities**:

Example 1094. Define $j : \mathbb{R} \to \mathbb{R}$ by

$$j(x) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$



We have $j(0) = 1$. However, $\lim_{x \to 0} j(x)$ does not exist.

Since $\lim_{x \to 0} j(x) \neq j(0)$, j is discontinuous (or has a discontinuity) at 0.

Like the last three examples, j is again continuous everywhere except at a single point (namely 0). And so again, j is **not** a continuous function.

But unlike the last three examples, here we have instead a **jump discontinuity**. Informally, a jump discontinuity is where the function "jumps".

A jump discontinuity is "worse" than a removable discontinuity because it's "harder to fix". A removable discontinuity can be "fixed" by simply "patching a hole".

Here we see (at least) two ways to "fix" the jump discontinuity: Shift **(i)** j's left half upwards; or **(ii)** its right half downwards.

Two more examples of jump discontinuities:

Example 1095. Define $i : \mathbb{R} \to \mathbb{R}$ by

$$i(x) = \begin{cases} -x, & \text{for } x < 1, \\ x, & \text{for } x \geq 1. \end{cases}$$

Figure to be inserted here.

The function i has a jump discontinuity at 1.

Example 1096. Define $k : \mathbb{R} \to \mathbb{R}$ by

$$k(x) = \begin{cases} x^2, & \text{for } x \leq 0, \\ x^2 - 1, & \text{for } x > 0. \end{cases}$$

Figure to be inserted here.

The function k has a jump discontinuity at 0.

We've just looked at removable and jump discontinuities. There's also a third type of discontinuity that's simply a catch-all category for "everything else"—if a discontinuity is neither removable nor jump, then we call it an **essential** (or **infinite**) **discontinuity**.

You don't need to know the formal definitions for A-Level Maths. But they aren't difficult to understand, so I'll just put them here:

Definition 201. Suppose the function f has a discontinuity at a. Then this discontinuity is called a

(a) *Removable discontinuity* if $\lim_{x \to a} f(x)$ exists;

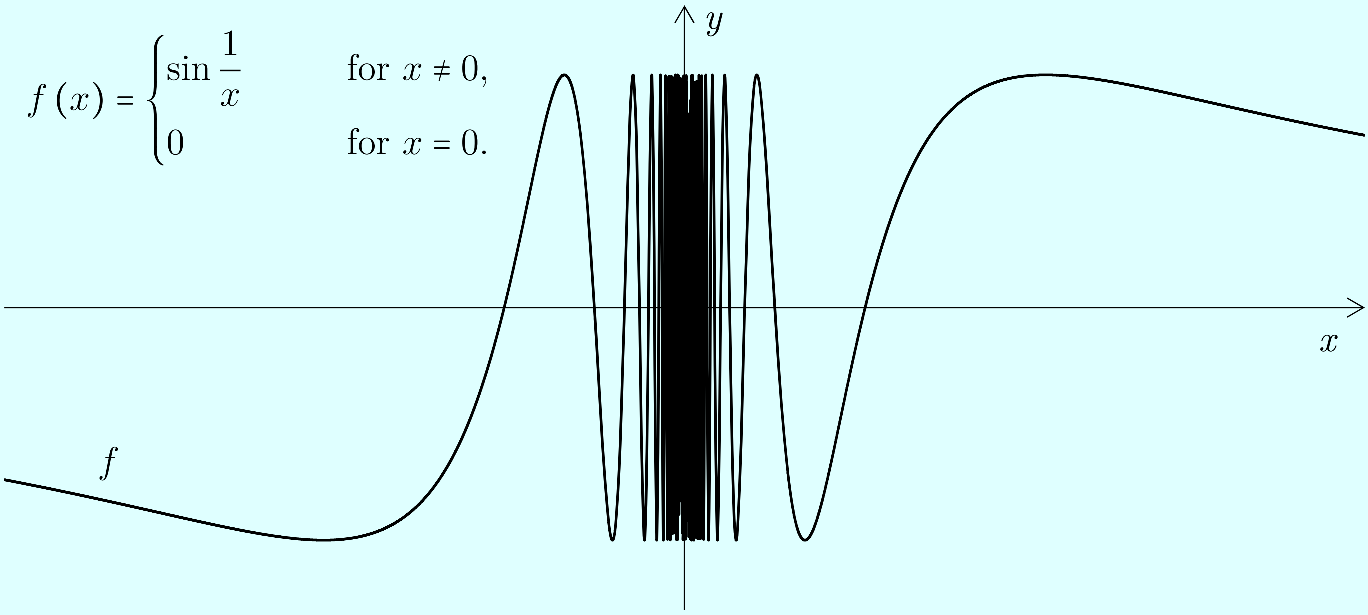
(b) *Jump discontinuity* if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal; or

(c) *Essential* (or *infinite*) *discontinuity* if it is neither removable nor jump.

Example of an **essential** (or **infinite**) **discontinuity**:

Example 1097. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$



As explained in Example 1087 (previous chapter),

$$\lim_{x \to 0} f(x) \text{ does not exist.}$$

So, f is discontinuous at 0 and is a discontinuous function.

Nonetheless, f is continuous everywhere except at a single point (namely 0). That is, f is continuous on $(-\infty, 0) \cup (0, \infty)$ or $\mathbb{R} \setminus \{0\}$.

It is possible to prove that the discontinuity at 0 is neither removable nor jump:

Since $\lim_{x \to 0} f(x)$ does not exist, by Definition 201, it is not a removable discontinuity.

It is also possible to show that $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ do not exist,[383] so that again by Definition 201, it is not a jump discontinuity either.

Since this discontinuity is neither removable nor jump, by Definition 201, it is an *essential* (or *infinite*) *discontinuity.*

From Definition 201, it is not difficult to prove this result:

Fact 197. *Suppose f is a nice function whose domain contains the point a. Then*

$$\begin{aligned} & f \textit{ has an essential discontinuity at } a \\ \iff \quad & \textit{At least one of } \lim_{x \to a^-} f(x) \textit{ and } \lim_{x \to a^+} f(x) \textit{ does not exist.} \end{aligned}$$

Proof. ($\Longleftarrow$) If at least one of $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ does not exist, then f has a discontinuity at a that is neither removable nor jump and is hence essential.

($\Longrightarrow$ by contrapositive) If both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist, then f is either continuous

[383] We do so in Example 1583 (Appendices).

at a or has either a removable or jump discontinuity at a. □

Two more examples of essential discontinuities:

Example 1098. Define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\begin{cases}\dfrac{1}{x} & \text{for } x\neq 0,\\ 0 & \text{for } x=0.\end{cases}$$

Figure to be
inserted here.

We have $g(0)=0$, but $\lim\limits_{x\to 0} g(x)$ does not exist.

Moreover, $\lim\limits_{x\to 0^-} g(x)$ and $\lim\limits_{x\to 0^+} g(x)$ do not exist (though we can write $\lim\limits_{x\to 0^-} g(x)=-\infty$ and $\lim\limits_{x\to 0^+} g(x)=\infty$).

And so by Fact 197, g has an essential discontinuity at 0.

Example 1099. Define $h:\mathbb{R}\to\mathbb{R}$ by

$$h(x)=\begin{cases}x & \text{for } x\leq 0,\\ \sin\dfrac{1}{x} & \text{for } x>0.\end{cases}$$

Figure to be
inserted here.

We have $h(0)=0$.

Here, the left-hand limit of h at 0 *does* exist: $\lim\limits_{x\to 0^-} h(x)=0$.

However, the right-hand limit of h at 0—or $\lim\limits_{x\to 0^+} h(x)$—does not exist.

And so by Fact 197, h has an essential discontinuity at 0.

Exercise 329. State whether each of the three graphed functions is continuous. If not, find all discontinuities and state their type. (Answer on p. 842.)

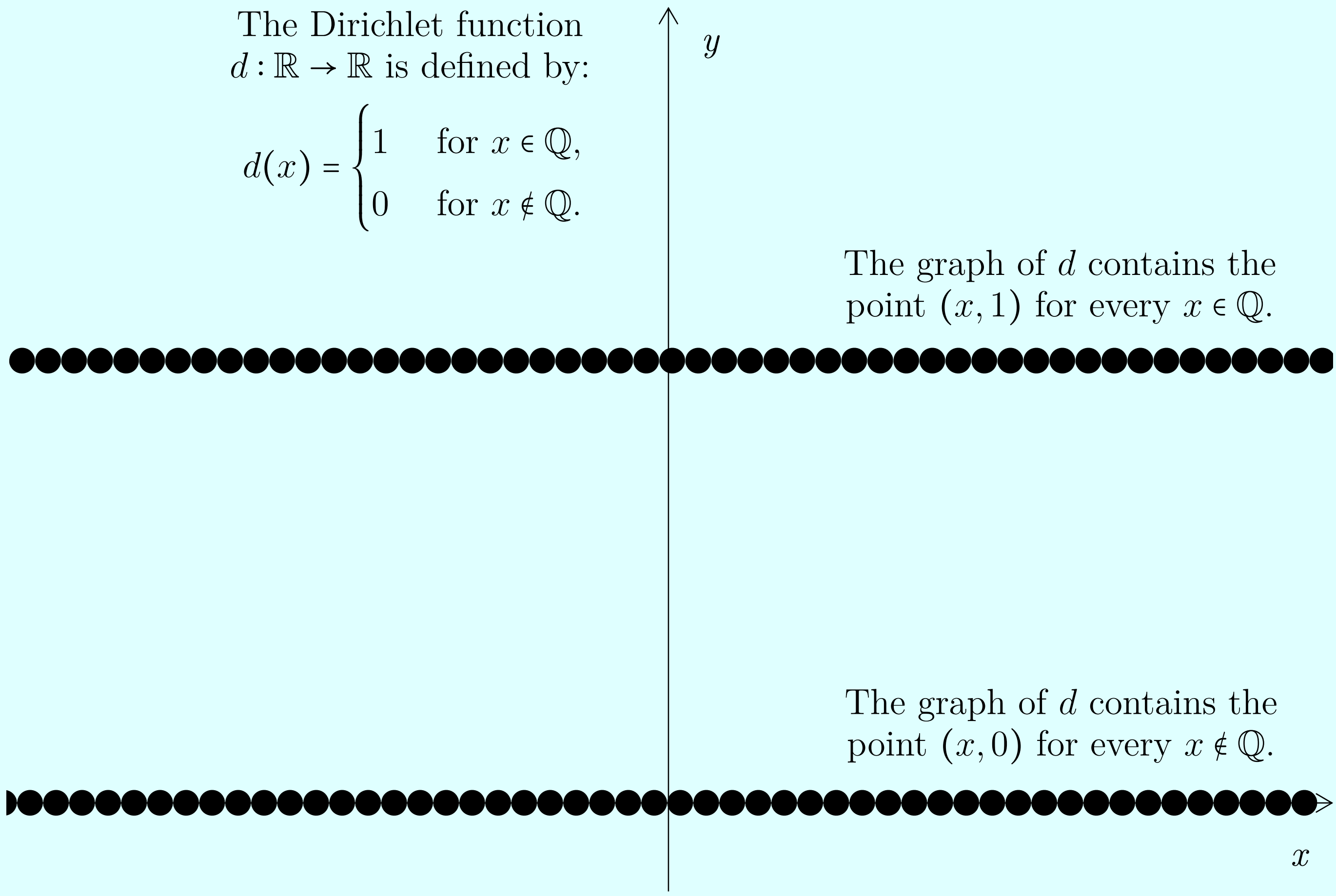
Figure to be
inserted here.

A329. XXX

87.2. An Example of a Function That Is Discontinuous Everywhere

Each function examined in the previous subchapter had exactly one discontinuity. Here now is an example of a function that is discontinuous everywhere:

Example 1100. We revisit the Dirichlet function d.[384]



Observe that for each $a \in \mathbb{R}$, $d(a)$ is defined (i.e. is equal to a real number); but $\lim_{x \to a} d(x)$ does not exist; so, $d(a) \neq \lim_{x \to a} d(x)$.

Hence, d is discontinuous at *every* point in $\mathbb{R}$. Equivalently, d is discontinuous *everywhere*.

Using Definition 201, it is possible to prove that *none* of these discontinuities is either removable or jump. So, each is an essential (or infinite) discontinuity.

Hence, we may say that d is **essentially** (or **infinitely**) **discontinuous everywhere**. Or equivalently, d has an essential (or infinite) discontinuity at every point $a \in \mathbb{R}$.

[384]Last examined on p. 829.

87.3. Functions That Seem Discontinuous But Aren't

Subtle point:

Continuity and discontinuity are defined only on a function's domain.

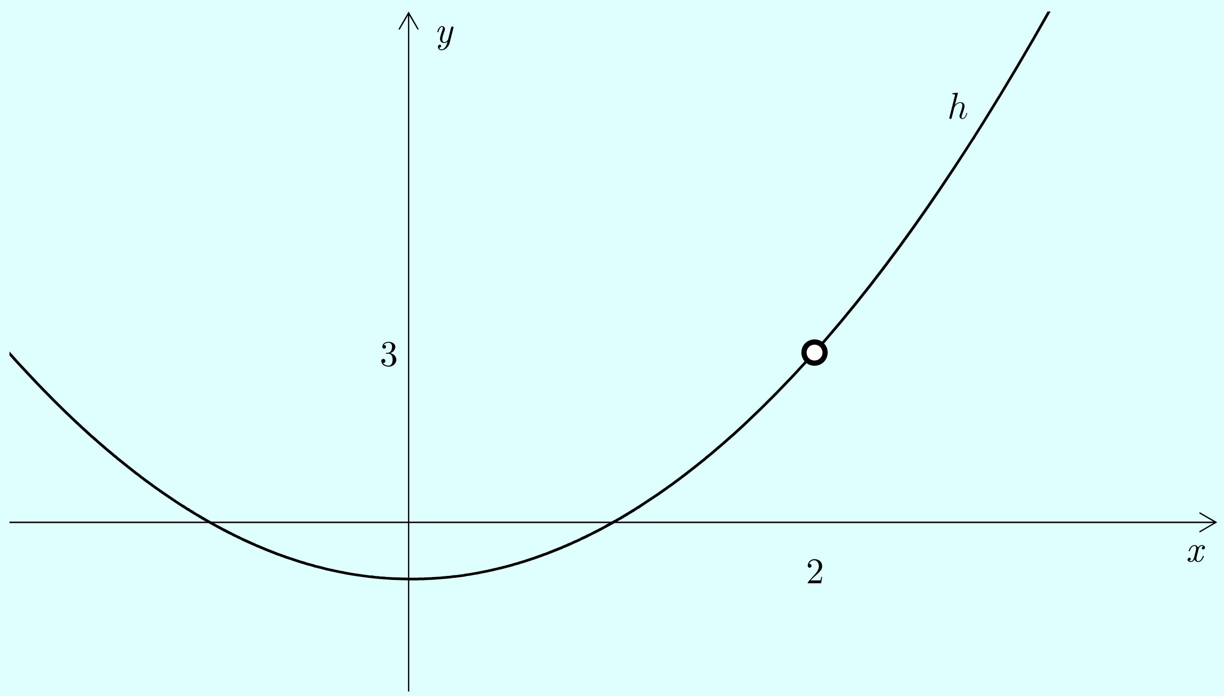
That is, a function may be said to be continuous or discontinuous *only* at points in its domain. If a point isn't in the function's domain, then the function is *neither continuous nor discontinuous* at that point.

In each of the next three examples, we are tempted to say that the function has at least one discontinuity. It turns out though that each function is continuous at every point in its domain and is therefore a continuous function!

Example 1101. Define the function $h : \mathbb{R} \setminus \{2\} \to \mathbb{R}$ by $h(x) = x^2 - 1$.

We are tempted to say that h is discontinuous at 2. But this would be wrong because 2 is **not** in the domain of h (equivalently, h is not defined at 2).

And so, h is neither continuous nor discontinuous at 2.



Indeed, perhaps surprisingly, h is a continuous function because h is continuous at every point in its domain.

Example 1102. Define the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = \dfrac{1}{x}$.

We are tempted to say that f is discontinuous at 0. But this would be wrong because 0 is not in the domain of f (equivalently, f is **not** defined at 0).

And so, f is neither continuous nor discontinuous at 0.

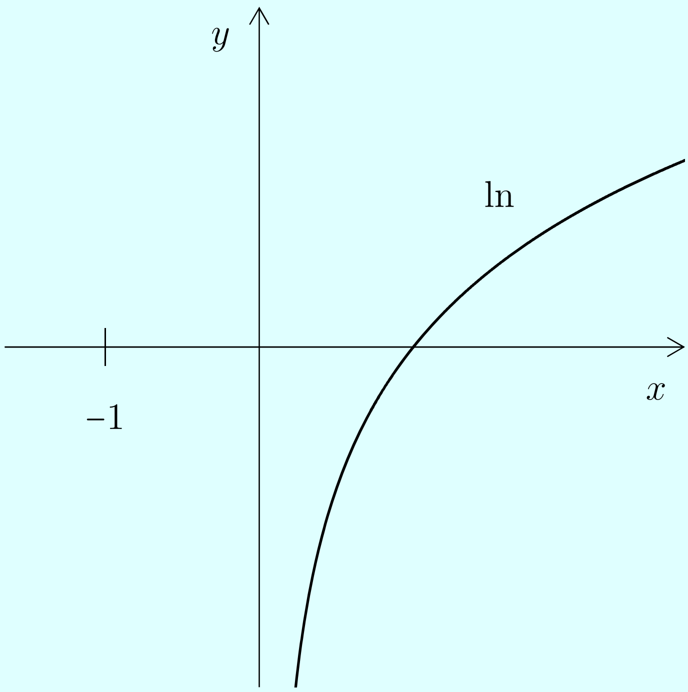
Figure to be
inserted here.

Indeed, perhaps surprisingly, f is a continuous function. That is, f is continuous at every point in its domain.

Example 1103. As noted in the previous chapter, $\lim_{x \to -1} \ln x$ **does not exist** or **is undefined**.

We are tempted to say that ln is discontinuous at -1. But this would be wrong because -1 is **not** in the domain of ln (which is $\mathbb{R}^+$). (Equivalently, ln is **not** defined at -1.)

And so, ln is neither continuous nor discontinuous at -1.



Indeed, given any $a \in (-\infty, 0]$, ln is neither continuous nor discontinuous at a.

Nonetheless, ln is a continuous function because it is continuous at every point in its domain $\mathbb{R}^+$.

Exercise 330. State whether each of the following graphed functions is continuous. If not, identify any discontinuities and their type. (Answer on p. 846.)

A330. XXX

87.4. Every Elementary Function Is Continuous

We reproduce from Ch. 37 this definition:

Definition 202. An *elementary function* is

a polynomial function, a trigonometric function,
an inverse trigonometric function, a natural logarithm function,
an exponential function, a power function,
any arithmetic combination of two elementary functions,
or any composition of two elementary functions.

Recall that *most* functions we'll encounter in H2 Maths are **elementary**. And very happily,

Theorem 24. *Every elementary function is continuous.*

Proof. See Ch. 87.6. □

In sum,

- Most functions we'll encounter in A-Level Maths are elementary.
- All elementary functions are continuous.
- Therefore, most functions we'll encounter in A-Level Maths are continuous.

Example 1104. XXX

Example 1105. Suppose we are asked to evaluate $\lim_{x\to 0} \sin x^2$.

We are tempted to do this:

$$\lim_{x\to 0} \sin x^2 \overset{\star}{=} \sin\left(\lim_{x\to 0} x^2\right) = \sin 0 = 0.$$

It turns out that the above *is* correct.

However, $\overset{\star}{=}$ requires justification. Why is it that we can simply "move" the limit in?

One of the (many) nice things about continuity is the next result: When taking the limit of a composite function, we can "move" the limit in if the "outer" function is continuous. This justifies $\overset{\star}{=}$ in the above example—since the "outer" function sin is continuous, we can "move" $\lim_{x\to 0}$ in.

Theorem 25. *Let f and g be nice functions such that the composite function fg is well-defined. Let $b \in \mathbb{R}$. Suppose $\lim_{x\to a} g(x) = b$ and f is continuous at b. Then*

$$\lim_{x\to a} f(g(x)) = f(b).$$

Proof. See p. 1667 (Appendices). □

In the previous subchapter, we saw that all elementary functions are continuous. Since (almost) all functions we'll ever encounter in H2 Maths are elementary, this means that happily enough, we can (almost) always apply the above result:

Example 1106. Consider $\lim_{x\to 1} \ln x$.

By the Power Rule, $\lim_{x\to 1} x \overset{\text{P}}{=} 1$.

Since ln is continuous at 1, by Fact 25, we can "move" the limit in:

$$\lim_{x\to 1} \ln x \varnothing 1 \ln\left(\lim_{x\to 1} x\right) = \ln 1 = 0.$$

Example 1107. Consider $\lim_{x\to 1} [\sin(\ln x)]$.

The previous example already showed that $\lim_{x\to 1} \ln x = 0$.

Since sin is continuous at 0, by Fact 25, we can "move" the limit in:

$$\lim_{x\to 1} [\sin(\ln x)] \varnothing 1 \sin\left(\lim_{x\to 1} \ln x\right) = \sin 0 = 0.$$

Example 1108. Consider $\lim_{x\to 1}(x+\ln x)^2$.

Using the Sum Rule for Limits and what we found in above two examples,

$$\lim_{x\to 1}(x+\ln x) \stackrel{\pm}{=} \lim_{x\to 1} x + \lim_{x\to 1}\ln x = 1+0 = 1.$$

Since the squaring function is continuous at 1, by Fact 25, we can "move" the limit in:

$$\lim_{x\to 1}(x+\ln x)^2 \varnothing 1 \left[\lim_{x\to 1}(x+\ln x)\right]^2 = 1^2 = 1.$$

Note that if the "outer" function is not continuous at the given point, then Fact 25 does not apply and we may not be able to "move" the limit in:

Example 1109. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x$ and

$$f(x) = \begin{cases} 1 & \text{for } x \neq 0, \\ 2 & \text{for } x = 0. \end{cases}$$

Figure to be
inserted here.

Then the composite function $f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f\circ g)(x) = f(g(x)) = f(x) = \begin{cases} 1 & \text{for } x \neq 0, \\ 2 & \text{for } x = 0. \end{cases}$$

Now, observe that

$$\lim_{x\to 0} f(g(x)) = \lim_{x\to 0} f(x) = 1.$$

However, if we try to "move" the limit in as usual, we get something else:

$$\lim_{x\to 0} f(g(x)) \varnothing ? f\left(\lim_{x\to 0} g(x)\right) = f(0) = 2. \qquad ✗$$

The step $\varnothing ?$ is wrong and does not work, because f is not continuous at $\lim_{x\to 0} g(x) = 0$.

Happily, most functions we'll encounter in H2 Maths are continuous. So the above example

is not likely to be a problem you'll ever encounter.

Exercise 331. Find **(a)** $\lim_{x \to 0} (x+1)^3$ and **(b)** $\lim_{x \to 0} \sin(\cos x^2)$. (Answer on p. 1892.)

87.6. Every Elementary Function is Continuous: Proof (Optional)

In this optional subchapter, we'll try to prove Theorem 24, i.e. that every elementary function is continuous.

"Obviously",

Fact 198. *Every constant function is continuous.*

Proof. This result may seem "obvious", but still requires a proof. This is given on p. 1665 (Appendices). □

Figure to be
inserted here.

Also "obviously",

Fact 199. *Every identity function is continuous.*

Proof. Again, this result may seem "obvious", but still requires a proof. This is given on p. 1665 (Appendices). □

The next result, which you may also find "obvious", says that continuity is preserved[385] under the four basic arithmetic operations and scalar multiplication:

Theorem 26. *Suppose f and g are continuous at $a \in \mathbb{R}$. Then so too is each of these functions:*

(a) $f \pm g$ **(b)** $f \cdot g$ **(c)** $\dfrac{f}{g}$ *(provided $g(a) \neq 0$)* **(d)** cf

Proof. See p. 1666 (Appendices). □

Using the above "obvious" results, we can easily prove that

Fact 200. *Every polynomial function is continuous.*

Proof. Let D be a set of real numbers; $c_0, c_1, \ldots, c_n$ be real numbers; and n be a non-negative integer.

Define $f : D \to \mathbb{R}$ by $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$. Our goal is to show that f is continuous.

[385] Or closed.

Define $g, h : D \to \mathbb{R}$ by $g(x) = x$ and $h(x) = c_0$.

By Fact 198, h is continuous.

By Fact 199, g is continuous.

By Theorem 28(**b**), $(g)^2 = g \cdot g$ is also continuous.

Similarly, for any $k = 1, 2, 3, \ldots$, the repeated application of Theorem 28(**b**) shows that $(g)^k = g \cdot g \cdot \cdots \cdot g$ is also continuous.

Now, observe that

$$f(x) = h(x) + c_1 g(x) + c_2 (g)^2 (x) + \cdots + c_n (g)^n (x).$$

Since f may be written as the sum of $n+1$ continuous functions, by the repeated application of Theorem 26(**a**), f is also continuous. □

Example 1110. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$ and $g(x) = -97x^7 + 10x^3 - x^2 + 5$.

Figure to be
inserted here.

Both f and g are polynomial functions. And so, by Fact 200, both f and g are continuous functions.

Fact 201. *The sine and cosine functions* (sin *and* cos) *are continuous.*

Proof. We will prove this on p. 1013. □

Figure to be
inserted here.

Fact 202. *The natural logarithm function,* ln, *is continuous.*

Proof. We will prove this in Ch. 111. □

The next result says that the **inverse of a continuous function defined on an interval is also continuous**:

Theorem 27. *Let D be an interval and $f : D \to \mathbb{R}$ be a one-to-one function. If f is continuous, then so too is its inverse f^{-1}.*

Proof. See p. 1670 (Appendices). □

Example 1111. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ is continuous.

Figure to be
inserted here.

It is also one-to-one, with inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(x) = x - 1$. By Theorem 27 then, f^{-1} should be continuous, as indeed it is.

Example 1112. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$ is continuous.

Figure to be
inserted here.

It is also one-to-one, with inverse $g^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $g^{-1}(x) = \sqrt[3]{x}$. By Theorem 27 then, g^{-1} should be continuous, as indeed it is.

Note the subtle but important requirement in Theorem 27 that the function's domain be an interval. If f is continuous and one-to-one but its domain is not an interval, then its inverse f^{-1} may not be continuous, as this counterexample shows:

Example 1113. Define $h : [0,1) \cup [2,3] \to [0,2]$ by

$$h(x) = \begin{cases} x & \text{for } x \in [0,1), \\ x-1 & \text{for } x \in [2,3]. \end{cases}$$

Figure to be inserted here.

The function h is continuous. Moreover, it is one-to-one, with inverse $h^{-1} : [0,2] \to [0,1) \cup [2,3]$ defined by

$$h^{-1}(x) = \begin{cases} x & \text{for } x \in [0,1), \\ x+1 & \text{for } x \in [1,2]. \end{cases}$$

Note though that the domain of h is not an interval, so that the conclusion of Theorem 27 may not hold—i.e., h^{-1} may not be continuous. And indeed, here we clearly see that h^{-1} isn't continuous.

Facts 201 and 202 said that sin, cos, and ln are continuous. Now, recall that

- $\sin^{-1}$ is the inverse of sin restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- $\cos^{-1}$ is the inverse of cos restricted to the interval $[0,\pi]$.
- exp is the inverse of ln, which is defined on the interval $(0,\infty)$.

And so, by Theorem 27,

Corollary 42. *The functions* $\sin^{-1}$, $\cos^{-1}$, *and* exp *are continuous.*

Corollary 43. *The functions* tan, cosec, sec, cot, *and* $\tan^{-1}$ *are continuous.*

Proof. See Exercise 332. □

Exercise 332. Prove Corollary 43. (You should carefully specify any definitions and results used at each step of the way.) (Answer on p. 1892.)

The **composition of two continuous functions is also continuous**:

Theorem 28. *Let f and g be nice functions with* $\text{Range}\, g \subseteq \text{Domain}\, f$. *If g is continuous at a and f is continuous at $g(a)$, then fg is also continuous at a.*

Proof. See p. 1665 (Appendices). □

Simple example to illustrate:

Example 1114. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(\cos x)$.

Since sin and cos are both continuous functions, by Theorem 28, f is also continuous.

Through the repeated application of Theorems 26 and 28, we can easily tell that even a complicated-looking function is continuous:

Example 1115. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \sin\left[\exp\left(x^2 + x + 1\right)\right] - 32x$.

The function g is built from the arithmetic combinations and compositions of continuous functions. And so, by Theorems 26 and 28, g is continuous.

Together, the results presented in this subchapter prove[386] Theorem 24: *Every* elementary function is continuous.

[386] Actually, there is one more thing we need to do: We need to also show that any function created from a continuous function through a domain restriction is itself also continuous. This is done in Theorem 52 (Appendices).

87.7. Continuity at Isolated Points (optional)

In Definition 198 of continuity, the fine print contained an annoying technicality, telling us that

If a is an isolated point of the domain of f, then f is continuous at a.

We now discuss why this additional fine print is necessary.

Recall that informally, an **isolated point** of a set is a point that isn't "close" to any other point in the set.[387]

1. It turns out that somewhat strangely, the formal definition of limits is such that the limit of a function is not defined at an isolated point.[388] And so, if a is an isolated point of the domain of f, then $\lim_{x \to a} f(x)$ is simply undefined.
2. Hence, $\lim_{x \to a} f(x) \neq f(a)$.
3. At the same time and also somewhat strangely, we'd like to say that a function is continuous at any isolated point (of its domain).[389]
4. Hence the need for the additional fine print, to ensure that a function is also continuous at its isolated points.

[387] For the formal definition, see Definition 276 (Appendices).

[388] See Definition 319 (Appendices)—in order for $\lim_{x \to a} f(x)$ to be defined, a must be a limit point and, in particular, not an isolated point.

Now, why might we leave $\lim_{x \to a} f(x)$ undefined in the case where a is an isolated point? Here's one reason: As discussed in Example 1085, "One key motivation for the concept of limits is to help us understand how a function behaves 'near' a point." If a is an isolated point, then there is no behaviour of f "near" a for us to understand and it might thus make sense to simply leave $\lim_{x \to a} f(x)$ undefined.

[389] Here are two reasons why we might want to do this:

First, we want to be able to say that the restriction of a continuous function to any subset of its domain is also continuous (see Theorem 52). Say $f : D \to \mathbb{R}$ is continuous. Let $E \subseteq D$ be a set of isolated points. Now consider the function $g : E \to \mathbb{R}$ defined by $g(x) = f(x)$. Perhaps naturally, we want to be able to say that g is also continuous.

Second, under more general definitions of continuity (e.g. the topological one where the pre-image of open sets are also open), a function is continuous at its isolated points. So here in our more specialised setting, we want to also include in our definition of continuity any isolated points.

Example 1116. Let $f:\{1,2,3\}\to\mathbb{R}$ be the function defined by $f(1)=0$, $f(2)=5$, and $f(3)=4$.

Figure to be
inserted here.

Perhaps surprisingly, the function f is continuous.

To see why, observe that the point 1 is not "close" to either 2 or 3 (the other two points in the domain). Similarly, 2 isn't "close" to 1 or 3; and 3 isn't "close" to 1 or 2. Hence, each of the three points in f's domain $\{1,2,3\}$ is an isolated point.

By our fine print then, f is automatically continuous at these three isolated points. And since f is continuous at every point in its domain, it is continuous.

Once again, this example demonstrates that our informal definition of continuity ("we can draw its entire graph without lifting our pencil") is actually not quite correct. The function f is continuous but to draw its graph, we must lift our pencil.

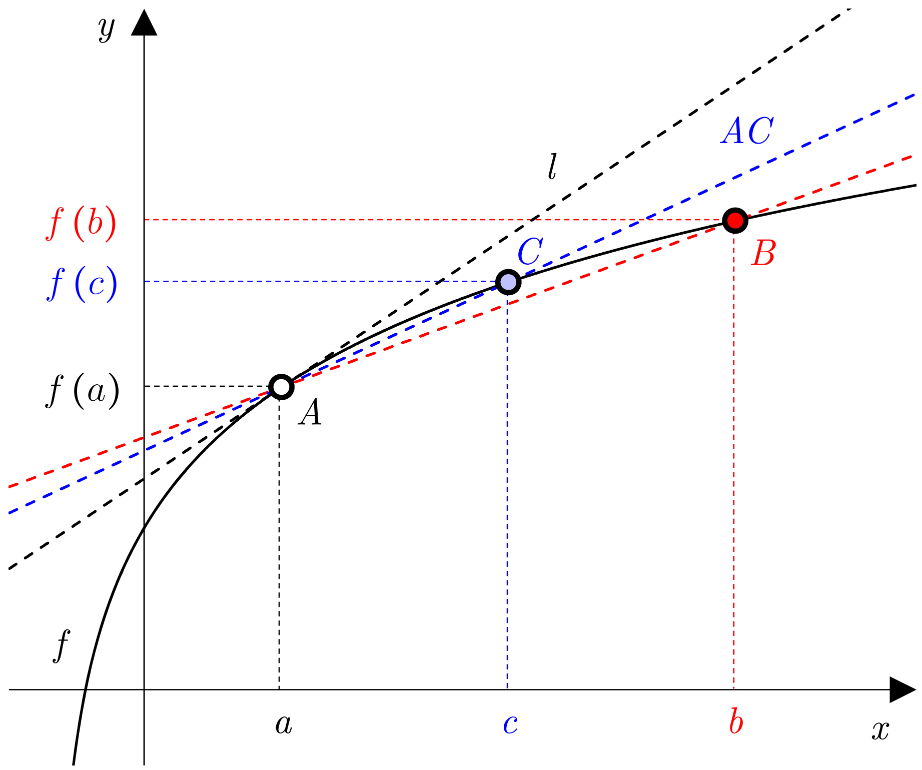
Definition 198 (of continuity) is perfectly good, but a bit inelegant due to the fine print. Writers of more advanced texts usually prefer definitions of continuity using ε-δ phrasing or open sets.

88. The Derivative, Revisited

Differentiation is the problem of finding the **gradient** of a curve at a point.

Graphed below is some function $f : \mathbb{R} \to \mathbb{R}$.

Consider the point $A = (a, f(a))$. Let l be the tangent line to the graph of f at A. (Informally, l is the line that just touches the graph of f at A.)



Now, how might we find the gradient of the line l? Unsure of how to proceed, we try a series of approximations:

1. We first pick some point $B = (b, f(b))$ on f.

Consider the line AB. We have

$$AB\text{'s gradient} = \frac{\text{Rise}}{\text{Run}} = \frac{f(b) - f(a)}{b - a}.$$

Clearly, our actual tangent line l is steeper than the line AB. Nonetheless, AB's gradient serves as our first approximation of l's gradient.

Now, can we improve on this first approximation? Sure. Simply

2. Pick some point $C = (c, f(c))$ that's also on f but that's is closer to A than B.

Consider the line AC. It's a little steeper than AB but still not as steep as l. Again, we have

$$AC\text{'s gradient} = \frac{\text{Rise}}{\text{Run}} = \frac{f(c) - f(a)}{c - a}.$$

The gradient of AC now serves as our second and slightly improved approximation of l's gradient.

We can keep repeating the above procedure to get ever-improved approximations or estimates of l's gradient:

- Pick a point $D = (d, f(d))$ that's on f but which is closer to A than C. The gradient of the line AD serves as our third and slightly improved approximation of l's gradient.
- Pick a point $E = (e, f(e))$ that's on f but which is closer to A than D. The gradient of the line AE serves as our fourth and slightly improved approximation of l's gradient.
- Etc.

The above discussion suggests that the gradient of l is this limit:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We are thus motivated to write down this formal definition:

Definition 203. Let f be a nice function with domain D and $a \in D$. Consider this limit:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

If the above limit exists (i.e. is equal to a real number), then we say that f is *differentiable at a*, call this limit the *derivative of f at a*, and denote it by $f'(a)$.

If not, then we say that f is *not differentiable at a*.

We call the expression $\dfrac{f(x) - f(a)}{x - a}$ the *difference quotient (of f at a)*.

Remark 126. The lines AB, AC, AD, and AE above are sometimes called **secant lines**. We may thus think of the tangent line l as **the limit of these secant lines**.

Remark 127. Observe that when considering the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

one possible fear is that $x - a = 0$, in which case we'd be committing the Cardinal Sin of Dividing by Zero (see Ch. 2.2).

Fortunately, with "$\lim_{x \to a}$", we are looking at values of x that are "near" *but not equal* to a. So, we have $x \neq a$ and hence $x - a \neq 0$. Hence, we can dismiss this fear.

Example 1117. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=|x|$.

Figure to be
inserted here.

In secondary school, we learnt that the derivative or gradient of f at a point a is

- 1 if $a>0$;
- -1 if $a<0$; and
- Undefined if $a=0$.

We did not however learn how to formally prove the above, which is what we'll now do.

We start by considering just the point 2. By Definition 203, $f'(2)$—the derivative of f at 2—is

$$\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}.$$

Using what we've learnt above, we can show that this limit equals 1:

$$\begin{aligned}
f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2} &= \lim_{x\to 2}\frac{|x|-|2|}{x-2} \\
&= \lim_{x\to 2}\frac{x-2}{x-2} && \text{(For all } x \text{ “near” } 2,\ x>0 \text{ and hence } |x|=x\text{)} \\
&= \lim_{x\to 2} 1 \\
&\overset{\text{C}}{=} 1 && \text{(Constant Rule for Limits).}
\end{aligned}$$

Since the derivative of f at 2 exists (it's equal to 1), we say that f is differentiable at 2.

We now consider any point $a>0$. We can similarly show that f is differentiable at a, with $f'(a)=1$:

$$\begin{aligned}
f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{|x|-|a|}{x-a} \\
&= \lim_{x\to a}\frac{x-a}{x-a} && \text{(For all } x \text{ “near” } a>0,\ x>0 \text{ and hence } |x|=x\text{)} \\
&= \lim_{x\to a} 1 \\
&\overset{\text{C}}{=} 1 && \text{(Constant Rule for Limits).}
\end{aligned}$$

Now do Exercise 333.

Exercise 333. Continue to define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$. Prove that f is differentiable at (Answer on p. 1893.)

(a) -3, with $f'(-3) = -1$;

(b) Any $a < 0$, with $f'(a) = -1$.

(... *Example continued from above.*)

We've shown that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is differentiable at any $a \neq 0$, with

$$f'(a) = \begin{cases} 1, & \text{for } a > 0, \\ -1, & \text{for } a < 0. \end{cases}$$

It turns out though that f is **not** differentiable at 0.

Informally and intuitively, this is because "near" 0, there is no single number that the gradient of f stays "near" to. To the left, the gradient is -1; while to the right, it is 1.

To formally show that f is not differentiable at 0, we can first show[390] that the left- and right-hand limits at 0 exist but are not equal:

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = -1 \qquad \text{and} \qquad \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 1.$$

Corollary 41 said that if the left- and right-hand limits are not equal, then the limit does not exist. And so, the following limit does not exist:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}.$$

We shall simply say that $f'(0)$ does not exist and that f is **not** differentiable at 0.

(*Example continues below ...*)

[390] The steps taken to compute these left- and right-hand limits are very similar to what we've just done:

$$\begin{aligned} \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} && \text{(Simply plug in for the function } f\text{)} \\ &= \lim_{x \to 0^-} \frac{-x - 0}{x - 0} && \text{(For all } x \text{ "near" but } \textit{less than } 0,\ x < 0 \text{ and hence } |x| = -x\text{)} \\ &= \lim_{x \to 0^-} -1 \stackrel{\text{C}}{=} 1. \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} && \text{(Simply plug in for the function } f\text{)} \\ &= \lim_{x \to 0^+} \frac{x - 0}{x - 0} && \text{(For all } x \text{ "near" but } \textit{more than } 0,\ x > 0 \text{ and hence } |x| = x\text{)} \\ &= \lim_{x \to 0^+} 1 \stackrel{\text{C}}{=} 1. \end{aligned}$$

Definition 204. Suppose the function f is differentiable at every point in a set S. Then we say that f is *differentiable on* S.

And if f is differentiable at every point in its domain, then we say that f is a *differentiable function.*

(... *Example continued from above.*)

Above, we've shown that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is differentiable everywhere *except* at 0. Equivalently, f is differentiable on $\mathbb{R} \setminus \{0\}$ or $(-\infty, 0) \cup (\infty, 0)$.

Since f fails to be differentiable at even a single point in its domain, by Definition 204, f is **not** a differentiable function.

Example 1118. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2$.

Figure to be
inserted here.

The derivative of g at 2 is

$$g'(2) = \lim_{x\to 2} \frac{g(x) - g(2)}{x - 2}.$$

Does this derivative exist? If so, what does it equal?

To find out, first simplify the difference quotient a little:

$$\frac{g(x) - g(2)}{x - 2} = \frac{x^2 - 2^2}{x - 2} = \frac{(x-2)(x+2)}{x - 2} = x + 2.$$

Then,[391]

$$\lim_{x\to 2} \frac{g(x) - g(2)}{x - 2} = \lim_{x\to 2}(x+2) \overset{+}{=} \overbrace{\lim_{x\to 2} x}^{2} + \overbrace{\lim_{x\to 2} 2}^{2} = 4.$$

We've just shown that yes, the derivative of g at 2 exists. Moreover, it equals 4.

Exercise 334 continues with this example.

[391] At $\overset{+}{=}$, we can use the Sum Rule for Limits because these two limits exist:

1. $\lim_{x\to 2} x = 2$ (by the Power Rule for Limits)
2. $\lim_{x\to 2} 2 = 2$ (by the Constant Rule for Limits)

If one of the above two limits does not exist, then it would be illegitimate and incorrect to apply the Sum Rule for Limits at $\overset{+}{=}$.

Exercise 334. Continue to define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2$. By mimicking the above example, determine whether each of the following derivatives exists; and if it does, find it. (Answer on p. 1893.)

(a) The derivative of g at -3.

(b) The derivative of g at 0.

(c) The derivative of g at any $a \in \mathbb{R}$.

88.1. Differentiable $\iff$ Approximately Linear

Informally,

A function f is differentiable at a point a $\overset{☺}{\iff}$ f is approximately linear at a.

Or even more informally, RHS of the above statement can be rewritten as

"Near" a (or, when we "zoom in" to a), f "looks" like a straight line.

Example 1119. The graph of sin doesn't "look" like a straight line anywhere.

Figure to be
inserted here.

However, if we pick any point, say 0, and "zoom in", then the graph does "look" increasingly like a straight line. We may say that sin is approximately linear at 0.

It turns out that sin is indeed differentiable at 0.

Example 1120. Consider the absolute value function. No matter how much we "zoom in" at the point 0, it never looks like a straight line.

We cannot say that the absolute value function is approximately linear at 0.

Figure to be
inserted here.

And as shown earlier, the absolute value function is indeed not differentiable at 0.

Remark 128. Note that instead of **approximately linear**, some writers say **locally linear**.[392]

Proposition 32 (Appendices) presents a more restatement of $\overset{☺}{\iff}$. Here we can provide a little heuristic justification of why $\overset{☺}{\iff}$ might be true.

[392]Still *other* writers object to the term **locally linear** for this reason: To say that f is **locally X** at a is to say that f *actually* satisfies property X, at least for points that are "near" a. However, if f is differentiable at a, then f may not actually be linear anywhere "near" a. Thus, according to these other writers, we should avoid the term phrase **locally linear**.

Start with the definition of the derivative:

$$f'(a) \varnothing 1 \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Let's hand wave and assume we can rewrite $\varnothing 1$ as

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}.$$

Rearranging, $$f(x) \approx f(a) + f'(a)(x - a).$$

Figure to be
inserted here.

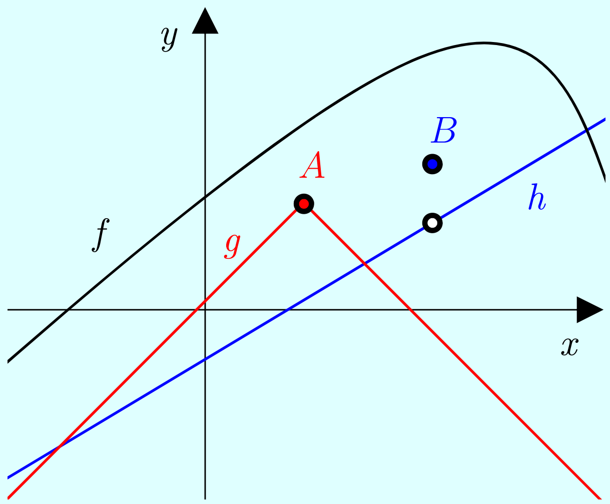
88.2. Continuity vs Differentiability

Informally, a function is

- **Continuous** if its graph contains no "holes" or "jumps" anywhere; and
- **Differentiable** if it is continuous *and* moreover, contains no "kinks" or other "abrupt turns" in its graph.

Informally, both **continuity** and **differentiability** tell us how **"smooth"** a function is.

Example 1121. The functions $f, g, h : \mathbb{R} \to \mathbb{R}$ are graphed below:



The function f is both continuous and differentiable.

The function g is continuous but not differentiable. In particular, it is differentiable at every point *except* A.

The function h is neither continuous nor differentiable. In particular, it is continuous and differentiable at every point *except* B.

Differentiability implies continuity. Equivalently, differentiability is a stronger condition than continuity. Equivalently,

Theorem 29. *If a function is differentiable at a point, then it is also continuous at that point.*

Proof. See p. 1673 (Appendices). □

Fun Fact

Many 19th-century mathematicians believed that continuity implied differentiability. Indeed, they often attempted to "prove" this "result".[393]

They were mistaken, as the function g in the above example clearly shows (g is continuous but not differentiable at the point A).

[393] Hawkins (1968, p. 74; 1975, pp. 43–44):

Example 1122. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$.

We showed earlier that f is differentiable everywhere except at 0, with $f'(a) = 1$ for $a > 0$ and $f'(a) = -1$ for $a < 0$.

Figure to be inserted here.

So far, we've been discussing only the object **the derivative of f at some point** a, which is a **(real) number**. For example, the derivative of f at -3 is -1 and the derivative of f at 2 is 1—$f'(-3) = -1$ and $f'(2) = 1$.

We'll now define a new object called the **derivative of** f—this is the **function** $f' : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$f'(x) = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases}$$

Note the difference in the domains: Domain $f = \mathbb{R}$, while Domain $f = \mathbb{R} \setminus \{0\}$.

Formally and generally,

Definition 205. Let f be a nice function. Let S be the set of all points at which f is differentiable. The *derivative of* f is the function $f' : S \to \mathbb{R}$ defined by

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

At each point $a \in S$, $f'(a)$ gives us the derivative (or the gradient of the tangent line) of f at that point.

And at each point $a \notin S$, f is not differentiable and so $f'(a)$ is simply left undefined (or does not exist).

> *Between the appearance of Ampère's paper [1806] and 1870, the proposition that any (continuous) function is differentiable in general was stated and proved in most of the leading texts on the calculus.*

The footnote at the end of the sentence lists several "leading texts" that "proved" this "result": "Lacroix 1810–19: I, 241n; Raabe 1839–47: I, 7ff; Duhamel 1847: I, 22; 1856: I, 94–97; Freycinet 1860: 39–42; Bertrand 1864–70: I, 2–4; Serret 1868: I, 16–21; Rubini 1868: I, 36–37. Two proofs were also attempted by Galois [1962: 383, 413ff]."

Example 1123. Define $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=x^2$.

We showed earlier that g is differentiable everywhere, with $g'(a)=2a$ for each $a\in\mathbb{R}$.

So, the derivative of g is the function $g':\mathbb{R}\to\mathbb{R}$ defined by $g'(x)=2x$.

Figure to be
inserted here.

(Note that in this example, $\text{Domain}\, g=\mathbb{R}=\text{Domain}\, g'$.)

In Ch. 17.3, we carefully distinguished between g and $g(a)$:

- The symbol g denotes a function.
- In contrast, the symbol $g(a)$ denotes a real number (namely, the value taken by the function g at the point a).

Here likewise, we must carefully distinguish between “the derivative of g” and “the derivative of g at a”:

- “The derivative of g” is a function and is denoted g'.
- In contrast, “the derivative of g at a” is a real number (namely, the value taken by the function g' at the point a) and is denoted $g'(a)$.

So, it makes sense to say “the derivative of g at 17 is 34”, because “the derivative of g at 17” is a real number, namely 34.

In contrast, the statement “the derivative of g is 34” makes no sense, because “the derivative of g” is a function, not a real number.

Example 1124. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^2 + x$. Let $a \in \mathbb{R}$.

The derivative of h at a is $h'(a) = \lim_{x \to a} \dfrac{h(x) - h(a)}{x - a}$.

Let's first simplify the difference quotient:

$$\frac{h(x) - h(a)}{x - a} = \frac{(x^2 + x) - (a^2 + a)}{x - a} = \frac{x^2 - a^2}{x - a} + \frac{x - a}{x - a} = \frac{(x - a)(x + a)}{x - a} + 1 = x + a + 1.$$

Now,[394] $\displaystyle \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} (x + a + 1) \overset{\pm}{=} \overbrace{\lim_{x \to a} x}^{a} + \overbrace{\lim_{x \to a} a}^{a} + \overbrace{\lim_{x \to a} 1}^{1} = 2a + 1.$

We've just shown that for any $a \in \mathbb{R}$, the derivative of h at a exists and equals $2a + 1$.

Hence, the derivative of h is the function $h' : \mathbb{R} \to \mathbb{R}$ defined by $h'(x) = 2x + 1$.

Figure to be inserted here.

Let us stress, emphasise, and repeat: Given a function f, its derivative f' is itself also a **function**. In particular, it is the function whose

- Domain is the set of points at which f is differentiable;
- Codomain is $\mathbb{R}$; and
- Mapping rule is $a \mapsto \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

[394] At $\overset{\pm}{=}$, we can apply the Sum Rule for Limits because these three limits exist:

1. $\lim_{x \to a} x = a$ (by the Power Rule for Limits);
2. $\lim_{x \to a} a = a$ (by the Constant Rule for Limits);
3. $\lim_{x \to a} 1 = 1$ (ditto).

Exercise 335. For each function, find its derivative (remember: this is a function). Then write down its derivative at 2 (remember: this is a number). (Answer on p. 1894.)

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7$.

(b) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 5x + 7$.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = 2x^2 + 5x + 7$.

Figure to be
inserted here.

88.4. “Most” Elementary Functions Are Differentiable

Recall that an *elementary function* is

a polynomial function,
a trigonometric function,
an inverse trigonometric function,
a natural logarithm function,
an exponential function,
a power function,
any arithmetic combination of two elementary functions,
or any composition of two elementary functions.

In Ch. 87.4, we learnt that *all* elementary functions are continuous. Now, is it likewise true that

“*All* elementary functions are differentiable”?

Unfortunately, no. Two examples of elementary functions that aren't differentiable:

Example 1125. The absolute value function As explained on p. 403 is an elementary function (we explained why on p. 403).

Figure to be
inserted here.

However, it is not differentiable at 0 and hence is not a differentiable function.

Example 1126. The arcsine function $\sin^{-1} : [-1, 1] \to \mathbb{R}$ is, by Definition 109, elementary.

Figure to be
inserted here.

In Example 1148, we'll learn that the derivative of $\sin^{-1}$ is the function $\left(\sin^{-1}\right)' : (-1, 1) \to \mathbb{R}$ defined by

$$\left(\sin^{-1}\right)'(x) = \frac{1}{\sqrt{1 - x^2}}.$$

Observe that $\sin^{-1}$ is not differentiable at the points ± 1 and, hence, is not differentiable. (What about $\cos^{-1}$?)[395]

Nonetheless and somewhat happily, we may make these imprecise claims:[396]

"Most" elementary functions are differentiable.
Moreover, their derivatives are themselves elementary.

So,

- "Most" functions we'll encounter in H2 Maths are elementary.
- "Most" elementary functions are differentiable.
- Therefore, "most" functions we'll encounter in H2 Maths are differentiable.

Exercise 336. Show that the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sqrt{|x|}$ is elementary but not differentiable. (Answer on p. 1895.)

[395] It's possible to show that $\cos^{-1}$ is likewise an elementary function, but isn't differentiable at the points ± 1 and, hence, is not differentiable.

[396] For some justification of these imprecise claims, see Ch. 146.10 (Appendices).

89. More on Leibniz's Notation

89.1. Using $\dfrac{\mathrm{d}}{\mathrm{d}x}$ as the Differentiation Operator

See Ch. 43.2 for a discussion of Lagrange's (f') vs Newton's ($\dot{f}$) vs Leibniz's ($\mathrm{d}f/\mathrm{d}x$) notation.

A function maps each object in its domain to (exactly) one object in its codomain.

Most functions we've looked at are *nice* (Ch. 17.4)—that is, their domain and codomain are both sets of real numbers and the function maps a real number to a real number.

But it is possible to imagine functions that are somewhat more abstract.

In particular, we can imagine a function whose domain and codomain are both *sets of functions*. Such a function would map a function (in its domain) to a function (in its codomain). Such functions go by the special name of *operators*. This sounds complicated, but isn't:

Example 1127. Define the functions $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$ and $g(x) = 3x - 1$.

Let S be the *squaring operator*.[397] Then

- $S(f) = h$, where $h : \mathbb{R} \to \mathbb{R}$ is the function defined by $h(x) = (x+1)^2$. That is, the squaring operator S maps the function f to the function h. And we have $S(f)(2) = h(2) = 9$.
- $S(g) = i$, where $i : \mathbb{R} \to \mathbb{R}$ is the function defined by $i(x) = (3x-1)^2$. That is, the squaring operator S maps the function g to the function i. And we have $S(g)(2) = i(2) = 25$.

Let T be the *divide-by-two operator*. Then

- $T(f) = j$, where $j : \mathbb{R} \to \mathbb{R}$ is the function defined by $j(x) = (x+1)/2$. That is, the divide-by-two operator T maps the function f to the function j. And we have $T(f)(2) = j(2) = 3/2$.
- $T(g) = k$, where $k : \mathbb{R} \to \mathbb{R}$ is the function defined by $k(x) = (3x-1)/2$. That is, the divide-by-two operator T maps the function g to the function k. And we have $T(g)(2) = k(2) = 5/2$.

Exercise 337. Continuing with the above example, what are $T(h)$, $T(i)$, $S(j)$, $S(k)$, $T(h)(2)$, $T(i)(2)$, $S(j)(2)$, and $S(k)(2)$? (Answer on p. 1896.)

Let f be a function. Then using Leibniz's notation, we can denote

- Its derivative by the symbol $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ (this is a function); and
- Its derivative at some point a by the symbol $\dfrac{\mathrm{d}f}{\mathrm{d}x}(a)$ (this is a number).

[397] Strictly and pedantically speaking, since S is also a function, we should also specify its domain and codomain. They could, for example, both be the set of all nice functions.

Let's now go a little further with Leibniz's notation and use the symbol $\frac{\mathrm{d}}{\mathrm{d}x}$ to denote the **differentiation operator**. That is, $\frac{\mathrm{d}}{\mathrm{d}x}$ will itself denote the function (or operator) that maps any function (f) to its derivative ($f' = \dot{f} = \frac{\mathrm{d}f}{\mathrm{d}x}$).

Example 1128. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 7x + 3$ and $g(x) = 2x + 7$.
Observe that the derivative of f is g. We may write

$$\frac{\mathrm{d}f}{\mathrm{d}x} \stackrel{1}{=} g.$$

Equation $\stackrel{1}{=}$ says that "the derivative of f is g".
Now, we shall also write

$$\frac{\mathrm{d}}{\mathrm{d}x} f \stackrel{2}{=} g.$$

Equation $\stackrel{2}{=}$ now says that "the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ maps f to g".

Of course, there isn't any practical difference between equations $\stackrel{1}{=}$ and $\stackrel{2}{=}$. All we've done is to introduce a new symbol $\frac{\mathrm{d}}{\mathrm{d}x}$ that denotes the differentiation operator. As we'll see, this new piece of notation will occasionally be convenient.

Example 1129. Define $h, i : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $h(x) = \frac{1}{x}$ and $i(x) = -\frac{1}{x^2}$.

The derivative of h is i
$$\frac{\mathrm{d}h}{\mathrm{d}x} = i.$$

Equivalently, the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ maps h to i:

$$\frac{\mathrm{d}}{\mathrm{d}x} h = i.$$

Example 1130. The derivative of the sine function is the cosine function:

$$\frac{\mathrm{d}\sin}{\mathrm{d}x} = \cos.$$

Equivalently, the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ maps sin to cos:

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin = \cos.$$

Example 1131. The derivative of the cosine function is the negative sine function:

$$\frac{\mathrm{d}\cos}{\mathrm{d}x} = -\sin.$$

Equivalently, the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ maps cos to $-\sin$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos = -\sin.$$

Example 1132. Given the functions f and g, the Product Rule says that

$$\frac{\mathrm{d}(f \cdot g)}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \cdot g + f \cdot \frac{\mathrm{d}g}{\mathrm{d}x}.$$

Equivalently, the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ maps the function $f \cdot g$ to the function $\frac{\mathrm{d}f}{\mathrm{d}x} \cdot g + f \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(f \cdot g) = \frac{\mathrm{d}f}{\mathrm{d}x} \cdot g + f \cdot \frac{\mathrm{d}g}{\mathrm{d}x}.$$

Again, it is merely customary to use "x" as our symbol for the dummy variable. We can replace "x" with any other symbol, such as y, t, θ, $\triangle$, or $\star$. So, the last five examples could've been rewritten as:

Example 1133. $\frac{\mathrm{d}f}{\mathrm{d}y} = g$ or $\frac{\mathrm{d}}{\mathrm{d}y}f = g$.

Example 1134. $\frac{\mathrm{d}h}{\mathrm{d}t} = i$ or $\frac{\mathrm{d}}{\mathrm{d}t}h = i$.

Example 1135. $\frac{\mathrm{d}\sin}{\mathrm{d}\theta} = \cos$ or $\frac{\mathrm{d}}{\mathrm{d}\theta}\sin = \cos$.

Example 1136. $\frac{\mathrm{d}\cos}{\mathrm{d}\triangle} = -\sin$ or $\frac{\mathrm{d}}{\mathrm{d}\triangle}\cos = -\sin$.

Example 1137. $\frac{\mathrm{d}(f \cdot g)}{\mathrm{d}\star} = \frac{\mathrm{d}f}{\mathrm{d}\star} \cdot g + f \cdot \frac{\mathrm{d}g}{\mathrm{d}\star}$ or $\frac{\mathrm{d}}{\mathrm{d}\star}(f \cdot g) = \frac{\mathrm{d}f}{\mathrm{d}\star} \cdot g + f \cdot \frac{\mathrm{d}g}{\mathrm{d}\star}$.

89.2. Using $\frac{\mathrm{d}}{\mathrm{d}x}$ as Shorthand

So far, whenever we've written

$$\frac{\mathrm{d}}{\mathrm{d}x}\square = \square,$$

each of the objects in the two blank $\square$s is a function.

So far, we have not written statements such as this:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 \overset{1}{=} 2x.$$

We will now go ahead and say that $\overset{1}{=}$ is a perfectly legal statement. In particular, $\overset{1}{=}$ shall serve as convenient shorthand for this precise but long-winded statement:[398]

> If a differentiable function maps each x to x^2, then its derivative maps each x to $2x$.

Example 1138. The statement

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^3 + x\right) \overset{1}{=} 6x^2 + 1$$

is convenient shorthand for this precise but long-winded statement:

> If a differentiable function maps each x to $2x^3 + x$, then its derivative maps each x to $6x^2 + 1$.

Note that while convenient, $\overset{1}{=}$ leaves out some information—in particular, the domain and codomain of both the function and the derivative.

Sometimes we may not have this information and so $\overset{1}{=}$ is all we can say.

But where we do have this information, we might prefer to explicitly state it. For example, instead of just the statement $\overset{1}{=}$, we might prefer to write

> The function $f : [-3, 5] \to \mathbb{R}$ defined by $f(x) = 2x^3 + x$ has derivative $f' : [-3, 5] \to \mathbb{R}$ defined by $f'(x) = 6x^2 + 1$.

[398]Definition 206 (Appendices) explains how such shorthand generally works.

Example 1139. The statement

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x \stackrel{1}{=} \cos x.$$

is convenient shorthand for this precise but long-winded statement:

> If a differentiable function maps each x to $\sin x$, then its derivative maps each x to $\cos x$.

Again, $\stackrel{1}{=}$ leaves out information about the domain and codomain of both the function and the derivative. Where we do have this information, instead of just the statement $\stackrel{1}{=}$, we might prefer to write, for example,

> The function $f:[-3,5]\to\mathbb{R}$ defined by $f(x)=\sin x$ has derivative $f':[-3,5]\to\mathbb{R}$ defined by $f'(x)=\cos x$.

Formally,

Definition 206. Let $f(x)$ and $g(x)$ be expressions containing the variable x. We shall write

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x)=g(x)$$

to mean the following:

> *If a differentiable function maps each x to $f(x)$, then its derivative maps each x to $g(x)$.*

Exercise 338. For each of the following statements, write down its corresponding precise but long-winded statement. (Answer on p. 1896.)

(a) $\frac{\mathrm{d}}{\mathrm{d}x}x^3=3x^2$.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin x^2=2x\cos x^2$.

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\ln x=\frac{1}{x}$.

Exercise 339. What's wrong with the following "proof" that $0=1$?

1. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^2-3x$.
2. The derivative of f is the function $f':\mathbb{R}\to\mathbb{R}$ defined by $f'(x)=2x-3$.
3. Observe that $f(2)=2^2-3\cdot 2=-2$.
4. So, $f'(2)=\frac{\mathrm{d}}{\mathrm{d}x}(-2)=0$.
5. But from Step 2, we also have $f'(2)=2\cdot 2-3=1$.
6. Hence, by Steps 4 and 5, we have $0=1$. (Answer on p. 1896.)

89.3. $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ Is Not a Fraction; Nonetheless ...

To repeat, the symbol f', $\dot{f}$, or $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ denotes a *function*—namely, the derivative of f:

$$\underset{\substack{\downarrow\\ \text{Newton}}}{\overset{\text{Lagrange}}{\overset{\uparrow}{f'}} = \dot{f}} = \overset{\substack{\text{Leibniz}\\ \uparrow}}{\frac{\mathrm{d}f}{\mathrm{d}x}} = \text{Derivative of } f$$

The symbol $f'(a)$, $\dot{f}(a)$, $\left.\dfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}$, or $\dfrac{\mathrm{d}f}{\mathrm{d}x}(a)$ denotes a *real number*—namely, the derivative of f at a, which is a limit (of a fraction):

$$\overset{\substack{\text{Lagrange}\\ \uparrow}}{f'(a)} = \underset{\substack{\downarrow\\ \text{Newton}}}{\dot{f}(a)} = \overbrace{\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a} = \frac{\mathrm{d}f}{\mathrm{d}x}(a)}^{\text{Leibniz}} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Leibniz's notation $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ looks very much like a fraction, with numerator $\mathrm{d}f$ and denominator $\mathrm{d}x$. However, $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ is not a fraction.

I repeat, $$\frac{\mathrm{d}f}{\mathrm{d}x} \text{ is not a fraction.}$$

Neither $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ nor $\dfrac{\mathrm{d}f}{\mathrm{d}x}(a)$ denotes a fraction. Instead and to repeat, $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ denotes a *function*, while $\dfrac{\mathrm{d}f}{\mathrm{d}x}(a)$ denotes a *real number*.

We just made it clear that $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ is not a fraction. Nonetheless, as an informal and intuitive aid to our understanding, it can be helpful to think of $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ as a fraction.

To illustrate, consider the **Chain Rule**. When formally stated in Lagrange's notation, it looks like this (see Theorem 32 below):

$$(f\circ g)' \overset{1}{=} (f'\circ g)\cdot g'.$$

With $\overset{1}{=}$, it is difficult to understand or remember the Chain Rule.

But when stated using Leibniz's notation, the Chain Rule becomes much easier to understand and remember:

$$\frac{\mathrm{d}z}{\mathrm{d}x} \overset{2}{=} \frac{\mathrm{d}z}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}x}.$$

With $\stackrel{2}{=}$, we can easily see the intuition behind the Chain Rule:

$$\begin{matrix}\text{The change in } z \text{ due to} \\ \text{a small unit change in } x\end{matrix} = \begin{matrix}\text{The change in } z \text{ due to} \\ \text{a small unit change in } y\end{matrix} \times \begin{matrix}\text{The change in } y \text{ due to} \\ \text{a small unit change in } x\end{matrix}$$

With $\stackrel{2}{=}$, we can also easily remember the Chain Rule, by simply thinking of three terms $\frac{\mathrm{d}z}{\mathrm{d}x}$, $\frac{\mathrm{d}z}{\mathrm{d}y}$, and $\frac{\mathrm{d}y}{\mathrm{d}x}$ as fractions and naïvely applying primary-school algebra:

$$\text{“}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\cancel{\mathrm{d}y}} \times \frac{\cancel{\mathrm{d}y}}{\mathrm{d}x}.\text{”}$$

Of course, formally, we aren't actually doing anything like cancelling the dy's. Nonetheless, the above is a nice way to remember the Chain Rule.

The **Inverse Function Theorem** (which we'll learn about in Ch. 91.2) is another example of the advantages of Leibniz's notation.

It turns out that **Leibniz *did* think of $\frac{\mathrm{d}f}{\mathrm{d}x}$ as a fraction**! For him,

- dx denoted an "infinitesimal change in x";
- df denoted the corresponding "infinitesimal change in f"; and
- $\frac{\mathrm{d}f}{\mathrm{d}x}$ really was a fraction with numerator df and denominator dx.

It turns out though that this approach ran into some technical difficulties. In particular, while the concept of *infinitesimals* was intuitive and plausible, it was difficult to give it precise and rigorous meaning.

In the 19th century, these difficulties were resolved (or perhaps simply sidestepped) by redefining the derivative as a limit (of a fraction).[399] This 19th-century solution remains the standard modern approach and is also the approach we've taken in this textbook.[400]

Subchapter summary:

- Under the modern approach, it is technically and formally **wrong** to think of $\frac{\mathrm{d}f}{\mathrm{d}x}$ as a fraction with numerator df and denominator dx.
- Nonetheless, as an informal and intuitive aid to our understanding, it can be helpful to think of $\frac{\mathrm{d}f}{\mathrm{d}x}$ as a fraction. For example, it helps us see the Chain Rule's underlying intuition and also remember the Rule more easily. (The same will be true of the Inverse Function Theorem.)

---

[399] See Ch. 146.2 (Appendices).

[400] In the 1960s, Abraham Robinson (1966) introduced what he called *Non-Standard Analysis*—he gave Leibniz's infinitesimals precise and rigorous meaning, so that $\frac{\mathrm{d}f}{\mathrm{d}x}$ could at last be legitimately treated as a fraction. However, non-standard analysis has not been widely taken up as it is not clearly superior to the standard approach.

- And actually, Leibniz *did* think of $\frac{\mathrm{d}f}{\mathrm{d}x}$ as a fraction—it was only in the 19th century that such thinking was definitively replaced by the modern approach.

Also, as explained in Ch. 89.1, another advantage of Leibniz's notation is that we can use the symbol $\frac{\mathrm{d}}{\mathrm{d}x}$ to clearly denote the differentiation operator. (It'd be harder to do the same with Lagrange's prime $'$ or Newton's dot notation $\dot{}$.)

Remark 129. Stack Exchange has at least 10 questions about whether we should think of $\frac{\mathrm{d}f}{\mathrm{d}x}$ as a fraction.

The most highly-voted question, "Is $\frac{\mathrm{d}y}{\mathrm{d}x}$ not a ratio?", has 20 answers. I recommend reading the answers by Arturo Magidin and Mikhail Katz.

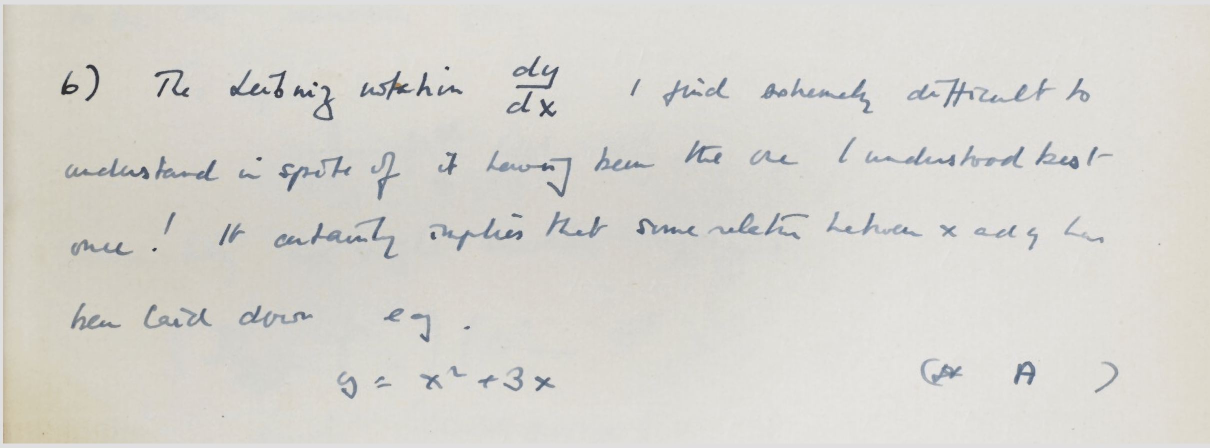
Another good question is "When not to treat dy/dx as a fraction in single-variable calculus?". As the answers to this question suggest, at least in single-variable calculus (which is what we cover in H2 Maths), there is little danger in thinking of $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a fraction. It is in multi-variable calculus where such thinking is more likely to produce errors and should be avoided.[401]

[401] Below are the other eight questions I've come across. They do not add much to the discussions already given in the two questions mentioned above.

- How misleading is it to regard $\frac{dy}{dx}$ as a fraction?
- Can I ever go wrong if I keep thinking of derivatives as ratios?
- If $\frac{dy}{dt}dt$ doesn't cancel, then what do you call it?
- Why do most of the operations treating dy/dx as a ratio works?
- What is wrong with treating $\frac{dy}{dx}$ as a fraction? [duplicate]
- When can't dy/dx be used as a ratio/fraction?
- When can we not treat differentials as fractions? And when is it perfectly OK?
- Why do we treat differentials as infinitesimals, even when it's not rigorous

Fun Fact

Here's what Alan Turing thought about Leibniz's notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ in his recently discovered wartime notebooks:



6) The Leibniz notation $\frac{dy}{dx}$ I find extremely difficult to understand in spite of it having been the one I understood best once! It certainly implies that some relation between x and y has been laid down e.g.

$$y = x^2 + 3x \qquad (A)$$

The Leibniz notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ *I find extremely difficult to understand in spite of it having been the one I understood best once! It certainly implies that some relation between x and y has been laid down e.g.*

$$y = x^2 + 3x$$

— Alan Turing (c. 1944).[402]

[402]Discussion of what Turing may have meant by this remark: .

90. Rules of Differentiation, Revisited

We reproduce from Ch. 43.6 and 43.7 these **Rules of Differentiation**:

Theorem 14. **(Rules of Differentiation)** *Let c be a constant and x, y, and z be variables. Then*

Constant Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}c \overset{\mathrm{C}}{=} 0$$

Constant Factor Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(cy) \overset{\mathrm{F}}{=} c\frac{\mathrm{d}y}{\mathrm{d}x}$$

Power Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}x^c \overset{\mathrm{P}}{=} cx^{c-1}$$

Sum and Difference Rules

$$\frac{\mathrm{d}}{\mathrm{d}x}(y \pm z) \overset{\pm}{=} \frac{\mathrm{d}y}{\mathrm{d}x} \pm \frac{\mathrm{d}z}{\mathrm{d}x}$$

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(yz) \overset{\times}{=} z\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}z}{\mathrm{d}x}$$

Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{y}{z} \overset{\div}{=} \frac{z\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\mathrm{d}z}{\mathrm{d}x}}{z^2}$$

Sine

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

Cosine

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

Natural Logarithm

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$

Exponential

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$$

Theorem 15 (informal). **(Chain Rule)** *Suppose y and z are functions. Then*

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Some easy exercises to get you reacquainted with the above Rules:

Exercise 340. XXX (Answer on p. 882.)

A340.

We now have a better understanding of what derivatives are. In particular, we now understand that

A derivative is itself a function.

Also, as discussed in Examples 1138 and 1139, the Rules of Differentiation as stated in Theorem 14 leave out information about the domain and codomain of both the function and its derivative.

Moreover, Theorem 15 is stated somewhat informally.

So, in the following four subchapters (optional), we'll restate and also prove these Rules of Differentiation.

90.1. Proving Some Basic Rules of Differentiation (optional)

We start with the simplest, the **Constant Rule**, which says that **the derivative of a constant function is a zero function**:

> **Fact 203. (Constant Rule for Differentiation)** *Let $D, E \subseteq \mathbb{R}$, $c \in E$, and $f : D \to E$ be a function. If f is defined by $f(x) = c$, then the derivative of f is the function $f' : D \to \mathbb{R}$ defined by*
>
> $$f'(x) \stackrel{\text{C}}{=} 0.$$

Proof. Let $a \in D$. At $\stackrel{1}{=}$, use the Constant Rule for Limits (Theorem 23):

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{c - c}{x - a} = \lim_{x \to a} 0 \stackrel{1}{=} 0,$$

We've just shown that for every $a \in D$, the derivative of f at a equals 0. Hence, the derivative of f is the function $f' : D \to \mathbb{R}$ defined by $f'(x) \stackrel{\text{C}}{=} 0$. □

By the way, a *partial*[403] converse of the above is also true—**provided the function's domain is an interval,** if that function's derivative is a zero function, then the original function is a constant function:

> **Proposition 8.** *Let $D, E \subseteq \mathbb{R}$ and $f : D \to E$ be a function. Suppose D is an interval. If the derivative of f is the function $f' : D \to \mathbb{R}$ defined by $f'(x) = 0$, then f is a constant function—i.e. f defined by $f(x) = c$ (for some $c \in E$).*

Proof. This result is perhaps intuitively "obvious". However, its proof requires some slightly more advanced tools and so we'll relegate it to the Appendices—see p. 1684. □

The **Constant Factor Rule** says that the **derivative of a constant multiple of a function is that constant multiple of that function's derivative**:

> **Fact 204. (Constant Factor Rule for Differentiation)** *Let $D, E \subseteq \mathbb{R}$, $c \in \mathbb{R}$, and $f : D \to E$ be a function. Suppose f is differentiable. Then the function cf is also differentiable and its derivative $(cf)' : D \to \mathbb{R}$ is defined by*
>
> $$(cf)'(x) \stackrel{\text{F}}{=} cf'(x).$$

Proof. Let $a \in D$. At $\stackrel{1}{=}$, use the Product Rule for Limits (why is this legitimate?):[404]

---

[403] This is a *partial* converse because we add a requirement (namely, that the function's domain is an interval).

[404] Because these two limits exist:

1. $\lim_{x \to a} c = c$ (by the Constant Rule for Limits);
2. $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = f'(a)$ (by the definition of the derivative).

$$
\begin{aligned}
(cf)'(a) &= \lim_{x \to a} \frac{cf(x) - cf(a)}{x-a} = \lim_{x \to a} \left[c \frac{f(x) - f(a)}{x-a} \right] \\
&\overset{1}{=} \underbrace{\left(\lim_{x \to a} c \right)}_{c} \underbrace{\left[\lim_{x \to a} \frac{f(x) - f(a)}{x-a} \right]}_{f'(a)} = cf'(a).
\end{aligned}
$$

We've just shown that for every $a \in D$, the derivative of cf at a equals $cf'(a)$. Hence, the derivative of cf is the function $(cf)' : D \to \mathbb{R}$ defined by $(cf)'(x) \overset{\text{F}}{=} cf'(x)$. □

The **Sum and Difference Rules** say that the **derivative of the sum (or difference) of two functions is the sum (or difference) of their derivatives**:

Fact 205. (Sum and Difference Rules for Differentiation) *Suppose $f, g : D \to \mathbb{R}$ are differentiable functions. Then the function $f \pm g$ is differentiable and its derivative $(f \pm g)' : D \to \mathbb{R}$ is defined by*

$$(f \pm g)'(x) \overset{\pm}{=} f'(x) \pm g'(x).$$

Proof. For any $a \in D$, we have[405]

$$
\begin{aligned}
(f \pm g)'(a) &= \lim_{x \to a} \frac{(f \pm g)(x) - (f \pm g)(a)}{x-a} = \lim_{x \to a} \frac{f(x) \pm g(x) - [f(a) \pm g(a)]}{x-a} \\
&= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x-a} \pm \frac{g(x) - g(a)}{x-a} \right] \\
&\overset{\pm}{=} \underbrace{\lim_{x \to a} \frac{f(x) - f(a)}{x-a}}_{f'(a)} \pm \underbrace{\lim_{x \to a} \frac{g(x) - g(a)}{x-a}}_{g'(a)} = f'(a) \pm g'(a).
\end{aligned}
$$

We've just shown that for every $a \in D$, the derivative of $f \pm g$ at a equals $f'(a) \pm g'(a)$. Hence, the derivative of $f \pm g$ is the function $(f \pm g)' : D \to \mathbb{R}$ defined by $(f \pm g)'(x) = f'(a) \pm g'(a)$. □

Fact 206. (Power Rule for Differentiation) *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a differentiable function. Suppose f is defined by $f(x) = x^c$ for some $c \in \mathbb{R}$. Then the derivative of f is the function $f' : D \to \mathbb{R}$ defined by*

$$f'(x) \overset{\text{P}}{=} cx^{c-1}.$$

If either of the above limits did not exist, then the application of the Product Rule for Limits would **not** be legitimate.

[405] At $\overset{\pm}{=}$, we can use the Sum and Difference Rules for Limits because these two limits exist:

1. $\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = f'(a)$ (by the definition of the derivative)
2. $\lim_{x \to a} \frac{g(x) - g(a)}{x-a} = g'(a)$ (ditto)

Sadly, a complete proof of the Power Rule is beyond the scope of H2 Maths and this textbook.

Nonetheless and very excitingly, we will now learn to prove the Power Rule in the special case where the exponent c is a non-negative integer.

Of course, we've already proven the Power Rule in the simplest case where the exponent is 0—this is simply the Constant Rule. And so, let us start with the next simplest case—namely, the case where the exponent is 1—and work our way up:

Example 1140. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$.

For any $a \in \mathbb{R}$, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = \lim_{x \to a} 1 \stackrel{\text{C}}{=} 1.$$

We've just shown that for any $a \in \mathbb{R}$, the derivative of f at a exists and equals 1.

Hence, the derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 1$.

Example 1141. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2$.

Let $a \in \mathbb{R}$. Simplify the difference quotient:

$$\frac{g(x) - g(a)}{x - a} = \frac{x^2 - a^2}{x - a} = \frac{(x - a)(x + a)}{x - a} = x + a.$$

So,[406]

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} (x + a) \stackrel{\pm}{=} \underbrace{\lim_{x \to a} a}_{a} + \underbrace{\lim_{x \to a} a}_{a} = 2a.$$

We've just shown that for any $a \in \mathbb{R}$, the derivative of g at a exists and equals $2a$.

Hence, the derivative of g is the function $g' : \mathbb{R} \to \mathbb{R}$ defined by $g'(x) = 2x$.

[406] We actually already did this in Exercise 334(c).

Example 1142. Define $h:\mathbb{R}\to\mathbb{R}$ by $h(x)=x^3$.

We are given this hint: $\qquad x^3-a^3\overset{1}{=}(x-a)\left(x^2+ax+a^2\right)$.

Let $a\in\mathbb{R}$. Simplify the difference quotient:

$$\frac{h(x)-h(a)}{x-a}=\frac{x^3-a^3}{x-a}\overset{1}{=}\frac{(x-a)\left(x^2+ax+a^2\right)}{x-a}=x^2+ax+a^2.$$

So,[407]

$$h'(a)=\lim_{x\to a}\frac{h(x)-h(a)}{x-a}=\lim_{x\to a}\left(x^2+ax+a^2\right)\overset{\pm}{=}\underbrace{\lim_{x\to a}x^2}_{a^2}+\underbrace{\lim_{x\to a}ax}_{a^2}+\underbrace{\lim_{x\to a}a^2}_{a^2}=3a^2.$$

We've just shown that for any $a\in\mathbb{R}$, the derivative of h at a exists and equals $3a^2$. Hence, the derivative of h is the function $h':\mathbb{R}\to\mathbb{R}$ defined by $h'(x)=3a^2$.

We can go on finding the derivatives of higher integer powers using this "hint":

$$\begin{aligned}
x^2-a^2&=(x-a)(x+a),\\
x^3-a^3&=(x-a)\left(x^2+ax+a^2\right),\\
x^4-a^4&\overset{\star}{=}(x-a)\left(x^3+ax^2+a^2x+a^3\right),\\
&\vdots\\
x^n-a^n&\overset{\circ}{=}(x-a)\left(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-1}+a^n\right)\\
&\vdots
\end{aligned}$$

You'll use $\overset{\star}{=}$ and $\overset{\circ}{=}$ in the next two Exercises:

Exercise 341. Define $i:\mathbb{R}\to\mathbb{R}$ by $i(x)=x^4$. Find the derivative of i. (Answer on p. 1897.)

Exercise 342. Define $j:\mathbb{R}\to\mathbb{R}$ by $j(x)=x^c$, where c is a positive integer. Find the derivative of j. (Answer on p. 1897.)

[407] At $\overset{\pm}{=}$, we can use the Sum Rule for Limits because these three limits exist:

1. $\lim_{x\to a}x^2=a^2$ (by the Power Rule for Limits)
2. $\lim_{x\to a}ax=a^2$ (by the Power and Constant Factor Rule for Limits)
3. $\lim_{x\to a}a^2=a^2$ (by the Constant Rule for Limits)

Remark 130. As stated above, a complete proof of the Power Rule is beyond the scope of H2 Maths and this textbook.

With Exercise 342, we've merely proven the Power Rule in the special case where the exponent c is a positive integer.

For a slightly more general but still incomplete proof of the Power Rule, see p. 1674 (Appendices).

90.2. Proving the Product Rule (optional)

Theorem 30. (Product Rule) *Let $D \subseteq \mathbb{R}$. Suppose $f, g : D \to \mathbb{R}$ are differentiable functions. Then $f \cdot g$ is differentiable and its derivative is*

$$f \cdot g' + f' \cdot g.$$

Proof. Let $a \in D$. Take the difference quotient and manipulate it:

$$\begin{aligned}
\frac{(f \cdot g)(x) - (f \cdot g)(a)}{x-a} &= \frac{f(x)g(x) - f(a)g(a)}{x-a} \\
&= \frac{f(x)g(x) \overbrace{- f(x)g(a) + f(x)g(a)}^{\text{Plus Zero Trick}} - f(a)g(a)}{x-a} \\
&= f(x)\frac{g(x)-g(a)}{x-a} + \frac{f(x)-f(a)}{x-a}g(a).
\end{aligned}$$

So,[408]

$$\begin{aligned}
(f \cdot g)'(a) &= \lim_{x \to a} \frac{(f \cdot g)(x) - (f \cdot g)(a)}{x-a} \\
&= \lim_{x \to a}\left[f(x)\frac{g(x)-g(a)}{x-a} + \frac{f(x)-f(a)}{x-a}g(a)\right] \\
&\overset{+,\times}{=} \underbrace{\lim_{x \to a} f(x)}_{f(a)} \underbrace{\lim_{x \to a}\frac{g(x)-g(a)}{x-a}}_{g'(a)} + \underbrace{\lim_{x \to a}\frac{f(x)-f(a)}{x-a}}_{f'(a)} \underbrace{\lim_{x \to a} g(a)}_{g(a)} \\
&= f(a)g'(a) + f'(a)g(a).
\end{aligned}$$

We've just shown that for any $a \in D$, the derivative of $f \cdot g$ at a exists and is equal to $f(a)g'(a) + f'(a)g(a)$.

Hence, the derivative of $f \cdot g$ is the function $(f \cdot g)' : D \to \mathbb{R}$ defined by $(f \cdot g)'(x) = f(x)g'(x) + f'(x)g(x)$.

Or equivalently, the derivative of $f \cdot g$ is the function $f \cdot g' + f' \cdot g$. □

[408] At $\overset{+,\times}{=}$, we can use the Sum and Product Rules for Limits. because these four limits exist:

1. $\lim_{x \to a} f(x) = f(a)$ (by the differentiability and hence continuity of f)
2. $\lim_{x \to a} g(a) = g(a)$ (by the Constant Rule for Limits)
3. $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ (by the definition of the derivative)
4. $\lim_{x \to a} \frac{g(x)-g(a)}{x-a} = g'(a)$ (ditto)

Fun Fact

The Product Rule (for differentiation) is sometimes called the Leibniz Rule.

But Leibniz initially got it wrong! He initially guessed (as one might) that analogous to the Sum and Difference Rules, the derivative of the product is equal to the product of the derivatives. That is,

$$(fg)' = f'g'.$$

He quickly realised though that this naïve Product Rule was wrong and arrived at the correct Product Rule.[409]

[409]See Ch. 146.11 (Appendices) for more about this story.

90.3. Proving the Quotient Rule (optional)

Theorem 31. (Quotient Rule) *Let $D \subseteq \mathbb{R}$. Suppose $f, g : D \to \mathbb{R}$ are differentiable functions. Then the function f/g is differentiable and its derivative is*

$$\frac{g \cdot f' - f \cdot g'}{(g)^2}.$$

Note that the domain of f/g and its derivative is $D \smallsetminus \{x : g(x) = 0\}$.

Proof. Let $a \in D \smallsetminus \{x : g(x) = 0\}$. Take the difference quotient and manipulate it:

$$\begin{aligned}
\frac{(f/g)(x) - (f/g)(a)}{x-a} &= \frac{f(x)/g(x) - f(a)/g(a)}{x-a} \\
&= \frac{f(x)/g(x) \overbrace{-f(x) + f(x)}^{\text{Plus Zero Trick}} - f(a)/g(a)}{x-a} \\
&= \frac{1}{g(a)} \frac{1}{g(x)} \frac{f(x)g(a) - f(x)g(x) + f(x)g(x) - f(a)g(x)}{x-a} \\
&= \frac{1}{g(a)} \frac{1}{g(x)} \left[g(x) \frac{f(x) - f(a)}{x-a} - f(x) \frac{g(x) - g(a)}{x-a} \right].
\end{aligned}$$

So,[410]

[410] At $\overset{-,\times}{=}$, we can use the Difference and Product Rules for Limits because these six limits exist:

1. $\lim\limits_{x \to a} \dfrac{1}{g(a)} = \dfrac{1}{g(a)}$ (by the Constant Rule for Limits)
2. $\lim\limits_{x \to a} f(x) = f(a)$ (by the continuity of f)
3. $\lim\limits_{x \to a} g(x) = g(a)$ (by the continuity of g)
4. $\lim\limits_{x \to a} \dfrac{1}{g(x)} = \dfrac{1}{g(a)}$ (by the Reciprocal Rule for Limits)
5. $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x-a} = f'(a)$ (by the definition of the derivative)
6. $\lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x-a} = g'(a)$ (ditto)

$$
\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \lim_{x\to a}\frac{(f/g)(x)-(f/g)(a)}{x-a} \\
&= \lim_{x\to a}\left\{\frac{1}{g(a)}\frac{1}{g(x)}\left[g(x)\frac{f(x)-f(a)}{x-a}-f(x)\frac{g(x)-g(a)}{x-a}\right]\right\} \\
&\overset{-,\times}{=} \underbrace{\lim_{x\to a}\frac{1}{g(a)}}_{\frac{1}{g(a)}}\underbrace{\lim_{x\to a}\frac{1}{g(x)}}_{\frac{1}{g(a)}}\left[\underbrace{\lim_{x\to a}g(x)}_{g(a)}\underbrace{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}_{f'(a)}-\underbrace{\lim_{x\to a}f(x)}_{f(a)}\underbrace{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}_{g'(a)}\right] \\
&= \frac{g(a)f'(a)-f(a)g'(a)}{[g(a)]^2}.
\end{aligned}
$$

We've just shown that for any $a \in D \setminus \{x : g(x) = 0\}$, the derivative of fg at a exists and is equal to $\dfrac{g(a)f'(a)-f(a)g'(a)}{[g(a)]^2}$.

Hence, the derivative of f/g is the function $(f/g)' : D \setminus \{x : g(x) = 0\} \to \mathbb{R}$ defined by

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}.$$

Or equivalently, the derivative of f/g is the function

$$\frac{g\cdot f'-f\cdot g'}{(g)^2}. \qquad \square$$

90.4. Proving the Chain Rule (optional)

Formal statement of the Chain Rule:

Theorem 32. (Chain Rule) *Let f and g be nice functions with* $\operatorname{Range} g \subseteq \operatorname{Domain} f$*. Suppose f and g are differentiable. Then the function $f \circ g$ is differentiable and its derivative is*

$$(f' \circ g) \cdot g'.$$

"*Proof.*" Let $E = \operatorname{Domain} g$ and $a \in E$.

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x - a} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
&\stackrel{1}{=} \lim_{x \to a} \left[\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a} \right] \\
&\stackrel{\times}{=} \underbrace{\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}}_{f'(g(a))} \underbrace{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}}_{g'(a)} \\
&= f'(g(a)) \, g'(a).
\end{aligned}$$

We've just shown that for any $a \in E$, the derivative of $f \circ g$ at a exists and is equal to $f'(g(a)) \, g'(a)$.

Hence, the derivative of $f \circ g$ is the function $(f \circ g)' : E \to \mathbb{R}$ defined by

$$(f \circ g)'(x) = f'(g(x)) \, g'(x).$$

Or equivalently, the derivative of $f \circ g$ is the function

$$(f' \circ g) \cdot g'. \quad \square$$

Unfortunately, the above "proof" is wrong! It contains two flaws:

1. At $\stackrel{1}{=}$, there is the possibility that $g(x) = g(a)$ for some values of x "near" a. There is thus the possibility that we're committing the Cardinal Sin of Dividing by Zero (see Ch. 2.2).[411]
2. At $\stackrel{\times}{=}$, we use $\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \stackrel{2}{=} f'(g(a))$—it turns out that some work must be done to show that this is actually true.

Nonetheless, these two flaws may be regarded as mere blemishes or technicalities that can be "easily" fixed (as we do (Appendices), Ch. 146.12). Though flawed, the above "proof" is "mostly" correct and "mostly" explains why the Chain Rule "works".

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[411] In contrast, it is correct to assert that $\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a)$, as discussed in Remark 127. The difference is that when considering "$\lim_{x \to a}$", we are assured that $x - a \neq 0$ but not that $g(x) - g(a) \neq 0$.

91. Some Techniques of Differentiation

91.1. Implicit Differentiation

Example 1143. Consider the curve described by

$$x^2 + \sqrt{y} \overset{1}{=} 1 \qquad \text{for } x \in [-1, 1],\ y \geq 0.$$

Figure to be
inserted here.

What's the gradient of this curve at each point?

Method 1. Take $\overset{1}{=}$ and do the algebra to express y in terms of x:

$$x^2 + \sqrt{y} \overset{1}{=} 1 \qquad \Longleftrightarrow \qquad \sqrt{y} = 1 - x^2 \qquad \Longleftrightarrow \qquad y \overset{2}{=} \left(1 - x^2\right)^2.$$

Now apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator to $\overset{2}{=}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - x^2\right)^2 = 2\left(1 - x^2\right)(-2x) = -4x\left(1 - x^2\right).$$

This last expression[412] gives us the gradient of the curve at each point x.

For example, at the point $x = 0$, the gradient of the curve is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = -4 \cdot 0\left(1 - 0^2\right) = 0.$$

While at $x = \dfrac{1}{2}$, it is $$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{1}{2}} = -4 \cdot \frac{1}{2}\left(1 - \frac{1}{2}^2\right) = -\frac{3}{2}.$$

(*Example continues on the next page ...*)

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[412] A bit more formally and in Leibniz's notation, we'd say that $\overset{2}{=}$ allows us to define a function $y : [-1, 1] \to \mathbb{R}$ by

$$y(x) \overset{2}{=} \left(1 - x^2\right)^2.$$

The derivative of y is then the function $y' : [0, \infty) \to \mathbb{R}$ defined by

$$y'(x) = -4x\left(1 - x^2\right).$$

(... *Example continued from the previous page.*)

Method 2. Take $\stackrel{1}{=}$ and directly apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+\sqrt{y}\right)=\frac{\mathrm{d}}{\mathrm{d}x}1 \qquad \stackrel{\text{CR}}{\Longrightarrow} \qquad 2x+\frac{1}{2\sqrt{y}}\frac{\mathrm{d}y}{\mathrm{d}x}=0 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x}\stackrel{5}{=}-4x\sqrt{y}.$$

This last expression gives us the gradient of the curve at each point x.

For example, by $\stackrel{1}{=}$, at $x=0$, we have $y=1$; so, the gradient of the curve is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,1)}=-4\cdot 0\sqrt{1}=0.$$

While at $x=\dfrac{1}{2}$, we have $y=\dfrac{9}{16}$; so, the gradient of the curve is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=\left(\frac{1}{2},\frac{9}{16}\right)}=-4\cdot\frac{1}{2}\sqrt{\frac{9}{16}}=-\frac{3}{2}.$$

Not surprisingly, these numbers agree with what we found in Method 1 (we'd be worried if they didn't).

By the way, if we'd like, we can plug $\stackrel{2}{=}$ into $\stackrel{5}{=}$ to get

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-4x\sqrt{\left(1-x^2\right)^2}=-4x\left(1-x^2\right).$$

Again, this agrees with what we found in Method 1.

In Method 1, we first expressed y explicitly in terms of x, then found $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

In contrast, in Method 2, we skipped the step of expressing y explicitly in terms of x. Instead, we directly applied the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator to the original equation $\stackrel{1}{=}$. This method is called **implicit differentiation** (because we skip the step of expressing y explicitly in terms of x).

Implicit differentiation will usually involve using the Chain Rule. Here for example, we used the Chain Rule at $\stackrel{\text{CR}}{\Longrightarrow}$.

Example 1144. Consider the curve described by

$$x^2 + y^2 \overset{1}{=} 1 \qquad \text{for } x \in [-1, 1],\ y \geq 0.$$

Figure to be
inserted here.

What's the gradient of this curve at each point?

Method 1. Take $\overset{1}{=}$ and do the algebra to express y in terms of x:

$$x^2 + y^2 \overset{1}{=} 1 \qquad \Longleftrightarrow \qquad y^2 = 1 - x^2 \qquad \Longleftrightarrow \qquad y = \pm\sqrt{1 - x^2}.$$

We were given that $y \geq 0$. So, we can discard any negative values of x to get

$$y \overset{2}{=} \sqrt{1 - x^2}.$$

Now apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator to $\overset{2}{=}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1 - x^2} = \frac{1}{2\sqrt{1 - x^2}}(-2x) = -\frac{x}{\sqrt{1 - x^2}}.$$

Note that whereas our curve is defined for all $x \in [-1, 1]$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is not defined at $x = \pm 1$.

Indeed, as we can see from the above graph, at $x = \pm 1$, the gradient of the curve is vertical (and thus considered undefined).

This last expression[413] gives us the gradient of the curve at each point x.

(*Example continues on the next page ...*)

[413] A bit more formally and in Leibniz's notation, we'd say that $\overset{2}{=}$ allows us to define a function $y : [-1, 1] \to \mathbb{R}$ by

$$y(x) \overset{2}{=} \sqrt{1 - x^2}.$$

The derivative of y is then the function $y' : (-1, 1) \to \mathbb{R}$ defined by

$$y'(x) = -\frac{x}{\sqrt{1 - x^2}}.$$

Again, note the subtle difference in the domains of y and y' (the latter excludes ± 1).

(… *Example continued from the previous page.*)

For example, at the point $x = 0$, the gradient of the curve is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = -\frac{0}{\sqrt{1-0^2}} = 0.$$

While at $x = \frac{1}{2}$, it is $$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{1}{2}} = -\frac{\frac{1}{2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

Method 2. Take $\overset{1}{=}$ and directly apply the $\frac{\mathrm{d}}{\mathrm{d}x}$ operator:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}1 \quad \overset{\text{CR}}{\Longrightarrow} \quad 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \Longleftrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} \overset{3}{=} -\frac{x}{y}.$$

By $\overset{1}{=}$, at $x = 0$, we have $y = 1$; so, the gradient of the curve is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,1)} = -\frac{0}{1} = 0.$$

By $\overset{1}{=}$, at $x = \frac{1}{2}$, we have $y = \frac{\sqrt{3}}{2}$; so, the gradient of the curve is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)} = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

If we'd like, we can plug $\overset{2}{=}$ into $\overset{3}{=}$ to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{1-x^2}}.$$

In the above two examples, Method 2 (implicit differentiation) was not obviously superior to Method 1. It will be in the next example, where Method 1 can't even be used:

Example 1145. Consider the curve described by

$$y^7 + y + x^5 + x \stackrel{1}{=} 0.$$

Figure to be
inserted here.

Q: What's the gradient of this curve at the point $x = 0$?

Here, Method 1 does not work at all. Try as we might, we are unable to express y in terms of x.

So, we turn to **Method 2** (implicit differentiation). Apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator to $\stackrel{1}{=}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y^7 + y + x^5 + x\right) = \frac{\mathrm{d}}{\mathrm{d}x}0 \quad \stackrel{\text{CR}}{\Longrightarrow} \quad 7y^6\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} + 5x^4 + 1 = 0 \quad \Longleftrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{2}{=} -\frac{5x^4 + 1}{7y^6 + 1}.$$

By $\stackrel{1}{=}$, at $x = 0$, we have $y^7 + y = 0$ or $y\left(y^6 + 1\right) = 0$ or $y = 0$. So, plug $(x, y) = (0, 0)$ into $\stackrel{2}{=}$ to get

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,0)} = -\frac{5 \cdot 0^4 + 1}{7 \cdot 0^6 + 1} = -1.$$

A: The gradient of the curve at $x = 0$ is -1.[414]

[414] In this example, we've used not just the Chain Rule, but also a slightly deeper result called the **Implicit Function Theorem**. The Implicit Function Theorem says that even though we may be unable to express y in terms of x, equation $\stackrel{1}{=}$ does nonetheless implicitly define a function y in terms of x and, moreover, this function y is differentiable.

For the Implicit Function Theorem to work, certain technical conditions need to be satisfied. But we needn't worry about this—in H2 Maths, we can safely assume that these conditions are always satisfied.

We won't say much more about the Implicit Function Theorem, which isn't covered in H2 Maths or most introductory calculus courses, as it requires a little knowledge of multi-variable calculus and partial derivatives. For more information, see Wikipedia or Krantz & Parks (1993, in particular Theorem 1.3.1).

Example 1146. Consider the curve described by

$$x^2 + \sin(xy) + y \overset{1}{=} 0.$$

Figure to be inserted here.

Q: What's the gradient of this curve at the point $x = 0$?

Again, Method 1 does not work at all. Try as we might, we are unable to express y in terms of x.

So, we turn to **Method 2**. Apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator to $\overset{1}{=}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 + \sin(xy) + y\right] = \frac{\mathrm{d}}{\mathrm{d}x}0 \qquad \overset{\text{CR}}{\Longrightarrow} \qquad 2x + \cos(xy)\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{\mathrm{d}y}{\mathrm{d}x} \overset{2}{=} 0.$$

By $\overset{1}{=}$, at $x = 0$, we have $y = 0$. So, plug $(x, y) = (0, 0)$ into $\overset{2}{=}$ to get

$$2 \cdot 0 + \cos(0 \cdot 0)\left(0 + 0 \cdot \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,0)}\right) + \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,0)} = 0 \qquad \Longleftrightarrow \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,0)} = 0.$$

A: The gradient of the curve at $x = 0$ is 0.

Example 1147. Consider the curve described by

$$x^3y^5 + x^2 + y \overset{1}{=} 0.$$

Figure to be
inserted here.

What's the gradient of this curve at the point $x = 0$?

Again, Method 1 does not work at all. Try as we might, we are unable to express y in terms of x.

So, we turn to **Method 2**. Apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator to $\overset{1}{=}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3y^5 + x^2 + y\right) = \frac{\mathrm{d}}{\mathrm{d}x}0 \quad \overset{\text{CR}}{\Longrightarrow} \quad 3x^2y^5 + 5x^3y^4\frac{\mathrm{d}y}{\mathrm{d}x} + 2x + \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \Longleftrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -x\frac{2 + 3xy^5}{5x^3y^4 + 1}.$$

And so, at $x = 0$, we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Conclude: The gradient of the curve at $x = 0$ is 0.

Exercise 343. Consider the curve described by

$$x^3 \sin y = 1, \qquad \text{for } x \in (-\infty, -1) \cup (1, \infty),\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Find the gradient of the curve at each point x, in terms of x. (Answer on p. 1899.)

Exercise 344. Consider the curve described by

$$y \sin x = \mathrm{e}^{x+y} - 1.$$

Find the gradient of the curve at the point $x = 0$. (Answer on p. 1899.)

The following derivatives are in List MF26 (p. 3), so no need to *mug*:[415]

[415] We also mentioned this earlier in Part I, Remark 87 (p. 493).

Fact 207. **(a)** $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$, where $x \in (-1,1)$,

(b) $\dfrac{d}{dx}\cos^{-1}x = \dfrac{-1}{\sqrt{1-x^2}}$, where $x \in (-1,1)$,

(c) $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$,

(d) $\dfrac{d}{dx}\operatorname{cosec}x = -\operatorname{cosec}x\cot x$, where x is not a multiple of π,

(e) $\dfrac{d}{dx}\sec x = \sec x\tan x$, where x is not an odd multiple of $\dfrac{\pi}{2}$.

Proof. **(a)** is proven in Example 1148 (below).

You're asked to prove **(b)** and **(c)** in Exercise 345.

As for **(d)** and **(e)**, in Exercise 186(f) and (g), we already showed that

$$\frac{d}{dx}\operatorname{cosec}x \overset{1}{=} -\frac{\cos x}{\sin^2 x} \qquad \text{and} \qquad \frac{d}{dx}\sec x \overset{2}{=} \frac{\sin x}{\cos^2 x}.$$

Can you explain why $\overset{1}{=}$ and $\overset{2}{=}$ are consistent with **(d)** and **(e)**?[416] □

Example 1148. We'll use implicit differentiation to show that $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$.

Let $y = \sin^{-1}x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, so that $x \overset{1}{=} \sin y$.

Apply the $\dfrac{d}{dx}$ operator to $\overset{1}{=}$:

$$\frac{d}{dx}x = \frac{d}{dx}\sin y \quad \overset{\text{CR}}{\Longrightarrow} \quad 1 = \cos y\frac{dy}{dx} \quad \Longleftrightarrow \quad \frac{dy}{dx} \overset{2}{=} \frac{1}{\cos y}, \quad \text{for } \cos y \neq 0.$$

By the identity $\sin^2 y + \cos^2 y = 1$, we have $\cos y \overset{3}{=} \pm\sqrt{1-x^2}$.

But since $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, we know that $\cos y \geq 0$. And so, we may discard negative values and rewrite $\overset{3}{=}$ as $\cos y \overset{4}{=} \sqrt{1-x^2}$.

Now plug $\overset{4}{=}$ into $\overset{2}{=}$ to get $\dfrac{d}{dx}\sin^{-1}x = \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$, which is defined for $x \in (-1,1)$.[417]

[416] Answer: $-\dfrac{\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\dfrac{\cos x}{\sin x} = -\operatorname{cosec}x\cot x$; $\dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\dfrac{\sin x}{\cos x} = \sec x\tan x$.

[417] So, $\sin^{-1}$ is *not* a differentiable function: Its domain is $[-1,1]$, but that of its derivative is $(-1,1)$.

Exercise 345. Show that **(a)** $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1,1)$; and **(b)** $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{1+x^2}$.

(Answer on p. 1899.)

Example 1149. Let x be the mass (g) of Milo powder in a cup and y be the volume (cm^3) of Milo powder in that cup.

Suppose that when we add 1 g of Milo to the cup, the volume of Milo powder increases by $\frac{1}{2}\,\mathrm{cm}^3$. That is, the rate of change of volume (y) with respect to mass (x) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\,\mathrm{cm}^3\,\mathrm{g}^{-1}.$$

Intuition suggests that if we had wanted to increase the volume of Milo in the cup by $1\,\mathrm{cm}^3$, then we should instead have added 2 g of Milo. That is, the rate of change of mass (x) with respect to volume (y) is

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 2\,\mathrm{g}^{-1}\,\mathrm{cm}^3.$$

In the above example, we used this piece of intuition:

$$\begin{array}{c}\text{The change in } y \text{ due to}\\ \text{a small unit change in } x\end{array} \overset{1}{=} \frac{1}{\begin{array}{c}\text{The change in } x \text{ due to}\\ \text{a small unit change in } y\end{array}}$$

This is the **Inverse Function Theorem (IFT)**:[418]

Theorem 33 (informal)**. (The Inverse Function Theorem)**

$$\frac{\mathrm{d}y}{\mathrm{d}x} \overset{2}{=} \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} \qquad \left(\text{provided } \frac{\mathrm{d}x}{\mathrm{d}y} \neq 0\right).$$

Earlier we saw that with the Chain Rule, Leibniz's notation shines. Here with the IFT, Leibniz's again notation shines. From $\overset{2}{=}$, we can easily see the IFT's underlying intuition (as given by $\overset{1}{=}$).[419]

Exercise 346. Suppose $x^2y + \sin x = 0$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Hence write down $\frac{\mathrm{d}x}{\mathrm{d}y}$. (You may leave your answers expressed in terms of both x and y.) (Answer on p. **1900**.)

[418]For a formal statement, see Theorem 54 (Appendices).

[419]In contrast, in Lagrange's notation, we'd have to state the IFT as

$$y' \overset{3}{=} \frac{1}{(y^{-1})'}.$$

From $\overset{3}{=}$, it is arguably harder to see the IFT's underlying intuition (as given by $\overset{1}{=}$).

Example 1150. Suppose $x = \sin y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Then $\frac{dx}{dy} = \frac{d}{dy}\sin y = \cos y$.

By the IFT, $\frac{dy}{dx} = \frac{1}{\cos y}$.

But $y = \sin^{-1} x$. So, $\frac{dy}{dx} = \frac{d}{dx}\sin^{-1} x = \frac{1}{\cos y} = \frac{1}{\cos\left(\sin^{-1} x\right)}$.

Exercise 347. The last example showed that $\frac{d}{dx}\sin^{-1} x \overset{1}{=} \frac{1}{\cos\left(\sin^{-1} x\right)}$. But Example 1148 also showed that $\frac{d}{dx}\sin^{-1} x \overset{2}{=} \frac{1}{\sqrt{1-x^2}}$. Why aren't $\overset{1}{=}$ and $\overset{2}{=}$ contradictory? (Answer on p. 1900.)

91.3. Term-by-Term Differentiation (optional)

Let $f_0, f_1, f_2, \ldots$ be nice, differentiable functions, whose derivatives are $f_0', f_1', f_2', \ldots$. Suppose we define a new function g by

$$g = f_0 + f_1.$$

Then by the Sum Rule (for Differentiation), g is also differentiable and its derivative is simply

$$g' = f_0' + f_1'.$$

Similarly, suppose we define a new function h by

$$h = f_0 + f_1 + f_2.$$

Then again by the Sum Rule, we know that h is also differentiable and its derivative is simply

$$h' = f_0' + f_1' + f_2'.$$

More generally, for any $n \in \mathbb{Z}^+$, suppose we define a new function i by

$$i = \sum_{i=0}^{n} f_i = f_0 + f_1 + \cdots + f_n.$$

Then by the Sum Rule, i is also differentiable, with

$$i' = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=0}^{n} f_i \stackrel{1}{=} \sum_{i=0}^{n} \frac{\mathrm{d}}{\mathrm{d}x} f_i = \sum_{i=0}^{n} f_i' = f_0' + f_1' + \cdots + f_n'.$$

In $\stackrel{1}{=}$, we

$$\text{interchange differentiation } \frac{\mathrm{d}}{\mathrm{d}x} \text{ and } \textbf{finite} \text{ summation } \sum_{i=0}^{n}.$$

We see that if a function i is the **finite** sum of n other differentiable functions, then i is also differentiable and its derivative is simply the sum of those n functions' derivatives.

Now, suppose instead we define a new function j by

$$j = \sum_{i=0}^{\infty} f_i = f_0 + f_1 + f_2 + \dots .$$

Q1. Is j also differentiable?

Q2. Is it the case that

$$j' = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=0}^{\infty} f_i \stackrel{2}{=} \sum_{i=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} f_i = \sum_{i=0}^{\infty} f_i' = f_0' + f_1' + f_2' + \dots ?$$

In particular, is $\stackrel{2}{=}$ legitimate? That is, can we

$$\text{interchange differentiation } \frac{\mathrm{d}}{\mathrm{d}x} \text{ and } \textbf{infinite} \text{ summation } \sum_{i=0}^{\infty}?$$

Is this interchange operation legitimate?

It turns out that the answer to Q1 and Q2 is,

Yes, but only if certain technical conditions are met.

In particular, *if* certain technical conditions are met, *then*, as was done in $\stackrel{2}{=}$, we can

$$\text{interchange differentiation } \frac{\mathrm{d}}{\mathrm{d}x} \text{ and } \textbf{infinite} \text{ summation } \sum_{i=0}^{\infty}.$$

Fortunately, these "certain technical conditions" are beyond the scope of H2 Maths and we shall not discuss them in this textbook. Based on past-year A-Level exams, it would seem that your A-Level examiners simply *assume* that the interchange operation is always legitimate—so, at least for H2 Maths, there is probably little danger if you also do likewise.

In Ch. 101.5, we'll revisit the questions just asked, but in the special case where j is an "infinite polynomial function".

Exercise 348. XXX (Answer on p. 905.)

A348.

92. Parametric Equations

We revisit Example 666 (from Part I):

(... *Example continued from the previous page.*)

One nice **interpretation** is that our parametric equations describe the **motion of some particle** P in the plane as **time** t progresses. P's **rightward and upward displacements** (metres) away from the origin are given by x and y.

As time progresses, P moves anti-clockwise along a unit circle centred on the origin O. At any instant of time t, P's position is $(\cos t, \sin t)$.

- At (the instant of) time $t = 0$, P is at $(x, y) = (\cos 0, \sin 0) = (1, 0)$. Thus, P's starting position is 1 m to the right of O.
- At time $t = 1$, P is at $(x, y) = (\cos 1, \sin 1) \approx (0.54, 0.84)$. Thus, after 1 s, P is 0.54 m right and 0.84 m above O.
- At time $t = 5\pi/4$, P is at $(x, y) = (\cos(5\pi/4), \sin(5\pi/4)) = \left(-\sqrt{2}/2, -\sqrt{2}/2\right) \approx (-0.71, -0.71)$. Thus, after $5\pi/4$ s, P is $\sqrt{2}/2$ m left of and $\sqrt{2}/2$ m below O.

Every 2π s, P travels one full circle and returns to its starting position $(1, 0)$.

Under this interpretation, we can also calculate the particle's **velocity** at each instant in time. We will decompose the particle's velocity into the x- and y-components.

Its velocity in the x-direction is the rate of change of displacement in the x-direction with respect to time t, i.e. the (first) derivative of x w.r.t. t:

$$v_x = \frac{dx}{dt} = -\sin t.$$

And its velocity in the y-direction is the rate of change of displacement in the y-direction with respect to time t, i.e. the (first) derivative of y w.r.t. t:

$$v_y = \frac{dy}{dt} = \cos t.$$

Altogether, $(v_x, v_y) = (-\sin t, \cos t)$. And so,

- At time $t = 0$, the particle P has velocity $(v_x, v_y) = (-\sin 0, \cos 0) = (0, 1)$—it is moving upwards at $1\,\mathrm{m\,s^{-1}}$ (and not moving rightwards at all).
- At time $t = 1$, P has velocity $(v_x, v_y) = (-\sin 1, \cos 1) \approx (-0.84, 0.54)$—it is moving *left*wards at $0.84\,\mathrm{m\,s^{-1}}$ and upwards at $0.54\,\mathrm{m\,s^{-1}}$.
- At time $t = 5\pi/4$, P has velocity

$$(v_x, v_y) = \left(-\sin\frac{5\pi}{4}, \cos\frac{5\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \approx (0.71, -0.71).$$

At time $t = 5\pi/4$, P is moving rightwards at $\sqrt{2}/2\,\mathrm{m\,s^{-1}}$ and *down*wards at $\sqrt{2}/2\,\mathrm{m\,s^{-1}}$.

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

We can similarly calculate the particle's **acceleration** at each instant in time. We will decompose the particle's acceleration into the x- and y-components.

Its acceleration in the x-direction is the rate of change of velocity in the x-direction with respect to time t or, in other words, the *second* derivative of x w.r.t. t:

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\cos t.$$

And its acceleration in the y-direction is the rate of change of velocity in the y-direction with respect to time t or, in other words, the *second* derivative of y w.r.t. t:

$$v_y = \frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\sin t.$$

Altogether, $(a_x, a_y) = (-\cos t, -\sin t)$. And so,

- At time $t = 0$, the particle P has acceleration $(a_x, a_y) = (-\cos 0, -\sin 0) = (-1, 0)$—it is accelerating *left*wards at $1\,\mathrm{m\,s^{-2}}$ (and not upwards at all).
- At time $t = 1$, P has acceleration $(a_x, a_y) = (-\cos 1, -\sin 1) \approx (-0.54, -0.84)$—it is accelerating *left*wards at $0.54\,\mathrm{m\,s^{-2}}$ and *down*wards at $0.84\,\mathrm{m\,s^{-2}}$. Note that at $t = 1$, we have $v_y = 0.54 > 0$ but $a_y = -0.84 < 0$—this means P is still moving upwards, but this upwards movement is slowing down.
- At time $t = 5\pi/4$, P has acceleration

$$(a_x, a_y) = (-\cos(5\pi/4), -\sin(5\pi/4)) = \left(\sqrt{2}/2, \sqrt{2}/2\right) \approx (0.71, 0.71).$$

At time $t = 5\pi/4$, P is moving rightwards at $\sqrt{2}/2\,\mathrm{m\,s^{-1}}$ and also upwards at $\sqrt{2}/2\,\mathrm{m\,s^{-1}}$. Note that at $t = 5\pi/4$, we have $v_y = -\sqrt{2}/2 < 0$ but $a_y = \sqrt{2}/2 > 0$—this means P is still moving downwards, but this downwards movement is slowing down.[420]

Exercise 349. Particle Q travels on the same plane as P (from the above example). The position of Q is described by $\{(x, y) : x = \sin t, y = \cos t, t \geq 0\}$. (Answer on p. 1828.)

(a) What is Q's starting position (i.e. at $t = 0$)?

(b) Is Q travelling clockwise or anticlockwise?

(c) Where will P and Q be at $t = 665\pi$?

(d) At what times t are P and Q at the exact same position?

(e) What are Q's velocity and acceleration in the x- and y-directions at each instant t?

[420] We will revisit this example when we study vectors in Part III. There, we will show that P's direction of movement is always tangent to the circle and its direction of acceleration is always towards the centre. Moreover, the magnitudes of the particle's (overall) velocity and acceleration are always constant.

We revisit Example 667 (from Part I):

Exercise 350. Continue with the above example. (Answer on p. 1828.)

(a) What are R's velocity and acceleration in the x- and y-directions at each instant t?

(b) At each of the following times, state R's position. State also its velocity and acceleration in the x- and y- directions. Describe all of these in words to a layperson, with reference to the figure in the above example.

(i) $t = \frac{\pi}{4}$. **(ii)** $t = \frac{\pi}{2}$. **(iii)** $t = 2\pi$.

(c) Above we specified that $a, b > 0$. How does R's starting position and direction of travel (clockwise or anticlockwise) change if

(i) $a, b < 0$? **(ii)** $a > 0$, $b < 0$? **(iii)** $a < 0$, $b > 0$?

We revisit

Exercise 351. The set $\{(x,y) : x = \tan t, y = \sec t, t \geq 0\}$ describes particle B's position in the plane. (Answer on p. 1830.)

(a) Rewrite the equations $x = \tan t$ and $y = \sec t$ into a single equation that does not contain the parameter t.

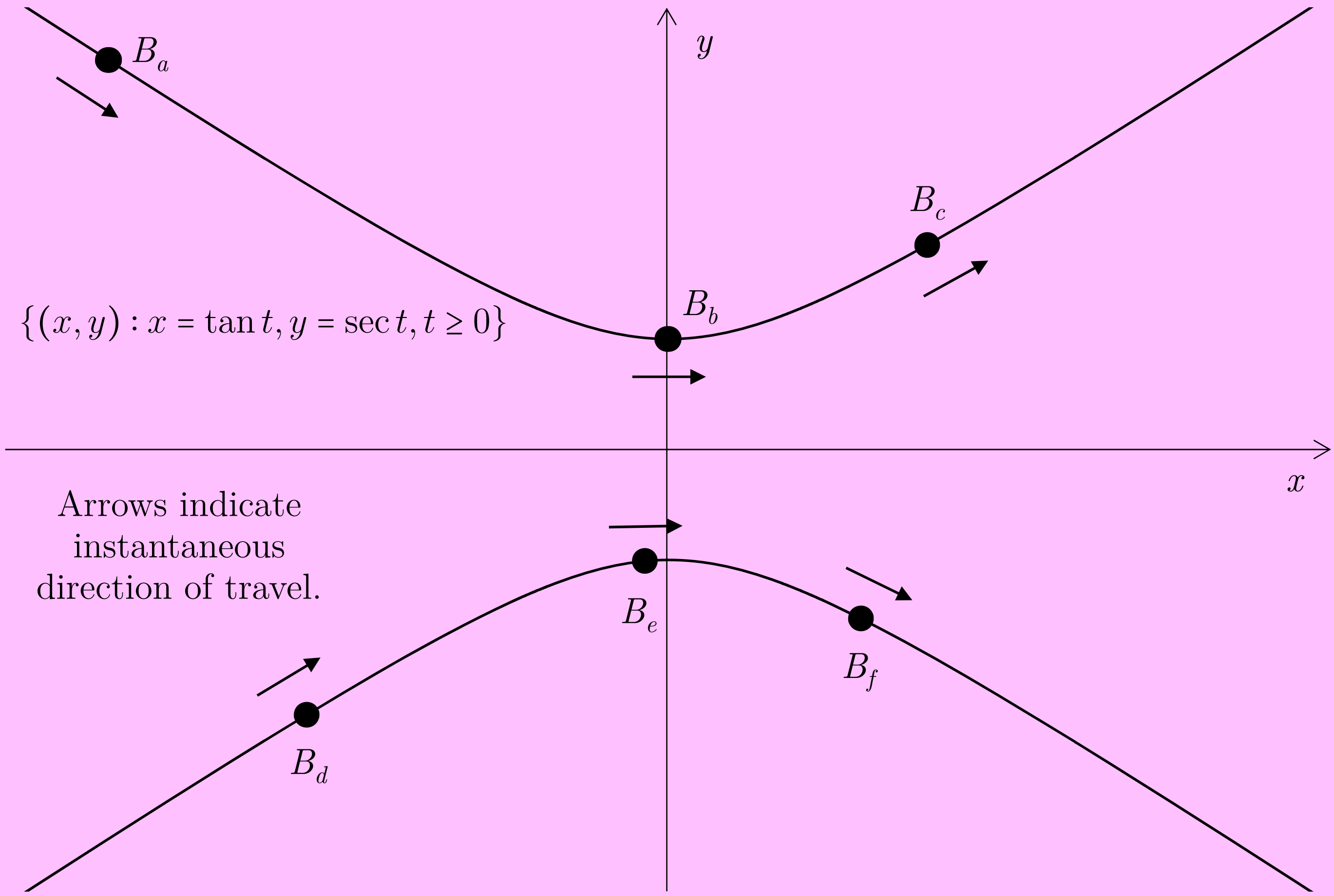
(b) What is particle B's starting position (i.e. at $t = 0$)?

(c) Compute $\mathrm{d}x/\mathrm{d}t$. And hence, conclude that the particle B always moves ________.

(d) Describe qualitatively what happens during each of these time intervals:

(i) $t \in \left[0, \frac{\pi}{2}\right)$. **(ii)** $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. **(iii)** $t \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$.

(e) Marked below are the positions of the particle B at six different instants in time $t = 0$, 1, 2, 3, 4, and 5. However, we do not know which position corresponds to which instant in time. **Without using a calculator**, match each of the six positions to the corresponding instant in time. (Hint: $0.5\pi \approx 1.57$, $\pi \approx 3.14$, $1.5\pi \approx 4.71$, and $2\pi \approx 6.28$.)



92.1. Parametric Differentiation

Theorem 34 (informal)**. (Parametric Differentiation Rule)**[421]

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \qquad \left(\text{provided } \frac{dx}{dt} \neq 0\right).$$

"*Proof.*" Simply use the Chain Rule (CR) and the Inverse Function Theorem (IFT):

$$\frac{dy}{dx} \overset{\text{CR}}{=} \frac{dy}{dt}\frac{dt}{dx} \overset{\text{IFT}}{=} \frac{dy}{dt} \div \frac{dx}{dt}. \quad \square$$

Example 1151. Let $x = t^5$ and $y = t^6$. Find $\frac{dy}{dx}$.

Method 1. First express y in terms of x: $y = x^{1.2}$.

Then compute $\frac{dy}{dx} = 1.2x^{0.2}$.

Method 2. Compute $\frac{dy}{dt} = 6t^5$ and $\frac{dx}{dt} = 5t^4$. By the Parametric Differentiation Rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 6t^5 \div \left(5t^4\right) = 1.2t = 1.2x^{0.2}.$$

Example 1152. Let $x = \sin t$ and $y = 2t$. Find $\frac{dy}{dx}$.

Method 1. First express y in terms of x: $y = 2\sin^{-1} x$.

Then compute $\frac{dy}{dx} \overset{1}{=} \frac{2}{\sqrt{1-x^2}}$ (for $x \neq \pm 1$).

Method 2. Compute $\frac{dy}{dt} = 2$ and $\frac{dx}{dt} = \cos t$. By the Parametric Differentiation Rule,

$$\frac{dy}{dx} \overset{2}{=} \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2}{\cos t} \qquad (\text{for } \cos t \neq 0).$$

Note that $\overset{1}{=}$ and $\overset{2}{=}$ are consistent with each other because

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos t} = \cos t.^{422}$$

In the last two examples, Method 2 (parametric differentiation) was not obviously superior to Method 1. In the following example, it is, because Method 1 can't even be used:

[422] And also, $x \neq \pm 1 \iff \cos t \neq 0$.

Example 1153. Let $x = t^5 + t$ and $y = t^6 - t$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Here it is difficult (or even impossible) to express y in terms of x. So, we can't use Method 1. Instead, we must use

Method 2. Compute $\frac{\mathrm{d}y}{\mathrm{d}t} = 6t^5 - 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 5t^4 - 1$. By the Parametric Differentiation Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{6t^5 - 1}{5t^4 - 1} \qquad (\text{for } 5t^4 - 1 \neq 0).$$

So, for example, at $t = 0$, we have $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=0} = 1$.

Exercise 352. Let $x = \cos t + t^2$ and $y = \mathrm{e}^t - t^3$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. (Answer on p. **1900**.)

93. The Second and Higher Derivatives

Example 1154. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.

The (first) derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 3x^2$.

The (first) derivative of f at -1 is the number $f'(-1) = 3(-1)^2 = 3$.

Figure to be
inserted here.

The *second derivative* of f is the function $f'' : \mathbb{R} \to \mathbb{R}$ defined by $f''(x) = 6x$.

The *second derivative* of f at -1 is the number $f''(-1) = 6(-1) = -6$.

Observe that f'' is itself

- the (first) derivative of f'; and
- a function.

The *second derivative* is simply the derivative of the (first) derivative. Importantly, just like the (first) derivative, the second derivative is itself a function.

Formal definitions concerning the **second derivative**:

Definition 207. Let D be a set of real numbers and $a \in D$. Suppose $f : D \to \mathbb{R}$ is a differentiable function whose derivative is $f' : D \to \mathbb{R}$. Consider this limit:

$$\lim_{x \to a} \frac{f'(x) - f'(a)}{x - a}.$$

If the above limit exists (i.e. is equal to a real number), then we say that f is *twice differentiable at* a, call this limit the *second derivative of* f *at* a, and denote it by $f''(a)$, $\ddot{f}(a)$, $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(a)$, or $\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=a}$.

If not, then we say that f is *not twice differentiable at* a.

The expression $\frac{f'(x) - f'(a)}{x - a}$ is called the *difference quotient (of* f' *at* a*)*.

If f is twice differentiable at every point in a set S, then we say that f is *twice differentiable on* S.

Suppose T is the set of points on which f is twice differentiable. Then the *second derivative of* f is the function denoted f'', $\ddot{f}$, or $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$, with domain T, codomain $\mathbb{R}$, and mapping rule

$$x \mapsto \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a}.$$

If $T = D$ (i.e. f is twice differentiable on its domain), then we call f a *twice-differentiable function*.

Remark 131. The above definition may seem long and intimidating. But in fact, it is no more than a formalisation of common sense. Indeed, it was already fully illustrated in the previous example. Once you take a moment to try to understand it (as always, the examples should help), you should find that it is in fact not at all difficult.

This will often be true in maths. You'll often encounter definitions and results that seem long and intimidating, but are in fact quite simple so long as you're willing to take a moment to try to understand them.

For comparison, we now reproduce Definitions 203, 204, and 205 (concerning the first derivative), which are very similar to Definition 207 (second derivative):

Definition 203. Let f be a nice function with domain D and $a \in D$. Consider this limit:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

If the above limit exists (i.e. is equal to a real number), then we say that f is *differentiable at* a, call this limit the *derivative of* f *at* a, and denote it by $f'(a)$.

If not, then we say that f is *not differentiable at* a.

We call the expression $\frac{f(x) - f(a)}{x - a}$ the *difference quotient (of* f *at* a*)*.

Definition 204. Suppose the function f is differentiable at every point in a set S. Then we say that f is *differentiable on* S.

And if f is differentiable at every point in its domain, then we say that f is a *differentiable function.*

Definition 205. Let f be a nice function. Let S be the set of all points at which f is differentiable. The *derivative of* f is the function $f' : S \to \mathbb{R}$ defined by

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Again, we have the notation of Lagrange, Newton, and Leibniz:

- The **second derivative of** f is a function and may be denoted

$$\underset{\textbf{Lagrange)}}{f''} \quad \text{or} \quad \underset{\textbf{Newton}}{\ddot{f}} \quad \text{or} \quad \underset{\textbf{Leibniz}}{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}}$$

- The **second derivative of** f **at a point** a is a number and may be denoted

$$\underset{\textbf{Lagrange}}{f''(a)} \quad \text{or} \quad \underset{\textbf{Newton}}{\ddot{f}(a)} \quad \text{or} \quad \underset{\textbf{Leibniz}}{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(a) \text{ or } \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=a}}$$

Under Leibniz's notation, the differentiation operator is denoted $\frac{\mathrm{d}}{\mathrm{d}x}$, while the repeated application of this operator is denoted $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Thus, the second derivative of f is denoted[423]

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \quad \text{and } \textbf{not} \quad \frac{\mathrm{d}f^2}{\mathrm{d}x^2}.$$

[423]Given our notation for the second derivative, the expression $\frac{\mathrm{d}f^2}{\mathrm{d}x^2}$ is confusing and should be avoided. However, if used, it would denote the derivative of the composite function $f^2 = f \circ f$ with respect to the variable x^2.

Example 1155. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3 + 2x$.

The (first) derivative of g is the <u>function</u> with domain $\mathbb{R}$, codomain $\mathbb{R}$, and mapping rule $x \mapsto 3x^2 + 2$.

We may denote this function by g', $\dot{g}$, or $\frac{\mathrm{d}g}{\mathrm{d}x}$; and write its mapping rule as

$$g'(x) = \dot{g}(x) = \frac{\mathrm{d}g}{\mathrm{d}x}(x) = 3x^2 + 2.$$

Note: In the case of Leibniz's notation, we may write $\frac{\mathrm{d}g}{\mathrm{d}x}(x) \overset{1}{=} 3x^2 + 2$. But as explained earlier, because the dummy variable x is already specified in "$\frac{\mathrm{d}g}{\mathrm{d}x}$", the "$(x)$" is actually superfluous. So, $\overset{1}{=}$ could be written more simply as (and is entirely equivalent to) $\frac{\mathrm{d}g}{\mathrm{d}x} \overset{2}{=} 3x^2 + 2$.

The (first) derivative of g at -1 is the <u>number</u> 5—we may write

$$g'(-1) = 5 \qquad \text{or} \qquad \dot{g}(-1) = 5 \qquad \text{or} \qquad \frac{\mathrm{d}g}{\mathrm{d}x}(-1) = 5 \quad \text{or} \quad \left.\frac{\mathrm{d}g}{\mathrm{d}x}\right|_{x=-1} = 5.$$

Figure to be
inserted here.

The second derivative of g is the <u>function</u> with domain $\mathbb{R}$, codomain $\mathbb{R}$, and mapping rule $x \mapsto 6x$.

We may denote this function by g'', $\ddot{g}$, or $\frac{\mathrm{d}^2g}{\mathrm{d}x^2}$; and write its mapping rule as

$$g''(x) = \ddot{g}(x) = \frac{\mathrm{d}^2g}{\mathrm{d}x^2}(x) = \frac{\mathrm{d}^2g}{\mathrm{d}x^2} = 6x.$$

The second derivative of g at -1 is the <u>number</u> -6—we may write

$$g''(-1) = -6 \qquad \text{or} \qquad \text{or} \qquad \ddot{g}(-1) = -6 \qquad \frac{\mathrm{d}^2g}{\mathrm{d}x^2}(-1) = -6 \quad \text{or} \quad \left.\frac{\mathrm{d}^2g}{\mathrm{d}x^2}\right|_{x=-1} = -6.$$

A twice-differentiable function must also be differentiable. However, the converse need not be true. That is, a differentiable function need not be twice differentiable:

Example 1156. Define $h : \mathbb{R}_0^+ \to \mathbb{R}$ by $h(x) = x^{3/2}$.

The (first) derivative of h is the function with domain $\mathbb{R}_0^+$, codomain $\mathbb{R}$, and mapping rule $x \mapsto \frac{3}{2}x^{1/2}$.

We may denote this function by h', $\dot{h}$, or $\frac{\mathrm{d}h}{\mathrm{d}x}$; and write its mapping rule as

$$h'(x) = \dot{h}(x) = \frac{\mathrm{d}h}{\mathrm{d}x}(x) = \frac{\mathrm{d}h}{\mathrm{d}x} = \frac{3}{2}x^{1/2}.$$

The (first) derivative of h at 4 is the number 3—we may write

$$h'(4) = 3 \qquad \text{or} \qquad \dot{h}(4) = 3 \qquad \text{or} \qquad \frac{\mathrm{d}h}{\mathrm{d}x}(4) = 3 \quad \text{or} \quad \left.\frac{\mathrm{d}h}{\mathrm{d}x}\right|_{x=4} = 3.$$

Since h is differentiable at every point in its domain $\mathbb{R}_0^+$, it is a differentiable function.

Figure to be
inserted here.

The second derivative of h is the function with domain $\mathbb{R}^+$, codomain $\mathbb{R}$, and mapping rule $x \mapsto \frac{3}{4}x^{-1/2}$.

We may denote this function by h'', $\ddot{h}$, or $\frac{\mathrm{d}^2h}{\mathrm{d}x^2}$; and write its mapping rule as

$$h''(x) = \ddot{h}(x) = \frac{\mathrm{d}^2h}{\mathrm{d}x^2}(x) = \frac{\mathrm{d}^2h}{\mathrm{d}x^2} = \frac{3}{4}x^{-1/2}.$$

The second derivative of h at 4 is the number 3/8—we may write

$$h''(4) = \frac{3}{8} \qquad \text{or} \qquad \ddot{h}(4) = \frac{3}{8} \qquad \text{or} \qquad \frac{\mathrm{d}^2h}{\mathrm{d}x^2}(4) = \frac{3}{8} \quad \text{or} \quad \left.\frac{\mathrm{d}^2h}{\mathrm{d}x^2}\right|_{x=4} = \frac{3}{8}.$$

Since h is **not** twice differentiable at every point in its domain $\mathbb{R}_0^+$ (in particular, it is not twice differentiable at 0), it is **not** a twice-differentiable function. Hence, h is an example of **a function that is differentiable but not twice differentiable**.

Exercise 353. Find each function's first and second derivatives. State if each function is differentiable and/or twice differentiable. Evaluate each function's derivative and second derivative at 1. (Answer on p. 1901.)

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \mathrm{e}^{\cos x}$.

(b) $g : [1, \infty) \to \mathbb{R}$ defined by $g(x) = \sqrt{x^2 - 1}$.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \sqrt{\sin(x^2 + 1) + 2}$. (Tedious and requires care.)

Exercise 354. Explain whether each statement is true or false. (As usual, one way to show that a statement is false is to provide a counterexample.) (Answer on p. 1902.)

(a) "f is twice differentiable $\iff$ f is differentiable."

(b) "If f is differentiable and its derivative f' is also differentiable, then f is twice differentiable."

(c) "If f is not differentiable at a, then f cannot be twice differentiable at a."

Exercise 355. Suppose $f : D \to \mathbb{R}$ is a differentiable function. Then what can we say about the domains of f' and f''? (Answer on p. 1902.)

93.1. Higher Derivatives

The third derivative is the derivative of the second derivative, the fourth is that of the third, etc. And in general, the nth derivative is the derivative of the $(n-1)$th derivative.

Again, we have the notation of Lagrange, Newton, and Leibniz:

- The **third derivative of f** is a function and may be denoted

$$\underset{\textbf{Lagrange}}{f'''} \quad \text{or} \quad \underset{\textbf{Newton}}{\dddot{f}} \quad \text{or} \quad \underset{\textbf{Leibniz)}}{\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}}$$

  (Again, we stress that **every derivative is a function**.)

- The **third derivative of f at a point** a is a number and may be denoted

$$\underset{\textbf{Lagrange}}{f'''(a)} \quad \text{or} \quad \underset{\textbf{Newton}}{\dddot{f}(a)} \quad \text{or} \quad \underset{\textbf{Leibniz}}{\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}(a) \text{ or } \left.\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right|_{x=a}}$$

With higher derivatives, Lagrange's prime and Newton's dot notation will start getting messy. And so, for $n \geq 4$,

- The **nth derivative of f** is a function and may be denoted

$$\underset{\textbf{Lagrange}}{f^{(n)}} \quad \text{or} \quad \underset{\textbf{(Newton}}{\overset{n}{\dot{f}}} \quad \text{or} \quad \underset{\textbf{Leibniz}}{\frac{\mathrm{d}^n f}{\mathrm{d}x^n}}$$

- The **nth derivative of f at a point** a is a number and may be denoted

$$\underset{\textbf{Lagrange}}{f^{(n)}(a)} \quad \text{or} \quad \underset{\textbf{Newton}}{\overset{n}{\dot{f}}(a)} \quad \text{or} \quad \underset{\textbf{Leibniz}}{\frac{\mathrm{d}^n f}{\mathrm{d}x^n}(a) \text{ or } \left.\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right|_{x=a}}$$

Example 1157. The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^5$ is (at least) seven-times differentiable. Its first seven derivatives have domain $\mathbb{R}$, codomain $\mathbb{R}$, and these mapping rules:

$$\begin{aligned}
f'(x) = \dot{f}(x) &= \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a} = 5x^4,\\
f''(x) = \ddot{f}(x) &= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) = \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=a} = 20x^3,\\
f'''(x) = \dddot{f}(x) &= \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}(x) = \left.\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right|_{x=a} = 60x^2,\\
f^{(4)}(x) = \overset{4}{\dot{f}}(x) &= \frac{\mathrm{d}^4 f}{\mathrm{d}x^4}(x) = \left.\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}\right|_{x=a} = 120x,\\
f^{(5)}(x) = \overset{5}{\dot{f}}(x) &= \frac{\mathrm{d}^5 f}{\mathrm{d}x^5}(x) = \left.\frac{\mathrm{d}^5 f}{\mathrm{d}x^5}\right|_{x=a} = 120,\\
f^{(6)}(x) = \overset{6}{\dot{f}}(x) &= \frac{\mathrm{d}^6 f}{\mathrm{d}x^6}(x) = \left.\frac{\mathrm{d}^6 f}{\mathrm{d}x^6}\right|_{x=a} = 0,\\
f^{(7)}(x) = \overset{7}{\dot{f}}(x) &= \frac{\mathrm{d}^7 f}{\mathrm{d}x^7}(x) = \left.\frac{\mathrm{d}^7 f}{\mathrm{d}x^7}\right|_{x=a} = 0.
\end{aligned}$$

Clearly, for any integer $n\geq 6$, f is n-times differentiable and its nth derivative is defined by

$$f^{(n)}(x) = \overset{n}{\dot{f}}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x) = \left.\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right|_{x=a} = 0.$$

As we'll learn in the next subchapter, f is an example of an "infinitely differentiable" or **smooth** function.

Example 1158. The sine function sin is (at least) six-times differentiable—its first six derivatives have domain $\mathbb{R}$, codomain $\mathbb{R}$, and these mapping rules:

$$\begin{aligned}
\sin' &= \dot{\sin} = \frac{\mathrm{d}\sin}{\mathrm{d}x} &&= \cos,\\
\sin'' &= \ddot{\sin} = \frac{\mathrm{d}^2\sin}{\mathrm{d}x^2} &&= -\sin,\\
\sin''' &= \dddot{\sin} = \frac{\mathrm{d}^3\sin}{\mathrm{d}x^3} &&= -\cos,\\
\sin^{(4)} &= \overset{4}{\dot{\sin}} = \frac{\mathrm{d}^4\sin}{\mathrm{d}x^4} &&= \sin,\\
\sin^{(5)} &= \overset{5}{\dot{\sin}} = \frac{\mathrm{d}^5\sin}{\mathrm{d}x^5} &&= \cos,\\
\sin^{(6)} &= \overset{6}{\dot{\sin}} = \frac{\mathrm{d}^6\sin}{\mathrm{d}x^6} &&= -\sin.
\end{aligned}$$

As you can probably tell, sin is actually "infinitely differentiable" or smooth. We'll continue with this example in the next subchapter.

Formal definitions concerning the nth derivative (once again, these seem long and intimidating, but are in fact just a mere formalisation of common sense):

Definition 208. Let D be a set of real numbers, $a \in D$, and $n \geq 3$ be an integer. Suppose $f : D \to \mathbb{R}$ is a $(n-1)$-times differentiable function whose $(n-1)$th derivative is $f^{(n-1)} : D \to \mathbb{R}$.

Consider this limit:

$$\lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a}.$$

If the above limit exists (i.e. is equal to a real number), then we say that f is *n-times differentiable at a*, call this limit the *nth derivative of f at a*, and denote it by $f^{(n)}(a)$ (or $f'''(a)$ for $n = 3$), $\overset{n}{\dot{f}}(a)$ (or $\dddot{f}(a)$ for $n = 3$), $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}(a)$, or $\left.\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right|_{x=a}$.

If not, then we say that f is *not n-times differentiable at a*.

The expression $\frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a}$ is called the *difference quotient (of $f^{(n-1)}$ at a)*.

If f is *n-times* differentiable at every point in a set S, then we say that f is *n-times differentiable on S*.

Suppose T is the set of points on which f is *n-times* differentiable. Then the *nth derivative of f* is the function denoted $f^{(n)}$ (or f''' for $n = 3$), $\overset{n}{\dot{f}}$ (or $\dddot{f}$ for $n = 3$), or $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$, with domain T, codomain $\mathbb{R}$, and mapping rule

$$x \mapsto \lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a}.$$

If $T = D$ (i.e. f is *n-times* differentiable on its domain), then we call f a *n-times-differentiable function*.

Remark 132. The above Definition is an example of a **recursive definition**.

Exercise 356. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^4 - x^3 + x^2 - x + 1$. Write down *all* of its derivatives. Evaluate each of these derivatives at 1. Write all of your answers in Lagrange's, Newton's, *and* Leibniz's notation. (Answer on p. **1903**.)

93.2. Smooth (or “Infinitely Differentiable”) Functions

A **smooth function** is simply one that’s “infinitely differentiable”:

Definition 209. Let f be a function. If for every $n \in \mathbb{Z}^+$, f is n-times differentiable at a, then we say that f is *smooth* at a.

We say that a function is *smooth on a set* S if it is smooth at every point in S.

If a function is smooth on its domain, then we call it a *smooth function*.[424]

Most functions you’ll encounter in H2 Maths are smooth. This includes, for example, *all* polynomial functions:

Example 1159. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^5$.

In Example 1157 (previous subchapter), we showed that for every positive integer n, f is n-times differentiable. Hence, f is a smooth (or “infinitely differentiable”) function.

Example 1160. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^4 - x^3 + x^2 - x + 1$.

In Exercise 356 (previous subchapter), we showed that for every positive integer n, g is n-times differentiable. Hence, g is a smooth (or “infinitely differentiable”) function.

Exercise 357. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^5 - x^4 + x^3 - x^2 + x - 1$. Find *all* the derivatives of h. Is h smooth? (Answer on p. 1903.)

As the above examples and exercise suggest, every polynomial function is smooth; moreover, if a polynomial function is of order k (i.e. the highest power is k), then its $(k+1)$th and subsequent derivatives are zero functions.[425]

The exponential function is also smooth:

Example 1161. Consider the exponential function exp. Its (first) derivative is itself:

$$\exp' = \exp.$$

So, the second derivative of exp is also itself:

$$\exp'' = \exp' = \exp.$$

And for any positive integer n, the nth derivative of exp is again itself:

$$\exp^{(n)} = \exp^{(n-1)} = \ldots \exp'' = \exp' = \exp.$$

We’ve just shown that for every positive integer n, the nth derivative of exp is exp itself and that exp is n-times differentiable. Hence, g is smooth.

The sine function is also smooth:

[424] More advanced texts may also say that a smooth function is *of class* C^∞.
[425] This assertion is formally stated as Fact 295 (Appendices).

Example 1162. Consider the sine function sin. Here are its first four derivatives:

$$\begin{aligned}\sin' &= \cos,\\ \sin'' &= -\sin,\\ \sin''' &= -\cos,\\ \sin^{(4)} &= \sin.\end{aligned}$$

The cycle then repeats. Specifically, for each $k = 0, 1, 2, 3, \ldots$, we have

$$\begin{aligned}\sin^{(1+4k)} &= \cos,\\ \sin^{(2+4k)} &= -\sin,\\ \sin^{(3+4k)} &= -\cos,\\ \sin^{(4k)} &= \sin.\end{aligned}$$

So, for example, the 8 603rd derivative of sin is

$$\sin^{(8\,603)} = \sin^{(3+4\times 2\,150)} = -\cos.$$

We've just shown that sin is n-times differentiable for every positive integer n. Hence, exp is smooth.

Exercise 358. Show that the cosine function cos is smooth. Find the 8 603rd derivative of cos. (Answer on p. 1904.)

In Ch. 88.4, we made these imprecise claims:[426]

"Most" elementary functions are differentiable.
Moreover, their derivatives are themselves elementary.

From the above imprecise claims, we have this imprecise corollary:

"Most" elementary functions are smooth.

[426]For some justification of these imprecise claims, see Ch. 146.10 (Appendices).

93.3. Confusing Notation

We reproduce from Ch. 22 (Composite Functions):

Definition 77. Let f be a function. The symbol f^2 denotes the composite function $f \circ f$. For any integer $n \geq 3$, the symbol f^n denotes the composite function $f \circ f^{n-1}$.

We just [427] defined $f^{(n)}$ to be the nth derivative of the function f.

We now also define the symbol $(f)^n$:

Definition 210. Given the nice function $f : D \to \mathbb{R}$ and $n \in \mathbb{R}$, the function $(f)^n : D \to \mathbb{R}$ is defined by $(f)^n (x) = [f(x)]^n$.

Hence, the three symbols f^n, $f^{(n)}$, and $(f)^n$ denote three different functions and we must take care to distinguish between them:

Example 1163. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.

Consider the three functions f^4, $f^{(4)}$, and $(f)^4$.

1. The function $f^4 : \mathbb{R} \to \mathbb{R}$ is the composite function $f \circ f \circ f \circ f$ and is defined by
$$f^4(x) = f(f(f(f(x)))) = f\left(f\left(f\left(x^3\right)\right)\right) = f\left(f\left(\left(x^3\right)^3\right)\right) = f\left(\left(\left(x^3\right)^3\right)^3\right) = \left(\left(\left(x^3\right)^3\right)^3\right)^3$$
$$= \left(\left(x^3\right)^3\right)^9 = \left(x^3\right)^{27} = x^{81}.$$
2. The function $f^{(4)} : \mathbb{R} \to \mathbb{R}$ is the fourth derivative of f and is defined by
$$f^{(4)}(x) = \frac{\mathrm{d}^4}{\mathrm{d}x^4} x^3 = \underset{\cdot}{\mathrm{x}}\left\{\underset{\cdot}{\mathrm{x}}\left[\underset{\cdot}{\mathrm{x}}\left(\underset{\cdot}{\mathrm{x}} x^3\right)\right]\right\} = \underset{\cdot}{\mathrm{x}}\left\{\underset{\cdot}{\mathrm{x}}\left[\underset{\cdot}{\mathrm{x}}\left(3x^2\right)\right]\right\} = \underset{\cdot}{\mathrm{x}}\left[\underset{\cdot}{\mathrm{x}}(6x)\right] = \underset{\cdot}{\mathrm{x}}6 = 0.$$
3. The function $(f)^4 : \mathbb{R} \to \mathbb{R}$ is the function f raised to the power of four and is defined by
$$(f)^4(x) = [f(x)]^4 = \left(x^3\right)^4 = x^{12}.$$

Clearly, the functions f^4, $f^{(4)}$, and $(f)^4$ are distinct. If we evaluate each of them at 2, we get wildly different values:

$$f^4(2) = 2^{81} \approx 2.417 \times 10^{24}, \qquad f^{(4)}(2) = 0, \qquad (f)^4(2) = 2^{12} = 8\,096.$$

Remark 133. The use of $(f)^n$ to denote a function raised to the nth power is not commonly used. Indeed, it does not appear on your H2 Maths syllabus or exams. We'll try not to use it. But as we'll see later, it can sometimes be convenient.

Separately, we must also take care to distinguish the following five symbols, which denote five different functions:

$$\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^n, \qquad \frac{\mathrm{d}^n f}{\mathrm{d}x^n}, \qquad \frac{\mathrm{d}f^n}{\mathrm{d}x}, \qquad \frac{\mathrm{d}f}{\mathrm{d}x^n}, \quad \text{and} \quad \frac{\mathrm{d}f^n}{\mathrm{d}x^n}.$$

[427] Definition 208.

Example 1164. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$.

Below we consider the five functions $\left(\dfrac{\mathrm{d}f}{\mathrm{d}x}\right)^2$, $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}$, $\dfrac{\mathrm{d}f^2}{\mathrm{d}x}$, $\dfrac{\mathrm{d}f}{\mathrm{d}x^2}$, and $\dfrac{\mathrm{d}f^2}{\mathrm{d}x^2}$.

But first, let's note that the derivative of f is the function $\dfrac{\mathrm{d}f}{\mathrm{d}x} : \mathbb{R} \to \mathbb{R}$ defined by

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} x^3 = 3x^2.$$

Also, the composite function $f^2 : \mathbb{R} \to \mathbb{R}$ is defined by

$$f^2(x) = f(f(x)) = f\left(x^3\right) = \left(x^3\right)^3 = x^9.$$

1. The function $\left(\dfrac{\mathrm{d}f}{\mathrm{d}x}\right)^2 : \mathbb{R} \to \mathbb{R}$ is defined by

$$\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2 = \left(3x^2\right)^2 = 9x^4.$$

2. The function $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} : \mathbb{R} \to \mathbb{R}$ is the second derivative of f and is defined by

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} x^3 = \underset{\cdot}{\mathrm{x}}\left(\underset{\cdot}{\mathrm{x}} x^3\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2\right) = 6x.$$

3. The function $\dfrac{\mathrm{d}f^2}{\mathrm{d}x} : \mathbb{R} \to \mathbb{R}$ is the first derivative of the composite function $f^2 = f \circ f$ and is defined by

$$\frac{\mathrm{d}f^2}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} x^9 = 9x^8.$$

4. Informally, the function $\dfrac{\mathrm{d}f}{\mathrm{d}x^2}$ gives us the rate of change of f with respect to x^2. By the Chain Rule,

$$\frac{\mathrm{d}f}{\mathrm{d}x^2} \underbrace{\frac{\mathrm{d}x^2}{\mathrm{d}x}}_{2x} = \underbrace{\frac{\mathrm{d}f}{\mathrm{d}x}}_{3x^2}.$$

Rearranging, $$\frac{\mathrm{d}f}{\mathrm{d}x^2} = \frac{\mathrm{d}f}{\mathrm{d}x} \div \frac{\mathrm{d}x^2}{\mathrm{d}x} = \frac{3x^2}{2x} = \frac{3}{2}x.$$

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

5. Informally, the function $\frac{\mathrm{d}f^2}{\mathrm{d}x^2}$ gives us the rate of change of the composite function f^2 with respect to x^2. By the Chain Rule,

$$\frac{\mathrm{d}f^2}{\mathrm{d}x^2}\underbrace{\frac{\mathrm{d}x^2}{\mathrm{d}x}}_{2x} = \underbrace{\frac{\mathrm{d}f^2}{\mathrm{d}x}}_{9x^8}.$$

Rearranging, $$\frac{\mathrm{d}f}{\mathrm{d}x^2} = \frac{\mathrm{d}f^2}{\mathrm{d}x} \div \frac{\mathrm{d}x^2}{\mathrm{d}x} = \frac{9x^8}{2x} = \frac{9}{2}x^7.$$

Clearly, the functions $\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2$, $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$, $\frac{\mathrm{d}f^2}{\mathrm{d}x}$, $\frac{\mathrm{d}f}{\mathrm{d}x^2}$, and $\frac{\mathrm{d}f^2}{\mathrm{d}x^2}$ are distinct.
If we evaluate each of them at 2, we get wildly different values:

$$\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2\Bigg|_{x=2} = 9\cdot 2^4 = 144,$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\Bigg|_{x=2} = 6\cdot 2 = 12,$$

$$\frac{\mathrm{d}f^2}{\mathrm{d}x}\Bigg|_{x=2} = 9\cdot 2^8 = 2\,304,$$

$$\frac{\mathrm{d}f}{\mathrm{d}x^2}\Bigg|_{x=2} = \frac{3}{2}\cdot 2 = 3,$$

$$\frac{\mathrm{d}f^2}{\mathrm{d}x^2}\Bigg|_{x=2} = \frac{9}{2}\cdot 2^7 = 576.$$

Remark 134. We shall assign no meaning to the symbols $\frac{\mathrm{d}^n f}{\mathrm{d}x}$ and $\frac{\mathrm{d}f}{\mathrm{d}^n x}$.

Exercise 359. In Lagrange's notation, why do we denote the nth derivative of f with parentheses? That is, why do we denote the nth derivative of f by $f^{(n)}$ rather than more simply f^n? (Answer on p. 1904.)

Exercise 360. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^2 + 1$. Find each of the following eight functions and evaluate each at 1: (Answer on p. 1904.)

(a) h' **(b)** h^2 **(c)** $(h)^2$ **(d)** $\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2$

(e) $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}$ **(f)** $\frac{\mathrm{d}h^2}{\mathrm{d}x}$ **(g)** $\frac{\mathrm{d}h}{\mathrm{d}x^2}$ **(h)** $\frac{\mathrm{d}h^2}{\mathrm{d}x^2}$

94. The Increasing/Decreasing Test

In Ch. 19, we asked, "When is a function increasing or decreasing?" We now revisit this question.

We reproduce (from Ch. 19) this definition:

Definition 73. Let f be a nice function with domain D. Suppose $S \subseteq D$. Then f is

(a)	*Increasing on S*	if for any $a, b \in S$ with $a < b$, we have	$f(a) \le f(b)$.
(b)	*Strictly increasing*	"	$f(a) < f(b)$.
(c)	*Decreasing*	"	$f(a) \ge f(b)$.
(d)	*Strictly decreasing*	"	$f(a) > f(b)$.

If f is increasing on D, then f is an *increasing function.*

If f is strictly increasing on D, then f is a *strictly increasing function.*

If f is decreasing on D, then f is a *decreasing function.*

If f is strictly decreasing on D, then f is a *strictly decreasing function.*

The problem of finding a function's derivative is the problem of finding the function's gradient. And so, not surprisingly, the derivative is closely related to whether a function is increasing or decreasing. We have, in particular, the intuitively "obvious" **Increasing/Decreasing Test (IDT)**:

Fact 208. (Increasing/Decreasing Test, IDT) *Let f be a function and D be an interval. Suppose f is differentiable on D.*

(a)	$f'(x) \ge 0$ for all $x \in D$	$\iff$	f is increasing on D.
(b)	$f'(x) > 0$ for all $x \in D$	$\implies$	f is strictly increasing on D.
(c)	$f'(x) \le 0$ for all $x \in D$	$\iff$	f is decreasing on D.
(d)	$f'(x) < 0$ for all $x \in D$	$\implies$	f is strictly decreasing on D.

Proof. See p. 1686 (Appendices). □

Example 1165. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

The derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 2x$.

Figure to be
inserted here.

(a) $f'(x) \geq 0$ for all $x \in [0, \infty)$ $\overset{\text{IDT}}{\Longleftrightarrow}$ f is increasing on $[0, \infty)$.

(b) $f'(x) > 0$ for all $x \in (0, \infty)$ $\overset{\text{IDT}}{\Longrightarrow}$ f is strictly increasing on $(0, \infty)$.

(c) $f'(x) \leq 0$ for all $x \in (-\infty, 0]$ $\overset{\text{IDT}}{\Longleftrightarrow}$ f is decreasing on $(-\infty, 0]$.

(d) $f'(x) < 0$ for all $x \in (-\infty, 0)$ $\overset{\text{IDT}}{\Longrightarrow}$ f is strictly decreasing on $(-\infty, 0)$.

Example 1166. Define $i : \mathbb{R} \to \mathbb{R}$ by $i(x) = 1$.

The derivative of i is the function $i' : \mathbb{R} \to \mathbb{R}$ defined by $i'(x) = 0$.

Figure to be
inserted here.

(a) $i'(x) \geq 0$ for all $x \in \mathbb{R}$ $\overset{\text{IDT}}{\Longleftrightarrow}$ i is increasing on $\mathbb{R}$.

(c) $i'(x) \leq 0$ for all $x \in \mathbb{R}$ $\overset{\text{IDT}}{\Longleftrightarrow}$ i is decreasing on $\mathbb{R}$.

Example 1167. Define $j : \mathbb{R} \to \mathbb{R}$ by $j(x) = x^3 - 2x^2 + x$.

Figure to be
inserted here.

The derivative of j is the function $j' : \mathbb{R} \to \mathbb{R}$ defined by $j'(x) = 3x^2 - 4x + 1$. We have

$$j'(a) = 0 \qquad \Longleftrightarrow \qquad 3a^2 - 4a + 1 = (3a-1)(a-1) = 0 \qquad \Longleftrightarrow \qquad a = \frac{1}{3} \text{ or } a = 1.$$

So, we can draw this sign diagram for j':

Figure to be
inserted here.

(a) $j'(x) \geq 0$ for all $x \in \left(-\infty, \frac{1}{3}\right] \cup [1, \infty)$ $\overset{\text{IDT}}{\Longleftrightarrow}$ j is increasing on $\left(-\infty, \frac{1}{3}\right] \cup [1, \infty)$.

(b) $j'(x) > 0$ for all $x \in \left(-\infty, \frac{1}{3}\right) \cup (1, \infty)$ $\overset{\text{IDT}}{\Longrightarrow}$ j is strictly increasing on $\left(-\infty, \frac{1}{3}\right) \cup (1, \infty)$.

(c) $j'(x) \leq 0$ for all $x \in \left[\frac{1}{3}, 1\right]$ $\overset{\text{IDT}}{\Longleftrightarrow}$ j is decreasing on $\left[\frac{1}{3}, 1\right]$.

(d) $j'(x) < 0$ for all $x \in \left(\frac{1}{3}, 1\right)$ $\overset{\text{IDT}}{\Longrightarrow}$ j is strictly decreasing on $\left(\frac{1}{3}, 1\right)$.

The following example shows that the converse of IDT**(b)** is false—that is, we **cannot** replace the $\Longrightarrow$ with $\Longleftrightarrow$.

Example 1168. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3$.

The derivative of g is the function $g' : \mathbb{R} \to \mathbb{R}$ defined by $g'(x) = 3x^2$.

Figure to be inserted here.

Observe that g is strictly increasing on $\mathbb{R}$ because if $a < b$, then $g(a) = a^3 < b^3 = g(b)$.

However, g' is not strictly positive on $\mathbb{R}$. In particular, g' is not strictly positive at 0:

$$g'(0) = 3 \cdot 0^2 = 0 \not> 0.$$

This shows that the converse of IDT**(b)** is false. (Similarly, the converse of IDT**(d)** is false, as Exercise 362 will show.)

Exercise 361. For each function, find its derivative and draw the sign diagram for the derivative. Explain how your findings are consistent with the Increasing/Decreasing Test (IDT). (Answer on p. 1906.)

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x - x$.

(b) $g : [0, 2\pi] \to \mathbb{R}$ defined by $g(x) = \sin x$.

Exercise 362. Define $i : \mathbb{R} \to \mathbb{R}$ by $i(x) = -x^3$. (Answer on p. 1907.)

(a) Find the derivative of i.

(b) Identify the intervals on which i is increasing, decreasing, strictly increasing, and/or strictly decreasing.

(c) *Fill in the blanks*: The function i shows that the ________ of ________ is false.

95. Determining the Nature of a Stationary Point

We reproduce (from Ch. **??**) these informal definitions of the eight types of **extrema** (i.e. **maximum** and **minimum points**):[428]

(a) A **global maximum** is at least as high as any other point.
(b) The **strict global maximum** is higher than any other point.
(c) A **local maximum** is at least as high as any "nearby" point.
(d) A **strict local maximum** is higher than any "nearby" point.
(e) A **global minimum** is at least as low as any other point.
(f) The **strict global minimum** is lower than any other point.
(g) A **local minimum** is at least as low as any "nearby" point.
(h) A **strict local minimum** is lower than any "nearby" point.

We reproduce (from Ch. 43.8) these formal definitions of **stationary** and **turning points**:

Definition 115. A point x is a *stationary point* of a function f if $f'(x) = 0$.

Definition 58. A *turning point* is a point that's both a stationary point and a strict local maximum or minimum.

Quick examples to remind you of how these work:

[428]For the formal definitions of extrema, see Definition 277 (Appendices).

Example 1169. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

The derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 2x$. We have

$$f'(a) = 2a \qquad \Longleftrightarrow \qquad 2a = 0 \qquad \Longleftrightarrow \qquad a = 0.$$

So, f has exactly one stationary point, 0.

Figure to be
inserted here.

The point 0 is also a strict local minimum of f, because it is lower than any "nearby" point: $f(0) < f(x)$ for all x "near" 0.

(Indeed, 0 is actually also a strict *global* minimum of f, because it is lower than *any* point: $f(0) < f(x)$ for all $x \neq 0$.)

Since 0 is both a stationary point and a strict extremum, by Definition 58, it is also a turning point of f.

Does f have any global maxima or strict global maximum?[429]

[429]No.

Example 1170. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3 - 2x^2 + x$.

The derivative of g is the function $g' : \mathbb{R} \to \mathbb{R}$ defined by $g'(x) = 3x^2 - 4x + 1$. We have

$$g'(a) = 0 \quad \iff \quad 3a^2 - 4a + 1 = (3a - 1)(a - 1) = 0 \quad \iff \quad a = \frac{1}{3} \text{ or } a = 1.$$

So, g has two stationary points, namely $\frac{1}{3}$ and 1.

Figure to be
inserted here.

From the graph, we can informally tell that $\frac{1}{3}$ is a strict local maximum of g, while 1 is a strict local minimum of g. (In the following subchapters, we'll learn about two tests that will formally confirm these informal observations.)

Each of $\frac{1}{3}$ and 1 is a turning point of g, because each is both a stationary point and a strict extremum of g.

Does g have any global maxima, global minima, strict global maximum, or strict global minimum?[430]

[430] No.

Example 1171. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^3$.

The derivative of h is the function $h' : \mathbb{R} \to \mathbb{R}$ defined by $h'(x) = 3x^2$. We have

$$h'(a) = 0 \qquad \iff \qquad 3a^2 = 0 \qquad \iff \qquad a = 0.$$

So, h has exactly one stationary point, 0.

Figure to be
inserted here.

Observe that $h(0) = 0$. However, $h(x) < 0$ for all $x < 0$, while $h(x) > 0$ for all $x > 0$.

Thus, 0 is neither a minimum nor a maximum of h. Indeed, h has no extrema.

The sole stationary point, 0, is not a strict local extremum. Hence, h has no turning points.

Example 1172. Define $i : \mathbb{R} \to \mathbb{R}$ by $i(x) = 1$.

The derivative of i is the function $i' : \mathbb{R} \to \mathbb{R}$ defined by $i'(x) = 0$.

For every $a \in \mathbb{R}$, we have $i'(a) = 0$. So, *every* point $a \in \mathbb{R}$ is a stationary point of i.

Figure to be
inserted here.

Note also that *every* point $a \in \mathbb{R}$ is a local maximum, local minimum, global maximum, *and* global minimum of i. (Why?)[431]

However, no point is a strict local maximum or strict local minimum of i. Hence, i has no turning points.

[431] Pick any $a \in \mathbb{R}$. We have $i(x) \le i(a) \le i(x)$ for all $x \in \mathbb{R}$. Hence, a is a local maximum, local minimum, global maximum, *and* global minimum of i.

95.1. The First Derivative Test for Extrema (FDTE)

The **First Derivative Test for Extrema (FDTE)** is one way to check if a stationary point (or indeed *any* point) is an extremum:

Proposition 9 (informal)**. (First Derivative Test for Extrema, FDTE)** *Let f be a continuous function and a be a point.*

(a) If $f' \geq 0$ on a's "immediate left" and $f' \leq 0$ on a's "immediate right", then a is a local maximum of f.

(b) If $f' > 0$ on a's "immediate left" and $f' < 0$ on a's "immediate right", then a is a strict local maximum of f.

(c) If $f' \leq 0$ on a's "immediate left" and $f' \geq 0$ on a's "immediate right", then a is a local maximum of f.

(d) If $f' < 0$ on a's "immediate left" and $f' > 0$ on a's "immediate right", then a is a strict local maximum of f.

"*Proof*". **(a)** If $f' \geq 0$ on a's left, then by the IDT, f is increasing on a's left.

Similarly, if $f' \leq 0$ on a's right, then by the IDT, f is decreasing on a's right.

So, "obviously", f attains a global maximum at a.

Figure to be
inserted here.

The proofs of **(b)**–**(d)** are similar. □

Remark 135. The above statement and proof of the FDTE are a little informal. For a formal (and rigorous) statement and proof, see Proposition 33 (Appendices).

Example 1173. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

Figure to be inserted here.

The derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 2x$.

The sole stationary point of f is 0.

At all points to the left of 0, $f' < 0$. And at all points to the right of 0, $f' > 0$.

Figure to be inserted here.

So, by FDTE**(d)**, 0 is a strict local minimum of f. (And since 0 is both a stationary point and a strict extremum, it is also a turning point of f.)

Example 1174. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3 - 2x^2 + x$.

Figure to be
inserted here.

The derivative of g is the function $g' : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 3x^2 - 4x + 1$.

The two stationary points of g are $\dfrac{1}{3}$ and 1.

At all points to the left of $\dfrac{1}{3}$ and right of 1, $g' > 0$. And at all points between $\dfrac{1}{3}$ and 1, $g' < 0$. So,

- By FDTE(**b**), $\dfrac{1}{3}$ is a strict local maximum of f. (And since $\dfrac{1}{3}$ is both a stationary point and a strict extremum, it is also a turning point of f.)
- By FDTE(**d**), 1 is a strict local minimum of f. (And since 1 is both a stationary point and a strict extremum, it is also a turning point of f.)

In the next example, the FDTE can tell us nothing whatsoever. In such cases, we say that the FDTE is **inconclusive**.

Example 1175. Define $h:\mathbb{R}\to\mathbb{R}$ by $h(x)=x^3$.

Figure to be
inserted here.

The derivative of h is the function $h':\mathbb{R}\to\mathbb{R}$ defined by $h'(x)=3x^2$.

The sole stationary point of h is 0.

Figure to be
inserted here.

At all points to the left of 0, $h'>0$. And at all points to the right of 0, we *also* have $h'>0$.

So, the FDTE can tell us nothing whatsoever. In cases such as this, we say that the FDTE is **inconclusive**.

Since the FDTE is inconclusive, we must determine the nature of the stationary point 0 using other methods.

One such method is to simply observe that at 0, we have $h(0)=0$. But at all points to the left of 0, we have $h<0$; while at all points to the right of 0, we have $h>0$. Hence, 0 is neither a minimum nor a maximum of h.

And since 0 is a stationary point but not a strict local extremum, it is *not* a turning point of h. (And h has no turning points.)

Example 1176. Define $i : \mathbb{R} \to \mathbb{R}$ by $i(x) = 1$.

Figure to be
inserted here.

The derivative of i is the function $i' : \mathbb{R} \to \mathbb{R}$ defined by $i'(x) = 0$.

For each real number a, we have $i'(a) = 0$. So, *every* point $a \in \mathbb{R}$ is a stationary point of i.

Figure to be
inserted here.

At each $a \in \mathbb{R}$, we have $i' = 0$ everywhere on its left and also everywhere on its right.

So, at each $a \in \mathbb{R}$, $i' \leq 0$ everywhere on its left and $i' \geq 0$ everywhere on its left. Hence, by FDTE(**a**), each $a \in \mathbb{R}$ is a local maximum of f.

Also, at each $a \in \mathbb{R}$, $i' \geq 0$ everywhere on its left and $i' \leq 0$ everywhere on its left. Hence, by FDTE(**c**), each $a \in \mathbb{R}$ is a local minimum of f.

Altogether, every $a \in \mathbb{R}$ is *both* a local maximum and a local minimum of f.

However, each $a \in \mathbb{R}$ is neither a strict local maximum nor a strict local minimum. This is because for any $a \in \mathbb{R}$, we can always find some "nearby" x for which $f(x) = f(a)$.

So, although *every* $a \in \mathbb{R}$ is a stationary point of i, i has no strict local extrema. Hence, i has no turning points.

Exercise 363. For each function, **(i)** find all stationary points; **(ii)** determine whether each of these stationary points is a local maximum, local minimum, strict local maximum, or strict local minimum; **(iii)** hence write down all turning points. (Answer on p. 1908.)

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -x^3 + 3x + 1$.

(b) $g : [0, 2\pi] \to \mathbb{R}$ defined by $g(x) = \sin x$.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = e^{x^2}$.

Remark 136. It turns out that the converses of FDTE**(a)**–**(d)** are false! See Example 1584 (Appendices) for a counterexample.

(Nonetheless, for H2 Maths, there is probably little danger in blithely assuming that these converses are true.)

In the next subchapter, we'll learn of a partial converse to the FDTE: **Fermat's Theorem on Extrema**.

95.2. A Situation Where a Stationary Point Is Also a Strict Global Extremum

As we've seen above, it is **not** generally true that a stationary point is an extremum.

But fortunately, we have the following result, which says this:

If on a closed interval, a function has only one stationary point and the value of the function at that stationary point is greater (or smaller) than at the interval's two endpoints, then that stationary point is a strict global maximum (or strict global minimum).

A bit more precisely,

Proposition 10. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function and $c \in (a, b)$. Suppose c is the only stationary point of f in (a, b).*

(a) *If $f(c)$ is greater than $f(a)$ and $f(b)$, then c is the strict global maximum of f.*

(b) *If $f(c)$ is smaller than $f(a)$ and $f(b)$, then c is the strict global minimum of f.*

Proof. See p. 1690 (Appendices). □

Here's an informal argument for why Proposition 10(a) might be true:

Since $f(c) > f(a)$, f must be increasing on (a, c) (otherwise we'd have another stationary point somewhere in (a, c)).

Similarly, since $f(c) > f(b)$, f must be decreasing on (c, b) (otherwise we'd have another stationary point somewhere in (c, b)).

Thus, c must also be the strict global maximum of f.

Figure to be inserted here.

Example 1177. XXX

Example 1178. XXX

Exercise 364. XXX (Answer on p. 941.)

A364.

95.3. Fermat's Theorem on Extrema

Loosely, the First Derivative Test for Extrema (FDTE) says that

If f' changes signs at c, then c is an extremum of f.

Fermat's Theorem on Extrema is a partial converse to the FDTE:

Theorem 35. (Fermat's Theorem on Extrema) *Let f be a function that is differentiable on (a, b). Suppose $c \in (a, b)$.*

If c is an extremum of f, then $f'(c) = 0$.

Proof. See p. 1683 (Appendices). □

Example 1179. XXX

Example 1180. XXX

Remark 137. A bit like Euler, Fermat was a stud whose name adorns many results and theorems. Most famously, we have Fermat's Last Theorem.

Here's an example where Fermat's Theorem seemingly fails:

Example 1181. Define $f : [2, 3] \to \mathbb{R}$ by $f(x) = x$.

The derivative of f is the function $f' : [2, 3] \to \mathbb{R}$ defined by $f'(x) = 1$.

Figure to be inserted here.

Then 2 is an extremum of f. (Indeed, 2 is a strict local and global minimum of f.) However, 2 is not a stationary point of f because $f'(2) = 1 \neq 0$—this seems to contradict Fermat's Theorem.

Similarly, 3 is an extremum of f. (Indeed, 3 is a strict local and global maximum of f.) However, 3 is not a stationary point of f because $f'(3) = 1 \neq 0$—this seems to contradict Fermat's Theorem.

Of course, there isn't actually anything wrong with Fermat's Theorem. In Fermat's Theorem, we have the subtle but important condition that $c \in (a, b)$—that is, the extremum c should be in the **interior** (and not be an endpoint) of the function's domain.

In our present example, 2 and 3 aren't in the interior (and are instead endpoints) of f's domain. So, Fermat's Theorem may not apply.

In the main text, we don't define what exactly the term **interior** means.[432] But if we did, then we could've written Fermat's Theorem on Extrema more simply:

Theorem 36. (Also Fermat's Theorem on Extrema) *Let f be a function that is differentiable. If c is an interior extremum of f, then $f'(c) = 0$.*

By the way, this is why Fermat's Theorem on Extrema is also called the **Interior Extremum Theorem**.

Example 1182. XXX

Example 1183. XXX

Exercise 365. XXX (Answer on p. 943.)

A365.

[432]For this, see Definition XXX (Appendices).

95.4. The Second Derivative Test for Extrema (SDTE)

Given a twice-differentiable function, the **Second Derivative Test for Extrema (SDTE)** can help us determine whether a stationary point is a strict local maximum or minimum.

Below, we'll formally state the SDTE as Proposition 11. But first, let's illustrate the SDTE with two quick examples. It turns out that the SDTE is an immediate consequence of the Increasing/Decreasing Test (IDT) and the First Derivative Test for Extrema (FDTE).

Example 1184. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

The derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 2x$.

The sole stationary point of f is 0.

Figure to be inserted here.

The second derivative of f is the function $f'' : \mathbb{R} \to \mathbb{R}$ defined by $f''(x) = 2$.

So, f'' is strictly positive everywhere and in particular at 0. By the IDT(b) then, f' is strictly increasing at 0.

Since $f'(0) = 0$, it must be that $f' < 0$ at all points to the left of 0 and $f' > 0$ at all points to the right of 0. By the FDTE(d) then, 0 must be a strict local minimum of f.

This illustrates SDTE(**b**) (to be formally stated below), which says that if $f'(c) = 0$ and $f''(c) > 0$, then c is a strict local minimum of f.

Example 1185. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3 - 2x^2 + x$.

The derivative of g is the function $g' : \mathbb{R} \to \mathbb{R}$ defined by $g'(x) = 3x^2 - 4x + 1$. Since $g'(a) = 0 \iff a = \frac{1}{3}, 1$, the two stationary points of g are $\frac{1}{3}$ and 1.

The second derivative of g is the function $g'' : \mathbb{R} \to \mathbb{R}$ defined by $g''(x) = 6x - 4$.

At $\frac{1}{3}$, we have $g''\left(\frac{1}{3}\right) = 6 \cdot \frac{1}{3} - 4 = -2 < 0$. So,[433] g'' is strictly negative "around" $\frac{1}{3}$. Hence, by the IDT(d), g' must be strictly decreasing "around" $\frac{1}{3}$.

Since $g'\left(\frac{1}{3}\right) = 0$, it must be $g' > 0$ to the left of $\frac{1}{3}$ and $g' < 0$ to the right of $\frac{1}{3}$. By the FDTE(b) then, $\frac{1}{3}$ must be a strict local maximum of g.

This illustrates SDTE**(a)** (to be formally stated below), which says that if $g'(c) = 0$ and $g''(c) < 0$, then c is a strict local maximum of g.

Figure to be inserted here.

Similarly, at 1, we have $g''(1) = 6 \cdot 1 - 4 = 2 > 0$. So, g'' is strictly positive "around" 1. Hence, by the IDT(b), g' must be strictly increasing "around" 1.

Since $g'(1) = 0$, it must be $g' < 0$ to the left of 1 and $g' > 0$ to the right of 1. By the FDTE(d) then, 1 must be a strict local minimum of g.

This illustrates SDTE**(b)** (to be formally stated below), which says that if $g'(c) = 0$ and $g''(c) > 0$, then c is a strict local maximum of g.

Proposition 11. (Second Derivative Test for Extrema, SDTE) *Let f be a function that is twice differentiable at c. Suppose $f'(c) = 0$ (i.e. c is a stationary point of f).*

(a) *If $f''(c) < 0$, then c is a strict local maximum of f.*

(b) *If $f''(c) > 0$, then c is a strict local minimum of f.*

(c) *If $f''(c) = 0$, then the SDTE is inconclusive—more specifically, c could be any of the following:*

- *a strict local maximum;*
- *a strict local minimum;*
- *an inflexion point; or*
- *none of the above.*

[433] By the continuity of g''.

Proof. See p. 1690 (Appendices). □

Exercise 366. For each function, **(i)** find all stationary points; **(ii)** determine whether each of these stationary points is a strict local maximum or strict local minimum; **(iii)** hence write down all turning points. (Answer on p. **1909**.)

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^8 + \frac{2}{7}x^7 - x^6$.

(b) $g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ defined by $g(x) = \tan x$.

(c) $h : [0, 2\pi] \to \mathbb{R}$ defined by $h(x) = \sin x + \cos x$.

We now illustrate SDTE**(c)** with four examples:

Example 1186. Define $k : \mathbb{R} \to \mathbb{R}$ by $k(x) = -x^4$.

The first derivative of k is the function $k' : \mathbb{R} \to \mathbb{R}$ defined by $k'(x) = -4x^3$. Observe:

$$k'(a) = 0 \quad \iff \quad -4a^3 = 0 \quad \iff \quad a = 0.$$

So, the sole stationary point of k is 0.

The second derivative of k is the function $k'' : \mathbb{R} \to \mathbb{R}$ defined by $k''(x) = -12x^2$.

Figure to be
inserted here.

At 0, we have $k''(0) = -12 \cdot 0^2 = 0$. So, by SDTE**(c)**, the SDTE is inconclusive—the stationary point 0 could be

- a strict local maximum;
- a strict local minimum;
- an inflexion point; or
- none of the above.

Here it so happens that the stationary point 0 is a strict local maximum. (For all $x \neq 0$, $k(x) < k(0) = 0$.)

By the way, the above example also shows that the converse of SDTE**(a)** is false—0 is a strict local maximum of k, but $k''(0) \nless 0$.

Example 1187. Define $j : \mathbb{R} \to \mathbb{R}$ by $j(x) = x^4$.

The first derivative of j is the function $j' : \mathbb{R} \to \mathbb{R}$ defined by $j'(x) = 4x^3$. Observe:

$$j'(a) = 0 \qquad \iff \qquad 4a^3 = 0 \qquad \iff \qquad a = 0.$$

So, the sole stationary point of j is 0.

The second derivative of j is the function $j'' : \mathbb{R} \to \mathbb{R}$ defined by $j''(x) = 12x^2$.

Figure to be
inserted here.

At 0, we have $j''(0) = 12 \cdot 0^2 = 0$. So, by SDTE(**c**), the SDTE is inconclusive—the stationary point 0 could be

- a strict local maximum;
- a strict local minimum;
- an inflexion point; or
- none of the above.

Here it so happens that 0 is a strict local minimum. (For all $x \neq 0$, $k(x) > k(0) = 0$.)

By the way, the above example shows that the converse of SDTE(**b**) is false—0 is a strict local minimum of j, but $j''(0) \not> 0$.

Example 1188. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^3$.

The derivative of h is the function $h' : \mathbb{R} \to \mathbb{R}$ defined by $h'(x) = 3x^2$. Since $h'(a) = 0 \iff a = 0$, the sole stationary point of h is 0.

The second derivative of h is the function $h'' : \mathbb{R} \to \mathbb{R}$ defined by $h''(x) = 6x$.

Figure to be
inserted here.

At 0, we have $h''(0) = 6 \cdot 0 = 0$. So, by SDTE**(c)**, the SDTE is inconclusive—the stationary point 0 could be

- a strict local maximum;
- a strict local minimum;
- an inflexion point; or
- none of the above.

Here it so happens that 0 is an inflexion point—we'll learn why this is so in Ch. 97 (below).

Example 1189. Define $i : \mathbb{R} \to \mathbb{R}$ by $i(x) = 1$.

The derivative of i is the function $i' : \mathbb{R} \to \mathbb{R}$ defined by $i'(x) = 0$. Since $i'(a) = 0$ for all $a \in \mathbb{R}$, *every* point $a \in \mathbb{R}$ is a stationary point of i.

The second derivative of i is the function $i'' : \mathbb{R} \to \mathbb{R}$ defined by $i''(x) = 0$.

Figure to be
inserted here.

At each $a \in \mathbb{R}$, we have $i''(a) = 0$. So, by SDTE(**c**), the SDTE is inconclusive—the stationary point a could be

- a strict local maximum;
- a strict local minimum;
- an inflexion point; or
- none of the above.

Here it so happens that a is "none of the above". We can already prove that a is neither a strict local maximum nor a strict local minimum.[434] In Ch. 97 (below), we'll also learn to prove that a is not an inflexion point.

[434]Let $a \in \mathbb{R}$. We have $i(a) = 0$. There exists some "nearby" point c such that $i(c) = 0$, so that $i(c) \leq i(a) \leq i(c)$. So, a is neither a strict local minimum nor a strict local minimum. (This proof is slightly informal in that the term "nearby" is slightly imprecise.)

Example 1190. Let $f : (0, 2) \to \mathbb{R}$ be the function defined by

$$f(x) = x - \sin \frac{\pi x}{2}.$$

Q. Find any maximum or minimum points of f.

Compute the first derivative:

$$f'(x) = 1 - \frac{\pi}{2} \cos \frac{\pi x}{2}.$$

Find any stationary points $c \in (0, 2)$ of f:

$$f'(c) = 0 \qquad \iff \qquad 1 - \frac{\pi}{2} \cos \frac{\pi c}{2} = 0 \qquad \iff \qquad c = \frac{2}{\pi} \cos^{-1} \frac{2}{\pi} \approx 0.561.$$

So, the sole stationary point of f is $c = \frac{2}{\pi} \cos^{-1} \frac{2}{\pi} \approx 0.561$.

By Fermat's Theorem on Extrema, any extremum of f must also be a stationary point. Hence, c is the only possible extremum of f. Note though that at this point, we have not yet shown that c is actually an extremum of f.

To do so, we can use the Second Derivative Test for Extrema (SDTE). Compute the second derivative:

$$f''(x) = 1 + \left(\frac{\pi}{2}\right)^2 \sin \frac{\pi x}{2}.$$

Evaluate the second derivative at c:

$$f''(c) = 1 + \left(\frac{\pi}{2}\right)^2 \sin \frac{\pi c}{2} \approx 2.90 > 0.$$

A. Hence, by the SDTE, c is a strict local minimum of f.

96. Concavity

Definition 211 (*informal*). *Let $l < r$.*[435] *We say that a function is*[436]

(a) Concave *on* (l, r) *if its gradient is decreasing on* (l, r).

(b) Strictly concave *on* (l, r) *if its gradient is strictly decreasing on* (l, r).

(c) Convex *on* (l, r) *if its gradient is increasing on* (l, r).

(d) Strictly convex *on* (l, r) *if its gradient is strictly increasing on* (l, r).

Example 1191. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = -x^2$.

Figure to be
inserted here.

The gradient of f is decreasing on $\mathbb{R} = (-\infty, \infty)$. So, f is concave on $\mathbb{R} = (-\infty, \infty)$.

Indeed, the gradient of f is *strictly* decreasing on $\mathbb{R} = (-\infty, \infty)$. So, f is *strictly* concave on $\mathbb{R} = (-\infty, \infty)$.

Example 1192. Consider the exponential function exp.

Figure to be
inserted here.

The gradient of exp is increasing on $\mathbb{R} = (-\infty, \infty)$. So, exp is convex on $\mathbb{R} = (-\infty, \infty)$.

Indeed, the gradient of exp is *strictly* increasing on $\mathbb{R} = (-\infty, \infty)$. So, exp is *strictly* convex on $\mathbb{R} = (-\infty, \infty)$.

[435] These definitions are "unofficial" because of these two flaws: They **(i)** define concavity only on open intervals, even though concavity may be more generally defined on any interval with more than one number; and **(ii)** require that the function be differentiable on (l, r). Definition 329 (Appendices) fixes these flaws (and gives this textbook's official definitions of concavity).

[436] We allow for the possibilities that $l = -\infty$ and/or $r = \infty$.

Example 1193. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3$.

Figure to be inserted here.

The gradient of g is decreasing on $(-\infty, 0)$. So, g is concave on $(-\infty, 0)$.

Indeed, the gradient of g is *strictly* decreasing on $(-\infty, 0)$. So, g is *strictly* concave on $(-\infty, 0)$.

The gradient of g is increasing on $(0, \infty)$. So, g is convex on $(0, \infty)$.

Indeed, the gradient of g is *strictly* increasing on $(0, \infty)$. So, g is *strictly* convex on $(0, \infty)$.

Mnemonic: Con**cave** functions look like a **cave**, while e^x is conve^x.

Figure to be inserted here.

Remark 138. In this textbook, we'll use the terms *concave* and *convex.* But just so you know, some writers instead use *concave downwards* and *concave upwards* (respectively).

Also, in place of *concavity*, still other writers use the word *curvature*.

Example 1194. Define $h : [0, 2\pi]$ by $h(x) = \sin x$.

Figure to be
inserted here.

The gradient of h is decreasing on $(0, \pi)$. So, h is concave on $(0, \pi)$.

Indeed, the gradient of h is *strictly* decreasing on $(0, \pi)$. So, h is *strictly* concave on $(0, \pi)$.

The gradient of h is increasing on $(\pi, 2\pi)$. So, h is convex on $(\pi, 2\pi)$.

Indeed, the gradient of h is *strictly* increasing on $(\pi, 2\pi)$. So, h is *strictly* convex on $(\pi, 2\pi)$.

Example 1195. Define $i : \mathbb{R} \to \mathbb{R}$ by $i(x) = 1$.

Figure to be
inserted here.

The gradient of i is decreasing but not strictly increasing on $\mathbb{R} = (-\infty, \infty)$.

The gradient of i is increasing but not strictly decreasing on $\mathbb{R} = (-\infty, \infty)$.

So, i is both concave and convex on $\mathbb{R} = (-\infty, \infty)$, but neither strictly concave nor strictly convex on $\mathbb{R} = (-\infty, \infty)$.

Example 1196. XXX

Remark 139. Concavity is only ever defined on an interval.

This has an important implication: We shall never speak of a function being concave or convex *at a point*, because we have not defined what this means.

Exercise 367. For each function, explain whether there are any intervals on which the function is concave, convex, strictly concave, or strictly convex. (Answer on p. 954.)

(a) The cosine function $\cos : \mathbb{R} \to \mathbb{R}$

(b) $k : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ defined by $k(x) = \cos x$.

(c) $l : \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R}$ defined by $l(x) = \cos x$.

(d) $i : \mathbb{R} \to \mathbb{R}$ defined by $i(x) = -x^4$

(e) $j : \mathbb{R} \to \mathbb{R}$ defined by $j(x) = x^4$

A367(a)

Figure to be
inserted here.

(b)

(c)

(d) The gradient of i is decreasing and indeed strictly decreasing on $\mathbb{R}$. Hence, i is concave and also strictly concave on $\mathbb{R}$.

Figure to be
inserted here.

(e) The gradient of j is increasing and indeed strictly increasing on $\mathbb{R}$. Hence, j is convex and also strictly convex on $\mathbb{R}$.

96.1. The First Derivative Test for Concavity (FDTC)

Take the unofficial definitions of concavity (p. 951) and replace each instance of the word "gradient" with "f'". This gives us the **First Derivative Test for Concavity (FDTC)**:

Proposition 12. (First Derivative Test for Concavity, FDTC) *Let D be an interval with endpoints l and r. Suppose $f : D \to \mathbb{R}$ is differentiable. Then*

(a) *f' is decreasing on (l, r) $\iff$ f is concave on D.*

(b) *f' is strictly decreasing on (l, r) $\iff$ f is strictly concave on D.*

(c) *f' is increasing on (l, r) $\iff$ f is convex on D.*

(d) *f' is strictly increasing on (l, r) $\iff$ f is strictly convex on D.*

Proof. See p. 1693 (Appendices). □

We illustrate the FDTC using the same examples as before:

Example 1197. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = -x^2$.

The first derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = -2x$.

Figure to be
inserted here.

Since f' is decreasing on $\mathbb{R}$, by FDTC**(a)**, f is concave on $\mathbb{R}$.

Indeed, f' is *strictly* decreasing on $\mathbb{R}$. So, by FDTC**(b)**, f is *strictly* concave on $\mathbb{R}$.

Example 1198. Consider the exponential function exp.

Its first derivative is itself: $\exp' = \exp$.

Figure to be
inserted here.

Since $\exp' = \exp$ is increasing on $\mathbb{R}$, by FDTC**(c)**, exp is convex on $\mathbb{R}$.

Indeed, $\exp' = \exp$ is *strictly* increasing on $\mathbb{R}$. So, by FDTC**(d)**, exp is *strictly* convex on $\mathbb{R}$.

Example 1199. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3$.

The first derivative of g is the function $g'(x) : \mathbb{R} \to \mathbb{R}$ defined by $g'(x) = 3x^2$.

Figure to be
inserted here.

Since g' is decreasing on $(-\infty, 0)$, by FDTC**(a)**, g is concave on $(-\infty, 0)$.

Indeed, g' is *strictly* decreasing on $(-\infty, 0)$. So, by FDTC**(b)**, g is *strictly* concave on $(-\infty, 0)$.

Since g' is increasing on $(0, \infty)$, by FDTC**(c)**, g is convex on $(0, \infty)$.

Indeed, g' is *strictly* increasing on $(0, \infty)$. So, by FDTC**(d)**, g is *strictly* convex on $(0, \infty)$.

Example 1200. Consider the sine function sin.

Example 1201. XXX

Example 1202. XXX

Exercise 368. XXX (Answer on p. 956.)

A368.

96.2. The Second Derivative Test for Concavity (SDTC)

For convenience, we reproduce the First Derivative Test for Concavity (FDTC) (Ch. 96.1) and the Increasing/Decreasing Test (IDT) (Ch. 94):

Proposition 12. (First Derivative Test for Concavity, FDTC) *Let D be an interval with endpoints l and r. Suppose $f : D \to \mathbb{R}$ is differentiable. Then*

(a) f' *is decreasing on* (l,r) $\iff$ f *is concave on* D.
(b) f' *is strictly decreasing on* (l,r) $\iff$ f *is strictly concave on* D.
(c) f' *is increasing on* (l,r) $\iff$ f *is convex on* D.
(d) f' *is strictly increasing on* (l,r) $\iff$ f *is strictly convex on* D.

Fact 208. (Increasing/Decreasing Test, IDT) *Let f be a function and D be an interval. Suppose f is differentiable on D.*

(a) $f'(x) \geq 0$ for all $x \in D$ $\iff$ f is increasing on D.
(b) $f'(x) > 0$ for all $x \in D$ $\implies$ f is strictly increasing on D.
(c) $f'(x) \leq 0$ for all $x \in D$ $\iff$ f is decreasing on D.
(d) $f'(x) < 0$ for all $x \in D$ $\implies$ f is strictly decreasing on D.

By applying the IDT to the FDTC, we get the **Second Derivative Test for Concavity (SDTC)**:

Corollary 44. (Second Derivative Test for Concavity, SDTC) *Let D be an interval with endpoints l and r ($l < r$). Suppose $f : D \to \mathbb{R}$ is differentiable. Then*

(a) $f'' \leq 0$ *on* (l,r) $\iff$ f *is concave on* D.
(b) $f'' < 0$ *on* (l,r) $\implies$ f *is strictly concave on* D.
(c) $f'' \geq 0$ *on* (l,r) $\iff$ f *is convex on* D.
(d) $f'' > 0$ *on* (l,r) $\implies$ f *is strictly convex on* D.

We illustrate the SDTC using the same examples as before:

Example 1203. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=-x^2$.

The second derivative of f is the function $f'':\mathbb{R}\to\mathbb{R}$ defined by $f''(x)=-2$.

Figure to be
inserted here.

Since $f''\leq 0$ on $\mathbb{R}$, by SDTC(**a**), f is concave on $\mathbb{R}$.

Indeed, $f''<0$ on $\mathbb{R}$. So, by SDTC(**b**), f is also *strictly* concave on $\mathbb{R}$.

Example 1204. Consider the exponential function exp.

Its second derivative is itself: $\exp''=\exp$.

Figure to be
inserted here.

Since $\exp''=\exp\geq 0$ on $\mathbb{R}$, by SDTC(**c**), exp is convex on $\mathbb{R}$.

Indeed, $\exp''=\exp>0$ on $\mathbb{R}$. So, by SDTC(**d**), exp is also *strictly* convex on $\mathbb{R}$.

Example 1205. Define $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=x^3$.

The second derivative of g is the function $g''(x):\mathbb{R}\to\mathbb{R}$ defined by $g''(x)=6x$.

Figure to be
inserted here.

Since $g''\leq 0$ on $(-\infty,0)$, by SDTC(**a**), g is concave on $(-\infty,0)$.

Indeed, $g''<0$ on $(-\infty,0)$. So, by SDTC(**b**), g is *strictly* concave on $(-\infty,0)$.

Since $g''\geq 0$ on $(0,\infty)$, by SDTC(**c**), g is convex on $(0,\infty)$.

Indeed, $g''>0$ on $(0,\infty)$, so that by SDTC(**d**), g is *strictly* convex on $(0,\infty)$.

Example 1206. Consider the sine function sin.

Example 1207. XXX

Example 1208. XXX

By the way, note the one-way $\Longrightarrow$'s in the SDTC**(b)** and **(d)**.[437] That is, the converses of **(b)** and **(d)** are false:

Example 1209. Define $i : \mathbb{R} \to \mathbb{R}$ defined by $i(x) = -x^4$.

The first derivative of i is the function $i' : \mathbb{R} \to \mathbb{R}$ defined by $i'(x) = -4x^3$.

The second derivative of i is the function $i'' : \mathbb{R} \to \mathbb{R}$ defined by $i''(x) = -12x^2$.

Figure to be
inserted here.

We already showed in Exercise 367(d) that i is strictly concave on $\mathbb{R}$.

However, contrary to the converse of SDTC**(b)**, it is not true that $i'' < 0$ on $\mathbb{R}$. In particular, at 0, we have $i''(0) = -12 \cdot 0^2 = 0 \nless 0$.

This shows that the converse of SDTC**(b)** is false—the function i is strictly concave on $\mathbb{R}$, but $i'' \nless 0$ on $\mathbb{R}$.

[437]These one-way $\Longrightarrow$'s are simply inherited from the Increasing/Decreasing Test (IDT).

Example 1210. Define $j : \mathbb{R} \to \mathbb{R}$ defined by $j(x) = -x^4$.

The first derivative of j is the function $j' : \mathbb{R} \to \mathbb{R}$ defined by $j'(x) = 4x^3$.

The second derivative of j is the function $j'' : \mathbb{R} \to \mathbb{R}$ defined by $j''(x) = 12x^2$.

Figure to be
inserted here.

We already showed in Exercise 367(d) that j is strictly convex on $\mathbb{R}$.

However, contrary to the converse of SDTC**(b)**, it is not true that $j'' > 0$ on $\mathbb{R}$. In particular, at 0, we have $j''(0) = 12 \cdot 0^2 = 0 \not> 0$.

This shows that the converse of SDTC**(b)** is false—the function j is strictly convex on $\mathbb{R}$, but $j'' \not> 0$ on $\mathbb{R}$.

Exercise 369. XXX (Answer on p. 960.)

A369.

96.3. The Graphical Test for Concavity (GTC)

Example 1211. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = -x^2$.

Figure to be
inserted here.

Below, we'll formally state the **Graphical Test for Concavity (GTC)**. Here the GTC says that

(a) Since f is concave, if we pick any two points, say A and B, then f is on or above the line segment AB.

(b) Since f is *strictly* concave, f is (strictly) above the line segment AB (excluding the two endpoints A and B).

Example 1212. Consider the exponential function exp.

Figure to be
inserted here.

The GTC says that

(a) Since exp is convex, if we pick any two points, say A and B, then exp is on or below the line segment AB.

(b) Since exp is *strictly* convex, exp is (strictly) above the line segment AB (excluding the two endpoints A and B).

Fact 209. (Graphical Test for Concavity, GTC) *Let D be an interval and f be a function. Then*

(a) *f is concave on D* $\iff$ *f is on or above the line segment connecting any two points in D.*

(b) *f is strictly concave on D* $\iff$ *f is above the line segment connecting any two points in D (excluding the two endpoints).*

(c) *f is convex on D* $\iff$ *f is on or below the line segment connecting any two points in D.*

(d) *f is strictly convex on D* $\iff$ *f is below the line segment connecting any two points in D (excluding the two endpoints).*

Proof. See p. 1695 (Appendices). □

Example 1213. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3$.

Figure to be
inserted here.

The GTC says that

(a) Since g is concave on $(-\infty, 0)$, if we pick any two points, say A and B in $(-\infty, 0)$, then g is on or above the line segment AB.

(b) Since g is *strictly* concave on $(-\infty, 0)$, g is (strictly) above AB (other than the endpoints A and B).

(c) Since g is convex on $(0, \infty)$, if we pick any two points, say C and D in $(0, \infty)$, then g is on or below the line segment CD.

(d) Since g is *strictly* concave on $(0, \infty)$, g is (strictly) below CD (except the endpoints C and D).

Example 1214. Define $h : [0, 2\pi]$ by $h(x) = \sin x$.

Figure to be
inserted here.

The GTC says that

(a) Since h is concave on $(0, \pi)$, if we pick any two points, say A and B in $(0, \pi)$, then h is on or above the line segment AB.

(b) Since h is *strictly* concave on $(0, \pi)$, h is (strictly) above AB (other than the endpoints A and B).

(c) Since h is convex on $(\pi, 2\pi)$, if we pick any two points, say C and D in $(\pi, 2\pi)$, then h is on or below the line segment CD.

(d) Since h is *strictly* convex on $(\pi, 2\pi)$, h is (strictly) below on CD (other than the endpoints C and D).

Example 1215. Define $i : \mathbb{R} \to \mathbb{R}$ by $i(x) = 1$.

Figure to be
inserted here.

We showed earlier that i is both concave and convex on $\mathbb{R}$, but neither strictly concave nor strictly convex on $\mathbb{R}$.

The GTC says that

(a) Since i is concave on $\mathbb{R}$, if we pick any two points, say A and B in $\mathbb{R}$, then i is on or above the line segment AB.

(b) Since i is **not** *strictly* concave on $\mathbb{R}$, we can find two points C and D such that i is not (strictly) above the line segment CD.

(c) Since i is convex on $\mathbb{R}$, if we pick any two points, say E and F in $\mathbb{R}$, then i is on or below the line segment EF.

(d) Since i is **not** *strictly* convex on $\mathbb{R}$, we can find two points G and H such that i is not (strictly) below the line segment GH.

Exercise 370. XXX (Answer on p. 964.)

A370.

96.4. A Linear Function Is One That's Both Concave and Convex

Informally, a **linear function** is one whose graph is a straight line—or one whose gradient is *both* decreasing and increasing. Hence, a linear function is a function that's *both* concave and convex.

Example 1216. Graphed below is the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x$.

Figure to be
inserted here.

"Obviously", h is linear on $\mathbb{R}$.

Less obviously, h is also concave on $\mathbb{R}$. Because XXX

Similarly, h is also convex on $\mathbb{R}$. Because XXX

Exercise 371. XXX (Answer on p. 965.)

A371.

97. Inflexion Points

Informally, an **inflexion point** is a point at which a function's concavity (or curvature) changes. A little more precisely, an inflexion point is a point at which the function "changes from"

(a) Strictly concave to strictly convex; OR

(b) strictly convex to strictly concave.

Even more precisely (but still not quite formally),

Definition 212 (*informal*)**.** We call c an *inflexion point of a function* if the function satisfies either statement **(a)** or **(b)**:[438]

(a) Strictly concave on c's "immediate left" & strictly convex on c's "immediate right".

(b) Strictly convex on c's "immediate left" & strictly concave on c's "immediate right".

Example 1217. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.

Figure to be
inserted here.

At 0, f "changes from" strictly concave to strictly convex. Hence, 0 is an inflexion point of f.

Example 1218. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x^3$.

Figure to be
inserted here.

At 0, g "changes from" strictly convex to strictly concave. Hence, 0 is an inflexion point of g.

[438]The definition given here is informal in that the phrases "immediate left" and "immediate right" are imprecise. For a formal definition, see Definition 330 (Appendices).

Remark 140. It's usually spelt *inflection* rather than *inflexion*.[439] But *inflexion* is what appears on your syllabus and so that's how we'll spell it too.

Example 1219. XXX

Example 1220. XXX

Exercise 372. XXX (Answer on p. 967.)

A372.

[439] According to Google Ngram, even when we restrict attention to British English, *inflection* is the more common spelling. It seems that *inflexion* is an archaic spelling (similar to *connexion*).

Proposition 13 (informal)**. (First Derivative Test for Inflexion Points, FDTI)** *Let f be a function and c be a point. If c is an inflexion point of f, then c is a strict local extremum of f'.*[440]

Proof. Since c is an inflexion point, by definition, f is either **(a)** strictly concave on c's "immediate left" and strictly convex on c's "immediate right"; OR **(b)** strictly convex on c's "immediate right" and strictly concave on c's "immediate left".

Suppose **(a)**. Then by the First Derivative Test for Concavity (FDTC), f' is strictly decreasing on c's "immediate left" and strictly increasing on c's "immediate left". And so, by the First Derivative Test for Extrema (FDTE), c is a strict local minimum of f'.

Suppose **(b)**. Then we can similarly show that c is a strict local maximum of f'. □

Example 1221. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x-1)^3$.

- Its first derivative is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 3(x-1)^2$.
- It has an inflexion point at 1.

Figure to be inserted here.

As promised by the FDTI, the point 1 is a strict local minimum of f'.[441]

[440] For a formal statement of the FDTI, see Proposition 35 (Appendices).

[441] Indeed, 1 is a strict *global* minimum of f'—for all $x \neq 1$, $f'(x) > 0 = f'(1)$.

Example 1222. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -(x-1)^3$.

- Its first derivative is the function $g' : \mathbb{R} \to \mathbb{R}$ defined by $g'(x) = -3(x-1)^2$.
- It has an inflexion point at 1.

Figure to be
inserted here.

As promised by the FDTI, the point 1 is a strict local maximum of f'.[442]

Example 1223. XXX

Example 1224. XXX

Exercise 373. XXX (Answer on p. 969.)

A373.

Remark 141. For most functions you'll encounter in H2 Maths, you may safely assume that the converse of the FDTI is also true. That is, you may assume that if c is a strict local extremum of f', then c is an inflexion point of f.

However, you should be aware that this is **not** generally true. See Example 1585 (Appendices) for an example where c is a strict local extremum of f' but is **not** an inflexion point of f,

[442]Indeed, 1 is a strict *global* maximum of g'—for all $x \neq 1$, $g'(x) < 0 = g'(1)$.

97.2. The Second Derivative Test for Inflexion Points (SDTI)

Proposition 14 (informal)**. (Second Derivative Test for Inflexion Points, SDTI)** *Let f be a function and c be a point. Suppose f is twice-differentiable "around" c. If c is an inflexion point of f, then* $f''(c) = 0$.[443]

Proof. By the First Derivative Test for Inflexion Points, c is a strict local extremum of f'. So, by Fermat's Theorem on Extrema, $f''(c) = 0$. □

Example 1225. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x-1)^3$.

- Its first derivative is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 3(x-1)^2$.
- Its second derivative is the function $f'' : \mathbb{R} \to \mathbb{R}$ defined by $f''(x) = 6(x-1)$.
- It has an inflexion point at 1.

Figure to be
inserted here.

As promised by the SDTI, $f''(1) = 6 \cdot (1-1) = 0$.

Example 1226. XXX

Example 1227. XXX

Exercise 374. XXX (Answer on p. 970.)
Exercise 375. Write down the statement that is the converse of the SDTI—is this statement true? Explain the relationship between the SDTI and SDTE**(c)**. (Answer on p. 970.)

A374.

A375. The statement that is the converse of the SDTI is this:

"*If $f''(c) = 0$, then c is an inflexion point of f.*"

The above statement is false. As we learnt in SDTE**(c)**, if $f''(c) = 0$, then c could be a strict local maximum, a strict local minimum, an inflexion point, or "none of the above".

The SDTI is a (partial) converse to SDTE**(c)**.

[443] For a formal statement of the SDTI, see Proposition 36(Appendices).

97.3. The Tangent Line Test (TLT)

In Ch. 96 (Concavity), we had three tests to help us investigate concavity—the FDTC, the SDTC, and the GTC.

In the present chapter, we have, analogously, three tests to help us investigate inflexion points—the FDTI (Ch. 97.1), the SDTI (Ch. 97.2), and now the **Tangent Line Test (TLT)**:[444]

Proposition 15 (informal)**. (Tangent Line Test, TLT)** *Let f be a function and c be a point. If c is an inflexion point of f, then the tangent line of f at c satisfies either* ***(a)*** *or* ***(b)****:*

(a) Strictly above f on c's "immediate left" & strictly below f on c's "immediate right".

(b) Strictly below f on c's "immediate left" & strictly above f on c's "immediate right".

Example 1228. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ has the inflexion point 0. Let l be the tangent line of f at 0.

Figure to be inserted here.

We can easily verify that the inflexion point 0 passes the TLT**(a)**—l is

- Strictly above f on c's "immediate left"; and
- Strictly below f on c's "immediate right".

Example 1229. XXX

Example 1230. XXX

Example 1231. XXX

Exercise 376. XXX (Answer on p. 971.)

A376.

[444]For a formal statement of the TLT, see Proposition 37 (Appendices).

Remark 142. Just so you know, the converse of the TLT is **not** generally true. That is, it is possible that a point that passes the TLT but is **not** an inflexion point. For such an example, see ✿.

(Nonetheless, for H2 Maths, it is probably safe for you to assume that the converse of the TLT is true. That is, you may blithely assume that if a point passes the TLT, then it is an inflexion point.)

97.4. Non-Stationary Points of Inflexion (optional)

In H2 Maths, you'll only ever be asked about **stationary points of inflexion**. That is, every inflexion point you'll ever be asked about will also be a stationary point.

Nonetheless, you should be aware that **non-stationary points of inflexion** exist. That is, an inflexion point can be a non-stationary point. Indeed, such points occur in commonly encountered functions:

Example 1232. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 + x$.

The first derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 3x^2 + 1$.

Figure to be inserted here.

Consider the point 0.

Observe that 0 is not a stationary point because $f'(0) = 3 \cdot 0^2 + 1 = 1 \neq 0$.

However, at 0, f is strictly concave to the "immediate left" and strictly convex to the "immediate right". In other words, at 0, f changes from being strictly concave to strictly convex. Hence, by our 212 informal definition of inflexion points, 0 is an inflexion point of f.

Exercise 377. Verify that the inflexion point in the last example satisfies **(a)** the FDTI; **(b)** the SDTI; and **(c)** the TLT. (Answer on p. 973.)

A377.

Example 1233. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^3 - 2x^2 + x$.

XXX

98. A Summary of Chapters 94, 95, 96, and 97

In the last four chapters, we covered *nine* tests. Quick recap:

1. A function can be increasing or decreasing on an interval.

   The **IDT** can help us determine which is the case.

2. A stationary point can be an extremum, inflexion point, or neither of the above.

   The **FDTE** can help us determine whether a stationary point is a local maximum, local minimum, strict local maximum, or strict local minimum.

   The **SDTE** can help us determine whether a stationary point is a strict local maximum or strict local minimum.

3. A function can be concave or convex on an interval.

   The **FDTC**, **SDTC**, and **GTC** can help us determine which is the case.

4. An inflexion point is a point at which a function's concavity (or curvature) changes.

   The **FDTI**, **SDTI**, and **TLT** can help us determine whether a point is an inflexion point.

Exercise 378. For each function, identify any intervals on which the function is **(i)** increasing, decreasing, strictly increasing, or strictly decreasing; and **(ii)** concave, convex, strictly concave, or strictly convex. Then also identify **(iii)** any stationary points; **(iv)** the nature of each stationary point; and **(v)** any turning points. (Answer on p. 974.)

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = XXX$

(b) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = XXX$

(c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = XXX$

A378(a) XXX

(b) XXX

(c) XXX

99. More Techniques of Differentiation

Your H2 Maths syllabus (p. 8) includes

- "relating the graph of $y = f'(x)$ to the graph of $y = f(x)$";
- "finding equations of tangents and normals to curves, including cases where the curve is defined implicitly or parametrically"; and
- "connected rates of change problems".

So, these are the three topics we'll cover in this chapter.

99.1. Relating the Graph of f' to That of f

Example 1234. XXX

Example 1235. XXX

Example 1236. XXX

Exercise 379. XXX (Answer on p. 975.)

A379.

99.2. Equations of Tangents and Normals

We reproduce from Ch. 7 the following fact and definition:

Fact 28. *The line that contains the point* (x_1, y_1) *and has gradient* m *is*

$$y - y_1 = m(x - x_1).$$

Definition ??. Two lines are *perpendicular* if

(a) Their gradients are negative reciprocals of each other; or

(b) One line is vertical while the other is horizontal.

If two lines l and m are perpendicular, then we will also write $l \perp m$.

Example 1237. XXX

Figure to be inserted here.

We now also write down the formal definitions of tangent and normal lines:

Definition 213. Let f be a nice function that is differentiable at the point a.

The *tangent line* (or simply *tangent*) *of* f *at* a is the line that has gradient $f'(a)$ and the point $(a, f(a))$.

The *normal line* (or simply *normal*) *of* f *at* a is the line that has gradient $-1/f'(a)$ and the point $(a, f(a))$ (provided $f'(a) \neq 0$).

Corollary 45. *Suppose* f *is a nice function that is differentiable at the point* a*. Then the tangent of* f *at* a *is described by*

$$y = f(a) + f'(a)(x - a).$$

And if $f'(a) \neq 0$*, then the normal of* f *at* a *is described by*

$$y = f(a) - \frac{1}{f'(a)}(x - a).$$

Proof. Apply Fact 28 to Definition 213. □

Example 1238. The curve C has parametric equations[445]

$$x = t^5 + t, \quad y = t^6 - t, \qquad \text{for } t \in \mathbb{R}.$$

Let l be the normal line of C at $t = 0$.

Figure to be
inserted here.

Q. Other than at $t = 0$, at which point(s) do l and C intersect?

First, at $t = 0$, we have $(x, y) = (0, 0)$.

Next, by implicit differentiation,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=0} = \left.\left(\frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}\right)\right|_{t=0} = \left.\frac{6t^5 - 1}{5t^4 - 1}\right|_{t=0} = 1.$$

Hence, l has gradient -1 and l is described by $y = 0 - (-1)(x - 0)$—or more simply, $y = x$.

The point(s) where l and C intersect is/are given by this system of equations:

$$y \overset{1}{=} x, \qquad y \overset{2}{=} t^6 - t, \quad \text{and} \quad x \overset{3}{=} t^5 + t.$$

Plug $\overset{2}{=}$ and $\overset{3}{=}$ into $\overset{1}{=}$ to get $t^5 + t \overset{4}{=} t^6 - t$. Rearranging, $t\left(t^5 - t^4 - 2\right) = 0$. So, $t = 0$ or $t \approx 1.45$ (calculator).[446]

A. Other than at $t = 0$, l and C intersect at $t \approx 1.45$ or at $(x, y) \approx (7.88, 7.88)$.

Exercise 380. The curve C has parametric equations

$$x = t^5 + t, \quad y = t^4 - t, \qquad \text{for } t \in \mathbb{R}.$$

Let l_1 and l_2 be the tangent lines to C at $t = 0$ and $t = 1$, respectively. Find the point where l_1 and l_2 intersect. (Answer on p. **1915**.)

[445] Here we again appeal to the Implicit Function Theorem (see n. 414 on p. 897)—we assume it is possible to express y as a function of x (even though we don't know how exactly to).

[446] By the Fundamental Theorem of Algebra, the sixth-degree polynomial equation $\overset{4}{=}$ must have six roots. In this case, only two are real (and these are the two that we've found), while the other four are complex.

99.3. Connected Rates of Change Problems

Example 1239. XXX

Example 1240. XXX

Example 1241. Sand lands on the floor at a steady rate of $1\,\mathrm{m}^3\,\mathrm{s}^{-1}$. (The floor is initially spotless.)

Magically, the sand always forms a perfect cone with volume V, height h, and base radius r. Moreover, the height and base diameter of the cone are always equal.

Figure to be
inserted here.

Q. At the instant $t = 2\,\mathrm{s}$ after sand begins landing on the floor, what is the rate at which r is increasing?

You will doubtless[447] recall the formula for the volume of a cone: $V = \dfrac{1}{3}\pi r^2 h$.

Since the height and base diameter are always equal, $h = 2r$. Plug this in: $V \overset{1}{=} \dfrac{2}{3}\pi r^3$.

Differentiate $\overset{1}{=}$ with respect to t: $\qquad \dfrac{\mathrm{d}V}{\mathrm{d}t} \overset{2}{=} 2\pi r^2 \dfrac{\mathrm{d}r}{\mathrm{d}t}$.

(What Rule did we just use?)[448]

We are given that sand lands at a steady rate of $1\,\mathrm{m}^3\,\mathrm{s}^{-1}$—$\dfrac{\mathrm{d}V}{\mathrm{d}t} \overset{3}{=} 1$ (for all $t \geq 0$).

At $t = 2$, the volume of sand that has landed is $2\,\mathrm{m}^3$—$V\Big|_{t=2} \overset{4}{=} 2$.

Plug $\overset{4}{=}$ into $\overset{1}{=}$ and rearrange—at $t = 2$, the base radius is $r\Big|_{t=2} \overset{5}{=} \left(\dfrac{3}{\pi}\right)^{1/3}$.

Plug $\overset{3}{=}$ and $\overset{5}{=}$ into $\overset{2}{=}$:

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{t=2} = 1 = 2\pi r^2\Big|_{t=2} \left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{t=2} = 2\pi\left(\frac{3}{\pi}\right)^{2/3} \left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{t=2} = 2\pi^{1/3}\cdot 3^{2/3} \left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{t=2}.$$

Rearranging, $$\left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{t=2} \overset{6}{=} \frac{1}{2\pi^{1/3}\cdot 3^{2/3}} \approx 0.164.$$

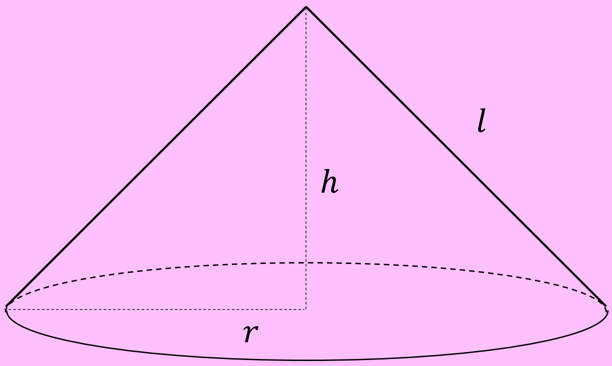
A. The base radius r is increasing at a rate of $0.164\,\mathrm{m}\,\mathrm{s}^{-1}$.

[447] Just kidding. You probably don't.

But unfortunately, you are supposed to remember *everything* you learnt in O-Level Maths. And one of those things is this formula! (See e.g. 2019 O-Level Maths syllabus, p. 8.) And even more unfortunately, this formula is **not** on List MF26. So yea, you may want to remember this formula (which we'll learn

Exercise 381. Continue with the last example. Let A be the base area of the cone. At $t = 2$, what is the rate at which A is increasing? (Answer on p. 1915.)

Exercise 382. A cone has lateral l, base radius r, and height h. (Answer on p. 1915.)



Let V be the cone's volume. Let S be the cone's external surface area *excluding the base*. From O-Level Maths, we know that[449]

$$V = \frac{1}{3}\pi r^2 h \qquad \text{and} \qquad S = \pi r l.$$

Suppose we want to construct a cone with volume $1\,\text{m}^3$ and whose external surface area (excluding the base) is minimised. What should this cone's height h be? (You can follow the steps below.)

(a) Express r in terms of h.

(b) Use Pythagoras' Theorem to express l in terms of r and h. Hence express l solely in terms of h.

(c) Now express S solely in terms of h.

(d) Show that the only stationary point of S (with respect to h) is at $h = \left(\frac{6}{\pi}\right)^{1/3}$.

(e) Explain why this stationary point is a strict global minimum and hence, conclude that $h = \left(\frac{6}{\pi}\right)^{1/3}$.

---

to derive in Ch. 109.6).

Note though that in N2016/II/1 (Exercise 685), they were nice enough to provide this cone volume formula. Maybe they'll be nice with you too. Or maybe not.

[448]Chain Rule.

[449]Again, both these two formulae are from O-Level Maths. Hence, you're supposed to know them.

100. More Fun With Your TI84

As part of your training as an obedient monkey, your H2 Maths syllabus (p. 8) includes:

- "locating maximum and minimum points using a graphing calculator"; and
- "finding the approximate value of a derivative at a given point using a graphing calculator".

So, these fascinating topics are what we'll cover in this chapter.

Example 1242. Let $f : (0,2) \to \mathbb{R}$ be the function defined by

$$f(x) = x - \sin\frac{\pi x}{2}.$$

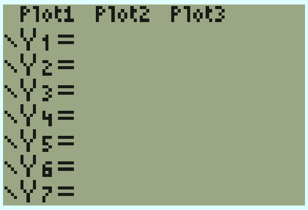
Q. Find any maximum or minimum points of f.

In Example 1190, we already solved this question without using our TI84.

But now as an exercise, let's also solve it using our TI84:

Step 1.

Step 2.



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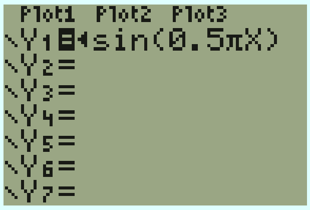
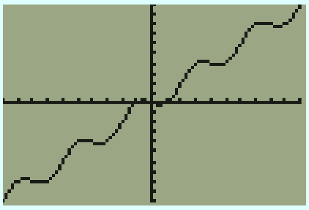
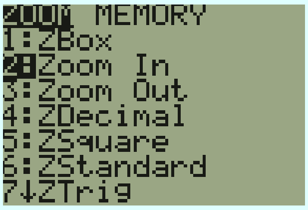


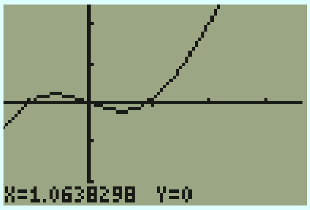
Step 4.



Step 5.



Step 6.



1. Press ON to turn on your calculator.
2. Press Y= to bring up the $Y =$ editor.
3. Press X,T,θ,n − SIN 0 . 5. To enter "π", press the blue 2ND button and then π (which corresponds to the ^ button). Now press X,T,θ,n) and altogether you will have entered "$x - \sin(0.5\pi x)$".
4. Now press GRAPH and the calculator will graph $y = x - \sin(0.5\pi x)$.

   Note that in our question, the domain of f is actually $(0,2)$, but we didn't bother telling the calculator this. So the calculator just went ahead and graphed the equation $y = x - \sin(0.5\pi x)$ for all possible real values of x and y.

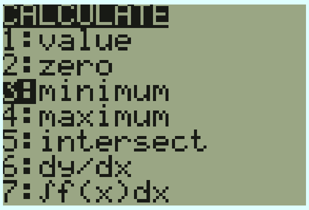
   No big deal, all we need to do is to zoom in to the region where $0 \le x \le 2$.
5. Press the ZOOM button to bring up a menu of ZOOM options.
6. Press 2 to select the Zoom In option. Using the ⟨ and ⟩ arrow keys, move the cursor to where $X = 1.0638298$, $Y = 0$. Now press ENTER and the TI will zoom in a little, centred on the point $X = 1.0638298$, $Y = 0$.

(Example continues on the next page ...)

(... *Example continued from the previous page.*)

It looks like starting at $x = 0$, the function is decreasing, hits a minimum point, then keeps increasing. Our goal now is to find this minimum point.

Step 7.



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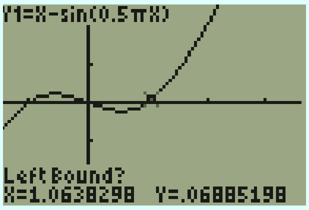
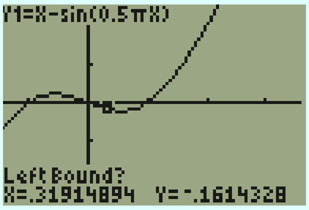


Step 9.



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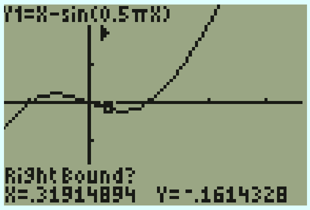

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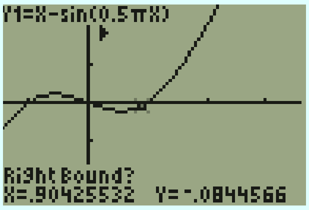
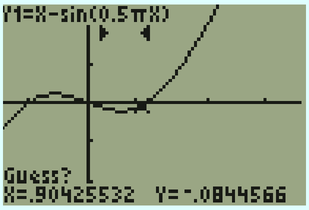
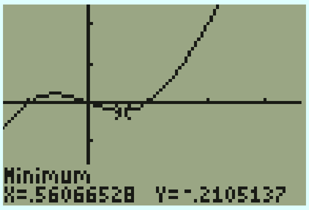


Step 12.



Step 13.



7. Press the blue 2ND button and then CALC (which corresponds to the TRACE button). This brings up the CALCULATE menu.
8. Press 3 to select the "minimum" option. This brings you back to the graph, with a cursor flashing. Also, the TI84 prompts you with the question: "Left Bound?"

   TI84's MINIMUM function works by you first choosing a "Left Bound" and a "Right Bound" for x. TI84 will then look for the minimum point **within your chosen bounds**.
9. Using the ⟨ and ⟩ arrow keys, move the blinking cursor until it is where you want your first "Left Bound" to be. For me, I have placed it a little to the left of where I believe the minimum point to be.
10. Press ENTER and you will have just entered your first "Left Bound".

    TI84 now prompts you with the question: "Right Bound?".
11. Now repeat: Using the ⟨ and ⟩ arrow keys, move the blinking cursor until it is where you want your first "Right Bound" to be. For me, I have placed it a little to the right of where I believe the minimum point to be.
12. Again press ENTER and you will have just entered your first "Right Bound".

    TI84 now asks you: "Guess?" This is just asking if you want to proceed and get TI84 to work out where the minimum point is. So go ahead and
13. Press ENTER. TI84 now informs you that there is a "Minimum" at "$X = .56066485$", "$Y = -.2105137$".

Exercise 383. XXX (Answer on p. 983.)

A383.

100.2. Find the Derivative at a Point Using Your TI84

This example will also illustrate how to graph parametric equations on the TI84.

Example 1243. The curve C has parametric equations

$$x = t^5 + t, \quad y = t^4 - t, \qquad \text{for } t \in \mathbb{R}.$$

Q. What is $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1}$?

We can solve this question easily without using our TI84:

A.
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1} = \left.\left(\frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}\right)\right|_{t=1} = \left.\frac{6t^5 - 1}{5t^4 - 1}\right|_{t=0} = \frac{5}{6}.$$

But as an exercise, let's try solving this question using our TI84:

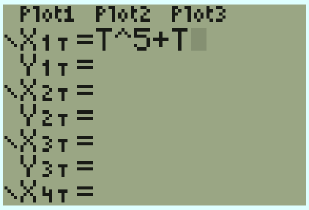
Step 1.

Step 2.

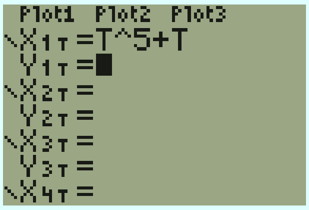
Step 3.

Step 4.

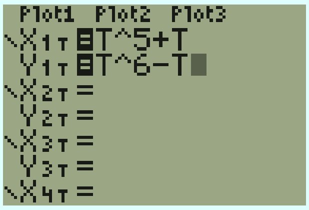
Step 5.



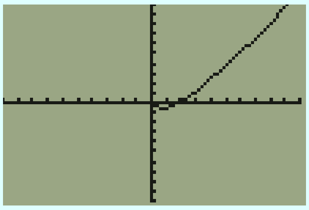
Step 6.



Step 7.



Step 8.



1. Press ON to turn on your calculator.
2. Press MODE to bring up a menu of settings that you can play with. In this example, all we want is to plot a curve based on parametric equations. So,
3. Using the arrow keys, move the blinking cursor to the word PAR (short for *parametric*) and press ENTER.
4. Now as usual, we'll input the equations of our curve. To do so, press Y= to bring up the "Y =" editor. Notice that this screen looks a little different from usual, because we are now under the parametric setting.
5. Press X,T,θ,n ^ 5 ⟩ + X,T,θ,n and altogether you will have entered "$T^5 + T$" in the first line.
6. Now press ENTER to go to the second line.
7. Press X,T,θ,n ^ 6 ⟩ − X,T,θ,n and altogether you will have entered "$T^6 - T$" in the second line.
8. Now press GRAPH and the calculator will graph the given pair of parametric equations.

   Notice that strangely enough, the graph seems to be empty for the region where $x < 0$. But clearly there are values for which $x < 0$—for example, if $t = -2$, then $(x, y) = (-34, 18)$. So why isn't the TI84 graphing this?

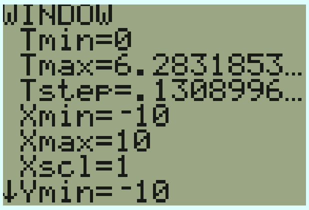
(*Example continues on the next page ...*)

(… *Example continued from the previous page.*)

The reason is that by default, the TI84 graphs only the region for where $0 \leq t \leq 2\pi$ (at least this is so for my particular calculator and operating system). We can easily adjust this:

9. Press the WINDOW button to bring up a menu of WINDOW options.
10. Using the arrow keys, the number pad, and the ENTER key as is appropriate, change Tmin and Tmax to your desired values (I use Tmin = −10 and Tmax = 10).
11. Then press GRAPH again and the calculator will graph the given pair of parametric equations, now for the region Tmin $\leq t \leq$ Tmax, where Tmin and Tmax are whatever you chose in the previous step.

Step 9.



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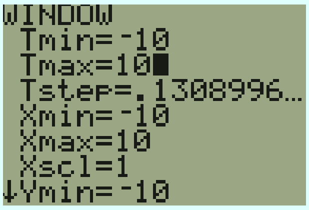
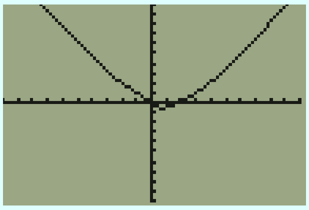


Step 11.



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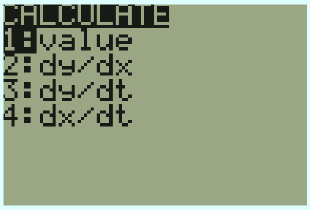

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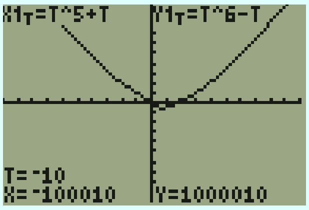

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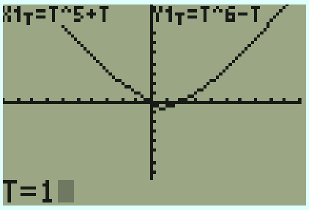
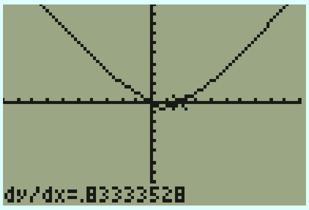


Step 15.



Actually, the last few steps were really not necessary, if all we wanted was to find $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1}$, as we do now:

12. Press the blue 2ND button and then CALC (which corresponds to the TRACE button). This brings up the CALCULATE menu, which once again looks a little different under the current parametric setting.
13. Press 2 to select the "dy/dx" option. This brings you back to the graph.

    Nothing seems to be happening. So,
14. Press 1 and now the bottom left of the screen changes to display "$T = 1$".
15. Hit ENTER. What you've just done is to ask the TI84 to compute $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1}$. It duly tells you that "dy/dx = .83333528".[450]

[450]Your TI84 will sometimes be a tiny bit off as it uses numerical (rather than exact) methods. Here for example, we know that the exact correct answer is $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1} = \frac{5}{6} = 0.8333\ldots$. So, the TI84's ".83333528" is a tiny bit off.

Exercise 384. XXX (Answer on p. 987.)

A384.

101. Power Series

101.1. A Power Series Is Simply an "Infinite Polynomial"

We reproduce from Ch. 5.8 our definition of **polynomials**:

Definition 33. Let $c_0, c_1, \ldots, c_n$ be constants, with $c_n \neq 0$. The expression

$$c_n x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \cdots + c_1 x + c_0$$

is called an *nth-degree polynomial (in one variable x)*. We also call

- Each $c_i x^i$ the *ith-degree term* (or more simply the *ith term*);
- Each c_i the *ith coefficient on x^i* (or the *ith-degree coefficient*, or the *ith coefficient*);
- The 0th coefficient c_0 the *constant term* (or, more simply, the *constant*).

A *(nth-degree) polynomial equation (in one variable x)* is any equation

$$c_n x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \cdots + c_1 x + c_0 = 0,$$

or any equation that can be rewritten in the above form.

Example 1244. The expression $7x - 3$ is a 1st-degree (or linear) polynomial.

The expression $3x^2 + 4x - 5$ is a 2nd-degree (or quadratic) polynomial.

The expression $-5x^3 + 2x + 9$ is a 3rd-degree (or cubic) polynomial.

The expression $18 + 5x - x^2 + x^4$ is a 4th-degree (or quartic) polynomial.

You can easily imagine what an "infinite-degree polynomial" is. Except we don't call it that. Instead, we call it a **power series**:

Example 1245. The following expression is a power series:

$$\sum_{n=0}^{\infty}(1+n)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \ldots$$

The 0th coefficient is 1. The constant term, constant, or 0th term is 1.

The 1st coefficient is 2. The 1st term is $2x$.

The 2nd coefficient is $3x^2$. The 2nd term is $3x^2$.

For each $n \in \mathbb{Z}_0^+$, the nth coefficient is $1+n$; and the nth term is $(1+n)x^n$.

Note that a power series is, by definition, an infinite series.[451]

By the way, we'll often find it convenient to give our power series a name—that is, denote it by a symbol.

Here for example, we might call the above power series P. This is convenient, because then we can write such things as

$$P(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \ldots$$

$$P\left(\frac{1}{4}\right) = 1 + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{4^2} + 4\cdot\frac{1}{4^3} + 5\cdot\frac{1}{4^4} + 6\cdot\frac{1}{4^5} + \ldots$$

Example 1246. The following expression is a power series:

$$\sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$$

The 0th coefficient is 1. The constant term, constant, or 0th term is 1.

The 1st coefficient is -1. The 1st term is $-x$.

The 2nd coefficient is 1. The 2nd term is x^2.

For each $n \in \mathbb{Z}_0^+$, the nth coefficient is $(-1)^n$; and the nth term is $(-1)^n x^n$.

If we call this power series Q, then we can write such things as

$$Q(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$$

$$Q\left(\frac{1}{3}\right) = 1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \frac{1}{3^5} + \ldots$$

Formal definitions:

[451] If you don't remember what an infinite series is, you should review Ch. 47.1.

Definition 214. Let $c_0, c_1, c_2, \ldots$ be constants. We call[452] the following expression a *power series*:[453]

$$\sum_{i=n}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

We also call

- Each $c_n x^n$ the *nth-degree term* (or more simply the *nth term*);
- Each c_n the *coefficient on* x^n (or the *nth-degree coefficient*, or the *nth coefficient*);
- c_0 the *constant term* (or, more simply, the *constant*).

Remark 143. Definition 214 mostly parallels Definition 33. The one exception is that we do **not** call the following a **power equation**:

$$1 + 2x + 3x^2 + 4x^3 + \cdots = 0.$$

The term *power equation* is rarely used in mathematics. And when it is used, it's usually for rather different purposes (example). So, in this textbook, we will never use the term *power equation.*

Exercise 385. XXX (Answer on p. 990.)

A385.

[452]To be clear, this is the definition for a power series **(a)** in one variable; and **(b)** centred on 0.

(a) Example of a power series in two variables: $2 + 3x + 4y + 5xy + 6x^2y + 7xy^2 + \ldots$.

(b) Example of a power series centred on 3: $1 + 2(x-3) + 3(x-3)^2 + 4(x-3)^3 + \ldots$. (Of course, a power series centred on 3 can always be rewritten as a power series centred on 0.)

[453]As usual, it is customary to use the symbol x to denote our variable. But as usual, this is merely a dummy variable that can be replaced by any other symbol, such as y, z, ☺, or $\star$.

101.2. A Power Series Can Converge or Diverge

In Part II (Sequences and Series), we briefly and informally discussed the concept of **convergence**.

Two quick examples to jog your memory and illustrate convergence:

Example 1247. The series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\dots$ converges to 1. Or equivalently,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\dots \stackrel{\star}{=} 1.$$

By the way, the expression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\dots$ is an infinite series, but is not a power series. This is because a power series involves a variable.[454]

Remark 144. As repeatedly emphasised in Part II (Sequences and Series), we must be *very careful* when dealing with infinite series.

In particular,

the equals sign $\stackrel{\star}{=}$ above is **not** the usual equals sign.

Instead, it means **converges to**, which has a precise and technical meaning.[455]

Fortunately, for H2 Maths, you are not required to know anything about this "precise and technical meaning". You need merely intuitively understand that $\stackrel{\star}{=}$ means that the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\dots$ "gets ever closer to" 1. And more generally, whenever we say that "a series converges to some number L", we simply mean that "the series gets ever closer to L".

This is just so you know. For H2 Maths, there will be little danger in simply and blithely assuming that $\stackrel{\star}{=}$ is the same as the usual equals sign (even if this is, strictly speaking, incorrect).

Example 1248. The expression $1+x+x^2+x^3+x^4+x^5+\dots$ is a power series.

As we learnt in Ch. 50.2 (Infinite Geometric Sequences and Series), for any $x \in (-1,1)$, this power series converges to $1/(1-x)$. That is,

$$1+x+x^2+x^3+x^4+x^5+\dots \stackrel{\star}{=} \frac{1}{1-x}.$$

(The above Remark also applies here.)

And so for example, for $x = 1/2$, we have

$$1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\dots = \frac{1}{1-1/2} = 2.$$

[453] And this variable is usually and customarily denoted by the symbol x (but could be denoted by any other symbol).

[454] See Ch. 143 (Appendices).

Three quick examples to illustrate **divergence**:

Example 1249. Informally, the series $1 + 1 + 1 + 1 + \dots$ "blows up to ∞".
So, it does not converge. Equivalently, it diverges.

Example 1250. Informally, the series $1 + 2 + 4 + 8 + 16 + \dots$ "blows up to ∞".
So, it does not converge. Equivalently, it diverges.

Example 1251. Informally, the series $-1 - 2 - 3 - 4 - 5 - \dots$ "blows up to $-\infty$".
So, it does not converge. Equivalently, it diverges.

Exercise 386. XXX (Answer on p. 992.)

A386.

101.3. A Power Series and Its Interval of Convergence

Definition 215. Given a power series, the set of values on which it converges is called its *interval of convergence*.

Example 1252. Consider the power series $1 + x + x^2 + x^3 + \ldots$.

We know that for every $x \in (-1, 1)$, this power series converges (to a real number).[456]

Conversely, for every $x \notin (-1, 1)$ or $x \in (-\infty, -1] \cup [1, \infty)$, it diverges (i.e. fails to converge to a real number).

Hence, this power series has **interval of convergence** $(-1, 1)$.

Example 1253. Consider the power series $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots$.

It can be shown that for every $x \in (-1, 1]$, this power series converges (to a real number).[457]

Conversely, for every $x \notin (-1, 1]$ or $x \in (-\infty, -1] \cup (1, \infty)$, it diverges (i.e. fails to converge to a real number).

Hence, this power series has interval of convergence $(-1, 1]$.

Note. In H2 Maths, we will not learn *why* this power series converges on $(-1, 1]$ but diverges elsewhere. Instead, this is simply something we mindless monkeys are to "know" without any actual understanding.

Example 1254. Consider the power series $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$.

It can be shown that for every $x \in \mathbb{R}$, this power series converges (to a real number).[458]

Hence, this power series has interval of convergence $\mathbb{R}$.

(**Note** from previous example applies here.)

Example 1255. Consider the power series $1 + x + 2!x^2 + 3!x^3 + \ldots$.

"Clearly", at $x = 0$, we have $1 \cdot 0 + 2! \cdot 0^2 + 3! \cdot 0^3 + \ldots$, which converges to 1.

It can be shown that, conversely, for every $x \neq 0$, $1 + x + 2!x^2 + 3!x^3 + \ldots$ diverges (i.e. fails to converge to a real number).

Hence, this power series has interval of convergence $\{0\}$. That is, it converges at 0 and no other point.

(**Note** from previous example applies here.)

The following result is beyond the scope of H2 Maths but is simple and good to know. It says that the set of values on which a power series converges is *always* an interval (which is why we call it the **interval of convergence**, rather than say the *set* of convergence). Moreover, this interval always has midpoint 0:

[456] Indeed, it converges to $1/(1-x)$.

[457] It converges to $\ln(1+x)$, as we'll learn in the next chapter.

[458] It converges to $\exp x$, as we'll learn in the next chapter.

Fact 210. *The set of values on which a power series*[459] *converges is* $(-R, R)$, $[-R, R]$, $(R, R]$, *or* $[R, R)$, *for some* $R \geq 0$ *(we also allow for the possibility that* $R = \infty$*).*

Proof. Omitted.[460] □

Remark 145. We call R in Fact 210 the *radius of convergence* (of that power series).

We revisit our last four examples of power series. We can easily verify each obeys Fact 210:

Example 1256. $1 + x + x^2 + x^3 + \ldots$ has interval of convergence $(-1, 1)$ or $(-R, R)$ with $R = 1$.

Example 1257. $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ has interval of convergence $(-1, 1]$ or $(-R, R]$ with $R = 1$.

Example 1258. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ has interval of convergence $\mathbb{R}$ or $(-R, R)$ with $R = \infty$.

Example 1259. $1 + x + 2!x^2 + 3!x^3 + \ldots$ has interval of convergence $\{0\}$ or $[-R, R]$ with $R = 0$.

Exercise 387. XXX (Answer on p. 994.)

A387.

[459] A power series centred on 0—see footnote to Definition 214.

[460] See e.g. Tao (*Analysis II*, 2016, Theorem 4.1.6, pp. 76–77).

101.4. (Some) Functions Can be Represented by a Power Series

Example 1260. Let $f : (-1, 1) \to \mathbb{R}$ be the function defined by $f(x) \stackrel{1}{=} \frac{1}{1-x}$.

Separately, let P be the power series defined by $P(x) = 1 + x + x^2 + x^3 + \ldots$.

Observe that for $x = 0$, we have

$$\underbrace{1 + 0 + 0^2 + 0^3 + \ldots}_{P(0)} = \underbrace{\frac{1}{1-0}}_{f(0)}.$$

So, we say that f can be **represented by** P at 0 (or that P is a **power series representation** of f at 0).

Observe that $$\underbrace{1 + x + x^2 + x^3 + \ldots}_{P(x)} = \underbrace{\frac{1}{1-x}}_{f(x)} \text{ for every } x \in (-1, 1) = \text{Domain } f.$$

That is, f can be **represented by** P at every $x \in \text{Domain } f$ (or P is a **power series representation** of f at every $x \in \text{Domain } f$).

So, we say that f can be **represented by** P (or that P is a **power series representation** of f).

Definition 216. Let P be a power series and f be a function.

If $P(a) = f(a)$ (for some $a \in \text{Domain } f$), then we say that f *can be represented by* P *at* a (or that P is *a power series representation of* f *at* a).

If f can be represented by P at every $x \in \text{Domain } f$, then we say that f *can be represented by* P (or that P is *a power series representation of* f).

Informally, given a function that can be represented by a power series, we can think of it as an "infinite polynomial function".

Example 1261. XXX

Example 1262. XXX

Exercise 388. XXX (Answer on p. 995.)

A388.

101.5. Term-by-Term Differentiation of a Power Series

By the Sum Rule for Differentiation, we can differentiate a polynomial function "term-by-term" to get its derivative:

Example 1263. Define the function $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = x^2 + x + 1.$$

By the Sum Rule (for Differentiation), we can simply differentiate the expression x^2+x+1 "term-by-term" to find that the derivative of g is the function $g': \mathbb{R} \to \mathbb{R}$ defined by

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}x^2 + \frac{\mathrm{d}}{\mathrm{d}x}x + \frac{\mathrm{d}}{\mathrm{d}x}1 = 2x + 1.$$

Example 1264. Define the function $h: \mathbb{R} \to \mathbb{R}$ by

$$h(x) = 17x^5 + 14x^4 + 11x^3 + 8x^2 + 5x + 2.$$

By the Sum Rule, we can simply differentiate the expression $17x^5+14x^4+11x^3+8x^2+5x+2$ "term-by-term" to find that the derivative of h is the function $h': \mathbb{R} \to \mathbb{R}$ defined by

$$h'(x) = \frac{\mathrm{d}}{\mathrm{d}x}17x^5 + \frac{\mathrm{d}}{\mathrm{d}x}14x^4 + \frac{\mathrm{d}}{\mathrm{d}x}11x^3 + \frac{\mathrm{d}}{\mathrm{d}x}8x^2 + \frac{\mathrm{d}}{\mathrm{d}x}5x + \frac{\mathrm{d}}{\mathrm{d}x}2 = 85x^4 + 56x^3 + 33x^2 + 16x + 5.$$

It turns out that happily, we can also differentiate "infinite polynomial functions"—or **power series**—term-by-term:

Example 1265. Define the function $f : (-1, 1) \to \mathbb{R}$ by

$$f(x) = \frac{1}{1-x}.$$

By proceeding as usual, we find that its derivative is the function $f' : (-1, 1) \to \mathbb{R}$ defined by

$$f'(x) \stackrel{1}{=} \frac{1}{(1-x)^2}.$$

We now introduce another method for finding the derivative of f.

We know that f can be represented by the power series $1 + x + x^2 + x^3 + \ldots$. That is,

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + . \text{ for all } x \in (-1, 1) = \text{Domain } f.$$

It turns out that by Theorem 37 (below), we can differentiate this power series term by term to find that f's derivative is the function $f' : (-1, 1) \to \mathbb{R}$ defined by

$$f'(x) \stackrel{2}{=} 1 + 2x + 3x^2 + \ldots.$$

By the way, as a bonus, together, $\stackrel{1}{=}$ and $\stackrel{2}{=}$ show that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots, \qquad \text{for all } x \in (-1, 1).$$

Example 1266. XXX

Example 1267. XXX

Here's the formal result that justifies term-by-term differentiation:

Theorem 37. (Term-by-Term Differentiation) *Let $R > 0$ and $c_0, c_1, c_2, \dots \in \mathbb{R}$. Suppose the function $f : (-R, R) \to \mathbb{R}$ is defined by*

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \cdots = \sum_{n=0}^{\infty} c_n x^n.$$

Then f is differentiable and its derivative is the function $f' : (-R, R) \to \mathbb{R}$ defined by

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

Proof. Omitted.[461] □

[461] See e.g. Abbott (2015, Theorem 6.5.7).

Given Theorem 37, we can easily prove that every function that can be represented by a power series is smooth (or infinitely differentiable):

Corollary 46. *Let $R > 0$ and $f : (-R, R) \to \mathbb{R}$ be a function. If f can be represented by a power series, then f is smooth.*

Proof. Let n be a positive integer.

By Theorem 37, f is differentiable and its derivative $f' : (-R, R) \to \mathbb{R}$ can be represented by a power series.

By Theorem 37, f' is differentiable and its derivative $f'' : (-R, R) \to \mathbb{R}$ can be represented by a power series.

$$\vdots$$

By Theorem 37, $f^{(n-1)}$ is differentiable.

We've just shown that for any positive integer n, f is n-times differentiable.[462] Hence, by Definition 209, f is smooth. □

Exercise 389. XXX (Answer on p. 998.)

A389.

[462] Here we use the **Principle of Mathematical Induction** (see e.g. Wikipedia), a simple and delightful topic that was sadly taken out from the H2 Maths syllabus in 2017.

101.6. Chapter Summary

1. A **power series** is simply an (infinite) series and an "infinite polynomial".

Example 1268. The following expression is a power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

2. Like any series, a power series can **converge** or **diverge**.

Example 1269. The power series $1 + x + x^2 + x^3 + \dots$ converges at $x = 0$, but diverges at $x = 2$.

3. Every power series has an **interval of convergence** centred on 0.

Example 1270. The power series $1 + x + x^2 + x^3 + \dots$ has interval of convergence $(-1, 1)$.

4. Some functions can be **represented** by a power series.

Example 1271. Let $f : (-1, 1) \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{1}{1-x}.$$

For all $x \in (-1, 1) = \text{Domain}\, f$, we have

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots.$$

So, we say that f can be represented by the power series $1 + x + x^2 + x^3 + \dots$.

5. **Term-by-term differentiation**: We've previously been differentiating polynomial functions term by term. It turns out that if a function can be represented by a power series, then we can also do the same.

Example 1272. Continuing with the last example, f's derivative is the function $f' : (-1, 1) \to \mathbb{R}$ defined by

$$f'(x) \stackrel{?}{=} 1 + 2x + 3x^2 + \dots.$$

102. The Maclaurin Series

We'll begin by learning to mechanically compute a special sort of power series that we call the **Maclaurin series (expansion)** (of a function). We'll initially provide no motivation for what the Maclaurin series is, where it comes from, and why it matters. We'll treat you like a mindless monkey. (It is only in the next subchapter that we'll learn why the Maclaurin series is important.)

Example 1273. Consider the function $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{1-x}$.

To compute the **Maclaurin series (expansion)** of f, we use these four steps:

1. Compute *all* of f's derivatives:

$$f'(x) = \frac{1}{(1-x)^2}, \qquad f''(x) = \frac{2}{(1-x)^3}, \qquad f'''(x) = \frac{3\cdot 2}{(1-x)^4},$$

$$f^{(4)}(x) = \frac{4\cdot 3\cdot 2}{(1-x)^5}, \qquad \ldots \qquad f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}.$$

2. Evaluate each of f, f', f'', etc. at 0:

$$f(0) = \frac{1}{1-0} = 1 = 0!, \qquad f'(0) = \frac{1}{(1-0)^2} = 1 = 1!, \qquad f''(0) = \frac{2}{(1-0)^3} = 2 = 2!,$$

$$f^{(3)}(0) = \frac{3\cdot 2}{(1-0)^4} = 3!, \qquad \ldots \qquad f^{(n)}(0) = \frac{n!}{(1-0)^{n+1}} = n!.$$

(*Example continues on the next page ...*)

Here we interrupt the example to introduce a definition that will prove convenient:

Definition 217. Let f be a function. We define the *zeroth derivative of* f as simply the function f itself and write

$$f^{(0)} = f.$$

(*... Example continued from the previous page.*)

3. For each $n \in \mathbb{Z}_0^+$, the n**th Maclaurin coefficient of** f is denoted m_n and is defined by

$$m_n = \frac{f^{(n)}(0)}{n!}.$$

   And so, the 0th, 1st, 2nd, 3rd, and nth Maclaurin coefficients of f are

$$m_0 = \frac{f(0)}{0!} = \frac{0!}{0!} = 1, \qquad m_1 = \frac{f'(0)}{1!} = \frac{1!}{1!} = 1, \qquad m_2 = \frac{f''(0)}{2!} = \frac{2!}{2!} = 1,$$

$$m_3 = \frac{f'''(0)}{3!} = \frac{3!}{3!} = 1, \qquad \dots \qquad m_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1.$$

   Here it so happens that every Maclaurin coefficient for f is simply 1. As we'll see, this will not generally be the case.

4. Given the function f, we now define its **Maclaurin series (expansion)** M to be the power series whose coefficients are simply the above Maclaurin coefficients:

$$M(x) = \sum_{n=0}^{\infty} m_n x^n = m_0 + m_1 x + m_2 x^2 \cdots = 1 + 1x + 1x^2 + \cdots = 1 + x + x^2 + \dots$$

In the above example, we were given a function f. We then used f to write down a power series that we denoted M and called the Maclaurin series of f. At no point did we claim that there was any relationship whatsoever between f and its Maclaurin series M. That such a relationship does exist is something we'll learn only in the next subchapter.

Remark 146. This textbook shall treat the terms **Maclaurin series** and **Maclaurin series expansion** as synonyms. Indeed, we'll often simply drop the word **expansion**.

Example 1274. Consider the sine function sin.

To compute the Maclaurin series for sin, we use the same four steps as before:

1. Compute *all* of sin's derivatives.[463]

   Unsure of how to proceed, we try writing down the first few derivatives:

$$\sin' = \cos, \qquad \sin'' = -\sin, \qquad \sin^{(3)} = -\cos, \qquad \sin^{(4)} = \sin, \qquad \sin^{(5)} = \cos.$$

   We observe a cycle after every four derivatives. And so, we have

$$\sin^{(n)} = \begin{cases} \sin, & \text{for } n = 0, 4, 8, \ldots, \\ \cos, & \text{for } n = 1, 5, 9, \ldots, \\ -\sin, & \text{for } n = 2, 6, 10, \ldots, \\ -\cos, & \text{for } n = 3, 7, 11, \ldots. \end{cases}$$

2. Evaluate each of sin, sin′, sin″, etc. at 0:

$$\sin^{(n)} 0 = \begin{cases} 0, & \text{for } n = 0, 4, 8, \ldots, \\ 1, & \text{for } n = 1, 5, 9, \ldots, \\ 0, & \text{for } n = 2, 6, 10, \ldots, \\ -1, & \text{for } n = 3, 7, 11, \ldots. \end{cases}$$

3. For each $n \in \mathbb{Z}_0^+$, the n**th Maclaurin coefficient** of sin is

$$m_n = \frac{\sin^{(n)}(0)}{n!} = \begin{cases} 0/n! = 0, & \text{for } n = 0, 4, 8, \ldots, \\ 1/n!, & \text{for } n = 1, 5, 9, \ldots, \\ 0/n! = 0, & \text{for } n = 2, 6, 10, \ldots, \\ -1/n!, & \text{for } n = 3, 7, 11, \ldots. \end{cases}$$

4. Thus, M, the **Maclaurin series** of sin, is defined by

$$M(x) = \sum_{n=0}^{\infty} m_n x^n = 0 + \frac{1}{1!}x + 0x^2 - \frac{1}{3!}x^3 + 0x^4 - \frac{1}{5!}x^5 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Formal definitions:

[463] Actually, we already did this in Example 1162.

Definition 218. Let $n \in \mathbb{Z}_0^+$. Suppose the function f is n-times differentiable at 0. Then the *nth Maclaurin coefficient* of f is denoted m_n and is the real number defined by

$$m_n = \frac{f^{(n)}(0)}{n!}.$$

If f is also smooth (infinitely differentiable) at 0, then the *Maclaurin series (expansion)* of f is the power series denoted M and defined by

$$M(x) \stackrel{1}{=} \sum_{n=0}^{\infty} m_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

Remark 147. List MF26 (p. 2) contains a version of $\stackrel{1}{=}$, so no need to *mug*.

For ease of reference, let's write down the **Four-Step Maclaurin Recipe** that we used in the above examples:

Four-Step Maclaurin Recipe (for Finding Maclaurin Series)

Let f be a function that is smooth at 0.

1. Find each of f's derivatives.
2. Evaluate each of f, f', f'', f''', etc. at 0.
3. For each $n = 0, 1, 2, \ldots$, the nth Maclaurin coefficient of f is

$$m_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}.$$

4. The Maclaurin series of f is

$$M(x) = \sum_{n=0}^{\infty} m_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

To repeat, so far, all we've learnt to do is this:

Take a function (that is smooth at 0) and compute its Maclaurin series.

We have not yet shown or even asserted that there is any relationship whatsoever between f and its Maclaurin series M. This will be done only in the next subchapter.

More examples:

Example 1275. Consider the exponential function exp.

1. We know that each derivative of exp is simply exp itself. That is, for every $n \in \mathbb{Z}_0^+$, we have $\exp^{(n)} x = \exp x$.
2. For every $n \in \mathbb{Z}_0^+$, we have $\exp^{(n)} 0 = \exp 0 = 1$.
3. So, for every $n \in \mathbb{Z}_0^+$, the **nth Maclaurin coefficient of** exp is

$$m_n = \frac{\exp^{(n)}(0)}{n!} = \frac{1}{n!}.$$

4. Thus, M, the **Maclaurin series of** exp, is defined by

$$M(x) = \sum_{n=0}^{\infty} m_n x^n = \sum_{n=0}^{\infty} \frac{\exp^{(n)}(0)}{n!} x^n = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Example 1276. XXX

Example 1277. XXX

Example 1278. XXX

Exercise 390. Find the Maclaurin series for each function. (Answer on p. 1919.)

(a) $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = (1 + x)^n$, where n is any real number.

(b) cos

(c) $g : (-1, 1] \to \mathbb{R}$ defined by $g(x) = \ln(1 + x)$.

Remark 148. Happily, your H2 Maths syllabus (p. 9) explicitly states, "Exclude derivation of the general term of the series". I take this to mean that they'll *never* ask you to derive the general nth term of a Maclaurin series.

Instead, what your A-Level exams usually do is to just ask for the "first few terms". And so, that's what we'll also usually do in this textbook.

(So far, they've kept their promise of not asking for the general nth term of a Maclaurin series. But of course, it is always possible that they'll break this promise and use this as their "creative" curveball question.)

Remark 149. Your H2 Maths syllabus and exams make no mention of the **Taylor series**. But just so you know, the Taylor series is simply a more general version of the Maclaurin series—namely, the Maclaurin series for a function f is simply the Taylor series for f **centred at** 0.

102.1. (Some) Functions Can Be Rep'd by Their Maclaurin Series

Given a function that's smooth at zero, we learnt to compute its Maclaurin series. We did not claim that there was any relationship whatsoever between a function and its Maclaurin series. We now learn that there *is* such a relationship (and this is why the Maclaurin series is important):

Theorem 38. *Let $R > 0$. Suppose the function $f : (-R, R) \to \mathbb{R}$ can be represented by a power series. Then f can also be represented by its Maclaurin series.*

Proof. Let $f : D \to \mathbb{R}$be our function and c_0, c_1, c_2, ... be constants such that

$$f(x) \stackrel{1}{=} c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots, \qquad \text{for every } x \in D.$$

By the Term-by-Term Differentiation Theorem, f is smooth and

$$\begin{aligned}
f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 \dots, \\
f''(x) &= (2 \cdot 1) c_2 + (3 \cdot 2) c_3 x + (4 \cdot 3) c_4 x^2 \dots, \\
f'''(x) &= (3 \cdot 2 \cdot 1) c_3 + (4 \cdot 3 \cdot 2) c_4 x + (5 \cdot 4 \cdot 3) c_5 x^2 + \dots, \\
f^{(4)}(x) &= (4 \cdot 3 \cdot 2 \cdot 1) c_4 + (5 \cdot 4 \cdot 3 \cdot 2) c_5 x + (6 \cdot 5 \cdot 4 \cdot 3) c_6 x^2 + \dots, \\
&\vdots \\
f^{(n)}(x) &= n! c_n + \frac{(n+1)!}{1!} c_{n+1} x + \frac{(n+2)!}{2!} c_{n+2} x^2 + + \frac{(n+3)!}{3!} c_{n+3} x^3 + \dots
\end{aligned}$$

Now evaluate each of f, f', f'', ..., and $f^{(n)}$ at the point 0:

$$\begin{aligned}
f(0) = f^{(0)}(0) &= c_0, & f^{(4)}(0) &= 4! c_4, \\
f'(0) &= c_1, & f^{(5)}(0) &= 5! c_5, \\
f''(0) &= 2c_2, & &\vdots \\
f^{(3)}(0) &= 3! c_3, & f^{(n)}(0) &= n! c_n.
\end{aligned}$$

Rearranging, we find that $c_n = \dfrac{f^{(n)}(0)}{n!}$ for each $n \in \mathbb{Z}_0^+$.

Hence, in $\stackrel{1}{=}$, each c_n is simply f's nth Maclaurin coefficient. Thus, f can be represented by its Maclaurin series. $\square$

Example 1279. Consider the function $f:(-1,1)\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{1-x}$.

In H2 Maths and this textbook, we will not learn when or why a function can be represented by a power series. Instead, we'll simply and blithely *assume* that this is true whenever the need arises.

So, here for example, let's simply assume f can be represented by a power series (without knowing why). Then by Theorem 38, f can also be represented by its Maclaurin series.

And as we already showed in the previous subchapter, this Maclaurin series is

$$1+x+x^2+\dots$$

Conclude: By Theorem 38, f can be represented by the above Maclaurin series.

That is, $$f(x)=1+x+x^2+x^3+\dots, \qquad \text{for every } x\in(-1,1).$$

Example 1280. Consider the function $g:(-1,1)\to\mathbb{R}$ defined by $g(x)=\dfrac{1}{(1-x)^3}$.

Again, let's simply assume that g can be represented by a power series. In which case, by Theorem 38, g can also be represented by its Maclaurin series.

So, let's find g's Maclaurin series (as usual, we use the Four-Step Maclaurin Recipe):

1. The first three derivatives of g are

$$g'(x)=\frac{3}{(1-x)^4},\qquad g''(x)=\frac{4\cdot 3}{(1-x)^5},\qquad g'''(x)=\frac{5\cdot 4\cdot 3}{(1-x)^6}.$$

2. Evaluate each of g, g', g'', g''' at 0:

$$g'(0)=1,\qquad g'(0)=3,\qquad g''(0)=12,\qquad g'''(0)=60.$$

3. The first four Maclaurin coefficients of g are

$$m_0=\frac{g(0)}{0!}=1,\qquad m_1=\frac{g'(0)}{1!}=3,\qquad m_2=\frac{g''(0)}{2!}=6,\qquad m_3=\frac{g'''(0)}{3!}=10.$$

4. The Maclaurin series of g is

$$1+3x+6x^2+10x^3+\dots$$

Conclude: By Theorem 38, g can be represented by the above Maclaurin series.

That is, $$g(x)=\frac{1}{(1-x)^3}=1+3x+6x^2+10x^3+\dots, \qquad \text{for all } x\in\mathbb{R}.$$

Example 1281. Let $n \in \mathbb{R}$ and define the function $h : (-1, 1) \to \mathbb{R}$ by $h(x) = (1 + x)^n$.

Again, let's simply assume that h can be represented by a power series. In which case, by Theorem 38, h can also be represented by its Maclaurin series.

In Exercise 390(a), we already showed that this Maclaurin series is

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Conclude: By Theorem 38, h can be represented by the above Maclaurin series.

That is, $$h(x) = (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \quad \text{for all } x \in (-1, 1).$$

Example 1282. Consider the exponential function exp.

Again, let's simply assume that exp can be represented by a power series. In which case, by Theorem 38, exp can also be represented by its Maclaurin series.

In Example 1275, we already showed that this Maclaurin series is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Conclude: By Theorem 38, exp can be represented by the above Maclaurin series.

That is, $$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \qquad \text{for all } x \in \mathbb{R}.$$

If we have in particular $x = 1$, then

$$\exp 1 = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \cdots = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

Also, by Fact 63, $e^1 = \exp 1$. And so, we also have

$$\exp 1 = e^1 = e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

We've just proven Theorem 5 (given long ago in Part I of this textbook).

Example 1283. Consider the sine function sin.

Again, let's simply assume that sin can be represented by a power series. In which case, by Theorem 38, sin can also be represented by its Maclaurin series.

In Example 1274, we already showed that this Maclaurin series is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Conclude: By Theorem 38, sin can be represented by the above Maclaurin series.

That is, $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \qquad \text{for all } x \in \mathbb{R}.$$

Example 1284. Consider the cosine function cos.

Again, let's simply assume that cos can be represented by a power series. In which case, by Theorem 38, cos can also be represented by its Maclaurin series.

In Exercise 390(b), we already showed that this Maclaurin series is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Conclude: By Theorem 38, cos can be represented by the above Maclaurin series.

That is, $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \qquad \text{for all } x \in \mathbb{R}.$$

Example 1285. Consider the function $i : (-1, 1] \to \mathbb{R}$ be the function defined by $i(x) = \ln(1 + x)$.

Again, let's simply assume that i can be represented by a power series. In which case, by Theorem 38, i can also be represented by its Maclaurin series.

In Exercise 390(c), we already showed that this Maclaurin series is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Conclude: By Theorem 38, i can be represented by the above Maclaurin series.[464]

That is, $$i(x) = \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \qquad \text{for all } x \in (-1, 1].$$

[464] There is actually a flaw in the argument here. Theorem 38 actually specifies that the function's domain must be an open interval of the form $(-R, R)$. In contrast, here our function's domain is the right-closed interval $(-1, 1]$. So, here Theorem 38 actually only enables us to conclude that $i(x) = \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$ for all $x \in (-1, 1)$.

Theorem 38 does not tell us whether we also have $i(1) = \ln(1 + 1) = 1 - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1^4}{4} + \dots$. That is, whether i can also be represented by its Maclaurin series at its domain's endpoint.

Exercise 391. For each function, find a power series representation, clearly stating any assumption(s) and result(s) you use. (Answer on p. 1009.)

(a) x

(b) x

(c) x

Exercise 392. Consider the function $f:(-1,1)\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{(1+x)^2}$. Find a power series representation of f, up to and including the x^3 term. (Hint: Simply make use of one of the above examples.) (Answer on p. 1921.)

A391(a) XXX

It turns out that this is a somewhat delicate technical issue. But for simplicity and for H2 Maths, we'll simply hand-wave and say that Theorem 38 also works at the endpoints. That is, we'll always just blithely do the same as in the above example.

102.2. The Five Standard Series and Their Intervals of Convergence

The following six equations are in List MF26 (p. 2), so no need to *mug*.

The first of these six equations is simply the Maclaurin series of a general function f.

The general Maclaurin series →

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots$$

The five "standard" Maclaurin series

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots \qquad (|x|<1)$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots \qquad (\text{all } x)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+\frac{(-1)^r x^{2r+1}}{(2r+1)!}+\ldots \qquad (\text{all } x)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+\frac{(-1)^r x^{2r}}{(2r)!}+\ldots \qquad (\text{all } x)$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+\frac{(-1)^{r+1}x^r}{r}+\ldots \qquad (-1<x\le 1)$$

The next five equations are the Maclaurin series for five specific functions. We already derived all five of these Maclaurin series on the previous pages.

Your H2 Maths syllabus (p. 9) and exams[465] call these five Maclaurin series the **"standard" series**.[466]

Observe that for each of these five "standard" series, List MF26 also lists on the right (in parentheses) the corresponding **interval of convergence**.

For each of the second, third, and fourth "standard" series (exp, sin, and cos), on the right, we have "(all x)". This means that each of these three power series has $\mathbb{R}$ as its interval of convergence. That is, each of these three statements is true *for all* $x \in \mathbb{R}$:

$$\exp x = 1 + x + \frac{x^2}{2!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

In contrast, for the first "standard" series, on the right, we have "($|x| < 1$)". This means that this power series has $(-1, 1)$ as its interval of convergence. That is,

$$(1+x)^n \stackrel{1}{=} 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \text{for, every } x \in (-1,1).$$

[465] See e.g. Exercise 677 (N2017/I/1).

[466] Few other writers do the same. There's nothing especially "standard" about these five series (other than that they frequently appear in your syllabus and exams).

But for any $x \notin (-1,1)$, $\stackrel{1}{=}$ is false.[467]

To repeat, in H2 Maths and this textbook, we shall not explain *why* $1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots$ converges only on $(-1,1)$. Instead, this is simply something that you the obedient Singaporean monkey are supposed to "know".

> *Remark* 150. The term **binomial series** was on the old 9740 syllabus but is no longer on the current 9758 syllabus.[468] Just so you know, the binomial series is simply the Maclaurin series for $(1+x)^n$.

Similarly, for the fifth "standard" series, on the right, we have "$(-1 < x \le 1)$". That is, this power series has interval of convergence $(-1,1]$. That is,

$$\ln(1+x) \stackrel{2}{=} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots, \qquad \text{for every } x \in (-1,1].$$

However, for any $x \notin (-1,1]$,$\stackrel{2}{=}$ is false.[469]

To repeat, in H2 Maths and this textbook, we shall not explain *why* $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dfrac{x^6}{6} + \ldots$ converges only on $(-1,1]$. Instead, this is simply something that you the obedient Singaporean monkey are supposed to "know".

---

[467]Suppose for example that $x = 2$ and $n = 1.5$. Then the LHS of $\stackrel{1}{=}$ is

$$(1+2)^{1.5} = 3^{1.5} \approx 5.20.$$

But the RHS of $\stackrel{1}{=}$ is the following, which "clearly" diverges:

$$1 + 1.5 \cdot 2 + \frac{1.5(1.5-1)}{2!}2^2 + \frac{1.5(1.5-1)(1.5-2)}{3!}2^3 \ldots.$$

So, $\stackrel{1}{=}$ is false in this case ($x = 2$, $n = 1.5$). More generally, $\stackrel{1}{=}$ is false for any $x \notin (-1,1)$.

[468]It does remain though on List MF26.

[469]Suppose for example that $x = 2$. Then the LHS of $\stackrel{2}{=}$ is

$$\ln(1+2) = \ln 3 \approx 1.10.$$

But the RHS of $\stackrel{2}{=}$ is the following, which "clearly" diverges:

$$2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \frac{2^5}{5} - \frac{2^6}{6} + \ldots$$

So, $\stackrel{2}{=}$ is false in this case ($x = 2$). More generally, $\stackrel{2}{=}$ is false for any $x \notin (-1,1)$.

102.3. Sine and Cosine, Formally Defined

In Ch. ??, we learnt about the right-triangle and unit-circle definitions of the sine and cosine functions. However, these definitions are considered informal. It turns out that to formally define the sine and cosine functions, there are several approaches. We'll use the most common approach, which is to use their power series:[470]

Definition 219. The *sine function* $\sin : \mathbb{R} \to \mathbb{R}$ is defined by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Definition 220. The *cosine function* $\cos : \mathbb{R} \to \mathbb{R}$ is defined by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Remark 151. For the above definitions to be valid, we must prove that the two power series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ and $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ are convergent (for all $x \in \mathbb{R}$). But such a proof is well beyond the scope of H2 Maths and so altogether omitted from this textbook.

With "some" work,[471] we can show that Definitions 219 and 220 correspond to our earlier unit-circle definitions. We can also show that under these definitions, all previous results involving sine and cosine still hold.

Given Definitions 219 and 220, and the Term-by-Term Differentiation Theorem, we can easily find the derivatives of sine and cosine:

Fact 211. $\sin' = \cos$.

Proof. By Definition 219, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ for every $x \in \mathbb{R}$.

By the Term-by-Term Differentiation Theorem, sin is differentiable and $\sin'$ is defined by

$$\sin' x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \dots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

For every $x \in \mathbb{R}$, $\sin' x = \cos x$ (Definition 220). Hence, $\sin' = \cos$. □

Fact 212. $\cos' = -\sin$.

Proof. By Definition 220, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$ for every $x \in \mathbb{R}$.

[470] Three other common approaches are to use the exponential function, and differential equations, or first define the arcsine function using an integral.

[471] See Ch. 146.19 (Appendices).

By the Term-by-Term Differentiation Theorem, cos is differentiable and $\cos'$ is defined by

$$\cos' x = -\frac{2x}{2!} + \frac{4x^3}{4!} - \cdots = -x + \frac{x^3}{3!} - \ldots .$$

For every $x \in \mathbb{R}$, $\cos' x = -\sin x$ (Definition 219). Hence, $\cos' = -\sin$. □

Facts 211 and 212 have just shown that sin and cos are differentiable. By Theorem 29 (Differentiability Implies Continuity) then, they are also continuous. We have thus proven the following result that was stated earlier in Ch. 87.4 and now reproduced:

Fact 201. *The sine and cosine functions* (sin *and* cos) *are continuous.*

We can now also provide a general proof for the First Pythagorean Identity (reproduced from p. 342):

Fact 71. (First or *The* Pythagorean Identity) *Suppose* $\theta \in \mathbb{R}$. *Then*

$$\sin^2 \theta + \cos^2 \theta = 1.$$

Proof. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(A) = \sin^2 A + \cos^2 A$.

Then f is differentiable and its derivative $f' : \mathbb{R} \to \mathbb{R}$ is defined by $f'(A) = 2\sin A \cos A - 2\cos A \sin A = 0$.

By Proposition 8, f is a constant function. That is, there exists some $c \in \mathbb{R}$ such that for all $A \in \mathbb{R}$,

$$f(A) = \sin^2 A + \cos^2 A = c.$$

Plug in $A = 0$ to find that $c = f(0) = \sin^2 0 + \cos^2 0 = 0 + 1 = 1$. Hence, for all $A \in \mathbb{R}$,

$$\sin^2 A + \cos^2 A = 1.$$ □

Exercise 393. XXX (Answer on p. 1013.)

A393.

102.4. Maclaurin Polynomials as Approximations

Given a function f, its n**th Maclaurin polynomial** is simply its Maclaurin series up to and including the x^n term. A bit more precisely,

Definition 221. Let $n \in \mathbb{Z}_0^+$. Suppose the function f is n-times differentiable at 0. Then the *nth Maclaurin polynomial of f* is defined to be the following polynomial:

$$M_n(x) = \sum_{i=0}^{n} m_i x^i = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Not surprisingly, if a function can be represented by its Maclaurin series, then it can also be approximated by its Maclaurin polynomials. This is yet another useful application of the Maclaurin series.

Example 1286. Consider the function $f : (-1, 1] \to \mathbb{R}$ defined by $f(x) = \ln(1 + x)$.

Previously, we already showed that the Maclaurin series of f is

$$M(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots.$$

So, by Definition 221, the 0**th**, 1**st**, 2**nd**, 3**rd**, 4**th**, and 5**th Maclaurin polynomials of** f are

$$\begin{aligned} M_0(x) &= 0, & M_3(x) &= x - \frac{x^2}{2} + \frac{x^3}{3}, \\ M_1(x) &= x, & M_4(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}, \\ M_2(x) &= x - \frac{x^2}{2}, & M_5(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}. \end{aligned}$$

Figure to be inserted here.

In Example 1285, we argued that g can be represented by its Maclaurin series. So, not surprisingly, the first six Maclaurin polynomials of g serve as (decent) approximations of g.

Remark 152. The term **Maclaurin polynomial** is not used in your H2 Maths syllabus or exams. Nor is it a standard and commonly used term. However, it is sufficiently convenient that I'll use it anyway in this textbook.

Example 1287. Consider the sine function sin.

Previously, we already showed that the Maclaurin series of sin is $M(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

So, the 0th, 1st, 2nd, 3rd, 4th, and 5th Maclaurin polynomials of sin are

$$\begin{aligned} M_0(x) &= 0, & M_3(x) &= x - \frac{x^3}{3!}, \\ M_1(x) &= x, & M_4(x) &= x - \frac{x^3}{3!}, \\ M_2(x) &= x, & M_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!}. \end{aligned}$$

Figure to be inserted here.

In Example 1283, we showed that sin can be represented by its Maclaurin series. So, not surprisingly, the first six Maclaurin polynomials of sin serve as (decent) approximations of sin.

By the way, observe that if $x \approx 0$, then we also have $\frac{x^3}{3!} \approx 0$ and $\frac{x^5}{5!} \approx 0$. Indeed, each term in the Maclaurin series grows smaller.

And so, if x is small, then even low-degree Maclaurin polynomials should serve as "good" approximations of sin.

Indeed, if x is very small (i.e. very close to zero), then we may simply assert that $\sin x \overset{1}{\approx} x$. We call $\overset{1}{\approx}$ the **small-angle approximation for sine**.

In Exercises 394(c) and 402, we'll similarly show that the **small-angle approximations for cosine** and **tangent** are

$$\cos x \approx 1 - \frac{x^2}{2} \qquad \text{and} \qquad \tan x \approx x.$$

One might think that that a higher-degree Maclaurin polynomial is always a better approximation (than a lower-degree one). But unfortunately, this is not generally true, especially if x is far from zero:

Example 1288. Let $x = 10$, so that $\sin x = \sin 10 \approx -0.544$.

Evaluating the 5th Maclaurin polynomial of sin at 10, we have

$$M_5(10) = 10 - \frac{10^3}{3!} + \frac{10^5}{5!} \approx 677.$$

Clearly, $M_5(10) \approx 677$ is a terrible approximation for $\sin 10 \approx -0.544$.

One might expect that the 9th Maclaurin polynomial does better. But this is not the case

$$M_9(10) = 10 - \frac{10^3}{3!} + \frac{10^5}{5!} - \frac{10^7}{7!} + \frac{10^9}{9!} \approx 1\,448.$$

While $M_5(10)$ was a terrible approximation for $\sin 10$, it turns out that $M_9(10)$ is even worse!

Figure to be inserted here.

It's true that the Maclaurin series (which is the "∞th Maclaurin polynomial") will eventually get it exactly right:

$$M(10) = 10 - \frac{10^3}{3!} + \frac{10^5}{5!} - \cdots = \sin 10 = -0.544.$$

However, as the present example shows, as approximations, the Maclaurin polynomials can get worse before they get better.

Exercise 394. For each function, write down its first four Maclaurin polynomials. Then sketch on a single figure the graphs of the function and the first four Maclaurin polynomials. (Answer on p. 1921.)

(a) exp

(b) $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = (1 + x)^n$, where n is any real number.

(c) cos

The **small-angle approximation for cosine** is given by the 2nd Maclaurin polynomial—write it down.

102.5. Creating New Series Using Substitution

Example 1289. From the second "standard" Maclaurin series, we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \qquad \text{for all } x \in \mathbb{R}.$$

We can take the above statement and substitute "x" with any other expression, such as "$2x$", "x^2+1", "$\sin x$", or "$x^3/5$":

$$\begin{aligned}
e^{2x} &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots, && \text{for all } \quad 2x \in \mathbb{R} \quad \text{or } x \in \mathbb{R}.\\
e^{x^2+1} &= 1 + \left(x^2+1\right) + \frac{\left(x^2+1\right)^2}{2!} + \frac{\left(x^2+1\right)^3}{3!} + \dots, && \text{for all } x^2+1 \in \mathbb{R} \text{ or } x \in \mathbb{R}.\\
e^{\sin x} &= 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \dots, && \text{for all } \ \sin x \in \mathbb{R} \ \text{ or } x \in \mathbb{R}.\\
e^{x^3/5} &= 1 + \frac{x^3}{5} + \frac{\left(x^3/5\right)^2}{2!} + \frac{\left(x^3/5\right)^3}{3!} + \dots, && \text{for all } \quad \frac{x^3}{5} \in \mathbb{R} \quad \text{or } x \in \mathbb{R}.
\end{aligned}$$

In the above example, it so happens that the exponential function can be represented by its power series everywhere. That is, the interval of convergence is $\mathbb{R}$.

And so, we could substitute "x" with any other expression and obtain a new statement that continued to be true for all $x \in \mathbb{R}$. That is, the new interval of convergence was still $\mathbb{R}$.

This is not the case in the next example:

Example 1290. From the fifth "standard" Maclaurin series, we know that

$$\ln(1+x) \overset{1}{=} x - \frac{x^2}{2} + \frac{x^3}{3} + \dots, \qquad \text{for all } x \in (-1,1].$$

Note that the interval of convergence is $(-1,1]$.

As in the previous example, we can substitute "x" with "$2x$", but now we must be careful about finding the new interval of convergence:

$$\ln(1+2x) \overset{2}{=} x - \frac{x^2}{2} + \frac{x^3}{3} + \dots, \qquad \text{for all } 2x \in (-1,1] \text{ or } x \in \left(-\frac{1}{2}, \frac{1}{2}\right].$$

Note that the new interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right]$. Although $\overset{1}{=}$ holds for all $x \in (-1,1]$, $\overset{2}{=}$ holds only for all $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$.

Similarly, we can substitute "x" with "x^2+1", "$\sin x$", or "$x^3/5$". But in each case, we must be careful about finding the new interval of convergence, which can get tricky:

$$\ln\left(1+x^2+1\right) \overset{3}{=} x^2+1 - \frac{\left(x^2+1\right)^2}{2} + \frac{\left(x^2+1\right)^3}{3} + \dots, \qquad \text{for all } x^2+1 \in (-1,1] \text{ or } x^2 \in (-2,0] \text{ or}$$

$$\ln(1+\sin x) \overset{4}{=} \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} + \dots, \qquad \text{for all } \sin x \in (-1,1].$$

$$\ln\left(1+\frac{x^3}{5}\right) \overset{5}{=} \frac{x^3}{5} - \frac{\left(x^3/5\right)^2}{2} + \frac{\left(x^3/5\right)^3}{3} + \dots, \qquad \text{for all } \frac{x^3}{5} \in (-1,1] \text{ or } x \in \left(-\sqrt[3]{5}, \sqrt[3]{5}\right].$$

Here's what the interval of convergence in each of these three statements says:

- $\overset{3}{=}$ holds for $x=0$ (and no other values)!
- $\overset{4}{=}$ holds for all real numbers x, *except* those for which $\sin x = -1$.[472]
- $\overset{5}{=}$ holds for all $x \in \left(-\sqrt[3]{5}, \sqrt[3]{5}\right]$.

[472] $\sin x = -1 \iff x = \frac{3\pi}{2} + 2k\pi$ for some integer k.

Example 1291. Consider the function $f : (0, 2) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x}.$$

It's not obvious whether f can be represented by any power series.

But take the first "standard" Maclaurin series:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots, \qquad \text{for all } x \in (-1, 1).$$

Substitute "x" with "$x - 1$":

$$\frac{1}{1+x-1} = \frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots \qquad \text{for all } x - 1 \in (-1, 1) \text{ or } x \in (0, 2).$$

And so, for all $x \in (0, 2) = \text{Domain } f$, we have

$$f(x) = \frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots$$

Thus, we've just shown that f can be represented by the power series $1-(x-1)+(x-1)^2-(x-1)^3+\ldots$.

Example 1292. XXX

Example 1293. XXX

Example 1294. XXX

Exercise 395. For each given Maclaurin series, create four new series by substituting "x" with **(i)** "$2x$"; **(ii)** "x^2+1"; **(iii)** "$\sin x$"; and **(iv)** "$x^3/5$". In each case, take care to specify the new interval of convergence. (Answer on p. 1923.)

(a) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots$ for all $x \in \mathbb{R}$.

(b) $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \dots$ for all $x \in (-1,1)$.

(c) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dots$ for all $x \in \mathbb{R}$.

Exercise 396. For each function, find a power series representation. (Answer on p. 1923.)

(a) The function $f : (0,2] \to \mathbb{R}$ defined by $f(x) = \ln x$.

(b) x

(c) x

Exercise 397. Let g be a nice function defined by $g(x) = \dfrac{1}{1+2x^2}$. Suppose the domain of g contains zero and is the largest possible such that g can be represented by a power series. Find this power series and the domain of g. (Answer on p. 1924.)

102.6. Creating New Series Using Multiplication

Given two (finite) polynomial functions, we know how to find their product function. Two quick examples:

Example 1295. Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(x) = x + 5 \qquad \text{and} \qquad g(x) = 2x^2 - 3.$$

Then the function $f \cdot g : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f \cdot g)(x) = f(x)\, g(x) = (x+5)\left(2x^2 - 3\right) = 2x^3 + 10x^2 - 3x - 15.$$

Example 1296. Let $h : [1,5] \to \mathbb{R}$ and $i : [3,7] \to \mathbb{R}$ be the functions defined by

$$h(x) = 2x^3 - x + 1 \qquad \text{and} \qquad i(x) = x^2 + 2x.$$

Then the product function $h \cdot i : [3,5] \to \mathbb{R}$ is defined by

$$(h \cdot i)(x) = h(x)\, i(x) = \left(2x^3 - x + 1\right)\left(x^2 + 2x\right) = 2x^5 + 4x^4 - x^3 - x^2 + 2x.$$

By the way, note the domain of the product function $h \cdot i$.[473]

Very happily, given two functions that can be represented by power series, we can do something similar: [474]

Theorem 39 (informal)**.** *Suppose the functions f and g can be represented by the power series A and B. Then the function $f \cdot g$ can also be represented by the power series that is the product of A and B.*[475]

[473]As explained in Ch. 20, Domain $h \cdot i$ = Domain $h \cap$ Domain $i = [1,5] \cap [3,7] = [3,5]$.

[474]This informally stated result omits an important technical condition. For a formal (and correct) statement of this result, see Theorem 57 (Appendices).

[475]Formally, we call this power series the **Cauchy product** of A and B.

Example 1297. Let $f : (-1,1) \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{1}{1-x}.$$

Let $g = \exp \cdot f$. That is, let $g : (-1,1) \to \mathbb{R}$ be the function defined by

$$g(x) = (\exp \cdot f)(x) = \frac{\exp x}{1-x}.$$

From the first two "standard" Maclaurin series, we know that for every $x \in (-1,1)$,

$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad \text{and} \qquad f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

So, by Theorem 39, g can be represented by a power series. Moreover, this power series is simply the product of the above two power series. To find it, let's write

$$\left(1 + 1x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots\right)\left(1 + 1x + 1x^2 + 1x^3 + \dots\right) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Comparing coefficients, we have

$$\begin{aligned} c_0 &= 1 \times 1 = 1, \\ c_1 &= 1 \times 1 + 1 \times 1 = 2, \\ c_2 &= \frac{1}{2} \times 1 + 1 \times 1 + 1 \times 1 = \frac{5}{2}, \\ c_3 &= \frac{1}{6} \times 1 + \frac{1}{2} \times 1 + 1 \times 1 + 1 \times 1 = \frac{8}{3}. \end{aligned}$$

Thus, we conclude that the function $g = \exp \cdot f$ can be represented by this power series:

$$1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \dots$$

That is, $$g(x) = \frac{\exp x}{1-x} = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \dots \quad \text{for all } x \in (-1,1) = \operatorname{Domain} g.$$

By the way, we could've arrived at this same conclusion using the Four-Step Maclaurin Recipe and Theorem 38, as was done in earlier subchapters, and which we'll do on the next page:

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

By Theorem 39, g can be represented by a power series. And so, by Theorem 38, g can be represented by its Maclaurin series, which we now find:

1. The first three derivatives of g are defined by

$$g'(x) = \frac{\exp x}{1-x} + \frac{\exp x}{(1-x)^2} = g(x)\left(1 + \frac{1}{1-x}\right),$$

$$g''(x) = g'(x)\left(1 + \frac{1}{1-x}\right) + \frac{g(x)}{(1-x)^2},$$

$$g'''(x) = g''(x)\left(1 + \frac{1}{1-x}\right) + \frac{g'(x)}{(1-x)^2} + \frac{g'(x)}{(1-x)^2} + 2\frac{g(x)}{(1-x)^3}.$$

2. Evaluate each of g, g', g'', g''' at 0:

$$g(0) = \frac{\exp 0}{1-0} = \frac{1}{1} = 1,$$

$$g'(0) = g(0)\left(1 + \frac{1}{1-0}\right) = 1 \cdot (1+1) = 2,$$

$$g''(0) = g'(0)\left(1 + \frac{1}{1-0}\right) + \frac{g(0)}{(1-0)^2} = 2 \cdot 2 + \frac{1}{1} = 5,$$

$$g'''(0) = g''(0)\left(1 + \frac{1}{1-0}\right) + 2\frac{g'(0)}{(1-0)^2} + 2\frac{g(0)}{(1-0)^3} = 5 \cdot 2 + 2 \cdot \frac{2}{1} + 2 \cdot \frac{1}{1} = 16.$$

3. The first four Maclaurin coefficients of g are

$$m_0 = \frac{g(0)}{0!} = \frac{1}{1} = 1, \qquad m_2 = \frac{g''(0)}{2!} = \frac{5}{2},$$

$$m_1 = \frac{g'(0)}{1!} = \frac{2}{1} = 2, \qquad m_3 = \frac{g'''(0)}{3!} = \frac{16}{6} = \frac{8}{3}.$$

4. The Maclaurin series of g is $1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \dots$.

So, g can be represented by the above Maclaurin series.

That is, $$g(x) = \frac{\exp x}{1-x} = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \dots, \quad \text{for all } x \in (-1,1) = \text{Domain}\, g.$$

Happily, this is the same conclusion as before (we'd be worried otherwise).

Example 1298. Define the functions $f : (-1, 1) \to \mathbb{R}$ and $g : \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \to \mathbb{R}$ by

$$f(x) = \frac{1}{(1-x)^2} \qquad \text{and} \qquad g(x) = \frac{1}{1+2x^2}.$$

Let $h = f \cdot g$. That is, define $h : \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \to \mathbb{R}$ by

$$h(x) = (f \cdot g)(x) = f(x)\,g(x) = \frac{1}{(1-x)^2}\frac{1}{1+2x^2}.$$

In Exercises 392 and 397, we already showed that the functions f and g can be represented by these power series (respectively):

$$1 + 2x + 3x^2 + 4x^3 + \ldots \qquad \text{and} \qquad 1 - 2x^2 + 4x^4 - 6x^6 + \ldots$$

And so, by Theorem 39, the function h can also be represented by a power series. Moreover, this power series is simply the product of the above two power series. To find it, let's write

$$\left(1 + 2x + 3x^2 + 4x^3 + \ldots\right)\left(1 - 2x^2 + 4x^4 - 6x^6 + \ldots\right) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

Comparing coefficients, we have

$$\begin{aligned} c_0 &= 1 \times 1 = 1, \\ c_1 &= 2 \times 1 = 2, \\ c_2 &= 1 \times (-2) + 3 \times 1 = 1, \\ c_3 &= 2 \times (-2) + 4 \times 1 = 0. \end{aligned}$$

Thus, we conclude that h can be represented by

$$1 + 2x + x^2 + 0x^3 + \ldots$$

That is, $$h(x) = \frac{1}{(1-x)^2}\frac{1}{1+2x^2} = 1 + 2x + x^2 + 0x^3 + \ldots \quad \text{for all } x \in \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \text{Domain } h.$$

By the way, just like in the last example, we could have arrived at this same conclusion by using the Four-Step Maclaurin Recipe and Theorem 38. You are asked to do so in Exercise 400.

Example 1299. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sin x \cos x.$$

To show that f can be represented by a power series, observe that $f = \sin \cdot \cos$. We already knew that sin and cos can be represented by power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \qquad \text{and} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad \text{for all } x \in \mathbb{R}.$$

And so, by Theorem 39, f can also be represented by a power series. Moreover, this power series is simply the product of the above two power series. To find it, write

$$\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Comparing coefficients, we have

$$\begin{aligned} c_0 &= 0 \times 1 = 0, \\ c_1 &= 0 \times 0 + 1 \times 1 = 1, \\ c_2 &= 0 \times \left(-\frac{1}{2}\right) + 1 \times 0 + 0 \times 1 = 0, \\ c_3 &= 0 \times 0 + 1 \times \left(-\frac{1}{2}\right) + 0 \times 0 + \left(-\frac{1}{6}\right) \times 1 = -\frac{2}{3}. \end{aligned}$$

Thus, we conclude that f can be represented by

$$0 + x + 0x^2 - \frac{2}{3}x^3 + \cdots = x - \frac{2}{3}x^3$$

That is, $$f(x) = \sin x \cos x = x - \frac{2}{3}x^3 + \dots, \qquad \text{for all } x \in \mathbb{R} = \text{Domain } f.$$

By the way, just like in the previous examples, we could have arrived at this same conclusion by using the Four-Step Maclaurin Recipe and Theorem 38. You are asked to do so in Exercise XXX.

Example 1300. XXX

Example 1301. XXX

Example 1302. XXX

Exercise 398. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x \exp x$.

(a) Explain why f can be represented by a power series.

(b) Use Theorem 57 to find a power series representation of f, up to and including the x^3 term.

(c) Do the same as in **(b)**, but now use the Four-Step Maclaurin Recipe and Theorem 38.

(d) Do your answers in **(b)** and **(c)** agree? (Answer on p. 1924.)

Exercise 399. Define $g : (-1, 1) \to \mathbb{R}$ by $g(x) = \cos x \ln(1 + x)$.

(a) Explain why g can be represented by a power series.

(b) Use Theorem 57 to find a power series representation of g, up to and including the x^3 term.

(c) Do the same as in **(b)**, but now use the Four-Step Maclaurin Recipe and Theorem 38.

(d) Do your answers in **(b)** and **(c)** agree? (Answer on p. 1924.)

Exercise 400. As in Example 1298, define $h : \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \to \mathbb{R}$ by

$$h(x) = \frac{1}{(1-x)^2} \frac{1}{1+2x^2}.$$

Use the Four-Step Maclaurin Recipe and Theorem 38 to find a power series representation of h, up to and including the x^3 term. How does your finding compare to the conclusion in Example 1298? (Answer on p. 1925.)

Exercise 401. As in Example 1299, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sin x \cos x.$$

Use the Four-Step Maclaurin Recipe and Theorem 38 to find a power series representation of f, up to and including the x^3 term. How does your finding compare to the conclusion in Example 1299? (Answer on p. 1926.)

102.7. Repeated Differentiation to Find a Maclaurin Series

Your H2 Maths syllabus explicitly includes “derivation of the first few terms of the Maclaurin series by repeated differentiation”. This isn’t really anything new or separate. Example:

Example 1303. Consider the secant function sec. We can find its Maclaurin series using, as usual, the Four-Step Maclaurin Recipe:

1. The first four derivatives of sec are defined thus:

$$\begin{aligned}
\sec' x &= \sec x \tan x, \\
\sec'' x &= \sec x \tan^2 x + \sec^3 x, \\
\sec''' x &= \sec x \tan^3 x + 2 \tan x \sec^3 x + 3 \sec^3 x \tan x = \sec x \tan^3 x + 5 \sec^3 x \tan x, \\
\sec^{(4)} x &= \sec x \tan^4 x + 3 \sec^3 x \tan^2 x + 15 \sec^3 x \tan^2 x + 5 \sec^5 x \\
&= \sec x \tan^4 x + 18 \sec^3 x \tan^2 x + 5 \sec^5 x.
\end{aligned}$$

2. Observe that $\sec 0 = 1$ and $\tan 0 = 0$. So,

$$\begin{aligned}
\sec 0 &= 1, & \sec''' 0 &= 0, \\
\sec' 0 &= 0, & \sec^{(4)} 0 &= 5. \\
\sec'' 0 &= 1,
\end{aligned}$$

3. The first five Maclaurin coefficient of sec are 1, 0, 1/2, 0, and $5/4! = 5/24$.
4. The Maclaurin series of sec is[476]

$$1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots$$

Exercise 402. Find the Maclaurin series of the tangent function tan, up to and including the x^5 term. The **small-angle approximation for** tan is given by the 1st Maclaurin polynomial—write it down. (Answer on p. 1926.)

[476]It turns out that the interval of convergence here is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. That is,

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots, \qquad \text{for all } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

(Don’t worry, this isn’t something you need to know for H2 Maths.)

102.8. Repeated Implicit Differentiation to Find a Maclaurin Series

Your H2 Maths syllabus includes “derivation of the first few terms of the Maclaurin series by repeated implicit differentiation”. We’ll try the example given in your syllabus:

Example 1304. Consider the curve described by $y^3 + y^2 + y \overset{1}{=} x^2 - 2x$.

The above equation implicitly defines y as a function of x. That is, even though we may not know how to, it is possible to express y as a function of x.[477]

Let’s try to find the first few terms of the Maclaurin series of this function y.

Plug $x = 0$ into $\overset{1}{=}$:

$$[y(0)]^3 + [y(0)]^2 + y(0) = 0^2 - 2 \cdot 0 = 0 \qquad \text{or} \qquad [y(0)]\left\{[y(0)]^2 + y(0) + 1\right\} \overset{2}{=} 0.$$

Observe that $z^2 + z + 1 = 0$ has no (real) solutions. So, from $\overset{2}{=}$, we have $y(0) \overset{3}{=} 0$.

Apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator to $\overset{1}{=}$:

$$3y^2 \cdot y' + 2y \cdot y' + y' = 2x - 2 \qquad \text{or} \qquad y'\left(3y^2 + 2y + 1\right) \overset{4}{=} 2x - 2.$$

Plug $x = 0$ and $y(0) \overset{3}{=} 0$ into $\overset{4}{=}$:

$$y'(0)\left(3 \cdot 0^2 + 2 \cdot 0 + 1\right) = 2 \cdot 0 - 2 = -2 \qquad \text{or} \qquad y'(0) \overset{5}{=} -2.$$

Apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator to $\overset{4}{=}$:

$$y''\left(3y^2 + 2y + 1\right) + y'\left(6y \cdot y' + 2y'\right) = 2 \qquad \text{or} \qquad y''\left(3y^2 + 2y + 1\right) + (y')^2(6y + 2) \overset{6}{=} 2.$$

Plug $x = 0$, $y(0) \overset{3}{=} 0$, and $y'(0) \overset{5}{=} -2$ into $\overset{6}{=}$:

$$y''(0)\left(3 \cdot 0^2 + 2 \cdot 0 + 1\right) + (-2)^2(6 \cdot 0 + 2) = 2 \qquad \text{or} \qquad y''(0) \overset{7}{=} -6.$$

Altogether, we have $y(0) \overset{3}{=} 0$, $y'(0) \overset{5}{=} -2$, and $y''(0) \overset{7}{=} -6$.

Hence, the first three Maclaurin coefficients of y are 0, -2, and -3. And the Maclaurin series of y is $-2x - 3x^2 + \ldots$.[478]

[477]This is the Implicit Function Theorem (see n. 414 on p. 897).

[478]This is a pointless and tedious exercise (and hence the sort of thing beloved by the Singapore education system).

There is little point in simply computing the Maclaurin series for its own sake. What we *might* be interested in is the set of values for which $y = -2x - 3x^2 + \ldots$ holds. Unfortunately, finding this set is well beyond the scope of H2 Maths (in fact, I’m not even sure it can be done analytically).

Example 1305. Suppose the function f satisfies this equation:

$$x\left[f(x)\right]^2+\mathrm{e}\overset{1}{=}\mathrm{e}^{f(x)}.$$

Using repeated implicit differentiation, we can find the first few terms of the Maclaurin series of f.

Apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator once:

$$\left[f(x)\right]^2+2xf(x)f'(x)\overset{2}{=}\mathrm{e}^{f(x)}f'(x).$$

Apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator again:

$$2f(x)f'(x)+2f(x)f'(x)+2x\left\{\left[f'(x)\right]^2+f(x)f''(x)\right\}\overset{3}{=}\mathrm{e}^{f(x)}\left[f'(x)\right]^2+\mathrm{e}^{f(x)}f''(x).$$

Plug $x=0$ into $\overset{1}{=}$:

$$0\left[f(0)\right]^2+\mathrm{e}=0+\mathrm{e}=\mathrm{e}=\mathrm{e}^{f(0)}\qquad\text{or}\qquad f(0)\overset{4}{=}1.$$

Next, plug $x=0$ and $f(0)\overset{4}{=}1$ into $\overset{2}{=}$:

$$1+0=1=\mathrm{e}^1f'(0)\qquad\text{or}\qquad f'(0)\overset{5}{=}\frac{1}{\mathrm{e}}.$$

Finally, plug $x=0$, $f(0)\overset{4}{=}1$, and $f'(0)\overset{5}{=}1/\mathrm{e}$ into $\overset{3}{=}$:

$$2\cdot1\cdot\frac{1}{\mathrm{e}}+2\cdot1\cdot\frac{1}{\mathrm{e}}+0=\frac{4}{\mathrm{e}}=\mathrm{e}^1\cdot\left(\frac{1}{\mathrm{e}}\right)^2+\mathrm{e}^1f''(0)=\frac{1}{\mathrm{e}}+\mathrm{e}f''(0)\qquad\text{or}\qquad f''(0)=\frac{3}{\mathrm{e}^2}.$$

Altogether, we have $f(0)\overset{4}{=}1$, $f'(0)\overset{5}{=}\dfrac{1}{\mathrm{e}}$, and $f''(0)=\dfrac{3}{\mathrm{e}^2}$.

Hence, the first three Maclaurin coefficients are 1, $1/\mathrm{e}$, and $3/\left(2\mathrm{e}^2\right)$. And the Maclaurin series of f is[479]

$$1+x+\frac{3}{2\mathrm{e}^2}x^2+\dots.$$

[479] This is again a pointless exercise.

Exam Tips for *Towkays*

Comparing the new 9758 syllabus (first examined 2017) with the old 9740 syllabus (last examined 2017), we have mostly subtractions and rarely any additions. One of the rare additions is this subchapter's topic. My suspicion is therefore that it will soon show up. (Note that it didn't appear on the 2017 9758 A-Level exams.)

Although very tedious, there is conceptually nothing difficult about repeated implicit differentiation—it's just a whole bunch of differentiation and algebra. So, just make sure you go slowly and carefully. Ensure that everything is correct at each step of the way.

Exercise 403. XXX (Answer on p. 1030.)

A403.

103. Antidifferentiation

Antidifferentiation is simply the inverse of differentiation.

Example 1306. Define the functions $f, g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2 \qquad \text{and} \qquad g(x) = 2x.$$

The derivative of f is g.

So, f is an **antiderivative** of g.

And g is an **antidifferentiable function** (because it has an antiderivative).

Definition 222. Let f and g be functions. If the derivative of f is g, then f is an *antiderivative of g*.[480]

A function that has an antiderivative is called an *antidifferentiable function*.

Example 1307. XXX

Remark 153. Just so you know, a synonym for *antiderivative* is *primitive*. (But *primitive* is not commonly used and we won't use it.)

Example 1308. XXX

[480]One perhaps subtle implication here is that f and g have the same domain.

103.1. Antiderivatives Are Not Unique ...

Example 1309. Consider the functions $g, h, i : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = x^2, \qquad h(x) = x^2 + 5, \quad \text{and} \quad i(x) = x^2 - 9.$$

Consider also the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$.

Observe that f is the derivative of g, h, and i.

Equivalently, the function f has antiderivatives g, h, and i.

This shows that **antiderivatives are not unique**.

Suppose g is an antiderivative of f. If h differs from g by a constant, then h is also an antiderivative of f. More precisely,

Fact 213. *Let f be a nice function. Suppose g is an antiderivative of f. If $h = g + C$ for some $C \in \mathbb{R}$, then h is also an antiderivative of f.*

Proof. Let D be the domain of f (and hence also of g and h).

For all $x \in D$, $h'(x) = g'(x) + \dfrac{\mathrm{d}}{\mathrm{d}x}C = g'(x) + 0 = g'(x) = f(x)$.

So, the derivative of h is f. Equivalently, h is an antiderivative of f. □

103.2. ... But An Antiderivative Is Unique Up to a COI

It turns out that the converse[481] of Fact 213 is also true. Though antiderivatives are not unique, they are **unique up to a constant**.

That is, suppose g is an antiderivative of f. If h is also an antiderivative of f, then h differs from g by a constant. A little more precisely,

Fact 214. *Let f be a nice function whose domain is an interval. Suppose g is an antiderivative of f. If h is also an antiderivative of f, then $h = g + C$ for some $C \in \mathbb{R}$.*

Proof. Let $i = g - h$. Since $g' = h' = f$, $i' = g' - h' = 0$.

Since the derivative of i is a zero function, by Proposition 8, i must be a constant function. That is, there exists $C \in \mathbb{R}$ such that i is defined by $i(x) = C$—or equivalently, $h = g + C$. □

Facts 213 and 214, combined:

[481] Actually, a *partial* converse, because we have now quietly added the requirement that f's domain is an interval.

Corollary 47. *Let f be a nice function whose domain is an interval. Suppose g is an antiderivative of f. Then*

$$\begin{array}{ll} & h \text{ is an antiderivative of } f \\ \iff & h = g + C \text{ for some } C \in \mathbb{R}. \end{array}$$

Example 1310. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$ and $g(x) = x^2$.

Observe that g is an antiderivative of f.

Suppose another function h is also an antiderivative of f. Then by Corollary 47, there exists $C \in \mathbb{R}$ such that

$$h(x) = g(x) + C = x^2 + C \qquad \text{for all } x \in \mathbb{R}.$$

Example 1311. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions.

Suppose g is an antiderivative of f and is defined by

$$g(x) = \sin\left(e^{x^2-3x+5}\right).$$

We haven't been told anything about f. But if told that h is also an antiderivative of f, then we know there must exist $C \in \mathbb{R}$ such that h may be defined by

$$h(x) = g(x) + C = \sin\left(e^{x^2-3x+5}\right) + C.$$

Remark 154. It would make sense to call C the *constant of antidifferentiation*. But by tradition and for reasons that will be given later, we don't. Instead, we call C the **constant of integration**.

Exercise 404. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x - 3$. (Answer on p. 1928.)

(a) Find three antiderivatives of f.

(b) Suppose you're told that some function $A : \mathbb{R} \to \mathbb{R}$ is also an antiderivative of f. Then how must the function A be related to each of the three functions you found in **(a)**?

Exercise 405. Define $f, g, h : \mathbb{R} \to \mathbb{R}$ by $f(x) = 4\sin 4x$, $g(x) = -\cos 4x$, and $h(x) = 8\sin^2 x \cos^2 x$.

(a) Show that g and h are both antiderivatives of f.

(b) The functions g and h are both antiderivatives of f, but look very different. Does this observation contradict Fact 214? (Answer on p. **1928**.)

Example 1312. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$.

Here are three of f's antiderivatives:

- $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.
- $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2 + 2$.
- $i : \mathbb{R} \to \mathbb{R}$ defined by $i(x) = x^2 - 1$.

More generally, the antiderivatives of f are *exactly* those functions $j : \mathbb{R} \to \mathbb{R}$ defined by $j(x) = x^2 + C$ (for $C \in \mathbb{R}$).

And even more generally,

> *Suppose a function is antidifferentiable and defined on an interval. If this function is defined by $x \mapsto 2x$, then its antiderivatives are* exactly *those functions defined by $x \mapsto x^2 + C$ (for $C \in \mathbb{R}$).*

The above italicised statement is precise, formal, and correct—but also *sibei* long-winded. So, let's declare it to be exactly equivalent to this shorthand statement:

$$\int 2x \, \mathrm{d}x = x^2 + C.$$

For the next three examples, assume D is an interval.

Example 1313. The statement

$$\int 3x^2 \, \mathrm{d}x = x^3 + C$$

says that if the function $f : D \to \mathbb{R}$ defined by $f(x) = 3x^2$ is antidifferentiable, then its antiderivatives are exactly those functions $g : D \to \mathbb{R}$ defined by $g(x) = x^3 + C$ (for $C \in \mathbb{R}$).

Example 1314. The statement

$$\int \cos x \, \mathrm{d}x = \sin x + C$$

says that if the function $f : D \to \mathbb{R}$ defined by $f(x) = \cos x$ is antidifferentiable, then its antiderivatives are exactly those functions $g : D \to \mathbb{R}$ defined by $g(x) = \sin x + C$ (for $C \in \mathbb{R}$).

Example 1315. The statement

$$\int 4x^3 - 2x \, \mathrm{d}x = x^4 - 2x + C$$

says that if the function $f : D \to \mathbb{R}$ defined by $f(x) = 4x^3 - 2x$ is antidifferentiable, then its antiderivatives are exactly those functions $g : D \to \mathbb{R}$ defined by $g(x) = x^4 - 2x + C$ (for $C \in \mathbb{R}$).

In general,

Definition 223. Let $f(x)$ and $g(x)$ be expressions containing the variable x. We shall write

$$\int f(x)\,\mathrm{d}x \stackrel{\star}{=} g(x) + C$$

to mean the following:

> *Suppose a function is antidifferentiable and defined on an interval. If this function is defined by $x \mapsto f(x)$, then its antiderivatives are* exactly *those functions defined by $x \mapsto g(x) + C$ (for $C \in \mathbb{R}$).*

In $\stackrel{\star}{=}$, we call

- The symbol $\int$ (an elongated S) the **antidifferentiation symbol**.

  (Spoiler: Later, we'll also call $\int$ the **integration symbol**.)
- $f(x)$ the **integrand**.
- x the **(dummy) variable of antidifferentiation**.

  (Spoiler: Later, we'll also call x the **(dummy) variable of integration**.)
- $\mathrm{d}x$ the **differential of the variable** x. (You can think of $\mathrm{d}x$ as a sort of punctuation mark.)

Remark 155. The symbol $\int f(x)\,\mathrm{d}x$ is often called the **indefinite integral of** f.

In this textbook, we'll avoid using the term **indefinite integral**, because it is highly confusing for two main reasons:

1. Confusingly, different writers give this term *many* different definitions.[482]
2. It makes the relationship between differentiation, antidifferentiation, and integration that much more confusing. In particular and importantly, it leaves students confused as to why the Fundamental Theorems of Calculus have any substance and don't just follow from definition.

Unfortunately, the term *indefinite integral* remains in common use by other writers. You're likely to encounter it elsewhere, which is why we even bothered briefly mentioning it here in this remark.

(Where the need arises in this textbook, the symbol $\int f(x)\,\mathrm{d}x$ shall simply be called **the antiderivative**.)

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[482] Here are just six:

1. Larson and Edwards (2010): "The term **indefinite integral** is a synonym for antiderivative."
2. Hass, Heil, and Weir (*Thomas' Calculus*, 2018, p. 236): "The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,\mathrm{d}x."$$

Remark 156. Earlier with limits, we warned[483] that in the statement $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots \overset{\blacklozenge}{=} 1$, the equals sign $\overset{\blacklozenge}{=}$ isn't our usual equals sign.

Here similarly, in the statement $\int f(x)\,\mathrm{d}x \overset{\star}{=} g(x)+C$, the equals sign "$\overset{\star}{=}$" isn't our usual equals sign.

Instead of thinking of "$\int f(x)\,\mathrm{d}x$" and "$g(x)+C$" as objects exist independently, it is better to think of $\overset{\star}{=}$ as a single inseparable statement that is shorthand for a more precise but long-winded statement (that we gave in Definition 223 above)

3. Shurman (2012, p. 293): "*Let I be a nonempty interval in $\mathcal{R}$, and consider a function*

$$f : I \to \mathcal{R}$$

such that $\int_a^b f$ exists for all $a, b \in I$. An ***indefinite integral of*** *f is a second function*

$$F : I \to \mathcal{R}$$

such that

$$\int_a^b f = F(b) - f(a) \qquad \textit{for all } a, b \in I."$$

4. Priestley (1997): "Let $g \in \mathrm{C}[a,b]$. Define the *indefinite integral* of g to be G, where

$$G(x) := \int_a^x g."$$

5. Hughes et al. (1998): "All antiderivatives of $f(x)$ are of the form $F(x)+C$. We introduce a notation for the general antiderivative that looks like the definite integral without the limits and is called the *indefinite integral*:

$$\int f(x)\,\mathrm{d}x = F(x) + C."$$

6. Herman & Strang (2017, p. 487): "Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)\,dx,$$

is the most general antiderivative of f."

[483] Remark 144.

Remark 157. We just cautioned against thinking of "$\int f(x)\,\mathrm{d}x$" as an object that exists independently.

Nonetheless, we will often ask questions like, "Find $\int f(x)\,\mathrm{d}x$." This last statement will once again be shorthand, this time for this precise but long-winded question:

> *Suppose a function is antidifferentiable, defined on an interval, and defined by $x \mapsto f(x)$. Find its antiderivatives.*

And our answer would then be something like, "$\int f(x)\,\mathrm{d}x = g(x) + C$", by which is meant:

> *Its antiderivatives are* exactly *those functions defined by $x \mapsto g(x)+C$ (for $C \in \mathbb{R}$).*

We've been using the symbol x as our (dummy) variable of antidifferentiation. As usual, we can replace x with any other symbol:

Example 1316. The statement $\int 2x\,\mathrm{d}x = x^2+C$ is equivalent to any of these statements:

$$\int 2y\,\mathrm{d}y = y^2 + C, \quad \int 2z\,\mathrm{d}x = z^2 + C, \quad \int 2☺\,\mathrm{d}☺ = ☺^2 + C, \quad \int 2\,\square\,\mathrm{d}\square = \square^2 + C.$$

Exercise 406. XXX (Answer on p. 1037.)

A406.

103.4. Rules of Antidifferentiation

On the next page, we'll formally state some **Rules of Antidifferentiation**. But first, we'll illustrate them with some quick examples. You should find these Rules familiar from secondary school:

Example 1317. (Constant Rule) $\int 5\,\mathrm{d}x = 5x + C$.

Example 1318. (Power Rule) $\int x^{17}\,\mathrm{d}x = \frac{1}{18}x^{18} + C$.

Example 1319. (Power Rule) $\int x^{-4}\,\mathrm{d}x = -\frac{1}{3}x^{-3} + C = -\frac{1}{3}\frac{1}{x^3} + C$.

Example 1320. (Reciprocal Rule) $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$.

Example 1321. (Exponential function) $\int \exp x\,\mathrm{d}x = \exp x + C$.

Example 1322. (Sine) $\int \sin x\,\mathrm{d}x = -\cos x + C$.

Example 1323. (Cosine) $\int \cos x\,\mathrm{d}x = \sin x + C$.

Example 1324. (Sum Rule) $\int \sin x + \exp x\,\mathrm{d}x = -\cos x + \exp x + C$.

Example 1325. (Difference Rule) $\int \frac{1}{x} - \cos x\,\mathrm{d}x = \ln|x| - \sin x + C$.

Example 1326. (Constant Factor Rule) $\int 5x^{-4}\,\mathrm{d}x = -\frac{5}{3}\frac{1}{x^3} + C$.

Example 1327. (LPC Rule) $\int \cos(2x+3)\,\mathrm{d}x = \frac{1}{2}\sin(2x+3) + C$.

Example 1328. (LPC Rule) $\int \exp(2x+3)\,\mathrm{d}x = \frac{1}{2}\exp(2x+3) + C$.

Example 1329. (LPC Rule) $\int (2x+3)^{17}\,\mathrm{d}x = \frac{1}{18}(2x+3)^{18} + C$.

Fact 215. *Let $f(x)$ and $g(x)$ be expressions containing the variable x. If $\frac{\mathrm{d}}{\mathrm{d}x}g(x) = f(x)$, then $\int f(x)\,\mathrm{d}x = g(x) + C$.*

Proof. See p. 1704 (Appendices). □

These Rules of Antidifferentiation are, of course, simply the inverse of our earlier Rules of Differentiation.

Theorem 40. (Rules of Antidifferentiation) *Suppose $k \in \mathbb{R}$. Then*

(a) $\int k\,\mathrm{d}x = kx + C$ **(Constant Rule)**

(b) $\int x^k\,\mathrm{d}x = \frac{x^{k+1}}{k+1} + C$, *for $k \neq -1$ and $x \neq 0$ if $k < 0$* **(Power Rule)**

(c) $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$, *for $x \neq 0$* **(Reciprocal Rule)**

(d) $\int \exp x\,\mathrm{d}x = \exp x + C$ **(Exponential Rule)**

(e) $\int \sin x\,\mathrm{d}x = -\cos x + C$ **(Sine)**

(f) $\int \cos x\,\mathrm{d}x = \sin x + C$ **(Cosine)**

Now suppose also that f and g are functions with antiderivatives F and G. Then

(g) $\int (f \pm g)(x)\,\mathrm{d}x = F(x) \pm G(x) + C$ **(Sum and Difference Rules)**

(h) $\int kf(x)\,\mathrm{d}x = kF(x) + C$ **(Constant Factor Rule)**

(i) $\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) + C$, *for $a, b \in \mathbb{R}$ with $a \neq 0$* **(LPC Rule)**

Proof. By Fact 215, to prove that $\int f(x)\,\mathrm{d}x = F(x) + C$, show that $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.

Here we'll do only **(c)**, which is a little trickier:

(c) By the Natural Logarithm Rule for Differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln|x| = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}, & \text{for } x > 0, \\ \frac{\mathrm{d}}{\mathrm{d}x}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}, & \text{for } x < 0. \end{cases}$$

Exercise **407** asks you to prove the remaining Rules of Antidifferentiation. □

Remark 158. In the Reciprocal Rule, there is, annoyingly enough, an **absolute value sign**. No, it is **not** OK to simply drop it.[484] Take care to always include it. ✿.

Remark 159. For lack of a better name, I shall call the last Rule the **Linear Polynomial Composition (LPC) Rule**.

When written out formally, it looks complicated. But as illustrated by the last three examples, it's jolly simple and you'll already have seen plenty of it in secondary school.

Remark 160. To repeat, each of the Rules of Antidifferentiation in Theorem 40 is merely shorthand for a precise but long-winded statement.

For example, Theorem 40(a) states, "$\int k\,\mathrm{d}x = kx + C$". This is merely shorthand for the following precise but long-winded statement:

> *Suppose a function is antidifferentiable and defined on an interval. If this function is defined by $x \mapsto k$, then its antiderivatives are* exactly *those functions defined by $x \mapsto kx + C$ (for $C \in \mathbb{R}$).*

Remark 161. In Theorem 40(a), in the case where $k = 1$, we have

$$\int 1\,\mathrm{d}x \varnothing\star x + C.$$

By convention, we may write $\int 1\,\mathrm{d}x$ more simply as $\int \mathrm{d}x$.

So, $\varnothing\star$ may also be rewritten more simply as

$$\int \mathrm{d}x = x + C.$$

Exercise 407. Prove Theorem 40(a), (b), and (d)–(i). State any Rules of Differentiation that you use. (Answer on p. **1928**.)

Exercise 408. Mimicking Remark 160, write out the precise but long-winded versions for each of Theorem 40(b)–(i). (Answer on p. 1928.)

Exercise 409. Let $a, b, c, d \in \mathbb{R}$. Find the following. (Answer on p. 1929.)

(a) $\int ax + b\,\mathrm{d}x$

(b) $\int ax^2 + bx + c\,\mathrm{d}x$

(c) $\int ax^3 + bx^2 + cx + d\,\mathrm{d}x$

(d) $\int (ax + b)^c\,\mathrm{d}x$ (for $c \neq -1$)

(e) $\int \frac{1}{ax + b}\,\mathrm{d}x$ (for $ax + b \neq 0$)

(f) $\int a \sin(bx + c) + d\,\mathrm{d}x$

(g) $\int a \exp(bx + c) + d\,\mathrm{d}x$

(h) $\int a \cos bx + c + \frac{1}{dx}\,\mathrm{d}x$ (for $d \neq 0$)

[484]Unless of course we already assume that $x > 0$.

103.5. Every Continuous Function Has an Antiderivative

Happily, **every continuous function is antidifferentiable**:

Theorem 41. *Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function. If f is continuous, then f is antidifferentiable.*

Proof. We'll prove this later (Exercise 418). □

Example 1330. XXX

Example 1331. XXX

Remark 162. Just so you know, the converse of Theorem 41 is false. That is, a function has an antiderivative need **not** be continuous—see Example 1587 (Appendices).

(But don't worry about this. For H2 Maths, most functions you'll ever encounter are continuous anyway, so this hardly matters.)

103.6. Be Careful with This Common Practice

One **Common Practice** is to use the symbol "$\int f(x)\,\mathrm{d}x$" to denote *any* antiderivative of f.

And so, under this Common Practice, we may write

$$\int 2x\,\mathrm{d}x \stackrel{1}{=} x^2, \qquad \int 2x\,\mathrm{d}x \stackrel{2}{=} x^2+1, \qquad \int \cos x\,\mathrm{d}x \stackrel{3}{=} \sin x, \quad \text{or} \quad \int \cos x\,\mathrm{d}x \stackrel{4}{=} \sin x - 5. ✗$$

And more generally, we may write

$$\int f(x)\,\mathrm{d}x \stackrel{5}{=} g(x). \qquad ✗$$

This Common Practice is convenient and perfectly fine, so long as one keeps in mind that antiderivatives are unique—*but only up to a constant.*

But if this is forgotten, then the Common Practice can lead to error, such as in this "proof" that $0=1$:

"Since $\int 2x\,\mathrm{d}x \stackrel{1}{=} x^2$ and $\int 2x\,\mathrm{d}x \stackrel{2}{=} x^2+1$, we have $x^2 = x^2+1$ and hence $0=1$." ✗

The error in the above "proof" is easy enough to spot and hence avoid. But when we have more complicated problems,[485] it is harder to avoid similar errors.

It's mostly OK to follow this Common Practice. But you should be careful. In particular, you should always bear in mind that antiderivatives are unique—*but only up to a constant.*

[485] See e.g. Exercises 431 and 432.

103.7. The "Punctuation Mark" $\mathrm{d}x$ (optional)

Actually, the "punctuation mark" $\mathrm{d}x$ is a little unnecessary:

Example 1332. Instead of $\int x\,\mathrm{d}x = \frac{x^2}{2} + C$, we could more simply write

$$\int x = \frac{x^2}{2} + C.$$

It is clear from the context that x is the (dummy) variable of antidifferentiation. Hence, the "$\mathrm{d}x$" is a little unnecessary.

Example 1333. Instead of $\int f(x)\,\mathrm{d}x = g(x) + C$, we could more simply write

$$\int f(x) = g(x) + C.$$

It is clear from the context that x is the (dummy) variable of antidifferentiation. Hence, the "$\mathrm{d}x$" is a little unnecessary.

Indeed, even the "(x)"s are a little unnecessary. And so, we could even more simply write

$$\int f = g + C.$$

Nonetheless, the "punctuation mark" $\mathrm{d}x$ does serve two purposes:

1. It makes clear that the (dummy) variable of antidifferentiation is x. (This is especially helpful for reducing confusion when we have more than one variable or symbol.)
2. It serves as a clear marker for where the integrand ends. (This can reduce the need for parentheses.)

Illustrations:

Example 1334. Instead of $\int k^3 x\,\mathrm{d}x \stackrel{1}{=} \frac{k^2x^2}{2} + C$, we *could* more simply write

$$\int k^3 x \stackrel{2}{=} \frac{k^3x^2}{2} + C.$$

However, with $\stackrel{2}{=}$, the reader is faced with an ambiguity: It's not obvious whether in "$\int k^3x$", k or x is the variable of antidifferentiation. She is expected to figure out for herself that it is x and not k that is the variable of integration.

To save the reader such troublesome ambiguity, we prefer to use the "punctuation mark" $\mathrm{d}x$. That is, we prefer to write $\stackrel{1}{=}$ rather than $\stackrel{2}{=}$.

Example 1335. Instead of $\int 2x + 1\,\mathrm{d}x \stackrel{1}{=} x^2 + x + C$, we *could* more simply write

$$\int 2x + 1 \stackrel{2}{=} x^2 + x + C.$$

However, with $\stackrel{2}{=}$, the reader is faced with an ambiguity: It's not obvious whether $\int 2x + 1 = \int 2x + 1\,\mathrm{d}x$ or $\int 2x + 1 = \int 2x\,\mathrm{d}x + 1$. To clarify matters, we may wish to add parentheses to $\stackrel{2}{=}$ and instead write

$$\int (2x + 1) \stackrel{3}{=} x^2 + x + C.$$

Statement $\stackrel{3}{=}$ is perfectly good and clear, but does require the use of an additional pair of parentheses. So, one might prefer to just stick with $\stackrel{1}{=}$, where the "punctuation mark" $\mathrm{d}x$ clearly where the integrand "$2x + 1$" ends and eliminates any need for the parentheses.

Calculus may be divided into two branches, the **differential calculus** and the **integral calculus**—or more simply, **differentiation** and **integration**. These two branches correspond to two geometric problems:

Differentiation is the problem of finding the **derivative**, or the **gradient** of a curve.

Integration is the problem of finding the **definite integral**, or the **area** under a curve.

Figure to be inserted here.

So far, we've been looking only at **differentiation**.

In the following chapters, we'll look at **integration**.

As we'll learn, differentiation and integration are two sides of the same coin. Specifically (*spoiler alert*), we'll learn that

> **Integration and antidifferentiation are the "same thing".**
>
> **Equivalently, integration and differentiation are inverse operations**.

104. The Definite Integral

Integration is the problem of finding the **definite integral**, which is the area under a curve.

Example 1336. Define $f : [0, 9] \to \mathbb{R}$ by $f(x) = \sqrt{x} + 1$.

Figure to be inserted here.

The *definite integral of f from 0 to 1* is the red area and equals $5/3$. There are three ways to denote a definite integral:

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{5}{3} \qquad \text{or} \qquad \int_0^1 f\,\mathrm{d}x = \frac{5}{3} \qquad \text{or} \qquad \int_0^1 f = \frac{5}{3}.$$

The *definite integral of f from 2 to 4* is the blue area and equals approximately 5.45:

$$\int_2^4 f(x)\,\mathrm{d}x \approx 5.45 \qquad \text{or} \qquad \int_2^4 f\,\mathrm{d}x \approx 5.45 \qquad \text{or} \qquad \int_2^4 f \approx 5.45.$$

How do we find the red or blue areas?

Right now, let's pretend we have no idea (even though we actually already learnt to do this in secondary school). We'll revisit this question only in the next subchapter.

In the previous chapter, we already used the symbol $\int$ as our **antidifferentiation symbol**. Here, we again use the same symbol.

But now, in this new context, this symbol $\int$ will instead be called the **integration symbol**.

The symbol $\int$ thus serves **double duty**, once as the antidifferentiation symbol and again as the integration symbol.

For now, you should find it deeply puzzling that we use the exact same symbol $\int$ in two completely different contexts. This puzzle will be resolved in the next chapter, when we learn about the Fundamental Theorems of Calculus.

Example 1337. Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x - 2$.

Figure to be
inserted here.

We treat any area under the x-axis as negative. (And so, the definite integral is actually the *signed* area under a curve.)

The *definite integral of g from* 0 *to* 1 is the red area and equals −1.5:

$$\int_0^1 g(x)\,\mathrm{d}x = -1.5 \qquad \text{or} \qquad \int_0^1 g\,\mathrm{d}x = -1.5 \qquad \text{or} \qquad \int_0^1 g = -1.5.$$

The *definite integral of g from* 2 *to* 4 is the blue area and equals 2:

$$\int_2^4 g(x)\,\mathrm{d}x = 2 \qquad \text{or} \qquad \int_2^4 g\,\mathrm{d}x = 2 \qquad \text{or} \qquad \int_2^4 g = 2.$$

We can actually compute these areas using primary-school geometry:

$$\int_0^1 g(x)\,\mathrm{d}x = \int_0^1 g\,\mathrm{d}x = \int_0^1 g = \frac{\text{Base} \times \text{Height}}{2} = \frac{1 \times 1}{2} = \frac{1}{2}.$$

$$\int_2^4 g(x)\,\mathrm{d}x = \int_2^4 g\,\mathrm{d}x = \int_2^4 g = \frac{\text{Base} \times \text{Height}}{2} = \frac{2 \times 2}{2} = 2.$$

You are probably most familiar with this notation for the definite integral:

$$\int_a^b f(x)\,\mathrm{d}x.$$

We call

- The symbol $\int$ the **integration symbol** (it is simply an elongated S);
- The numbers a and b the **lower** and **upper limits of integration**;
- The function f to be integrated the **integrand**; and
- The symbol $\mathrm{d}x$ the **differential of the variable** x—it tells us that the **(dummy) variable of integration** is x.

To repeat, the symbol $\int$ does double duty: In the previous chapter, $\int$ was the antidifferentiation symbol; in this chapter, $\int$ is the antidifferentiation symbol. This is something you should find puzzling and which we'll resolve only in the next chapter.

As usual, x is merely a **dummy variable** that can be replaced with any other symbol.

When describing the definite integral of f from a to b, what really matters is that we specify **(i)** the function f; and **(ii)** the lower and upper limits a and b. The symbol we use to denote the dummy variable or variable of integration doesn't really matter—it is customarily x but could be any other symbol like y, t, u, or even ☺. So, we could write

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(y)\,\mathrm{d}y = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(u)\,\mathrm{d}u = \int_a^b f(☺)\,\mathrm{d}☺.$$

Indeed, the "(x)" and even the "$\mathrm{d}x$" are somewhat superfluous. Instead of $\int_a^b f(x)\,\mathrm{d}x$, we could write

$$\int_a^b f\,\mathrm{d}x, \qquad \text{or even more simply,} \qquad \int_a^b f.$$

Example 1338. XXX

Example 1339. XXX

Example 1340. XXX

Exercise 410. XXX (Answer on p. 1048.)

A410.

104.1. Notation for Integration

As we saw earlier (Ch. 89), in the differential calculus, there are (at least) three commonly used pieces of notation (due to Leibniz, Newton, and Lagrange).

In contrast, in the integral calculus, the only notation that is still commonly used (and which we use) is **Leibniz's**:

$$\int_a^b f(x)\,\mathrm{d}x \qquad \text{or} \qquad \int_a^b f\,\mathrm{d}x \qquad \text{or} \qquad \int_a^b f.$$

Fun Fact

To denote the derivative of f, Newton placed a dot above—$\dot{f}$.

To denote the integral of f, he similarly placed a vertical line above—$\overset{|}{f}$. But as just mentioned, this notation is now rarely (never?) used.

104.2. The Definite Integral, Informally Defined

We formally define the **definite integral** only in Ch. 146.22 (Appendices). Nonetheless, just to provide a little clarity and precision, here's an informal definition anyway:

Definition 224 (*informal*)**.** Let $a < b$ and f be a function that is continuous on $[a,b]$. Consider the "area" bounded by f, the x-axis, and the vertical lines $x = a$ and $x = b$. This "area" is a real number that we call the *definite integral of f from a to b* and denote by

$$\int_a^b f(x)\,\mathrm{d}x, \qquad \int_a^b f\,\mathrm{d}x, \quad \text{or} \quad \int_a^b f.$$

Figure to be
inserted here.

We call the symbol $\int$ the *integration symbol*; the numbers a and b the *lower* and *upper limits of integration*; the function f the *integrand*; x the *variable of integration*; and the symbol $\mathrm{d}x$ the *differential of x*.

The above definition is informal because we haven't formally defined what "area" is or how it might be computed. In the next subchapter, we'll make a sketch of how this might be done.

Above, we've defined $\int_a^b f$ in those cases where $a < b$. It will be convenient to also cover those cases $a = b$ or $a > b$:

Definition 225. Let $a < b$ and f be a function that is continuous on $[a,b]$.

(a) The *definite integral of f from a to a* is denoted $\int_a^a f$ and is defined by

$$\int_a^a f = 0.$$

(b) The *definite integral of f from b to a* is denoted $\int_b^a f$ and is defined by

$$\int_b^a f = -\int_a^b f.$$

Example 1341. XXX

Example 1342. XXX

Exercise 411. XXX (Answer on p. 1051.)

A411.

104.3. An Important Warning

In O-Level Additional Maths, you may have been taught that integration is, *by definition*, simply the inverse of differentiation.[486]

That approach is **wrong**,[487] confusing, and detrimental to your understanding of calculus. Indeed, that approach often leaves students confused and unable to understand why the Fundamental Theorems of Calculus (to be covered only in the next chapter) have any substance.

We will not use that incorrect approach in this textbook. To repeat,

Differentiation is the problem of finding the **gradient** of a curve.

Integration is the problem of finding the **area** under a curve.

Figure to be
inserted here.

A priori,[488] there is no reason to think that a curve's gradient is in any way related to the area under the curve. Equivalently, *a priori*, there is no reason to think that differentiation and integration are in any way related.

That they *are* related is established only with the two **Fundamental Theorems of Calculus (FTCs)**. The FTCs are an important landmark in the history of mathematics and indeed the history of humanity.

We will now work our way towards the FTCs. Don't worry, we'll omit most of the yucky technical details. The goal here is merely to provide you with some intuition and hence a better understanding of *why* the FTCs work.

[486] Indeed, on your 4047 A Maths syllabus (and again on your H2 Maths syllabus), the very first mention of integration states, "integration as the reverse of differentiation".

[487] Of course, a definition can never be wrong (unless self-contradictory). We can after all assign to any object or idea any name we like. What is wrong is to favour an approach that causes more confusion.

[488] *A priori* is just a fancy Latin phrase for *beforehand*.

Example 1343. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$.

The definite integral of f from 0 to 16 is the red area and is denoted

$$\int_0^{16} f(x)\,\mathrm{d}x, \qquad \int_0^{16} f\,\mathrm{d}x, \quad \text{or} \quad \int_0^{16} f.$$

Figure to be
inserted here.

Suppose we're asked to show that $\int_0^{16} f = 1\,381\frac{1}{3}$. How might we proceed?

One possibility is to try a crude approximation: Construct a rectangle with width 16 and height $f(8) = 8^2 + 1 = 65$. Call its area A_1. We have

$$A_1 = 16 \times f(8) = 16 \times 65 = 1\,040.$$

So, our first crude approximation of $\int_0^{16} f$ is $A_1 = 1\,040$.

We call the difference between A_1 and $\int_0^{16} f$ the **approximation error** and denote it by e_1:

$$e_1 = \left| \int_0^{16} f - A_1 \right| = \left| 1\,381\frac{1}{3} - 1\,040 \right| = 341\frac{1}{3}.$$

Can we improve on this first crude approximation? (Equivalently, can we reduce the approximation error?)

Sure. One obvious possibility is to use more rectangles:

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Figure to be inserted here.

Construct *two* rectangles, each with width 8, but one with height $f(4) = 4^2 + 1 = 17$ and the other with height $f(12) = 12^2 + 1 = 145$. Call their areas A_{21} and A_{22}. We have

$$A_{21} = 8 \times f(4) = 8 \times 17 = 136 \qquad \text{and} \qquad A_{22} = 8 \times f(12) = 8 \times 145 = 1\,200.$$

So, the total area of these two rectangles is

$$A_2 = A_{21} + A_{22} = 136 + 1\,200 = 1\,336.$$

Previously, our approximation error was $e_1 = 341\frac{1}{3}$. Our new approximation error is smaller:

$$e_2 = \left| \int_0^{16} f - A_2 \right| = \left| 1\,381\frac{1}{3} - 1\,336 \right| = 45\frac{1}{3}.$$

(*Example continues on the next page ...*)

(*... Example continued from the previous page.*)

Figure to be
inserted here.

For our third crude approximation, we construct *four* rectangles, each with width 4, but whose heights are

$$f(2) = 2^2 + 1 = 5,\ f(6) = 6^2 + 1 = 37,\ f(10) = 10^2 + 1 = 101, \text{ and } f(14) = 14^2 + 1 = 197.$$

Call their areas A_{41}, A_{42}, A_{43}, and A_{44}. We have

$$A_{41} = 4 \times f(2) = 20,\ A_{42} = 4 \times f(6) = 148,\ A_{43} = 4 \times f(10) = 404, \text{ and}$$
$$A_{44} = 4 \times f(14) = 788.$$

So, the total area of these four rectangles is

$$A_4 = A_{41} + A_{42} + A_{43} + A_{44} = 20 + 148 + 404 + 788 = 1\,360.$$

Our previous two approximation errors were $e_1 = 341\frac{1}{3}$ and $e_2 = 45\frac{1}{3}$. Our new approximation error is smaller:

$$e_4 = \left| \int_0^{16} f - A_3 \right| = \left| 1\,381\frac{1}{3} - 1\,360 \right| = 21\frac{1}{3}.$$

As you can probably tell, we can use ever more rectangles to find ever better approximations for $\int_0^{16} f$. Exercise 412 continues with this example.

Exercise 412. Continue to define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$. Approximate $\int_0^{16} f$ using *eight* rectangles, each with width 2. Let A_8 be the total area of these eight rectangles and e_8 be the corresponding approximation error. Is this fourth approximation an improvement over our previous three approximations? (Answer on p. 1930.)

Sketched in the above example and exercise is the main idea underlying **integration**:

Use thin rectangles to approximate the area under a curve.

With more (and thinner) rectangles, our approximations improve.

Figure to be
inserted here.

More precisely, integration (which is the problem of finding the area under a curve) involves these steps:

Suppose we want to find $\int_a^b f$, the definite integral of f from a to b.

1. Partition the interval $[a, b]$ into n rectangles of equal width. Each rectangle's height is the value taken by the function at the rectangle's midpoint.[489]
2. Let the total area of these n rectangles be A_n.[490] We observe that A_n serves as an approximation for $\int_a^b f$.
3. As n (the number of rectangles) increases, A_n becomes an ever better approximation for $\int_a^b f$.
4. Indeed, it is possible to prove that as $n \to \infty$, $A_n \to \int_a^b f$, or equivalently,

$$\lim_{n\to\infty} A_n = \int_a^b f.$$

It turns out that formally, we shall simply *define* $\int_a^b f$—the definite integral of f from a to b—to be $\lim_{n\to\infty} A_n$.

But don't worry. For H2 Maths, these are technical details that you needn't worry about. [491] The above informal and intuitive explanation of how integration works should more than suffice. (For a recent A-Level exam question that asks for such an explanation, see N2015-I-3.)

[489]Specifically, each rectangle has width $\frac{b-a}{n}$ and the ith rectangle has height $f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{n}\right)$ (for $i = 1, 2, \ldots, n$). Hence, the ith rectangle has area $\frac{b-a}{n} f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{n}\right)$.

[490]$A_n = \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{n}\right)$.

[491]But see Ch. 146.22 (Appendices) if you're interested.

104.5. If a Function Is Continuous, Then Its Definite Integral Exists

Figure to be inserted here.

Formally,

Theorem 42. *Let $a < b$. If the function f is continuous on $[a, b]$, then $\int_a^b f$ exists.*

Proof. See p. 1706 (Appendices). □

104.6. Rules of Integration

Theorem 43. (Rules of Integration) *Let $a < b$, $c \in (a,b)$, and $d, e \in \mathbb{R}$. Suppose the functions f and g are continuous on $[a,b]$, so that by Theorem 42, $\int_a^b f$, $\int_a^b g$, $\int_a^c f$, and $\int_c^b f$ exist. Then*

(a) $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$. **(Sum and Difference Rules)**

(b) $\int_a^b f = \int_a^c f + \int_c^b f$. **(Adjacent Intervals Rule)**

(c) $\int_a^b (df) = d \int_a^b f$. **(Constant Factor Rule)**

(d) $\int_a^b d = (b-a)\,d$. **(Constant Rule)**

(e) *If $f \geq g$ on $[a,b]$, then* $\int_a^b f \geq \int_a^b g$. **(Comparison Rule I)**

(f) *If $d \leq f(x) \leq e$ for every $x \in [a,b]$, then*

$$(b-a)\,d \leq \int_a^b f \leq (b-a)\,e.$$

(Comparison Rule II)

Proof. For the formal proofs, see p. 1707 (Appendices). Here we give only informal "proofs"-by-picture:

(a) Consider the function $h = f + g$. "Clearly", the area under the graph of h must be equal to the sum of the areas under the graphs of f and g.

Figure to be
inserted here.

Similarly, consider the function $i = f - g$. "Clearly", the area under the graph of i must be equal to the difference of the areas under the graphs of f and g.

(b) "Obviously", $\int_a^b f = \int_a^c f + \int_c^b f$.[492]

[492]While this may seem intuitively "obvious", formally proving it is somewhat less so.

Figure to be inserted here.

(c) Stretch the graph of f upwards by a factor d. The area under the new graph must be d times the area under the graph of f.

Figure to be inserted here.

(d) "Clearly", $\int_a^b c$ is simply the area of a rectangle with base $b-a$ and height c. So $\int_a^b c = (b-a)\,c$.

Figure to be inserted here.

(e) If f is everywhere on or above g, then the area under f cannot be smaller than that under g.

Figure to be inserted here.

(f) The numbers c and d serve as lower and upper bounds for f on the relevant interval (a,b). And so "obviously", $\int_a^b f$, the area under the graph of f from a to b, is bounded from below and above by the rectangles with base $b-a$ and heights c and d.

Figure to be inserted here.

Actually, it's not difficult to formally prove **(f)** and you are asked to do so in Exercise XXX. □

Exercise 413. This exercise will guide you through a proof of Theorem 43(f) (Comparison Rule II).

Let $d, e \in \mathbb{R}$. Define the functions $h, i : [a, b] \to \mathbb{R}$ by $h(x) = d$ and $i(x) = e$.

(a) What are $\int_a^b h$ and $\int_a^b i$?

(b) What can we say about $\int_a^b f$, $\int_a^b h$, and $\int_a^b i$?

(c) Hence complete the proof of Comparison Rule II. (Answer on p. 1930.)

Exercise 414. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $f \geq 0$ on $[a, b]$. Prove that $\int_a^b f \geq 0$. (Hint: Define $h : [a, b] \to \mathbb{R}$ by $h(x) = 0$.) (Answer on p. 1930.)

Exercise 415. The Constant Factor Rule of Antidifferentiation (Theorem 40) states that

$$\int kf(x)\,\mathrm{d}x = kF(x) + C.$$

The Constant Factor Rule of Integration (Theorem 43) states that

$$\int_a^b (df) = d\int_a^b f.$$

Are these two Rules the exact same thing? Why or why not? (Answer on p. 1930.)

104.7. Term-by-Term Integration (optional)

In Ch. 91.3, we discussed **term-by-term differentiation**.

In this subchapter, we will analogously discuss **term-by-term integration**.

Let $a < b$ and $f_0, f_1, f_2, \ldots$ be nice functions that are continuous on $[a,b]$.

By Theorem 42, $\int_a^b f_0$, $\int_a^b f_1$, $\int_a^b f_2$, ... exist (i.e. are equal to real numbers).

Suppose we define a new function g by

$$g = f_0 + f_1.$$

Then by the Sum Rule (for Integration), $\int_a^b g$ exists, with

$$\int_a^b g = \int_a^b f_0 + \int_a^b f_1.$$

Similarly, suppose we define a new function h by

$$h = f_0 + f_1 + f_2.$$

Then again by the Sum Rule, $\int_a^b h$ exists, with

$$\int_a^b h = \int_a^b f_0 + \int_a^b f_1 + \int_a^b f_2.$$

More generally, for any $n \in \mathbb{Z}^+$, suppose we define a new function i by

$$i = \sum_{i=0}^{n} f_i = f_0 + f_1 + \cdots + f_n.$$

Then by the Sum Rule, $\int_a^b i$ exists, with

$$\int_a^b i = \int_a^b \sum_{i=0}^{n} f_i \stackrel{1}{=} \sum_{i=0}^{n} \int_a^b f_i = \int_a^b f_0 + \int_a^b f_1 + \cdots + \int_a^b f_n.$$

In $\stackrel{1}{=}$, we

$$\text{interchange integration } \int_a^b \text{ and } \textbf{finite} \text{ summation } \sum_{i=0}^{n}.$$

We see that if a function i is the **finite** sum of n other continuous functions defined on $[a,b]$, then $\int_a^b i$ also exists and is equal to the sum of those n functions' definite integrals from a to b.

Now, suppose instead we define a new function j by

$$j = \sum_{i=0}^{\infty} f_i = f_0 + f_1 + f_2 + \ldots.$$

Q1. Does $\int_a^b j$ exist?

Q2. Is it the case that

$$\int_a^b j = \int_a^b \sum_{i=0}^{\infty} f_i \stackrel{2}{=} \sum_{i=0}^{\infty} \int_a^b f_i = \int_a^b f_0 + \int_a^b f_1 + \int_a^b f_2 + \ldots ?$$

In particular, is $\stackrel{2}{=}$ legitimate? That is, can we

$$\text{interchange integration } \frac{\mathrm{d}}{\mathrm{d}x} \text{ and } \textbf{infinite} \text{ summation } \sum_{i=0}^{\infty}?$$

Is this interchange operation legitimate?

It turns out that the answer to Q1 and Q2 is,

Yes, but only if certain technical conditions are met.

In particular, *if* certain technical conditions are met, *then*, as was done in $\stackrel{2}{=}$, we can

$$\text{interchange differentiation } \frac{\mathrm{d}}{\mathrm{d}x} \text{ and } \textbf{infinite} \text{ summation } \sum_{i=0}^{\infty}.$$

Fortunately, these "certain technical conditions" are beyond the scope of H2 Maths and we shall not discuss them in this textbook. Based on past-year A-Level exams, it would seem that your A-Level examiners simply *assume* that the interchange operation is always legitimate—so, at least for H2 Maths, there is probably little danger if you also do likewise.

In Ch. 108, we'll revisit the questions just asked, but in the special case where j is an "infinite polynomial function".

Exercise 416. This exercise guides you through a counterexample showing that we cannot always interchange integration $\int_a^b$ and **infinite** summation $\sum_{i=0}^{\infty}$. (Answer on p. 1062.)

A416.

105. The Fundamental Theorems of Calculus

105.1. Definite Integral Functions

So far, we've only defined an object denoted $\int_a^b f$ and called the definite integral of f from a to b. This object, $\int_a^b f$, is a **real number**.

We now use this number to define a **function** that we call the **definite integral function**:

Example 1344. Define $f:[0,8]\to\mathbb{R}$ by $f(x)=3x^2+2$.

Figure to be
inserted here.

The **definite integral function of** f **from** 1 is the function $g:[0,8]\to\mathbb{R}$ defined by

$$g(x)=\int_1^x f.$$

So for example, $g(3)=\int_1^3 f$, $g(5)=\int_1^5 f$, and $g(0)=\int_1^0 f=-\int_0^1 f$ are the areas depicted above.

Example 1345. Define $f:[-5,7]\to\mathbb{R}$ by $f(x)=2x+1$.

Figure to be
inserted here.

The **definite integral function of f from 0** is the function $g:[-5,7]\to\mathbb{R}$ defined by

$$g(x)=\int_0^x f.$$

So for example, $g(1)=\int_0^1 f=1$, $g(3)=\int_0^3 f=12$, and $g(5)=\int_0^5 f=30$. Also,

$$g(0)=\int_0^0 f=0,\qquad g(-1)=\int_0^{-1} f=-\int_{-1}^0 f=-1,\quad \text{and}$$
$$g(-2)=\int_0^{-2} f=-\int_{-2}^0 f=30.$$

Definition 226. Let D be a closed interval,[493] $f:D\to\mathbb{R}$ be a continuous function, and $c\in D$. The *definite integral function of f from c* is the function $g:D\to\mathbb{R}$ defined by

$$g(x)=\int_c^x f.$$

Remark 163. The term **definite integral function** is not standard.[494]

Exercise 417. Define $f:[-2,5]\to\mathbb{R}$ by $f(x)=2x+1$. Let h be the definite integral function of f from 2 and i be the definite integral of f from 2 to 3.

(a) What is h?

(b) Evaluate h at 3, 5, 0, and -2.

(c) What is i?

(d) Evaluate i at 3, 5, 0, and -2. (Answer on p. 1932.)

[493] Somewhat strangely, we include $\mathbb{R}=(-\infty,\infty)$ as a closed interval. Also, if $a\in\mathbb{R}$, then $(-\infty,a]$ and $[a,\infty)$ are also included as closed intervals. You can think of these as conventions that we'll simply follow. (Formally, an interval or set is closed if it includes its limit points.)

[494] Unfortunately there does not seem to be any standard name for this object. This object is however key to understanding *why* the FTC1 is true, which is why I decided it should be given a name, even if that name is non-standard.

Example 1346. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$.[495]

The definite integral function of f from 0 is the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \int_0^x f.$$

Now, our goal is to

Solve the problem of integration.

That is, Find the area under the graph of f

Or equivalently, Find a general expression for $g(x)$.

Instead of approaching this goal directly, we'll take a strange and indirect approach. We'll pose a seemingly unrelated question:

What is the derivative of g? ☺

This is a strange question to ask. Right now, we don't even know what g is—that is, we don't even know what a general expression for $g(x)$ is. So, how could we possibly know what the derivative of g is? This question is thus strange—it is like asking someone who has no idea where Singapore is to locate Sentosa.

Let us nonetheless see where the above question might lead us. For concreteness, let's consider some arbitrary point, say, 4. We'll try to find $g'(4)$, the derivative of g at 4.

Figure to be
inserted here.

In the above figure, we've picked some point c that is "near" 4.

The red area is $g(c) - g(4)$, while the blue area is $(c-4) f(c)$. Observe that these red and blue areas are approximately equal. That is,

$$g(c) - g(4) \approx (c-4) f(4) \qquad \text{or} \qquad \frac{g(c) - g(4)}{c-4} \overset{1}{\approx} f(4).$$

The closer c is to 4, the better the approximation $\overset{1}{\approx}$. Hence,[496]

$$\lim_{c \to 4} \frac{g(c) - g(4)}{c - 4} = f(4).$$

Or equivalently,

$$g'(4) \overset{1}{=} f(4).$$

(... *Example continued from the previous page.*)

Figure to be
inserted here.

Here's a second way to see why $\overset{1}{=}$ is true:

Recall that informally and intuitively, g', the derivative of g, answers this question:

Given a small unit increase in x, by how much does g increase?

Well, we know that g measures the area under the graph of f. So, when x increases by a small unit, we'd expect $g(x)$ to increase by approximately $f(x)$ (see above figure). So, at the point 4 in particular, we should have

$$g'(4) \overset{1}{=} f(4).$$

We've just shown that $g'(4) \overset{1}{=} f(4)$. But since the point 4 was arbitrarily chosen, the foregoing argument should also work for any other point $x \in \mathbb{R}$. That is, for any $x \in \mathbb{R}$, we also have

$$g'(x) = f(x).$$

In other words, $g' = f$.

In ☺, we asked, "What is the derivative of g?" We now have an answer:

The derivative of g is f.

This answer is also exactly what the **First Fundamental Theorem of Calculus (FTC1)** says:

Very loosely, the **First Fundamental Theorem of Calculus (FTC1)** says that definite integrals are also antiderivatives. A bit more precisely, if g is a definite integral function of f, then g is also an antiderivative of f. Most precisely,

[496]Here we are hand-waving.

Theorem 44. (First Fundamental Theorem of Calculus, FTC1) *Let $a < b$, $f : [a,b] \to \mathbb{R}$ be a continuous function, and $c \in [a,b]$. Suppose the function $g : [a,b] \to \mathbb{R}$ is defined by*

$$g(x) = \int_c^x f.$$

Then $g' = f$ (i.e. the derivative of g is f or equivalently, g is an antiderivative of f).

Proof. The above example already discussed the intuition behind the FTC1. For a formal proof, see p. 1714 (Appendices). □

Remark 164. The FTC1 establishes that

Integration and antidifferentiation are the "same thing".

Equivalently, integration and differentiation are inverse operations.

Again, we must stress, emphasise, and repeat that **this is a genuinely surprising result** that you should **not** take for granted. In particular, it is **wrong** to believe that integration is *by definition* the inverse of differentiation.

Let's illustrate the FTC1 with a familiar example from physics:

Example 1347. A car is moving. Below we graph its **velocity** v ($\mathrm{m\,s^{-1}}$) as a function of **time** t (s).

Figure to be
inserted here.

Recall that the **distance** d (m) travelled by the car is the area under the graph.

- For example, after 5 s, the distance travelled by the car is $\int_0^5 v$.
- And after 8 s, the distance travelled by the car is $\int_0^8 v$.
- In general, after x s, the distance travelled by the car is $\int_0^x v$.

But we already know that

The derivative of distance (with respect to time) is velocity.

That is,
$$d' = v.$$

What we've just shown is precisely the FTC1:

The derivative of the area under a function's graph is the function itself.

Exercise 418. This exercise guides you through a proof of Theorem 41: Every continuous function is antidifferentiable. (Answer on p. 1932.)

Let $a < b$, $f : [a, b] \to \mathbb{R}$ be a continuous function, and $c \in [a, b]$. Let g be the definite integral of f from c.

(a) What is g?

(b) What does the FTC1 say about f and g?

(c) Hence, we have shown that f has ________, namely ________ (*fill in the blanks*).

105.3. The Second Fundamental Theorem of Calculus (FTC2)

Theorem 45. (Second Fundamental Theorem of Calculus, FTC2) *Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose g is an antiderivative of f. Then*

$$\int_a^b f = g(b) - g(a).$$

Proof. See Exercise 419. □

Exercise 419. This exercise guides you through a proof of the FTC2.

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function. suppose g is an antiderivative of f.

Define the function $h : [a, b] \to \mathbb{R}$ by $h(x) = \int_a^x f$.

(a) What does the FTC1 say about h and f?

(b) So, how are h and g related?

(c) Now prove the FTC2 by considering $g(b) - g(a)$. (Answer on p. 1932.)

The FTC2 gives us a powerful tool for solving the problem of integration (i.e. the problem of finding the area under a curve):

Three-Step Integration Recipe

Let $a < b$ and f be a nice function that is continuous on $[a, b]$. To find $\int_a^b f$,

1. Find *any* antiderivative g of f.
2. Plug the upper limit b and lower limit a into g—that is, evaluate $g(b)$ and $g(a)$.
3. Conclude: $\int_a^b f = g(b) - g(a)$.

Example 1348. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Find $\int_0^3 f$.

Figure to be inserted here.

Use the Three-Step Integration Recipe:

1. Find *any* antiderivative g of f.

One such antiderivative is $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{1}{3}x^3$.

2. Plug the upper limit 3 and lower limits 0 into g:

$$g(3) = \frac{1}{3} \cdot 3^3 = 9 \qquad \text{and} \qquad g(0) = \frac{1}{3} \cdot 0^3 = 0.$$

3. Conclude: $\int_0^3 f = g(3) - g(0) = 9 - 0 = 9.$

Example 1349. XXX

Example 1350. XXX

Exercise 420. Use the Three-Step Integration Recipe to evaluate each definite integral. (Answer on p. 1071.)

(a) xxx

(b) xxx

A420(a) xxx

(b) xxx

105.4. Why Integration and Antidifferentiation Use the Same Notation

To repeat,

Differentiation is the problem of finding the **gradient** of a curve.

Integration is the problem of finding the **area** under a curve.

Figure to be
inserted here.

A priori,[497] there is no reason to think that the area under a curve is in any way related to its gradient. Equivalently, *a priori*, there is no reason to think that integration is in any way related to differentiation (or antidifferentiation).

That they *are* in fact related was only just established by the two FTCs:

- The FTC1 says that any definite integral function of f is an antiderivative of f. And so, Loosely speaking, integration and antidifferentiation are the "same thing". Or equivalently, integration and differentiation are inverse operations.
- The FTC2 says that if g is *any* antiderivative of f, then

$$\int_a^b f = g(b) - g(a).$$

  And so, to compute definite integrals, we can use the Three-Step Integration Recipe.

Earlier, we were puzzled by why we made the symbol $\int$ do **double duty**, i.e. why $\int$ serves as both the **antidifferentiation symbol** and the **integration symbol**. We can now solve this puzzle:

By the FTCs, antidifferentiation and integration are in fact the "same thing". And so, it makes (some) sense to use the same symbol $\int$ fot both antidifferentiation and integration.

We now also see why $\int f(x)\,\mathrm{d}x$ is sometimes called the **indefinite integral**:

- $\int f(x)\,\mathrm{d}x$ tells us about the antiderivatives of f. But since antidifferentiation and integration are in fact the "same thing", it makes (some) sense to also call $\int f(x)\,\mathrm{d}x$ (or indeed any antiderivative) an *integral*.
- $\int f(x)\,\mathrm{d}x$ is *indefinite* in the sense that it can vary by up to a constant (i.e. the constant of integration, C).

[497] *A priori* is just a fancy Latin phrase for *beforehand*.

These though are mostly matters of historical convention. In my opinion, it is better to avoid using the term *indefinite integral*—and this is exactly what I've done in this textbook.

It would also be better if there were simply a different symbol for antidifferentiation. By making the symbol $\int$ do double duty, generations of students have been confused into thinking that

- Integration is *by definition* the same thing as antidifferentiation. ✗
- And hence, the FTCs have no substance. ✗

I hope that in this textbook, we have left absolutely no room for confusion. In particular, we've repeatedly emphasised these important points:

- Differentiation is the problem of finding the gradient of a curve.
- Integration is the problem of finding the area under a curve.
- Antidifferentiation is the inverse of differentiation.
- *A priori*, there is no reason to believe that integration is in any way related to differentiation (or antidifferentiation).
- The FTCs establish that—remarkably enough—integration is the "same thing" as antidifferentiation.
- (Which is why historical convention has us using the exact symbol $\int$ for both integration and antidifferentiation.)

Exercise 421. Let $a < b$ and f be a function that is continuous on $[a, b]$. Explain what $\int f(x)\,\mathrm{d}x$, $\int_a^b f(x)\,\mathrm{d}x$, and $\int_c^x f(x)\,\mathrm{d}x$ are and why they are different. (Answer on p. 1073.)

A421.

105.5. Some New Notation and More Examples and Exercises

Let f be a nice function whose domain contains the points a and b. Then we introduce this new piece of notation:

$$[f(x)]_a^b = f(b) - f(a).$$

Example 1351. XXX

Instead of $[f(x)]_a^b$, we will sometimes also more simply write $[f]_a^b$. That is,

$$[f]_a^b = [f(x)]_a^b = f(b) - f(a).$$

Example 1352. XXX

This notation we've just introduced is particularly convenient when using the Three-Step Integration Recipe:

Example 1353. XXX

Example 1354. XXX

Remark 165. Just so you know, instead of $[f(x)]_a^b$, some other writers may instead write $f(x)\big]_a^b$ or $f(x)\Big|_a^b$.

Many more exercises for you to practise the Three-Step Integration Recipe and make use of the new notation just introduced:

Exercise 422. XXX (Answer on p. 1074.)

A422.

106. More Techniques of Antidifferentiation

To find the area under the graph of a function f, we will often use the Three-Step Integration Recipe, in which the only step that might be challenging is the first: Find an antiderivative of f.

This is the reason why it's so important to know how to find antiderivatives. In Ch. 103, we already learnt several rules and techniques of antidifferentiation. In this chapter and the next, we'll learn more.

106.1. Factorisation

Example 1355. Find $\int \frac{1}{x^2+2x+1}\,\mathrm{d}x$ (for $x^2+2x+1 \neq 0$).

Looks tricky. But observe that $x^2+2x+1 = (x+1)^2$. And so, by the Power and LPC Rules of Integration (Theorem 41),

$$\int \frac{1}{x^2+2x+1}\,\mathrm{d}x = \int \frac{1}{(x+1)^2}\,\mathrm{d}x = -\frac{1}{x+1} + C.$$

Example 1356. Find $\int \frac{1}{x^3+3x^2+3x+1}\,\mathrm{d}x$ (for $x^3+3x^2+3x+1 \neq 0$).

Observe that $x^2+3x^2+3x+1 = (x+1)^3$. So,

$$\int \frac{1}{x^3+3x^2+3x+1}\,\mathrm{d}x = \int \frac{1}{(x+1)^3}\,\mathrm{d}x = -\frac{1}{2}\frac{1}{(x+1)^2} + C.$$

Exercise 423. Find the following. (Answer on p. 1933.)

(a) $\int \frac{1}{4x^2-4x+1}\,\mathrm{d}x$ (for $4x^2-4x+1 \neq 0$). **(b)** $\int \frac{1}{9x^2+30x+25}\,\mathrm{d}x$ (for $9x^2 + 30x$

106.2. Partial Fractions: Finding $\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x$ where $b^2-4ac>0$

In the last subchapter, we learnt to find $\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x$ in those cases where $b^2-4ac=0$, so that ax^2+bx+c was a **perfect square**.

We now learn to also find $\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x$ in those cases where $b^2-4ac>0$. In such cases, ax^2+bx+c is no longer a perfect square but is still **factorisable**. So, this is really just more factorisation, except that now we'll also make use of **partial fractions** (Ch. **??**):

Example 1357. Find $\int \frac{1}{x^2-1}\,\mathrm{d}x$ (for $x^2-1\neq 0$).

We observe that $x^2-1=(x+1)(x-1)$ and so write

$$\frac{1}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1}=\frac{A(x-1)+B(x+1)}{(x+1)(x-1)}=\frac{(A+B)x-A+B}{x^2-1}.$$

Comparing coefficients, we have $A+B=0$ and $-A+B=1$. Solving, we have $A=-1/2$ and $B=1/2$. So,

$$\begin{aligned}
\int \frac{1}{x^2-1}\,\mathrm{d}x &= \int \frac{-1/2}{x+1}+\frac{1/2}{x-1}\,\mathrm{d}x \\
&= \int \frac{-1/2}{x+1}\,\mathrm{d}x+\int \frac{1/2}{x-1}\,\mathrm{d}x && \text{(Sum Rule)} \\
&= -\frac{1}{2}\int \frac{1}{x+1}\,\mathrm{d}x+\frac{1}{2}\int \frac{1}{x-1}\,\mathrm{d}x && \text{(Constant Factor Rule)} \\
&\overset{\varnothing\star}{=} -\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C && \text{(Reciprocal and LPC Rules)} \\
&= \frac{1}{2}(\ln|x-1|-\ln|x+1|)+C \\
&= \frac{1}{2}\ln\frac{|x-1|}{|x+1|}+C && \text{(Law of Logarithm)} \\
&= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C. && \text{(Fact 10)}
\end{aligned}$$

Note: It would've been perfectly fine to leave our answer at $\varnothing\star$. The last three steps were nice but not necessary.

Example 1358. Find $\int \frac{1}{x^2+x-6}\,\mathrm{d}x$ (for $x^2+x-6\neq 0$).

Observing that $x^2+x-6=(x+3)(x-2)$, write

$$\frac{1}{x^2+x-6}=\frac{A}{x+3}+\frac{B}{x-2}=\frac{A(x-2)+B(x+3)}{(x+3)(x-2)}=\frac{(A+B)x+3B-2A}{x^2+x-6}.$$

Comparing coefficients, we have $A+B=0$ and $3B-2A=1$. Solving, we have $A=-1/5$ and $B=1/5$. Thus,

$$\begin{aligned}
\int \frac{1}{x^2+x-6}\,\mathrm{d}x &= \int \frac{-1/5}{x+3}+\frac{1/5}{x-2}\,\mathrm{d}x \\
&= \int \frac{-1/5}{x+3}\,\mathrm{d}x+\int \frac{1/5}{x-2}\,\mathrm{d}x && \text{(Sum Rule)} \\
&= -\frac{1}{5}\int \frac{1}{x+3}\,\mathrm{d}x+\frac{1}{5}\int \frac{1}{x-2}\,\mathrm{d}x && \text{(Constant Rule)} \\
&\varnothing\star -\frac{1}{5}\ln|x+3|+\frac{1}{5}\ln|x-2|+C && \text{(Reciprocal and LPC Rules)} \\
&= \frac{1}{5}\left(\ln|x-2|-\ln|x+3|\right)+C \\
&= \frac{1}{5}\ln\frac{|x-2|}{|x+3|}+C && \text{(Law of Logarithm)} \\
&= \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|+C. && \text{(Fact 10)}
\end{aligned}$$

Again, $\varnothing\star$ would've sufficed as our answer.

Exercise 424. Find the following. (Answer on p. 1933.)

(a) $\int \frac{1}{5x^2-2x-3}\,\mathrm{d}x$ (for $5x^2-2x-3\neq 0$).

(b) $\int \frac{1}{x^2-a^2}\,\mathrm{d}x$ (for $x^2-a^2\neq 0$).

(c) $\int \frac{1}{a^2-x^2}\,\mathrm{d}x$ (for $a^2-x^2\neq 0$).

Example 1359. Find $\int \frac{x}{x^2+2x+1}\,\mathrm{d}x$ (assume $x^2+2x+1 \neq 0$).

Observe that $x^2+2x+1=(x+1)^2$. So,

$$\begin{aligned}
\int \frac{x}{x^2+2x+1}\,\mathrm{d}x &= \int \frac{x}{(x+1)^2}\,\mathrm{d}x \\
&= \int \frac{x+1-1}{(x+1)^2}\,\mathrm{d}x && \text{(Plus Zero Trick)} \\
&= \int \frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2}\,\mathrm{d}x \\
&= \int \frac{1}{x+1} - \frac{1}{(x+1)^2}\,\mathrm{d}x \\
&= \int \frac{1}{x+1}\,\mathrm{d}x - \int \frac{1}{(x+1)^2}\,\mathrm{d}x. && \text{(Difference Rule)} \\
&= \ln|x+1| + \frac{1}{x+1} + C. && \text{(Reciprocal, Power, and LPC Rules)}
\end{aligned}$$

Example 1360. Find $\int \frac{x}{x^3 - 3x^2 + 3x - 1} \, \mathrm{d}x$ (assume $x^3 - 3x^2 + 3x - 1 \neq 0$).

Observe that $x^3 - 3x^2 + 3x - 1 = (x-1)^3$. So,

$$
\begin{aligned}
\int \frac{x}{x^3 - 3x^2 + 3x - 1} \, \mathrm{d}x &= \int \frac{x}{(x-1)^3} \, \mathrm{d}x \\
&= \int \frac{x - 1 + 1}{(x-1)^3} \, \mathrm{d}x && \text{(Plus Zero Trick)} \\
&= \int \frac{x-1}{(x-1)^3} + \frac{1}{(x-1)^3} \, \mathrm{d}x \\
&= \int \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} \, \mathrm{d}x \\
&= \int \frac{1}{(x-1)^2} \, \mathrm{d}x + \int \frac{1}{(x-1)^3} \, \mathrm{d}x. && \text{(Sum Rule)} \\
&= -\frac{1}{x-1} - \frac{1}{2} \frac{1}{(x-1)^2} + C && \text{(Power and LPC Rules)} \\
&= -\frac{2x-1}{2(x-1)^2} + C.
\end{aligned}
$$

The last step is nice but not necessary.

Exercise 425. Find the following. (Answer on p. 1934.)

(a) $\int \frac{7x+2}{4x^2 - 4x + 1} \, \mathrm{d}x$ (for $4x^2 - 4x + 1 \neq 0$).

(b) $\int \frac{7x+2}{x^2 + x - 6} \, \mathrm{d}x$ (for $x^2 + x - 6 \neq 0$; Hint: Use Example 1358).

(c) $\int \frac{7x+2}{5x^2 - 2x - 3} \, \mathrm{d}x$ (for $5x^2 - 2x - 3 \neq 0$; Hint: Use your answer from Exercise 424(a).)

106.4. More Rules of Antidifferentiation

The following rules of antidifferentiation are in List MF26 (p. 4),[498] so no need to *mug*:

Proposition 16. *Let $a \neq 0$.*

(a) $\displaystyle\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(b) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{|a|} + C,$ *for* $|x| < |a|$

(c) $\displaystyle\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C,$ *for* $x \neq \pm a$

(d) $\displaystyle\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C,$ *for* $x \neq \pm a$

(e) $\displaystyle\int \tan x\,\mathrm{d}x = \ln|\sec x| + C,$ *for x not an odd multiple of* $\dfrac{\pi}{2}$

(f) $\displaystyle\int \cot x\,\mathrm{d}x = \ln|\sin x| + C,$ *for x not a multiple of* π

(g) $\displaystyle\int \operatorname{cosec} x\,\mathrm{d}x = -\ln|\operatorname{cosec} x + \cot x| + C,$ *for x not a multiple of* π

(h) $\displaystyle\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + C,$ *for x not an odd multiple of* $\dfrac{\pi}{2}$

Proof. By Fact 215, to prove that $\displaystyle\int f(x)\,\mathrm{d}x = F(x) + C$, show that $\dfrac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.

(a) By Fact 207(c), $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \dfrac{1}{x^2+1}$. So,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{a}\frac{1}{\left(\frac{x}{a}\right)^2+1}\cdot\frac{1}{a} = \frac{1}{x^2+a^2}.$$

(b) By Fact 207(a), $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$ for $x \in (-1,1)$.

So, for $\dfrac{x}{|a|} \in (-1,1)$ or $x \in (-|a|,|a|)$ or $|x| < |a|$, if $a \geq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\frac{x}{|a|} = \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\frac{1}{a} = \frac{a}{\sqrt{a^2-x^2}}\frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}.$$

And if $a < 0$, then $\sqrt{a^2} \stackrel{1}{=} -a$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\frac{x}{|a|} = \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\frac{x}{-a} = \frac{1}{\sqrt{1-\left(\frac{x}{-a}\right)^2}}\left(-\frac{1}{a}\right) \stackrel{1}{=} \frac{-a}{\sqrt{a^2-x^2}}\left(-\frac{1}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}.$$

[498] Proposition 16 is slightly more general than what's on List MF26. Specifically, it allows for $a < 0$ and in **(b)**–**(h)**, a wider range of values of x.

For **(c)** and **(d)**, see Exercise 424.

For **(e)**–**(h)**, see Exercise **426**. (We'll also prove **(e)** again in Example 1390.) □

Exercise 426. Prove Proposition 16**(e)**–**(h)**. (Hint: In each, you'll have to examine two cases, similar to **(b)**.) (Answers on p. **1935**)

106.5. Completing the Square: $\int \frac{1}{ax^2+bx+c}\,dx$ where $b^2-4ac<0$

Find
$$\int \frac{1}{ax^2+bx+c}\,dx.$$

In Chs. 106.2, we learnt to solve the above problem in those cases where $b^2-4ac=0$ or $b^2-4ac>0$.

We now learn to also solve the above problem in those cases where $b^2-4ac<0$. It turns out that in these cases, the trick is to **complete the square**, so that we can make use of Proposition 16(a):

$$\int \frac{1}{x^2+a^2}\,dx \varnothing\star \frac{1}{a}\tan^{-1}\frac{x}{a}+C.$$

Example 1361. Find $\int \frac{1}{x^2+x+1}\,dx$.

Complete the square: $$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}.$$

So, $$\int \frac{1}{x^2+x+1}\,dx=\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx.$$

Now, let $x+1/4$ and $\sqrt{3/4}$ take the places of "x" and "a" in $\varnothing\star$. Then

$$\begin{aligned}\int \frac{1}{x^2+x+1}\,dx &= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx\\ &\varnothing\star \frac{1}{\sqrt{3/4}}\tan^{-1}\frac{x+\frac{1}{2}}{\sqrt{3/4}}+C\\ &= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C.\end{aligned}$$

In Ch. ??, we reviewed how to **complete the square**. In general,

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}.$$

But rather than to try memorise the above general formula, it's probably easier to try to understand and thus easily "see" how you can complete the square in each specific case.

Example 1362. Find $\displaystyle\int \frac{1}{2x^2+3x+5}\,\mathrm{d}x$.

Let's first rewrite the integrand so that the polynomial in the denominator has leading coefficient 1:

$$\int \frac{1}{2x^2+3x+5}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{x^2+1.5x+2.5}\,\mathrm{d}x.$$

Complete the square: $\qquad x^2+1.5x+2.5 = \left(x+\frac{3}{4}\right)^2 + \frac{31}{16}.$

Let $x+3/4$ and $\sqrt{31/16}$ take the places of "x" and "a" in $\varnothing\star$. Then

$$\begin{aligned}\int \frac{1}{2x^2+3x+5}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{4}\right)^2+\frac{31}{16}}\,\mathrm{d}x \\ &\overset{\varnothing\star}{=} \frac{1}{2}\frac{1}{\sqrt{31/16}}\tan^{-1}\frac{x+3/4}{\sqrt{31/16}}+C \\ &= \frac{2}{\sqrt{31}}\tan^{-1}\frac{4x+3}{\sqrt{31}}+C.\end{aligned}$$

Remark 166. In Ch. 107, we'll learn another technique for finding the antiderivative covered in this subchapter.

In Ch. 106.1, Ch. 106.2, and this subchapter, we've learnt to find

$$T = \int \frac{1}{ax^2+bx+c}\,\mathrm{d}x$$

in all three possible cases ($b^2-4ac=0$, $b^2-4ac>0$, and $b^2-4ac<0$). Fact 216 summarises what we've learnt. It looks intimidating, but is really just what we've been doing, except that now we have a, b, and c instead of actual numbers.

Fact 216. *Let $a,b,c\in\mathbb{R}$ with $a\neq 0$ and $ax^2+bx+c\neq 0$. If $d=\sqrt{|b^2-4ac|}$, then*

$$\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x = \begin{cases} \dfrac{1}{d}\ln\left|\dfrac{x+\frac{b-d}{2a}}{x+\frac{b+d}{2a}}\right|+C, & \text{for } b^2-4ac>0, \\ \dfrac{-2}{2ax+b}+C, & \text{for } b^2-4ac=0, \\ \dfrac{2}{d}\tan^{-1}\dfrac{2ax+b}{d}+C, & \text{for } b^2-4ac<0. \end{cases}$$

Proof. See p. 1716 (Appendices). □

Remark 167. Fact 216 is for your reference only. Rather than try to memorise this result, it is probably wiser to understand (and hence be able to execute) the steps that produce it.

Exercise 427. Find the following. (Answer on p. 1084.)

(a) xxx

(b) xxx

A427.

106.6. $\displaystyle\int \frac{1}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x$ in the Special Case where $a<0$

Example 1363. Find $\displaystyle\int \frac{1}{\sqrt{-x^2+x+1}}\,\mathrm{d}x$ (for $-x^2+x+1>0$).

Complete the square:$-x^2+x+1 = \dfrac{5}{4} - \left(x-\dfrac{1}{2}\right)^2 = \left(\dfrac{\sqrt{5}}{2}\right)^2 - \left(x-\dfrac{1}{2}\right)^2$.

By Proposition 16(**b**),

$$\int \frac{1}{\sqrt{d^2-y^2}}\,\mathrm{d}y = \sin^{-1}\frac{y}{|d|} + C.$$

Replace d and y with $\sqrt{5}/2$ and $x-1/2$:

$$\begin{aligned}\int \frac{1}{\sqrt{-x^2+x+1}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}}\,\mathrm{d}x \\ &= \sin^{-1}\frac{x-\frac{1}{2}}{\left|\sqrt{5}/2\right|} + C \\ &= \sin^{-1}\frac{2x-1}{\sqrt{5}} + C.\end{aligned}$$

Example 1364. Find $\int \frac{1}{\sqrt{-2x^2+3x+5}}\,\mathrm{d}x$ (for $-2x^2+3x+5>0$).

Complete the square: $-2x^2+3x+5 = 2\left(-x^2+\frac{3}{2}x+\frac{5}{2}\right) = 2\left[\frac{49}{16}-\left(x-\frac{3}{4}\right)^2\right] = 2\left[\left(\frac{7}{4}\right)^2-\left(x-\frac{3}{4}\right)^2\right]$.

By Proposition 16(**b**),

$$\int \frac{1}{\sqrt{d^2-y^2}}\,\mathrm{d}y = \sin^{-1}\frac{y}{|d|}+C.$$

Replace d and y with $7/4$ and $x-3/4$:

$$\begin{aligned}\int \frac{1}{\sqrt{-2x^2+3x+5}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{2}\sqrt{\left(\frac{7}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2}}\sin^{-1}\frac{x-3/4}{|7/4|}+C \\ &= \frac{1}{\sqrt{2}}\sin^{-1}\frac{4x-3}{7}+C.\end{aligned}$$

Exercise 428. Find the following. (Answer on p. 1936.)

(a) $\int \frac{1}{\sqrt{-3x^2+x+6}}\,\mathrm{d}x$ (for $-3x^2+x+6>0$).

(b) $\int \frac{1}{\sqrt{-7x^2-x+2}}\,\mathrm{d}x$ (for $-7x^2-x+2>0$).

Fact 217 is the general formula. Again, this is for your reference only and not meant for *mugging* (which would be foolish).

Fact 217. *Suppose $a,b,c\in\mathbb{R}$ with $a<0$ and $ax^2+bx+c>0$. Then*

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x = \frac{1}{\sqrt{-a}}\sin^{-1}\frac{-(2ax+b)}{\sqrt{b^2-4ac}}+C.$$

Proof. See p. 1718 (Appendices). □

Remark 168. In this subchapter, we've learnt to find the antiderivative $\int \frac{1}{\sqrt{ax^2+bx+c}} \, \mathrm{d}x$, but **only in the special case** where $a < 0$.

Happily, your H2 Maths syllabus does **not** include finding this antiderivative in the case where $a > 0$.[499]

[499] But see Fact 299 (Appendices) if you're interested.

106.7. Using Trigonometric Identities

The following rules of antidifferentiation are **not** on List MF26, but **are** explicitly listed on your H2 Maths syllabus. Which means you'll have to know how to derive them.

Proposition 17. *Let $m, n \in \mathbb{R}$ such that $m + n \neq 0$, and $m - n \neq 0$.*

(a) $$\int \sin^2 x \, \mathrm{d}x = \frac{1}{2}x - \frac{\sin 2x}{4} + C,$$

(b) $$\int \cos^2 x \, \mathrm{d}x = \frac{1}{2}x + \frac{\sin 2x}{4} + C,$$

(c) $$\int \tan^2 x \, \mathrm{d}x = \tan x - x + C,$$ *where x is not an odd m*

(d) $$\int \sin mx \cos nx \, \mathrm{d}x = -\frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n}\right] + C,$$

(e) $$\int \sin mx \sin nx \, \mathrm{d}x = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right] + C,$$

(f) $$\int \cos mx \cos nx \, \mathrm{d}x = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right] + C.$$

Exam Tip for *Towkays*

Whenever you see a question with trigonometric functions, put MF26 (p. 3) next to you!

Proof. We could simply verify these, i.e. differentiate the expression on RHS to get the integrand on LHS. However, these do not appear in List MF26 and you're required to know how to derive these. So, here our proof will be the "proper" one, going from LHS to RHS.

Here we prove only **(a)** and **(d)**. You're asked to prove **(b)**, **(c)**, **(e)**, and **(f)** in Exercise 429.

(a) Use the identity $\cos 2x = 1 - 2\sin^2 x$. So,

$$\int \sin^2 x \, \mathrm{d}x = \int \frac{1 - \cos 2x}{2} \, \mathrm{d}x = \frac{1}{2}x - \frac{\sin 2x}{4} + C.$$

(d) As stated on List MF26 (p. 3), $\sin P + \sin Q \varnothing 12 \sin \dfrac{P+Q}{2} \cos \dfrac{P-Q}{2}$.

So, let $\quad mx = \dfrac{P+Q}{2} \quad$ and $\quad nx = \dfrac{P-Q}{2}$.

Hence, $\quad P \varnothing 2 \, (m+n)\, x \quad$ and $\quad Q \varnothing 3 \, (m-n)\, x$.

Plug in $\varnothing 1$, $\varnothing 2$, and $\varnothing 3$:

$$\int \sin mx \cos nx \, \mathrm{d}x = \int \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right] \mathrm{d}x$$
$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right] + C.$$

□

Exercise 429. Prove Proposition 17 **(b)**, **(c)**, **(e)**, and **(f)**. (Answer on p. **1936**.)

106.8. Integration by Parts (IBP)

Let u and v be differentiable functions.[500]

By the Product Rule, $$(u \cdot v)' = u' \cdot v + u \cdot v'.$$

Equivalently, $$u \cdot v = \int u' \cdot v + u \cdot v' \, \mathrm{d}x = \int u' \cdot v \, \mathrm{d}x + \int u \cdot v' \, \mathrm{d}x.$$

Rearranging, $$\int u \cdot v' \, \mathrm{d}x = u \cdot v - \int u' \cdot v \, \mathrm{d}x.$$

This last equation is our **Integration by Parts (IBP)** formula. We see then that **IBP** is simply the inverse of the **Product Rule**. For future reference, let's jot it down as a formal result:

Theorem 46. (Integration by Parts) *Suppose u and v are differentiable functions. Then*

$$\int u \cdot v' \, \mathrm{d}x = u \cdot v - \int u' \cdot v \, \mathrm{d}x.$$

Example 1365. Find $\int x\mathrm{e}^x \, \mathrm{d}x$.

To use IBP, we must first decide, Which of x or e^x is u and which is v'?

Let's choose $u = x$ and $v' = \mathrm{e}^x$, so that $u' = 1$ and $v = \mathrm{e}^x$. So,

$$\int \overbrace{x}^{u} \overbrace{\mathrm{e}^x}^{v'} \, \mathrm{d}x = \overbrace{x}^{u} \overbrace{\mathrm{e}^x}^{v} - \int \overbrace{1}^{u'} \overbrace{\mathrm{e}^x}^{v} \, \mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x + C = \mathrm{e}^x (x - 1) + C.$$

Example 1366. Find $\int x \sin x \, \mathrm{d}x$.

Let's choose $u = x$ and $v' = \sin x$, so that $u' = 1$ and $v = -\cos x$. And now,

$$\int \overbrace{x}^{u} \overbrace{\sin x}^{v'} \, \mathrm{d}x = \overbrace{x}^{u} \overbrace{(-\cos x)}^{v} - \int \overbrace{1}^{u'} \overbrace{(-\cos x)}^{v} \, \mathrm{d}x = -x \cos x + \sin x + C.$$

To choose v', use the mnemonic and rule of thumb **dETAIL**. That is, choose the **d**erivative v' in this order:

Exponential, **T**rig., **A**lgebraic, **I**nverse trig., **L**ogarithmic.

The above rule of thumb (usually) works because it is easiest to find an antiderivative of an **E**xponential function and hardest to find one of a **L**ogarithmic function.[501]

Example 1367. XXX

[500] Following common practice, when discussing IBP, our functions shall be u and v (rather than the more typical f and g).

[501] Kasube (1983) first gave this mnemonic as LIATE. By reversing the letters and adding the letter d in front, we get the actual English word *dETAIL*, which is probably more memorable.

Example 1368. XXX

Example 1369. Find $\int \sqrt{1-x^2}\,\mathrm{d}x$ for $x \in [-1,1]$.

XXX

Conclude:
$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2} + C$$

We'll revisit this problem in Example 1397.

Example 1370. Find $\int \sin x \cos x\,\mathrm{d}x$.

An easy way to find this antiderivative is to use the identity $\sin x \cos x = \dfrac{1}{2}\sin 2x$:

$$\int \sin x \cos x\,\mathrm{d}x = \frac{1}{2}\int \sin 2x\,\mathrm{d}x \overset{1}{=} -\frac{1}{4}\cos 2x + C.$$

But here as an exercise, let's use IBP instead.

This time the dETAIL rule of thumb doesn't help because we have two trigonometric functions. So, let's just choose $u = \sin x$ and $v' = \cos x$, so that $u' = \cos x$ and $v = \sin x$:

$$\int \overbrace{\sin x}^{u}\,\overbrace{\cos x}^{v'}\,\mathrm{d}x = \overbrace{\sin x}^{u}\,\overbrace{\sin x}^{v} - \int \overbrace{\cos x}^{u'}\,\overbrace{\sin x}^{v}\,\mathrm{d}x.$$

Rearranging,

$$2\int \sin x \cos x\,\mathrm{d}x = \sin^2 x + \hat{C} \qquad \text{or} \qquad \int \sin x \cos x\,\mathrm{d}x = \frac{1}{2}\sin^2 x + C.$$

Can you explain why ∅1 and ∅2 are consistent with each other?[502]

Sometimesm we need to apply IBP more than once:

Example 1371. Find $\int x^2 \mathrm{e}^x\,\mathrm{d}x$.

By dETAIL, choose $v' = \mathrm{e}^x$ (and $u = x^2$), so that $v = \mathrm{e}^x$ (and $u' = 2x$):

$$\int \overbrace{x^2}^{u}\,\overbrace{\mathrm{e}^x}^{v'}\,\mathrm{d}x \overset{1}{=} \overbrace{x^2}^{u}\,\overbrace{\mathrm{e}^x}^{v} - \int \overbrace{2x}^{u'}\,\overbrace{\mathrm{e}^x}^{v}\,\mathrm{d}x.$$

At this point, we would apply IBP a second time. But we already did this in an earlier example and found that $\int x\mathrm{e}^x\,\mathrm{d}x \overset{2}{=} \mathrm{e}^x(x-1)$. So let's just plug $\overset{2}{=}$ into $\overset{1}{=}$:

$$\int x^2 \mathrm{e}^x\,\mathrm{d}x = x^2\mathrm{e}^x - 2\mathrm{e}^x(x-1) + C = \mathrm{e}^x\left(x^2 - 2x + 2\right) + C.$$

Sometimes, we can use IBP together with the Times One Trick:

Example 1372. To find $\int \ln x \, \mathrm{d}x$, use the Times One Trick and IBP:

$$\int \ln x \, \mathrm{d}x = \int \ln x \cdot 1 \ln x \cdot 1 = (\ln x)\, x - \int \frac{1}{x} x \, \mathrm{d}x = x \ln x - \int 1 \, \mathrm{d}x = x \ln x - x + C.$$

Remark 169. The IBP formula (Theorem 46) is **not** on List MF26. And unfortunately, I have not found any good mnemonic for it.

But perhaps this is all for the best, since this may occasionally force you to derive it from the Product Rule (and hence understand where it comes from):

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$
$$u \cdot v = \int u' \cdot v \, \mathrm{d}x + \int u \cdot v' \, \mathrm{d}x$$
$$\int u \cdot v' \, \mathrm{d}x \overset{1}{=} u \cdot v - \int u' \cdot v \, \mathrm{d}x.$$

Exercise 430. Find **(a)** $\int x^3 e^x \, dx$ (you may use our result from Example 1371); and **(b)** $\int x^2 \sin x \, dx$. (Answer on p. **1937**.)

Exercise 431. Below is a five-step "proof" that $0 = 1$. Identify the error.

1. Consider $\int \frac{1}{x} \, dx$.
2. By the Times One Trick, $\int \frac{1}{x} \, dx = \int \frac{1}{x} \cdot 1 \, dx$.
3. Apply IBP, choosing $v' = 1$ (and $u = \frac{1}{x}$), so that $v = x$ (and $u' = \frac{-1}{x^2}$):

$$\int \frac{1}{x} \, dx = \int \frac{1}{x} \cdot 1 \, dx = \frac{1}{x} \cdot x - \int \frac{-1}{x^2} x \, dx = 1 + \int \frac{1}{x} \, dx.$$

4. So, $\int \frac{1}{x} \, dx = 1 + \int \frac{1}{x} \, dx$.
5. Subtracting $\int \frac{1}{x} \, dx$ from both sides, we get $0 = 1$. (Answer on p. 1938.)

Exercise 432. Below is a five-step "proof" that $0 = 1$. Identify the error.

1. Consider $\int_1^2 \frac{1}{x} \, dx$.
2. By the Times One Trick, $\int_1^2 \frac{1}{x} \, dx = \int_1^2 \frac{1}{x} \cdot 1 \, dx$.
3. Apply IBP, choosing $v' = 1$ (and $u = \frac{1}{x}$), so that $v = x$ (and $u' = \frac{-1}{x^2}$):

$$\int_1^2 \frac{1}{x} \, dx = \int_1^2 \frac{1}{x} \cdot 1 \, dx = \left[\frac{1}{x} \cdot x - \int \frac{-1}{x^2} x \, dx\right]_1^2 = \left[1 + \int \frac{1}{x} \, dx\right]_1^2 = 1 + \int_1^2 \frac{1}{x} \, dx.$$

4. So, $\int_1^2 \frac{1}{x} \, dx = 1 + \int_1^2 \frac{1}{x} \, dx$.
5. Subtract $\int_1^2 \frac{1}{x} \, dx$ from both sides to get $0 = 1$. (Answer on p. 1938.)

107. The Substitution Rule

The **Substitution Rule** is simply another technique of antidifferentiation, but is of sufficient heft that it gets its own chapter.

For H2 Maths, you are merely required to *mug* and mindlessly regurgitate what I'll call the **Five-Step Substitution Rule Recipe**, which we'll state below. But first, some examples to illustrate how it works:

Example 1373. Using the substitution $u \stackrel{s}{=} \sin x$, find $\int \cot x \, dx$ (for x **not** an integer multiple of π).[503]

Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{du}{dx} \stackrel{1}{=} \cos x$.
2. Using $\stackrel{s}{=}$ and $\stackrel{1}{=}$, rewrite the integrand $\cot x$ into an expression involving u and $\dfrac{du}{dx}$:
$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx \stackrel{s}{=} \int \frac{\cos x}{u} \, dx \stackrel{1}{=} \int \frac{1}{u} \frac{du}{dx} \, dx.$$
3. Magically cancel out the dx's:
$$\int \frac{1}{u} \frac{du}{\cancel{dx}} \cancel{dx} = \int \frac{1}{u} \, du.$$
4. Find this last antiderivative:
$$\int \frac{1}{u} \, du = \ln |u| + C.$$
5. Plug back $\stackrel{s}{=}$ to get rid of u:
$$\ln |u| + C = \ln |\sin x| + C.$$

Conclude: $$\int \cot x \, dx = \ln |\sin x| + C.$$

[503] The parenthetical requirement ensures that $\sin x \neq 0$ and $\cot x$ is well-defined.

Example 1374. Find $\int e^{x^2} 2x \, dx$ using the substitution $u \stackrel{s}{=} x^2$.

Five-Step Substitution Rule Recipe:

1. Compute $\frac{du}{dx} \stackrel{1}{=} 2x$.
2. Using $\stackrel{s}{=}$ and $\stackrel{1}{=}$, rewrite the integrand $e^{x^2} 2x$ into an expression involving u and $\frac{du}{dx}$:

$$\int e^{x^2} 2x \, dx \stackrel{s}{=} \int e^u 2x \, dx \stackrel{1}{=} \int e^u \frac{du}{dx} \, dx.$$

3. Magically cancel out the dx's:

$$\int e^u \frac{du}{\cancel{dx}} \cancel{dx} = \int e^u \, du.$$

4. Find this last antiderivative:

$$\int e^u \, du = e^u + C.$$

5. Plug back $\stackrel{s}{=}$ to get rid of u:

$$e^u + C \stackrel{s}{=} e^{x^2} + C.$$

Conclude:

$$\int e^{x^2} 2x \, dx = e^{x^2} + C.$$

Example 1375. Find $\int \left(\cos x^2\right) 2x \, \mathrm{d}x$ using the substitution $u \stackrel{\text{s}}{=} x^2$.

Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} \stackrel{1}{=} 2x$.
2. Using $\stackrel{\text{s}}{=}$ and $\stackrel{1}{=}$, rewrite the integrand $\left(\cos x^2\right) 2x$ into an expression involving u and $\dfrac{\mathrm{d}u}{\mathrm{d}x}$:
$$\int \left(\cos x^2\right) 2x \, \mathrm{d}x \stackrel{\text{s}}{=} \int (\cos u)\, 2x \, \mathrm{d}x \stackrel{1}{=} \int \cos u \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x.$$
3. Magically cancel out the dx's:
$$\int \cos u \frac{\mathrm{d}u}{\cancel{\mathrm{d}x}} \cancel{\mathrm{d}x} = \int \cos u \, \mathrm{d}u.$$
4. Find this last antiderivative:
$$\int \cos u \, \mathrm{d}u = \sin u + C.$$
5. Plug back $\stackrel{\text{s}}{=}$ to get rid of u:
$$\sin u + C \stackrel{\text{s}}{=} \sin x^2 + C$$

Conclude: $$\int \left(\cos x^2\right) 2x \, \mathrm{d}x = \sin x^2 + C$$

Remark 170. Just so you know, the **Substitution Rule** is also called **integration by substitution**, **change of variables**, **substitution by parts**, or u**-substitution**.

More examples, but now without explicitly explaining each step:

Example 1376. Find $\int \left[\sin\left(x^2+x\right)\right](2x+1)\,\mathrm{d}x$ using the substitution $u \overset{s}{=} x^2+x$.

Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} \overset{1}{=} 2x+1$.

$$
\begin{aligned}
&2. \quad \int \left[\sin\left(x^2+x\right)\right](2x+1)\,\mathrm{d}x \overset{s}{=} \int (\sin u)(2x+1)\,\mathrm{d}x \overset{1}{=} \int \sin u \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \\
&3. \quad = \int \sin u \frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int \sin u\,\mathrm{d}u \\
&4. \quad = -\cos u + C \\
&5. \quad \overset{s}{=} -\cos\left(x^2+x\right) + C.
\end{aligned}
$$

Conclude: $\int \left[\sin\left(x^2+x\right)\right](2x+1)\,\mathrm{d}x = -\cos\left(x^2+x\right)+C$

Example 1377. Find $\int \left(x^3+2x^2\right)\left(3x^2+4x\right)\mathrm{d}x$.

One possible method is to expand the integrand, then antidifferentiate term by term:

$$
\int \left(x^3+2x^2\right)\left(3x^2+4x\right)\mathrm{d}x = \int 3x^5+4x^4+6x^4+8x^3\,\mathrm{d}x
$$

$$
= \frac{1}{2}x^6 + \frac{4}{5}x^5 + \frac{6}{5}x^5 + 2x^4 + C = \frac{1}{2}x^6 + 2x^5 + 2x^4 + C.
$$

Another method is to use the substitution $u \overset{s}{=} x^3+2x^2$—**Five-Step Substitution Rule Recipe**:

1. Compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} \overset{1}{=} 3x^2+4x$.

$$
\begin{aligned}
&2. \quad \int \left(x^3+2x^2\right)\left(3x^2+4x\right)\mathrm{d}x \overset{s}{=} \int u\left(3x^2+4x\right)\mathrm{d}x \overset{1}{=} \int u\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \\
&3. \quad = \int u\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int u\,\mathrm{d}u \\
&4. \quad = \frac{1}{2}u^2 + C \\
&5. \quad \overset{s}{=} \frac{1}{2}\left(x^3+2x^2\right)^2 + C.
\end{aligned}
$$

Conclude: $\int \left(x^3+2x^2\right)\left(3x^2+4x\right)\mathrm{d}x = \frac{1}{2}\left(x^3+2x^2\right)^2 + C$

The Five-Step Substitution Rule Recipe

Find $\int f(x)\,\mathrm{d}x$ using the substitution $u \overset{\mathrm{s}}{=} u(x)$:

1. Find $\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

2. Use $\overset{\mathrm{s}}{=}$ and **Step 1** to rewrite the integrand $f(x)$ into an expression in terms of u and $\dfrac{\mathrm{d}u}{\mathrm{d}x}$:

$$\int f(x)\,\mathrm{d}x = \int g(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.$$

3. Magically cancel out the $\mathrm{d}x$'s:

$$\int g(u)\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int g(u)\,\mathrm{d}u.$$

4. Find this last antiderivative:

$$\int g(u)\,\mathrm{d}u = G(u) + C.$$

5. Plug back $\overset{\mathrm{s}}{=}$ to get rid of u:

$$G(u) = G(u(x)).$$

Remark 171. The Key Magical Step in the above Recipe is **Step 3**, where we "cancel out the $\mathrm{d}x$'s".

This is a strange and questionable move. As we repeatedly warned earlier, $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ is **not** a fraction and the $\mathrm{d}x$'s are **not** numbers. So, how is it that we can just "cancel out the $\mathrm{d}x$'s"?

It turns out that formally, we are not actually doing anything like "cancelling out the $\mathrm{d}x$'s"—instead, this is simply an informal and convenient mnemonic.

Indeed, strictly and pedantically speaking, Step 3 and hence also our Five-Step Substitution Rule Recipe are wrong!

But this isn't something you need to worry about for H2 Maths. What matters for H2 Maths is that the above Recipe "works" and enables us to find the correct antiderivative.

For why Step 3 is wrong and what it should instead be, see Ch. 146.26 (Appendices).

More examples:

Example 1378. XXX

Example 1379. XXX

Example 1380. XXX

Remark 172. Your H2 Maths syllabus includes "integration by a given substitution". I take this to mean that they'll always give you the appropriate substitution to make (as was done in the above examples) and that they'll never ask you to figure out what the appropriate substitution should be.

Exercise 433. XXX (Answer on p. 1099.)

A433.

107.1. The Substitution Rule Is the Inverse of the Chain Rule

It turns out that the **Substitution Rule is simply the inverse of the Chain Rule** (for Differentiation). Let's illustrate why this is so with a few examples.

Example 1381. XXX

Example 1382. XXX

For a formal proof that the Substitution Rule is simply the inverse of the Chain Rule, see Ch. 146.26 (Appendices).

Exercise 434. XXX (Answer on p. 1100.)

A434.

107.2. Skipping Steps

Since the Substitution Rule is really just the inverse of the Chain Rule, if we can "see" how the Chain Rule works, then we can often skip the entire Five-Step Substitution Rule Recipe:

Example 1383. Find $\int (\cos x^2)\, 2x \,\mathrm{d}x$.

In Example 1375, we solved this using the Five-Step Substitution Rule Recipe and the substitution $u = x^2$.

We can now solve this more quickly. Observe that by the Chain Rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x^2 = (\cos x^2)\, 2x$$

Thus,

$$\int (\cos x^2)\, 2x \,\mathrm{d}x = \sin x^2 + C.$$

Implicitly, what we've just done is no different from the Five-Step Substitution Rule Recipe. Just that we're doing it much more quickly (indeed, in a single line).

Example 1384. XXX

Example 1385. XXX

Exercise 435. XXX (Answer on p. 1101.)

A435.

107.3. $\int f' \exp f \, \mathrm{d}x = \exp f + C$

By the Exponential and Chain Rules (of Differentiation),

$$\underset{\cdot}{\mathrm{x}} \exp f = f' \exp f.$$

Hence,

Fact 218. *Suppose f is a differentiable function. Then*

$$\int f' \exp f \, \mathrm{d}x = \exp f + C.$$

And so, by recognising that an integrand is of the form $f' \exp f$, we can simply skip the substitution altogether and thus solve the problem more quickly. We now revisit the above two examples:

Example 1386. Find $\int \mathrm{e}^{\sin x} \cos x \, \mathrm{d}x$.

Earlier in Example XXX, we solved this using the Five-Step Substitution Rule Recipe and the substitution $u = \sin x$.

We can now solve this more quickly. Observe that $\underset{\cdot}{\mathrm{x}} \sin x = \cos$. Hence, by the Chain Rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{\sin x} = \mathrm{e}^{\sin x} \cos x.$$

Thus, $$\int \mathrm{e}^{\sin x} \cos x \, \mathrm{d}x = \mathrm{e}^{\sin x} + C.$$

Again, implicitly, what we've just done is no different from the Five-Step Substitution Rule Recipe. Just that we're doing it much more quickly (indeed, in a single line).

Example 1387. Find $\int \frac{\mathrm{e}^{\sqrt{x}}}{\sqrt{x}} \, \mathrm{d}x.$.

Earlier in Example XXX, we solved this using the Five-Step Substitution Rule Recipe and the substitution $u = \sqrt{x}$.

We can now solve this more quickly. Observe that $\underset{\cdot}{\mathrm{x}} \sqrt{x} = -\frac{1}{2\sqrt{x}}$. Hence, by the Chain Rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} - 2\mathrm{e}^{\sqrt{x}} = \frac{\mathrm{e}^{\sqrt{x}}}{\sqrt{x}}.$$

Thus, $$\int \frac{\mathrm{e}^{\sqrt{x}}}{\sqrt{x}} \, \mathrm{d}x = -2\mathrm{e}^{\sqrt{x}} + C.$$

Exercise 436. XXX (Answer on p. 1103.)

A436.

107.4. $\displaystyle\int \frac{f'}{f}\,\mathrm{d}x = \ln|f| + C$

By the Natural Logarithm and Chain Rules,

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln|f| = \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\ln f = \dfrac{f'}{f}, & \text{for } f > 0, \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}x}\ln(-f) = \dfrac{-f'}{-f} = \dfrac{f'}{f}. & \text{for } f < 0. \end{cases}$$

Hence,

Fact 219. *Suppose f is a differentiable function. Then*

$$\int \frac{f'}{f}\,\mathrm{d}x = \ln|f| + C, \qquad \textit{for } f(x) \neq 0.$$

Example 1388. Find $\displaystyle\int \cot x\,\mathrm{d}x$ (for x not an integer multiple of π)

Earlier in Example 1373, we solved this using the Five-Step Substitution Rule Recipe and the substitution $u = \sin x$.

To solve this problem more quickly, simply observe that

$$\cot x = \frac{\cos x}{\sin x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x.$$

And so by Fact 219, $\displaystyle\int \cot x\,\mathrm{d}x = \ln|\sin x| + C$.

Example 1389. Find $\displaystyle\int \frac{2x + \cos x}{x^2 + \sin x}\,\mathrm{d}x$.

One possible method is to use the Five-Step Substitution Rule Recipe and the substitution $u \varnothing 1 x^2 + \sin x$:

1. Compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} \overset{1}{=} 2x + \cos x$.

$$
\begin{aligned}
&2. \quad \int \frac{2x+\cos x}{x^2+\sin x}\,\mathrm{d}x \overset{\mathrm{s}}{=} \int \frac{2x+\cos x}{u}\,\mathrm{d}x \overset{1}{=} \int \int \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \\
&3. \quad = \int \frac{1}{u}\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = \int \frac{1}{u}\,\mathrm{d}u \\
&4. \quad = \ln|u| + C \\
&5. \quad \overset{\mathrm{s}}{=} \ln\left|x^2 + \sin x\right| + C.
\end{aligned}
$$

Conclude: $$\int \frac{2x+\cos x}{x^2+\sin x}\,\mathrm{d}x = \ln\left|x^2+\sin x\right| + C$$

A second quicker method would be to simply observe that in the integrand, the derivative of the denominator is the numerator:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + \sin x\right) = 2x + \cos x.$$

So, we can use Fact 219:

$$\int \frac{2x+\cos x}{x^2+\sin x}\,\mathrm{d}x = \int \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+\sin x\right)}{x^2+\sin x}\,\mathrm{d}x = \ln\left|x^2+\sin x\right| + C.$$

We now prove Proposition 16(e) using Fact 219:

Example 1390. Find $\displaystyle\int \tan x\,\mathrm{d}x$.

Observing that $\tan x = \dfrac{\sin x}{\cos x}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$, we can use Fact 219:

$$\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x = -\int \frac{\frac{\mathrm{d}}{\mathrm{d}x}\cos x}{\cos x}\,\mathrm{d}x = -\ln|\cos x| + C = \ln|\sec x| + C.$$

Exercise 437. Find the following. (Answer on p. 1939.)

(a) $\displaystyle\int \frac{3x^2 + \sin x}{x^3 - \cos x}\,\mathrm{d}x$ (for $x^3 - \cos x \neq 0$).

(b) $\displaystyle\int \frac{\sin 2x}{\sin^2 x + 1}\,\mathrm{d}x$.

(c) $\displaystyle\int \frac{10x + \cos x}{5x^2 + \sin x}\,\mathrm{d}x$.

(d) $\displaystyle\int \frac{3x^2 + 2x + 1}{x^3 + x^2 + x + 1}\,\mathrm{d}x$ (for $x^3 + x^2 + x + 1 \neq 0$).

(e) $\displaystyle\int \cot x\,\mathrm{d}x$ (for x not an odd multiple of $\pi/2$).[504]

(f) $\displaystyle\int \sec x\,\mathrm{d}x$ (for x not an odd multiple of $\pi/2$).[505] (**Hint:** Multiply by $\dfrac{\sec x + \tan x}{\sec x + \tan x}$.)

[504]This was also Proposition 16(f).

[505]This was also Proposition 16(h).

107.5. $\displaystyle\int (f)^n \cdot f' \, \mathrm{d}x = \frac{1}{n+1} (f)^{n+1} + C$

We reproduce from Ch. 93.3 this definition:

Definition 210. Given the nice function $f : D \to \mathbb{R}$ and $n \in \mathbb{R}$, the function $(f)^n : D \to \mathbb{R}$ is defined by $(f)^n(x) = [f(x)]^n$.

By the Power and Chain Rules, $\underset{\cdot}{\text{x}} (f)^2 = 2f \cdot f'$.

Hence, $$\int f \cdot f' \, \mathrm{d}x \overset{1}{=} \frac{1}{2} (f)^2 + C.$$

Example 1391. Find $\displaystyle\int \left(x^3 + 2x^2\right)\left(3x^2 + 4x\right) \mathrm{d}x$.

In Example 1377, we solved this using two methods:

1. Expand the integrand, get a fifth-degree polynomial, then integrate term-by-term.
2. Use the Five-Step Substitution Rule Recipe (and in particular the substitution $u = x^3 + 2x^2$).

We now solve the above problem using a third method:

Observe that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^3 + 2x^2\right) = 3x^2 + 4x$. Hence, the integrand is of the form $f \cdot f'$ and we can use $\overset{1}{=}$:

$$\int \left(x^3 + 2x^2\right)\left(3x^2 + 4x\right) \mathrm{d}x = \int \underbrace{\left(x^3 + 2x^2\right)}_{f(x)} \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 + 2x^2\right)}_{f'(x)} \mathrm{d}x \overset{1}{=} \frac{1}{2} \underbrace{\left(x^3 + 2x^2\right)^2}_{[f(x)]^2} + C.$$

Again, this “third” method is secretly the same as our second method. The only difference is that we skip the step of making the explicit substitution and thus arrive at our answer more quickly.

By the Power and Chain Rules, $\underset{\cdot}{\text{x}} (f)^3 = 3(f)^2 \cdot f'$.

Hence, $$\int (f)^2 \cdot f' \, \mathrm{d}x \overset{2}{=} \frac{1}{3} (f)^3 + C.$$

Example 1392. Find $\int \left(x^3+2x^2\right)^2\left(3x^2+4x\right)\,\mathrm{d}x$.

Again, we can solve this using two methods:

1. Expand the integrand, get a fifth-degree polynomial, then integrate term-by-term.
2. Use the Five-Step Substitution Rule Recipe (and in particular the substitution $u = x^3+2x^2$).

But let's do it using the third and quickest method:

Observe that $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3+2x^2\right) = 3x^2+4x$. Hence, the integrand is of the form $(f)^2 \cdot f'$ and we can use $\stackrel{2}{=}$:

$$\int \left(x^3+2x^2\right)^2\left(3x^2+4x\right)\,\mathrm{d}x = \int \underbrace{\left(x^3+2x^2\right)^2}_{[f(x)]^2}\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3+2x^2\right)}_{f'(x)}\mathrm{d}x \stackrel{2}{=} \frac{1}{3}\underbrace{\left(x^3+2x^2\right)^3}_{[f(x)]^3}+C.$$

More generally, by the Power and Chain Rules, for *any* $n \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(f)^{n+1} = (n+1)(f)^n \cdot f'.$$

Hence,

Fact 220. *Suppose f is a differentiable function and $n \in \mathbb{R}$. Then*

$$\int (f)^n f'\,\mathrm{d}x = \frac{1}{n+1}(f)^{n+1}+C, \qquad \textit{for } n \neq -1.$$

Example 1393. Find $\int \left(x^3+5x^2-3x+2\right)^{50}\left(3x^2+10x-3\right)\mathrm{d}x$.

Again, we can solve this using two methods:

1. Expand the integrand to get a 152nd-degree polynomial(!), then integrate term-by-term. This is doable, but absurdly tedious.
2. Use the Five-Step Substitution Rule Recipe (and in particular the substitution $u = x^3+5x^2-3x+2$).

But let's do it using the third and quickest method:

Observe that $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3+5x^2-3x+2\right) = 3x^2+10x-3$. Hence, the integrand is of the form $(f)^{50}\cdot f'$ and we can use Fact 220:

$$\int \left(x^3+5x^2-3x+2\right)^{50}\left(3x^2+10x-3\right)\mathrm{d}x = \left(x^3+5x^2-3x+2\right)^{51}+C.$$

Exercise 438. XXX (Answer on p. 1108.)

A438.

107.6. Building a Derivative

Sometimes, the integrand is secretly of the form examined in the past few subchapters. However, it may not be obvious and we may first have to do a little work. In particular, we may first have to **build a derivative**.

Example 1394. Find $\int \frac{x}{x^2+x+1}\,\mathrm{d}x$.

Observe that $\dot{x}\left(x^2+x+1\right) = 2x+1$. This suggests the possibility of using Fact 219:

$$\int \frac{f'}{f}\,\mathrm{d}x \overset{1}{=} \ln|f| + C.$$

In particular, we can build the numerator so that it's a derivative of the denominator:

$$\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x \overset{1}{=} \ln\left|x^2+x+1\right| + C \overset{2}{=} \ln\left(x^2+x+1\right) + C.$$

(At $\overset{2}{=}$, we can remove the absolute value operator because $x^2+x+1 \geq 0$ for all $x \in \mathbb{R}$.)

Now,

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\,\mathrm{d}x &= \frac{1}{2}\int \frac{2x}{x^2+x+1}\,\mathrm{d}x && \text{(Times One Trick)} \\
&= \frac{1}{2}\int \frac{2x+1-1}{x^2+x+1}\,\mathrm{d}x && \text{(Plus Zero Trick)} \\
&= \frac{1}{2}\left(\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x - \int \frac{1}{x^2+x+1}\,\mathrm{d}x\right) \\
&\overset{2}{=} \frac{1}{2}\left[\ln\left(x^2+x+1\right) - \int \frac{1}{x^2+x+1}\,\mathrm{d}x\right].
\end{aligned}$$

We now have the separate problem of finding $\int \frac{1}{x^2+x+1}\,\mathrm{d}x$. But we already learnt to do this in Ch. 106.5:

$$\int \frac{1}{x^2+x+1}\,\mathrm{d}x = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}x = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C.$$

Altogether then,

$$\int \frac{x}{x^2+x+1}\,\mathrm{d}x = \frac{1}{2}\ln\left(x^2+x+1\right) + -\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \bar{C}.$$

Example 1395. XXX

Exercise 439. XXX (Answer on p. 1109.)

A439.

Example 1396. Find $\int \tan^{-1} x \, \mathrm{d}x$.

$$\int \tan^{-1} x \, \mathrm{d}x = \int \overbrace{\tan^{-1} x}^{u} \cdot \overbrace{1}^{v'} \, \mathrm{d}x \qquad \text{(Times One Trick)}$$

$$= \overbrace{\tan^{-1} x}^{u} \cdot \overbrace{x}^{v} - \int \overbrace{\frac{1}{1+x^2}}^{u'} \cdot \overbrace{x}^{v} \, \mathrm{d}x \qquad \text{(IBP)}$$

$$= x \tan^{-1} x - \frac{1}{2} \ln \left|1 + x^2\right| + C \qquad \left(\int \frac{f'}{f} \, \mathrm{d}x = \ln |f| \right)$$

$$= x \tan^{-1} x - \frac{1}{2} \ln \left(1 + x^2\right) + C \qquad (1 + x^2 \geq 0)$$

Exercise 440. Find the following. (Answer on p. 1939.)

(a) $\int \sin^{-1} x \, \mathrm{d}x$.

(b) $\int \cos^{-1} x \, \mathrm{d}x$

107.8. Integration with the Substitution Rule

XXX To be written.

107.9. More Challenging Applications of the Substitution Rule

The examples and exercises in this subchapter are harder than what you can expect to find on your A-Level exams. So, if you're able to go through these

Example 1397. Find $\int \sqrt{1-x^2}\,\mathrm{d}x$ for $x \in [-1,1]$.

We actually solved this problem in Example 1369 using Integration by Parts (IBP).

Let's now solve it again using the substitution $x \overset{\text{s}}{=} \sin u$ or $u \overset{\text{s}}{=} \sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{\mathrm{d}x}{\mathrm{d}u} \overset{1}{=} \cos u$.

$$
\begin{aligned}
2.\quad \int \sqrt{1-x^2}\,\mathrm{d}x &\overset{\text{s}}{=} \int \sqrt{1-\sin^2 u}\,\mathrm{d}x && \left(\text{Plug in } \overset{\text{s}}{=}\right) \\
&= \int \sqrt{\cos^2 u}\,\mathrm{d}x && \left(\sin^2 u + \cos^2 u = 1\right) \\
&= \int |\cos u|\,\mathrm{d}x && \left(\sqrt{t^2} = |t|\right) \\
&= \int \cos u\,\mathrm{d}x && \left(u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies \cos u \geq 0\right) \\
&= \int \cos u \frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x && (\text{Times One Trick})^{506} \\
&\overset{1}{=} \int \cos^2 u \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \\
3.\quad &= \int \cos^2 u \frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = \int \cos^2 u\,\mathrm{d}u \\
4.\quad &= \int \frac{\cos 2u + 1}{2}\,\mathrm{d}u && (\text{Double Angle Formula for Cosine}) \\
&= \frac{\sin 2u}{4} + \frac{u}{2} + C \\
&= \frac{\sin u \cos u}{2} + \frac{u}{2} + C && (\text{Double Angle Formula for Sine}) \\
5.\quad &\overset{\text{s}}{=} \frac{x \cos u}{2} + \frac{\sin^{-1} x}{2} + C \\
&= \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2} + C. && \cos u = \sqrt{1-\sin^2 u} = \sqrt{1-x^2}
\end{aligned}
$$

Conclude:
$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2} + C$$

[506] If $u = \pm\pi/2$, then $\cos u = 0$ and $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{\cos u}$ involves the Cardinal Sin of Dividing by Zero (see Ch. 2.2). This is a legitimate problem that we'll simply ignore or hand-wave away.

Example 1398. Find $\int \frac{x^2}{\sqrt[3]{1+2x}}\,\mathrm{d}x$ (for $1+2x \neq 0$) using the substitution $u \stackrel{s}{=} \sqrt[3]{1+2x} \neq 0$ or $x \stackrel{s}{=} \frac{1}{2}\left(u^3-1\right)$.

Five-Step Substitution Rule Recipe:

1. Compute $\frac{\mathrm{d}x}{\mathrm{d}u} \stackrel{1}{=} \frac{3}{2}u^2$.

$$2.\quad \int \frac{x^2}{\sqrt[3]{1+2x}}\,\mathrm{d}x \stackrel{s}{=} \int \frac{x^2}{u}\,\mathrm{d}x \stackrel{s}{=} \int \frac{\left(u^3-1\right)^2}{4u}\,\mathrm{d}x = \int \frac{\left(u^3-1\right)^2}{4u}\frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \qquad \text{(Times One Tr}$$

$$\stackrel{1}{=} \int \frac{\left(u^3-1\right)^2}{4u}\frac{3}{2}u^2\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \frac{3}{8}\int \left(u^3-1\right)^2 u\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

$$3.\quad = \frac{3}{8}\int \left(u^3-1\right)^2 u\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x}$$

$$= \frac{3}{8}\int \left(u^6-2u^3+1\right)u\,\mathrm{d}u = \frac{3}{8}\int u^7-2u^4+u\,\mathrm{d}u \qquad \text{(Expand)}$$

$$4.\quad = \frac{3}{8}\left(\frac{1}{8}u^8-\frac{2}{5}u^5+\frac{1}{2}u^2\right)+C = \frac{3}{320}\left(5u^8-16u^5+20u^2\right)+C$$

$$5.\quad \stackrel{s}{=} \frac{3}{320}\left[5(1+2x)^{8/3}-16(1+2x)^{5/3}+20(1+2x)^{2/3}\right]+C$$

$$= \frac{3}{320}(1+2x)^{2/3}\left[5(1+2x)^2-16(1+2x)+20\right]+C$$

$$= \frac{3}{320}(1+2x)^{2/3}\left(20x^2-12x+9\right)+C$$

The last two steps (algebra) are nice but not necessary.

Conclude: $$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{3}{320}(1+2x)^{2/3}\left(20x^2-12x+9\right)+C$$

If we want to be really anal, correct, and proper about it, we can simply treat the above as mere guesswork, then formally verify that our guess or conclusion is correct, like this:

Let D be an interval in $[-1,1]$ and the function $f: D \to \mathbb{R}$ be defined by $f(x) = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2}$. We can verify that f is differentiable and $f': D \to \mathbb{R}$ is defined by $f'(x) = \sqrt{1-x^2}$. And hence, by Fact 215, our conclusion holds.

Example 1399. Find $\displaystyle\int \frac{1}{1+\cos^2 x}\,\mathrm{d}x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,[507] using the substitution $u \overset{\text{s}}{=} \tan x$.

Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} \overset{1}{=} \sec^2 x$.

$$
\begin{array}{lrll}
2. & \displaystyle\int \frac{1}{1+\cos^2 x}\,\mathrm{d}x & = \displaystyle\int \frac{\sec^2 x}{\sec^2 x+1}\,\mathrm{d}x & \left(\times\dfrac{\sec^2 x}{\sec^2 x}\right) \\
 & & \overset{1}{=} \displaystyle\int \frac{1}{\sec^2 x+1}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x & \\
3. & & \displaystyle\int \frac{1}{\sec^2 x+1}\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int \frac{1}{\sec^2 x+1}\,\mathrm{d}u & \\
 & & = \displaystyle\int \frac{1}{\tan^2 x+2}\,\mathrm{d}u & \left(\tan^2 x+1=\sec^2 x\right) \\
 & & \overset{\text{s}}{=} \displaystyle\int \frac{1}{u^2+2}\,\mathrm{d}u & \\
4. & & = \dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{u}{\sqrt{2}}+C & (\text{Proposition 16(a)}) \\
5. & & \overset{\text{s}}{=} \dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{\tan x}{\sqrt{2}}+C &
\end{array}
$$

Conclude:
$$\int \frac{1}{1+\cos^2 x}\,\mathrm{d}x = \frac{1}{\sqrt{2}}\tan^{-1}\frac{\tan x}{\sqrt{2}}+C$$

[507] Not something you need to worry about for H2 Maths, but just so you know, the condition that $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is important and should not be omitted. (Observe that for example $\tan\frac{\pi}{2}$ would be undefined.)

It is possible to find $\displaystyle\int \frac{1}{1+\cos^2 x}\,\mathrm{d}x$ for the more general case where $x \in \mathbb{R}$—but this requires more work and produces a different answer. See ✿, Jeffrey (1994), and Jeffrey & Rich (1994).

Exercise 441. Consider $\int \frac{1}{x^2\sqrt{x^2-1}}\,\mathrm{d}x$ for $x \in (1,\infty)$. (Answer on p. **1939**.)

(a) By using the substitution $x \overset{a}{=} \sec u$ or $u \overset{a}{=} \sec^{-1} x$ for $u \in \left(0, \frac{\pi}{2}\right)$, show that

$$\int \frac{1}{x^2\sqrt{x^2-1}}\,\mathrm{d}x = \sin\left(\sec^{-1} x\right) + C_1, \qquad \text{for } x \in (1,\infty).$$

(b) By using the substitution $u \overset{b}{=} \frac{x^2-1}{x^2}$ for $u \in (0,1)$, show that

$$\int \frac{1}{x^2\sqrt{x^2-1}}\,\mathrm{d}x = \frac{\sqrt{x^2-1}}{x} + C_2, \qquad \text{for } x \in (1,\infty).$$

(c) Show that $\sin\left(\sec^{-1} x\right) \overset{3}{=} \frac{\sqrt{x^2-1}}{x}$ for $x \in (1,\infty)$. (**Hint:** Plug in $x = 2$.)

Exercise 442. Consider $\int \frac{x^3}{(x^2+1)^{3/2}}\,\mathrm{d}x$ for $x \in \mathbb{R}$. (Answer on p. 153.21.)

(a) By using the substitution $u \overset{a}{=} x^2+1$, show that

$$\int \frac{x^3}{(x^2+1)^{3/2}}\,\mathrm{d}x = \frac{1}{\sqrt{x^2+1}} + \sqrt{x^2+1} + C_1, \qquad \text{for } x \in \mathbb{R}.$$

(b) By using the substitution $x \overset{b}{=} \tan u$ for $u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, show that

$$\int \frac{x^3}{(x^2+1)^{3/2}}\,\mathrm{d}x = \frac{1}{\cos\left(\tan^{-1} x\right)} + \cos\left(\tan^{-1} x\right) + C_2, \qquad \text{for } x \in \mathbb{R}.$$

(c) Prove that $\cos\left(\tan^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$, for all $x \in \mathbb{R}$. (**Hint 1:** Plug in $x = 0$.)

(**Hint 2:** If $a, b \neq 0$, then

$$\frac{1}{a} + a = \frac{1}{b} + b \qquad \Longleftrightarrow \qquad a = b \text{ or } a = \frac{1}{b}.)$$[508]

[508]Here's a proof of this assertion:

$$\frac{1}{a} + a = \frac{1}{b} + b \overset{\times ab}{\Longleftrightarrow} b + a^2 b = a + ab^2 \Longleftrightarrow ba^2 - \left(1+b^2\right)a + b = 0$$

$$\Longleftrightarrow \quad a = \frac{1+b^2 \pm \sqrt{\left(1+b^2\right)^2 - 4b^2}}{2b} = \frac{1+b^2 \pm \sqrt{\left(b^2-1\right)^2}}{2b} = \frac{1+b^2 \pm \left|b^2-1\right|}{2b} = \frac{1+b^2 \pm \left(b^2-1\right)}{2b} = b, \frac{1}{b}.$$

108. Term-by-Term Antidifferentiation and Integration of a Power Series

By the Sum Rule for Antidifferentiation, we can antidifferentiate a polynomial function term-by-term to get an antiderivative.[509]

Example 1400. Define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = x^2 + x + 1.$$

By the Sum Rule for Antidifferentiation, we can simply antidifferentiate the expression $x^2 + x + 1$ "term-by-term" to find that the antiderivatives of g are exactly those functions $G : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} G(x) &= \int x^2 + x + 1 \, \mathrm{d}x \\ &= \int x^2 \, \mathrm{d}x + \int x \, \mathrm{d}x + \int 1 \, \mathrm{d}x \\ &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C, \qquad \text{for } C \in \mathbb{R}. \end{aligned}$$

And as usual, by the FTC2, we can pick any of the antiderivatives G that we just found[510] to compute definite integrals. For example, the definite integral of g from 0 to 1 is

$$\int_0^1 g = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right]_0^1 = \frac{11}{6} - 0 = \frac{11}{6}.$$

[509]This chapter is the integration analogue of Ch. 101.5 (Term-by-Term Differentiation of a Power Series).
[510]As usual, the most convenient choice will be $C = 0$.

Example 1401. Define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = 17x^5 + 14x^4 + 11x^3 + 8x^2 + 5x + 2.$$

By the Sum Rule, we can simply antidifferentiate the expression $17x^5 + 14x^4 + 11x^3 + 8x^2 + 5x + 2$ "term-by-term" to find that the antiderivatives of h are exactly those functions $H : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} H(x) &= \int 17x^5 + 14x^4 + 11x^3 + 8x^2 + 5x + 2 \, \mathrm{d}x \\ &= \int 17x^5 \, \mathrm{d}x + \int 14x^4 \, \mathrm{d}x + \int 11x^3 \, \mathrm{d}x + \int 8x^2 \, \mathrm{d}x + \int 5x \, \mathrm{d}x + \int 2 \, \mathrm{d}x \\ &= \frac{17}{6}x^6 + \frac{14}{5}x^5 + \frac{11}{4}x^4 + \frac{8}{3}x^3 + \frac{5}{2}x^2 + 2x + C, \qquad \text{for } C \in \mathbb{R}. \end{aligned}$$

And as usual, by the FTC2, we can pick any of the antiderivatives H that we just found to compute definite integrals. For example, the definite integral of h from 0 to 1 is

$$\int_0^1 h = \left[\frac{17}{6}x^6 + \frac{14}{5}x^5 + \frac{11}{4}x^4 + \frac{8}{3}x^3 + \frac{5}{2}x^2 + 2x\right]_0^1 = \frac{311}{20} - 0 = \frac{311}{20}.$$

It turns out that happily, we can also antidifferentiate "infinite polynomial functions"—or **power series**—term-by-term:

Example 1402. Define the function $f:(-1,1)\to\mathbb{R}$ by

$$f(x)=\frac{1}{1-x}.$$

Proceeding as usual,

$$\int\frac{1}{1-x}\,\mathrm{d}x=-\ln|1-x|+C=-\ln(1-x)+C.^{511}$$

Hence, the antiderivatives of f are the functions $F:(-1,1)\to\mathbb{R}$ defined by

$$F(x)\overset{1}{=}-\ln(1-x)+C,\qquad\text{for } C\in\mathbb{R}.$$

We now introduce another method for finding the antiderivatives of f.

We know from Ch. 101.4 that f can be represented by the power series $1+x+x^2+x^3+\ldots$. That is,

$$f(x)=\frac{1}{1-x}=1+x+x^2+x^3+\ldots,\quad\text{for all } x\in(-1,1)=\text{Domain } f$$

It turns out that thanks to Corollary 48 (below), happily, we can antidifferentiate this power series term-by-term:

$$\begin{aligned}&\int 1+x+x^2+x^3+\ldots\,\mathrm{d}x\\&=\int 1\,\mathrm{d}x+\int x\,\mathrm{d}x+\int x^2\,\mathrm{d}x+\int x^3\,\mathrm{d}x+\ldots\\&=x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots+\bar{C};\end{aligned}$$

and conclude that f's antiderivatives are the functions $F:(-1,1)\to\mathbb{R}$ defined by

$$F(x)\overset{2}{=}x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots+\bar{C},\qquad\text{for all } C\in\mathbb{R}.$$

And as usual, by the FTC2, we can pick any of the antiderivatives F that we just found to compute definite integrals. For example, the definite integral of f from 0 to 1/2 is

$$\int_0^{1/2}f=\left[x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\ldots\right]_0^{1/2}\approx 0.693-0=0.693.$$

By the way, as a bonus, together, $\overset{1}{=}$ and $\overset{2}{=}$ show that $-\ln(1-x)$ and $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\ldots$ differ by at most a constant. That is, for all $x\in\mathbb{R}$, there exists some $\hat{C}\in\mathbb{R}$ such that

$$-\ln(1-x)=x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots+\hat{C}.$$

In fact, $\hat{C}=0$ and the two expressions are equal for all $x\in\mathbb{R}$:

More examples:

Example 1403. XXX

Example 1404. XXX

For two more examples, see Exercises 696(iii) (N2014/I/8) 712(ii)(a) (N2011/I/4).

Exercise 443. XXX (Answer on p. 1119.)

A443.

[511] In the last step, we remove the absolute value operator because $1 - x > 0$ for $x \in (-1, 1)$.

108.1. Formal Results (optional)

Below are the two formal results justifying the term-by-term integration and antidifferentiation that we did in the above examples and exercises. (Note that above, we did term-by-term antidifferentiation first, *then* used FTC2 to justify also term-by-term integration. The two formal results below reverse this order.)

Theorem 47. (Term-by-Term Integration) *Let* $R > 0$, $a, b \in (-R, R)$ *with* $a < b$, *and* $c_0, c_1, c_2, \cdots \in \mathbb{R}$. *Suppose the function* $f : (-R, R) \to \mathbb{R}$ *is defined by*

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \cdots = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$\int_a^b f(x) \, \mathrm{d}x = c_0 \frac{b-a}{1} + c_1 \frac{b^2 - a^2}{2} + c_2 \frac{b^3 - a^3}{3} + c_3 \frac{b^4 - a^4}{4} + \cdots = \sum_{n=0}^{\infty} c_n \frac{b^{n+1} - a^{n+1}}{n+1}.$$

Proof. Omitted.[512] □

Corollary 48. (Term-by-Term Antidifferentiation) *Let* $R > 0$, $a, b \in (-R, R)$ *with* $a < b$, *and* $c_0, c_1, c_2, \cdots \in \mathbb{R}$. *Suppose the function* $f : (-R, R) \to \mathbb{R}$ *is defined by*

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \cdots = \sum_{n=0}^{\infty} c_n x^n.$$

If $g : [a, b] \to \mathbb{R}$ *is defined by* $g(x) = f(x)$, *then an antiderivative of* g *is the function* $G : [a, b] \to \mathbb{R}$ *defined by*

$$G(x) = c_0 \frac{x-a}{1} + c_1 \frac{x^2 - a^2}{2} + c_2 \frac{x^3 - a^3}{3} + c_3 \frac{x^4 - a^4}{4} + \cdots = \sum_{n=0}^{\infty} c_n \frac{x^{n+1} - a^{n+1}}{n+1}.$$

Proof. Apply FTC1 to Theorem 47. □

[512] See e.g. Tao (*Analysis II*, 2016, pp. 76–77, Theorem 4.1.6(e))

109. More Definite Integrals

Your H2 Maths syllabus explicitly includes:

- "finding the area of a region"
  - "bounded by a curve and lines parallel to the coordinate axes",
  - "between a curve and a line", or
  - "between two curves";
- "area below the x-axis";
- "finding the area under a curve defined parametrically";
- "finding the volume of revolution about the x- or y-axis"; and
- "finding the approximate value of a definite integral using a graphing calculator".

So, these fascinating topics are what we'll cover in this chapter.

Example 1405. Find the area bounded by the curve $y = x^2$ and the lines $y = 1$ and $y = 2$.

It's often helpful (and worth the time spent) to first make a quick sketch. (You can use your calculator, though this shouldn't be necessary here.)

Figure to be inserted here.

Here are two methods for finding the requested area A.

Method 1. The area of the rectangle $A + B + C + D$ is

$$2 \times 2\sqrt{2} = 4\sqrt{2}.$$

The area of C is 2.

The area of each of B and D is

$$\int_{-\sqrt{2}}^{-1} x^2 \, dx = \left[\frac{x^3}{3}\right]_{-\sqrt{2}}^{-1} = -\frac{1}{3} - \left(-\frac{2\sqrt{2}}{3}\right) = \frac{2\sqrt{2}-1}{3}.$$

Hence, the area of A is

$$4\sqrt{2} - \left(2 + 2 \times \frac{2\sqrt{2}-1}{3}\right) = \frac{4}{3}\left(2\sqrt{2}-1\right).$$

Method 2. The right branch of the parabola $y = x^2$ has equation $x = \sqrt{y}$.

Hence, the area of the right half of A is

$$\int_{y=1}^{y=2} x \, dy = \int_{y=1}^{y=2} \sqrt{y} \, dy = \frac{2}{3}\left[y^{3/2}\right]_1^2 = \frac{2}{3}\left(2\sqrt{2}-1\right).$$

Hence, the area of A is

$$2 \times \frac{2}{3}\left(2\sqrt{2}-1\right) = \frac{4}{3}\left(2\sqrt{2}-1\right).$$

Happily, our two answers coincide (we'd be worried if they didn't).

Exercise 444. Find the exact area bounded by the curve $y = x^3$, the lines $y = 1$ and $y = 2$, and the y-axis. (Answer on p. **1943**.)

Example 1406. Find the area bounded by the curve $y \overset{1}{=} x^2$ and the line $y \overset{2}{=} 4x + 5$.

Find the points at which the line and curve intersect by combining $\overset{1}{=}$ and $\overset{2}{=}$:

Figure to be inserted here.

To find the points at which the line and curve intersect, combine $\overset{1}{=}$ and $\overset{2}{=}$:

$$4x + 5 = x^2 \iff -x^2 + 4x + 5 = 0 \iff$$
$$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)5}}{2(-1)} = \frac{4 \mp \sqrt{36}}{2} = 5, -1.$$

Hence, the requested area is

$$\int_{-1}^{5} -x^2 + 4x + 5 \, \mathrm{d}x = \left[-\frac{x^3}{3} + 2x^2 + 5x \right]_{-1}^{5} = \left[-\frac{125}{3} + 50 + 25 \right] - \left[-\frac{-1}{3} + 2 - 5 \right] = -\frac{126}{3} + 78 = 3$$

Exercise 445. Consider the curve $y = \sin x$ for $x \in [\pi, 2\pi]$. Find the exact area bounded by this curve and the line $y = 1/2$. (Answer on p. **1943**.)

Example 1407. Find the area bounded by the curves $y \overset{1}{=} x^2 - 8x - 3$ and $y \overset{2}{=} 7 - x^2$.

As usual, first make a quick sketch:

Figure to be
inserted here.

Find the points at which the line and curve intersect by combining $\overset{1}{=}$ and $\overset{2}{=}$:

$$x^2 - 8x - 3 = 7 - x^2 \iff 2x^2 - 8x - 10 = 0 \iff x^2 - 4x - 5 = 0$$

$$\iff x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)} = \frac{4 \pm \sqrt{36}}{2} = 5, -1.$$

Hence, the requested area is

$$\int_{-1}^{5} 2x^2 - 8x - 10 \, \mathrm{d}x = \left[\frac{2}{3}x^3 - 4x^2 - 10x\right]_{-1}^{5} = \left[\frac{250}{3} - 100 - 50\right] - \left[-\frac{2}{3} - 4 + 10\right] = \frac{252}{3} - 156 = -72.$$

Exercise 446. Find the exact area bounded by the curves $y = 2 - x^2$ and $y = x^2 + 1$. (Answer on p. **1944**.)

109.4. Area below the x-Axis

Example 1408. Find the area bounded by the curve $y = x^2 - 4$ and the x-axis.

As usual, first make a quick sketch:

Figure to be
inserted here.

The curve intersects the x-axis at

$$x^2 - 4 = 0 \qquad \iff \qquad x = \pm 2.$$

As stated earlier, the definite integral gives us the **signed area**. So if the curve is under the x-axis (as is the case here), then the computed definite integral will be negative:

$$\int_{-2}^{2} x^2 - 4\,\mathrm{d}x = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right) = -\frac{32}{3}.$$

But of course, an area is simply a magnitude (i.e. we don't care about the sign). So, our answer is that the requested area is $\dfrac{32}{3}$.

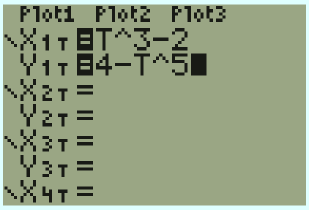
Exercise 447. Find the exact area bounded by $y = x^4 - 16$ and the x-axis. (Answer on p. **1944**.)

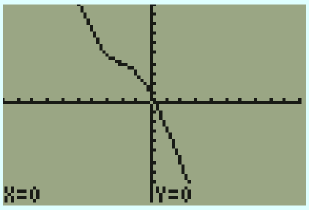
Example 1409. The curve C has parametric equations

$$x = t^3 - 2, \qquad y = 4 - t^5, \qquad \text{for } t \in \mathbb{R}.$$

Find the exact area bounded by C, the x-axis, and the lines $x = -2$ and $x = -1$.

As usual, first make a quick sketch of C:





Now, the line $x = -2$ intersects C at $-2 = t^3 - 2$ or $t = 0$.

Similarly, the line $x = -1$ intersects C at $-1 = t^3 - 2$ or $t = 1$.

So the requested area is

$$\begin{aligned}
\int_{-2}^{-1} y \, \mathrm{d}x &= \int_{-2}^{-1} 4 - t^5 \, \mathrm{d}x \\
&= \int_{-2}^{-1} \left(4 - t^5\right) \frac{\mathrm{d}x}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}x} \, \mathrm{d}x && \text{(Times One Trick)} \\
&= \int_{-2}^{-1} \left(4 - t^5\right) 3t^2 \frac{\mathrm{d}t}{\mathrm{d}x} \, \mathrm{d}x && \left(\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2\right) \\
&= \int_{x=-2}^{x=-1} \left(4 - t^5\right) 3t^2 \frac{\mathrm{d}t}{\cancel{\mathrm{d}x}} \cancel{\mathrm{d}x} && \text{(Substitution Rule)} \\
&= \int_{t=0}^{t=1} \left(4 - t^5\right) 3t^2 \, \mathrm{d}t && \text{(Change of limits)} \\
&= \int_{0}^{1} 12t^2 - 3t^7 \, \mathrm{d}t \\
&= \left[4t^3 - \frac{3}{8}t^8\right]_0^1 = 3\frac{5}{8}.
\end{aligned}$$

Exercise 448. The curve C has parametric equations

$$x = t^2 + 2t, \qquad y = t^3 - 1, \qquad \text{for } t \in \mathbb{R}.$$

Find the exact area bounded by the curve, the y-axis, and the lines $y = 1$ and $y = 2$. (Answer on p. **1944**.)

Example 1410. Consider the line segment $y = 2$, $x \in [0, 3]$.

Rotate this line segment about the x-axis to form a 3D cylinder with height 3 and base radius 2.

Q. Find the cylinder's volume V.

Figure to be inserted here.

Method 1 (Primary-school formula). The cylinder's base area is $\pi \cdot 2^2 = 4\pi$. So, by a primary-school formula,

$$V = \text{Base} \times \text{Height} = 4\pi \times 3 = 12\pi. \checkmark$$

It turns out that we can also obtain this volume using integration:

Method 2 (Integration). In Ch. 104.4, we dealt with the 2D case. We wanted to find the area under a curve. To do so, we used the intuitive idea of slicing the area under the curve into infinitely many infinitely thin rectangles—then adding up the areas of these rectangles.

Here in the 3D case, we want to find the volume of a cylinder. To do so, we'll use the similar idea of slicing the cylinder into infinitely many infinitely thin slices—then adding up the volumes of these slices:

Figure to be inserted here.

At each x, the corresponding infinitely thin slice has area 4π.

Let $f(x)$ be the volume of the cylinder from 0 to x. We now use the same intuition that led to the two FTCs:

FTC1: The derivative of f must be one of these thin slices. That is,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 4\pi.$$

FTC2: We can use an antiderivative of f to find V:

$$V = \int_0^3 \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = \int_0^3 4\pi\,\mathrm{d}x = [4\pi x]_0^3 = 12\pi. \checkmark$$

Example 1411. Consider the line segment $y = 3x$, $x \in [0, 1]$.

Rotate this line segment about the x-axis to form a cone with height 1 and base radius 3.

Q. Find the cone's volume V.

Figure to be
inserted here.

Method 1 (Secondary-school formula). The cone's base area is $\pi \cdot 3^2 = 9\pi$. By a secondary-school formula,

$$V = \frac{1}{3}\text{Base} \times \text{Height} = \frac{1}{3}9\pi \times 1 = 3\pi. \checkmark$$

It turns out that we can also obtain this volume using integration:

Method 2 (Integration). Again, we'll slice the cone into infinitely many infinitely thin slices, then add up the volumes of these slices:

Figure to be
inserted here.

At each x, the corresponding infinitely thin slice has area $\pi y^2 = \pi (3x)^2 = 9\pi x^2$.

Let $f(x)$ be the volume of the cone from 0 to x.

FTC1: The derivative of f must be one of these thin slices. That is,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 9\pi x^2.$$

FTC2: We can use an antiderivative of f to find V:

$$V = \int_0^1 \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = \int_0^1 9\pi x^2\,\mathrm{d}x = \left[3\pi x^3\right]_0^1 = 3\pi. \qquad \checkmark$$

Happily, using integration, we've arrived at the same volume as when we used our secondary-school formula.

Example 1412. Consider the curve $y = x^2$. Rotate it about the x-axis to form a solid. Let V be the volume of the portion of the solid that is between the vertical lines $x = 1$ and $x = 2$.

Q. Find V.

Figure to be
inserted here.

This time, we do not know of any formulae from primary or secondary school that will help us find V. Instead, we can only use integration:

Again, we'll slice the solid into infinitely many infinitely thin slices, then add up the volumes of these slices:

Figure to be
inserted here.

At each x, the corresponding infinitely thin slice has area $\pi y^2 = \pi\left(x^2\right)^2 = \pi x^4$.

Let $f(x)$ be the volume of the cylinder from 0 to x.

FTC1: The derivative of f must be one of these thin slices. That is,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \pi x^4.$$

FTC2: We can use an antiderivative of f to find V:

$$V = \int_1^2 \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = \int_1^2 \pi x^4\,\mathrm{d}x = \left[\frac{\pi}{5}x^5\right]_1^2 = \frac{\pi}{5}(32-1) = \frac{31}{5}\pi.$$

Definition 227 (*informal*)**.** Given a curve, rotate it about the x-axis to form a solid. We call the volume of the solid between $x = a$ and $x = b$ the *curve's volume of rotation about the x-axis from a to b.*

Figure to be
inserted here.

Here's the general result or formula whose intuition was illustrated in the above examples. This isn't on List MF26, so yea you'll have to *mug* it.

Proposition 18. *Given a curve $y = f(x)$, its volume of rotation about the x-axis from a to b is*

$$\int_a^b \pi y^2 \, \mathrm{d}x \qquad \text{or} \qquad \int_a^b \pi \left[f(x) \right]^2 \, \mathrm{d}x.$$

Proof. Omitted.[513] □

We can now easily find the general formulae for volumes of spheres and cones:

[513]We cat prove this only after we've formally and precisely defined such terms as *volume* and *volume of rotation*. And given that this subchapter is the only place in H2 Maths Calculus where volume is discussed, I decided not to bother adding another 10 pages to the Appendices for that task (and that no one will read anyway).

Example 1413. Find the volume of a sphere with radius r.

Figure to be
inserted here.

To do so, consider the curve $x^2 + y^2 = r^2$, for $x \in [-r, r]$ and $y \geq 0$. Rotating this curve about the x-axis produces our sphere:

Figure to be
inserted here.

Hence, by Proposition 18, the volume of the sphere is

$$\int_{-r}^{r} \pi y^2 \, dx = \pi \int_{-r}^{r} r^2 - x^2 \, dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right] = \frac{4}{3} \pi r^3.$$

Example 1414. Find the volume of a cone with base radius r and height h.

Figure to be inserted here.

To do so, consider the line segment $y = \frac{r}{h}x$, for $x \in [0, h]$. Rotating this curve about the x-axis produces our cone:

Figure to be inserted here.

Hence, by Proposition 18, the volume of the cone is

$$\int_0^h \pi \left(\frac{r}{h}x\right)^2 \mathrm{d}x = \pi\frac{r^2}{h^2}\int_0^h x^2\,\mathrm{d}x = \pi\frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \pi\frac{r^2}{h^2}\frac{h^3}{3} = \frac{1}{3}\pi r^2 h.$$

Remark 173. What we've just learnt is sometimes called the **disc method**, the **method of discs**, or **disc integration**.

(There are other methods for finding volumes, but happily, these aren't in H2 Maths.)

More examples:

Example 1415. XXX

Example 1416. XXX

Example 1417. XXX

Exercise 449. Consider the curve $y = \sin x$. Find its volume of rotation about the x-axis from 0 to π. (Answer on p. **1945**.)

109.7. Volume of Rotation about the y-axis

In the previous subchapter, we learnt to find the volume of rotation about the x-axis. Of course, finding the volume of rotation about the y-axis is exactly analogous:

Definition 228 (*informal*)**.** Given a curve, rotate it about the y-axis to form a solid. We call the volume of the solid between $y = a$ and $y = b$ the *curve's volume of rotation about the y-axis from a to b.*

Figure to be inserted here.

Proposition 19. *Given a curve $x = f(y)$, its volume of rotation about the y-axis from a to b is*

$$\int_a^b \pi x^2 \, dy \qquad \text{or} \qquad \int_a^b \pi [f(y)]^2 \, dy.$$

Proof. Omitted. □

Example 1418. Consider the curve $y = x^2$. Rotate it about the y-axis to form a solid. Let V be the volume of the portion of the solid that is between the horizontal lines $y = 1$ and $y = 2$.

Figure to be inserted here.

$$V = \int_1^2 \pi x^2 \, dy = \int_1^2 \pi y \, dy = \pi \left[\frac{y^2}{2} \right]_1^2 = \frac{3}{2}\pi.$$

Example 1419. XXX

Example 1420. XXX

Example 1421. XXX

Exercise 450. Consider the curve $y = \sin x$, for $x \in [0, \pi/2]$. Let V be its volume of rotation about the y-axis from 0 to 1.

(a) Find $\int u^2 \cos u \, du$. (Hint: Use Integration by Parts twice.)

(b) Find $\int \left(\sin^{-1} y\right)^2 dy$. (Hint: Use the substitution $y = \sin u$ or $u = \sin^{-1} y \in [0, 1]$, **(a)**, and Lemma **??**(b).)

(c) Hence, find V. (Answer on p. 1945.)

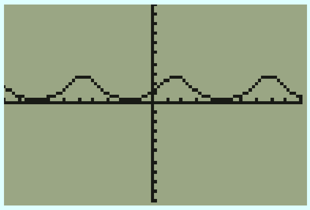
Example 1422. Let A be the area bounded by the curve $y = e^{\sin x}$, the x-axis, and the vertical lines $x = 1$ and $x = 2$. Find A using your TI84.

Step 1.

Step 2.

Step 3.

Step 4.



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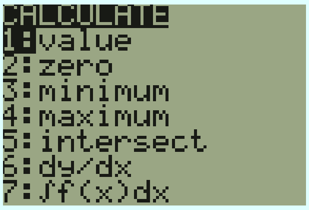

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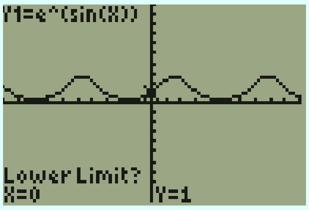

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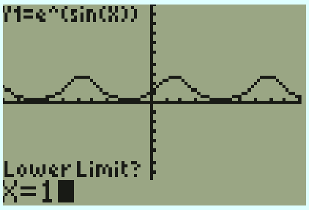

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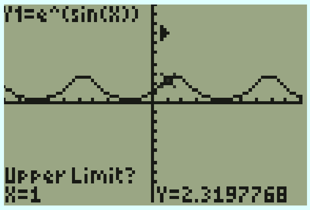
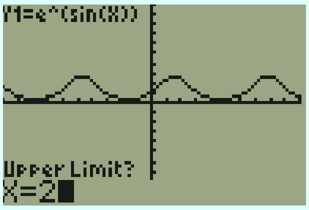
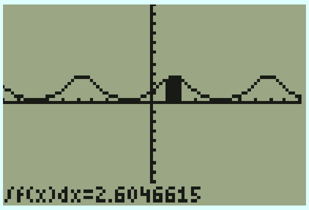


Step 9.



Step 10.



1. Press ON to turn on your calculator.
2. Press Y=.
3. Press blue 2ND button and then e^x (which corresponds to the LN button). Then press SIN X,T,θ,n) and altogether you will have entered $e^{\sin x}$.
4. Now press GRAPH and the calculator will (very slowly) graph the given equation.
5. Press the blue 2ND button and then CALC (which corresponds to the TRACE button), to bring up the CALCULATE menu.
6. Press 7 to select the "$\int f(x)\,dx$" option. This brings you back to the graph.
7. The TI84 is now prompting you for "Lower Limit?" Simply press 1.
8. Now press ENTER and you will have told the TI84 that your lower limit is $x = 1$.
9. The TI84 is now similarly prompting you for "Upper Limit?" Simply press 2.
10. Now press ENTER and you will have told the TI84 that your upper limit is $x = 2$. The TI84 also informs you that "$\int f(x)\,dx = 2.6046615$". This is our desired area A (which our TI84 also crudely shades in black).

110. Introduction to Differential Equations

110.1. $\dfrac{dy}{dx} = f(x)$

We won't actually be doing anything new in this subchapter. Instead, all we'll do is introduce the four terms **differential equation**, **general solution**, **initial condition**, and **particular solution**.

For convenience, we reproduce from Ch. 103.4 this result:

Fact 215. *Let $f(x)$ and $g(x)$ be expressions containing the variable x. If $\frac{d}{dx}g(x) = f(x)$, then $\int f(x)\,dx = g(x) + C$.*

Example 1423. Here's a **differential equation**:

$$\frac{dy}{dx} \stackrel{1}{=} x^2.$$

Q. Solve $\stackrel{1}{=}$. (That is, express y in terms of x.)

By Fact 215, $y = \int x^2\,dx = \frac{x^3}{3} + C$. We call

$$y \stackrel{2}{=} \frac{x^3}{3} + C.$$

the **general solution** to the differential equation $\stackrel{1}{=}$. It is *general* because the constant of integration C is free to vary (it can be any real number), so that there are infinitely many possible solutions for y.

Suppose we are now also told that if $x = 0$, then $y = 1$. That is,

$$(x, y) \stackrel{3}{=} (0, 1).$$

This additional piece of information $\stackrel{3}{=}$ is often called an **initial condition**. Here's why: Say y is the mass (g) of bacteria in a Petri dish and x is time (s). Then the initial condition $\stackrel{3}{=}$ says that at time $x = 0$ (i.e. "initially"), the mass of bacteria $y = 1$. And over time, the mass of bacteria changes according to the differential equation $\stackrel{1}{=}$.

Plugging the initial condition $\stackrel{3}{=}$ into the general solution $\stackrel{2}{=}$, we get $1 = \frac{0^3}{3} + C = C$. Hence,

$$y \stackrel{4}{=} \frac{x^3}{3} + 1.$$

We call $\stackrel{4}{=}$ the ***particular*** **solution** to "the differential equation $\frac{dy}{dx} \stackrel{1}{=} x^2$ with initial condition $(x, y) \stackrel{3}{=} (0, 1)$". (It is *particular* in that it corresponds to a specific, given initial condition.)

Example 1424. Solve this differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \overset{1}{=} \cos x.$$

The general solution to $\overset{1}{=}$ is

$$y \overset{2}{=} \int \cos x \, \mathrm{d}x = \sin x + C.$$

We are now given this initial condition:

$$(x, y) \overset{3}{=} (0, 2).$$

Plugging $\overset{3}{=}$ into $\overset{2}{=}$, we get $2 = \sin 0 + C = C$. Hence, the particular solution to $\overset{1}{=}$ is

$$y \overset{4}{=} \sin x + 2.$$

Example 1425. Solve this differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \overset{1}{=} x^2 - \sin x.$$

The general solution to $\overset{1}{=}$ is

$$y \overset{2}{=} \int x^2 - \sin x \, \mathrm{d}x = \frac{x^3}{3} + \cos x + C.$$

We are now given this initial condition:

$$(x, y) \overset{3}{=} (0, 3).$$

Plugging $\overset{3}{=}$ into $\overset{2}{=}$, we get $3 = \frac{0^3}{3} + \cos 0 + C = 1 + C$ or $C = 2$. Hence, this particular solution to $\overset{1}{=}$ is

$$y \overset{4}{=} \frac{x^3}{3} + \cos x + 2.$$

In the above examples, our initial condition has $x = 0$. But this need not generally be so:

Example 1426. Solve this differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{1}{=} \frac{1}{x} + \exp x, \qquad \text{for } x \neq 0.$$

The general solution to $\stackrel{1}{=}$ is

$$y \stackrel{2}{=} \int \frac{1}{x} + \exp x \, \mathrm{d}x = \ln|x| + \exp x + C$$

We are now given this initial condition:

$$(x, y) \stackrel{3}{=} (1, 2) .$$

Plugging $\stackrel{3}{=}$ into $\stackrel{2}{=}$, we get $2 = \ln|1| + \exp 1 + C = \mathrm{e} + C$ or $C = 2 - \mathrm{e}$. Hence, the particular solution to $\stackrel{1}{=}$ is

$$y \stackrel{4}{=} \frac{x^3}{3} + \cos x + 2.$$

Exercise 451. Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \exp x \sin x$. Given also the initial condition $(x, y) = (0, 1)$, find the particular solution. (Answer on p. **1947**.)

110.2. $\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$

Again, this subchapter will involve nothing new. We'll just put together stuff we've already learnt:

Example 1427. Solve this differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{1}{=} y^2.$$

By the Inverse Function Theorem (Ch. 91.2),

$$\frac{\mathrm{d}x}{\mathrm{d}y} \stackrel{1}{=} \frac{1}{y^2}, \qquad \text{for } y \neq 0.$$

The general solution to $\stackrel{1}{=}$ is

$$x \stackrel{2}{=} \int \frac{1}{y^2}\,\mathrm{d}y = -\frac{1}{y} + C \qquad \text{or} \qquad y \stackrel{2}{=} \frac{1}{C-x} \text{ (for } x \neq C\text{)}.$$

We are now given this initial condition:

$$(x, y) \stackrel{3}{=} (0, 1).$$

Plugging $\stackrel{3}{=}$ into $\stackrel{2}{=}$, we get $C = 1$. Hence, the particular solution to $\stackrel{1}{=}$ is

$$x \stackrel{4}{=} -\frac{1}{y} + 1 \qquad \text{or} \qquad y \stackrel{2}{=} \frac{1}{1-x} \text{ (for } x \neq 1\text{)}.$$

Example 1428. Solve this differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}y} \overset{1}{=} \frac{1}{\sin y}, \qquad \text{for } y \in (0, \pi).$$

By the Inverse Function Theorem (Ch. 91.2),

$$\frac{\mathrm{d}x}{\mathrm{d}y} \overset{1}{=} \sin y, \qquad \text{for } y \in (0, \pi).$$

The general solution to $\overset{1}{=}$ is

$$x \overset{2}{=} \int \sin y \, \mathrm{d}y = -\cos y + C \qquad \text{or} \qquad y \overset{2}{=} \cos^{-1}(C - x).$$

We are now given this initial condition:

$$(x, y) \overset{3}{=} \left(0, \frac{\pi}{2}\right).$$

Plugging $\overset{3}{=}$ into $\overset{2}{=}$, we get $C = 0$. Hence, the particular solution to $\overset{1}{=}$ is

$$x \overset{2}{=} -\cos y \qquad \text{or} \qquad y \overset{2}{=} \cos^{-1}(-x).$$

Example 1429. Solve this differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \overset{1}{=} 1 + y^2.$$

By the Inverse Function Theorem (Ch. 91.2),

$$\frac{\mathrm{d}x}{\mathrm{d}y} \overset{1}{=} \frac{1}{1 + y^2}.$$

By Proposition 16(a), the general solution to $\overset{1}{=}$ is

$$x \overset{2}{=} \int \frac{1}{1 + y^2} \, \mathrm{d}y = \tan^{-1} y + C \qquad \text{or} \qquad y \overset{2}{=} \tan(x - C).$$

We are now given this initial condition:

$$(x, y) \overset{3}{=} (0, 1).$$

Plugging $\overset{3}{=}$ into $\overset{2}{=}$, we get $C = -\pi/4$. Hence, the particular solution to $\overset{1}{=}$ is

$$x \overset{2}{=} \tan^{-1} y - \frac{\pi}{4} \qquad \text{or} \qquad y \overset{2}{=} \tan\left(x + \frac{\pi}{4}\right).$$

Example 1430. Solve this differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{1}{=} \sin y, \qquad \text{for } y \in (0, \pi).$$

By the Inverse Function Theorem (Ch. 91.2),

$$\frac{\mathrm{d}x}{\mathrm{d}y} \stackrel{1}{=} \frac{1}{\sin y} = \operatorname{cosec} y, \qquad \text{for } y \in (0, \pi).$$

By Proposition 16(g),

$$x = \int \operatorname{cosec} y \,\mathrm{d}y = -\ln|\operatorname{cosec} y + \cot y| + C.$$

We are now given this Hint: $\operatorname{cosec} y + \cot y = \cot \frac{y}{2}$. (Can you show this?)[514]

Hence, $\quad x = -\ln|\operatorname{cosec} y + \cot y| + C = -\ln\left|\cot \frac{y}{2}\right| + C \stackrel{2}{=} -\ln\left(\cot \frac{y}{2}\right) + C.$

(We can remove the absolute value operator because $\cot \frac{y}{2} \geq 0$ for $y \in (0, \pi)$.)

Now rearrange $\stackrel{2}{=}$ to also get

$$\exp(C - x) = \cot \frac{y}{2} \qquad \text{or} \qquad \exp(x - C) = \tan \frac{y}{2} \qquad \text{or} \qquad y \stackrel{2}{=} 2\tan^{-1}(\exp(x - C)).$$

Altogether, the general solution to $\stackrel{1}{=}$ is

$$x \stackrel{2}{=} -\ln\left(\cot \frac{y}{2}\right) + C \qquad \text{or} \qquad y \stackrel{2}{=} 2\tan^{-1}(\exp(x - C)).$$

Suppose we are now given this initial condition:

$$(x, y) \stackrel{3}{=} \left(1, \frac{\pi}{2}\right).$$

Plug $\stackrel{3}{=}$ into $\stackrel{2}{=}$ to get $1 = -\ln\left(\cot \frac{\pi}{4}\right) + C = -\ln 1 + C = C$. Hence, the particular solution to $\stackrel{1}{=}$ is

$$x \stackrel{2}{=} -\ln\left(\cot \frac{y}{2}\right) + 1 \qquad \text{or} \qquad y \stackrel{2}{=} 2\tan^{-1}(\exp(x - 1)).$$

[514] $\operatorname{cosec} y + \cot y = \frac{1}{\sin y} + \frac{\cos y}{\sin y} = \frac{1 + \cos y}{\sin y} = \frac{1 + 2\cos^2 \frac{y}{2} - 1}{2\sin \frac{y}{2}\cos \frac{y}{2}} = \frac{\cos \frac{y}{2}}{\sin \frac{y}{2}} = \cot \frac{y}{2}$.

Exercise 452. Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \exp y$. Given also the initial condition $(x, y) = (0, 0)$, find the particular solution. (Answer on p. **1947**.)

110.3. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x)$

In the last two subchapters, we've been looking at **first-order differential equations**. We now look at **second-order differential equations**:

Example 1431. Solve this differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \overset{1}{=} x^2.$$

By Fact 215, $\frac{\mathrm{d}y}{\mathrm{d}x} = \int x^2 \,\mathrm{d}x = \frac{x^3}{3} + C$.

Again by Fact 215, $y = \int \frac{x^3}{3} + C \,\mathrm{d}x = \frac{x^4}{12} + Cx + \bar{C}$.

Hence, the general solution to $\overset{1}{=}$ is

$$y \overset{2}{=} \frac{x^4}{12} + Cx + \bar{C}.$$

Observe that with a second-order differential equation, our general solution will (usually) contain two constants of integration. And so, to get a particular solution, we need to be given two initial conditions.

Suppose we're now given these two initial conditions:

$$(x,y) \overset{3}{=} (0,1) \qquad \text{and} \qquad (x,y) \overset{4}{=} (1,0).$$

Plugging $\overset{3}{=}$ into $\overset{2}{=}$, we get $1 = 0 + 0 + \bar{C}$ or $\bar{C} \overset{5}{=} 1$.

Plugging $\overset{4}{=}$ and $\overset{5}{=}$ into $\overset{2}{=}$, we get $0 = \frac{1}{12} + C + 1$ or $C = -\frac{13}{12}$.

Hence, the particular solution to $\overset{1}{=}$ is

$$y \overset{6}{=} \frac{x^4}{12} - \frac{13}{12}x + 1.$$

Example 1432. Solve this differential equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \stackrel{1}{=} \sin x.$$

The general solution to $\stackrel{1}{=}$ is

$$y \stackrel{2}{=} \int \left(\int \sin x \,\mathrm{d}x \right) \mathrm{d}x = \int (-\cos x + C)\, \mathrm{d}x = -\sin x + Cx + \bar{C}.$$

We're now given these two initial conditions:

$$(x, y) \stackrel{3}{=} (0, 1) \qquad \text{and} \qquad (x, y) \stackrel{4}{=} (\pi, 0).$$

Plugging $\stackrel{3}{=}$ into $\stackrel{2}{=}$, we get $1 = 0 + 0 + \bar{C}$ or $\bar{C} \stackrel{5}{=} 1$.

Plugging $\stackrel{4}{=}$ and $\stackrel{5}{=}$ into $\stackrel{2}{=}$, we get $0 = 0 + C\pi + 1$ or $C = -\dfrac{1}{\pi}$.

Hence, the particular solution to $\stackrel{1}{=}$ is

$$y \stackrel{6}{=} -\sin x - \frac{x}{\pi} + 1.$$

Exercise 453. Solve $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \exp x \sin x$. Given also the initial conditions $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$, find the particular solution.x (Answer on p. **1947**.)

110.4. Word Problems

The H2 Maths syllabus includes:

- "formulating a differential equation from a problem situation"; and
- "interpreting a differential equation and its solution in terms of a problem situation".[515]

So these are what we'll cover in this subchapter.

Example 1433. The mass of bacteria on a plate grows at a rate that is inversely proportional to the present mass of bacteria. Express the mass of bacteria as a function of time.

Let x be the mass of bacteria. Let t be time. We are given that the growth rate of x is inversely proportional to x. That is,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{k}{x}, \qquad \text{for some } k \in \mathbb{R} \text{ (and } x > 0\text{)}.$$

Or,
$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{x}{k}$$

Hence,
$$t = \int \frac{x}{k}\,\mathrm{d}x = \frac{x^2}{2k} + C.$$

Further rearranging, we have $x = \pm\sqrt{2k(t-C)}$. Since the mass of bacteria can't be negative, we can reject the negative root and conclude that

$$x = \sqrt{2k(t-C)}.$$

We are now further given the initial conditions $(t, x) = (0, 1)$ and $(t, x) = (1, 2)$. Plugging these into the above equation, we get

$$1 = \sqrt{2k(-C)} \qquad \text{and} \qquad 2 = \sqrt{2k(1-C)}.$$

Solving, $C = -1/3$ and $k = 3/2$. Thus,

$$x = \sqrt{3t+1}.$$

[515] I understand "problem situation" to mean more simply "word problem".

Exercise 454. Suppose a small object is initially on the surface of the Earth. It is then shot upwards with initial velocity V. The minimum initial velocity such that the object travels upwards forever is called that object's **escape velocity** from Earth (so called because the object *escapes* Earth's gravity).

In this exercise, we will work towards finding the escape velocity, V_e.[516] (Answer on p. 1948.)

(a) Newton's Law of Gravitation states that the force of attraction $\mathbf{F}$ (unit: N or kg m s^{-2}) between two point masses M and m (kg) is proportional to the product of their masses and inversely proportional to the square of the distance r (m) between them.

Write down this law as an equation. Your equation should contain a constant denoted G (this is the *gravitational constant*).

(b) Momentum is defined as the product of mass m and velocity $\mathbf{v}$. Newton's Second Law of Motion states that force $\mathbf{F}$ is the rate of change of momentum.

(i) Write down this law as an equation.

(ii) Show that if mass m is constant, then $\mathbf{F} = m\dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$.

Let m be the mass of our small object and M be the mass of the Earth. Assume

- The Earth is a fixed perfect sphere with radius R m and its mass is concentrated at a single point at its centre.
- Our small object is initially on the surface of the earth. At time $t = 0$ s, it is shot upwards with initial velocity $V > 0$.
- Once the ball is shot upwards, the sole force acting on it is gravity. (In particular, there is no air resistance or any other form of friction.)

Figure to be inserted here.

Let r be the distance (m) the ball is above the centre of the Earth and h be the height (m) is above the surface of the Earth. (Hence, $r = R + h$.)

(c) Explain why $\dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\dfrac{GM}{r^2}$.

(d) From **(c)**, we have $\displaystyle\int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}r = \int -\frac{GM}{r^2}\,\mathrm{d}r$. By the Substitution Rule, $\displaystyle\int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}r = \int \frac{\mathrm{d}r}{\mathrm{d}t}\,\mathrm{d}\mathbf{v}$.[517]

(i) Show that $\displaystyle\int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}r = \frac{1}{2}\mathbf{v}^2 + C$.

(ii) Find $\displaystyle\int -\frac{GM}{r^2}\,\mathrm{d}r$.

(iii) What initial condition were we given for $(r, \mathbf{v})$?

110.5. Yet Another Way to Define Sine and Cosine (optional)

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because they are vectors.

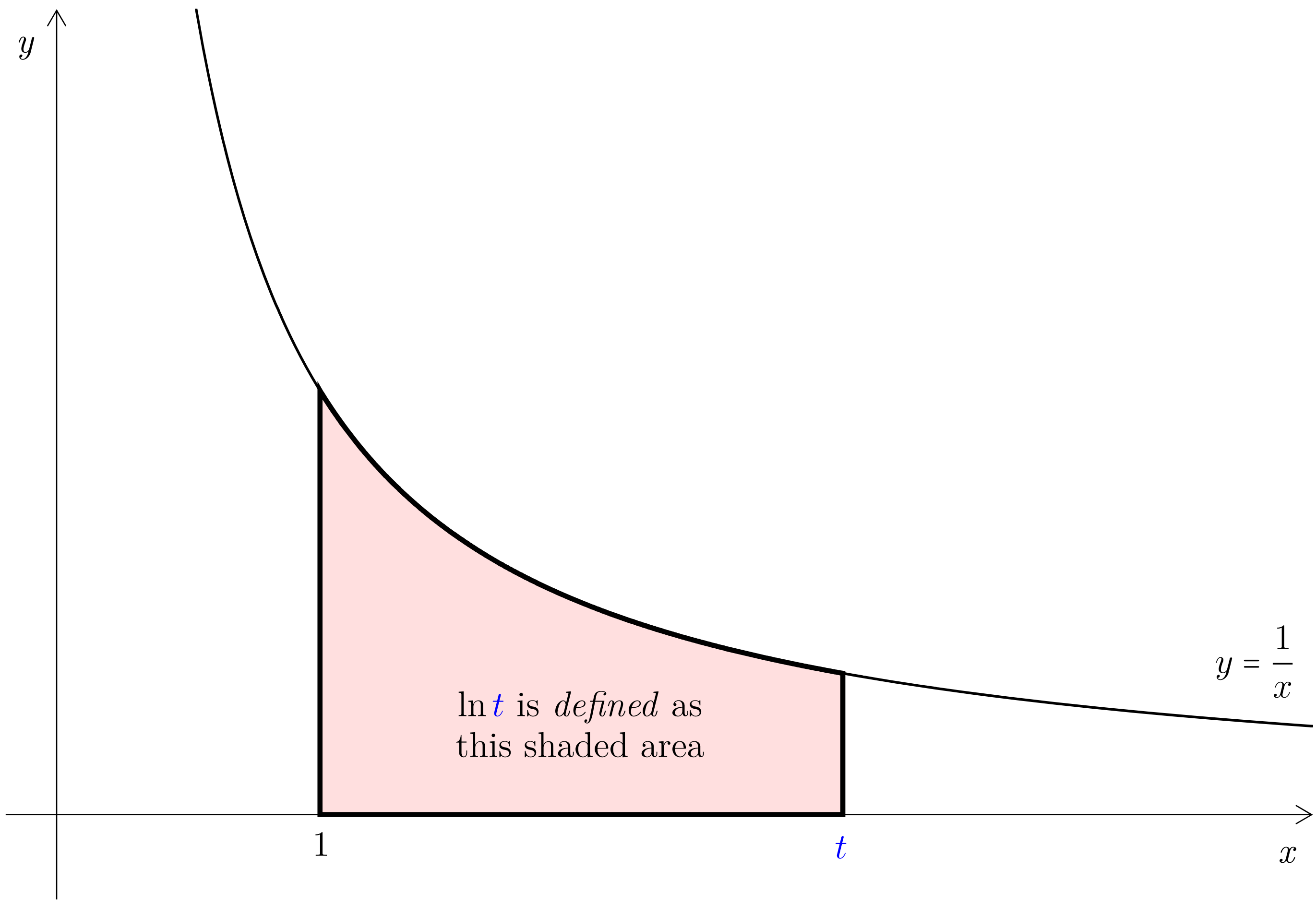
[517] "*Proof*". We have $\int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}r = \int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}\,\mathrm{d}r = \int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\cancel{\mathrm{d}r}}\cancel{\mathrm{d}r} = \int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}\,\mathrm{d}t$.

Similarly, $\int \frac{\mathrm{d}r}{\mathrm{d}t}\,\mathrm{d}\mathbf{v} = \int \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\mathbf{v}}\,\mathrm{d}\mathbf{v} = \int \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\frac{\mathrm{d}t}{\cancel{\mathrm{d}\mathbf{v}}}\cancel{\mathrm{d}\mathbf{v}} = \int \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}t$.

Hence, $\int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,\mathrm{d}r = \int \frac{\mathrm{d}r}{\mathrm{d}t}\,\mathrm{d}\mathbf{v}$. □

111. Revisiting ln, $\log_b$, and exp (optional)

In Ch. 28, we informally defined the **natural logarithm function** ln as giving us a particular area under a particular graph:



Now that we've learnt about integration, we can (at long last) write down its formal definition:

Definition 229. The *natural logarithm (function)* $\ln : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t} \, \mathrm{d}t.$$

Remark 174. In the above equation, students are often confused by the presence of the two variables x and t. *Both* are dummy variables that could be replaced by any other symbol. So for example, the following three equations are *exactly* equivalent:

$$\ln x = \int_1^x \frac{1}{t} \, \mathrm{d}t, \qquad \ln \star = \int_1^\star \frac{1}{\circ} \, \mathrm{d}\circ, \qquad \ln ☺ = \int_1^{☺} \frac{1}{\blacksquare} \, \mathrm{d}\blacksquare.$$

However, we must be careful not to mix up x and t. The dummy variable x is used to tell us about the mapping rule of ln, while the dummy variable t is the variable of integration.

By the above definition and the FTC1, we have this result:

Fact 221. *The derivative of* $\ln : \mathbb{R}^+ \to \mathbb{R}$ *is the function* $f : \mathbb{R}^+ \to \mathbb{R}$ *defined by* $f(x) = \frac{1}{x}$.

Proof. By Definition 229, for any $x \in \mathbb{R}^+$,

$$\ln x = \int_1^x f.$$

And so, by the FTC1, $\frac{\mathrm{d}}{\mathrm{d}x} \ln = f$. □

Since ln is differentiable, by Theorem 29, it is also continuous. Hence the following result that was given long ago in Ch. 87.4 and now reproduced:

Fact 202. *The natural logarithm function,* ln*, is continuous.*

With the above definition, it is not difficult to prove some basic properties of the natural logarithm function:

Fact 222. *Suppose* $x, y > 0$. *Then*

(a) $\ln 1 = 0$.

(b) $\ln(xy) = \ln x + \ln y$.

(c) $\ln \frac{1}{x} = -\ln x$.

(d) $\ln \frac{x}{y} = \ln x - \ln y$.

Proof. **(a)** Simply plug $x = 1$ into Definition 229, then apply Definition 225(a):

$$\ln 1 = \int_1^1 \frac{1}{x} \, \mathrm{d}x = 0.$$

(b) We will play a little trick. Differentiate each side with respect to x to get

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(xy) = \frac{1}{xy}\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{1}{x} + \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x},$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x + \ln y) = \frac{1}{x} + \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

By Corollary 47, $\ln(xy)$ and $\ln x + \ln y$ differ by at most a constant:

$$\ln(xy) = \ln x + \ln y + C, \qquad \text{for some } C \in \mathbb{R}.$$

Plug in $x = 1$ to get $\ln y = \ln 1 + \ln y + C = \ln y + C$, or $C = 0$. Thus, $\ln(xy) = \ln x + \ln y$.

(c) See Exercise 455(a).

(d) follows from **(b)** and **(c)**. □

Exercise 455. Prove Fact 222(c) (use the same trick as in the proof of Fact 222(b)). (Answer on p. 1950.)

Fact 223. *Suppose $x > 0$ and $n \in \mathbb{R}$. Then* $\ln x^n = n \ln x$.

Proof. See p. 1724 (Appendices). □

We can now prove the following result (given long ago in Ch. 28.2 and now reproduced):

Theorem 6. $\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Proof. By Fact 221, $\ln' 1 \stackrel{1}{=} 1/1 = 1$.

Also, by Definition 203 (of the derivative),

$$\ln' 1 = \lim_{x\to 1}\frac{\ln x - \ln 1}{x-1} = \lim_{x\to 1}\frac{\ln x}{x-1} \stackrel{3}{=} \lim_{1+\frac{1}{n}\to 1} n\ln\left(1+\frac{1}{n}\right) \stackrel{2}{=} \lim_{n\to\pm\infty}\ln\left(1+\frac{1}{n}\right)^n,$$

where at $\stackrel{3}{=}$, we let $x = 1+\frac{1}{n}$ and hence also $x \to 1 \iff 1+\frac{1}{n} \to 1 \iff n \to \pm\infty$.

Together, $\stackrel{1}{=}$ and $\stackrel{2}{=}$ show that $\lim_{n\to\pm\infty}\ln\left(1+\frac{1}{n}\right)^n \stackrel{4}{=} 1$. Thus,

$$\mathrm{e} = \exp 1 \stackrel{4}{=} \exp\left(\lim_{n\to\pm\infty}\ln\left(1+\frac{1}{n}\right)^n\right) \stackrel{5}{=} \lim_{n\to\pm\infty}\left[\exp\left(\ln\left(1+\frac{1}{n}\right)^n\right)\right] \stackrel{6}{=} \lim_{n\to\pm\infty}\left(1+\frac{1}{n}\right)^n,$$

where $\stackrel{5}{=}$ uses the continuity of exp and Theorem 25 (continuity allows us to "move" limits), while $\stackrel{6}{=}$ uses Definition 85 (exp is inverse of ln).

The above proof has also shown that $\mathrm{e} = \lim_{n\to-\infty}\left(1+\frac{1}{n}\right)^n$. □

111.1. Revisiting Logarithms

In Ch. 5.7, Informal Definition 32 of logarithms said that

$$b^x = n \qquad \Longrightarrow \qquad x = \log_b n.$$

Now that we've formally defined the natural logarithm ln, we can replace Definition 32 with the following definition:

Definition 230. Suppose $b, n > 0$ with $b \neq 1$. Then the *base b logarithm of n* is denoted $\log_b n$ and defined by

$$\log_b n = \frac{\ln n}{\ln b}.$$

In Ch. 5.7, we gave informal proofs of the Laws of Logarithms. With the above Definitions and results, we can now prove these rigorously and indeed more easily:

Proposition 2. (Laws of Logarithms) *Let $x \in \mathbb{R}$ and $a, b, c > 0$ with $b \neq 1$. Then*

(a) $\log_b 1 = 0$

(b) $\log_b b = 1$

(c) $\log_b b^x = x$

(d) $b^{\log_b a} = a$

(e) $c \log_b a = \log_b a^c$

(f) $\log_b \frac{1}{a} = -\log_b a$ **(Logarithm of Reciprocal)**

(g) $\log_b (ac) = \log_b a + \log_b c$ **(Sum of Logarithms)**

(h) $\log_b \frac{a}{c} = \log_b a - \log_b c$ **(Difference of Logarithms)**

(i) $\log_b a = \frac{\log_c a}{\log_c b}$, *for* $c \neq 1$ **(Change of Base)**

(j) $\log_{a^b} c = \frac{1}{b} \log_a c$, *for* $a \neq 1$

Proof. Below, $\stackrel{\star}{=}$ indicates the use of Definition 230

(a) $\log_b 1 \stackrel{\star}{=} \frac{\ln 1}{\ln b} = 0.$

(b) $\log_b b \stackrel{\star}{=} \frac{\ln b}{\ln b} = 1.$

(c) $\log_b b^x \stackrel{\star}{=} \frac{\ln b^x}{\ln b} = \frac{x \ln b}{\ln b} = x$. (The middle step uses Fact 223.)

(d) $\log_b a \stackrel{\star}{=} \frac{\ln a}{\ln b}$. Rearranging, $(\log_b a) \ln b = \ln a$.

By Fact 223, $\ln b^{\log_b a} = \ln a$.

Apply exp: $\exp\left(\ln b^{\log_b a}\right) = \exp(\ln a)$ or $b^{\log_b a} = a$ (exp is defined as inverse of ln).

(e)–(j) See Exercise 456. $\square$

Exercise 456. Prove Proposition 2(e)–(j). (Answer on p. 1950.)

111.2. Revisiting the Exponential Function

We reproduce from Ch. 28 our formal definition of the exponential function:

Definition 85. The *exponential function*, denoted exp, is the inverse of the natural logarithm function.

We now prove certain basic properties about the exponential function:

Fact 224. *Suppose $x, y \in \mathbb{R}$. Then*

(a) $\exp 0 = 1$.

(b) $\exp 1 = \mathrm{e}$.

(c) $\exp(x+y) = \exp x \exp y$.

(d) $\exp(-x) = \dfrac{1}{\exp x}$.

(e) $\exp(x-y) = \dfrac{\exp x}{\exp y}$.

Proof. **(a)** By Fact 222(a), $\ln 1 = 0$. So, $\exp 0 = 1$.

(b) This is actually not a result but our definition of Euler's number (Definition 86).

(c) Let $a = \exp x$ and $b = \exp y$. Then by definition, $x = \ln a$ and $y = \ln b$. So,

$$\exp(x+y) = \exp(\ln a + \ln b) \stackrel{1}{=} \exp[\ln(ab)] = ab = (\exp x)(\exp y),$$

where $\stackrel{1}{=}$ uses Fact 222(b).

(d) $\exp(-x)\exp x \stackrel{\text{(c)}}{=} \exp(-x+x) = \exp 0 \stackrel{\text{(a)}}{=} 1$. Rearranging (note that $\exp x > 0$ for all $x \in \mathbb{R}$),

$$\exp(-x) = \frac{1}{\exp x}.$$

(e) is immediate from **(c)** and **(d)**. □

We can now also easily prove that **the derivative of the exponential function is itself**:

Fact 225. $\dfrac{\mathrm{d}}{\mathrm{d}x}\exp = \exp$.

Proof. By the Chain Rule, $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(\exp x) \stackrel{1}{=} \dfrac{1}{\exp x}\dfrac{\mathrm{d}}{\mathrm{d}x}\exp x$.

But observing that $\ln(\exp x) = x$, we also have $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(\exp x) \stackrel{2}{=} 1$.

Put $\stackrel{1}{=}$ and $\stackrel{2}{=}$ together and rearrange to get $\dfrac{\mathrm{d}}{\mathrm{d}x}\exp x = \exp x$. □

Interestingly, the only function that is its own derivative is the exponential function (or constant multiples thereof):

Proposition 20. *Let $f : D \to \mathbb{R}$ be a nice function. Suppose f is its own derivative, i.e. $f = f'$. Then there exists some $c \in \mathbb{R}$ such that for all $x \in D$,*

$$f(x) = c \exp x.$$

Proof. Define $g : D \to \mathbb{R}$ by $g(x) = f(x) \exp(-x)$. Then for each $x \in D$,

$$g'(x) = f'(x) \exp(-x) - f(x) \exp(-x) = 0.$$

Since the derivative of g is the zero function, by Proposition 8, g must be a constant function. That is, there exists some $c \in \mathbb{R}$ such that for all $x \in D$, or

$$g(x) = c \qquad \text{or} \qquad f(x) \exp(-x) = c \qquad \text{or} \qquad f(x) = c \exp x.$$ □

Part VI.
Probability and Statistics

Exam Tips for *Towkays*

Probability and Statistics takes up 60 (out of 200) points and hence 30% of your A-Level exam.

Remark 175. This Part was mostly written in 2016. Since early 2018, I've been working to completely rewrite this textbook. Unfortunately, I haven't gotten to this Part yet and don't know when I'll have time to do so.

Two holes are better than one. Any mouse will tell you that.

— Willy Wonka (1972).

If you don't get this elementary, but mildly unnatural, mathematics of elementary probability into your repertoire, then you go through a long life like a one-legged man in an ass-kicking contest. You're giving a huge advantage to everybody else.

— Charlie Munger (1994).

112. How to Count: Four Principles

How many arrangements or **permutations** are there of the three letters in CAT? For example, one possible permutation of CAT is TCA.

To solve this problem, one possible method is **the method of enumeration**. That is, simply list out (enumerate) all the possible permutations.

ACT, ATC, CAT, CTA, TAC, TCA.

We see that there are 6 possible permutations.

Enumeration works well enough when we have just three letters, as in CAT. Indeed, enumeration is sometimes the quickest method.

In contrast, the 13 letters in the word UNPREDICTABLY have $6,227,020,800$ possible permutations. So enumeration is probably not practical.

To help us count more efficiently, we'll learn about four basic principles of counting:

1. The Addition Principle (AP);
2. The Multiplication Principle (MP);
3. The Inclusion-Exclusion Principle (IEP); and
4. The Complements Principle (CP).

112.1. How to Count: The Addition Principle

The **addition principle** (AP) is very simple.

Example 1434. For lunch today, I can either go to the food court or the hawker centre. At the food court, I have 2 choices: ramen or briyani. At the hawker centre, I have 3 choices: bak chor mee, nasi lemak, or kway teow.

Altogether then, I have $2 + 3 = 5$ choices of what to eat for lunch today.

Here's an informal statement of the AP:[518]

The Addition Principle (AP). *I have to choose a destination, out of two possible areas. At area #1, there are p possible destinations to choose from. At area #2, there are q possible destinations to choose from.*

The Addition Principle (AP) simply states that I have, in total, $p + q$ different choices.

(Just so you know, the AP is sometimes also called **the Second Principle of Counting** or **the Rule of Sum** or **the Disjunctive Rule**.)

Of course, the AP generalities to cases where there are more than just 2 "areas". It may seem a little silly, but just to illustrate, let's use the AP to tackle the CAT problem:

[518]See section 147.1 (Appendices) (optional) for a more precise statement of the AP.

Example 1435. Problem: *How many permutations are there of the letters in the word CAT?*

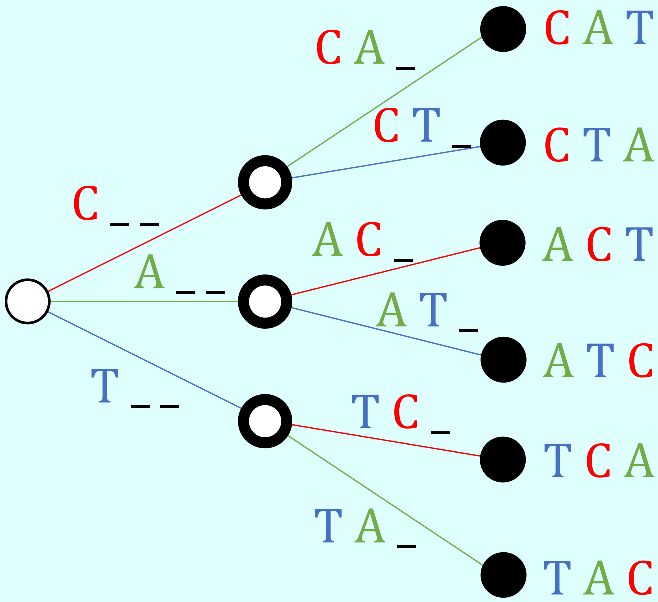
We can divide the possibilities into three cases:

Case #1. First letter is an A. Then the next two letters are either CT or TC—2 possibilities.

Case #2. First letter is a C. Then the next two letters are either AT or TA—2 possibilities.

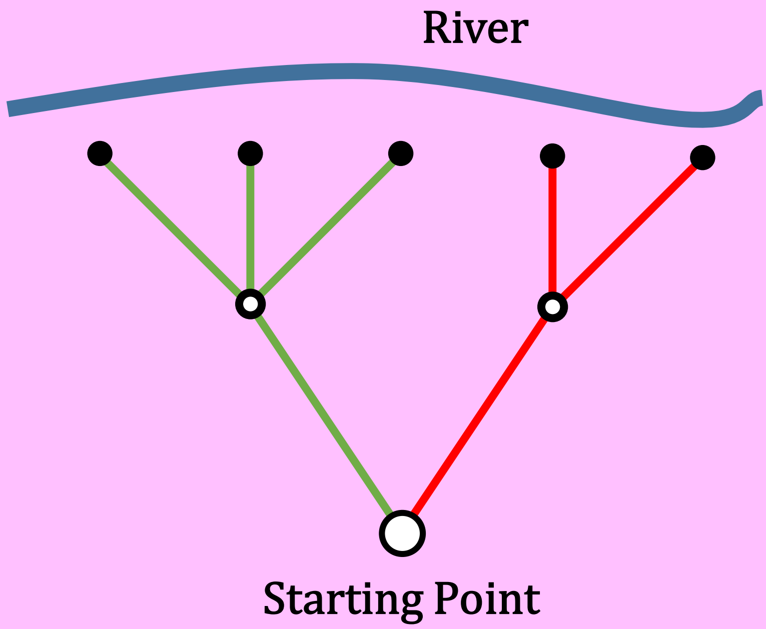
Case #3. First letter is a T. Then the next two letters are either AC or CA—2 possibilities.

Altogether then, by the AP, there are $2 + 2 + 2 = 6$ possibilities. That is, there are 6 possible permutations of the letters in CAT. These are illustrated in the **tree diagram** below.



The next exercise is very simple and just to illustrate again the AP.

Exercise 457. Without retracing your steps, how many ways are there to get from the Starting Point to the River (see figure below)? (Answer on p. **1951**.)



Exercise 458. How many permutations are there of the letters in the word DEED? Illustrate your answer with a tree diagram similar to that given in the CAT example above. (Answer on p. **1951**.)

112.2. How to Count: The Multiplication Principle

Example 1436. For lunch today, I can either have prata or horfun. For dinner tonight, I can have McDonald's, KFC, or Pizza Hut.

Enumeration shows that I have a total of 6 possible choices for my two meals today:

(Prata, McDonald's), (Prata, KFC), (Prata, Pizza Hut),

(Horfun, McDonald's), (Horfun, KFC), (Horfun, Pizza Hut).

Alternatively, we can use the **Multiplication Principle** (MP). I have 2 choices for lunch and 3 choices for dinner. Hence, for my two meals today, I have in total $2 \times 3 = 6$ possible choices.

Here's an informal statement of the MP:[519]

The Multiplication Principle (MP). *I have to choose two destinations, one from each of two possible areas. At area #1, there are p possible destinations to choose from. At area #2, there are q possible destinations to choose from.*

The Multiplication Principle (AP) simply states that I have, in total, $p \times q$ different choices.

(The MP is sometimes also called **the Fundamental** or **First Principle of Counting** or **the Rule of Product** or **the Sequential Rule**.)

Of course, the MP generalities to cases where there are more than just 2 "areas". Here's an example where we have to make 3 decisions:

[519]See section 147.1 (Appendices) (optional) for a more precise statement of the MP.

Example 1437. For breakfast tomorrow, I can have shark's fin or bird's nest (2 choices). For lunch, I can have black pepper crab or curry fishhead (2 choices). For dinner, I can have an apple, a banana, or a carrot (3 choices). By the MP, for tomorrow's meals, I have a total of $2 \times 2 \times 3 = 12$ possible choices. We can enumerate these (I'll use abbreviations):

$$\begin{array}{cccc} \text{(SF, BPC, A)}, & \text{(SF, BPC, B)}, & \text{(SF, BPC, C)}, & \text{(SF, CF, A)}, \\ \text{(SF, CF, B)}, & \text{(SF, CF, C)}, & \text{(BN, BPC, A)}, & \text{(BN, BPC, B)}, \\ \text{(BN, BPC, C)}, & \text{(BN, CF, A)}, & \text{(BN, CF, B)}, & \text{(BN, CF, C)}. \end{array}$$

Example 1438. Problem: *How many four-letter words can be formed using the letters in the 26-letter alphabet?*

Let's rephrase this problem so that it is clearly in the framework of the MP. We have 4 blank spaces to be filled:

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}.$$

These 4 blanks spaces correspond to 4 decisions to be made. Decision #1: What letter to put in the first blank space? Decision #2: What letter to put in the second blank space? Decision #3: What letter to put in the third blank space? Decision #4: What letter to put in the fourth blank space?

How many choices have we for each decision?

For Decision #1, we can put A, B, C, ..., or Z. So we have 26 choices for Decision #1.

For Decision #2, we can again put A, B, C, ..., or Z. So we *again* have 26 choices for Decision #2.

We likewise have 26 choices for Decision #3 and also 26 choices for Decision #4.

Altogether then, by the MP, there are $26 \times 26 \times 26 \times 26 = 26^4 = 456,976$ ways to make our four decisions.

Solution: *There are* $26^4 = 456,976$ *possible four-letter words that can be formed using the* 26-*letter alphabet.*

Example 1439. One 18-sided die has the numbers 1 through 18 printed on each of its sides. Another six-sided die has the letters A, B, C, D, E, and F printed on each of its sides. We roll the two dice. How many distinct possible outcomes are there?

Again, let's rephrase this problem in the framework of the MP. Consider 2 blank spaces:

$$\underset{1}{\_}\ \underset{2}{\_}.$$

These 2 blank spaces correspond to 2 decisions to be made. Decision #1: What number to put in the first blank space? Decision #2: What letter to put in the second blank space?

Again we ask: How many choices have we for each decision?

For Decision #1, we can put 1, 2, 3, ..., or 18. So we have 18 choices for Decision #1.

For Decision #2, we can put A, B, C, D, E, or F. So we have 6 choices for Decision #2.

Altogether then, by the MP, there are $18 \times 6 = 108$ ways to make our two decisions. In other words, there are 108 possible outcomes from rolling these two dice.

(If necessary, it is tedious but not difficult to enumerate them: 1A, 1B, 1C, 1D, 1E, 1F, 2A, 2B, ..., 17E, 17F, 18A, 18B, 18C, 18D, 18E, and 18F.)

Exercise 459. A club as a shortlist of 3 men for president, 5 animals for vice-president, and 10 women for club mascot. How many possible ways are there to choose the president, the vice-president, and the mascot? (Answer on p. **1952**.)

Exercise 460. (Answer on p. **1952**.) The highly stimulating game of 4D consists of selecting a four-digit number, between 0000 and 9999 (so there are $10,000$ possible numbers).

Your mother tells you to go to the nearest gambling den (also known as a Singapore Pools outlet) to buy any three numbers, subject to these two conditions:

- The four digits in each number are distinct.
- Each four-digit number is distinct.

How many possible ways are there to fulfil your mother's request?

112.3. How to Count: The Inclusion-Exclusion Principle

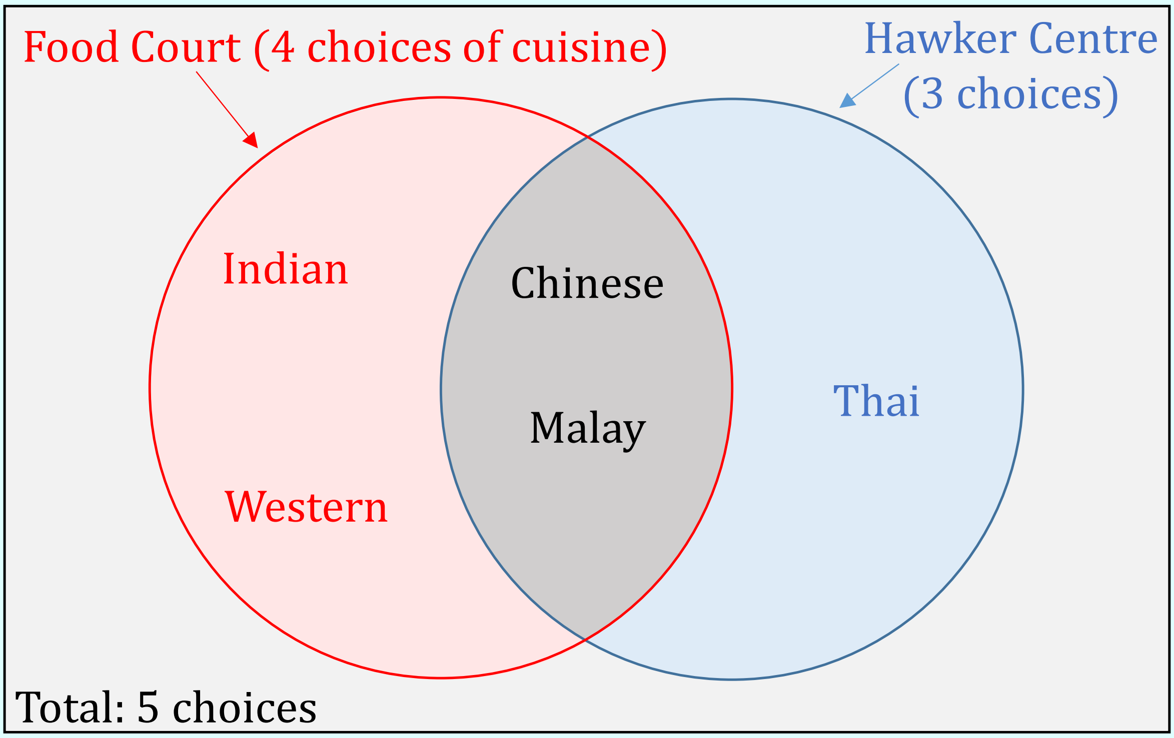
The **Inclusion-Exclusion Principle (IEP)** is another very simple principle.

Example 1440. For lunch today, I can either go to the food court or the hawker centre. At the food court, I have 4 choices of cuisine: Chinese, Indian, Malay, and Western. At the hawker centre, I have 3 choices of cuisine: Chinese, Malay, and Thai.

There are 2 choices of cuisine that are common to both the food court and the hawker centre (Chinese and Malay).

And so by the Inclusion-Exclusion Principle (IEP), I have in total $4 + 3 - 2 = 5$ choices of cuisine. The Venn diagram below illustrates.

Why do we subtract 2? If we simply added the 4 choices available at the food court to the 3 available at the hawker centre, then we'd double-count the Chinese and Malay cuisines, which are available at both the food court and the hawker centre. And so we must subtract the 2 cuisines that are at both locations.



Example 1441. Problem: *How many integers between* 1 *and* 20 *are divisible by* 2 *or* 5*?*

There are 10 integers divisible by 2, namely 2, 4, 6, 8, 10, 12, 14, 16, 18, and 20.

There are 4 integers divisible by 5, namely 5, 10, 15, and 20.

There are 2 integers divisible by BOTH 2 and 5, namely 10 and 20.

Hence, by the IEP, there are $10 + 4 - 2 = 12$ integers that are divisible by either 2 or 5. (These are namely 2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18, and 20.)

Here's an informal statement of the IEP:[520]

[520]See section 147.1 (Appendices) (optional) for a more precise statement of the IEP.

The Inclusion-Exclusion Principle (IEP). *I have to choose a destination, out of two possible areas. At area #1, there are p possible destinations to choose from. At area #2, there are q possible destinations to choose from. Areas #1 and #2 overlap—they have r destinations in common.*

The IEP simply states that I have, in total, $p + q - r$ different choices.

Exercise 461. (Answer on p. 1953.) The food court has 4 types of cuisine: Chinese, Indonesian, Korean, and Western. The hawker centre has 3: Chinese, Malay, and Western. A restaurant has 3: Chinese, Japanese, or Malay.

In total, how many different types of cuisine are there? Illustrate your answer with a Venn diagram.

112.4. How to Count: The Complements Principle

The **Complements Principle (CP)** is another very simple principle.

Example 1442. The food court has 4 types of cuisine: Chinese, Malay, Indian, and Other.

I'm at the food court but don't feel like eating Malay or Chinese. So by the Complements Principle (CP), I have $4 - 2 = 2$ possible choices of cuisine (Indian and Other).

Here's an informal statement of the CP:[521]

The Complements Principle (CP). *There are p possible destinations. I must choose one. I rule out q of the possible destinations.*

The Complements Principle says that I am left with $p - q$ possible choices.

Exercise 462. There are 10 Southeast Asian countries, of which 3 (Brunei, Indonesia, and the Philippines) are not on the mainland. How many mainland Southeast Asian countries are there that a European tourist can visit? (Answer on p. **1953**.)

[521] See section 147.1 (Appendices) (optional) for a more precise statement of the CP.

113. How to Count: Permutations

In this chapter, we'll use the MP to generate several more methods of counting.

But first, some notation you should find familiar from secondary school:

Definition 231. Let $n \in \mathbb{Z}_0^+$. Then *n-factorial*, denoted $n!$, is defined by $n! = n \times (n-1) \times \cdots \times 1$ for $n \geq 1$ and $0! = 1$.

Example 1443. $0! = 1$, $1! = 1$, $2! = 2\times = 2$, $3! = 3 \times 2 \times 1 = 6$, $4! = 4 \times 3 \times 2 \times 1 = 24$, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Exercise 463. Compute 6!, 7!, and 8!. (Answer on p. 1954.)

We now revisit the CAT problem, using the MP:

Example 1444. Problem: *How many permutations (or arrangements) are there of the three letters in the word CAT?*

Let's rephrase this problem in the framework of the MP. Consider three blank spaces:

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}.$$

These 3 blank spaces correspond to 3 decisions to be made. Decision #1: What letter to put in the first blank space? Decision #2: What letter to put in the second blank space? Decision #3: What letter to put in the third blank space?

Again we ask: How many choices have we for each decision?

For Decision #1, we can put C, A, or T. So we have 3 choices for Decision #1.

Having already used up a letter in Decision #1, we are left with two letters. So we have 2 choices for Decision #2.

Having already used up a letter in Decision #1 and another in Decision #2, we are left with just one letter. So we have only 1 choice for Decision #3.

Altogether then, by the MP, there are $3 \times 2 \times 1 = 3! = 6$ possible ways of making our decisions. This is also the number of ways there are to arrange the three letters in the word CAT.

Let's now try the UNPREDICTABLY problem.

Example 1445. Problem: *How many ways permutations are there of the* 13 *letters in the word UNPREDICTABLY?*

Again, let's rephrase this problem in the framework of the MP. Consider 13 blank spaces:

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{\_}\ \underset{6}{\_}\ \underset{7}{\_}\ \underset{8}{\_}\ \underset{9}{\_}\ \underset{10}{\_}\ \underset{11}{\_}\ \underset{12}{\_}\ \underset{13}{\_}.$$

These 13 blanks spaces correspond to 13 decisions to be made. Decision #1: What letter to put in the first blank space? Decision #2: What letter to put in the second blank space? ... Decision #13: What letter to put in the 13th blank space?

Again we ask: How many choices have we for each decision?

First an important note: In the word UNPREDICTABLY, no letter is repeated. (Indeed, UNPREDICTABLY is the longest "common" English word without any repeated letters.)

For Decision #1, we can put U, N, P, R, E, D, I, C, T, A, B, L, or Y. So we have 13 choices for Decision #1.

For Decision #2, having already used up a letter in Decision #1, we are left with 12 letters. So we have 12 choices for Decision #2.

For Decision #3, having already used up a letter in Decision #1 and another letter in Decision #2, we are left with 11 letters. So we have 11 choices for Decision #3.

$\vdots$

For Decision #13, having already used up a letter in Decision #1, another in Decision #2, another in Decision #3, ..., and another in Decision #12, we are left with one letter. So we have 1 choice for Decision #13.

Altogether then, by the MP, there are $13 \times 12 \times \cdots \times 2 \times 1 = 13! = 6,227,020,800$ possible ways of making our decisions. This is also the number of ways there are to arrange the 13 letters in the word UNPREDICTABLY.

The next fact simply summarises what should already be obvious from the above examples:

Fact 226. *There are $n!$ possible permutations of n distinct objects.*

Here is an informal proof of the above fact.[522]

Consider n empty spaces. We are to fill them with the n distinct objects.

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \cdots\ \underset{n}{\_}.$$

For space #1, we have n possible choices. For space #2, we have $n-1$ possible choices (because one object was already placed in space #1). ... And finally for space #n, we have only 1 object left and thus only 1 choice. By the MP then, there are $n \times (n-1) \times \cdots \times 1 = n!$ possible ways of filling in these n spaces with the n distinct objects.

Example 1446. The word COWDUNG has seven distinct letters. Hence, there are $7! = 5040$ permutations of the letters in the word COWDUNG.

[522]This is informal because, amongst other omissions, we haven't yet given a precise definition of the term *permutation*.

113.1. Permutations with Repeated Elements

In the previous section, we saw that there are 3! permutations of the three letters in the word CAT and 13! permutations of the 13 letters in the word UNPREDICTABLY. We made an important note: In each of these words, there was no repeated letter.

We now consider permutations of a set where some elements are repeated.

Example 1447. *How many permutations are there of the three letters in the word SEE?*

A naïve application of the MP would suggest that the answer is 3! = 6. This is wrong. Enumeration shows that there are only 3 possible permutations:

EES, ESE, SEE.

To see why a naïve application of the MP fails, set up the problem in the framework of the MP. Consider 3 blank spaces:

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}.$$

These 3 blanks spaces correspond to 3 decisions to be made. Decision #1: What letter to put in the first blank space? Decision #2: What letter to put in the second blank space? Decision #3: What letter to put in the third blank space?

Again we ask: How many choices have we for each decision?

For Decision #1, we can put E or S. So we have 2 choices for Decision #1.

But now the number of choices available for Decision #2 **depends on what we chose for Decision #1**! (If we chose E in Decision #1, then we again have 2 choices for Decision #2. But if instead we chose S in Decision #2, then we now have only 1 choice for Decision #2.) This violates the implicit but important assumption in the MP that the number of choices available in one decision is independent on the choice made in the other decision. **Hence, the MP does not directly apply**.

The reason SEE has only 3 possible permutations (instead of 3! = 6) is that it contains a repeated element, namely E. But why would this make any difference?

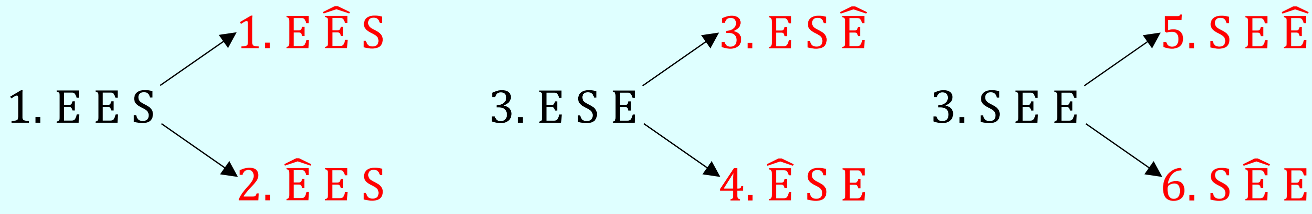
To understand why, let's rename the second E as Ê, so that the word SEE is now transformed into a new word SEÊ. From the three letters of this new word, we'd again have 3! = 6 possible permutations:

EÊS, ÊES, ESÊ, ÊSE, SEE, SÊE.

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Restricting attention to the two letters E$\hat{\text{E}}$, we see that there are 2! = 2 ways to permute these two letters. Hence, any single permutation (in the case where we do not distinguish between the two E's) corresponds to 2 possible permutations (in the case where we do). The figure below illustrates how the 3 permutations of SEE correspond to the 6 permutations in SE$\hat{\text{E}}$.



Hence, when we do *not* distinguish between the two E's, there are only half as many possible permutations.

We next consider permutations of SASS.

Example 1448. *How many permutations are there of the four letters in the word SASS?*

The answer is 4!/3! = 4. Let's see why.

If we distinguish between the three S's, perhaps by calling them S, $\hat{\text{S}}$, and $\bar{\text{S}}$, then we'd have 4! = 24 possible permutations of the letters in the word SA$\hat{\text{S}}\bar{\text{S}}$.

But amongst the three S's themselves, we have 3! = 6 possible permutations: S$\hat{\text{S}}\bar{\text{S}}$, S$\bar{\text{S}}\hat{\text{S}}$, $\hat{\text{S}}$S$\bar{\text{S}}$, $\bar{\text{S}}$S$\hat{\text{S}}$, $\hat{\text{S}}\bar{\text{S}}$S, and $\bar{\text{S}}\hat{\text{S}}$S. So distinguishing between the three S's increases by 6-fold the number of possible permutations. Working backwards, the word SASS thus has one-sixth as many permutations as SA$\hat{\text{S}}\bar{\text{S}}$. That is, SASS has 4!/3! = 4 possible permutations.

The figure below illustrates how the 4 possible permutations of SASS correspond to the 24 possible permutations of SA$\hat{\text{S}}\bar{\text{S}}$.

Example 1449. *How many permutations are there of the four letters in the word DEED?*

$$\text{Answer:} \quad \frac{4!}{2!2!}.$$

In the numerator, the 4! corresponds to the total of 4 letters. In the denominator, the 2! corresponds to the 2 D's and the 2! corresponds to the 2 E's. Where do these numbers come from?

Let x be the number of permutations of DEED (i.e. x is our desired answer).

If we distinguish between the two D's, then we'd increase by 2!-fold the number of possible permutations, to $x \cdot 2!$. If, in addition, we distinguish between the 2 E's, then we'd increase again by 2!-fold the number of possible permutations, to $x \cdot 2! \cdot 2!$. But we know that if all 4 letters are distinct, then there are 4! possible permutations. Therefore,

$$x \cdot 2! \cdot 2! = 4!$$

Rearrangement yields the answer:

$$x = \frac{4!}{2!2!} = 6.$$

You can go back and check that this answer is consistent with our answer for Exercise 458 (above).

We next consider permutations of ASSESSES.

Example 1450. Problem: *How many permutations are there of the eight letters in the word ASSESSES?*

$$\text{Answer: } \frac{8!}{2!5!}.$$

In the numerator, the 8! corresponds to the total of 8 letters. In the denominator, the 2! corresponds to the 2 E's and the 5! corresponds to the 5 S's. Where do these come from?

Let y be the number of permutations of ASSESSES (i.e. y is our desired answer).

If we distinguish between the two E's, then we'd increase by 2!-fold the number of possible permutations, to $y \cdot 2!$. If, in addition, we distinguish between the 5 S's, then we'd increase again by 5!-fold the number of possible permutations, to $y \cdot 2! \cdot 5!$. But we know that if all 8 letters are distinct, then there are 8! possible permutations. Therefore,

$$y \cdot 2! \cdot 5! = 8!$$

Rearrangement yields the answer:

$$y = \frac{8!}{2!5!}.$$

In general,

Fact 227. *Consider n objects, only k of which are distinct. Let r_1, r_2, ..., and r_k be the numbers of times the 1st, 2nd, ..., and kth distinct objects appear. (So $r_1+r_2+\cdots+r_k = n$.)*

Then the number of possible ways to permute these n objects is

$$\frac{n!}{r_1!r_2!\ldots r_k!}.$$

More examples:

Example 1451. *How many permutations are there of the six letters in the word BANANA?*

We have three distinct letters—B, A, and N. The letter B appears 1 time. The letter A appears 3 times. The letter N appears 2 times. Hence, by the above Fact, the number of possible permutations of these 6 letters is

$$\frac{6!}{1!3!2!} = 60.$$

Of course, 1! is simply equal to 1. So for the denominator, we shall usually not bother to write out any 1!. So we will normally instead write that the number of permutations of BANANA is

$$\frac{6!}{3!2!} = 60.$$

Example 1452. *How many permutations are there of the 11 letters in the word MISSISSIPPI?*

We have four distinct letters—M, I, S, and P. The letter M appears 1 time. The letter I appears 4 times. The letter S appears 4 times. The letter P appears 2 times. Hence, by the above Fact, the number of possible permutations of these 11 letters is

$$\frac{11!}{4!4!2!} = 34,650.$$

Exercise 464. There are 3 identical white tiles and 4 identical black tiles. How many ways are there of arranging these 7 tiles in a row? (Answer on p. **1954**.)

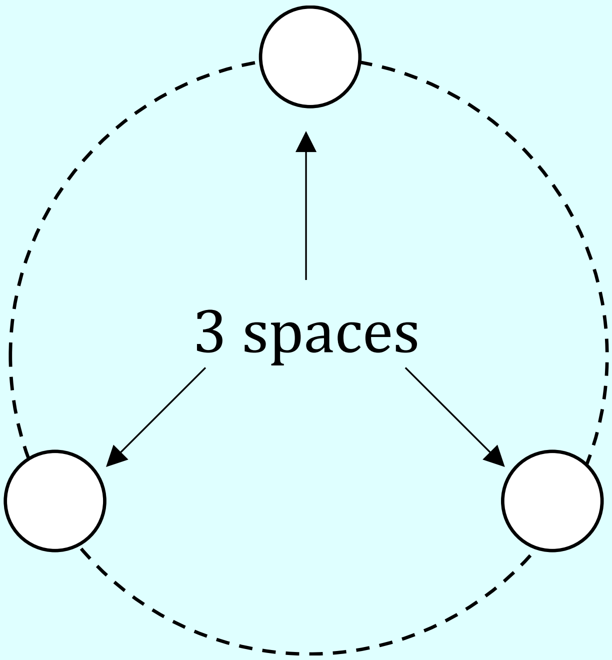
113.2. Circular Permutations

Informal Definition. *Two circular permutations are equivalent* if one can be transformed into another by means of a rotation.

Example 1453. There are 3! = 6 (linear) permutations of CAT. That is, there are 3! = 6 possible ways to fill them into these 3 linearly arranged spaces:

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}$$

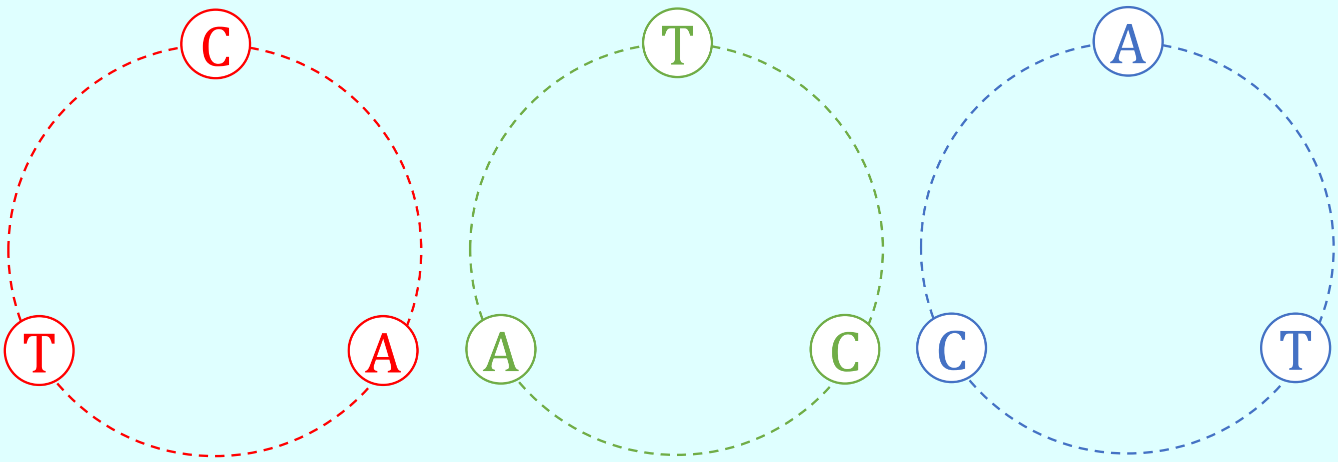
In contrast, there are only 2! = 2 circular permutations of CAT. That is, there are only 2! = 2 possible ways to fill them into these 3 circularly arranged spaces:



Let's see why there are only 2 circular permutations of CAT.

(*Example continues on the next page ...*)

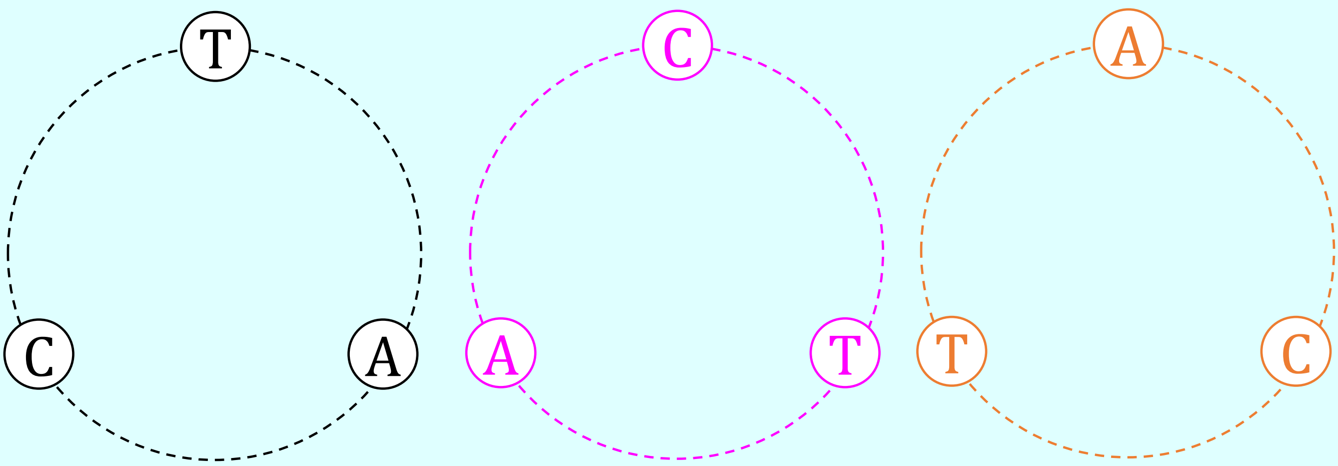
(... *Example continued from the previous page.*)



The three seemingly different arrangements above are considered to be the same circular permutation. This is because any arrangement is simply a rotation of another. Take the left red arrangement, rotate it clockwise by one-third of a circle to get the middle green arrangement. Repeat the rotation to get the right blue arrangement.

The second and only other circular arrangement of CAT is shown below. Again, these three seemingly different arrangements are considered to be the same circular permutation. This is because any arrangement is simply a rotation of another. Take the **left black arrangement**, rotate it clockwise by one-third of a circle to get the middle pink arrangement. Repeat the rotation to get the right orange arrangement.

Note importantly, that the arrangement (or three arrangements) below **cannot** be rotated to get the arrangement (or three arrangements) above. Hence, the arrangement below is indeed distinct from the arrangement above.



It turns out that in general, if we have n distinct objects, there are $(n-1)!$ ways to arrange them in a circle. So here there are only $(3-1)! = 2! = 2$ ways to arrange CAT in a circle.

In general:

Fact 228. *n distinct objects have $(n-1)!$ circular permutations.*

Proof. Given n distinct objects, any 1 circular permutation can be rotated n times to obtain

n distinct (linear) permutations. Hence, there are n times as many (linear) permutations as there are circular permutations.

But we already know that there are $n!$ (linear) permutations of n distinct objects. Hence, there are $n!/n = (n-1)!$ circular permutations of n distinct objects. □

Exercise 465. How many ways are there to seat 10 people in a circle? (Answer on p. **1954**.)

Note that if there are repeated objects, then the problem is considerably more difficult. See Ch. 147.2 (Appendices) for a brief discussion.

113.3. Partial Permutations

Example 1454. *Using the 26-letter alphabet, how many* 3*-letter words can we form that have no repeated letters?* This, of course, is simply the problem of filling in these 3 empty spaces using 26 distinct elements. For space #1, we have 26 possible choices. For space #2, we have 25. And for space #2, we have 24.

$$\underset{1}{\_\_}\ \underset{2}{\_\_}\ \underset{3}{\_\_}$$

By the MP then, the number of ways to fill the three spaces is $26 \times 25 \times 24$. This is also the number of three-letter words with no repeated letters.

Problems like the above example crop up often enough to motivate a new piece of notation:

Definition 232. Let n, k be positive integers with $n \geq k$. Then $P(n, k)$, read aloud as n *permute* k, is defined by

$$P(n,k) = \frac{n!}{(n-k)!}.$$

$P(n,k)$ answers the following question: "Given n distinct objects and k spaces (where $k \leq n$), how many ways are there to fill the k spaces?"

Just so you know, $P(n,k)$ is also variously denoted nPk, P_k^n, ${}^nP_k$, etc., but we'll stick solely with the $P(n,k)$ in this textbook.

Example 1179 (continued from above). The number of 3-letter words without repeated letters is simply $P(26,3) = 26!/23! = 26 \times 25 \times 24$.

Example 1455. Problem: *Using the 22-letter Phoenician alphabet, how many 4-letter words can we form that have no repeated letters?*

This, of course, is simply the problem of filling in these 4 empty spaces using 22 distinct elements. So the answer is $P(22,4) = 22!/18! = 22 \times 20 \times 19 \times 18$ words.

Exercise 466. Out of a committee of 11 members, how many ways are there to choose a president and a vice-president? (Answer on p. **1954**.)

113.4. Permutations with Restrictions

Example 1456. At a dance party, there are 7 heterosexual married couples (and thus 14 people in total). **Problem #1.** *How many ways are there of arranging them in a line, with the restriction that every person is next to his or her partner?*

Think of there as being 7 units (each unit being a couple). There are 7! ways to arrange these 7 units in a line. Within each unit, there are 2 possible arrangements. Hence, in total, there are $7! \times 2^7$ possible arrangements.

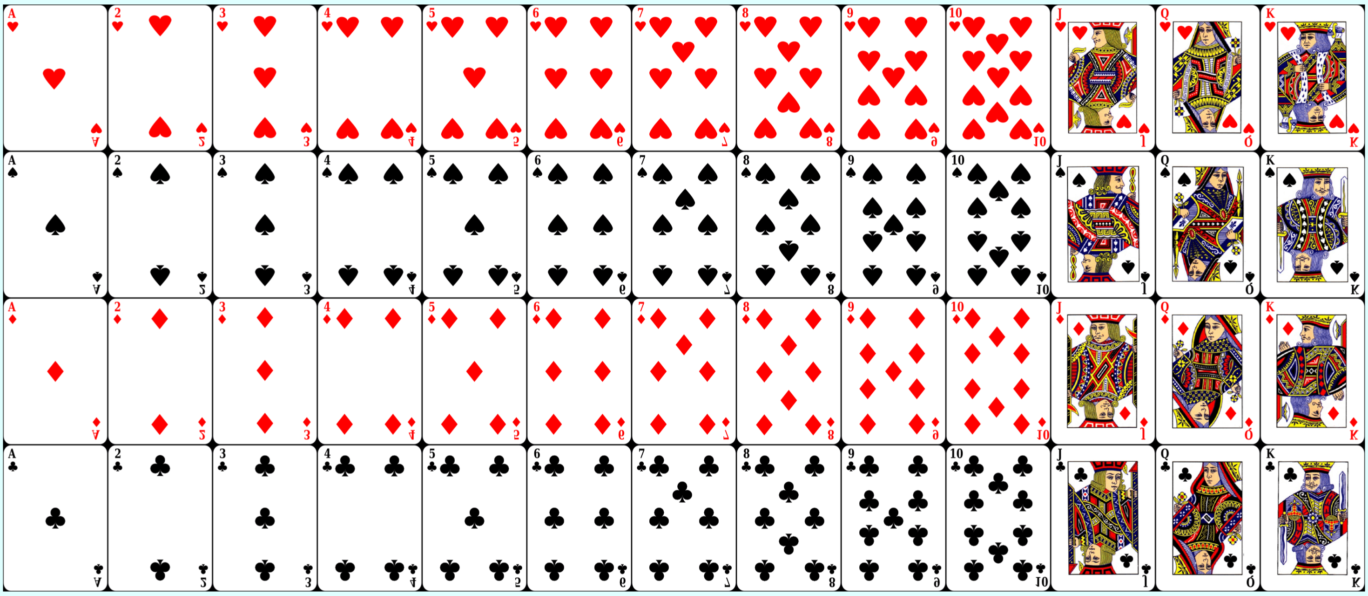
Problem #2. *Repeat the above problem, but now for a* **circle**, *rather than a line.*

There are 6! ways to arrange the 7 units in a circle. Within each unit, there are 2 possible arrangements. Hence, in total, there are $6! \times 2^7$ possible arrangements.

Problem #3. *How many ways are there of arranging them in a circle, with the restriction that every man is to the right of his wife?*

There are 6! ways to arrange the 7 units in a circle. Within each unit, there is only 1 possible arrangement. Hence, in total, there are 6! possible arrangements.

Example 1457. (I assume you're familiar with the standard 52-card deck.)



(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Problem #1. *Using a standard 52-card deck, how many ways are there of arranging any 3 cards in a line, with the restriction that no two cards of the same suit are next to each other?*

This is the problem of filling in 3 spaces with 52 distinct objects. For space #1, we have 52 possible choices.

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}.$$

For space #2, having picked a card of suit X for space #1, we must pick a card from some other suit Y. And so there are only 39 possible choices (we have three suits available—that's $3 \times 13 = 39$).

For space #3, having picked a card of suit Y for space #2, we must pick a card from some other suit Z. Note that suit Z can be the same as suit X. And so there are 38 possible choices (we have three suits available, less the card used for space #1—that's $3 \times 13 - 1 = 38$).

Altogether then, there are $52 \times 39 \times 38$ possible arrangements.

Problem #2. *Repeat the above problem, but now for a* circle*, rather than a line.*

One subtle thing is that, in addition to space #1 being of a different suit from space #2 and space #2 being of a different suit from space #3, we must also have that space #3 is of a different suit from space #1. Thus, there are $52 \times 39 \times 26$ possible ways to fill in these three spaces, if they were in a line.

Since they are instead in a circle, there are $52 \times 39 \times 26 \div 3$ possible ways to arrange three cards in a circle, with the condition that no two cards of the same suit are next to each other.

Exercise 467. (Answer on p. 1954.) There are 4 brothers and 3 sisters. In how many ways can they be arranged ...

(a) in a line, without any 2 brothers being next to each other?

(b) in a line, without any 2 sisters being next to each other?

(c) in a circle, without any 2 brothers being next to each other?

(d) in a circle, without any 2 sisters being next to each other?

114. How to Count: Combinations

$P(n,k)$ is the number of ways we can fill k (ordered) spaces using n distinct objects.

In contrast, $C(n,k)$ is the number of ways of choosing k out of n distinct objects. Equivalently, it is the same problem of filling k spaces using n distinct objects, **except that now order does not matter**.

Example 1458. Suppose we have a committee of 13 members and wish to select a president and a vice-president. This is equivalent to the problem of filling in 2 spaces, given 13 distinct objects.

$$\underset{1}{\text{—}}\ \underset{2}{\text{—}}$$

The answer is thus simply $P(13,2) = 13 \times 12$.

Suppose instead that we want to choose two co-presidents. How many ways are there of doing so?

This is simply the same problem as before—again we want to fill in 2 spaces, given 13 distinct objects. The only difference now is that **the order of the 2 chosen objects does not matter**. So the answer must be that there are $P(13,2)/2!$ ways of choosing the two co-presidents.

Example 1459. How many ways are there of choosing 5 cards out of a standard 52-card deck?

$$\underset{1}{\text{—}}\ \underset{2}{\text{—}}\ \underset{3}{\text{—}}\ \underset{4}{\text{—}}\ \underset{5}{\text{—}}$$

First, how many ways are there to fill 5 spaces using 52 distinct objects **(where order matters)**? Answer: $P(52,5) = 52 \times 51 \times 50 \times 49 \times 48 = 311,875,200$.

And so if we don't care about order, we must adjust this number by dividing by 5! to get $P(52,5)/5! = 2,598,960$. So the answer is that to choose 5 cards out of a 52-card deck, there are $2,598,960$ ways.

The above examples suggest that, in general, *to choose k out of n given distinct objects, there are $P(n,k)/k!$ possible ways.* This motivates the following definition:

Definition 233. Let n, k be positive integers with $n \geq k$. Then $C(n, k)$, read aloud as n *choose* k, is defined by

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{(n,k)!k!}.$$

It turns out that $C(n, k)$ appears so often in maths that it has many alternative notations—one of the most common is $\binom{n}{k}$.

"n choose k" also has several names, such as **the combination**, **the combinatorial number**, and even **the binomial coefficient**. Shortly, we'll see why the name **binomial coefficient** makes sense.

Exercise 468 gives an alternate expression for $C(n, k)$ which you'll often find very useful.

Exercise 468. (Answer on p. 1956.) Show that

$$C(n,k) = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)}{k!}.$$

Exercise 469. Compute $C(4, 2)$, $C(6, 4)$, and $C(7, 3)$. (Answer on p. 1956.)

Exercise 470. We wish to form a basketball team, consisting of 1 centre, 2 forwards, and 2 guards. We have available 3 centres, 7 forwards, and 5 guards. How many ways are there of forming a team? (Answer on p. **1956**.)

Here's a nice symmetry property:

Fact 229. (Symmetry.) $C(n, k) = C(n, n-k)$.

Proof. Choosing k out of n objects is the same as choosing which $n-k$ out of n objects *to ignore.* □

Example 1460. We have a group of 100 men. 70 are needed for a task. The number of ways to choose these 70 men is

$$C(100, 70) = \frac{100!}{30!70!}.$$

This is the same as the number of ways to choose the 30 men that will *not* be used for the task:

$$C(100, 30) = \frac{100!}{70!30!}.$$

114.1. Pascal's Triangle

Pascal's Triangle consists of a triangle of numbers. If we adopt the convention that the topmost row is row 0 and the leftmost term of each row is the 0th term, then the n^{th} row, k^{th} term is the number $C(n,k)$:

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 25 \quad 35 \quad 21 \quad 7 \quad 1 \\
\vdots
\end{array}$$

It turns out that beautifully enough, each term is equal to the sum of the two terms above it. The next exercise asks you to verify several instances of this:

Exercise 471. Verify the following: (a) $C(1,0)+C(1,1)=C(2,1)$; (b) $C(4,2)+C(4,3)=C(5,3)$; (c) $C(17,2)+C(17,3)=C(18,3)$. (Answer on p. **1956**.)

Fact 230. (Pascal's Rule/Identity/Relation.) $C(n+1,k)=C(n,k)+C(n,k-1)$.

Proof. $C(n+1,k)$ is the number of ways of choosing k out of $n+1$ distinct objects.

Suppose we do *not* choose the last object, i.e. the $(n+1)$th object. Then we have to choose our k objects out of the first n objects. There are $C(n,k)$ ways of doing so.

Suppose we *do* choose the last object. Then we have to choose another $k-1$ objects, out of the first n objects. There are $C(n,k-1)$ ways of doing so.

Altogether then, by the Addition Principle, there are $C(n,k)+C(n,k-1)$ ways of choosing k out of $n+1$ distinct objects. □

114.2. The Combination as Binomial Coefficient

> [L]a mathématique est l'art de donner le même nom à des choses différentes.
>
> *[M]athematics is the art of giving the same name to different things.*
>
> — Henri Poincaré (1908, 1914)

Poincaré's quote is especially true in combinatorics. In this section, we'll learn why $C(n,k)$ can be called the **combination** and also the **binomial coefficient**.

Verify for yourself that the following equations are true:

$$
\begin{aligned}
&(1+x) = 1, \\
&(1+x) = 1 + x, \\
&(1+x) = 1 + 2x + x^2, \\
&(1+x) = 1 + 3x + 3x^2 + x^3, \\
&(1+x) = 1 + 4x + 6x^2 + 4x^3 + x^4, \\
&(1+x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5, \\
&(1+x) = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6, \\
&(1+x) = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7. \\
&\qquad\vdots
\end{aligned}
$$

Each of the expressions on the RHS is called a **binomial series**. Each can also be called the **binomial expansion of** $(1+x)$.

Notice anything interesting? No? Try this exercise:

Exercise 472. Compute $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \binom{7}{3}, \binom{7}{4}, \binom{7}{5}, \binom{7}{6}, \binom{7}{7}$. Compare these to the coefficients of the binomial expansion of $(1+x)$. What do you notice? (Answer on p. **1957**.)

It turns out that somewhat surprisingly, the coefficients of the binomial expansions of $(1+x)$ are simply $\binom{n}{0}, \binom{n}{1}, \dots \binom{n}{n}$. As an additional exercise, you should verify for yourself that this is also true for $n = 0$ through $n = 6$.

There are several ways to explain why the combinatorial numbers also happen to be the binomial coefficients. Here we'll give only the combinatorial explanation:

Consider $(1+x)$. Expanding, we have

$$(1+x) = (1+x)(1+x) = 1 \cdot 1 + 1 \cdot x + x \cdot 1 + x \cdot x.$$

Consider the 4 terms on the right.

For $1 \cdot 1$, we "chose" 1 from the first $(1+x)$ and 1 from the second $(1+x)$.	$\longrightarrow$	From the two $(1+x)$'s in the product, there is $C(2,0) = 1$ way to choose 0 of the x's.
For $1 \cdot x$, we "chose" 1 from the first $(1+x)$ and x from the second $(1+x)$. For $x \cdot 1$, we "chose" x from the first $(1+x)$ and 1 from the second $(1+x)$.	$\}$	From the two $(1+x)$'s in the product, there are $C(2,1) = 2$ ways to choose 1 of the x's.
Finally, for $x \cdot x$, we "chose" x from the first $(1+x)$ and x from the second $(1+x)$.	$\longrightarrow$	From the two $(1+x)$'s in the product, there is $C(2,2) = 1$ way to choose 2 of the x's.

Altogether then, the coefficient on x^0 is $C(2,0)$ ("choose 0 of the x's"), that on x^1 is $C(2,1)$ ("choose 1 of the x's"), and that on x^2 is $C(2,1)$ ("choose 2 of the x's"). That is,

$$(1+x) = C(2,0)x^0 + C(2,1)x^1 + C(2,2)x^2 = 1 + 2x + x^2.$$

Exercise 473. (Answer on p. **1957**.) Mimicking what was just done above, explain why

$$(1+x) = C(3,0)x^0 + C(3,1)x^1 + C(3,2)x^2 + C(3,3)x^3.$$

More generally, we have

Fact 231. *Let $n \in \mathbb{Z}^+$. Then*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n.$$

114.3. The Number of Subsets of a Set is 2^n

By plugging $x = 1$, $y = 1$ into the last fact, we see that $(1+1) = 2^n$ is the sum of the terms in the nth row of Pascal's triangle:

Fact 232. *Let $n \in \mathbb{Z}^+$. Then*

$$2^n = \sum_{i=0}^{n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

There's a nice combinatorial interpretation of the above fact (Poincaré's quote at work again).

Consider the set $S = \{A, B\}$. S has $2^2 = 4$ subsets: $\varnothing = \{\}$, $\{A\}$, $\{B\}$, and $S = \{A, B\}$.

Now consider the set $T = \{A, B, C\}$. T has $2^3 = 8$ subsets: $\varnothing = \{\}$, $\{A\}$, $\{B\}$, $\{C\}$, $\{A, B\}$, $\{A, C\}$, $\{B, C\}$, and $T = \{A, B, C\}$.

In general, if a set has n elements, how many subsets does it have? We can couch this in the framework of the Multiplication Principle—this is really a sequence of n decisions of whether or not to include each element in the subset. There are 2 choices for each decision. Thus, there are 2^n choices altogether. In other words, using a set of n elements, we can form 2^n subsets.

But of course, this must in turn be equal to the sum of the following:

- $C(n, 0)$ ways to form subsets with 0 elements;
- $C(n, 1)$ ways to form subsets with 1 element;
- $C(n, 2)$ ways to form subsets with 2 elements;

...

- $C(n, n)$ ways to form subsets with n elements.

Thus,

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

Exercise 474. Verify that $2^7 = \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \cdots + \binom{7}{7}$. (Answer on p. 1957.)

Exercise 475. Using what you've learnt, write down $(3+x)^4$. (Answer on p. 1958.)

Exercise 476. (Answer on p. 1958.) (a) The Tan family has 4 sons and the Wong family has 3 daughters. Using the sons and daughters from these two families, how many ways are there of forming 2 heterosexual couples?

(b) The Lee family has 6 sons and the Ho family has 9 daughters. Using the sons and daughters from these two families, how many ways are there of forming 5 heterosexual couples?

115. Probability: Introduction

115.1. Mathematical Modelling

All models are wrong, but some are useful.

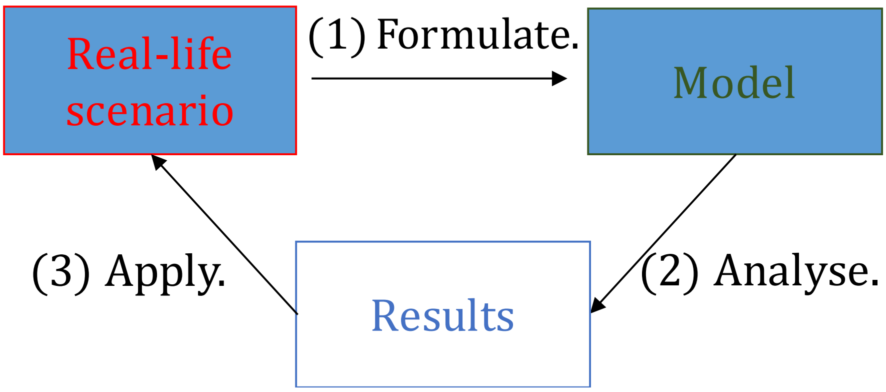
— George Box (1979).

Whenever we use maths in a real-world scenario, we have some mathematical model in mind. Here's a very simple example just to illustrate:

Example 1461. We want to know how much material to purchase, in order to build a fence around a field. We might go through these steps:

1. **Formulate** a mathematical model: Our field is the shape of a rectangle, with length 100 m and breadth 50 m.
2. **Analyse**: The rectangle has perimeter $100 + 50 + 100 + 50 = 300$ m.
3. **Apply the results of our analysis**: We need to buy enough material to build a 300-metre long fence.

The figure below depicts how mathematical modelling works.



Starting with some **real-world scenario**, we go through these steps:

1. **Formulate** a mathematical model.

That is, describe the real-world scenario in mathematical language and concepts.

This first step is arguably the most important. It is often subjective—not everyone will agree that your mathematical model is the most appropriate for the scenario at hand.

To use the above example, the field may not be a perfect rectangle, so some may object to your description of the field as a rectangle. Nonetheless, you may decide that all things considered, the rectangle is a good mathematical model.

2. **Analyse** the model.

This involves using maths and the rules of logic. (A-Level maths exams tend to be mostly concerned with this second step.)

In the above example, this second step simply involved computing the perimeter of the rectangle—100 + 50 + 100 + 50 = 300 m. Of course, for the A-Levels, you can expect the analysis to be more challenging than this.

Note that this second step, in contrast to the first, is supposed to be completely watertight, non-subjective, and with no room for disagreement. After all, hardly anyone reasonable could disagree that a perfect rectangle with length 100 m and breadth 50 m has perimeter 300 m.

3. **Apply** your results.

Now apply the results of your analysis to the real-world scenario.

In the above example, pretend you're a mathematical consultant hired by the fence-builder. Then your final report might simply say, "We recommend the purchase of 300 m worth of fence material."

This third and last step is, like the first, subjective and open to debate. It involves your interpretation of what the results of your analysis mean (in the real world) and your recommendation of what actions to take.

For example, you find that the fence will have perimeter 300 m and thus recommend that 300 m of fence material be purchased. However, someone else, looking at the same result, might point out that the corners of the fence require additional or special material; she might thus make a slightly different recommendation.

Secretly, we've always been using mathematical modelling; we just haven't always been terribly explicit about it. The foregoing discussion was placed here, because with probability and statistical models, we want to be especially clear about that we are doing mathematical modelling.

115.2. The Experiment as a Model of Scenarios Involving Chance

Real-world scenarios often involve chance. We can model such scenarios mathematically. For this purpose, we'll use a mathematical object named the **experiment**, typically denoted E.[523]

An experiment $E = (S, \Sigma, \mathrm{P})$ is an ordered triple[524] composed of three objects, called the **sample space** S, the **event space** Σ (upper-case sigma), and the **probability function** P, where

- The **sample space** S is simply the set of possible outcomes.
- An event is simply any set of possible outcomes. In turn, the **event space** Σ is simply the set of all events.
- The **probability function** P simply assigns to each event some probability between 0 and 1. This probability is interpreted as the likelihood of that particular event occurring.

Examples:

[523] An experiment is often instead called a **probability triple** or **probability space** or **(probability) measure space**.

[524] Previously, in the only ordered triples we encountered, the three terms were always simply real numbers. Here however, the first two terms are sets and the third is a function. Nonetheless, this is all the same an ordered triple, albeit a more complicated one.

Example 1462. We model a coin-flip with the experiment $E = (S, \Sigma, \mathrm{P})$. What are the sample space S, the event space Σ, and the probability function P?

1. $S = \{H, T\}$.

The **sample space** is simply the set of possible outcomes.

The choice of the sample space belongs to Step #1 (Formulate a mathematical model) in the process of mathematical modelling. It is subjective and open to disagreement.

For example, John (another scientist) might argue that the coin sometimes lands exactly on its edge. This is exceedingly unlikely but nonetheless possible—one empirical estimate is that the US 5-cent coin has probability 1 in 6000 of landing on its edge when flipped (source). So John might denote this third possible outcome X and his sample space would instead be $S = \{H, T, X\}$.

2. **Event space** $\Sigma = \{\varnothing, \{H\}, \{T\}, \{H, T\}\}$**.**

An **event** is simply any subset of S. In other words, an event is simply some set of possible outcomes. So here, $\{H\}$ is an event. So too is $\{T\}$. But there are also two other events, namely $\varnothing = \{\}$ (this is the event that never occurs) and $S = \{H, T\}$ (this is the event that always occurs).

The **event space** is simply the set of events. In other words, the event space is the set of all subsets of S.[525]

As we saw in Ch. 114.2, given any finite set S, there are $2^{|S|}$ possible subsets of S. In general, given a finite sample space S, the corresponding event space Σ always simply contains $2^{|S|}$ events. And so here, since there are 2 possible outcomes, there are, altogether, $2^2 = 4$ possible events.

If the real-world outcome of the coin flip is Heads, then our interpretation (in terms of our model) is that "the events $\{H\}$ and $\{H, T\}$ occur". If the real-world outcome of the coin flip is Tails, then our interpretation (in terms of our model) is that "the events $\{T\}$ and $\{H, T\}$ occur".

The event $\varnothing$ never occurs, whatever the real-world outcome is. And the event $S = \{H, T\}$ always occurs, whatever the real-world outcome is.

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

The mathematical modeller is free to select the sample space S she deems most appropriate. However, once she has selected the sample space S, the event space Σ is automatically determined by the rules of maths. There is no room for interpretation. Hence, the selection of the event space Σ belongs to Step #2 (Analysis) in the process of mathematical modelling.

So likewise, John, who chooses $S = \{H, T, X\}$ as his sample space, has no freedom to choose his event space Σ. It is automatically $\Sigma = \{\varnothing, \{H\}, \{T\}, \{X\}, \{H, T\}, \{H, X\}, \{T, X\}, S\}$ (consists of 8 elements).

3. **Probability function $\mathrm{P} : \Sigma \to \mathbb{R}$.**

The **probability function** simply assigns to each event a number (between 0 and 1) called a **probability**. So here, if heads and tails are "equally likely" (or the coin is "unbiased" or "fair"), then it makes sense to assign

$$\mathrm{P}(\varnothing) = 0, \quad \mathrm{P}(\{H\}) = \mathrm{P}(\{T\}) = 0.5, \quad \mathrm{P}(s) = 1.$$

The mathematical modeller has no freedom over the domain Σ and codomain $\mathbb{R}$ of the probability function. However, she does have freedom to choose the mapping rule she deems most appropriate. Hence, the act of choosing the mapping rule belongs to Step #1 (Formulation) in the process of mathematical modelling.

So here, if told that heads and tails are "equally likely" (or that the coin is "unbiased" or "fair"), the mathematical modeller would naturally choose to assign probability 0.5 to each of the events $\{H\}$ and $\{T\}$.

John, who chooses $S = \{H, T, X\}$ as his sample space, might instead assign probability 1/6000 to the event $\{X\}$ and probability 5999/12000 to each of the events $\{H\}$ and $\{T\}$.

Remark 176. It is correct and proper to write $\mathrm{P}(\{H\}) = \mathrm{P}(\{T\}) = 0.5$. It is incorrect and improper to write $\mathrm{P}(H) = \mathrm{P}(T) = 0.5$. This is because the function P is of events (sets of outcomes) and NOT of outcomes themselves.

Nonetheless, we will often allow ourselves to be sloppy and write the "incorrect and improper" $\mathrm{P}(H) = \mathrm{P}(T) = 0.5$. This is because the notation $\mathrm{P}(\{H\}) = \mathrm{P}(\{T\}) = 0.5$ can get rather messy. But you should always remember, even as you write $\mathrm{P}(H) = \mathrm{P}(T) = 0.5$, that this is technically incorrect.

Example 1463. A real-world die-roll can be modelled by an experiment $E = (S, \Sigma, \mathrm{P})$, where

1. $S = \{1, 2, 3, 4, 5, 6\}$.

2. **Event space**:

$\Sigma = \{\varnothing, \{1\}, \{2\}, \dots, \{6\}, \{1,2\}, \{1,3\}, \dots, \{5,6\}, \{1,2,3\}, \{1,2,4\}, \dots, \{4,5,6\}, \dots\dots, S\}$

There are 6 possible outcomes and thus $2^6 = 64$ possible events. The event space, given above, is simply the set of all possible events.

If the real-world outcome of the die roll is 3, then the interpretation (in terms of our model) is that the following 32 events occur: $\{3\}$, $\{1,3\}$, $\{2,3\}$, $\dots$, $\{1,2,3\}$, $\{1,3,4\}$, $\dots$, $S = \{1, 2, 3, 4, 5, 6\}$. (These are simply the events that contain the outcome 3.)

Similarly, if the real-world outcome of the die roll is 5, then the interpretation is that 32 events occur. You should be able to list all 32 of these events on your own.

3. **Probability function $\mathbf{P} : \Sigma \to \mathbb{R}$.**

If the die is "unbiased" or "fair", then it makes sense to assign

$$\mathrm{P}(\{1\}) = \mathrm{P}(\{2\}) = \mathrm{P}(\{3\}) = \mathrm{P}(\{4\}) = \mathrm{P}(\{5\}) = \mathrm{P}(\{6\}) = \frac{1}{6}.$$

What about the other 58 events? It makes sense to assign, for example, $\mathrm{P}(\{1, 3, 5, 6\}) = \dfrac{4}{6}$.

In general, the mapping rule of the probability function can be fully specified as: For any event $A \in \Sigma$,

$$\mathrm{P}(A) = \frac{|A|}{|S|} = \frac{|A|}{6}.$$

In words, given any event A, its probability $\mathrm{P}(A)$ is simply the number of elements it contains, divided by 6.

Here's the formal definition of an **experiment**:

Definition 234. An *experiment* is an ordered triple (S, Σ, P), where

- S, the *sample space*, is simply any set (interpreted as the set of possible outcomes in a real-world scenario involving chance).
- Σ, the *event space*, is the set of possible events.
- P, the *probability function*, has domain Σ, codomain $\mathbb{R}$, and must satisfy the three Kolmogorov axioms (to be discussed below in Definition 235).

Given any event $A \in \Sigma$, the number $\mathrm{P}(A)$ is called the *probability of A*.

For the probability function P, the mathematical modeller is free to choose the mapping rule she deems most appropriate. The only restriction is that P satisfies three axioms, called **the Kolmogorov Axioms**, to be discussed in the next section.

Exercise 477. (Answers on pp. **1959**, **1960**, and **1961**.) Consider each of the following real-world scenarios.

(a) You pick, at random, a card from a standard 52-card deck.

(b) You flip two fair coins.

(c) You roll two fair dice.

Model each of the above real-world scenarios as an experiment, by following steps (i) - (iii):

(i) Write down the appropriate sample space S.

(ii) How many possible events are there? Hence, how many elements does the event space Σ contain? If it is not too tedious, write out Σ in full.

(iii) What are the domain and codomain of the probability function P? Write down the probabilities of any three events. Given any event $A \in \Sigma$, what is $\mathrm{P}(A)$?

(iv) In each scenario, explain briefly how John, another scientist, might justify choosing a different sample space, event space, and probability function.

115.3. The Kolmogorov Axioms

An **axiom** (or **postulate**) is a statement that is simply accepted as being true, without justification or proof.

Example 1464. Euclid's **parallel axiom** says that "Two non-parallel lines in the plane eventually intersect". Historically, this axiom was accepted as a "self-evident truth", without need for justification or proof.

However, in the 19th century, mathematicians discovered "non-Euclidean geometries", in which the parallel axiom did not hold. These turned out to have significant implications for maths, philosophy, and physics.

The above example illustrates that an axiom is not an eternal and immutable truth. Instead, it is merely a statement that some mathematicians tentatively accept as being true. Having listed a bunch of axioms, mathematicians then study their implications.

In probability theory, we impose three axioms on the probability function. These can be thought of as restrictions on what the probability function looks like. Informally:

1. *Probabilities can't be negative.*
2. *The probability of an outcome occurring is* 1*.*
3. *The probability that one of two disjoint events occurs is the sum of the their individual probabilities.*

Formally:

Definition 235. We say that a function P satisfies the three *Kolmogorov axioms* if:

1. **Non-Negativity Axiom.** For any event $E \subseteq S$, we have $\mathrm{P}(E) \geq 0$.
2. **Normalisation Axiom.** $\mathrm{P}(s) = 1$.
3. **Additivity Axiom.**[526] Given any two disjoint events $E_1, E_2 \subseteq S$, we have $\mathrm{P}(E_1 \cup E_2) = \mathrm{P}(E_1) + \mathrm{P}(E_2)$.

In case you've forgotten, two sets are disjoint if they have no elements in common.

115.4. Implications of the Kolmogorov Axioms

Obviously, $\mathrm{P}(\varnothing) = 0$ (the probability that the empty event occurs is 0). Previously, you've probably taken this and other "obvious" properties for granted. Now we'll prove that they follow from the Kolmogorov axioms.

Recall that given any set A, its complement A^c (sometimes also denoted A') is defined to be "everything else"—more precisely, A^c is the set of all elements that are not in A.

Proposition 21. *Let* P *be a probability function and A, B be events. Then* P *satisfies the following properties:*

1. **Complements.** $\mathrm{P}(A) = 1 - \mathrm{P}(A^c)$.
2. **Probability of Empty Event is Zero.** $\mathrm{P}(\varnothing) = 0$.
3. **Monotonicity.** If $B \subseteq A$, then $\mathrm{P}(B) \leq \mathrm{P}(A)$.
4. **Probabilities Are At Most One.** $\mathrm{P}(A) \leq 1$.
5. **Inclusion-Exclusion.** $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.

You may recognise that the Complements and the Inclusion-Exclusion properties are analogous to the CP and IEP from counting.

Proof. **1. Complements.** By definition, $A \cap A^c$ are disjoint. And so by the Additivity Axiom, $\mathrm{P}(A) + \mathrm{P}(A^c) = \mathrm{P}(A \cup A^c)$.

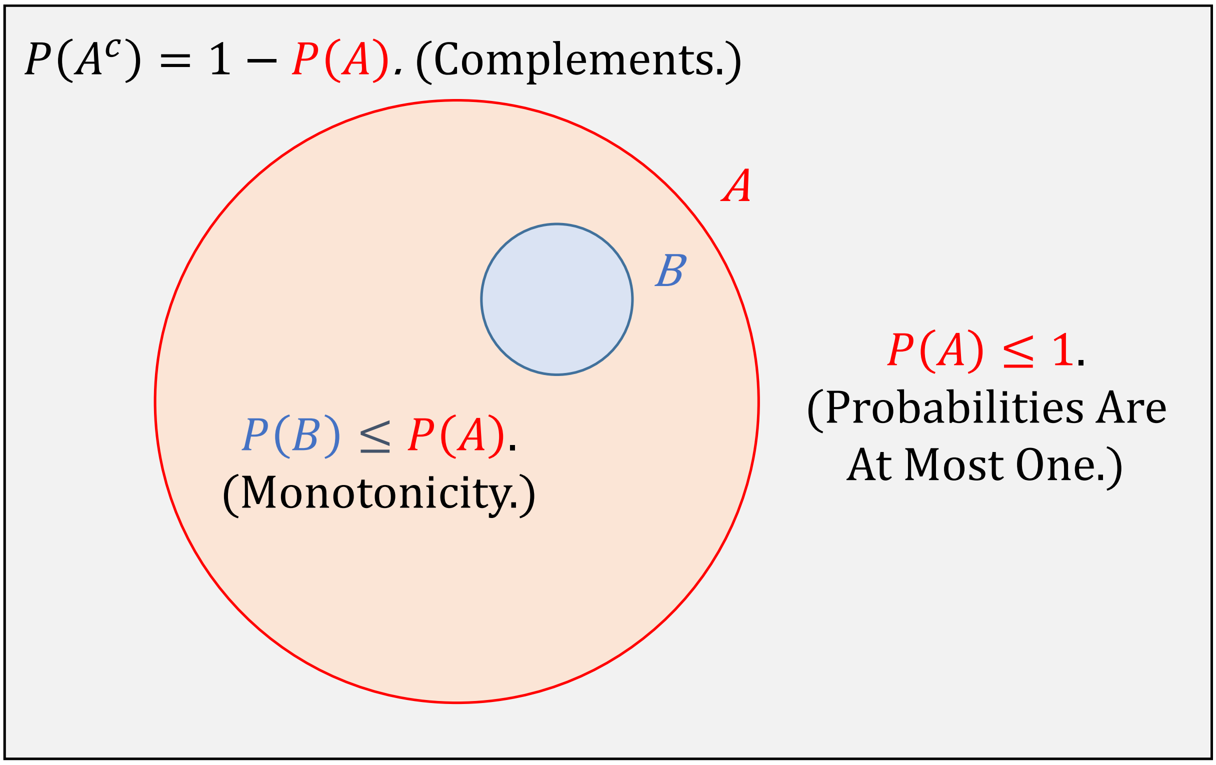
Also by definition, $A \cup A^c = S$. And so $\mathrm{P}(A \cup A^c) = \mathrm{P}(s)$.

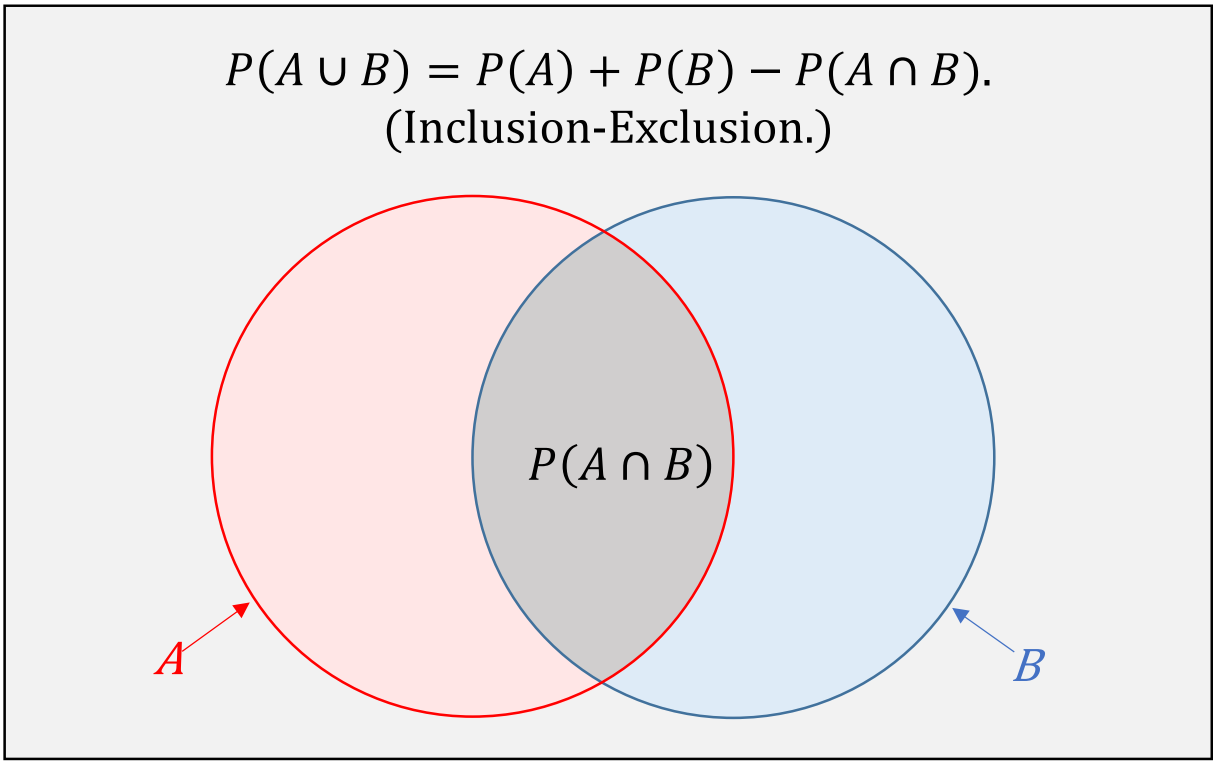
By the Normalisation Axiom, $\mathrm{P}(s) = 1$.

Altogether then, $\mathrm{P}(A) + \mathrm{P}(A^c) = \mathrm{P}(A \cup A^c) = \mathrm{P}(s) = 1$. Rearranging, $\mathrm{P}(A) = 1 - \mathrm{P}(A^c)$, as desired.

The remainder of the proof is continued on p. **1730** (Appendices). □

Venn diagrams are helpful for illustrating probabilities. Those below help to illustrate the four of the above five properties.





Exercise 478. Prove each of the following properties and illustrate with a Venn diagram: (a) "If two events A and B are mutually exclusive, then $\mathrm{P}(A \cap B) = 0$." (b) "Let A, B, and C be events. Then $\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(A^c \cap B) + \mathrm{P}(A^c \cap B^c \cap C)$." (Answer on p. **1962**.)

116. Probability: Conditional Probability

Example 1465. Flip three fair coins. Model this as an experiment $E = (S, \Sigma, \mathrm{P})$, where

- The sample space is $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
- The event space Σ has $2^8 = 256$ elements.
- The probability function $\mathrm{P} : \Sigma \to \mathbb{R}$ has mapping rule:

$$\mathrm{P}(HHH) = \mathrm{P}(HHT) = \cdots = \mathrm{P}(TTT) = \frac{1}{8},$$

and more generally, for any event $A \in \Sigma$, $\mathrm{P}(E) = \dfrac{|A|}{8}$.

Problem: *Suppose there is at least* 1 *tail. Find the probability that there are at least* 2 *heads.*

There are 7 possible outcomes where there is at least 1 tail: HHT, HTH, HTT, THH, THT, TTH, and TTT. Each is equally likely to occur. Of these, 3 outcomes involve at least 2 heads (HHT, HTH, and THH). Thus, *given there is at least* 1 *tail*, the probability that there are at least 2 heads is simply 3/7.

The above analysis was somewhat informal. Here is a more formal analysis.

Let A be the event that there are at least 2 heads: $A = \{HHT, HTH, THH, HHH\}$.

Let B be the event that there is at least 1 tail: $B = \{HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

$A \cap B$ is thus the event that there are at least 2 heads and 1 tail: $A \cap B = \{HHT, HTH, THH\}$.

Our problem is equivalent to finding $\mathrm{P}(A|B)$—**the conditional probability of A given B**, which is given by

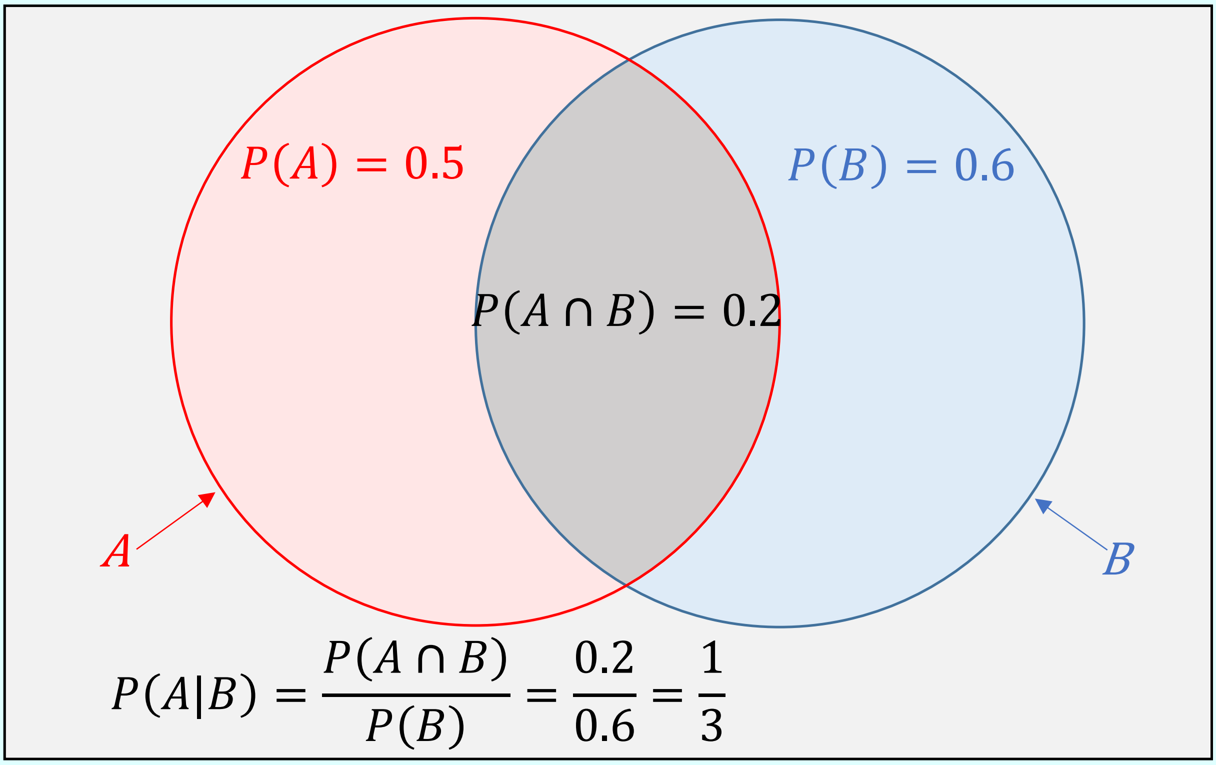
$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{3/8}{7/8} = \frac{3}{7}.$$

Example 1466. Let P be a probability function and $A, B \in \Sigma$ be events.

- $\mathrm{P}(A) = 0.5$ (the probability that A occurs is 0.5).
- $\mathrm{P}(B) = 0.6$ (the probability that B occurs is 0.6).
- $\mathrm{P}(A \cap B) = 0.2$ (the probability that both A and B occur is 0.2).

Hence, given that B has occurred, the probability that A has also occurred is simply $0.2/0.6 = 1/3$. (The information that $\mathrm{P}(A) = 0.5$ is irrelevant.) Formally:

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{0.2}{0.6} = \frac{1}{3}.$$



The foregoing examples motivate the following definition:

Definition 236. Let P be a probability function and $A, B \in \Sigma$ be events. Then *the conditional probability of A given B* is denoted $\mathrm{P}(A|B)$ and is defined by

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Exercise 479. Roll two dice. Given that the sum of the two dice rolls is 8, what is the probability that we rolled at least one even number? (Answer on p. **1963**.)

116.1. The Conditional Probability Fallacy (CPF)

Definition 237. The *conditional probability fallacy (CPF)* is the mistaken belief that

$$\mathrm{P}(A|B) = \mathrm{P}(B|A)$$

is always true.

Informally, the **CPF** is the fallacy of leaping from

"If A, then probably B" to "Since B, then probably A."

But in general, it is not true that $\mathrm{P}(A|B) = \mathrm{P}(B|A)$. Instead:

Fact 233. *(a) If $P(A) < P(B)$, then $P(A|B) < P(B|A)$.*
(b) If $P(A) > P(B)$, then $P(A|B) > P(B|A)$.
(c) If $P(A) = P(B)$, then $P(A|B) = P(B|A)$.

Proof. By definition, $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ and $\mathrm{P}(B|A) = \dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$.

Thus, $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A)}{\mathrm{P}(B)}\mathrm{P}(B|A)$. And so,

$$\begin{aligned}
\mathrm{P}(A) < \mathrm{P}(B) &\Longrightarrow \mathrm{P}(A|B) < \mathrm{P}(B|A), \\
\mathrm{P}(A) > \mathrm{P}(B) &\Longrightarrow \mathrm{P}(A|B) > \mathrm{P}(B|A), \\
\mathrm{P}(A) = \mathrm{P}(B) &\Longrightarrow \mathrm{P}(A|B) = \mathrm{P}(B|A).
\end{aligned}$$

□

The **CPF** is also known as the **confusion of the inverse** or the **inverse fallacy**. In different contexts, it is also known variously as the **base-rate fallacy**, **false-positive fallacy**, or **prosecutor's fallacy**.

Example 1467. Suppose the following statement is true: "If Mary has Ebola, then Mary will probably vomit today." Formally, we might write $\mathrm{P}(\text{Vomit}|\text{Ebola}) = 0.99$.

Mary vomits today. One might then reason, "Since $\mathrm{P}(\text{Vomit}|\text{Ebola}) = 0.99$, by the CPF, we also have $\mathrm{P}(\text{Ebola}|\text{Vomit}) = 0.99$. Thus, Mary probably has Ebola."

Formally, this reasoning is flawed because $\mathrm{P}(\text{Vomit})$ is probably much larger than $\mathrm{P}(\text{Ebola})$. Thus, $\mathrm{P}(\text{Vomit}|\text{Ebola})$ is probably much larger than $\mathrm{P}(\text{Ebola}|\text{Vomit})$.

Informally, the reasoning is flawed because:

- Ebola is extremely rare, so it is extremely unlikely that Mary has Ebola in the first place.
- Besides Ebola, there are many other alternative explanations for why Mary might have vomitted. For example, she might have had motion sickness or food poisoning.

Example 1468. Sally buys a 4D ticket every week. One day, she wins the first prize. To her astonishment, she wins the first prize again the following week.

Her jealous cousin Ah Kow makes a police report, based on the following reasoning:

"Without cheating, the probability that Sally wins the first prize two weeks in a row is 1 in 100 million. Given that she did win first prize two weeks in a row, the probability that she didn't cheat must likewise be 1 in 100 million. In other words, there is almost no chance that Sally didn't cheat."

Let's rephrase Ah Kow's reasoning more formally. Let A and B be the events "Sally wins the first prize two weeks in a row" and "Sally didn't cheat", respectively. We know that $\mathrm{P}(A|B) = 0.00000001$. By the CPF, we have $\mathrm{P}(A|B) = \mathrm{P}(B|A)$. Hence, $\mathrm{P}(B|A) = 0.00000001$. Equivalently, there is probability 0.99999999 that Sally cheated.

Formally, this reasoning is flawed because $\mathrm{P}(B)$ is probably much larger than $\mathrm{P}(A)$. Thus, $\mathrm{P}\left(B\middle|A\right)$ is probably much larger than $\mathrm{P}\left(A\middle|B\right)$.

Informally, the reasoning is flawed because:

- Cheating in 4D is extremely rare (and difficult), so it is extremely unlikely that Sally cheated in the first place.
- Besides cheating, there are many other alternative explanations for why there exists an individual who won first prize two weeks in a row.

One important alternative explanation is that so many individuals buy 4D tickets regularly that there will invariably be someone as lucky as Sally. Suppose that only $100,000$ Singaporeans (less than 2% of Singapore's population) buy one 4D number every week. Then we'd expect that about once every 20 years, one of these $100,000$ Singaporeans will have the fortune of winning the first prize on consecutive weeks. Rare, but hardly impossible.

The next example uses concrete numbers to illustrate how large the discrepancy between $\mathrm{P}(A|B)$ and $\mathrm{P}(B|A)$ can be.

Example 1469. A randomly chosen person is given a free smallpox screening. We know that 1 out of every $1,000,000$ people has smallpox. The test is very accurate: If you have smallpox, it correctly tells you so 99% of the time. (Equivalently, it gives a false negative only 1% of the time.) And if you don't have smallpox, it also correctly tells us so 99% of the time. (Equivalently, it gives a false positive only 1% of the time.)

Formally, let S, $+$, and $-$ denote the events "the randomly chosen person has smallpox", "the test returns positive", and "the test returns negative". Then

$$\mathrm{P}(S) = \frac{1}{1000000}, \qquad \mathrm{P}\left(S^C\right) = \frac{999999}{1000000},$$

$$\mathrm{P}(+|S) = 0.99, \qquad \mathrm{P}(-|S) = 0.01,$$

$$\mathrm{P}\left(-|S^C\right) = 0.99, \qquad \mathrm{P}\left(+|S^C\right) = 0.01.$$

The test result returns positive (i.e. it says that the randomly chosen person has smallpox). *What is the probability that this person actually has smallpox?*

In words, it is easy to confuse "the probability of a positive test result conditional on having smallpox" with "the probability of having smallpox conditional on a positive test result". Formally, this is the CPF. One starts with $\mathrm{P}(+|S) = 0.99$ and confusedly concludes that $\mathrm{P}\left(S\middle|+\right) = 0.99$—this person almost certainly has smallpox.

In fact, as we now show, despite testing positive, the person is very unlikely to have smallpox. The correct answer is $\mathrm{P}\left(S\middle|+\right) \approx \dfrac{1}{10,000}$! In the steps below, each $\stackrel{*}{=}$ simply uses the definition of conditional probability (Definition 236):

$$\begin{aligned}
\mathrm{P}\left(S\middle|+\right) &\stackrel{*}{=} \frac{\mathrm{P}(S \cap +)}{\mathrm{P}(+)} \stackrel{*}{=} \frac{\mathrm{P}(S)\,\mathrm{P}(+|S)}{\mathrm{P}(+)} = \frac{\mathrm{P}(S)\,\mathrm{P}(+|S)}{\mathrm{P}(+\cap S) + \mathrm{P}\left(+\cap S^C\right)} \\
&\stackrel{*}{=} \frac{\mathrm{P}(S)\,\mathrm{P}(+|S)}{\mathrm{P}(S)\,\mathrm{P}(+|S) + \mathrm{P}\left(S^C\right)\mathrm{P}\left(+|S^C\right)} \\
&= \frac{\frac{1}{1000000}0.99}{\frac{1}{1000000}0.99 + \frac{999999}{1000000}0.01} = 0.00009899029 \approx \frac{1}{10,000}.
\end{aligned}$$

This example illustrates how far off the CPF can lead one astray.

Now an actual, real-world example:

Example 1470. The British mother who murdered her two babies. In 1996, Sally Clark's first-born died suddenly within a few weeks of birth. In 1998, the same happened to her second child. Clark was then arrested on suspicion of murdering her babies.

At her trial, an "expert" witness claimed that in an affluent, non-smoking family such as Sally Clark's, the probability of an infant suddenly dying with no explanation was 1/8543. Hence, he concluded, the probability of *two* sudden infant deaths in the same family was $(1/8543)^2$ or approximately 1 in 73 *million*.

The "expert" then committed the CPF. He argued that since

$$\mathrm{P}\left(\text{Two babies suddenly die}|\text{Mother did not murder babies}\right) = \frac{1}{73,000,000},$$

it therefore follows that

$$\mathrm{P}\left(\text{Mother did not murder babies}|\text{Two babies suddenly die}\right) = \frac{1}{73,000,000}.$$

This erroneous reasoning led to Sally Clark being convicted for murdering her two babies.

(Some of you may have noticed that the "expert" actually also made another mistake. But we'll examine this only in the next chapter.)

It turns out that not only laypersons and court prosecutors commit the CPF. As we'll see later, even academic researchers also often commit the CPF, when it comes to interpreting the results of a **null hypothesis significance test** (Chapter 131).

Exercise 480. (Answer on p. **1963**.) At a murder scene, a sample of a blood stain is collected. Its DNA is analysed and compared to a database of DNA profiles. A match with one John Brown is found. Say there is only a 1 in 10 million chance that two random individuals have a DNA match.

Does this mean that there is probability 1 in 10 million that the DNA match with John Brown is merely a coincidence, and thus a near-certainty that the blood stain is really his? Explain why or why not, with reference to the following conditional probabilities:

$$\mathrm{P}\left(\text{Blood stain is not John Brown's}|\text{DNA match}\right),$$

$$\mathrm{P}\left(\text{Blood stain is not John Brown's}|\text{DNA match}\right).$$

116.2. Two-Boys Problem (Fun, Optional)

This is a famous puzzle, first popularised by Martin Gardner in 1959.

Example 1471. Consider all the families in the world that have two children, of whom **at least** one is a boy. Randomly pick one of these families. *What is the probability that both children in this family are boys?*

Think about it (set aside this book) before reading the answer below.

We already know that one child is a boy. So intuition might suggest that "obviously",

$$\mathrm{P}\left(\text{Both boys}\right) = \mathrm{P}\left(\text{The other child is a boy}\right) = 0.5.$$

Intuition would be wrong. Intuition goes astray by failing to recognise that there are three equally likely ways that a family with two children can have at least one boy: BB, BG, or GB. The answer is in fact 1/3:

$$\mathrm{P}\left(BB|\text{At least one boy}\right) = \frac{\mathrm{P}(BB \cap \text{"At least one boy"})}{\mathrm{P}(\text{At least one boy})} = \frac{\mathrm{P}(BB)}{\mathrm{P}(\text{At least one boy})}$$

$$= \frac{\mathrm{P}(BB)}{\mathrm{P}(BB) + \mathrm{P}(BG) + \mathrm{P}(GB)} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{1}{3}.$$

In 2010, the following variant of the above Martin Gardner problem was presented.

Example 1472. Consider all the families in the world that have two children, of whom **at least** one is a boy born on a Tuesday. Randomly pick one of these families. *What is the probability that both children in this family are boys?*

Those familiar with the previous problem might think, "Well, this is exactly the same as the two-boys problem, except with an obviously irrelevant bit of information about the boy being born on a Tuesday. So the answer must be the same as before: 1/3."

It turns out though that, surprisingly, the Tuesday bit of information makes a big difference. The answer is $13/27 = 0.\overline{481}$. This is much closer to 0.5 than to 1/3!

Consider all the "two-child, at-least-one-boy-born-on-a-Tuesday" families in the world. The four mutually exclusive possibilities are

	Child #1	**Child #2**	**Probability**
B_TB	Boy born on Tuesday	Boy (born on any day)	$\mathrm{P}(B_TB) = \frac{1}{14} \cdot \frac{1}{2} = \frac{7}{196}$
B_TG	Boy born on Tuesday	Girl	$\mathrm{P}(B_TG) = \frac{1}{14} \cdot \frac{1}{2} = \frac{7}{196}$
B_NB_T	Boy **not** born on Tuesday	Boy born on Tuesday	$\mathrm{P}(B_NB_T) = \frac{6}{14} \cdot \frac{1}{14} = \frac{6}{196}$
GB_T	Girl	Boy born on Tuesday	$\mathrm{P}(GB_T) = \frac{1}{2} \cdot \frac{1}{14} = \frac{7}{196}$

Altogether then, amongst two-child families with at least one boy born on a Tuesday, the proportion that have **two boys** is

$$\mathrm{P}(BB|\text{"At least one Tuesday boy"})$$

$$= \frac{\mathrm{P}(\text{Both boys, at least one of whom born on Tuesday})}{\mathrm{P}(\text{At least one Tuesday boy})}$$

$$= \frac{\mathrm{P}(B_TB) + \mathrm{P}(B_NB_T)}{\mathrm{P}(B_TB) + \mathrm{P}(B_TG) + \mathrm{P}(B_NB_T) + \mathrm{P}(GB_T)}$$

$$= \frac{\frac{7}{196} + \frac{6}{196}}{\frac{7}{196} + \frac{7}{196} + \frac{6}{196} + \frac{7}{196}} = \frac{13}{27}.$$

117. Probability: Independence

Informally, two events A and B are **independent** if the probability that both occur is simply the product of the probabilities that each occurs. **Independence** is thus analogous to the MP from counting. Formally:

Definition 238. Two events $A, B \in \Sigma$ are *independent* if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B).$$

There is a second, equivalent perspective of independence. Informally, two events A and B are independent if the probability that A occurs is independent of whether B has occurred. Formally:

Fact 234. *Suppose $P(B) \neq 0$. Then A, B are independent events $\iff$ $P(A|B) = P(A)$.*

Proof. By definition of conditional probabilities, $\mathrm{P}(A|B) \stackrel{1}{=} \mathrm{P}(A \cap B)/\mathrm{P}(B)$. By definition of independence, $\mathrm{P}(A \cap B) \stackrel{2}{=} \mathrm{P}(A)\mathrm{P}(B)$. Plugging $\stackrel{2}{=}$ into $\stackrel{1}{=}$, we have $\mathrm{P}(A|B) = \mathrm{P}(A)$, as desired. □

Example 1473. Flip two fair coins. Model this with the usual experiment, where

- $S = \{HH, HT, TH, TT\}$,
- Σ contains $2^4 = 16$ elements, and
- $\mathrm{P}(\{HH\}) = \mathrm{P}(\{HT\}) = \mathrm{P}(\{TH\}) = \mathrm{P}(\{TT\}) = 1/4$.

Let H_1 be the event that the first coin flip is Heads—that is, $H_1 = \{HH, HT\}$. Analogously define T_1, H_2, and T_2.

The intuitive idea of independence is easy to grasp. If we say that the two coin flips are independent, what we mean is that the following four conditions are true:

1. H_1 and H_2 are independent. (*The probability that the second flip is heads is independent of whether the first flip is heads.*)
2. H_1 and T_2 are independent. (*The probability that the second flip is tails is independent of whether the first flip is heads.*)
3. T_1 and H_2 are independent. (*The probability that the second flip is heads is independent of whether the first flip is tails.*)
4. T_1 and T_2 are independent. (*The probability that the second flip is tails is independent of whether the first flip is tails.*)

Formally:

1. $\mathrm{P}(H_1 \cap H_2) = \mathrm{P}(\{HH\}) = \mathrm{P}(H_1)\mathrm{P}(H_2) = \mathrm{P}(\{HH, HT\})\mathrm{P}(\{HH, TH\}) = 0.5 \cdot 0.5 = 0.25$.
2. $\mathrm{P}(H_1 \cap T_2) = \mathrm{P}(\{HT\}) = \mathrm{P}(H_1)\mathrm{P}(T_2) = \mathrm{P}(\{HH, HT\})\mathrm{P}(\{HT, TT\}) = 0.5 \cdot 0.5 = 0.25$.
3. $\mathrm{P}(T_1 \cap H_2) = \mathrm{P}(\{TH\}) = \mathrm{P}(T_1)\mathrm{P}(H_2) = \mathrm{P}(\{TH, TT\})\mathrm{P}(\{HH, TH\}) = 0.5 \cdot 0.5 = 0.25$.
4. $\mathrm{P}(T_1 \cap T_2) = \mathrm{P}(\{TT\}) = \mathrm{P}(T_1)\mathrm{P}(T_2) = \mathrm{P}(\{TH, TT\})\mathrm{P}(\{HT, TT\}) = 0.5 \cdot 0.5 = 0.25$.

Example 1474. Flip a fair coin and roll a fair die. This can be modelled by an experiment, where

- $S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.
- Σ consists of 2^{12} events.
- $\mathrm{P}(A) = |A|/12$, for any event $A \in \Sigma$.

Now consider the event "Heads" $E_1 = \{H1, H2, H3, H4, H5, H6\}$, and the event "Roll an odd number" $E_2 = \{H1, H3, H3, T1, T3, T5\}$. These two events E_1 and E_2 are independent, as we now verify:

$$\mathrm{P}(E_1|E_2) = \frac{\mathrm{P}(E_1 \cap E_2)}{\mathrm{P}(E_2)} = \frac{3/12}{6/12} = \frac{1}{2} = \mathrm{P}(E_1).$$

More broadly, we can even say that the coin flip and die roll are independent. Informally, this means that the outcome of the coin flip has no influence on the outcome of the die roll, and vice versa.

The idea of independence is a little tricky to illustrate on a Venn diagram. I'll try anyway.

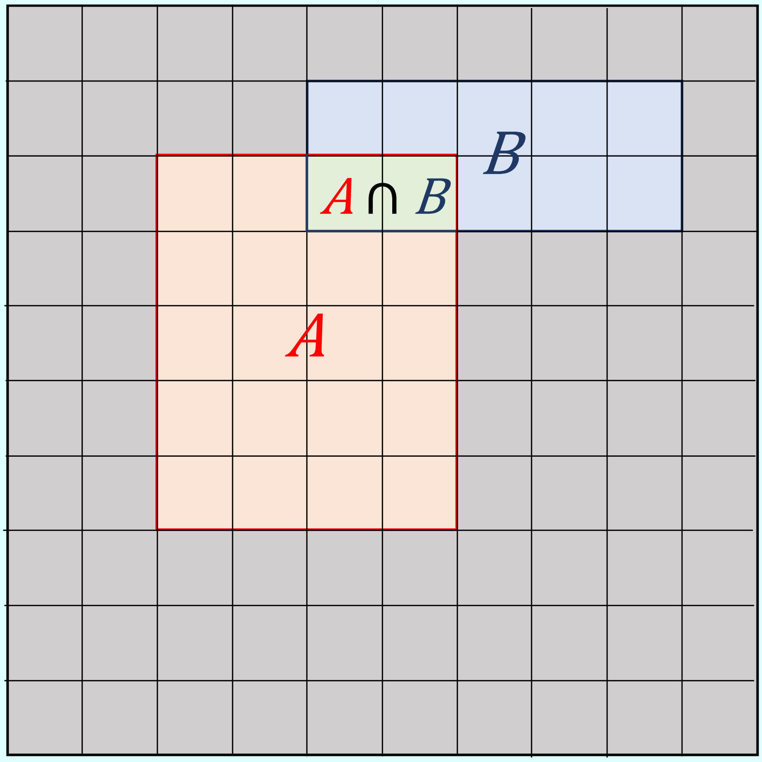
Example 1475. The Venn diagram below illustrates a sample space with 100 equally likely outcomes (represented by 100 small squares). The event A is highlighted in red. The event B is highlighted in blue.

$\mathrm{P}(A) = 0.2$ (A is made of 20 small squares). $\mathrm{P}(B) = 0.1$ (B is made of 10 small squares). The event $A \cap B$, coloured in green, is made of 2 small squares, so $\mathrm{P}(A \cap B) = 0.02$.

We compute

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{0.02}{0.1} = 0.2.$$

We observe that $\mathrm{P}(A) = 0.2 = \mathrm{P}(A|B)$. And so by Fact 234, we conclude that the events A and B are independent.



Exercise 481. Symmetry of Independence. In Fact 234, we showed that "A, B independent $\iff$ $\mathrm{P}(A|B) = \mathrm{P}(A)$". Now prove that "$A, B$ are independent events $\iff$ $\mathrm{P}(B|A) = \mathrm{P}(B)$." (Answer on p. **1964**.)

Exercise 482. (Answer on p. **1964**.) An example of a **transitive relation** is equality: If $A = B$ and $B = C$, then $A = C$. Another example is $\leq$: If $A \leq B$ and $B \leq C$, then $A \leq C$.

In contrast, **independence is not transitive**, as this exercise will demonstrate. That is, even if A and B are independent, and B and C are independent, it may not be that A and C are also independent.

Flip two fair coins. Let H_1 be the event that the first coin flip is heads, H_2 be the event that the second is heads, and T_1 be the event that the first flip is tails. Show that

(a) H_1 and H_2 are independent.

(b) H_2 and T_1 are independent.

(c) H_1 and T_1 are **not** independent.

117.1. Warning: Not Everything is Independent

The idea of independence is intuitively easy to grasp. Indeed, so much so that students often assume that "everything is independent". This is a mistake. Unless you're explicitly told, NEVER assume that two events are independent.

Here are two examples where the assumption of independence is plausible:

Example 1476. The event "coin-flip #1 is heads" and the event "coin-flip #2 is heads" are probably independent.

Example 1477. The event "die-roll #1 is 3" and the event "die-roll #2 is 6" are probably independent.

Here are two examples where the assumption of independence is **not** plausible:

Example 1478. The event "Google's share price rises today" is probably not independent of the event "Apple's share price rises today".

Example 1479. The event "it rains in Singapore today" is probably not independent of the event "it rains in Kuala Lumpur today".

Nonetheless, the assumption of independence is frequently—and incorrectly—made even when it is implausible. One reason is that the maths is easy if we assume independence—we can simply multiply probabilities together.

We now revisit the Sally Clark case. Previously, we saw that the court's "expert" witness committed the CPF. Now, we'll see that he also made a second mistake—that of assuming independence.

Example 1480. The "expert" witness claimed that in an affluent, non-smoking family such as Sally Clark's, the probability of an infant suddenly dying with no explanation was 1/8543. Hence, he concluded, the probability of *two* sudden infant deaths in the same family was $(1/8543)^2$ or approximately 1 in 73 *million*.

Can you spot the error in the reasoning?

By simply multiplying together probabilities, the "expert" implicitly assumed that the two events—"sudden death of baby #1" and "sudden death of baby #2"—are independent.

But as any doctor will tell you, if your family has a history of heart attack, diabetes, or pretty much any other ailment, then you may be at higher risk (than the average person) of suffering the same.

And so, it may well be that in any given year, a random person has probability 0.001 of dying of a heart attack. It does not however follow that in any given year, a random family has probability $0.001^2 = 0.000001$ of two deaths by heart attack.

Similarly, it may be that if one baby in a family has already suddenly died, a second baby is at higher risk (than the average baby) of suddenly dying.

Exercise 483. (Answer on p. **1964**.) Say the probability that a randomly chosen person is or was an NBA player is one in a million. (This is probably about right, since there've only ever been $4,000$ or so NBA players, since the late 1940s.)

The Barry family had *four* players in the NBA—the father Rick Barry and three of his four sons Jon, Brent, and Drew. (The oldest son Scooter didn't make the NBA but was still good enough to play professionally in other basketball leagues around the world.)

A journalist concludes that the probability of a Barry family ever occurring is

$$\left(\frac{1}{1,000,000}\right)^4 = \frac{1}{1,000,000,000,000,000,000,000,000}.$$

This is equal to the probability of buying a 4D number on six consecutive weeks, and winning first prize every time. Is the journalist correct?

117.2. Probability: Independence of Multiple Events

Definition 239. Let P be a probability function and $A, B, C \in \Sigma$ be events.

A, B, C are *pairwise independent* if all three of the following conditions are true:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B),$$
$$\mathrm{P}(B \cap C) = \mathrm{P}(B)\mathrm{P}(C),$$
$$\mathrm{P}(A \cap C) = \mathrm{P}(A)\mathrm{P}(C).$$

A, B, C are *independent* if in addition to the above three conditions being true, it is also true that

$$\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C).$$

It is tempting to believe that pairwise independence implies independence. That is, if the first three conditions listed above are true, then so is the fourth. *Alas*, this is false, as the next exercise demonstrates:

Exercise 484. (Pairwise independence does not imply independence.) (Answer on p. **1964**.)

Flip two fair coins. Let H_1 be the event that the first coin flip is heads, T_2 be the event that the second is tails, and X be the event that the two coin flips are different. Show that

(a) These three events are pairwise independent.

(b) These three events are **not** independent.

118. Fun Probability Puzzles

118.1. The Monty Hall Problem

The Monty Hall Problem is probably the world's most famous probability puzzle. It takes less than a minute to state. Yet its counter-intuitive answer confuses nearly everyone.

You're at a gameshow. There are three boxes, labelled #1, #2, and #3. One box contains one year's worth of a Singapore minister's salary. The other two are empty.

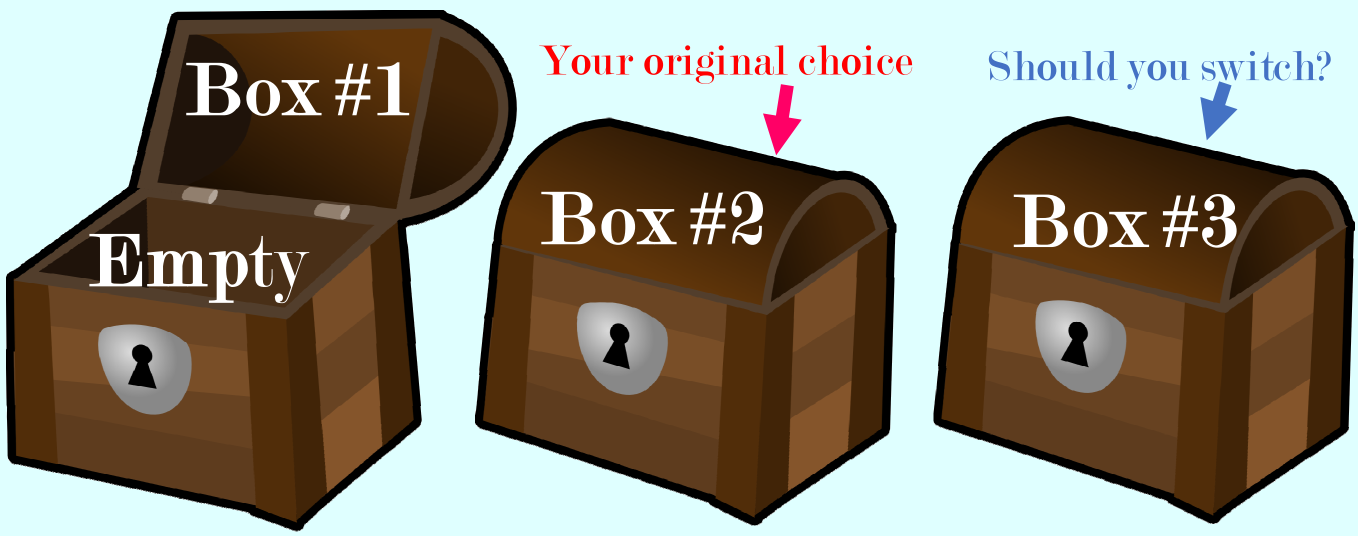
You are asked to pick one box (but you are not allowed to open it yet).

The host, who knows where the minister's salary is, opens one of the other two boxes, to reveal that it is empty. Important: The host is not allowed to open the box that contains the minister's salary; he must always open a box that is empty.

You're now given a choice: Stay (with your original choice) or switch (to the other unopened box). What should you do?

To illustrate:

Example 1481. Say you pick Box #2. The host then opens an empty Box #1. You're now given a choice: Stay (with Box #2) or switch (to Box #3). Which do you choose?



Take as long as you need to think about this problem, before turning to the next page for the answer.

A magazine columnist named Marilyn vos Savant[527] gave the correct answer:

> *Yes; you should switch. The first door has a 1/3 chance of winning, but the second door has a 2/3 chance.*

Here are two informal explanations:

1. The probability that the minister's salary is in the box you picked is 1/3. The probability that the minister's salary is in either of the other two boxes is 2/3. Of the other two boxes, the gameshow host (who knows where the salary is) helps you eliminate one of them. So the remaining unopened box still has probability 2/3 of containing the minister's salary.
2. Imagine instead that there are 100 boxes, of which one contains the minister's salary and the others are empty. You pick one. Of the remaining 99, the gameshow host opens 98. You are again given the choice: Should you stay or switch? In this more extreme version of the game, it is perhaps more obvious that your originally picked box has only probability 1/100 of containing the minister's salary, while the only other unopened box has probability 99/100 of the same. Therefore, you should switch.

Here's a more formal explanation using the method of enumeration:

3. Say you originally pick Box #1. There are three possible cases, each occurring with probability 1/3:

Case	Box #1	Box #2	Box #3	Host opens
A	Minister's salary	Empty	Empty	Box #2 or Box #3
B	Empty	Minister's salary	Empty	Box #3
C	Empty	Empty	Minister's salary	Box #2

Not switching wins you the minister's salary only in Case A (1/3 probability).

Switching wins you the minister's salary in Cases B and C (2/3 probability).

[527] Marilyn vos Savant was, briefly, on the Guinness Book of Records as the person with the world's highest IQ, until Guinness retired this category because IQ tests were considered to be too unreliable.

Even with the above explanations, some of you may remain unconvinced. Don't worry, you are not alone. After Marilyn's initial response, 10,000 readers sent in letters telling her she was wrong. Some were from Professors of Mathematics and PhDs. A few examples:[528]

> *As a professional mathematician, I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error and in the future being more careful.*

> *There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propagating more. Shame!*

> *Maybe women look at math problems differently than men.*

Unfortunately for the above letter writers, Marilyn was correct and they were wrong.

The best way to convince the sceptical is through simulations—try this Google spreadsheet. Or if you don't trust computers, do an actual experiment:

Class Activity

Form pairs. One person is the gameshow host and the other is the contestant. The host decides where the prize is (Box #1, #2, or #3). The contestant then picks a box. The host then tells the contestant which one of the other two boxes is empty. The contestant then decides whether to stay or switch.

Repeat as many times as you have time for. Record the proportion of times that the contestant should have switched. You should find that this proportion is about 2/3.

[528]You can read more of these letters at her website.

118.2. The Birthday Problem

Example 1482. (The birthday problem.) *What is the smallest number n of people in a room, such that it is more likely than not, that at least 2 people in the room share the same birthday?*[529]

Fix person #1's birthday. Then

- The probability that person #2's birthday is different (from person #1) is 364/365.
- The probability that person #3's birthday is different (from persons #1 and #2) is 363/365.
- The probability that person #4's birthday is different (from persons #1, #2, and #3) is 362/365.
- … …
- The probability that person #n's birthday is different (from persons #1 through #n−1) is $(366 - n)/365$.

Altogether, the probability that **no** 2 persons share the same birthday is

$$\frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{366-n}{365}.$$

Hence, the probability that **at least** 2 persons share the same birthday is

$$1 - \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{366-n}{365}.$$

The smallest integer n for which the above probability is at least 0.5 is 23. That is, perhaps surprisingly, with just 23 people, it is more likely than not that at least 2 persons share a birthday.

119. Random Variables: Introduction

Informally, a **random variable** is a function that assigns a real number (you can think of this as a "numerical code") to each possible outcome s. We call any such real number an **observed value** of X.

Example 1483. Model a fair coin-flip with the usual experiment $E = (S, \Sigma, \mathrm{P})$, where

- $S = \{H, T\}$.
- $\Sigma = \{\varnothing, \{H\}, \{T\}, S\}$.
- $\mathrm{P} : \Sigma \to \mathbb{R}$ is defined by $\mathrm{P}(\varnothing) = 0$, $\mathrm{P}(\{H\}) = \mathrm{P}(\{H\}) = 0.5$, and $\mathrm{P}(s) = 1$.

Let $X : S \to \mathbb{R}$ be the **random variable** that indicates whether the coin-flip is heads. That is, the **observed value** of X is $X(H) = 1$ if the outcome is heads and $X(t) = 0$ if the outcome is tails.

Formally:

Definition 240. Let $E = (S, \Sigma, \mathrm{P})$ be an experiment. A *random variable* X (on the experiment E) is any function with domain S and codomain $\mathbb{R}$.

Given any random variable X and any outcome $s \in S$, we call $X(s)$ the *observed* (or *realised*) *value* of the random variable X. We often denote a generic observed value $X(s)$ by the lower-case letter x.

119.1. A Random Variable vs. Its Observed Values

Students often confuse a random variable with an observed value of the random variable. This confusion is, of course, simply the confusion between a function and the value taken by the function.

Example 1483 (continued from above). X is a function with domain S and codomain $\mathbb{R}$. X is therefore a random variable.

If the outcome of the coin-flip is heads, we do **not** say that X is 1. Instead, we say that *the observed value of* X is 1.

If the outcome of the coin-flip is tails, we do **not** say that X is 0. Instead, we say that *the observed value of* X is 0.

Remember: A random variable X is a function that can take on many possible real number values. Each such value $x = X(s)$ is called an observed value of X.

119.2. $X = k$ Denotes the Event $\{s \in S : X(s) = k\}$

Definition 241. Given a random variable $X : S \to \mathbb{R}$, the notation "$X = k$" denotes the event $\{s \in S : X(s) = k\}$.

The notation "$X \geq k$", "$X > k$", "$X \leq k$", "$X < k$", "$a \leq X \leq b$", etc. are similarly defined.

Example 1483 (continued from above). $X(H) = 1$ and $X(t) = 0$. So we can write

$$X = 1 \text{ denotes the event } \{s \in S : X(s) = 1\} = \{H\},$$

$$X = 0 \text{ denotes the event } \{s \in S : X(s) = 0\} = \{T\}.$$

Moreover, $\mathrm{P}(\{H\}) = 0.5$ and $\mathrm{P}(\{T\}) = 0.5$. So we can also write

$$\mathrm{P}(X = 1) = 0.5 \quad \text{and} \quad \mathrm{P}(X = 0) = 0.5.$$

Now let's try some other arbitrary number like 13.71. Notice there is no outcome s such that $X(s) = 13.71$. Thus,

$$X = 13.71 \text{ denotes the event } \{s \in S : X(s) = 13.71\} = \varnothing, \quad \text{and} \quad \mathrm{P}(X = 13.71) = 0.$$

Indeed, for any $k \neq 0, 1$, there is no outcome s such that $X(s) = k$. Thus,

$$X = k \text{ denotes the event } \{s \in S : X(s) = k\} = \varnothing, \quad \text{and} \quad \mathrm{P}(X = k) = 0.$$

Since $\mathrm{P}(\varnothing) = 0$, we also have $\mathrm{P}(X = k) = \mathrm{P}(\varnothing) = 0$, for any $k \neq 0, 1$.

Define $Y : S \to \mathbb{R}$ by $Y(H) = 15.5$, $Y(t) = 15.5$. Y is an example of a **constant random variable**. We may write

$$Y = 15.5 \text{ denotes the event } \{s \in S : X(s) = 15.5\} = \{H, T\}, \quad \text{and} \quad \mathrm{P}(X = 15.5) = 1.$$

Moreover, for any $k \neq 15.5$,

$$Y = k \text{ denotes the event } \{s \in S : X(s) = k\} = \varnothing, \quad \text{and} \quad \mathrm{P}(Y = k) = 0.$$

119.3. The Probability Distribution of a Random Variable

We call a complete specification of $\mathrm{P}(X = k)$ for all values of k the **probability distribution** (or **probability law** or **probability mass function**) of X. In the above example, we gave the probability distributions of both X and Y.

More examples of random variables and their probability distributions:

Example 1484. Flip two fair coins. Model this with the usual experiment, where $S = \{HH, HT, TH, TT\}$.

Let $X : S \to \mathbb{R}$ indicate whether the two coin flips are the same and $Y : S \to \mathbb{R}$ count the number of heads. That is,

$$X(HH) = 1,\ X(HT) = 0,\ X(TH) = 0,\ X(TT) = 1,$$

$$Y(HH) = 2,\ Y(HT) = 1,\ Y(TH) = 1,\ Y(TT) = 0.$$

And

$$\mathrm{P}(X = 0) = 0.5,\ \mathrm{P}(X = 1) = 0.5,\ \text{and } \mathrm{P}(X = k) = 0,\ \text{for any } k \neq 0, 1.$$

$$\mathrm{P}(Y = 0) = 0.25,\ \mathrm{P}(Y = 1) = 0.5,\ \mathrm{P}(Y = 2) = 0.25,\ \text{and } \mathrm{P}(X = k) = 0,\ \text{for any } k \neq 0, 1, 2.$$

Another example:

Example 1485. Pick a random card from the standard 52-card deck. Model this with the usual experiment, where

$$S = \{A\spadesuit, K\spadesuit, , \dots, 2\spadesuit, A\heartsuit, K\heartsuit, \dots, 2\heartsuit, A\diamondsuit, K\diamondsuit, \dots, 2\diamondsuit, A\clubsuit, K\clubsuit, \dots, 2\clubsuit\}.$$

$X : S \to \mathbb{R}$ is the High Card Point count (used in the game of bridge). I.e.,

$$X(\text{A of any suit}) = 4, \qquad X(\text{K of any suit}) = 3, \qquad X(\text{Q of any suit}) = 2,$$
$$X(\text{J of any suit}) = 1, \qquad X(\text{Any other card}) = 0.$$

Thus,

$$\text{P}(X = 0) = \frac{36}{52}, \qquad \text{P}(X = 1) = \frac{4}{52}, \qquad \text{P}(X = 2) = \frac{4}{52},$$

$$\text{P}(X = 3) = \frac{4}{52}, \qquad \text{P}(X = 4) = \frac{4}{52}, \qquad \text{P}(X = k) = 0,$$

for any $k \neq 0, 1, 2, 3, 4$.

$Y : S \to \mathbb{R}$ indicates whether the picked card is a spade ($\spadesuit$). I.e.,

$$Y(\text{Any } \spadesuit) = 1, \quad Y(\text{Any other card}) = 0.$$

Thus,

$$\text{P}(Y = 0) = \frac{39}{52}, \qquad \text{P}(Y = 1) = \frac{13}{52}, \qquad \text{P}(Y = k) = 0, \text{ for any } k \neq 0, 1.$$

Example 1486. Roll two fair dice. Model this with the usual experiment, where

$$S = \left\{ \begin{matrix}⚀\\⚀\end{matrix}, \begin{matrix}⚀\\⚁\end{matrix}, \dots, \begin{matrix}⚁\\⚀\end{matrix}, \begin{matrix}⚁\\⚁\end{matrix}, \dots, \begin{matrix}⚅\\⚀\end{matrix}, \begin{matrix}⚅\\⚁\end{matrix}, \dots, \begin{matrix}⚅\\⚅\end{matrix} \right\}.$$

$X : S \to \mathbb{R}$ is the sum of the two dice. And so for example,

$$X\left(\begin{matrix}⚁\\⚄\end{matrix}\right) = 7 \quad \text{and} \quad X\left(\begin{matrix}⚃\\⚀\end{matrix}\right) = 5.$$

The table below says that $\mathrm{P}(X = 2) = 1/36$, because there is only one way the event $X = 2$ can occur. And $\mathrm{P}(X = 3) = 2/36$, because there are two ways the event $X = 3$ can occur.

You are asked to complete the table in the next exercise.

k	s such that $X(s) = k$	$\mathrm{P}(X = k)$
2	$\begin{matrix}⚀\\⚀\end{matrix}$	$\frac{1}{36}$
3	$\begin{matrix}⚀\\⚁\end{matrix}, \begin{matrix}⚁\\⚀\end{matrix}$	$\frac{2}{36}$
4		
5		
6		
7		
8		
9		
10		
11		
12		

Exercise 485. (Continuation of the above example.) (Answer on p. **1965**.) (a) Complete the above table.

Consider the event E, described in words as "the sum of the two dice is at least 10".

(b) Write down the event E in terms of X.

(c) Calculate $\mathrm{P}(E)$.

Example 1486 (continued from above). Continue with the same the roll-two-fair-dice example, with X again being the random variable that is the sum of the two dice. We had

$$X\begin{pmatrix}\epsdice{2}\\ \epsdice{5}\end{pmatrix} = 7 \quad \text{and} \quad X\begin{pmatrix}\epsdice{4}\\ \epsdice{1}\end{pmatrix} = 5.$$

Let $Y : S \to \mathbb{R}$ be the product of the two dice. And so for example,

$$Y\begin{pmatrix}\epsdice{2}\\ \epsdice{5}\end{pmatrix} = 10 \quad \text{and} \quad Y\begin{pmatrix}\epsdice{4}\\ \epsdice{1}\end{pmatrix} = 4.$$

Remember: random variables are simply functions. And thus, we can manipulate random variables just like we manipulate any functions.

So for example, consider the function $X + Y : S \to \mathbb{R}$. It is also a random variable. We have

$$(X+Y)\begin{pmatrix}\epsdice{2}\\ \epsdice{5}\end{pmatrix} = 17 \quad \text{and} \quad (X+Y)\begin{pmatrix}\epsdice{4}\\ \epsdice{1}\end{pmatrix} = 9.$$

Similarly, consider the function $XY : S \to \mathbb{R}$. It is also a random variable. We have

$$(XY)\begin{pmatrix}\epsdice{2}\\ \epsdice{5}\end{pmatrix} = 70 \quad \text{and} \quad (XY)\begin{pmatrix}\epsdice{4}\\ \epsdice{1}\end{pmatrix} = 20.$$

Finally, consider the function $4X - 5Y : S \to \mathbb{R}$. It is also a random variable. We have

$$(4X-5Y)\begin{pmatrix}\epsdice{2}\\ \epsdice{5}\end{pmatrix} = -22 \quad \text{and} \quad (4X-5Y)\begin{pmatrix}\epsdice{4}\\ \epsdice{1}\end{pmatrix} = 0.$$

Exercise 486. Continue with the above roll-two-fair-dice example. Let $P : S \to \mathbb{R}$ be the greater of the two dice. Let $Q : S \to \mathbb{R}$ be the difference of the two dice. Evaluate the functions P, Q, and PQ at ⚁⚄ and ⚃⚀. (Answer on p. **1966**.)

Exercise 487. (Answer on p. **1966**.) Model a fair die-roll with the usual experiment $E = \{S, \Sigma, \mathrm{P}\}$. Define the function $X : S \to \mathbb{R}$ by the mapping rule $X(1) = 1$, $X(2) = 2$, $X(3) = 3$, $X(4) = 4$, $X(5) = 5$, and $X(6) = 6$.

Is X a random variable on E? Why or why not?

If X is indeed a random variable on E, then write down also $\mathrm{P}(X = k)$, for all possible k.

Exercise 488. For each of the following real-world scenarios, write down, in precise mathematical notation (i) the experiment $E = \{S, \Sigma, \mathrm{P}\}$; (ii) what the random variable X is; and (iii) $\mathrm{P}(X = k)$, for all possible k. (Answers on pp. **1966** and **1967**.)

(a) Flip 4 (fair) coins. Let the random variable X be a count of the number of heads.

(b) Roll 3 (fair) dice. Let the random variable X be the sum of the three dice. (Tedious.)

120. Random Variables: Independence

Definition 242. Given random variables $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$, the notation "$X = x, Y = y$" denotes the event $\{s \in S : X(s) = x, Y(s) = y\}$.

Example 1487. Flip two fair coins. Model this with the usual experiment where $S = \{HH, HT, TH, TT\}$.

Let $X : S \to \mathbb{R}$ indicate whether the two coin flips were the same and $Y : S \to \mathbb{R}$ count the number of heads. That is,

$$\begin{array}{rcccc} & X(HH) = 1, & X(HT) = 0, & X(TH) = 0, & X(TT) = 1, \\ \text{and} & Y(HH) = 2, & Y(HT) = 1, & Y(TH) = 1, & Y(TT) = 0. \end{array}$$

Then $X = 0, Y = 0$ is the event that the two coin flips were not the same AND the number of heads was 0. By observation, this event is the empty set. Thus, $\mathrm{P}(X = 0, Y = 0) = \mathrm{P}(\varnothing) = 0$.

$X = 1, Y = 0$ is the event that the two coin flips were the same AND the number of heads was 0. By observation, this event is $\{TT\}$. Thus, $\mathrm{P}(X = 1, Y = 0) = \mathrm{P}(\{TT\}) = 0.25$.

Exercise: Verify for yourself that

$$\mathrm{P}(X = 0, Y = 1) = 0.5, \quad \mathrm{P}(X = 1, Y = 1) = 0,$$

$$\mathrm{P}(X = 0, Y = 2) = 0, \quad \mathrm{P}(X = 1, Y = 2) = 0.25.$$

Informally, two random variables are **independent** if knowing the value of one does not tell us anything about the value of the other.

Example 1487 (continued from above). Flip two fair coins. We say the two coin-flips are **independent**. Informally, the outcome of one doesn't affect the other. Knowing that the first coin-flip is heads tells us nothing about the second coin-flip.

A little more formally, let A and B be the random variables indicating whether the first and second coin-flip are heads (respectively). That is, $A = 1$ if the first coin-flip is heads and $A = 0$ otherwise; and $B = 1$ if the second coin-flip is heads and $B = 0$ otherwise. Then the informal statement "the two coin-flips are independent" may be translated into the formal statement "the random variables A and B are independent".

Informally, knowing the observed value of A tells us nothing about whether $B = 0$ or $B = 1$. (And vice versa.)

Formally:

Definition 243. Given random variables $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$, we say that X and Y are *independent* if for all x, y,

$$\mathrm{P}\left(X = x, Y = y\right) = \mathrm{P}(X = x)\mathrm{P}(Y = y).$$

Let's restate the above definition more explicitly. Suppose X can take on values $x_1, x_2, \ldots, x_n$ and Y can take on values $y_1, y_2, \ldots, y_m$. Then to say that X and Y are **independent** is to say that all of the following $n \times m$ pairs of events are independent

$$\begin{array}{cccc} X = x_1,\ Y = y_1, & X = x_1,\ Y = y_2, & \ldots & X = x_1,\ Y = y_m, \\ X = x_2,\ Y = y_1, & X = x_2,\ Y = y_2, & \ldots & X = x_2,\ Y = y_m, \\ \vdots & \vdots & \ldots & \vdots \\ X = x_n,\ Y = y_1, & X = x_n,\ Y = y_2, & \ldots & X = x_n,\ Y = y_m. \end{array}$$

Independence between two random variables is thus equivalent to independence between many pairs of events.

Example 1487 (continued from above). We now verify, in more formal and precise language, that "the two coin-flips are indeed independent".

Again, A and B are the random variables indicating whether the first and second coin-flips are heads (respectively).

We now verify that indeed, $P(A = a, B = b) = P(A = a)P(B = b)$ for all possible values of a and b:

	$P(A = a, B = b)$	$P(A = a)P(B = b)$	
$a = 0, b = 0$	$P(\{TT\}) = 0.25$	$P(\{TH, TT\})\,P(\{HT, TT\}) = 0.5 \times 0.5,$	✓
$a = 1, b = 0$	$P(\{HT\}) = 0.25$	$P(\{HH, HT\})\,P(\{HT, TT\}) = 0.5 \times 0.5,$	✓
$a = 0, b = 1$	$P(\{TH\}) = 0.25$	$P(\{TH, TT\})\,P(\{HH, TH\}) = 0.5 \times 0.5,$	✓
$a = 1, b = 1$	$P(\{HH\}) = 0.25$	$P(\{HH, HT\})\,P(\{HH, TH\}) = 0.5 \times 0.5.$	✓

Exercise 489. Flip two fair coins. Let $X : S \to \mathbb{R}$ indicate whether the two coin flips were the same and $Y : S \to \mathbb{R}$ count the number of heads. Are X and Y independent random variables? (Answer on p. **1969**.)

Earlier we warned against blithely assuming that any two events are independent. Here we can repeat this warning: Unless explicitly told (or you have a good reason), do not assume that two random variables are independent.

The assumption of independence is a strong one. There are many scenarios where it is plausible. For example, the flips of two coins are probably independent. The rolls of two dice are probably independent.

There are, however, also many scenarios where it is *not* plausible. Today's changes in the share prices of Google and Apple are probably not independent. Today's rainfall in Singapore and in Kuala Lumpur are probably not independent.

Nonetheless, the assumption of independence is frequently—and incorrectly—made even when it is implausible. The reason is that the maths is easy if we assume independence—we can simply multiply probabilities together. Unfortunately, incorrectly assuming independence has sometimes had tragic consequences, as we saw in the Sally Clark case.

121. Random Variables: Expectation

Example 1488. Let X be the outcome of a fair die roll.

What is the **expected value** (or the **mean**) of X? In other words, on average, what's the expected outcome of a fair die roll?

Note that X takes on a value 1 with probability 1/6. Similarly, it takes on a value 2 with probability 1/6. Etc. Hence, the expected value of X, denoted $\mathrm{E}[X]$ is given by

$$\mathbf{E}[X] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5.$$

$\mathrm{E}[X]$ is thus simply a weighted average of the possible values of X, where the weights are the probability weights.

We'll use the following slightly incorrect definition of a **discrete random variable**:[530]

Slightly Incorrect Definition. A random variable is *discrete* if its range is finite.

That is, a random variable is discrete if it takes on finitely many possible values.

We can now formally define the expected value of a discrete random variable:

Definition 244. Let $E = (S, \Sigma, \mathrm{P})$ be an experiment. Then the corresponding *expectation operator*, denoted E, is the function that maps any discrete random variable $X : S \to \mathbb{R}$ to a real number, according to the mapping rule

$$\mathbf{E}[X] = \sum_{k \in \text{Range } X} \mathrm{P}(X = k) \cdot k.$$

We call $\mathrm{E}[X]$ the *expected value* (or *mean*) of X. We often write $\mu_X = \mathrm{E}[X]$ or even $\mu = \mathrm{E}[X]$ (if it is clear from the context that we're talking about the mean of X).

Example 1489. Let X be the outcome of a fair die roll. The range of X is $\text{Range}(x) = \{1, 2, 3, 4, 5, 6\}$. So

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{k \in \text{Range}(x)} \mathrm{P}(X = k) \cdot k \\
&= \mathrm{P}(X=1)\cdot 1 + \mathrm{P}(X=2)\cdot 2 + \mathrm{P}(X=3)\cdot 3 + \mathrm{P}(X=4)\cdot 4 + \mathrm{P}(X=5)\cdot 5 + \mathrm{P}(X=6)\cdot 6. \\
&= \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = 3.5.
\end{aligned}$$

[530] The correct definition is this: A random variable is *discrete* if its range is finite or **countably infinite**. I avoid giving this correct definition because this would require explaining what "countably infinite" means.

Example 1490. Let Y be the sum of two fair die-rolls.

The range of Y is $\text{Range}(y) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. In Exercise 485, we worked out that $\text{P}(Y = 2) = 1/36$, $\text{P}(Y = 3) = 2/36$, etc. Thus,

$$\begin{aligned}
\mathbf{E}[Y] &= \sum_{k \in \text{Range}(y)} \text{P}(Y = k) \cdot k \\
&= \text{P}(Y = 2) \cdot 2 + \text{P}(Y = 3) \cdot 3 + \text{P}(Y = 4) \cdot 4 + \text{P}(Y = 5) \cdot 5 + \cdots + \text{P}(Y = 12) \cdot 12 \\
&= \frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \frac{4}{36} \cdot 5 + \frac{5}{36} \cdot 6 + \frac{6}{36} \cdot 7 + \frac{5}{36} \cdot 8 + \frac{4}{36} \cdot 9 + \frac{3}{36} \cdot 10 + \frac{2}{36} \cdot 11 + \frac{1}{36} \cdot 12 \\
&= \frac{2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12}{36} = \frac{252}{36} = 7.
\end{aligned}$$

Example 1491. XXFlip two fair coins and roll two fair dice. Let X be the number of heads and Y be the number of sixes.

Problem: *What is $E[X + Y]$?*

As it turns out, it is generally true that $\text{E}[X + Y] = \text{E}[X] + \text{E}[Y]$ (as we'll see in the next section). So if we knew this, then the problem would be very easy:

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y] = 1 + \frac{1}{3} = \frac{4}{3}.$$

But as an exercise, let's pretend we don't know that $\text{E}[X + Y] = \text{E}[X] + \text{E}[Y]$. We thus have to work out $\text{E}[X + Y]$ the hard way:

First, note that $\text{Range}(X + Y) = \{0, 1, 2, 3, 4\}$. $\text{P}(X + Y = 0)$ is the probability of 0 heads and 0 sixes. And $\text{P}(X + Y = 1)$ is the probability of 1 head and 0 sixes OR 0 heads and 1 six. We can compute

$$\text{P}(X + Y = 0) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{144},$$

$$\text{P}(X + Y = 1) = \binom{2}{1} \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{5}{6} + \frac{1}{2} \cdot \frac{1}{2} \binom{2}{1} \frac{5}{6} \frac{1}{6} = \frac{50}{144} + \frac{10}{144} = \frac{60}{72}.$$

You are asked to complete the rest of this problem in the exercise below.

Exercise 490. Complete the above example by following these steps: (a) Compute $\text{P}(X + Y = 2)$. (b) Compute $\text{P}(X + Y = 3)$. (c) Compute $\text{P}(X + Y = 4)$. (d) Now compute $\text{E}[X + Y]$. (Answer on p. **1969**.)

121.1. The Expected Value of a Constant R.V. is Constant

Example 1492. Let 5 be a constant random variable on some experiment $E = (S, \Sigma, \mathrm{P})$. That is, $5 : S \to \mathbb{R}$ is the function defined by $s \mapsto 5$. (Note that the symbol 5 does double duty by denoting both a function and a real number.) Then not surprisingly,

$$\mathbf{E} \quad \overset{\substack{\text{Function} \\ \downarrow}}{[5]} \quad = \quad \overset{\substack{\text{Number} \\ \downarrow}}{5} \quad .$$

That is, on average, we expect the random variable 5 to take on the value 5.

We can easily prove the above observation:

Fact 235. *If the constant random variable c maps every outcome to the number c, then $E[c] = c$.*

Proof. The PMF of the constant random variable c is given by $\mathrm{P}(c = c) = 1$ and $\mathrm{P}(c = k) = 0$ for any $k \neq c$. Hence, $\mathrm{E}[c] = \mathrm{P}(c = c) \cdot c = 1 \cdot c = c$. □

Exercise 491. In the game of 4D, you pay \$1 to pick any four-digit number between 0000 and 9999 (there are thus $10,000$ possible choices). There are two variants of the 4D game—"big" and "small". The prize structures are as given below. Let X be the prize received from a \$1 stake in the "big" game and Y be the prize received from a \$1 stake in the "small" game. (Answer on p. **1970**.)

(a) Write down the range of X and the range of Y.

(b) Write down the probability distributions of X and Y.

(c) Hence find $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.

(d) Which game—"big" or "small"—is expected to lose you less money?

Prize Structure for Ordinary, 4-D Roll and System Entries

(a) Prize Amounts and Winning Numbers for 4-D Game (Big)

Prize	Number of 4-digit Winning Numbers	Prize Amount (for every \$1 stake)
1st Prize	one number	\$ 2,000
2nd Prize	one number	\$ 1,000
3rd Prize	one number	\$ 490
Starter Prizes	ten numbers	\$ 250
Consolation Prizes	ten numbers	\$ 60

(b) Prize Amounts and Winning Numbers for 4-D Game (Small) – applicable to Ordinary Entry, 4-D Roll and System Entry

Prize	Number of 4-digit Winning Numbers	Prize Amount (for every \$1 stake)
1st Prize	one number	\$ 3,000
2nd Prize	one number	\$ 2,000
3rd Prize	one number	\$ 800

There shall be no starter prize or consolation prize for 4-D Game (Small).

(Source: Singapore Pools, "Rules for the 4-D Game", Version 1.11, 17/11/15. PDF.)

Definition 245. Let $f : A \to B$ be a function, $x, y \in A$, and $k \in \mathbb{R}$. We say that f is a *linear transformation* if it satisfies the following two conditions:

(a) Additivity: $f(x+y) = f(x) + f(y)$*; and*

(b) Homogeneity of degree 1*:* $f(kx) = kf(x)$.

Example 1493. The summation operator $\sum$ is an example of a linear transformation. Because it satisfies both additivity and homogeneity of degree 1:

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \quad \text{and} \quad \sum_{i=1}^{n}(ka_i) = k\sum_{i=1}^{n} a_i.$$

Example 1494. The differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ is an example of a linear transformation. Because it satisfies both additivity and homogeneity of degree 1:

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}(kf(x)) = k\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

A common mistake made by students is to believe that "everything is linear". Here are two examples of operators that are not linear transformations.

Example 1495. The square-root operator $\sqrt{\cdot}$ is **not** a linear transformation. In general, we do not have

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y} \quad \text{or} \quad \sqrt{kx} = k\sqrt{x}.$$

Example 1496. The square operator $\cdot^2$ is **not** a linear transformation. In general, we do not have

$$(x+y)^2 = x^2 + y^2 \quad \text{or} \quad (kx)^2 = kx^2.$$

It turns out that the expectation operator is a linear transformation.

Proposition 22. *The expectation operator E is linear. That is, if X and Y are random variables and c is a constant, then*

(a) Additivity: $E[X+Y] = E[X] + E[Y]$,

(b) Homogeneity of degree 1*:* $E[cX] = cE[X]$.

Proof. Optional, see p. **1731** (Appendices). □

The linearity of the expectation operator is a powerful property, especially because it is true even if independence is not satisfied.

Example 1497. I stake \$100 on each of two different 4D numbers for Saturday's drawing ("big" game). (So that's \$200 total.)

Let X and Y be my winnings (excluding my original stake) from the first and second numbers (respectively). Now, X and Y are certainly not independent because for example, if my first number wins first prize, then my second number cannot possibly also win first prize.

Nonetheless, despite X and Y not being independent, the linearity of the expectation operator tells us that

$$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y] = \$65.90 + \$65.90 = \$131.80.$$

Example 1498. Consider a random variable X that is equally likely to take on one of 5 possible values: $0, 1, 2, 3, 4$. Its mean is

$$\mu_X = \sum \mathrm{P}(X = k) \cdot k = \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot 1 + \frac{1}{5} \cdot 2 + \frac{1}{5} \cdot 3 + \frac{1}{5} \cdot 4 = 2.$$

Now consider another random variable Y that is equally likely to take on one of 5 possible values: $-8, -3, 2, 7, 12$. Coincidentally, its mean is the same:

$$\mu_Y = \sum \mathrm{P}(Y = k) \cdot k = \frac{1}{5} \cdot (-8) + \frac{1}{5} \cdot (-3) + \frac{1}{5} \cdot 2 + \frac{1}{5} \cdot 7 + \frac{1}{5} \cdot 12 = 2.$$

The random variables X and Y share the same mean. However, there is an obvious difference: Y is "more spread out".

What, precisely, do we mean when we say that one random variable is "more spread out" than another?

Our goal in this section is to invent a measure of "spread-outness". We'll call this the **variance** and denote the variance of any random variable X by $\mathrm{Var}[X]$.

It's not at all obvious how the variance should be defined. One possibility is to define the variance as **the weighted average of the deviations from the mean**.

Example 1234 (continued from above). (Our first proposed definition of variance.)

For X, the weighted average of the deviations from the mean is

$$\begin{aligned}
\mathbf{V}[X] &= \sum \mathrm{P}(X = k) \cdot (k - \mu) \\
&= \frac{1}{5} \cdot (0 - \mu) + \frac{1}{5} \cdot (1 - \mu) + \frac{1}{5} \cdot (2 - \mu) + \frac{1}{5} \cdot (3 - \mu) + \frac{1}{5} \cdot (4 - \mu) \\
&= \frac{1}{5} \cdot (0 - 2) + \frac{1}{5} \cdot (1 - 2) + \frac{1}{5} \cdot (2 - 2) + \frac{1}{5} \cdot (3 - 2) + \frac{1}{5} \cdot (4 - 2) \\
&= -\frac{2}{5} - \frac{1}{5} + 0 + \frac{1}{5} + \frac{2}{5} = 0.
\end{aligned}$$

Hmm. This works out to be 0. Is that just a weird coincidence? Let's try the same for Y:

$$\begin{aligned}
\mathbf{V}[Y] &= \sum \mathrm{P}(Y = k) \cdot (k - \mu) \\
&= \frac{1}{5} \cdot (-8 - \mu) + \frac{1}{5} \cdot (-3 - \mu) + \frac{1}{5} \cdot (2 - \mu) + \frac{1}{5} \cdot (7 - \mu) + \frac{1}{5} \cdot (12 - \mu) \\
&= \frac{1}{5} \cdot (-8 - 2) + \frac{1}{5} \cdot (-3 - 2) + \frac{1}{5} \cdot (2 - 2) + \frac{1}{5} \cdot (7 - 2) + \frac{1}{5} \cdot (12 - 2) \\
&= -2 - 1 + 0 + 1 + 2 = 0.
\end{aligned}$$

Hmm. Again it works out to be 0.

This is no mere coincidence. It turns out that $\sum_k \mathrm{P}(X = k) \cdot (k - \mu)$ is always equal to 0.

This is because

$$\sum_k \mathrm{P}(X=k)\cdot(k-\mu) = \overbrace{\sum_k \mathrm{P}(X=k)\cdot k}^{=\mu} - \sum_k \mathrm{P}(X=k)\cdot\mu$$
$$= \mu - \mu\underbrace{\sum_k \mathrm{P}(X=k)}_{=1} = 0.$$

So our first proposed definition of the variance—the weighted average of the deviations from the mean—is always equal to 0. Intuitively, the reason is that the negative deviations (corresponding to those values below the mean) exactly cancel out the positive deviations (corresponding to those values above the mean).

This proposed definition is thus quite useless. We cannot use it to say things like Y is "more spread out" than X.

This suggests a second approach: define the variance to be **the weighted average of the absolute deviations from the mean**.

Example 1234 (continued from above). (Our second proposed definition of variance.)

For X, the weighted average of the absolute deviations from the mean is

$$\begin{aligned}
\mathbf{V}[X] &= \sum \mathrm{P}(X=k)\cdot|k-\mu| \\
&= \frac{1}{5}\cdot|0-\mu| + \frac{1}{5}\cdot|1-\mu| + \frac{1}{5}\cdot|2-\mu| + \frac{1}{5}\cdot|3-\mu| + \frac{1}{5}\cdot|4-\mu| \\
&= \frac{1}{5}\cdot|0-2| + \frac{1}{5}\cdot|1-2| + \frac{1}{5}\cdot|2-2| + \frac{1}{5}\cdot|3-2| + \frac{1}{5}\cdot|4-2| \\
&= \frac{2}{5} + \frac{1}{5} + 0 + \frac{1}{5} + \frac{2}{5} = \frac{6}{5}.
\end{aligned}$$

And now let's work out the same for Y:

$$\begin{aligned}
\mathbf{V}[Y] &= \sum \mathrm{P}(Y=k)\cdot(k-\mu) \\
&= \frac{1}{5}\cdot|-8-\mu| + \frac{1}{5}\cdot|-3-\mu| + \frac{1}{5}\cdot|2-\mu| + \frac{1}{5}\cdot|7-\mu| + \frac{1}{5}\cdot|12-\mu| \\
&= \frac{1}{5}\cdot|-8-2| + \frac{1}{5}\cdot|-3-2| + \frac{1}{5}\cdot|2-2| + \frac{1}{5}\cdot|7-2| + \frac{1}{5}\cdot|12-2| \\
&= 2+1+0+1+2 = 6.
\end{aligned}$$

Wonderful! So we can now use this second proposed definition of the variance to say things like "Y is more spread out than X".

This second proposed definition seems perfectly satisfactory. Yet for some bizarre reason, we won't use it! Instead, we'll define the variance to be the **weighted average of the squared deviations from the mean**.

Example 1234 (continued from above). (The actual definition of variance.)

For X, the weighted average of the squared deviations from the mean is

$$\begin{aligned}\mathbf{V}[X] &= \sum \mathrm{P}(X=k)\cdot(k-\mu)^2 \\ &= \frac{1}{5}\cdot(0-\mu)^2 + \frac{1}{5}\cdot(1-\mu)^2 + \frac{1}{5}\cdot(2-\mu)^2 + \frac{1}{5}\cdot(3-\mu)^2 + \frac{1}{5}\cdot(4-\mu)^2 \\ &= \frac{1}{5}\cdot(0-2)^2 + \frac{1}{5}\cdot(1-2)^2 + \frac{1}{5}\cdot(2-2)^2 + \frac{1}{5}\cdot(3-2)^2 + \frac{1}{5}\cdot(4-2)^2 \\ &= \frac{4}{5} + \frac{1}{5} + 0 + \frac{1}{5} + \frac{4}{5} = 2.\end{aligned}$$

And now let's work out the same for Y:

$$\begin{aligned}\mathbf{V}[Y] &= \sum \mathrm{P}(Y=k)\cdot(k-\mu)^2 \\ &= \frac{1}{5}\cdot(-8-\mu)^2 + \frac{1}{5}\cdot(-3-\mu)^2 + \frac{1}{5}\cdot(2-\mu)^2 + \frac{1}{5}\cdot(7-\mu)^2 + \frac{1}{5}\cdot(12-\mu)^2 \\ &= \frac{1}{5}\cdot(-8-2)^2 + \frac{1}{5}\cdot(-3-2)^2 + \frac{1}{5}\cdot(2-2)^2 + \frac{1}{5}\cdot(7-2)^2 + \frac{1}{5}\cdot(12-2)^2 \\ &= 20+5+0+5+20 = 50.\end{aligned}$$

Formally,

Definition 246. Let $\mu = \mathrm{E}[X]$. Then the *variance operator* is denoted Var and is the function that maps each random variable X to a real number c, given by the mapping rule

$$\mathbf{V}[X] = \mathbf{E}\left[(X-\mu)^2\right].$$

We call $\mathrm{Var}[X]$ the *variance* of X. This is often also instead written as σ_X^2 or even more simply as σ^2 (if it is clear from the context that we're talking about the variance of X).

So to calculate the variance, we do this: Consider all the possible values that X can take. Take the difference between these values and the mean of X. Square them. Then take the probability-weighted average of these squared numbers.

More examples:

Example 1499. Let the random variable X be the outcome of the roll of a fair die. We already know that $\mu = 3.5$. Hence,

$$\begin{aligned}
\mathbf{V}[X] &= \mathbf{E}\left[(X-\mu)^2\right] = \mathbf{E}\left[(X-3.5)^2\right] \\
&= \mathrm{P}(X=1)\cdot(1-3.5)^2 + \mathrm{P}(X=2)\cdot(2-3.5)^2 + \cdots + \mathrm{P}(X=6)\cdot(6-3.5)^2 \\
&= \frac{1}{6}\left(2.5^2 + 1.5^2 + 0.5^2 + 0.5^2 + 1.5^2 + 2.5^2\right) = \frac{35}{12} \approx 2.92.
\end{aligned}$$

So the variance of the die roll is $\dfrac{35}{12} \approx 2.92$. This means that the expected squared deviation of X from its mean $\mu = 3.5$ is $\dfrac{35}{12} \approx 2.92$.

Example 1500. Roll two fair dice. Let the random variable Y be the sum of the two dice. We already know from Example 1490 that $\mu = 7$. So, using also our findings from Exercise 485,

$$\begin{aligned}
\mathbf{V}[Y] &= \mathbf{E}\left[(Y-\mu)^2\right] = \mathbf{E}\left[(Y-7)^2\right] \\
&= \mathrm{P}(Y=2)\cdot(2-7)^2 + \mathrm{P}(Y=3)\cdot(3-7)^2 + \cdots + \mathrm{P}(Y=12)\cdot(12-7)^2 \\
&= \frac{1}{36}\cdot 5^2 + \frac{2}{36}\cdot 4^2 + \frac{3}{36}\cdot 3^2 + \frac{4}{36}\cdot 2^2 + \frac{5}{36}\cdot 1^2 + \frac{6}{36}\cdot 0^2 + \frac{5}{36}\cdot 1^2 \\
&\quad + \frac{4}{36}\cdot 2^2 + \frac{3}{36}\cdot 3^2 + \frac{2}{36}\cdot 4^2 + \frac{1}{36}\cdot 5^2 \\
&= \frac{2(25+32+27+16+5)}{36} = \frac{210}{36} = \frac{70}{12} \approx 5.83.
\end{aligned}$$

So the variance of the sum of two dice is $\dfrac{70}{12} \approx 5.83$. This means that on average, the square of the deviation of Y from its mean $\mu = 7$ is $\dfrac{70}{12} \approx 5.83$.

As the above examples suggest, calculating the variance can be tedious. Fortunately, there is a shortcut:

Fact 236. *Let X be a random variable with mean μ. Then $Var[X] = E[X^2] - \mu^2$.*

Proof. Using the definition of variance, the linearity of the expectation operator (Proposition 22), and the fact that μ is a constant, we have

$$\mathbf{V}[X] = \mathbf{E}\left[(X-\mu)^2\right] = \mathbf{E}\left[X^2 + \mu^2 - 2X\mu\right] = \mathbf{E}\left[X^2\right] + \mathbf{E}\left[\mu^2\right] - 2\mathbf{E}[X\mu]$$

$$= \mathbf{E}\left[X^2\right] + \mu^2 - 2\mu\mathbf{E}[X] = \mathbf{E}\left[X^2\right] + \mu^2 - 2\mu\cdot\mu = \mathbf{E}\left[X^2\right] - \mu^2. \qquad \square$$

We now redo the previous two examples using this shortcut:

Example 1499 (continued from above). Let the random variable X be the outcome of the roll of a fair die. We already know that $\mu = 3.5$. So compute

$$\mathbf{E}\left[X^2\right] = \mathrm{P}(X=1)\cdot 1^2 + \mathrm{P}(X=2)\cdot 2^2 + \cdots + \mathrm{P}(X=6)\cdot 6^2 = \frac{1}{6}\left(1^2 + 2^2 + \cdots + 6^2\right) = \frac{91}{6}.$$

Hence, $\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - \mu^2 = \frac{91}{6} - 3.5^2 = \frac{182}{12} - \frac{147}{12} = \frac{35}{12}$.

Example 1500 (continued from above). Let the random variable Y be the sum of two rolled dice. We already know from Example 1490 that $\mu = 7$. So, using also our findings from Exercise 485,

$$\mathbf{E}\left[Y^2\right] = \mathrm{P}(Y=2)\cdot 2^2 + \mathrm{P}(Y=3)\cdot 3^2 + \cdots + \mathrm{P}(Y=12)\cdot 12^2$$

$$= \frac{1}{36}\cdot 2^2 + \frac{2}{36}\cdot 3^2 + \frac{3}{36}\cdot 4^2 + \cdots + \frac{1}{36}\cdot 12^2$$

$$= \frac{4+18+48+100+294+320+324+300+242+144}{36} = \frac{1974}{36} = \frac{658}{12}.$$

Hence, $\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right] - \mu^2 = \frac{658}{12} - 7^2 = \frac{658}{12} - \frac{588}{12} = \frac{70}{12}$.
This is still tedious, but arguably quicker than before.

Exercise 492. Let the random variable Z be the sum of three rolled dice. Find $\mathrm{Var}[Z]$. (Answer on p. **1971**.)

122.1. The Variance of a Constant R.V. is 0

A constant random variable cannot vary. So not surprisingly, the variance of a constant random variable is 0.

Fact 237. *Let c be a constant random variable (i.e. it maps every outcome to the real number c). Then*

$$\mathbf{V}[c] = 0.$$

Proof. Use Fact 1238: $\text{Var}[c] = \text{E}\left[c^2\right] - (\text{E}[c])^2 = c^2 - c^2 = 0.$ $\square$

122.2. Standard Deviation

Let X be a random variable. Then $\mathrm{E}[X]$ has the same unit of measure as X. In contrast, $\mathrm{Var}[X]$ uses the **squared unit**.

Example 1501. There are 100 dumbbells in a gym, of which 30 have weight 5 kg and the remaining 70 have weight 10 kg. Let X be the weight of a randomly chosen dumbbell. Then the mean of X is

$$\mathbf{E}[X] = \mu = 0.3 \times 5 \text{ kg} + 0.7 \times 10 \text{ kg} = 8.5 \text{ kg}.$$

And the variance of X is

$$\begin{aligned}\mathbf{V}[X] &= 0.3 \times (5 \text{ kg} - 8.5 \text{ kg})^2 + 0.7 \times (10 \text{ kg} - 8.5 \text{ kg})^2 \\ &= 0.3 \times 12.25 \text{ kg}^2 + 0.7 \times 2.25 \text{ kg}^2 = 5.25 \text{ kg}^2.\end{aligned}$$

To get a measure of "spread" that uses the original unit of measure, we simply take the square root of the variance. This is called the **standard deviation** as a measure of spread.

Definition 247. Let X be a random variable and $\mathrm{Var}[X]$ be its variance. Then the *standard deviation* of X is defined as

$$\mathbf{SD}[X] = \sqrt{\mathbf{V}[X]}.$$

The variance of a random variable X is often denoted σ_X^2 or even more simply as σ^2 (if it is clear from the context that we're talking about the variance of X).

Correspondingly, the standard deviation of X is often denoted σ_X or σ.

Example 1240 (continued from above). We calculated the variance of X to be $\mathrm{Var}[X] = \sigma^2 = 5.25 \text{ kg}^2$.

Hence, the standard deviation of X is simply $\sigma = \sqrt{5.25} \approx 2.29$ kg.

Exercise 493. There are 100 rulers in a bookstore, of which 35 have length 20 cm and the remaining 65 have length weight 30 cm. Let Y be the weight of a randomly chosen dumbbell. Find the mean, variance, and standard deviation of Y. (Be sure to include the units of measurement.)(Answer on p. **1971**.)

122.3. The Variance Operator is Not Linear

The variance operator is not linear. However, given independence, the variance operator does satisfy additivity and homogeneity of degree 2.

Proposition 23. *Let X and Y be independent random variables and c be a constant. Then*

(a) Additivity: $Var[X+Y] = Var[X] + Var[Y]$,

(b) Homogeneity of degree 2: $Var[cX] = c^2 Var[X]$.

Proof. Optional, see p. 1732 (Appendices). □

With the above, it becomes much easier than before to find the variance of the sum of 2 dice, 3 dice, or indeed n dice.

Example 1502. Let X be the outcome of a fair die-roll. We showed earlier that $\text{Var}[X] = \frac{35}{12}$.

Now roll two fair dice. Let X_1 and X_2 be the respective outcomes. Let Y be the sum of the two dice (i.e. $Y = X_1 + X_2$). Assuming independence, we have

$$\mathbf{V}[Y] = \mathbf{V}[X_1 + X_2] = \mathbf{V}[X_1] + \mathbf{V}[X_2] = \frac{70}{12}.$$

Compare this quick computation to the work we did in Example 1500!

Now roll three fair dice. Let X_3, X_4, and X_5 be the respective outcomes. Let Z be the sum of the three dice (i.e. $Z = X_3 + X_4 + X_5$). Again, assuming independence, we have

$$\mathbf{V}[Z] = \mathbf{V}[X_3 + X_4 + X_5] = \mathbf{V}[X_3] + \mathbf{V}[X_4] + \mathbf{V}[X_5] = \frac{105}{12}.$$

Again, compare this quick computation to the work you had to do in Exercise 492!

Now, let A be double the outcome of a die roll (i.e. $A = 2X$). Note importantly that $A \neq Y$. Y is the sum of two independent die rolls. In contrast, A is <u>double the outcome of a single die roll</u>. Indeed, by Proposition 23, we see that

$$\mathbf{V}[A] = \mathbf{V}[2X] = 4\mathbf{V}[X] = \frac{140}{12} \neq \mathbf{V}[Y].$$

Similarly, let B be triple the outcome of a die roll (i.e. $B = 3X$). Note importantly that $B \neq Z$. Z is the sum of three independent die rolls. In contrast, B is <u>triple the outcome of a single die roll</u>. Indeed, by Proposition 23, we see that

$$\mathbf{V}[B] = \mathbf{V}[3X] = 9\mathbf{V}[X] = \frac{315}{12} \neq \mathbf{V}[Z].$$

Exercise 494. The weight of a fish in a pond is a random variable with mean μ kg and variance σ^2 kg^2. (Include the units of measurement in your answers.) (Answer on p. **1971**.)

(a) If two fish are caught and the weights of these fish are independent of each other, what are the mean and variance of the total weight of the two fish?

(b) If one fish is caught and an exact clone is made of it, what are the mean and variance of the total weight of the fish and its clone?

(c) If two fish are caught and the weights of these fish are **not** independent of each other, what are the mean and variance of the total weight of the two fish?

122.4. The Definition of the Variance (Optional)

Why is the variance defined as the weighted average of squared deviations from the mean?

1. First, we tried defining the variance as the weighted average of deviations from the mean, i.e. $\text{Var}[X] = \text{E}[X - \mu]$. But this was no good, because this quantity would always be equal to 0.[531]
2. Next, we tried defining the variance as the weighted average of absolute deviations from the mean, i.e. $\text{Var}[X] = \text{E}[|X - \mu|]$. This seemed to work well enough. But yet for some bizarre reason, we choose not to use this definition.
3. Instead, we choose to use this definition:

$$\mathbf{V}[X] = \mathbf{E}\left[(X - \mu)^2\right].$$

Why do we prefer using squared (rather than absolute) deviations as our definition of variance? The conventional view is that the squared deviations definition is superior to the absolute deviations definition (but see Gorard (2005) and Taleb (2014) for dissenting views). Here are some reasons for believing the squared deviations definition to be superior:

- The maths works out more nicely. For example:
  - The algebra is easier when dealing with squares than with absolute values.
  - Differentiation is easier (serve that x^2 is differentiable but $|x|$ is not).
  - Variances are additive: If X and Y are independent, then $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$. In contrast, if we use the definition $\text{Var}[X] = \text{E}[|X - \mu|]$, then variances are no longer additive.
- Tradition (inertia).
  - A century or two ago, some Europeans preferred using squared to absolute deviations. And so we're stuck with using this.

See also these five SE discussions: , , mathoverflow, , .

[531] This is easily proven: $\text{E}[X - \mu] = \text{E}[X] - \text{E}[\mu] = \mu - \mu = 0$.

123. The Coin-Flips Problem (Fun, Optional)

Here's another example of a probability problem that can be stated very simply, yet have counter-intuitive results.

Example 1503. Keep flipping a fair coin until you get a sequence of HH (two heads in a row). Let X be the number of flips taken.

Now, keep flipping a fair coin until you get a sequence of HT. Let Y be the number of flips taken.

Which is larger $\mu_X = \mathrm{E}[X]$ or $\mu_Y = \mathrm{E}[Y]$?

Intuition might suggest that "obviously", $\mu_X = \mu_Y$. Intuition would be wrong. It turns out that, surprisingly enough, $\mu_X = 6$ and $\mu_Y = 4$!

Example 1504. Now suppose we flip a fair coin 10,001 times. This gives us a sequence of 10,000 pairs of consecutive coin-flips.

For example, if the 10,001 coin-flips are H**HT**$HT\ldots$, then the first four pairs of consecutive coin-flips are H**H**, **HT**, **T**H, and HT.

Let A be the proportion of the 10,000 consecutive coin-flips that are HH. Let B be the proportion of the 10,000 consecutive coin-flips that are HT.

Which is larger $\mu_A = \mathrm{E}[A]$ or $\mu_B = \mathrm{E}[B]$?

In the previous example, we saw that it took, on average, 6 flips before getting HH and 4 flips before getting HT. So "obviously", we'd expect a smaller proportion to be HH's. That is, $\mu_A < \mu_B$.

Sadly, we would again be wrong! It turns out that $\mu_A = \mu_B = 1/4$! This Google spreadsheet simulates 10,001 coin-flips and calculates A and B.

If you're interested, the results given in the above two examples are formally proven in Fact 300 (Appendices).

124. The Bernoulli Trial and the Bernoulli Distribution

A **Bernoulli trial** is an experiment (S, Σ, P). A coin flip is an example of a Bernoulli trial.

Example 1505. Flip a coin. We can model this with a Bernoulli trial with probability of success (heads) 0.5:

- Sample space $S = \{T, H\}$,
- Event space $\Sigma = \{\varnothing, \{T\}, \{H\}, S\}$,
- Probability function $\mathrm{P}(\{T\}) = 0.5$ and $\mathrm{P}(\{H\}) = 0.5$.

The corresponding **Bernoulli random variable** is simply the random variable $X : S \to \mathbb{R}$ defined by $X(\{T\}) = 0$ and $X(\{H\}) = 1$. Its probability distribution is given by $\mathrm{P}(X = 0) = 0.5$ and $\mathrm{P}(X = 1) = 0.5$.

Formally:

Definition 248. A *Bernoulli trial with probability of success* p is an experiment (S, Σ, P) where

- $S = \{0, 1\}$. (The sample space contains 2 elements.)
- $\Sigma = \{\varnothing, \{0\}, \{1\}, S\}$.
- $\mathrm{P} : \Sigma \to \mathbb{R}$ is defined by $\mathrm{P}(\{0\}) = 1 - p$ and $\mathrm{P}(\{1\}) = p$. (And as usual $\mathrm{P}(\varnothing) = 0$ and $\mathrm{P}(S) = 1$.)

The corresponding *Bernoulli random variable* is simply the random variable $X : S \to \mathbb{R}$ defined by $X(\{0\}) = 0$ and $X(\{1\}) = 1$. Its probability distribution is given by $\mathrm{P}(X = 0) = 1 - p$ and $\mathrm{P}(X = 1) = p$.

Note that we can denote the two elements of the sample space with any symbols. We could use 0—standing for failure—and 1—standing for success. Or we could use T and H, as was done in the example above.

Example 1506. On any given day, our refrigerator at home has probability 0.001 of breaking down. We can model this with a Bernoulli trial with probability of success 0.001:

- Sample space $S = \{0, 1\}$,
- Event space $\Sigma = \{\varnothing, \{0\}, \{1\}, S\}$,
- Probability function $\mathrm{P}(\{0\}) = 0.999$ and $\mathrm{P}(\{1\}) = 0.001$.

The corresponding **Bernoulli random variable** is simply the random variable $T : S \to \mathbb{R}$ defined by $T(\{0\}) = 0$ and $T(\{1\}) = 1$.

Its probability distribution is given by $\mathrm{P}(T = 0) = 0.999$ and $\mathrm{P}(T = 1) = 0.001$. In words, the probability of no failure is 0.999 and the probability of a failure is 0.001.

Example 1507. 90% of H2 Maths students pass their H2 Maths A-Level exams. We randomly pick a H2 Maths student and see if she passes her H2 Maths A-Level exam.

We can model this with a Bernoulli trial with probability of success 0.9:

- Sample space $S = \{F, P\}$,
- Event space $\Sigma = \{\varnothing, \{F\}, \{P\}, S\}$,
- Probability function $\mathrm{P}(\{F\}) = 0.1$ and $\mathrm{P}(\{P\}) = 0.9$.

The corresponding **Bernoulli random variable** is simply the random variable $Y : S \to \mathbb{R}$ defined by $Y(\{F\}) = 0$ and $Y(\{P\}) = 1$. Its probability distribution is given by $\mathrm{P}(Y = 0) = 0.1$ and $\mathrm{P}(Y = 1) = 0.9$.

The following two statements are equivalent:

1. T is a **Bernoulli random variable with probability of success** p.
2. The random variable T has **Bernoulli distribution with probability of success** p.

124.1. Mean and Variance of the Bernoulli Random Variable

Fact 238. *A Bernoulli random variable T with probability of success p has mean p and variance $p(1-p)$.*

Proof. $\mathrm{E}[T] = \mathrm{P}(T=0)\cdot 0 + \mathrm{P}(T=1)\cdot 1 = (1-p)\cdot 0 + p\cdot 1 = p$.

For the variance, first compute

$$\mathbf{E}\left[T^2\right] = \mathrm{P}(T=0)\cdot 0^2 + \mathrm{P}(T=1)\cdot 1^2 = (1-p)\cdot 0 + p\cdot 1^2 = p.$$

Hence, $\mathrm{Var}[T] = \mathrm{E}\left[T^2\right] - (\mathrm{E}[T])^2 = p - p^2 = p(1-p)$. □

125. The Binomial Distribution

Informally, the **binomial random variable** simply counts the number of successes in a sequence of n identical, but independent Bernoulli trials.

Example 1508. Flip 3 fair coins. Let X be the number of heads.

X is an example of a **binomial random variable** X with parameters 3 and $\frac{1}{2}$.

X can take on values 0, 1, 2, or 3 (corresponding to the number of heads).

The probability distribution of X is given by

$$\mathrm{P}(X=0)=\binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3=\frac{1}{8}, \qquad \mathrm{P}(X=1)=\binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2=\frac{3}{8},$$

$$\mathrm{P}(X=2)=\binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1=\frac{3}{8}, \qquad \mathrm{P}(X=3)=\binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0=\frac{1}{8}.$$

Formally:

Definition 249. Let T_1, T_2, ..., T_n be n identical, but independent Bernoulli random variables, each with probability of success p. Then the *binomial random variable X with parameters n and p* is defined as:

$$X = T_1 + T_2 + \cdots + T_n.$$

The following three statements are entirely equivalent:

1. X is a binomial random variable with parameters n and p.
2. The random variable X has the **binomial distribution with parameters n and p**.
3. $X \sim \mathrm{B}(n, p)$.

Example 1509. 90% of H2 Maths students pass their A-Level exams.

Let Y be the number of passes among two randomly chosen students. Then Y is a binomial random variable with parameters 2 and 0.9. Its probability distribution is given by

$$P(Y=0) = \binom{2}{0} 0.9^0 0.1^2 = 0.01,$$

$$P(Y=1) = \binom{2}{1} 0.9^1 0.1^1 = 0.18,$$

$$P(Y=2) = \binom{2}{2} 0.9^2 0.1^0 = 0.81.$$

In words, the probability that both fail is 0.01, the probability that exactly one passes is 0.18, and the probability that both pass is 0.81.

125.1. Probability Distribution of the Binomial R.V.

Let $X \sim \mathrm{B}(n,p)$. What is $\mathrm{P}(X=k)$?

Observe that $\mathrm{P}(X=k)$ is simply the probability that in a sequence of n independent Bernoulli trials, each with probability of success p, there are exactly k successes.

First consider instead the probability that in a sequence of n trials, the first k trials are successes and the remaining $n-k$ are failures. We know that the probability of a success is p and the probability of a failure is $1-p$. Hence, by the Multiplication Principle, this probability is simply $p^k(1-p)^{n-k}$.

The above is the probability of k successes and $n-k$ failures, but where exactly the first k trials are successes and exactly the last $n-k$ trials are failures. But we don't care about where the successes are. We only care that there are k successes. And there are $C(n,k)$ ways to have exactly k successes in n trials. Thus,

$$\mathrm{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}.$$

In summary:

Fact 239. *Let $X \sim B(n,p)$. Then for any $k=0,1,\dots,n$,*

$$\mathrm{P}(X=k)=\binom{n}{k}p^k(1-p)^{1-k}.$$

Example 1510. Let X be the number of heads when 10 fair coins are flipped.

Then $X \sim B(10,0.5)$. And the probability that exactly 8 coins are heads is

$$\mathrm{P}(X=8)=\binom{10}{8}0.5^8 0.5^2=\frac{45}{1024}.$$

Example 1511. 90% of H2 Maths students pass their A-Level exams.

Let Y be the number of passes among 20 randomly chosen students. Then $Y \sim B(20,0.9)$. And the probability that <u>at least</u> 18 pass is

$$\mathrm{P}(Y \geq 18)=\mathrm{P}(Y=18)+\mathrm{P}(Y=19)+\mathrm{P}(Y=20)$$

$$=\binom{20}{18}0.9^{18}0.1^2+\binom{20}{19}0.9^{19}0.1^1+\binom{20}{20}0.9^{20}0.1^0 \approx 0.677.$$

Example 1512. Problem: *Three machines each have, independently, probability* 0.3 *of failure. What is the expected number of failures? What is the variance of the number of failures?*

Solution: Let $Z \sim \mathrm{B}(3, 0.3)$ be the number of failures. Then

$$\mathrm{P}(Z=1) = \binom{3}{1} 0.3^1 0.7^2, \qquad \mathrm{P}(Z=2) = \binom{3}{2} 0.3^2 0.7^1, \qquad \mathrm{P}(Z=3) = \binom{3}{3} 0.3^3 0.7^0.$$

$$\begin{aligned} \text{Hence,} \quad \mathbf{E}[Z] &= \mathrm{P}(Z=1)\cdot 1 + \mathrm{P}(Z=2)\cdot 2 + \mathrm{P}(Z=3)\cdot 3 \\ &= \binom{3}{1} 0.3^1 0.7^2 \cdot 1 + \binom{3}{2} 0.3^2 0.7^1 \cdot 2 + \binom{3}{3} 0.3^3 0.7^0 \cdot 3 \\ &= 0.441 + 0.378 + 0.081 = 0.9. \end{aligned}$$

That is, the expected number of failures is 0.9.

$$\begin{aligned} \text{Now,} \mathbf{E}\left[Z^2\right] &= \mathrm{P}(Z=1)\cdot 1^2 + \mathrm{P}(Z=2)\cdot 2^2 + \mathrm{P}(Z=3)\cdot 3^2 \\ &= \binom{3}{1} 0.3^1 0.7^2 \cdot 1^2 + \binom{3}{2} 0.3^2 0.7^1 \cdot 2^2 + \binom{3}{3} 0.3^3 0.7^0 \cdot 3^2 \\ &= 0.441 + 0.756 + 0.243 = 1.44. \end{aligned}$$

$$\text{Hence,} \quad \mathbf{V}[Z] = \mathbf{E}\left[Z^2\right] - (\mathbf{E}[Z])^2 = 1.44 - 0.9^2 = 0.63.$$

That is, the variance of the number of failures is 0.63.

It turns out though that there is a much quicker formula for finding the mean and variance of any binomial random variable.

Fact 240. *If* $X \sim B(n,p)$, *then* $E[X] = np$ *and* $Var[X] = np(1-p)$.

(You can verify that this formula works for the last example: $n = 3$, $p = 0.3$, and thus $\mathrm{E}[Z] = np = 0.9$.)

Proof. Let T_1, T_2, ..., T_n be identical, but independent Bernoulli random variables with parameter p. Then $X = T_1 + T_2 + \cdots + T_n$. Hence,

$$\mathbf{E}[X] = \mathbf{E}[T_1 + T_2 + \cdots + T_n] = \mathbf{E}[T_1] + \mathbf{E}[T_2] + \cdots + \mathbf{E}[T_n] = p + p + \cdots + p = np.$$

$$\begin{aligned}\mathbf{V}[X] = \mathbf{V}[T_1 + T_2 + \cdots + T_n] &= \mathbf{V}[T_1] + \mathbf{V}[T_2] + \cdots + \mathbf{V}[T_n] \\ &= p(1-p) + p(1-p) + \cdots + p(1-p) = np(1-p).\end{aligned}$$ $\square$

Exercise 495. (Answer on p. **1972**.) Plane engine #1 contains 20 components, each of which has probability 0.01 of failure. Plane engine #2 contains 35 components, each of which has probability 0.005 of failure. The probability that any component fails is independent of whether any other component has failed.

An engine fails if and only if at least 2 of its components fail. What is the probability that both engines fail?

126. The Continuous Uniform Distribution

So far, all examples of random variables we've seen have been **discrete**. For example, the binomial random variable $X \sim \mathrm{B}(n, p)$ is discrete, because $\mathrm{Range}(X) = \{0, 1, 2, \ldots, n\}$ is finite.

We'll now look at **continuous** random variables. Informally, a random variable Y is continuous if its range takes on a continuum of values.

For H2 Maths, you need only learn about one continuous random variable: the **normal random variable** (subject of the next chapter).

Nonetheless, we'll first look at another continuous random variable that is not in the syllabus. This is the **continuous uniform random variable**. It is much simpler than the normal random variable and can thus help build up your intuition of how continuous random variables work.

126.1. The Continuous Uniform Distribution

A line measuring exactly 1 metre in length is drawn on the floor. It is about to rain. Let X be the position of the first rain-drop that hits the line. X is measured as the distance (in metres) from the left-most point of the line.

So for example, if the first rain-drop hits the left-most point of the line, then $x = 0$. If it hits the exact midpoint of the line, then $x = 0.5$. And if it hits the right-most point, then $x = 1$.

Assume we can measure X to infinite precision.

Then, assuming the first rain-drop is equally likely to hit any point of the line, we can model X as a **continuous uniform random variable on** $[0, 1]$. This says that

- The range of X is $[0, 1]$ (the first rain-drop can hit any point along the line); and
- X is equally likely to take on any value in the interval $[0, 1]$ (the first rain-drop is equally likely to hit any point along the line).

The following three statements are entirely equivalent:

1. X is a **continuous uniform random variable on** $[0, 1]$.
2. X is a random variable with the **continuous uniform distribution on** $[0, 1]$.
3. $X \sim U[0, 1]$.

Recall that previously with any discrete random variable Y, we could find its **probability distribution**. That is, we could find $\mathrm{P}(Y = k)$ (the probability that Y takes on the value k). For example, if $Y \sim \mathrm{B}(3, 0.5)$ modelled the number of heads in three coin-flips, then the probability that there was one heads was $\mathrm{P}(Y = 1) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} 0.5^1 0.5^2 = \frac{3}{8}$.

Now, in contrast, for any **continuous random variable** X, strangely enough, there is zero probability that X takes on any particular value! For example, if $X \sim U[0, 1]$, then $\mathrm{P}(X = 0.37) = 0$. That is, there is zero probability that X takes on the value of 0.37!

At first glance, this may seem strange.

But remember: There are *infinitely many* real numbers in the interval $[0, 1]$. So it makes sense to say that the probability of X taking on any particular value is zero.[532]

So for any continuous random variable X, it is pointless to try to write down $\mathrm{P}(X = k)$ for different possible values of k, because $\mathrm{P}(X = k)$ is always equal to zero (regardless of what k is). Instead, we shall try to write down $\mathrm{P}(a \le X \le b)$, for different possible values of a and b.

Now, if $X \sim \mathrm{U}[0, 1]$, then the probability that X takes on values between 0.3 and 0.7 is simply $0.7 - 0.3 = 0.4$. That is,

[532]But strangely enough, **zero probability is not the same thing as impossible**. For example, we'd say that

- There is zero probability, but it is not impossible that $X \sim U[0, 1]$ takes on the value 0.37.
- There is zero probability *and* it is impossible that $X \sim U[0, 1]$ takes on the value 1.2.

(Actually, rather than use the word "impossible", mathematicians prefer saying "almost never", which has a precise definition.)

$$\mathrm{P}(0.3 \leq X \leq 0.7) = 0.7 - 0.3 = 0.4.$$

Similarly, the probability that X takes on values between 0.16 and 0.35 is simply $0.35-0.16 = 0.19$. That is,

$$\mathrm{P}(0.16 \leq X \leq 0.35) = 0.35 - 0.16 = 0.19.$$

The above observations suggest that it may be useful to define a new concept, called the **cumulative distribution function**.

126.2. The Cumulative Distribution Function (CDF)

The CDF simply tells us the probability that X takes on values less than or equal to k, for every $k \in \mathbb{R}$. Formally:

Definition 250. The *cumulative distribution function (CDF)* of a random variable X is the function $F_X : \mathbb{R} \to \mathbb{R}$ given by the mapping rule

$$F_X(k) = \mathrm{P}(X \leq k).$$

It turns out that **every random variable can be uniquely defined by giving its CDF**. For example, the continuous uniform random variable is formally defined thus:

Definition 251. X is the *continuous uniform random variable on* $[0,1]$ if its CDF $F_X : \mathbb{R} \to \mathbb{R}$ is defined by

$$F_X(k) = \begin{cases} 0, & \text{if } k < 0, \\ k, & \text{if } k \in [0,1], \\ 1, & \text{if } k > 1. \end{cases}$$

Armed with the concept of the CDF, the formal definition of a continuous random variable can be simply stated:

Definition 252. A random variable X is *continuous* if its CDF F_X is continuous.

We can now summarise the three possible types of random variables.

1. **Discrete random variables.** A random variable is discrete if its range is finite.[533] Examples: Bernoulli, binomial.
2. **Continuous random variables.** A random variable is continuous if its CDF is continuous. Examples: Continuous uniform, normal.
3. **Other random variables.** There are random variables that are neither discrete nor continuous. But you will not study any of these for the A-Levels.

Note that *every* random variable (discrete, continuous, or otherwise) has a **cumulative distribution function (CDF)**.

[533] Or countably infinite.

126.3. Important Digression: $\mathbf{P}(X \le k) = \mathbf{P}(X < k)$

For any continuous random variable X, we have

$$\mathrm{P}(X \le k) = \mathrm{P}(X < k).$$

That is, whether an inequality is strict makes no difference. The reason is that by the third Kolmogorov axiom (additivity),

$$\mathrm{P}(X \le k) = \mathrm{P}(X < k) + \mathrm{P}(X = k) = \mathrm{P}(X < k) + 0 = \mathrm{P}(X < k).$$

Thus, for continuous random variables, it doesn't matter whether inequalities are strict or weak.

Example 1513. Let $X \sim \mathrm{U}[0,1]$. Then

$$\mathrm{P}(0.2 \le X \le 0.5) = \mathrm{P}(0.2 < X \le 0.5) = \mathrm{P}(0.2 \le X < 0.5) = \mathrm{P}(0.2 < X < 0.5).$$

126.4. The Probability Density Function (PDF)

The PDF is simply defined as the derivative of the CDF.[534]

Definition 253. Let X be a random variable whose CDF F_X is differentiable. Then the *probability density function (PDF) of* X is the function $f_X : \mathbb{R} \to \mathbb{R}$ defined by

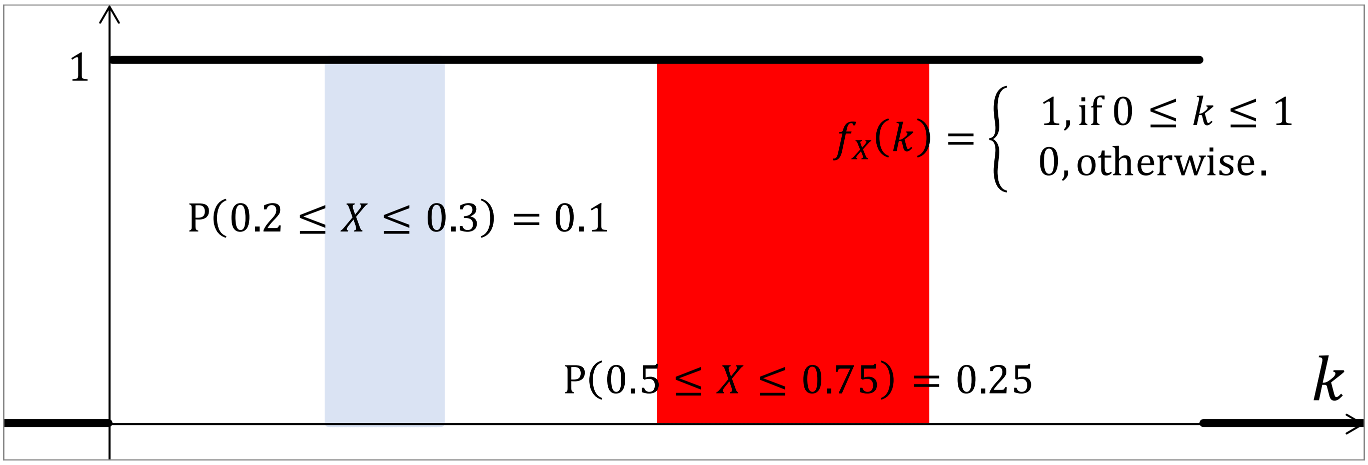
$$f_X(k) = \frac{d}{dk} F_X(k).$$

The PDF has an intuitive interpretation. The area under the PDF between points a and b is equal to $\mathrm{P}(a \le X \le b)$. This, of course, is simply a consequence of the Fundamental Theorems of Calculus:

$$\int_a^b f_X(k)dk = \int_a^b \frac{d}{dk} F_X(k)dk \overset{\text{FTC}}{=} F_X(b) - F_X(a) = \mathrm{P}(X \le b) - \mathrm{P}(X \le a) = \mathrm{P}(a \le X \le b).$$

The PDF of $X \sim \mathrm{U}[0,1]$ (graphed below) is simply the function $f_X : \mathbb{R} \to \mathbb{R}$ defined by

$$f_X(k) = 1, \quad \text{if } k \in [0,1], \quad \text{and} \quad f_X(k) = 1, \quad \text{otherwise}.$$



For any $a \le b$, the area under the PDF between a and b is precisely $\mathrm{P}(a \le X \le b)$. For example, there is probability 0.25 (red area) that X takes on values between 0.5 and 0.75. There is probability 0.1 (blue area) that X takes on values between 0.2 and 0.3.

Exercise 496. The continuous uniform random variable $Y \sim \mathrm{U}[3,5]$ is equally likely to take on values between 3 and 5, inclusive. (a) Write down its CDF F_Y. (b) Write down and graph its PDF f_Y. (c) Compute, and also illustrate on your graph, the quantities $\mathrm{P}(3.1 \le Y \le 4.6)$ and $\mathrm{P}(4.8 \le Y \le 4.9)$. (Answer on p. **1973**.)

[534]Note that although every random variable has a CDF, **not** every random variable has a PDF. In particular, if the random variable's CDF is not differentiable, then by our definition here, the random variable does not have a PDF.

127. The Normal Distribution

The **standard normal** (or **Gaussian**) **random variable (SNRV)** is very important. In fact, it is so important that we usually reserve the letter Z for it, and the Greek letters ϕ and Φ (lower- and upper-case phi) for its PDF and CDF.

The following three statements are entirely equivalent:

1. Z is a **SNRV**.
2. Z is **a random variable with the standard normal distribution**.
3. $Z \sim \mathrm{N}(0, 1)$.

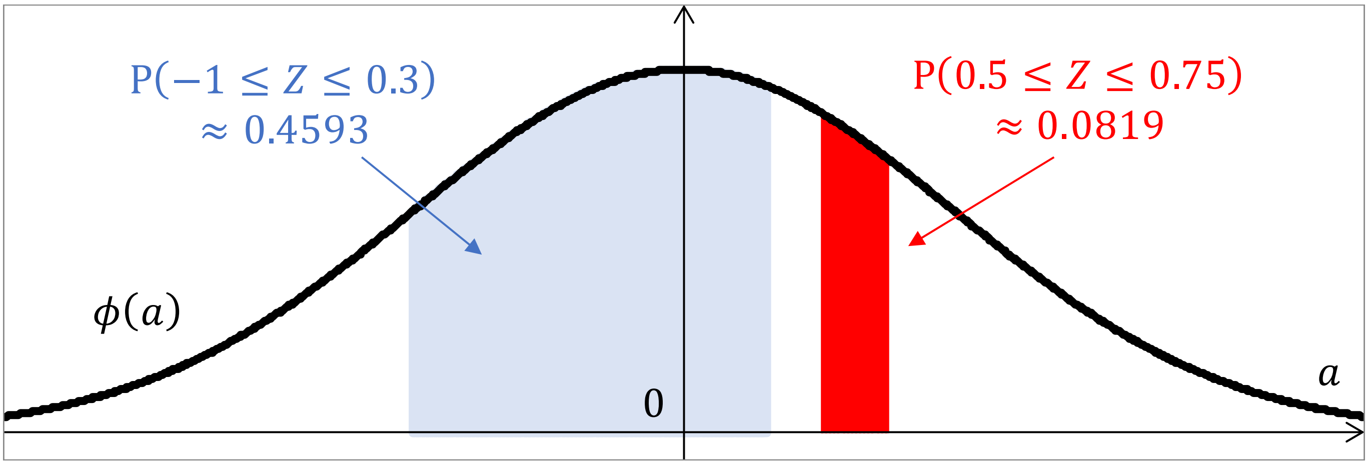
Here's the formal definition:

Definition 254. Z is called a *standard normal random variable (SNRV)* if its PDF $\phi : \mathbb{R} \to \mathbb{R}$ is defined by

$$\phi(a) = \frac{1}{\sqrt{2\pi}} e^{-0.5a^2}.$$

For the A-Levels, you need not remember this complicated-looking PDF. Nor need you understand where it comes from.

The normal PDF is often also referred to as **the bell curve**, due to its resemblance to a bell (kinda).



As with the continuous uniform, for any $a \leq b$, the area under the normal PDF between a and b gives us precisely $\mathrm{P}(a \leq X \leq b)$. For example, there is probability 0.25 (red area) that X takes on values between 0.5 and 0.75. There is probability 0.1 (blue area) that X takes on values between 0.2 and 0.3.

As usual, the CDF $\Phi : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Phi(a) = \mathrm{P}(Z \leq a) = \int_{-\infty}^{a} \phi(x)\, dx = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-0.5x^2}\, dx.$$

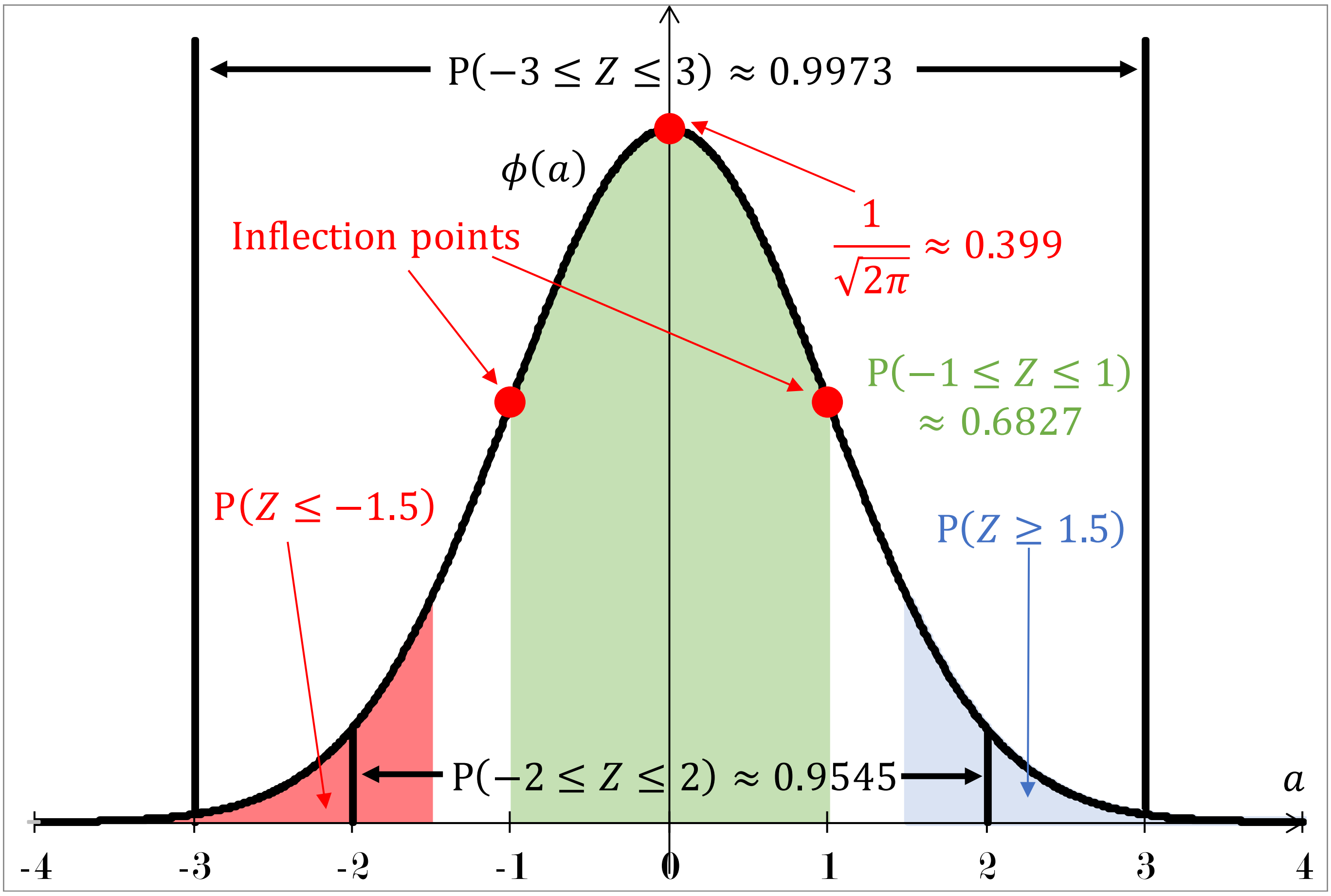
Unfortunately, this last integral has no simpler expression (mathematicians would say that it has no "closed-form expression"). Instead, as we'll soon see, we have to use the so-called Z-tables (or a graphing calculator) to look up values of $\Phi(k)$.

The next fact summarises the properties of the normal distribution. Some of these properties are illustrated in the figure that follows.

Fact 241. *Let $Z \sim N(0,1)$ and ϕ and Φ be its PDF and CDF.*

1. $\Phi(\infty) = 1$. *(As with any random variable, the area under the entire PDF is 1.)*
2. $\phi(a) > 0$, *for all* $a \in \mathbb{R}$. *(The PDF is positive <u>everywhere</u>. This has a surprising implication: however large a is, there is always some non-zero probability that $Z \geq a$.)*
3. $E[Z] = 0$. *(The mean of Z is 0.)*
4. *The PDF ϕ reaches a global maximum at the mean 0. (In fact, we can go ahead and compute* $\phi(0) = \dfrac{1}{\sqrt{2\pi}} \approx 0.399$.*)*
5. $Var[Z] = 1$. *(The variance of Z is 1.)*
6. $P(Z \leq a) = P(Z < a)$. *(We've already discussed this earlier. It makes no difference whether the inequality is strict. This is because $P(Z = a) = 0$.)*
7. *The PDF ϕ is symmetric about the mean. This has several implications:*
   (a) $P(Z \geq a) = P(Z \leq -a) = \Phi(-a)$.
   (b) *Since* $P(Z \geq a) = 1 - P(Z \leq a) = 1 - \Phi(a)$, *it follows that* $\Phi(-a) = 1 - \Phi(a)$ *or, equivalently,* $\Phi(a) = 1 - \Phi(-a)$.
   (c) $\Phi(0) = 1 - \Phi(0) = 0.5$.
8. $P(-1 \leq Z \leq 1) = \Phi(1) - \Phi(-1) \approx 0.6827$. *(There is probability 0.6827 that Z takes on values within 1 standard deviation of the mean.)*
9. $P(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2) \approx 0.9545$. *(There is probability 0.9545 that Z takes on values within 2 standard deviations of the mean.)*
10. $P(-3 \leq Z \leq 3) = \Phi(3) - \Phi(-3) \approx 0.9973$. *(There is probability 0.9973 that Z takes on values within 3 standard deviations of the mean.)*
11. *The PDF ϕ has two points of inflexion, namely at* ± 1. *(The points of inflexion are one standard deviation away from the mean.)*

Proof. Optional, see p. 1735 (Appendices). □



P(−3 ≤ Z ≤ 3) ≈ 0.9973
ϕ(a)
Inflection points
1/√2π ≈ 0.399
P(−1 ≤ Z ≤ 1) ≈ 0.6827
P(Z ≤ −1.5)
P(Z ≥ 1.5)
P(−2 ≤ Z ≤ 2) ≈ 0.9545
a
-4
-3
-2
-1
0
1
2
3
4

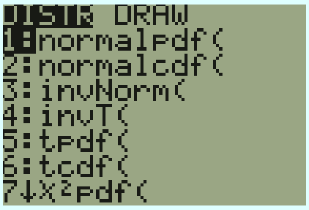
Example 1514. Let's use the TI84 to find $\Phi(2.51)$.

1. Press the blue 2ND button and then DISTR (which corresponds to the VARS button). This brings up the DISTR menu.
2. Press 2 to select the "normalcdf" option.

The TI84 is now asking for your lower and upper bounds. Since $\Phi(2.51) = \Phi(2.51) - \Phi(-\infty)$, your lower bound is $-\infty$ and your upper bound is 2.51.

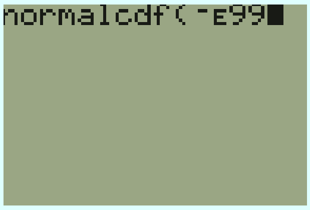
3. But there's no way to enter $-\infty$ on your TI84. So instead, you'll enter -10^{99}, which is simply a very large negative number. To do so, press (-), the blue 2ND button, EE (which corresponds to the , button), and then 9 9. (Don't press ENTER yet!)
4. Now to enter your upper bound. First press , (this simply demarcates your lower and upper bounds). Then enter your upper bound 2.51 by pressing 2 . 5 1. Then press ENTER. Your TI84 says that the answer is $\Phi(2.51) \approx 0.99396$.

Step 1.

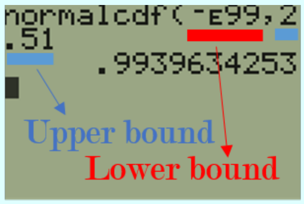


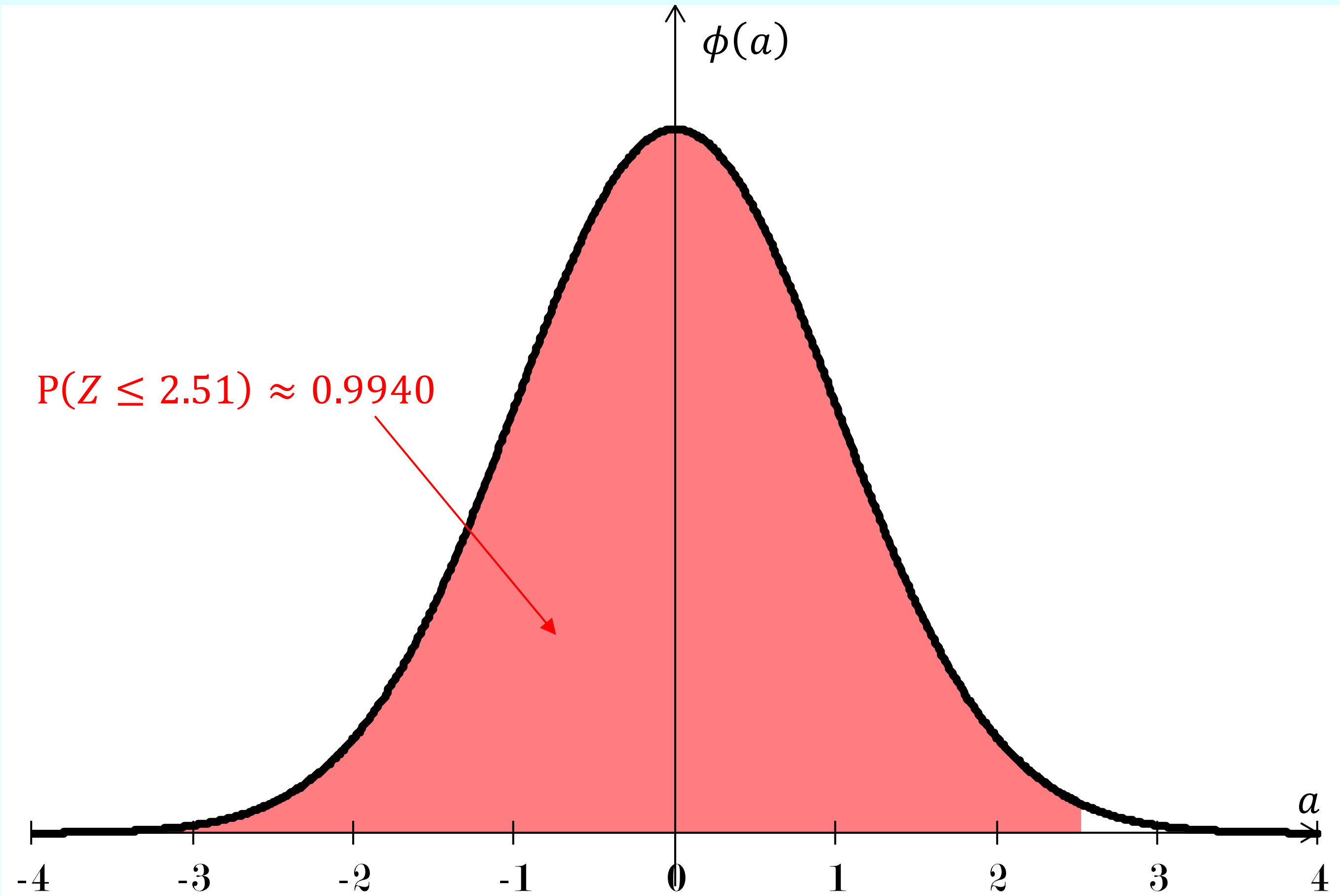
Step 2.

Step 3.



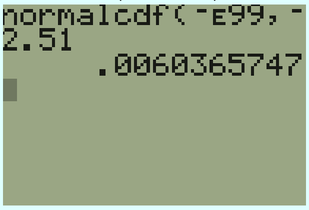
Step 4.



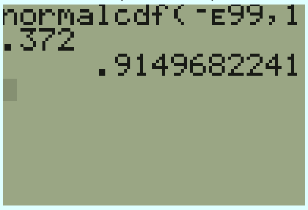


Example 1515. To find $\Phi(-2.51)$, $\Phi(1.372)$, and $\mathrm{P}(-4 \leq Z \leq 4)$, the steps are very similar. So for each, I'll simply give the screenshot from the TI84:

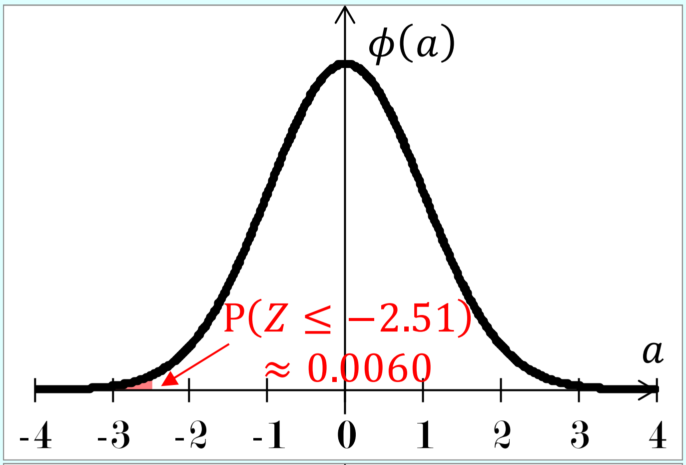
$\Phi(-2.51)$

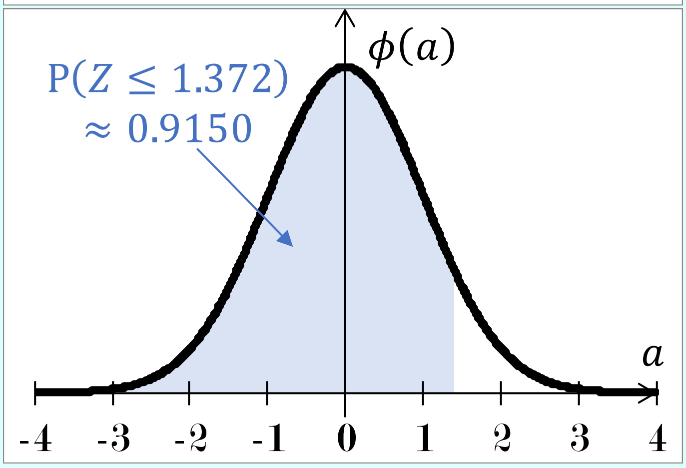


$\Phi(1.372)$



$\mathrm{P}(-4 \leq Z \leq 4)$





Example 1516. We'll find $\Phi(2.51)$, $\Phi(-2.51)$, $\Phi(1.372)$, and $\mathrm{P}(-4 \le Z \le 4)$ using Z-tables.

Refer to the Z-tables on p. 1265. (These are the exact same tables that appear on the List of Formulae (MF26).)

- To find $\Phi(2.51)$, look at the row labelled 2.5 and the column labelled 1—read off the number 0.9940. We thus have $\Phi(2.51) = 0.9940$.
- To find $\Phi(-2.51)$, note that the table does not explicitly give values of $\Phi(z)$, if $z < 0$. But we can exploit the fact that the standard normal is symmetric about the mean $\mu = 0$. This fact implies that $\Phi(-z) = 1 - \Phi(z)$. Hence, $\Phi(-2.51) = 1 - \Phi(2.51) = 0.0060$.
- To find $\Phi(1.372)$, first look at the row labelled 1.3 and the column labelled 7—read off the number 0.9147. This tells us that $\Phi(1.37) = 0.9147$. Now look at the right end of the table (where it says "ADD"). Since the third decimal place of 1.372 is 2, we look under the column labelled 2—this tells us to ADD 3. Thus, $\Phi(1.372) = 0.9147 + 0.003 = 0.9150$.
- To find $\mathrm{P}(-4 \le Z \le 4)$, the Z-tables printed are actually useless, because they only go to 2.99. So you can just write $\mathrm{P}(-4 \le Z \le 4) \approx 1$.

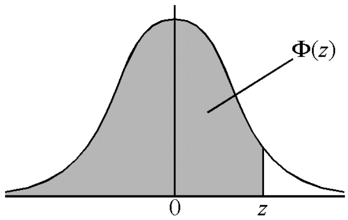
Exercise 497. Using both the Z-tables and your graphing calculator, find the following: (a) $\mathrm{P}(Z \ge 1.8)$. (b) $\mathrm{P}(-0.351 < Z < 1.2)$. (Answer on p. **1974**.)

THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1 then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = \mathrm{P}(Z \le z).$$

For negative values of z use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
															ADD				
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Critical values for the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that

$$\mathrm{P}(Z \le z) = p.$$

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

127.1. The Normal Distribution, in General

Let $Z \sim \mathrm{N}(0,1)$ be the SNRV and $\sigma, \mu \in \mathbb{R}$ be constants.

Consider $\sigma Z + \mu$, itself a random variable. We know that since $\mathrm{E}[Z] = 0$ and $\mathrm{Var}[Z] = 1$, it follows that

$$\mathbf{E}[\sigma Z + \mu] = \sigma \mathbf{E}[Z] + \mu = \mu \quad \text{and} \quad \mathbf{V}[\sigma Z + \mu] = \sigma^2 \mathbf{V}[Z] = \sigma^2.$$

It turns out that $\sigma Z + \mu$ is a **normal random variable with mean μ and variance σ^2**:

Definition 255. X is called a *normal random variable with mean μ and variance σ^2* if its PDF $f_X : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_X(a) = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5\left(\frac{a-\mu}{\sigma}\right)^2}.$$

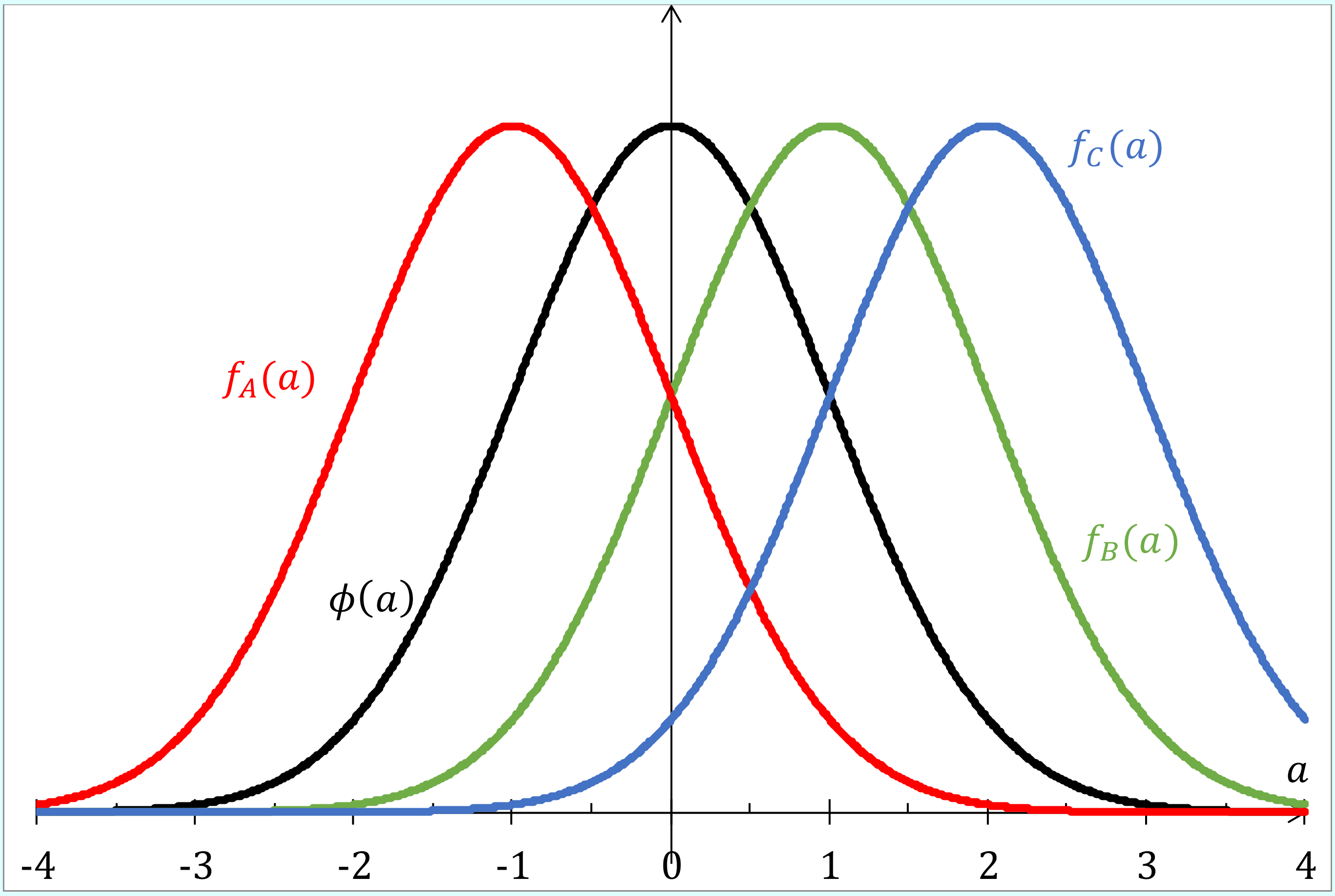
Once again, for the A-Levels, you need not remember this complicated-looking PDF. Nor need you understand where it comes from.

The following three statements are entirely equivalent:

1. X is a **normal random variable with mean μ and variance σ^2**.
2. X is a **random variable** with **normal distribution of mean μ and variance σ^2**.
3. $X \sim \mathrm{N}(\mu, \sigma^2)$.

Example 1517. The normal random variables $A \sim N(-1, 1)$, $B \sim N(1, 1)$, and $C \sim N(2, 1)$ have variance 1 (just like the SNRV), but non-zero means. Their PDFs are graphed below. (Included for reference is **the standard normal PDF in black**.)

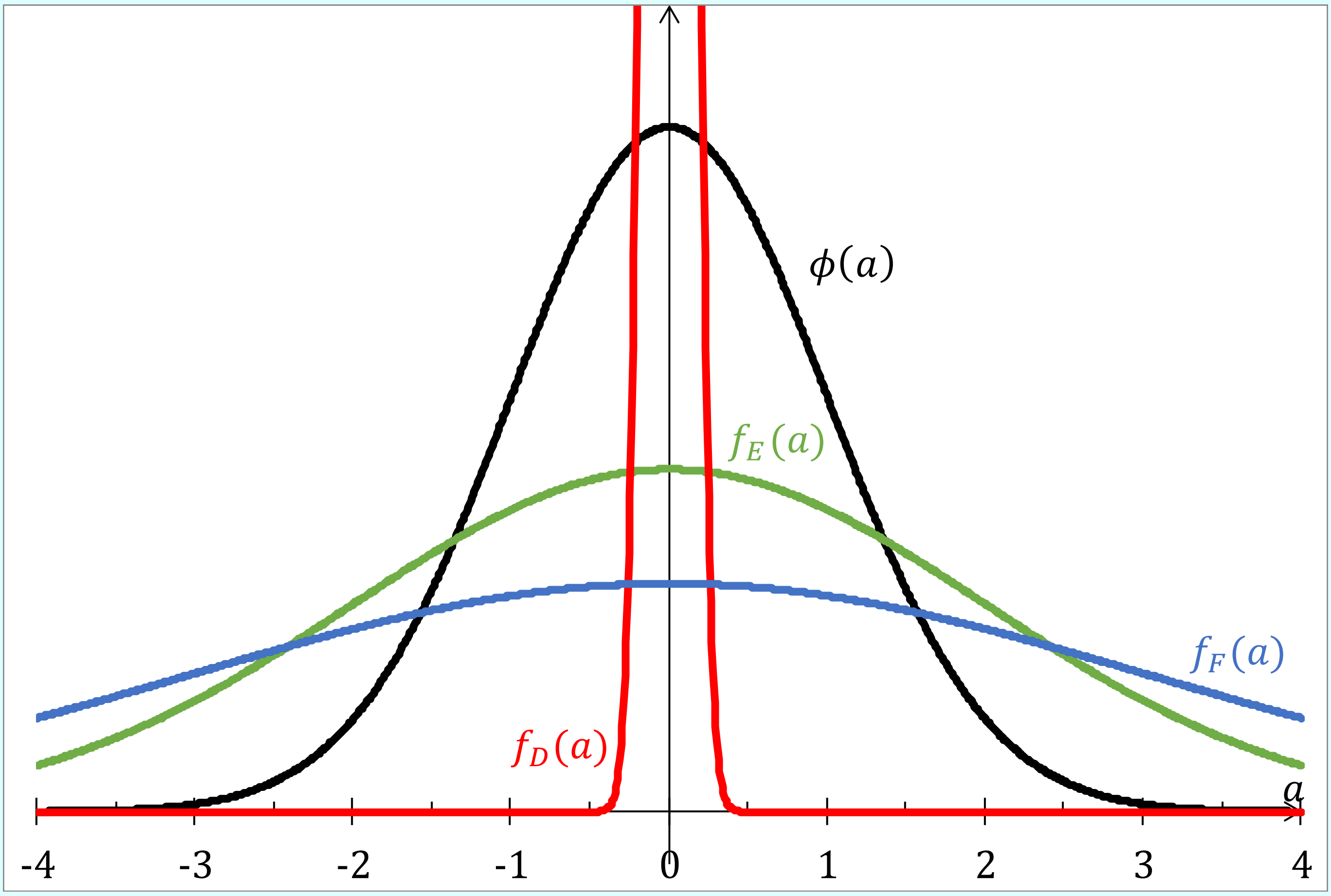
We see that the effect of increasing the mean μ is to move the graph of the PDF rightwards. And decreasing the mean moves it leftwards.



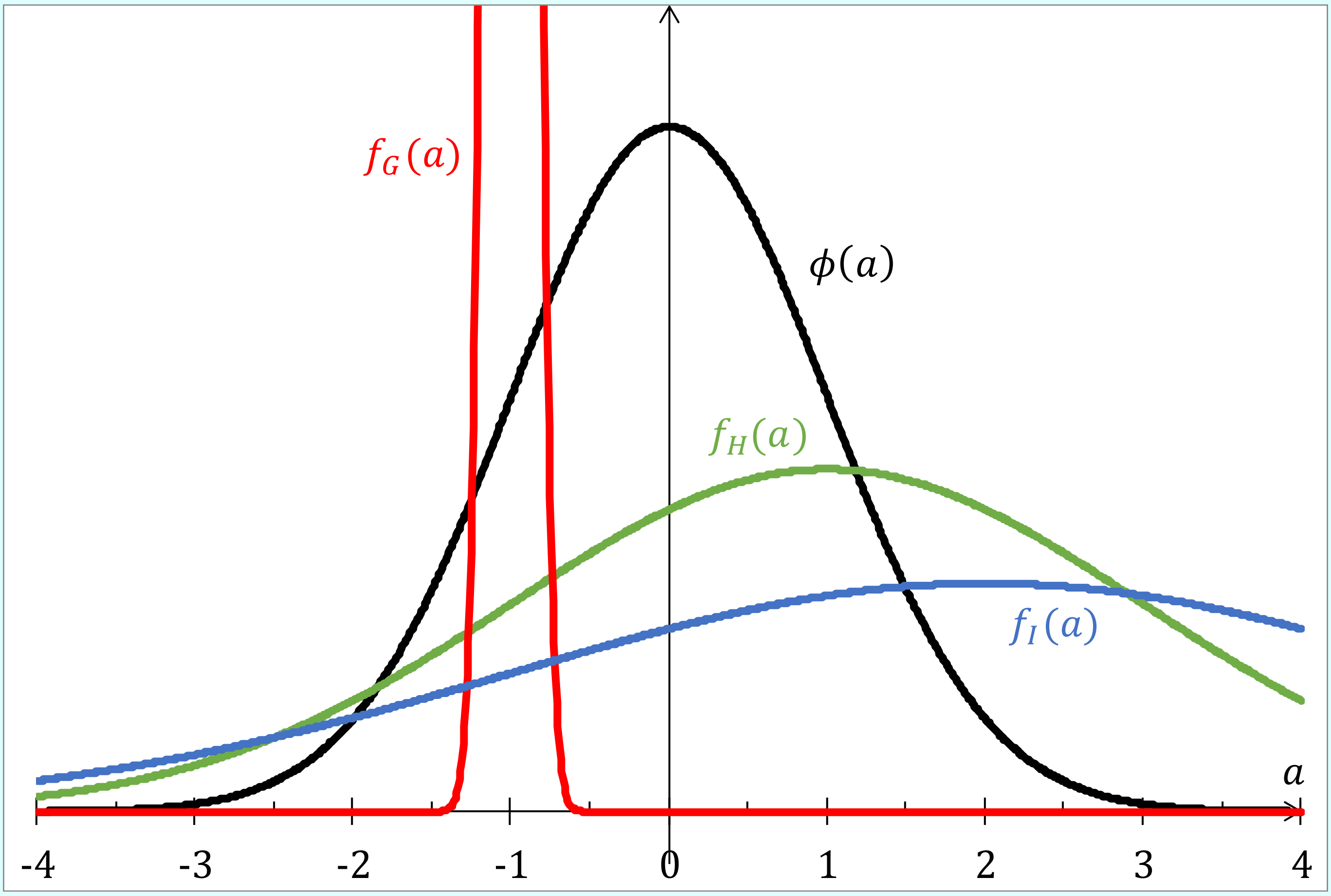
Example 1518. The normal random variables $D \sim \mathrm{N}(0, 0.1)$, $E \sim \mathrm{N}(0, 2)$, and $F \sim \mathrm{N}(0, 3)$ have mean 0 (just like the SNRV), but non-unit variances. Their PDFs are graphed below. (Included for reference is **the standard normal PDF in black**.)

The effect of changing the variance σ^2 is this:

- The larger the variance, the "fatter" the "tails" of the PDF and the shorter the peak.
- Conversely, the smaller the variance, the "thinner" the "tails" of the PDF and the taller the peak.



Example 1519. The normal random variables $G \sim \mathrm{N}(-1, 0.1)$, $H \sim \mathrm{N}(1, 2)$, and $I \sim \mathrm{N}(2, 3)$ have non-zero means and non-unit variances. Their PDFs are graphed below. (Included for reference is **the standard normal PDF in black**.)



Exercise 498. Let $X \sim \mathrm{N}(\mu, \sigma^2)$. Verify that if $\mu = 0$ and $\sigma^2 = 1$, then for all $a \in \mathbb{R}$, we have $f_X(a) = \phi(a)$. What can you conclude? (Answer on p. **1975**.)

In general, normality is preserved under linear transformations:

Fact 242. *Let $X \sim N\left(\mu, \sigma^2\right)$ and $a, b \in \mathbb{R}$ be constants. Then $aX + b \sim N\left(a\mu + b, a^2\sigma^2\right)$.*

Proof. Optional, see p. **1737** (Appendices). □

Thus, we can easily transform *any* normal random variable into the SNRV:

Corollary 49. *If $X \sim N\left(\mu, \sigma^2\right)$, then $\dfrac{X - \mu}{\sigma} = Z \sim N(0, 1)$. Equivalently, $X = \sigma Z + \mu$.*

(Just to be clear, two random variables are identical if their CDFs are identical.)

Proof. The next exercise asks you to prove this corollary. □

Exercise 499. Using Fact 242, prove that if $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$, then $\dfrac{X - \mu}{\sigma} = Z \sim \mathrm{N}(0, 1)$. (Answer on p. **1975**.)

The above corollary gives us an alternative method for computing probabilities associated with normal random variables. In general, if $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$, then

$$\mathrm{P}\left(X \leq c\right) = \mathrm{P}\left(\sigma Z + \mu \leq c\right) = \mathrm{P}\left(Z \leq \frac{c - \mu}{\sigma}\right) = \Phi\left(\frac{c - \mu}{\sigma}\right).$$

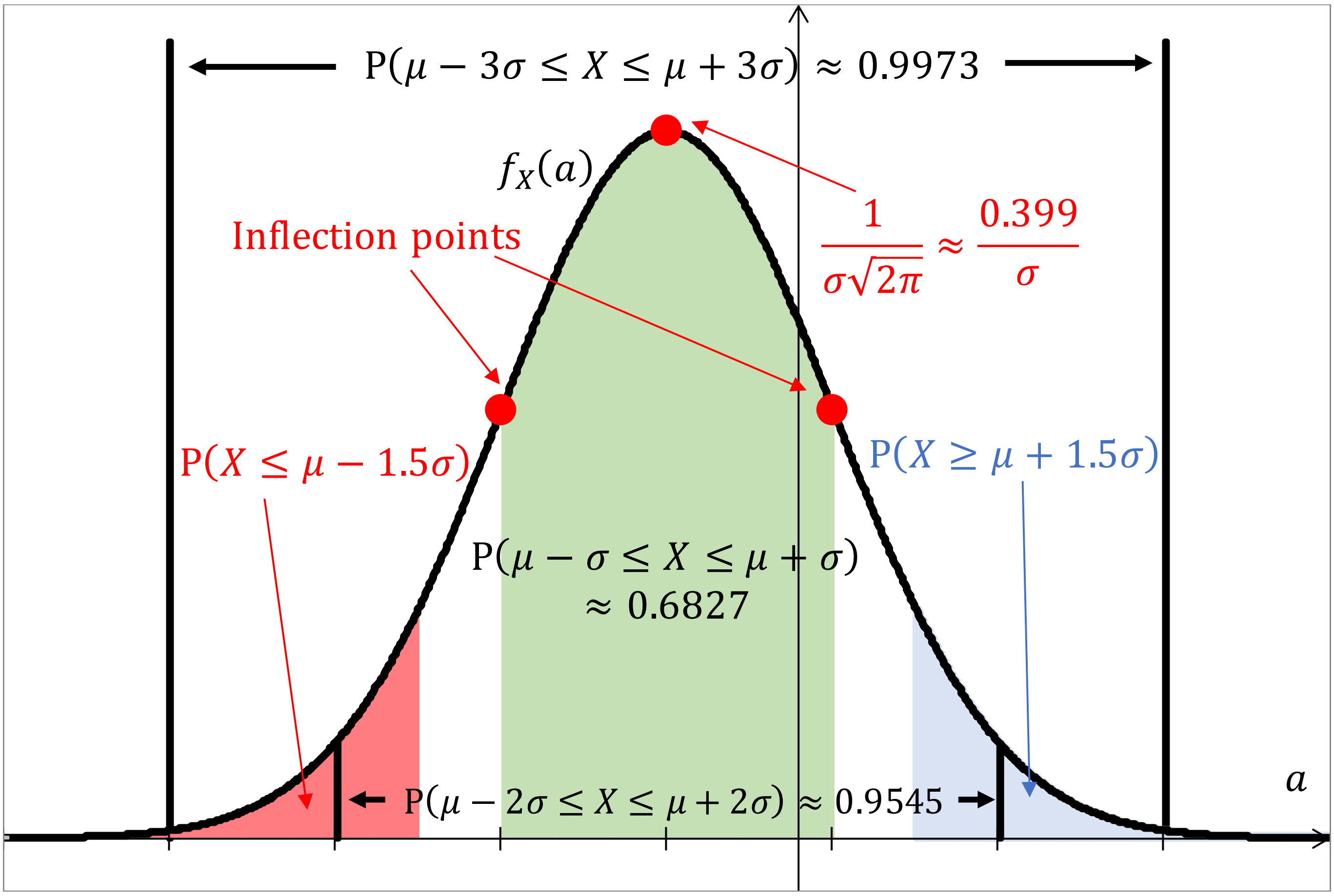
The properties that we listed for the SNRV also apply, with only a few modifications, to any NRV. I highlight any differences in red. The figure that follows illustrates.

Fact 243. *Let $X \sim N(\mu, \sigma^2)$ and let f_X and F_X be the PDF and CDF of X.*

1. $\Phi(\infty) = 1$. *(The area under the entire PDF is* 1. *This, of course, is true of any random variable.)*
2. $\phi(a) > 0$, *for all* $a \in \mathbb{R}$. *(The PDF is positive* everywhere. *This has the surprising implication that no matter how large* a *is, there is always some non-zero probability that* $Z \geq a$.*)*
3. $E[X] = \mu$. *(The mean of* Z *is* μ.*)*
4. *The PDF* f_X *reaches a global maximum at the mean* μ. *(In fact, we can go ahead and compute* $f_X(\mu) = \dfrac{1}{\sigma\sqrt{2\pi}} \approx \dfrac{0.399}{\sigma}$.*)*
5. $Var[X] = \sigma^2$. *(The variance of* X *is* σ^2.*)*
6. $P(Z \leq a) = P(Z < a)$. *(We've already discussed this earlier. It makes no difference whether the inequality is strict. This is because* $P(Z = a) = 0$.*)*
7. *The PDF* ϕ *is symmetric about the mean. This has several implications:*
   (a) $P(X \geq \mu + a) = P(X \leq \mu - a) = F_X(\mu - a)$.
   (b) *Since* $P(X \geq \mu + a) = 1 - P(X \leq \mu + a) = 1 - F_X(\mu + a)$, *it follows that* $F_X(\mu - a) = 1 - F_X(\mu + a)$ *or, equivalently,* $F_X(\mu + a) = 1 - F_X(\mu - a)$.
   (c) $F_X(\mu) = 1 - F_X(\mu) = 0.5$.
8. $P(\mu - \sigma \leq X \leq \mu + \sigma) = \Phi(1) - \Phi(-1) \approx 0.6827$. *(There is probability* 0.6827 *that* X *takes on values within* 1 *standard deviation of the mean.)*
9. $P(\mu - \sigma \leq X \leq \mu + \sigma) = \Phi(2) - \Phi(-2) \approx 0.9545$. *(There is probability* 0.9545 *that* X *takes on values within* 2 *standard deviations of the mean.)*
10. $P(\mu - \sigma \leq X \leq \mu + \sigma) = \Phi(3) - \Phi(-3) \approx 0.9973$. *(There is probability* 0.9973 *that* X *takes on values within* 3 *standard deviations of the mean.)*
11. *The PDF* ϕ *has two points of inflexion, namely at* $\pm\sigma$. *(The points of inflexion are one standard deviation away from the mean.)*

Proof. See the next exercise. □

Exercise 500. Prove all of the properties listed in Fact 243. (Hint: Use Corollary 49 to convert X into the SNRV. Then simply apply Fact 241.) (Answer on p. **1976**.)

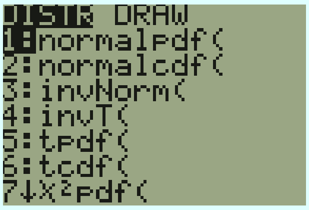


P($\mu - 3\sigma \leq X \leq \mu + 3\sigma$) ≈ 0.9973
$f_X(a)$
$\frac{1}{\sigma\sqrt{2\pi}} \approx \frac{0.399}{\sigma}$
Inflection points
P($X \leq \mu - 1.5\sigma$)
P($X \geq \mu + 1.5\sigma$)
P($\mu - \sigma \leq X \leq \mu + \sigma$)
≈ 0.6827
P($\mu - 2\sigma \leq X \leq \mu + 2\sigma$) ≈ 0.9545
a

Example 1520. Let $G \sim \mathrm{N}(-1, 0.1)$, $H \sim \mathrm{N}(1, 2)$, and $I \sim \mathrm{N}(2, 3)$. We'll find $\mathrm{P}(G < 2)$ using our TI84. The first few steps are similar to before:

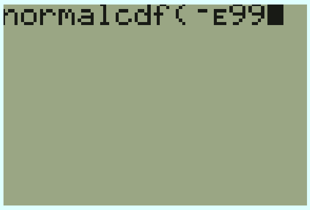
1. Press the blue 2ND button and then VARS (which corresponds to the DISTR button). This brings up the DISTR menu.
2. Press 2 to select the "normalcdf" option.
3. Enter the lower bound -10^{99} by pressing (-), the blue 2ND button, EE (which corresponds to the , button), and then 9 9. (Don't press ENTER yet!)
4. Enter the upper bound 2 by pressing , and 2. (Don't press ENTER yet!!).

Step 1.



Step 2.

Step 3.



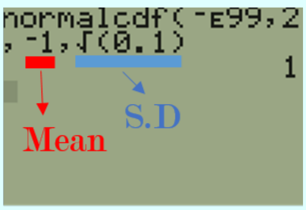
Step 4.

Previously, we didn't bother telling the TI84 our mean μ and **standard deviation** σ. And so by default, if we pressed ENTER at this point, the TI84 simply assumed that we wanted the SNRV $Z \sim \mathrm{N}(0, 1)$. Now we'll tell the TI84 what μ and σ are:

5. First enter the mean $\mu = -1$. Press , (-) 1.
6. Now enter the **standard deviation** $\sigma = \sqrt{0.1}$ (and **not** the variance). Press , √ 0 . 1). Finally, press ENTER. The TI84 says that $\mathrm{P}(G < 2) \approx 1$.

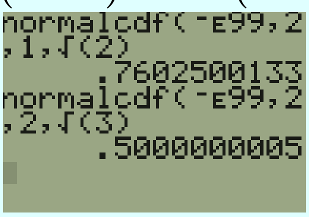
Step 5.

Step 6.

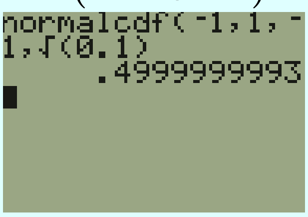


Finding $\mathrm{P}(H < 2)$, $\mathrm{P}(I < 2)$, $\mathrm{P}(-1 < G < 1)$, $\mathrm{P}(-1 < H < 1)$, and $\mathrm{P}(-1 < I < 1)$ is similar:

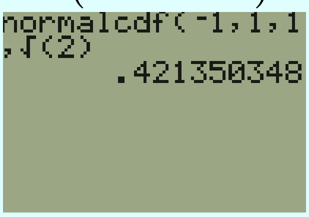
$\mathrm{P}(H < 2)$ and $\mathrm{P}(I < 2)$



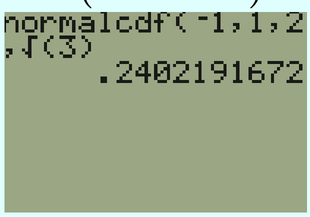
$\mathrm{P}(-1 < G < 1)$



$\mathrm{P}(-1 < H < 1)$



$\mathrm{P}(-1 < I < 1)$



Since I has mean $\mu = 2$, we should have exactly $\mathrm{P}(I < 2) = 0.5$. So here the TI84 has actually made a small error in reporting instead that $\mathrm{P}(I < 2) \approx 0.5000000005$.

Example 1521. We now redo the previous two examples, but use Z-tables:

$$\mathrm{P}(G<2)=\mathrm{P}\left(Z<\frac{2-\mu_G}{\sigma_G}=\frac{2-(-1)}{\sqrt{0.1}}\approx 9.4868\right)=\Phi(9.4868)\approx 1,$$

$$\mathrm{P}(H<2)=\mathrm{P}\left(Z<\frac{2-\mu_H}{\sigma_H}=\frac{2-1}{\sqrt{2}}\approx 0.7071\right)=\Phi(0.7071)\approx 0.7601,$$

$$\mathrm{P}(I<2)=\mathrm{P}\left(Z<\frac{2-\mu_I}{\sigma_I}=\frac{2-2}{\sqrt{3}}=0\right)=\Phi(0)=0.5,$$

$$\begin{aligned}\mathrm{P}(-1<G<1)&=\mathrm{P}\left(0=\frac{-1-(-1)}{\sqrt{0.1}}<Z<\frac{1-(-1)}{\sqrt{0.1}}\approx 6.3246\right)\\&=\Phi(6.3246)-\Phi(0)\approx 1-\Phi(0)=0.5.\end{aligned}$$

$$\begin{aligned}\mathrm{P}(-1<H<1)&=\mathrm{P}\left(-1.4142\approx\frac{-1-1}{\sqrt{2}}<Z<\frac{1-1}{\sqrt{2}}=0\right)\\&=\Phi(0)-\Phi(-1.4142)\approx 0.5-[1-\Phi(1.4142)]\\&=\Phi(1.4142)-0.5\approx 0.9213-0.5=0.4213,\end{aligned}$$

$$\begin{aligned}\mathrm{P}(-1<I<1)&=\mathrm{P}\left(-1.7321\approx\frac{-1-2}{\sqrt{3}}<Z<\frac{1-2}{\sqrt{3}}\approx -0.5774\right)\\&=\Phi(-0.5774)-\Phi(-1.7321)=1-\Phi(0.5774)-[1-\Phi(1.7321)]\\&\approx 0.9584-0.7182=0.2402.\end{aligned}$$

Exercise 501. Let $X \sim \mathrm{N}(2.14, 5)$ and $Y \sim \mathrm{N}(-0.33, 2)$. Using both the Z-tables and your graphing calculator, find the following: (a) $\mathrm{P}(X \geq 1)$ and $\mathrm{P}(Y \geq 1)$. (b) $\mathrm{P}(-2 \leq X \leq -1.5)$ and $\mathrm{P}(-2 \leq Y \leq -1.5)$. (Answer on p. **1977**.)

127.2. Sum of Independent Normal Random Variables

Theorem 48. *If X and Y are independent normal random variables, then $X+Y$ is also a normal random variable. Moreover, $X-Y$ is also a normal random variable.*

Proof. Omitted. □

We already knew from before that $\mathrm{E}[X \pm Y] = \mathrm{E}[X] \pm \mathrm{E}[Y]$. Moreover, if X and Y are independent, then $\mathrm{Var}[X \pm Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$. Thus, the above theorem implies:

Corollary 50. *Let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ be independent and $a, b \in \mathbb{R}$ be constants. Then $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$ and more generally, $aX + bY \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2)$.*

Moreover, $X - Y \sim N(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2)$ and more generally, $aX - bY \sim N(a\mu_X - b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2)$.

Examples:

Example 1522. The weight (in kg) of a sumo wrestler is modelled by $X \sim \mathrm{N}(200, 50)$. Assume that the weight of each sumo wrestler is independent of the weight of any other sumo wrestler.

We randomly choose two sumo wrestlers.

(a) What is the probability that their total weight is greater than 405 kg?

(b) What is the probability that one is more than 10% heavier than that the other?

(a) Let $X_1 \sim \mathrm{N}(200, 50)$ and $X_2 \sim \mathrm{N}(200, 50)$ be the weight of the first and second sumo wrestler. Then $X_1 + X_2 \sim \mathrm{N}(400, 100)$. Thus,

$$\mathrm{P}(X_1 + X_2 > 405) = \mathrm{P}\left(Z > \frac{405 - 400}{\sqrt{100}}\right) = \mathrm{P}(Z > 0.5) = 1 - \Phi(0.5) \approx 1 - 0.6915 = 0.3085.$$

(b) Our goal is to find $p = \mathrm{P}(X_1 > 1.1X_2) + \mathrm{P}(X_2 > 1.1X_1)$. This is the probability that the first sumo wrestler is more than 10% heavier than the second, *plus* the probability that the second is more than 10% heavier than the first. Of course, by symmetry, these two probabilities are equal. Thus, $p = 2 \times \mathrm{P}(X_1 > 1.1X_2)$. Now,

$$\mathrm{P}(X_1 > 1.1X_2) = \mathrm{P}(X_1 - 1.1X_2 > 0).$$

But $X_1 - 1.1X_2 \sim \mathrm{N}\left(200 - 1.1 \cdot 200, 50 + 1.1^2 \cdot 50\right) = \mathrm{N}(-20, 110.5)$. Thus,

$$\mathrm{P}(X_1 > 1.1X_2) = \mathrm{P}(X_1 - 1.1X_2 > 0) = \mathrm{P}\left(Z > \frac{0 - (-20)}{\sqrt{110.5}}\right)$$

$$\approx \mathrm{P}(Z > 1.9026) = 1 - \Phi(1.9026) \approx 1 - 0.9714 = 0.0286.$$

Altogether then, $p = 2\mathrm{P}(X_1 > 1.1X_2) = 2 \times 0.0286 = 0.0572$.

Example 1523. The weight (in kg) of a caught fish is modelled by $X \sim \mathrm{N}(1, 0.4)$. The weight (in kg) of a caught shrimp is modelled by $Y \sim \mathrm{N}(0.1, 0.1)$. Assume that the weights of any caught fish and shrimp are independent.

(a) What is the probability that the total weight of 4 caught fish and 50 caught shrimp is greater than 10 kg?

(b) What is the probability that a caught fish weighs more than 9 times as much as a caught shrimp?

(a) Let S be the total weight of 4 caught fish and 50 caught shrimp. Note, importantly, that it would be **wrong** to write $S = 4X + 50Y$, because $4X + 50Y$ would be 4 times the weight of a single caught fish, *plus* 50 times the weight of a single caught shrimp.

In contrast, we want Z to be the sum of the weights of 4 independent fish and 50 independent shrimp. Thus, we should instead write $S = X_1 + X_2 + X_3 + X_4 + Y_1 + Y_2 + \cdots + Y_{50}$, where

- $X_1 \sim \mathrm{N}(1, 0.4)$, $X_2 \sim \mathrm{N}(1, 0.4)$, $X_3 \sim \mathrm{N}(1, 0.4)$, and $X_4 \sim \mathrm{N}(1, 0.4)$ are the weights of each caught fish.
- $Y_1 \sim \mathrm{N}(0.1, 0.1)$, $Y_2 \sim \mathrm{N}(0.1, 0.1)$, $\ldots$, and $Y_{50} \sim \mathrm{N}(0.1, 0.1)$ are the weights of each caught shrimp.

Now, $S \sim \mathrm{N}(4 \times 1 + 50 \times 0.1, 4 \times 0.4 + 50 \times 0.1) = \mathrm{N}(9, 6.6)$.

(Note by the way that in contrast, $4X + 50Y \sim \mathrm{N}\left(9, 4^2 \times 0.4 + 50^2 \times 0.1\right) = \mathrm{N}(9, 256.4)$, which has a rather different variance!)

Thus, $\mathrm{P}(S > 10) \approx 0.3485$ (calculator).

(b) $\mathrm{P}(X > 9Y) = \mathrm{P}(X - 9Y > 0)$. But $X - 9Y \sim \mathrm{N}\left(1 - 9 \times 0.1, 0.4 + 9^2 \times 0.1\right) = \mathrm{N}(0.1, 8.5)$. Thus, $\mathrm{P}(X - 9Y > 0) \approx 0.5137$ (calculator).

Exercise 502. (Answer on p. 1978.) Water and electricity usage are billed, respectively, at \$2 per 1,000 litres (l) and \$0.30 per kilowatt-hour (kWh). Assume that each month, the amount of water used by Ahmad (and his family) at their HDB flat is normally distributed with mean 25,000 l and variance 64,000,000 l^2. Similarly, the amount of electricity they use is normally distributed with mean 200 kWh and variance 10,000 kWh^2.

Assume that monthly water usage and electricity usage are independent.

(a) Find the probability that their total water and electricity utility bill in any given month exceeds \$100.

(b) Find the probability that their total water and electricity utility bill in any given year exceeds \$1,000.

Suppose instead that electricity usage is billed at \$$x$ per kWh.

(c) Then what is the maximum value of x, in order for the probability that the total utility bill in a given month exceeds \$100 is 0.1 or less?

127.3. The Central Limit Theorem and The Normal Approximation

Suppose we have n independent random variables, each identically distributed with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}$. Then informally, the Central Limit Theorem (CLT) says that if n is "large enough", then their sum (which is also a random variable) has the approximate distribution $\mathrm{N}\left(n\mu, n\sigma^2\right)$. Formally:

Theorem 49. (The Central Limit Theorem.) *Let $X_1, X_2, \ldots, X_n$ be random variables. Suppose (i) they are independent; and (ii) they are identically distributed, with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}$.*

Then the sum $\sum_{i=1}^{n} X = X_1 + X_2 + \cdots + X_n$ converges in distribution to $N\left(n\mu, n\sigma^2\right)$.

Proof. The proof is a little advanced and thus entirely omitted from this book. □

What does it mean for one random variable to "converge in distribution" to another? This is a little beyond the scope of the A-Levels, but informally, this means that as $n \to \infty$, the random variable $\sum_{i=1}^{n} X_i$ becomes "ever more" like the random variable with distribution $\mathrm{N}\left(n\mu, n\sigma^2\right)$.

One big use of the CLT is this:

If n is "large enough", then the sum of n independent, identically distributed random variables can be approximated by a normal distribution.

How large is "large enough"? The most common rule-of-thumb is that $n \geq 30$ is "large enough", so that's what we'll use in this book, even though this is somewhat arbitrary.

Indeed, if the original distribution from which the random variables are drawn are not "nice enough", then $n \geq 30$ may not be "large enough". (Informally, a distribution is "nice enough" if it is—among other things—fairly symmetric, fairly unimodal, and not too skewed.)

You can safely assume that all distributions you'll ever encounter in the A-Levels are "nice enough", so that the $n \geq 30$ rule-of-thumb works. But whenever you use the CLT normal approximation, you should be clear to **state that you assume the distribution is "nice enough"**.

Example 1524. Let X be the random variable that is the sum of 100 rolls of a fair die. From our earlier work, we know that each die roll has mean 3.5 and variance 35/12. Problem: *Find* $\mathrm{P}(X \geq 360)$ *and* $\mathrm{P}(X > 360)$.

The CLT says that since $n = 100 \geq 30$ is large enough and the distribution is "nice enough" (we are assuming this), the random variable X can be approximated by the normal random variable $Y \sim \mathrm{N}(100 \times 3.5, 100 \times 35/12) = \mathrm{N}(350, 3500/12)$.

Now, in using Y as an approximation for X, we might be tempted to simply write

$$\mathrm{P}(X \geq 360) \approx \mathrm{P}(Y \geq 360) \quad \text{and} \quad \mathrm{P}(X > 360) \approx \mathrm{P}(Y > 360).$$

Note however that X is a discrete random variable, so that $\mathrm{P}(X \geq 360) \neq \mathrm{P}(X > 360)$. More specifically,

$$\mathrm{P}(X \geq 360) = \mathrm{P}(X = 360) + \mathrm{P}(X > 360).$$

In contrast, Y is a continuous random variable, so that $\mathrm{P}(Y \geq 360) = \mathrm{P}(Y > 360)$. Hence, if we simply use the approximations $\mathrm{P}(X \geq 360) \approx \mathrm{P}(Y \geq 360)$ and $\mathrm{P}(X > 360) \approx \mathrm{P}(Y > 360)$, then implicitly we'd be saying that $\mathrm{P}(X = 360) = 0$, which is blatantly false.

To correct for this, we perform the so-called **continuity correction**. This says that we'll instead use the approximations

$$\mathrm{P}(X \geq 360) \approx \mathrm{P}(Y \geq 359.5) \quad \text{and} \quad \mathrm{P}(X > 360) \approx \mathrm{P}(Y \geq 360.5).$$

Thus, $\mathrm{P}(X \geq 360) \approx \mathrm{P}(Y \geq 359.5) \approx 0.2890$ (calculator) and $\mathrm{P}(X > 360) \approx \mathrm{P}(Y \geq 360.5) \approx 0.2693$.

Continuity Correction. If X is a discrete random variable that is to be approximated by a continuous random variable Y, then

- $\mathrm{P}(X \geq k) \approx \mathrm{P}(Y \geq k - 0.5)$,
- $\mathrm{P}(X \leq k) \approx \mathrm{P}(Y \leq k + 0.5)$,
- $\mathrm{P}(X > k) \approx \mathrm{P}(Y > k + 0.5)$,
- $\mathrm{P}(X < k) \approx \mathrm{P}(Y < k - 0.5)$.

Note that if the random variable to be approximated is itself continuous, then there is no need to perform the continuity correction. This is illustrated in Exercise 504 below.

Exercise 503. Let X be the random variable that is the sum of 30 rolls of a fair die. Find $\mathrm{P}(100 \leq X \leq 110)$. (Answer on p. **1979**.)

Exercise 504. The weight of each Coco-Pop is independently and identically distributed with mean 0.1 g and variance 0.004 g^2. A box of Coco-Pops has exactly 5,000 Coco-Pops. It is labelled as having a net weight of 500 g. Find the probability that that the actual net weight of the Coco-Pops in this box is less than or equal to 499 g. (Answer on p. **1979**.)

128. The CLT is Amazing (Optional)

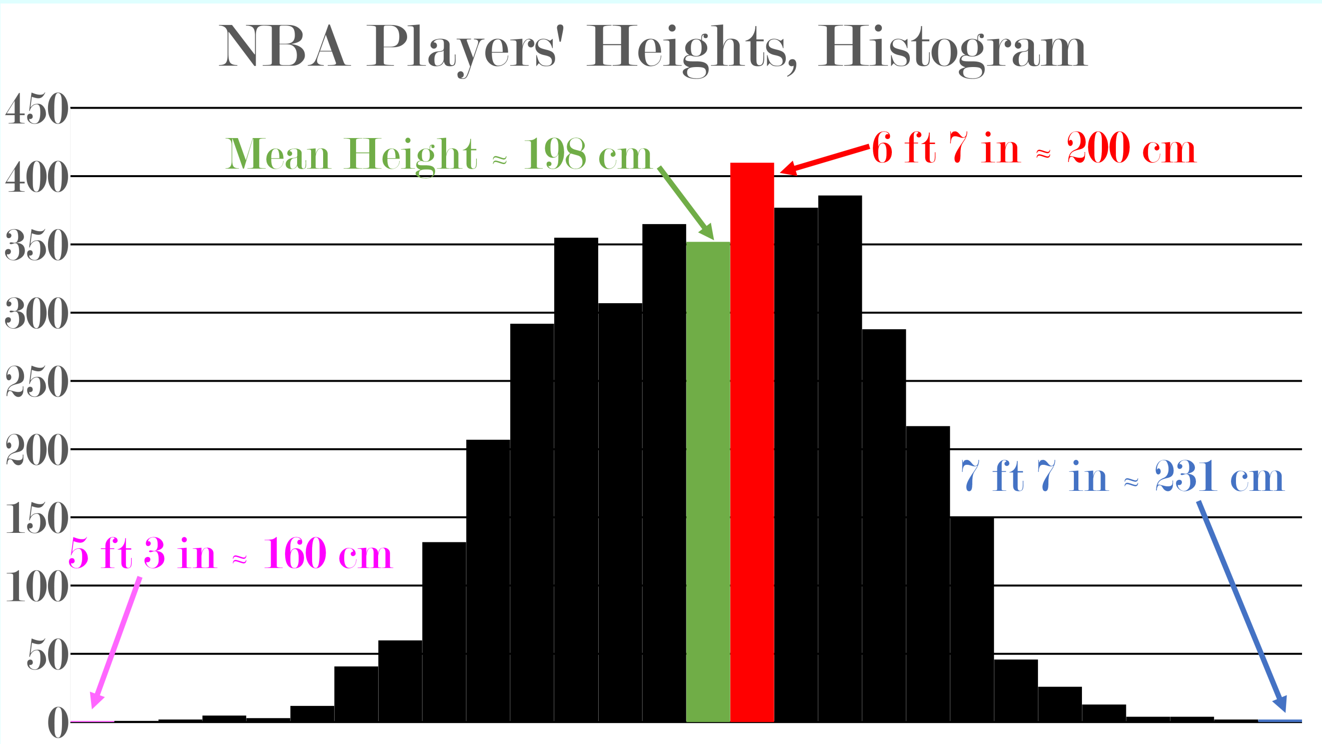
The Fundamental Theorems of Calculus and the CLT are the most profound and amazing results you'll learn in H2 Maths. This chapter briefly explains why the CLT is so amazing and why the normal distribution is ubiquitous.

128.1. The Normal Distribution in Nature

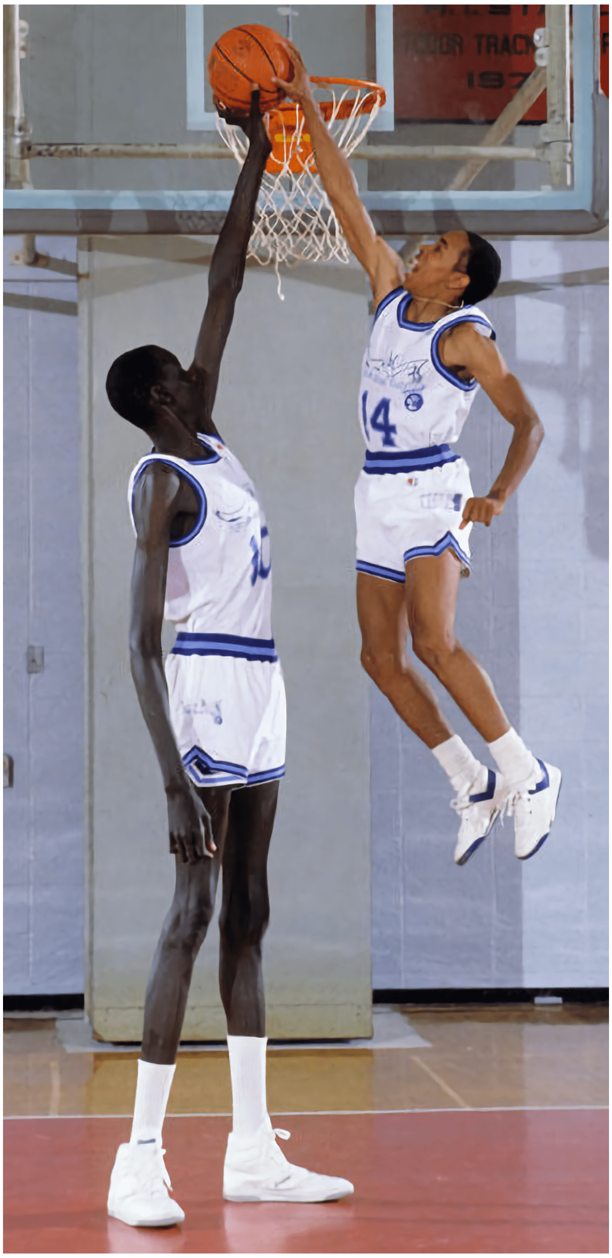
The normal distribution is ubiquitous in nature. The classic example is human height.[535]

Example 1525. Below is a **histogram** of the heights of the 4,060 NBA players who ever played in an NBA game (through the end of the 2016 season). (Heights are reported in feet and a whole number of inches, where 1 in = 2.54 cm and 1 ft = 12 in, so that 1 ft = 30.48 cm.) The histogram has 28 bins and (arguably) looks normal (bell-shaped).

The width of each bin is 1 inch. For example, the red bin says 410 players have had reported heights of 6 ft 7 in (approx. 200 cm). The pink (leftmost) bin is barely visible and says only 1 player has had a reported height of 5 ft 3 in (approx. 160 cm). The blue (rightmost) bin is also barely visible and says that only 2 players have had reported heights of 7 ft 7 in (approx. 231 cm). The average or mean height is approx. 6 ft 6 in (approx. 198 cm).



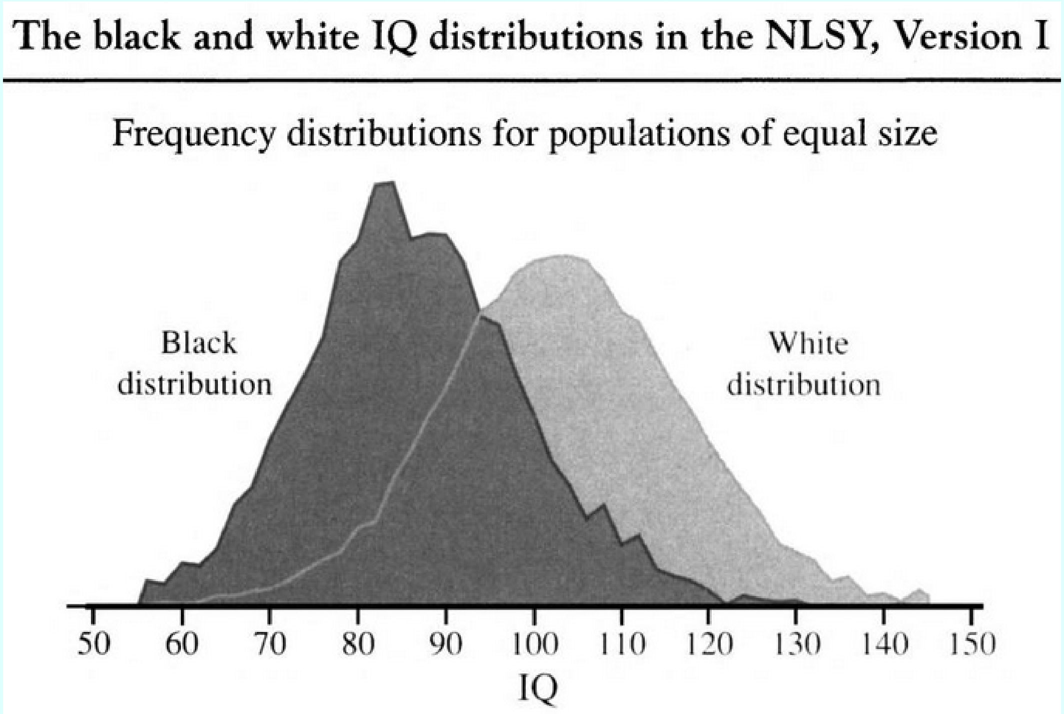
[535]Data: Excel spreadsheet. Source: Basketball-Reference.com (retrieved June 15th, 2016). Caveats: (1) For some reason, out of the 4060 players in that database at the time of retrieval, there was exactly one player (George Karl) whose height was not listed. WIKIPEDIA lists George Karl's height as 6 ft 2 in, so that is what I have used for his height. (2) By NBA, I actually mean the BAA (1946-1949), the NBA (1949-present), and the ABA (1967-1976), combined. (3) As is well-known among basketball fans, the listed heights of NBA players are not accurate and can sometimes be off by as much as 2 to 3 inches (5 to 7.5 cm). (See this recent Wall Street Journal article.)



Manute Bol (approx 231 cm) and Muggsy Bogues (approx 160 cm) were briefly on the same team. (YouTube highlights.)

An infamous example of the normal distribution concerns human intelligence:

Example 1526. The 1994 book *The Bell Curve: Intelligence and Class Structure in American Life* was named after the observation that the Intelligence Quotient (IQ) seems to be normally distributed. This observation was neither new nor controversial (though some scholars would dispute the usefulness of IQ measures).



What made the book especially controversial were its claims that intelligence was largely heritable and that black Americans had lower intelligence than whites. The figure above is taken from p. 279 of the book. It suggests that

- Black IQ is normally distributed, with a mean of around 80.
- White IQ is normally distributed, with a mean of around 105.

Another example—the Galton box:

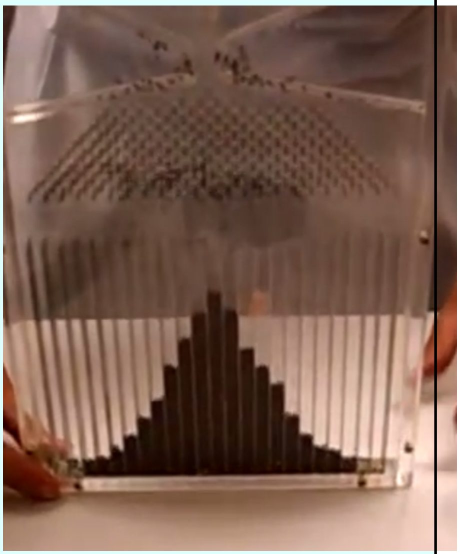
Example 1527. (Galton box.) Small beans are released from the top, through a narrow passage. There are numerous pegs that tend to randomly divert the path of the beans.

At the bottom of the box, there are many different narrow slots of equal width, into which the beans can drop. The beans will tend to form a bell shape at the bottom.

Beans just released.

Pegs divert beans.

Beans fill slots.



(Source: YouTube.)

Question:

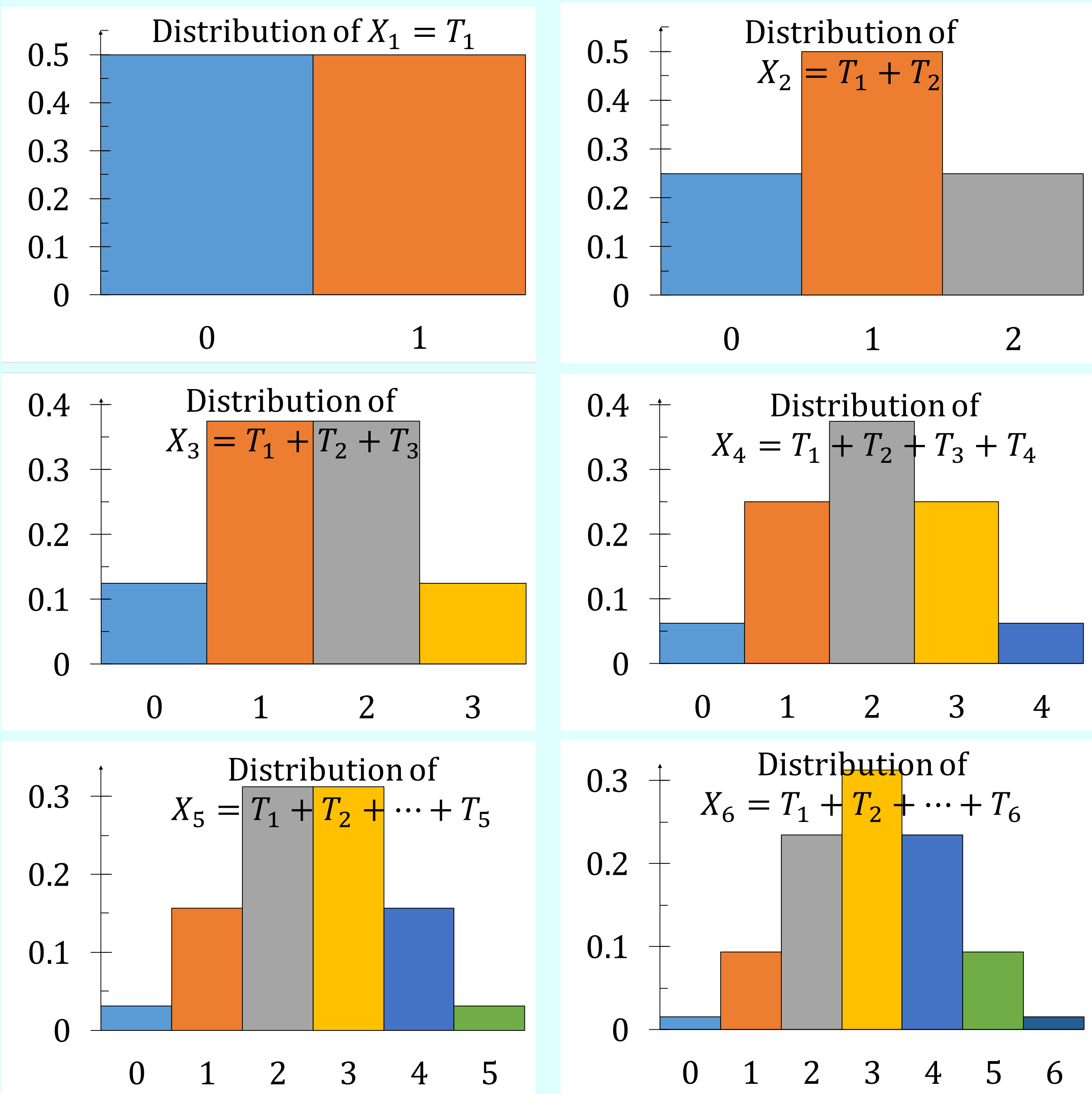
Many things in nature seem to be normally distributed. *Why*?

We will try to answer this question, but only after we've illustrated how the **Central Limit Theorem** works.

Example 1528. Flip many fair coins. Model each with the Bernoulli random variables T_1, T_2, T_3, ..., each with probability of success (heads) 0.5.

Let $X_n = T_1 + T_2 + \cdots + T_n \sim \mathrm{B}(n, 0.5)$.

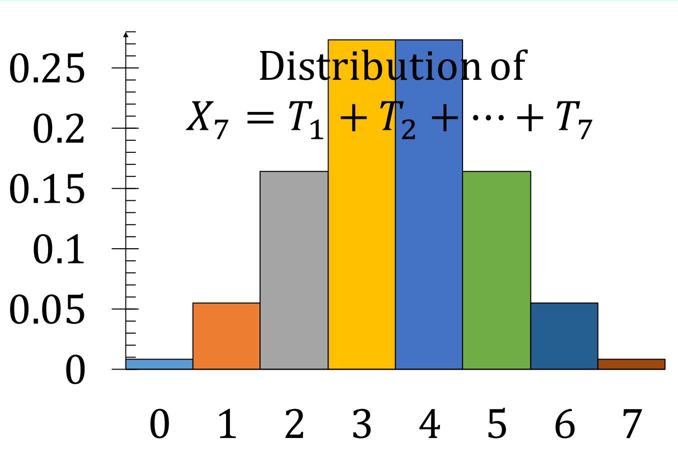
Below are the histograms of the distributions of X_1, X_2, ..., and X_6. X_1 has probability 0.5 of taking on each of the values 0 and 1. X_2 has probability 0.5 of taking on the value of 1; and probability of 0.25 of taking on each of the values 0 and 2. Etc.

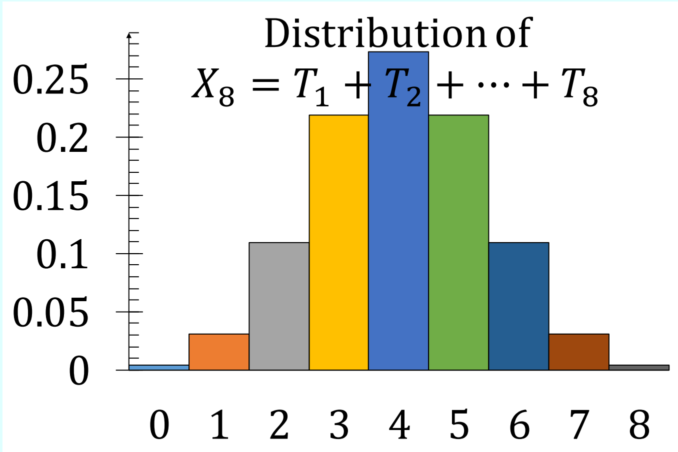


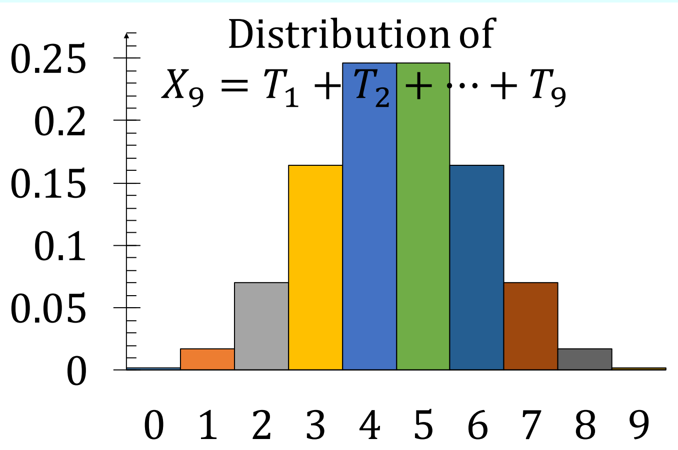
(*Example continues on the next page ...*)

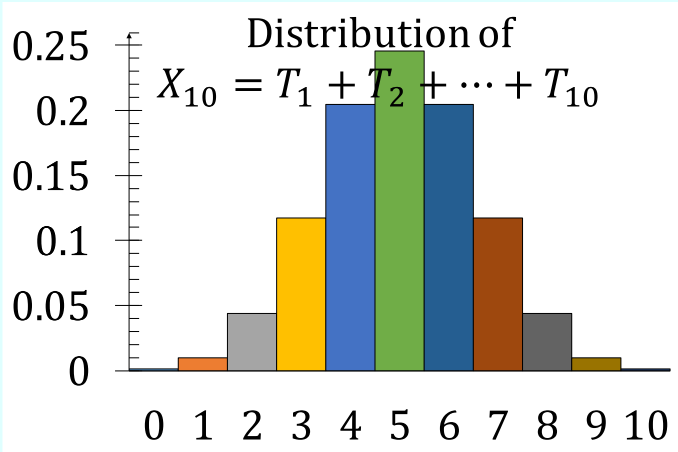
(... *Example continued from the previous page.*)

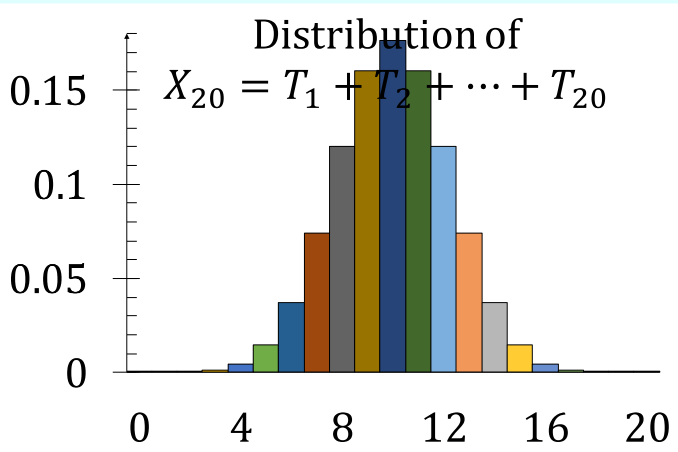
On this and the next page are the histograms of the distributions of X_7, X_8, X_9, X_{10}, X_{20}, X_{30}, X_{40}, X_{50}, and X_{100}. Observe that as n grows, the shape of the probability distribution of X_n looks ever more bell-shaped. This is exactly what the CLT says.

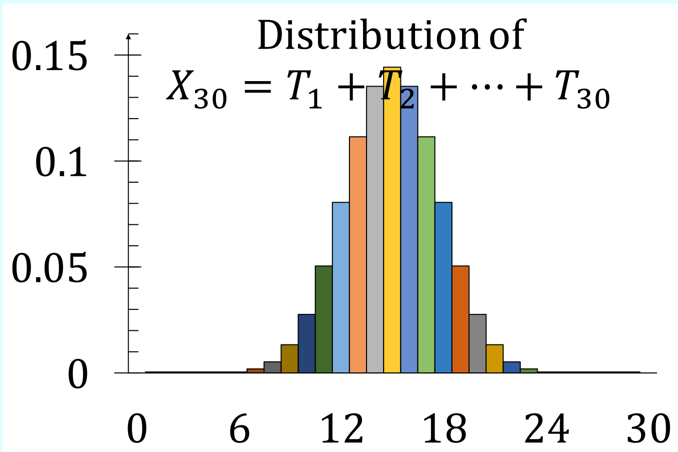












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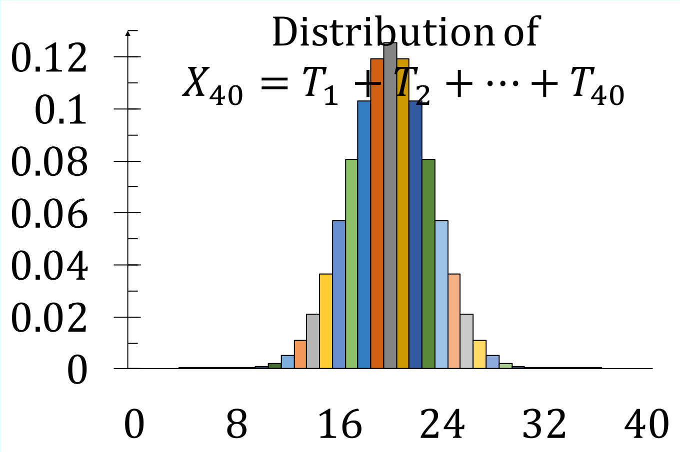
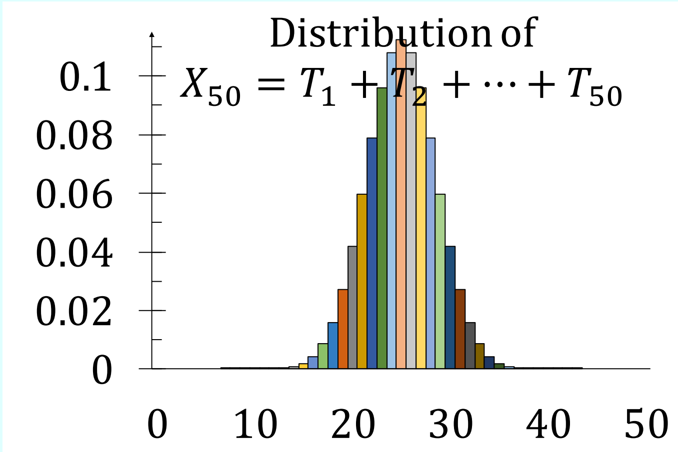
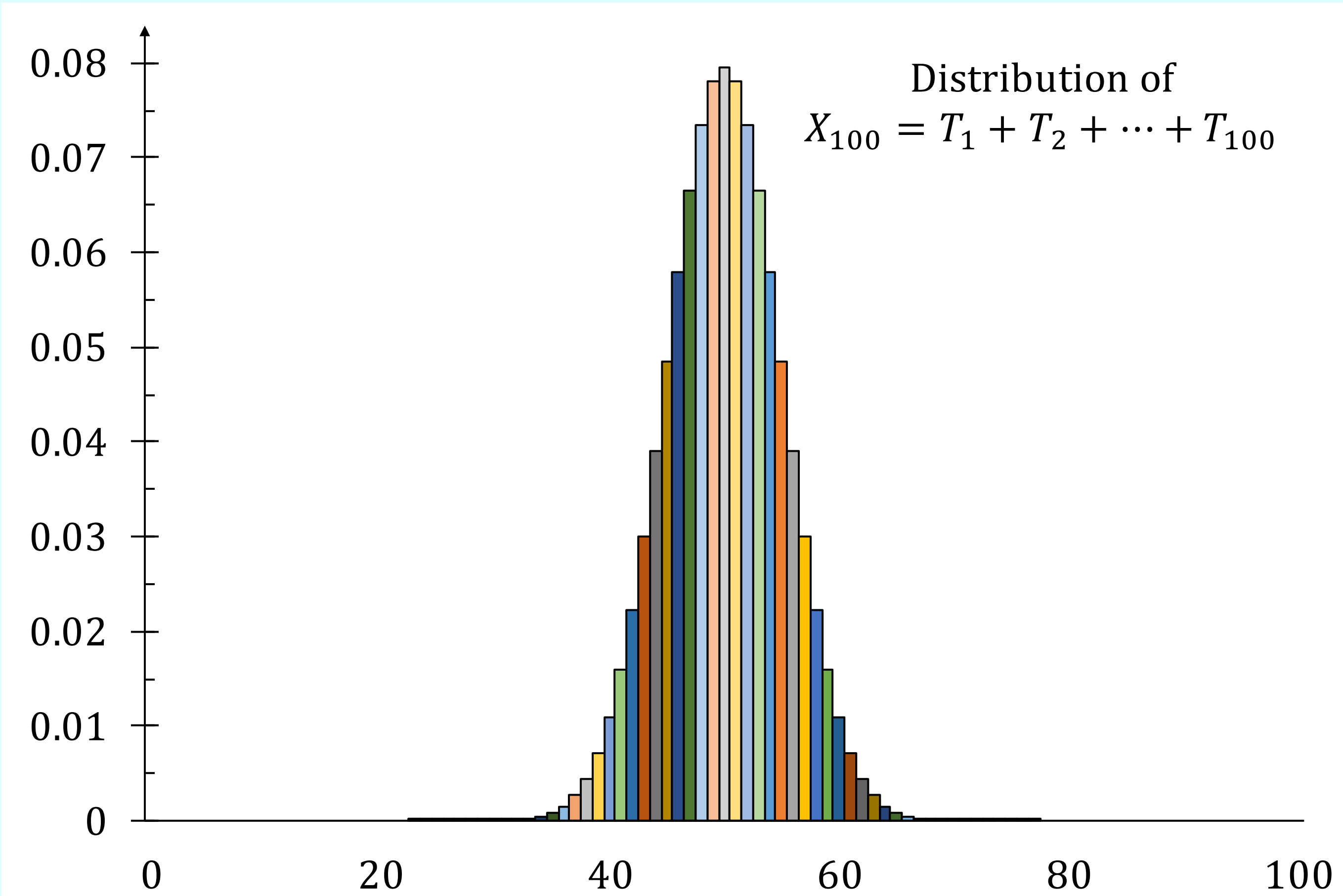






The CLT says the following:

1. Draw a sufficiently large number of independent and identical random variables from **ANY** distribution.[536]
2. Add them up to get another random variable S.
3. The probability distribution of S will look normal.

[536] I should say *nearly* any distribution. For the classical CLT to apply, the variance must be finite.

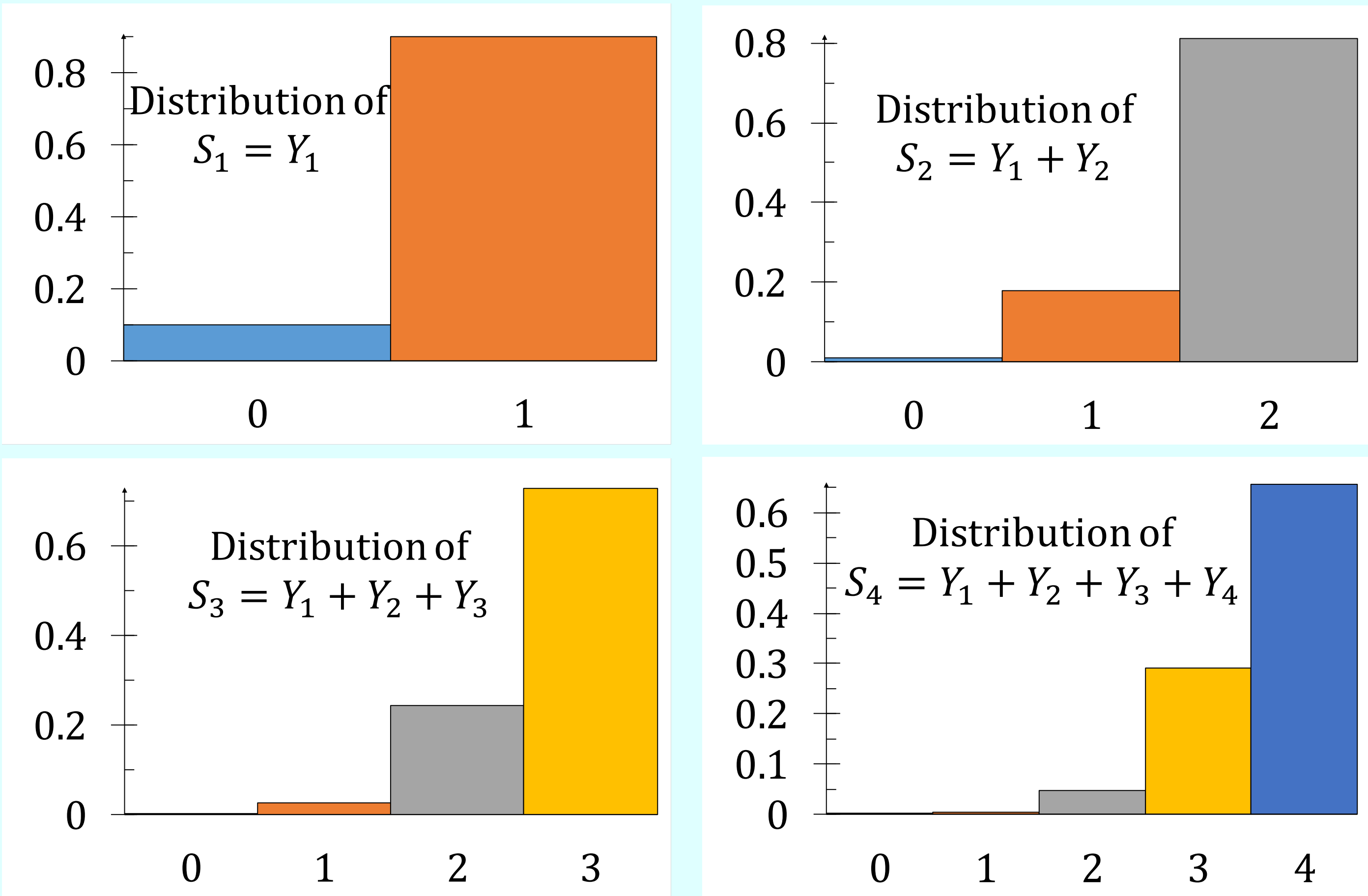
What makes the CLT particularly amazing is that it works with **ANY** distribution.

To illustrate, next up is an example where the original distribution is highly skewed and does not look at all bell-curved. Nonetheless, the CLT still works out nicely.

Example 1529. Flip many biased coins, each with probability 0.9 of heads. Model each with the Bernoulli random variables Y_1, Y_2, Y_3, ..., each with probability of success (heads) 0.9.

Let $S_n = Y_1 + Y_2 + \cdots + Y_n$ be the number of heads in the first n coin-flips. (By the way, $S_n \sim \mathrm{B}(n, 0.9)$.)

On this and the next page are the histograms of the distributions of S_1, S_2, ..., and S_{10}. S_1 has probability 0.1 of taking on value 0 and 0.9 of taking on value 1. S_2 has probability 0.01 of taking on the value of 0, 0.18 of taking on the value 1, and 0.81 of taking on the value 2. S_3 has probability 0.001 of taking on the value of 0, 0.036 of taking on the value 1, 0.486 of taking on the value 2, 0.2916 of taking on the value 3, 0.6561 of taking on the value 4. Etc.



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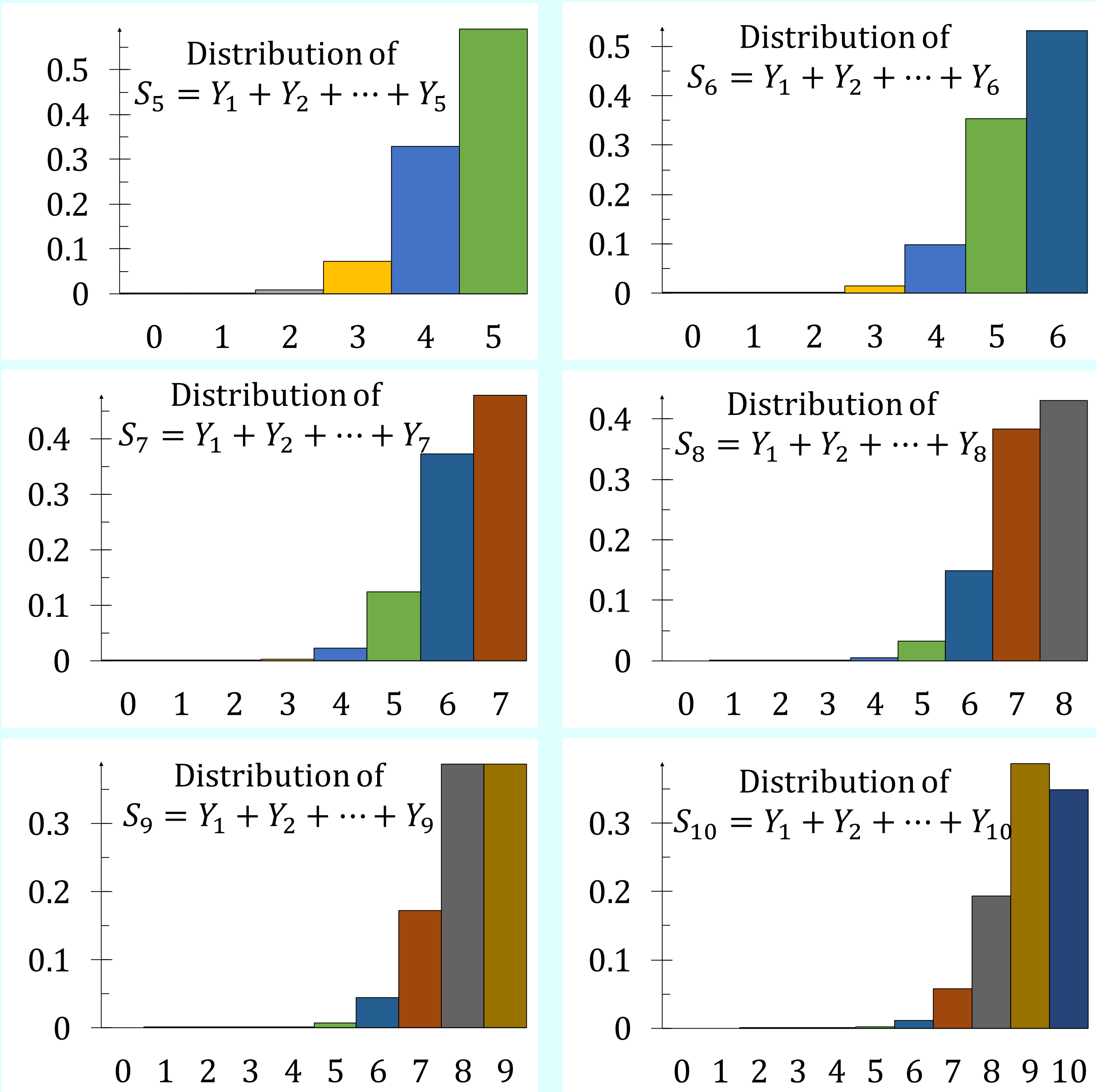
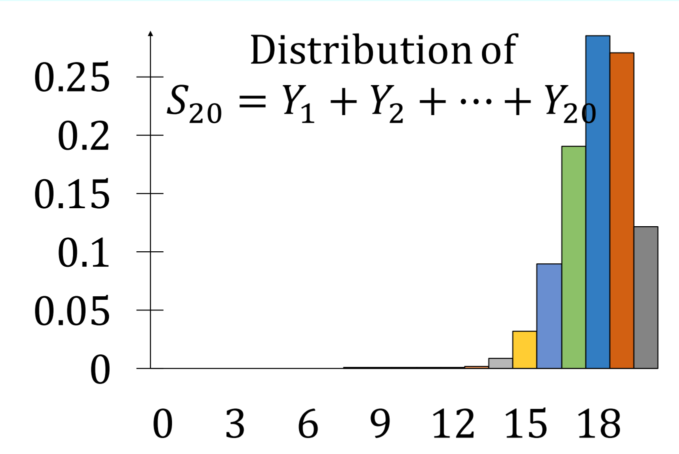


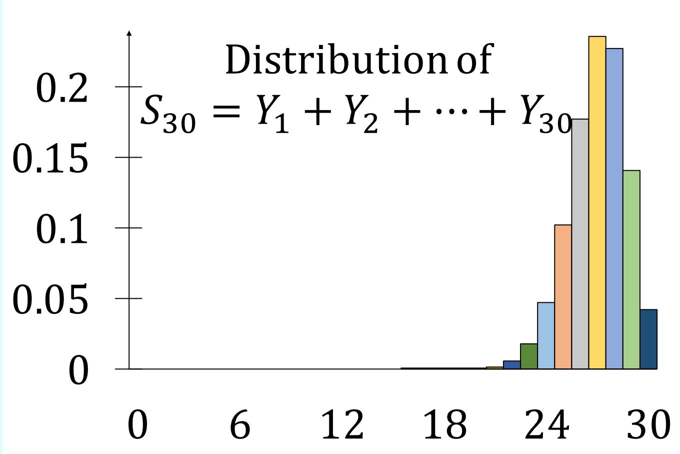
It certainly does not look like the distribution S_n is becoming increasingly bell-curved. Well, let's see.

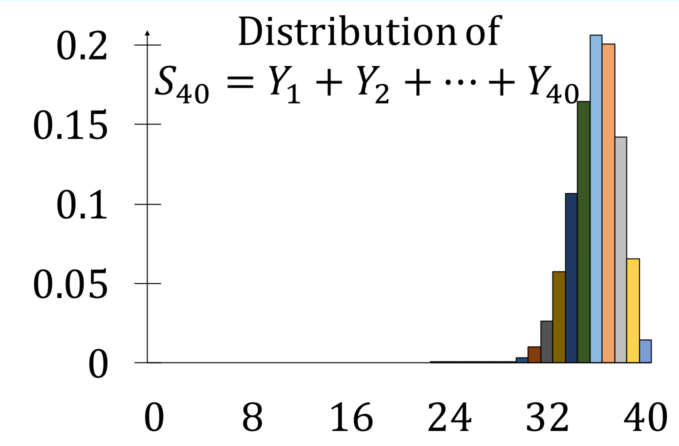
(*Example continues on the next page ...*)

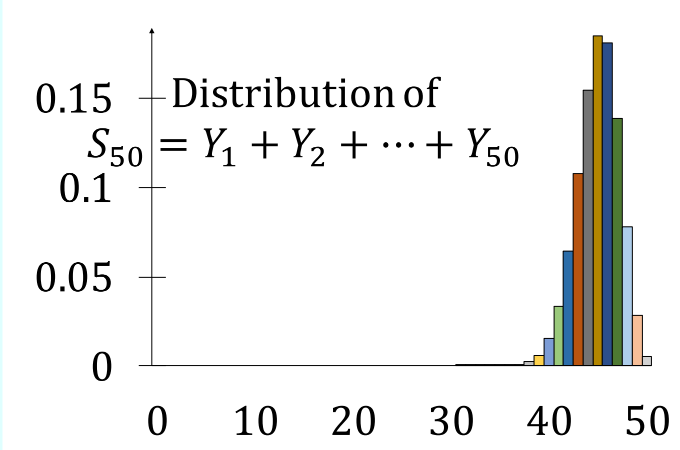
(... *Example continued from the previous page.*)

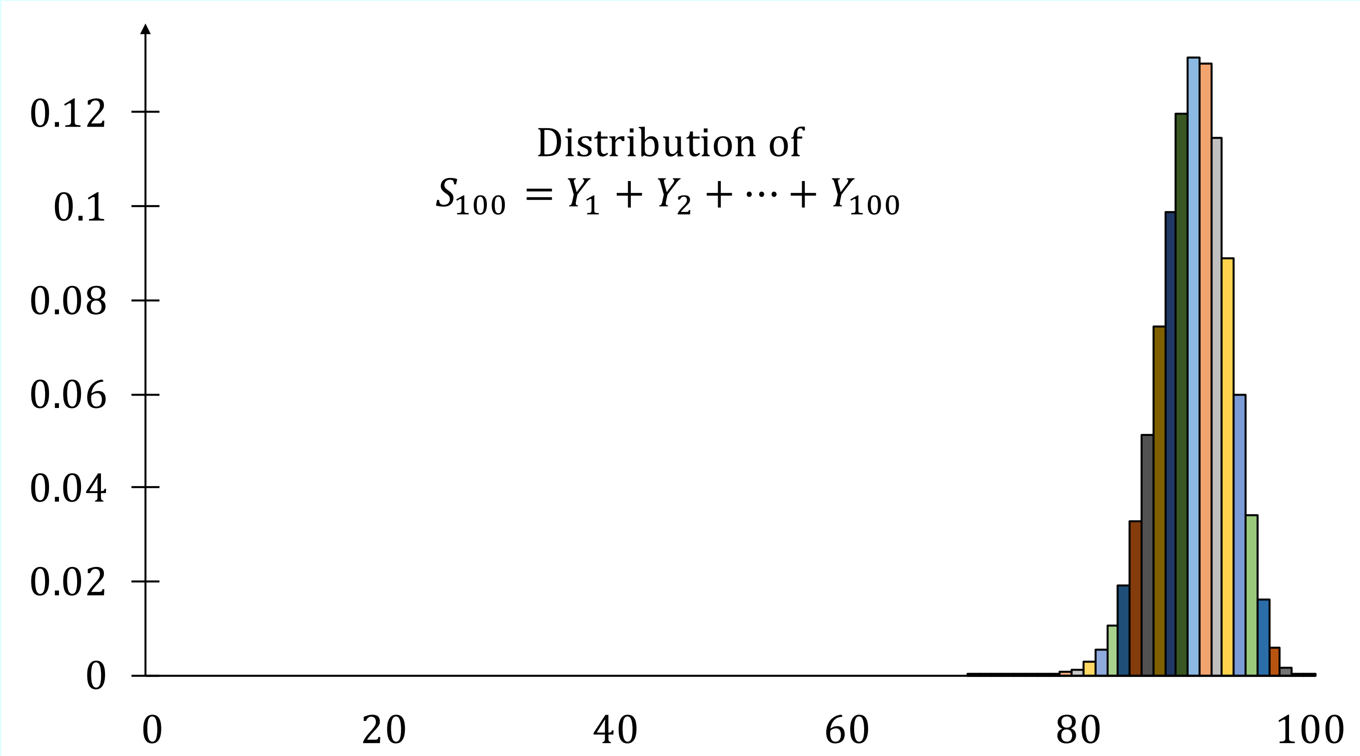
Below are the histograms of the distributions of S_{20}, S_{30}, S_{40}, S_{50}, and S_{100}. Remarkably enough, as n grows, the shape of the probability distribution of S_n looks ever more bell-shaped. As promised by the CLT.











128.3. Why Are So Many Things Normally Distributed?

We now return to the question posed earlier:

Many things in nature seem to be normally distributed. *Why*?

This is a deep question. The standard quick answer is this:

If S is the sum of a very large number of independent random variables, then by the CLT S is (approximately) normally distributed.

Examples to illustrate:

Example 1530. Assume that human height is entirely determined by 1000 independent genes (assume all human beings have these 1000 genes).

Assume that each of these 1000 genes is associated with an independent random variable $X_1, X_2, \ldots, X_{1000}$, each identically distributed with mean μ_X and variance σ_X^2. Assume also that human height is simply equal to the sum of these random variables. That is, a human being's height is simply given by $H = X_1 + X_2 + \cdots + X_{1000}$.

Then the CLT says that since $n = 1000$ is "large enough", H will be approximately normally distributed, with mean $1000\mu_X$ and variance $1000\sigma_X^2$. Amongst the world's 7.4 billion people, there will be some very short people and some very tall people, but most people will be near the mean height $1000\mu_X$.

Example 1531. Ah Kow's Mooncake Factory manufactures mooncakes. Each mooncake is supposed to weigh exactly 185 g, if the standard recipe is followed with absolute precision.

However, the exact weight of each mooncake will usually vary, due to myriad factors, such as whether the baker was paying attention, how much water the baker added, how long the mooncake was left in the oven, the room temperature that day, etc.

Assume there are 300 independent factors that determine the exact weight of a mooncake. Assume that each of these 300 factors is associated with an independent random variable $Y_1, Y_2, \ldots, Y_{300}$, each identically distributed with mean μ_Y and variance σ_Y^2. Assume also that the weight of a mooncake is simply given by $W = Y_1 + Y_2 + \cdots + Y_{300}$.

Then the CLT says that since $n = 300$ is "large enough", W will be approximately normally distributed, with mean $300\mu_Y$ and variance $300\sigma_Y^2$. Amongst the millions of mooncakes produced, there will be some very light mooncakes and some very heavy mooncakes, but most mooncakes will be near the mean weight $300\mu_Y$.

128.4. Don't Assume That Everything is Normal

Mathematical modellers often assume that "everything is normal". There are three justifications for this:

1. We have strong empirical evidence that many things in nature are normally distributed.
2. We have a strong theoretical reason (the CLT) for why this might be so.
3. The normal distribution is easy to handle (because e.g. the maths is easy, compared to some other distributions).

However, many things are **not** normally distributed. It is thus a mistake to assume that "everything is normal".

One example of a common but non-normal distribution found in nature is the **Pareto distribution**. We'll skip the formal details. Informally, it is called the Pareto Principle or the 80-20 Rule and businesspersons say things like:

"80% of your output is produced by 20% of your employees."

"80% of your sales come from 20% of your clients."

It is believed the Pareto distribution is a good description of many aspects of human performance (though apparently not of height or IQ). By the way, it was named after Vilfredo Pareto, who in 1896 found that approximately 80% of the land in Italy was owned by 20% of the population.

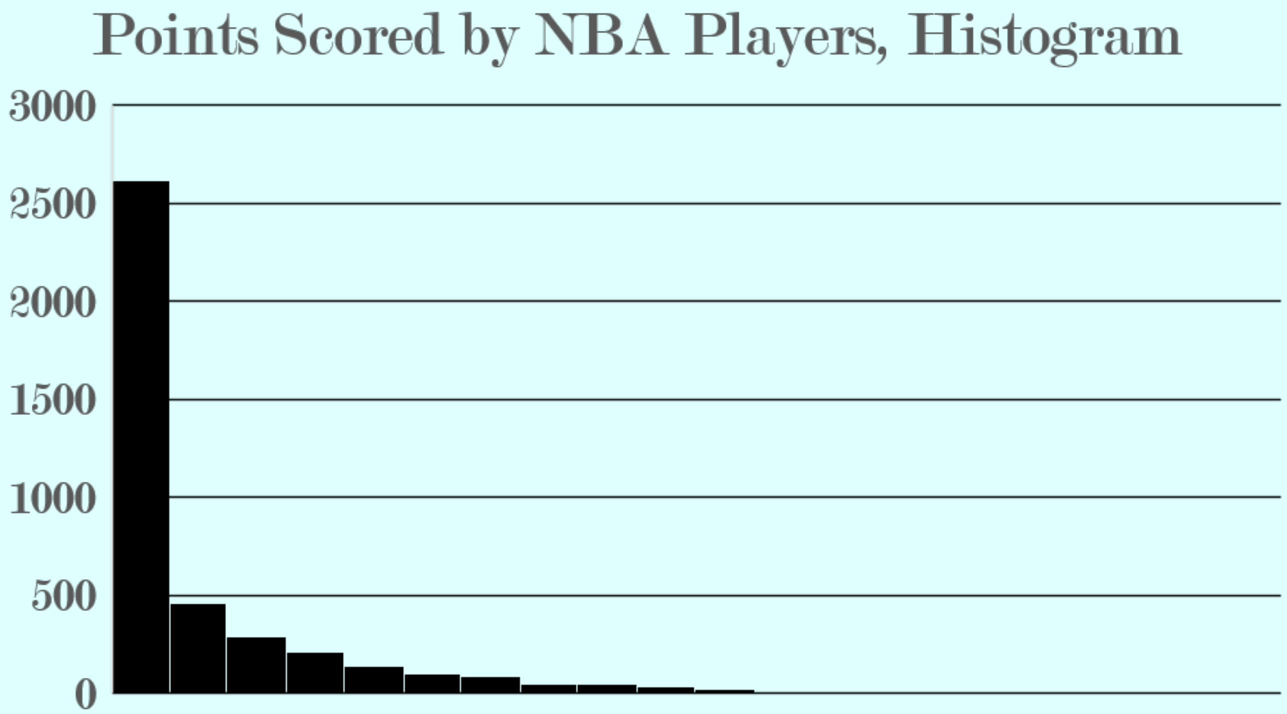
Let's see if the points scored in the NBA resembles the Pareto distribution.[537] In particular, is it the case that 20% of NBA players have scored 80% of the points?

[537]Source: Basketball-Reference.com. Caveats: (1) The data were retrieved on June 15th, 2016, so the points scored are between 1946 and that date. (2) By NBA, I actually mean the BAA (1946-1949), the NBA (1949-present), and the ABA (1967-1976), combined. Data: Excel spreadsheet.

Example 1532. Below is a **histogram** of the total points scored by each of 4,060 NBA players. Clearly, the total points scored by each player is not normally distributed.

The histogram has 20 bins of equal width. The leftmost bin says that $2,615$ players scored 0 to 1919 points. The rightmost bin says that only 2 players scored $36,468$ to $38,387$ points.

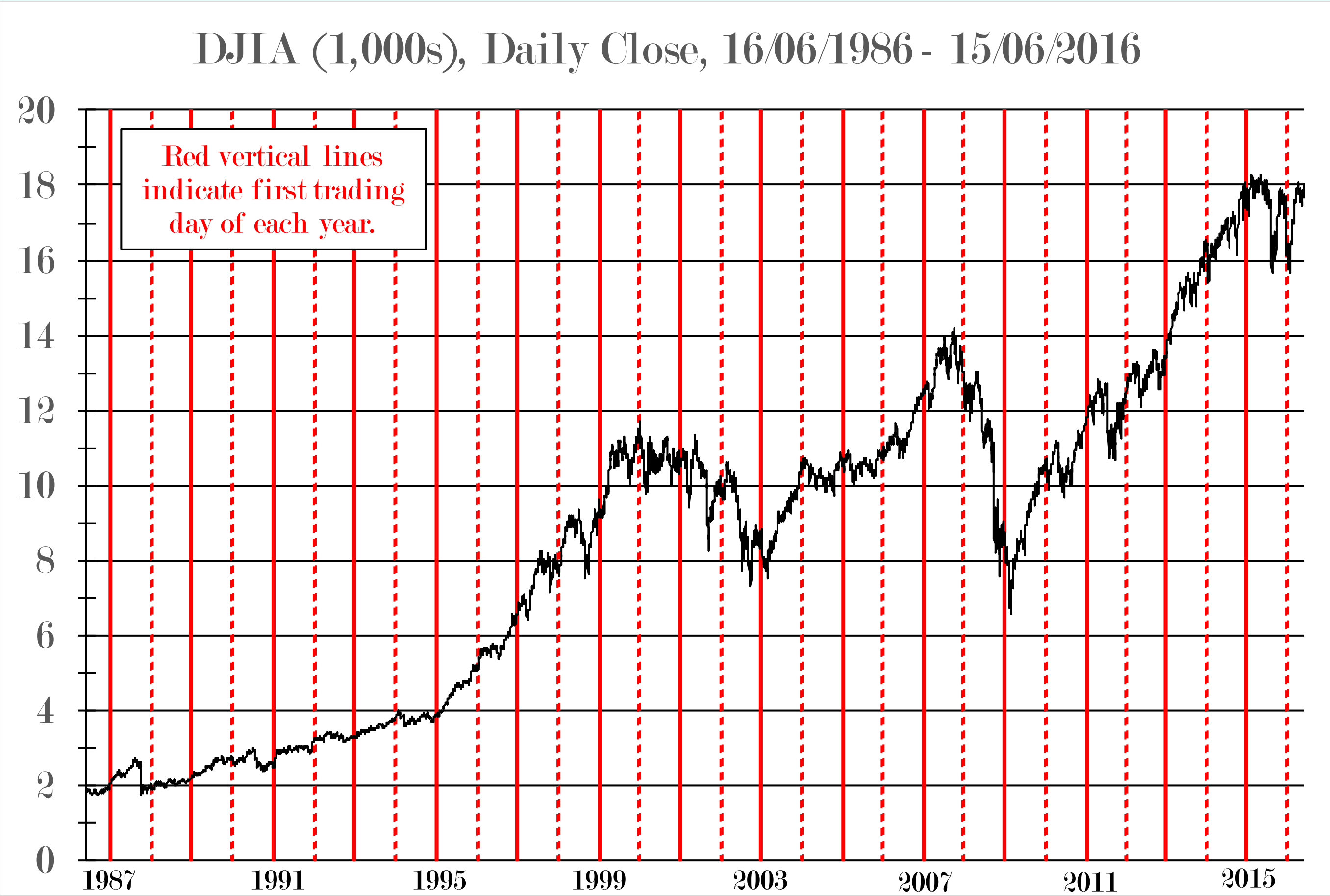
The grand total number of points ever scored in the NBA is $11,565,923$. Of which, $8,424,242$ (or 72.8%) were scored by the top 20% (812). So it appears that the 80-20 Rule is a reasonably good description of the distribution of total points scored by players! In contrast, the normal distribution is obviously not a good description.



It's fairly obvious to anyone who bothers graphing the data that "points scored in the NBA" is not normally distributed. There are however instances where this is less obvious. One is thus more likely to mistakenly assuming a normal distribution. A famous and tragic example of this is given by the financial markets.

Example 1533. The Dow Jones Industrial Average (DJIA) is one of the world's leading stock market indices. It is a weighted average of the share prices of 30 of the largest US companies (e.g. Apple, Coca-Cola, McDonald's).

Trading starts in the morning and closes in the afternoon (right now, the trading hours are 9:30 am to 4 pm). The next graph is of the daily closing values for the past 30 years.



Let q_i be the % change in closing value on day i, as compared to day $i-1$. For example, on June 14th, 2016, the DJIA closed at $17,674.82$. On June 15th, 2016, it closed at $17,640.17$, 34.65 points lower than the previous day's close. Thus,

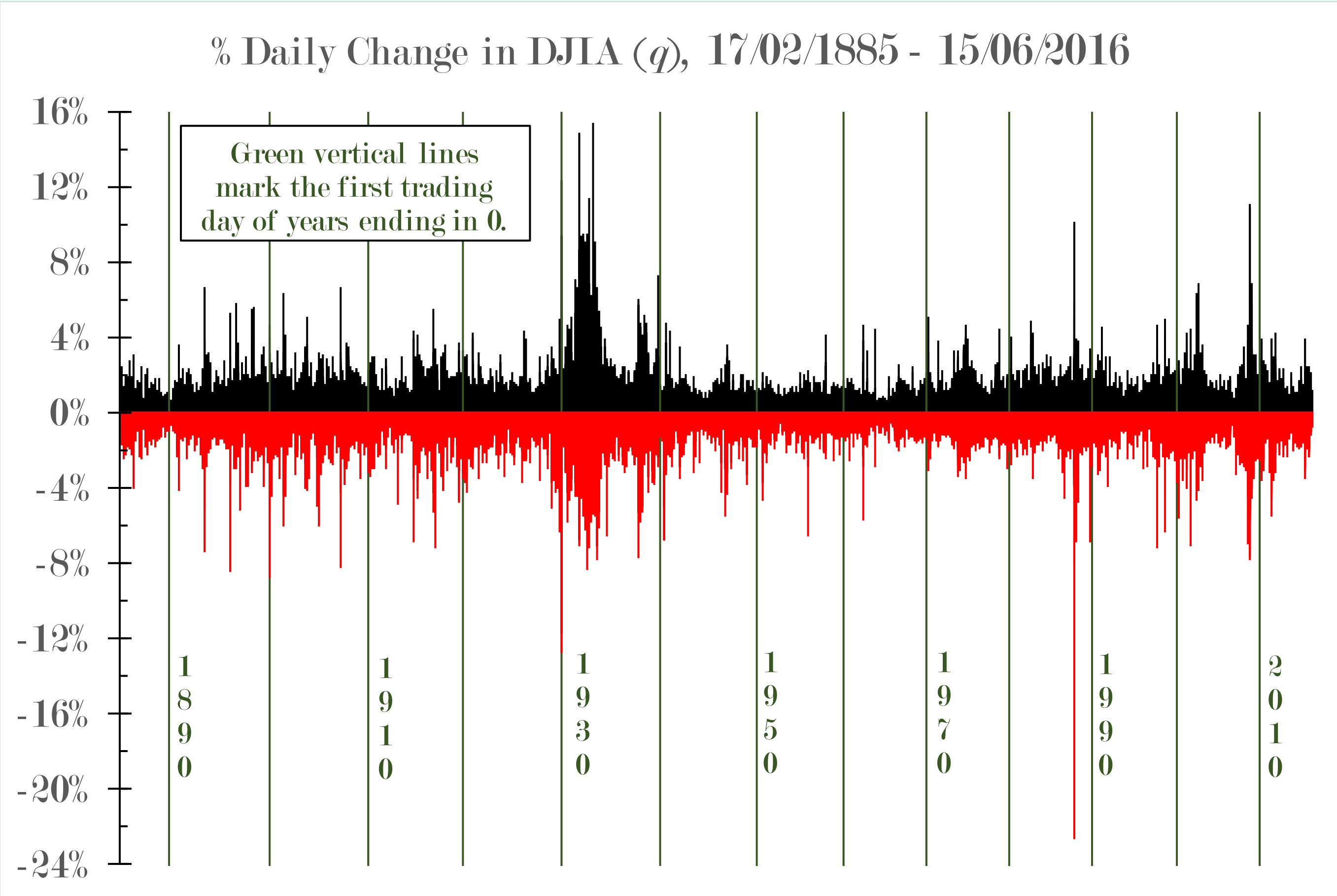
$$q_{20160615} = \frac{-34.65}{17,674.82} \approx -0.20\%.$$

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

The graph here is of q, on 36,044 consecutive trading days (over 131 years). **In black are those days when the DJIA rose**; in red are when it fell.

Can you spot the single largest one-day fall in the DJIA? (We'll talk about this singular day shortly.)

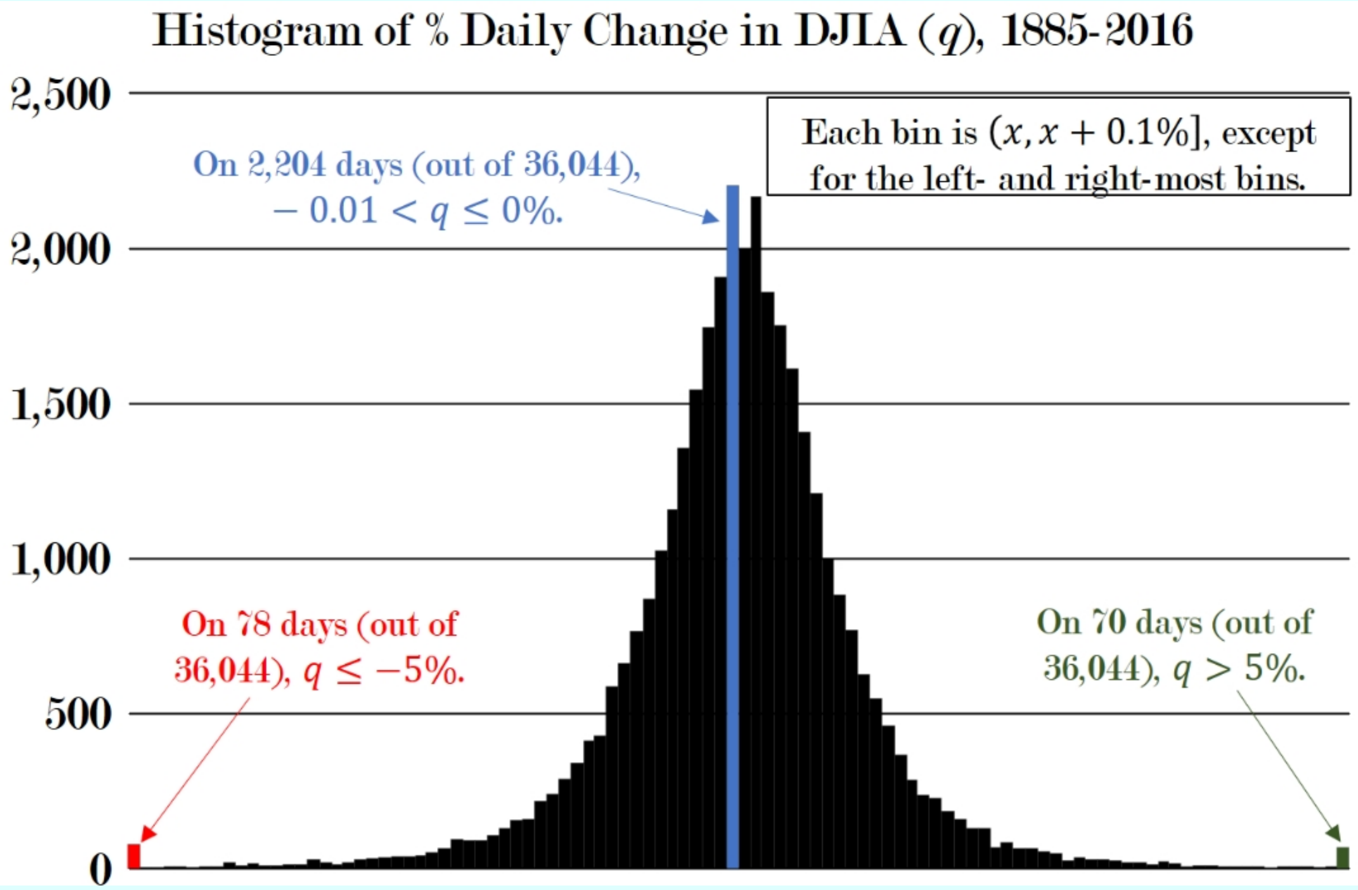


(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

The graph here is *also* of q, but in the form of a histogram. Each bin has width 0.1% (except the leftmost and rightmost bins). For example, on 2,204 days (out of 36,044), $q \in (-0.1\%, 0\%]$ (the DJIA fell by between 0.1% and 0%).

On 78 days, $q \leq -5\%$ (the DJIA fell by more than 5%). On 70 days, $q > 5\%$ (the DJIA rose by more than 5%).



It seems reasonable to say that q is normally distributed (at least if we ignore the leftmost and rightmost bins).

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

The sample mean and standard deviation (from the $36,044$ observations) are $\mu \approx 0.023\%$ and $\sigma \approx 1.064\%$. So let's suppose q were normally distributed with mean μ and variance σ^2.

If so, then we'd predict (as per the properties of the normal distribution) that

1. 0.6827 of the time, q is within 1 standard deviation of the mean, i.e. $q \in (-1.04\%, 1.09\%)$.
2. 0.954 of the time, q is within 2 standard deviations of the mean, i.e. $q \in (-2.10\%, 2.15\%)$.

As it actually turned out,

1. 0.7965 of the time ($28,709$ out of $36,044$ days), q was within 1 standard deviation of the mean. (A bit off, but not too bad.)
2. 0.9536 of the time ($34,373$ out of $36,044$ days), q was within 2 standard deviations of the mean. (Almost exactly correct!)

In addition to the above "evidence", we might make the following theoretical argument: Share prices are affected by a myriad random and arguably independent factors. Hence, by the CLT, we'd expect share prices (and thus q as well) to be normally distributed.

Unfortunately, modelling q as a normal random variable would be a disastrous mistake, especially when it comes to predicting rare events. If q is indeed normally distributed, then we'd predict that the DJIA rises or falls by more than ...

1. ... 5% less than once every $2,000$ years.
2. ... 7% less than once every 100 million years.
3. ... 10% less than once every $480,000,000$ billion years. (For comparison, the universe is estimated to be 13.8 billion years old.)

And so, during the 131 years for which we have data, it should have been very unlikely that the DJIA ever rose or fell by more than 5%.

But as it actually turned out, during these 131 years, the DJIA rose or fell by more than ...

1. ... 5% 148 times.
2. ... 7% 40 times.
3. ... 10% 10 times.

(Data: Excel spreadsheet.)

129. Statistics: Introduction (Optional)

129.1. Probability vs. Statistics

Probability	Statistics
Given a known model, what can we say about the data we'll observe?	*Given observed data, what can we say about the model?*

Example 1534. Let p be the probability of a coin-flip turning up heads. (p is an example of a **parameter**.) Flip a coin thrice.

Probability question: "Suppose we know that $p = 1/2$. Then what can we say about the probability of observing three heads (i.e. $\mathrm{P}(HHH)$)?" (For most probabilists, this is a simple question with a straightforward answer: $\mathrm{P}(HHH) = 1/8$.)

Statistics question: "Suppose we observe HHH. Then what can we say about p?" (Different statisticians will give different answers.)

In the real world, we will almost never know what p "truly" is. Instead, we usually only have some limited data observations (such as observing HHH).

Probability is about making heroic assumptions about what p is, in order to draw inferences about what the observed data will look like.

In contrast, statistics is about using limited, observed data to draw **statistical inferences** about the model and its parameters.

129.2. Objectivists vs Subjectivists

Example 1535. Ann and Bob are two infinitely intelligent persons. Ann believes that the probability of rain tomorrow is 0.2 and Bob believes that it is 0.6.

- **Objectivist** view: There is some single, "correct" probability p of rain tomorrow. Perhaps no one (except some Supreme Being up above) will ever know what exactly p is. But in any case, we can say that exactly one of the following must be true:

1. Ann is correct (and Bob is wrong);
2. Bob is correct (and Ann is wrong); or
3. Both Ann and Bob are wrong.

- **Subjectivist** view: A probability is not some objective, rational thing that exists outside the mind of any human being. There is no "correct" probability. Instead, a probability is merely

> the *degree of belief* in the occurrence of an event attributed by a given person at a given instant and with a given set of information. (De Finetti, infra, pp. 3-4.)

Thus, Ann and Bob can legitimately disagree about the probability of rain tomorrow, without either being wrong. After all, the numbers 0.2 and 0.6 are merely their personal, subjective degrees of belief in the likelihood of rain tomorrow.

Bruno de Finetti(1906–1985) was perhaps the most famous and extreme **subjectivist** ever. In the preface to his *Theory of Probability* (1970)[538], he wrote:

> *My thesis, paradoxically, and a little provocatively, but nonetheless genuinely, is simply this:*
>
> *PROBABILITY DOES NOT EXIST*
>
> *The abandonment of superstitious beliefs about the existence of Phlogiston, the Cosmic Ether, Absolute Space and Time, ..., or Fairies and Witches was an essential step along the road to scientific thinking. Probability, too, if regarded as something endowed with some kind of objective existence, is no less a misleading misconception, an illusory attempt to exteriorize or materialize our true probabilistic beliefs.*

In this textbook (and also in H2 Maths), we will be strict objectivists. The main practical implications of being an objectivist are illustrated in the following examples:

[538] Originally published in 1970 in Italian as *Teoria delle probabilità*. The link is to a recent 2017 English edition.

Example 1536. Judge Ann says the murder suspect is probably innocent. Judge Bob says the suspect is probably guilty.

Objectivist interpretation: Ann and Bob cannot both be correct. The suspect is either innocent (with probability 1) or guilty (with probability 1).

In fact, we can go even further and say that both Ann and Bob are talking nonsense. It is nonsensical to say things like the suspect is "probably" innocent (or "probably" guilty), because the suspect either is innocent or not.

Subjectivist interpretation: Ann and Bob are perfectly well-entitled to their beliefs.

Moreover, it is perfectly meaningful to say things like the suspect is "probably" innocent (or "probably" guilty). Ann and Bob do not know for sure whether the suspect is innocent or guilty. They are thereby perfectly well-entitled to speak probabilistically about the innocence or guilt of the suspect.

Example 1537. We flip a coin 100 times and get 100 heads.

Given these observed data (100 heads out of 100 flips), what can we say (what statistical inference can we make) about whether or not the coin is fair?

Subjectivist answer: The coin is probably not fair. (This is perhaps the answer that most laypersons would give.)

Objectivist answer: The coin either is fair (with probability 1) or isn't fair (with probability 1). Subjectivist statements like the coin is "probably" not fair are nonsensical.

Most untrained laypersons are innately subjectivist. Yet in this book (and also for the A-Levels), you'll be trained to think like strict objectivists.

Note though that it is **not** the case that one school of thought is correct and the other wrong. Both the objectivist and subjectivist schools of thought have merit. The growing consensus amongst statisticians is to take the best of both worlds.

Nonetheless, in this textbook, we learn only the objectivist interpretation. Not because it is necessarily superior, but rather because

1. The maths is easier.
2. Tradition: For most of the 20th century, the objectivist interpretation was favoured.

130. Sampling

130.1. Population

Definition 256. A *population* is any ordered set (i.e. vector) of objects we're interested in.

A population can be finite or infinite. But to keep things simple, we'll look at examples where it is finite.

Example 1538. The two candidates for the 2016 Bukit Batok SMC By-Election are Dr. Chee Soon Juan and PAP Guy. It is the night of the election and voting has just closed.

Our objects-of-interest are the $23,570$ valid ballots cast. (A **ballot** is simply a piece of paper on which a vote is recorded. The words *ballot* and *vote* are often used interchangeably.)

Arrange the ballots in any arbitrary order. Let $v_1 = 1$ if the first ballot is in favour of Dr. Chee and $v_1 = 0$ otherwise. Similarly and more generally, for any $i = 2, 3, \ldots, 23570$, let $v_i = 1$ if the ith ballot is in favour of Dr. Chee and $v_1 = 0$ otherwise.

Our population here is simply the ordered set $P = (v_1, v_2, \ldots, v_{23570})$. So in this example, the population is simply an ordered set of 1s and 0s.

130.2. Population Mean and Population Variance

The **population mean** μ is simply the average across all population values. The **population variance** σ^2 is a measure of the variation across all population values. Formally:[539]

Definition 257. Given a finite population $P = (v_1, v_2, \ldots, v_k)$, the population mean μ and population variance σ^2 are defined by

$$\mu = \frac{\sum_{i=1}^{k} v_i}{k} = \frac{v_1 + v_2 + \cdots + v_k}{k} \quad \text{and} \quad \sigma^2 = \frac{\sum_{i=1}^{k} (v_i - \mu)^2}{k} = \frac{(v_1 - \mu)^2 + (v_2 - \mu)^2 + \cdots + (v_k - \mu)^2}{k}.$$

Example 1300 (continued from above). Suppose that of the 23,570 votes, 9,142 were for Dr. Chee and the remaining against. So the vector $(v_1, v_2, \ldots, v_{23570})$ contains 9,142 1s and 14,428 0s.

Then the population mean is

$$\mu = \frac{v_1 + v_2 + \cdots + v_n}{n} = \frac{9142 \times 1 + 14428 \times 0}{23570} = \frac{9142}{23570} \approx 0.3879.$$

In this particular example, the population values are binary (either 0 or 1). And so we have a nice alternative interpretation: the population mean is also the **population proportion**. In this case, it is the proportion of the population who voted for Dr. Chee. So here the proportion of votes for Dr. Chee is about 0.3879.

The population variance is

$$\sigma^2 = \frac{(v_1 - \mu)^2 + (v_2 - \mu)^2 + \cdots + (v_n - \mu)^2}{n} = \frac{9142 \cdot \left(1 - \frac{9142}{23570}\right)^2 + 14428 \cdot \left(0 - \frac{9142}{23570}\right)^2}{23570} \approx 0.2374.$$

As usual, the **variance** tells us about the degree to which the v_i's **vary**. Of course, in this example, we already know that the v_i's can take on only two values—0 and 1. So the variance isn't terribly interesting or informative in this example. In particular, it doesn't tell us anything more that the population mean didn't already tell us (indeed, it can be shown that in this example, $\sigma^2 = \mu - \mu^2$).

[539]In the case of an infinite population, the definitions of μ and σ^2 must be adjusted slightly, but the intuition is the same.

130.3. Parameter

Informally, a **parameter** is some number we're interested in and which may be calculated based on the population.

Example 1300 (continued from above). A **parameter** we might be interested in is the population mean μ—this is also the proportion of votes in favour of Dr. Chee. (Another parameter we might be interested in is the population variance σ^2, but let's ignore that for now.)

Voting has just closed. In a few hours' time (after the vote-counting is done), we will know what exactly μ is. But right now, we still don't know what μ is.

Suppose we are impatient and want to know right away what μ might be. In other words, suppose we want to get an **estimate** of the true value of μ. What are some possible methods of getting a quick estimate of μ?

One possibility is to **observe a random sample** of 100 votes and count the proportion of these 100 votes that are in favour of Dr. Chee. So for example, say we do this and observe that 39 out of the 100 votes are for Dr. Chee. That is, we find that **the observed sample mean** (which in this context can also be called **the observed sample proportion**) is 0.39. Then we might conclude:

> Based on this observed random sample of 100 votes, we estimate that μ is 0.39.

The layperson might be content with this. But the statistician digs a little deeper and asks questions such as:

- How do we know if this estimate is "good"?
- What are the criteria to determine whether an estimate is "good"?

We'll now try to address, if only to a limited extent, these questions. But to do so, we must first precisely define terms like **sample** and **estimate**.

130.4. Distribution of a Population

Informally,[540] the distribution of a population tells us

1. The range of possible values taken on by the objects in the population; and
2. The proportion of the population that takes on each possible value.

Example 1300 (continued from above). The population is $P = (v_1, v_2, \ldots, v_{23570})$, the ordered set of 23570 ballots. Suppose that of these, $9,142$ are votes for Dr. Chee (hence recorded as 1s) and the remaining $14,428$ are for PAP Guy (hence recorded as 0s).

Then the distribution of the population can informally be described in words as:

- A proportion 9142/23570 of the population are 1s, and
- A proportion 14428/23570 of the population are 0s.

Example 1539. The population is $P = (3, 4, 7, 7, 2, 3)$.

Then the distribution of the population can informally be described in words as:

- A proportion 1/6 of the population are 2s;
- A proportion 2/6 of the population are 3s;
- A proportion 1/6 of the population are 4s; and
- A proportion 2/6 of the population are 7s.

[540] Formally, we'd define the population distribution as a function. Indeed, some writers define the population itself as the distribution function.

130.5. A Random Sample

Informally, to observe a random sample of size n, we follow this procedure: Imagine the $23,570$ ballots are in a single big bag.

1. Randomly pull out one ballot. Record the vote (either we write $x_1 = 1$, if the vote was for Dr. Chee, or we write $x_1 = 0$, if it wasn't).
2. Put this ballot back in (this second step is why we call it **sampling with replacement**).
3. Repeat the above n times in total, so as to record down the values of x_1, x_2, ..., x_n.

We call $(x_1, x_2, \ldots, x_n)$ an **observed random sample of size** n. Note that this is an ordered set (or vector) of numbers. Formally:

Definition 258. Let P be a population. Then the random vector (i.e. ordered set of random variables) $(X_1, X_2, \ldots, X_n)$ is a *random sample of size n from the population P* if

- $X_1, X_2, \ldots, X_n$ are independent; and
- $X_1, X_2, \ldots, X_n$ are identically distributed, with the same distribution as P.

As always, we must be careful to distinguish between a function and a value taken on by the function. This table summarises.

Function	Value taken by the function
f is a function	$f(x)$ is a possible value taken on by the function
X is a random variable	x is a possible observed value of the random variable
$(X_1, X_2, \ldots, X_n)$ is a random sample	$(x_1, x_2, \ldots, x_n)$ is a possible observed random sample

An example to illustrate:

Example 1300 (continued from above). To repeat, the distribution of the population $P = (v_1, v_2, \dots, v_{23570})$ can informally be described in words as:

- 9142/23570 of the population were 1s; and
- 14428/23570 of the population were 0s.

Let X_1, X_2, and X_3 be independent random variables, each with the same distribution as the population. That is, for each $i = 1, 2, 3$,

$$\mathrm{P}(X_i = 0) = \frac{14428}{23570} \quad \text{and} \quad \mathrm{P}(X_i = 1) = \frac{9142}{23570}.$$

The ordered set (or vector) (X_1, X_2, X_3) is **a random sample of size** 3.

An example of **an observed random sample of size** 3 might be $(x_1, x_2, x_3) = (1, 1, 0)$—this would be where we randomly sample 3 ballots (with replacement) and find that the first two are votes for Dr. Chee but the third is not.

Another example of an observed random sample of size 3 might be $(x_1, x_2, x_3) = (0, 0, 0)$—this would be where we randomly sample 3 ballots (with replacement) and find that none of the three are for Dr. Chee.

As another example, $(X_1, X_2, X_3, X_4, X_5)$ is **a random sample of size** 5.

An example of **an observed random sample of size** 5 might be $(x_1, x_2, x_3, x_4, x_5) = (0, 1, 0, 1, 0)$—this would be where we randomly sample 5 ballots (with replacement) and find that only the second and fourth are votes for Dr. Chee.

Another example of an observed random sample of size 5 might be $(x_1, x_2, x_3, x_4, x_5) = (1, 1, 0, 1, 1)$—this would be where we randomly sample 5 ballots (with replacement) and find that only the third is not a vote for Dr. Chee.

In this textbook, we'll be very careful to distinguish between a random sample (which is a vector of random variables) and an observed random sample (which is a vector of real numbers).

This may be contrary to the practice of your teachers or indeed even the A-Level exams.

130.6. Sample Mean and Sample Variance

Definition 259. Let $(X_1, X_2, \ldots, X_n)$ be a random sample of size n. Then the corresponding *sample mean* $\bar{X}$ and the *sample variance* S^2 are the random variables defined by

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n},$$

$$S^2 = \frac{\left(X_1 - \bar{X}\right)^2 + \left(X_2 - \bar{X}\right)^2 + \cdots + \left(X_n - \bar{X}\right)^2}{n-1} = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{n-1}.$$

(The List of Formulae (MF26) will contain the observed sample variance.)

Note that strangely enough, the denominator of S^2 is $n-1$, rather than n as one might expect. As we'll see later, there is a good reason for this.

By the way, there are two other formulae for calculating the sample variance:

Fact 244. *Let $S = (X_1, X_2, \ldots, X_n)$ be a random sample of size n. Let $\bar{X}$ be the sample mean and S^2 be the sample variance. Let $a \in \mathbb{R}$ be a constant. Then*

$$\textbf{(a)}\ S^2 = \frac{\sum_{i=1}^{n} X_i^2 - \frac{\left[\sum_{i=1}^{n} X_i\right]^2}{n}}{n-1} \quad \text{and} \quad (b)\ S^2 = \frac{\sum_{i=1}^{n}\left(X_i - a\right)^2 - \frac{\left[\sum_{i=1}^{n}(X_i - a)\right]^2}{n}}{n-1}.$$

(The List of Formulae (MF26) has **a** but not **b**.)

Proof. Optional, see p. **1738** (Appendices). □

Once again, it is important to distinguish between

- The **sample mean** $\bar{X}$ (a random variable) vs. the **observed sample mean** $\bar{x}$ (a real number).
- The **sample variance** S^2 (a random variable) vs. the **observed sample variance** s^2 (a real number).

Example 1300 (continued from above). Let (X_1, X_2, X_3) be a random sample of size 3. The corresponding sample mean $\bar{X}$ and sample variance S^2 are these **random variables**:

$$\bar{X} = \frac{X_1 + X_2 + X_3}{3}, \quad S^2 = \frac{\left(X_1 - \bar{X}\right)^2 + \left(X_2 - \bar{X}\right)^2 + \left(X_3 - \bar{X}\right)^2}{3 - 1}.$$

Suppose our observed random sample of size 3 is $(1, 0, 0)$. Then the corresponding **observed sample mean** $\bar{x}$ and **observed sample variance** s^2 are these **real numbers**:

$$\bar{x} = \frac{x_1 + x_2 + x_3}{n} = \frac{1 + 0 + 0}{3} = \frac{1}{3},$$

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{n - 1} = \frac{\left(1 - \frac{1}{3}\right)^2 + \left(0 - \frac{1}{3}\right)^2 + \left(0 - \frac{1}{3}\right)^2}{3 - 1} = \frac{1}{3}.$$

Let $(X_1, X_2, X_3, X_4, X_5)$ be a random sample of size 5. The corresponding sample mean $\bar{X}$ and sample variance S^2 are these **random variables:**

$$\bar{X} = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{5}, \quad S^2 = \frac{\left(X_1 - \bar{X}\right)^2 + \left(X_2 - \bar{X}\right)^2 + \cdots + \left(X_5 - \bar{X}\right)^2}{5 - 1}.$$

Suppose our observed random sample of size 5 is $(0, 1, 0, 0, 1)$. Then the corresponding **observed sample mean** $\bar{x}$ and **observed sample variance** s^2 are these **real numbers**:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{n} = \frac{0 + 1 + 0 + 0 + 1}{5} = \frac{2}{5} = 0.4,$$

$$\begin{aligned} s^2 &= \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + (x_4 - \bar{x})^2 + (x_5 - \bar{x})^2}{n - 1} \\ &= \frac{\left(0 - \frac{1}{5}\right)^2 + \left(1 - \frac{1}{5}\right)^2 + \left(0 - \frac{1}{5}\right)^2 + \left(0 - \frac{1}{5}\right)^2 + \left(1 - \frac{1}{5}\right)^2}{5 - 1} = 0.35. \end{aligned}$$

We call a random variable an **estimator** if it is used to generate **estimates** ("guesses") for some parameter. Example:

Example 1300 (continued from above). It is the night of the election and polling has just closed. We still do **not** know the true proportion μ that voted for Dr. Chee.

We decide to get a random sample of size 3: (X_1, X_2, X_3). The corresponding sample mean $\bar{X}_3 = (X_1 + X_2 + X_3)/3$ shall be an **estimator** for μ. (Informally, an **estimator** is a method for generating "guesses" for some unknown parameter, in this case μ.)

This **estimator** is used to generate **estimates** ("guesses") for μ. For every observed random sample, the estimator generates an estimate.

Suppose our observed random sample of size 3 is $(1, 0, 0)$. We calculate the corresponding observed sample mean to be $\bar{x} = 1/3$. We say that $\bar{x} = 1/3$ is an **estimate** for μ.

(By the way, unless we are extremely lucky, it is highly unlikely that the true value of the unknown parameter μ is precisely $1/3$. After all, $1/3$ is merely an estimate obtained from a single observed random sample of size 3.)

Suppose instead that our observed random sample of size 3 were $(0, 1, 1)$. Then the corresponding observed sample mean would be $\bar{x} = 2/3$. We'd instead say that $\bar{x} = 2/3$ is our **estimate** for μ.

There is also more than one estimator we can use. For example, suppose instead that we decide to get a random sample of size 5: $(X_1, X_2, X_3, X_4, X_5)$. We shall instead use the corresponding sample mean $\bar{X} = (X_1 + X_2 + X_3 + X_4 + X_5)/5$ as our **estimator** for μ. And so for example suppose our observed random sample of size 5 is is $(0, 1, 0, 0, 1)$. Then the corresponding observed sample mean $\bar{x} = 0.4$ and $\bar{x} = 0.4$ would be our **estimate** for μ.

Now, are these **estimators** and **estimates** "good" or "reliable"? How much should we trust them? These are questions that we'll address in the next section.

A different example:

Example 1540. Suppose we wish to find the average height μ (in cm) of an adult male.

As a practical matter, it would be quite difficult to locate and record the height of every adult male in the world. So instead, what we might do is to randomly pick 4 adult males and record their heights. This gives us a random sample (H_1, H_2, H_3, H_4) of heights. The corresponding sample mean is the random variable $\bar{H} = (H_1 + H_2 + H_3 + H_4)/4$. $\bar{H}$ shall serve as our **estimator** for μ.

Suppose our observed random sample is $(h_1, h_2, h_3, h_4) = (178, 165, 182, 175)$.

Then the corresponding observed sample mean is

$$\bar{h} = \frac{h_1 + h_2 + h_3 + h_4}{n} = \frac{178 + 165 + 182 + 175}{4} = 175.$$

Thus, $\bar{h} = 175$ serves as an estimate (or "guess") of the true average male height μ.

Again, are the **estimator** $\bar{H}$ and **estimate** $\bar{h} = 175$ "good" or "reliable"? How much should we trust them? These are questions that we'll address in the next section.

Example 1541. Let X be the random variable that is the height (in cm) of an adult female Singaporean. Our parameters-of-interest are the true population mean μ and true population variance σ^2 of X. We wish to generate estimates for μ and σ^2.

To this end, we get a random sample of size 8: $(X_1, X_2, \ldots, X_8)$. The corresponding sample mean $\bar{X} = (X_1 + X_2 + \cdots + X_8)/8$ will serve as our **estimator** for μ. And the corresponding sample variance $S^2 = \sum_{i=1}^{8} \left(X_i - \bar{X}\right)^2 / (8-1)$ will serve as our **estimator** for σ^2.

(a) Suppose our observed random sample is such that

$$\sum_{i=1}^{8} x_i = 1,320 \quad \text{and} \quad \sum_{i=1}^{8} x_i^2 = 218,360.$$

Then the observed sample mean $\bar{x}$ and the observed sample variance s^2 are

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{1320}{8} = 165,$$

$$s^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n-1} = \frac{218360 - \frac{1320^2}{8}}{7} = 80.$$

And our **estimates** for μ and σ^2 are, respectively, 165 cm and 80 cm^2.

(b) Suppose instead our observed random sample is such that

$$\sum_{i=1}^{8} (x_i - 160) = 72 \quad \text{and} \quad \sum_{i=1}^{8} (x_i - 160)^2 = 1,560.$$

Then the observed sample mean $\bar{x}$ and the observed sample variance s^2 are

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum_{i=1}^{n} (x_i - 160 + 160)}{n} = \frac{\sum_{i=1}^{n} (x_i - 160)}{n} + 160 = \frac{72}{8} + 160 = 169,$$

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - 160)^2 - \frac{\left[\sum_{i=1}^{n} (x_i - 160)\right]^2}{n}}{n-1} = \frac{1,560 - \frac{72^2}{8}}{7} \approx 130.3.$$

And our **estimates** for μ and σ^2 are, respectively, 169 cm and 130.3 cm^2.

Exercise 505. Calculate the observed sample mean and variance for the following observed random sample of size 7: $(3, 14, 2, 8, 8, 6, 0)$. (Answer on p. **1980**.)

Exercise 506. (Answer on p. **1980**.) Let X be the random variable that is the weight (in kg) of an American. Suppose we are interested in estimating the true population mean μ and variance σ^2 of X. We get an observed random sample of size 10: $(x_1, x_2, \ldots, x_{10})$.

(a) Suppose you are told that $\sum_{i=1}^{10} x_i = 1,885$ and $\sum_{i=1}^{10} x_i^2 = 378,265$. Find the observed sample mean $\bar{x}$ and observed sample variance s^2.

(b) Suppose you are instead told that $\sum_{i=1}^{10}(x_i - 50) = 1,885$ and $\sum_{i=1}^{10}(x_i - 50)^2 = 378,265$. Find the observed sample mean $\bar{x}$ and observed sample variance s^2.

130.7. Sample Mean and Sample Variance are Unbiased Estimators

Earlier we asked: How do we decide if an estimator and the estimates it generates are "good"? How do we know whether to trust any given estimate?

For H2 Maths, we'll learn only about one (important) criterion for deciding whether an estimator is "good". This is **unbiasedness**. Informally, an estimator is unbiased if on average, the estimator "gets it right". Formally:

Definition 260. Let X be a random variable and $\theta \in \mathbb{R}$ be a parameter (i.e. just some real number). We say that X is *an unbiased estimator* for θ if

$$\mathbf{E}[X] = \theta.$$

If x is an estimate generated by an unbiased estimator X, then we call x *an unbiased estimate.*

The next proposition says that **the sample mean $\bar{X}$ is an unbiased estimator for the population mean μ**; and **the sample variance S^2 is an unbiased estimator for the population variance σ^2**.

Proposition 24. *Let $(X_1, X_2, \ldots, X_n)$ be a random sample of size n drawn from a distribution with population mean μ and population variance σ^2. Let $\bar{X}$ be the sample mean and S^2 be the sample variance. Then*

(a) $E\left[\bar{X}\right] = \mu$. And

(b) $E\left[S^2\right] = \sigma^2$.

Proof. You are asked to prove (a) in Exercise 508. For the proof of (b), see p. **1739** (Appendices) (optional). □

Proposition 24(b) is the reason why, strangely enough, we define the sample variance with $n-1$ in the denominator:

$$S^2 = \frac{\left(X_1 - \bar{X}\right)^2 + \left(X_2 - \bar{X}\right)^2 + \cdots + \left(X_n - \bar{X}\right)^2}{n-1}.$$

As defined, S^2 is an unbiased estimator for the population variance σ^2. This, then, is the reason why we define it like this.

Some writers call S^2 the *unbiased* sample variance, but we shall not bother doing so. We'll simply call S^2 the sample variance.

Example 1538 (continued from above). (Chee Soon Juan election.)

Suppose two observed random samples of size 3 are $(x_1, x_2, x_3) = (1, 0, 0)$ and $(x_1, x_2, x_3) = (1, 0, 1)$. The corresponding observed sample means are $\bar{x}_1 = 1/3$ and $\bar{x}_2 = 2/3$. These are two possible estimates ("guesses") of the true sample proportion μ.

Unless we're extremely lucky, it's unlikely that either of these two estimates is *exactly* correct. Nonetheless, what the above unbiasedness proposition tells us is this:

Suppose the unknown population mean is $\mu = 0.39$. We draw the following 10 observed random samples of size 3 (table below). For each sample i, we calculate the corresponding observed sample mean $\bar{x}_i$.

Sample i	x_1	x_2	x_3	$\bar{x}_i$
1	1	0	1	2/3
2	0	0	0	0
3	0	1	0	2/3
4	1	0	0	1/3
5	0	1	1	2/3
6	1	0	0	1/3
7	0	0	0	0
8	0	0	0	0
9	0	0	1	1/3
10	1	1	0	2/3

Note that every estimate $\bar{x}_i$ is wrong. Indeed, since the sample mean $\bar{X}_i$ can only take on values 0, 1/3, 2/3, or 1, the estimates can never possibly be equal to the true $\mu = 0.39$.

Nonetheless, what the above proposition says informally is that *on average*, the estimate gets it correct. Formally, $\mathrm{E}\left[\bar{X}\right] = \mu = 0.39$.

For a demonstration that you can play around with, try this Google spreadsheet.

Exercise 507. (Answer on p. **1980**.) We are interested in the weight (in kg) of Singaporeans. We have an observed random sample of size 5: $(32, 88, 67, 75, 56)$.

(a) Find unbiased estimates for the population mean μ and variance σ^2 of the weights of Singaporeans. (State any assumptions you make.)

(b) What is the average weight of a Singaporean?

Exercise 508. Prove that $\mathrm{E}\left[\bar{X}\right] = \mu$. (This is part (a) of Proposition 24). (Answer on p. **1981**.)

Exercise 509. Suppose we flip a coin 10 times. The first 7 flips are heads and the next 3 are tails. Let 1 denote heads and 0 denote tails. (Answer on p. **1981**.)

(a) Write down, in formal notation, our observed random sample, the observed sample mean, and observed sample variance.

(b) Are these observed sample mean and variance unbiased estimates for the true population mean and variance?

(c) Can we conclude that this a biased coin (i.e. the true population mean is not 0.5)?

130.8. The Sample Mean is a Random Variable

This section is just to repeat, stress, and emphasise that the sample mean $\bar{X}$ is itself a random variable. This is an important point.

Indeed, the **sample mean** $\bar{X}$ is both (i) a **random variable**; and (ii) an **estimator**. In contrast, an **observed sample mean** $\bar{x}$ is both (i) a **real number**; and (ii) an **estimate**.

We've showed that $\mathrm{E}\left[\bar{X}\right] = \mu$. This equation can be interpreted in two equivalent ways:

- The expected value of the sample mean equals the population mean μ.
- The sample mean is an unbiased estimator for the population mean μ.

We now give the variance of the sample mean. It turns out to be equal to the population variance σ^2, divided by the sample size n.

Fact 245. $Var\left[\bar{X}\right] = \dfrac{\sigma^2}{n}$.

Proof. You are asked to prove this fact in Exercise 510 . □

Exercise 510. Prove Fact 245. (Hint: Note that $\bar{X} = \dfrac{1}{n}(X_1 + X_2 + \cdots + X_n)$ and X_1, X_2, …, X_n are independent.) (Answer on p. **1981**.)

Exercise 511. For each of the following terms, give a formal definition and an intuitive explanation. (State whether each term is a random variable or a real number.) For simplicity, you may assume that the finite population is given by $P = (x_1, x_2, \ldots, x_k)$. (Answer on p. **1982**.)

(a) The population mean.

(b) The population variance.

(c) The sample mean.

(d) The sample variance.

(e) The mean of the sample mean.

(f) The variance of the sample mean.

(g) The mean of the sample variance.

(h) The observed sample mean.

(i) The observed sample variance.

130.9. The Distribution of the Sample Mean

Fact 246. *Let* $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$ *be independent random variables. Then*

$$\bar{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right).$$

Proof. Corollary 50 tells us that the sum of normal random variables is itself a normal random variable. So $X_1 + X_2 + \cdots + X_n$ is a normal random variable.

Fact 242 tells us that a linear transformation of a normal random variable is itself a normal random variable. So $\bar{X}_n = (X_1 + X_2 + \cdots + X_n)/n$ is a normal random variable.

In the previous sections, we already showed that $\bar{X}_n$ has mean μ and variance σ^2/n.

Altogether then, $\bar{X}_n \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$. □

Now, suppose instead $X_1, X_2, \ldots, X_n$ are **not** normally distributed. Surprisingly, a similar result still holds, thanks to the CLT. Informally, draw $X_1, X_2, \ldots, X_n$ from *any* distribution. Then thanks to the CLT, it will still be the case that—provided n is "large enough"—$\bar{X}_n$ is (approximately) normally distributed. Formally:

Fact 247. *Let* $X_1, X_2, \ldots, X_n$ *be independent random variables, each identically distributed with mean* $\mu \in \mathbb{R}$ *and variance* $\sigma^2 \in \mathbb{R}$. *Let*

$$\bar{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

Then $\lim_{n\to\infty} \bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.

Proof. The CLT says that if n is "large enough", then $X_1 + X_2 + \cdots + X_n$ is well-approximated by the normal distribution $\mathrm{N}(n\mu, n\sigma^2)$.

And so it follows from Fact 242 (a linear transformation of a normal random variable is itself a normal random variable) that $\bar{X} = (X_1 + X_2 + \cdots + X_n)/n$ is well-approximated by the normal distribution $\mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$. □

In the next chapter, we'll make greater use of the two results given in this section.

130.10. Non-Random Samples

Some examples to illustrate the concept of a non-random sample:

Example 1542. Suppose we're interested in the average height of a Singaporean. The only way to know this for sure is to survey every single Singaporean. This, however, is not practical.

Instead, we have only the resources to survey 100 individuals. We decide to go to a basketball court and measure the heights of 100 people there. We thereby gather an observed sample of size 100: $(x_1, x_2, \dots, x_{100})$. We find that the average individual's height is $\bar{x} = \sum x_i / 100 = 179$ cm.

Is $\bar{x} = 179$ cm an unbiased estimate of the average Singaporean's height? Intuitively, we know that the answer is obviously no.

The reason is that our observed sample of size 100 was non-random. We picked a basketball court, where the individuals are overwhelmingly (i) male; and (ii) taller than average. Our estimate $\bar{x} = 179$ cm is thus probably biased upwards.

Example 1543. Suppose we're interested in what the average Singaporean family spends on food each month. The only way to know this for sure is to survey every single family in Singapore. This, however, is not practical.

Instead, we have only the resources to survey 100 families. We decide to go to Sixth Avenue and randomly ask 100 families living there what they reckon they spend on food each month. We thereby gather an observed sample of size 100: $(x_1, x_2, \dots, x_{100})$. We find that the average family spends $\bar{x} = \sum x_i / 100 = \$2,700$ on food each month.

Is $\bar{x} = \$2,700$ an unbiased estimate of the average monthly spending on food by a Singaporean family? Intuitively, we know that the answer is obviously no.

The reason is that our observed sample of size 100 was non-random. We picked an unusually affluent neighbourhood. Our estimate $\bar{x} = \$2,700$ is thus probably biased upwards.

131. Null Hypothesis Significance Testing (NHST)

Here's a quick sketch of how **Null Hypothesis Significance Testing (NHST)** works:

Example 1544. A piece of equipment has probability θ of breaking down. We have many pieces of the same type of equipment. Assume the rates of breakdown across the pieces of equipment are identical and independent.

1. Write down a **null hypothesis** H_0. In this case, it might be "H_0: $\theta = 0.6$".
2. Write down an **alternative hypothesis** H_A. In this case, it might be "H_A: $\theta < 0.6$".

(This is a **one-tailed test**—to be explained shortly.)

3. Observe a **random sample**. For example, we might have an observed random sample of size 5, where only the fourth piece of equipment breaks down. And so we'd write $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 1, 0)$.
4. Write down a **test statistic**. In this case, an obvious test statistic is the sample number of failures $T = X_1 + X_2 + X_3 + X_4 + X_5$. Our observed test statistic is thus $t = x_1 + x_2 + x_3 + x_4 + x_5 = 0 + 0 + 0 + 1 + 0 = 1$.
5. Now ask, how likely is it that—*if* H_0 *were true*—our test statistic would have been "at least as extreme as" that actually observed? That is, what is the probability

$$\mathrm{P}\left(\text{Observe data as extreme as that observed}\middle|H_0\right)?$$

The above probability is called the **p-value** of the observed sample.

In this case, the p-value is the probability of observing a random sample where 1 or fewer pieces of equipment broke down, *assuming* $H_0 : \theta = 0.6$ *were true*. That is,

$$p = \mathrm{P}\left(T \le t = 1\middle|H_0\right).$$

Now, remember that T is a random variable. In fact, it's a binomial random variable. Assuming H_0 to be true, we have $T \sim \mathrm{B}(n, \theta) = \mathrm{B}(5, 0.6)$. Thus,

$$p = \mathrm{P}\left(T \le 1\middle|H_0\right) = \mathrm{P}\left(T = 0\middle|H_0\right) + \mathrm{P}\left(T = 1\middle|H_0\right) = \begin{pmatrix} 5 \\ 0 \end{pmatrix} 0.6^0 0.4^5 + \begin{pmatrix} 5 \\ 1 \end{pmatrix} 0.6^1 0.4^4 = 0.08704.$$

This says that *if* H_0 were true, *then* the probability of observing a test statistic as extreme as the one we actually observed is only 0.08704. We might interpret this relatively **small p-value as casting doubt on or providing evidence against** H_0.

Here is the full list of the ingredients that go into NHST.

Null Hypothesis Significance Testing (NHST)

1. **Null hypothesis** H_0 (e.g. "this equipment has probability 0.6 of breaking down").
2. **Alternative hypothesis** H_A (e.g. "this equipment has probability less than 0.6 of breaking down"). The test is either **one-tailed** or **two-tailed**, depending on H_A.
3. A **random sample** of size n: $(X_1, X_2, \ldots, X_n)$.
4. A **test statistic** T (which simply maps each observed random sample to a real number.)
5. The p**-value** of the observed sample. This is the probability that—*assuming* H_0 *were true*—T takes on values that are at least "as extreme as" the actual observed test statistic t.
6. The **significance level** α. This is a pre-selected threshold, usually chosen to be some small value. The **conventional significance levels** are $\alpha = 0.1$, $\alpha = 0.05$, or $\alpha = 0.01$.

We then conclude qualitatively that

- **A small p-value casts doubt on or provides evidence against** H_0.
- **A large p-value fails to cast doubt on or provide evidence against** H_0.

In particular, if $p < \alpha$, then we say that **we reject H_0 at the significance level** α. And if $p \geq \alpha$, then we say that **we fail to reject H_0 at the significance level** α.

Note importantly that to reject H_0 (at some significance level α) does **NOT** mean that H_0 is false and H_A is true. Similarly, failure to reject H_0 does **NOT** mean that H_0 is true and H_A is false. More on this below.

Another example of NHST, now slightly more formally and carefully presented.

Example 1300. (Dr. Chee election example.) Our parameter of interest is μ, the proportion of votes for Dr. Chee. We guess that Dr. Chee won only 30% of the votes. We might thus write down two competing hypotheses:

$$H_0 : \mu = 0.3,$$
$$H_A : \mu > 0.3.$$

We call H_0 the **null hypothesis** and H_A the **alternative hypothesis**.

We pre-select $\alpha = 0.05$ as our **significance level**. This is the arbitrary threshold at which we'll say we reject (or fail to reject) H_0.

We gather a **random sample** of 100 votes: $(X_1, X_2, \ldots, X_{100})$. Our **test statistic** is the number of votes in favour of Dr. Chee, given by

$$T = X_1 + X_2 + \cdots + X_{100}.$$

Suppose that in our observed random sample $(x_1, x_2, \ldots, x_{100})$, we find that 39 are in favour of Dr. Chee. Our observed test statistic is thus $t = 39$.

We now ask: What is the probability that—*assuming H_0 were true*—T takes on values that are at least "as extreme as" the actual observed test statistic t? That is, what is the p**-value** of the observed sample?

Now, assuming H_0 were true, T is a binomial random variable with parameters 100 and 0.3. That is, $T \sim \mathrm{B}(n, p) = \mathrm{B}(100, 0.3)$. So,

$$p = \mathrm{P}\left(T \geq 39 \middle| H_0\right) = \mathrm{P}\left(T = 39 \middle| H_0\right) + \mathrm{P}\left(T = 40 \middle| H_0\right) + \cdots + \mathrm{P}\left(T = 100 \middle| H_0\right)$$
$$= \begin{pmatrix} 100 \\ 39 \end{pmatrix} 0.3^{39} 0.7^{61} + \begin{pmatrix} 100 \\ 40 \end{pmatrix} 0.3^{40} 0.7^{60} + \cdots + \begin{pmatrix} 100 \\ 100 \end{pmatrix} 0.3^{100} 0.7^{0} \approx 0.03398.$$

The small p-value casts doubt on or provides evidence against H_0.

And since $p \approx 0.03398 < \alpha = 0.05$, we can also say that **we reject H_0 at the $\alpha = 0.05$ significance level**.

Let θ be the parameter we're interested in. Under the objectivist interpretation, the value of θ may be unknown, but it is fixed. This has two consequences:

1. We never speak probabilistically about θ, because θ is a fixed number. For example, we never say "θ is probably less than 0.6" or "θ has probability 0.8 of being between 0.4 and 0.7". Such statements are nonsensical.
2. The null hypothesis, which is always written as an equality (e.g. "$H_0 : \theta = 0.6$"), is almost certainly false. After all, θ can (usually) take on a continuum of values. So **do NOT interpret "we fail to reject H_0" to mean "H_0 is true"**. This is because H_0 is almost certainly false.

When performing NHST, we will assiduously avoid saying things like "H_0 is true", "H_0 is false", "H_A is true", or "H_A is false". Instead, we will stick strictly to saying either "we reject H_0 at the significance level α" or "we fail to reject H_0 at the significance level α". Each of these two statements has a very precise meaning. The first says that $p < \alpha$. The second says that $p \geq \alpha$. Nothing more and nothing less.

Exercise 512. We flip a coin 20 times and get 17 heads. Test, at the 5% significance level, whether the coin is biased towards heads. (Answer on p. **1983**.)

131.1. One-Tailed vs Two-Tailed Tests

In the previous section, all the NHST we did were **one-tailed tests**.[541] For example, in the NHST done for Dr. Chee, we had

$$H_0 : \mu = 0.3,$$
$$H_A : \mu > 0.3.$$

This was a one-tailed test because the alternative hypothesis H_A was that μ was to the right of 0.3.

If instead we changed the alternative hypothesis to:

$$H_0 : \mu = 0.3,$$
$$H_A : \mu \neq 0.3.$$

Then this would be called a **two-tailed test**, because the alternative hypothesis H_A is that μ is either to the left or to the right of 0.3.

We now repeat the examples done in the previous section, but with H_A tweaked so that we instead have two-tailed tests. The difference is that the p-value is calculated differently.

[541] By the way, the more common convention is to say "one-tail**ed**" and "two-tail**ed**" tests, rather than "one-tail" and "two-tail" tests, as is the norm in Singapore (similar to those "Close for break" signs you sometimes see). But after some consultation with my grammatical experts, I have been told that both are equally correct.

Example 1544 (equipment breakdown).

Everything is as before, except that we now change the alternative hypothesis:

$$H_0 : \theta = 0.6,$$
$$H_A : \theta \neq 0.6.$$

Say we observe the same random sample as before: $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 1, 0)$.

Again our test statistic is the sample number of failures $T = X_1 + X_2 + X_3 + X_4 + X_5$. And so again our observed test statistic is $t = x_1 + x_2 + x_3 + x_4 + x_5 = 0 + 0 + 0 + 1 + 0 = 1$.

The difference now is how the p-value (of the observed sample) is calculated. In words, the p-value gives the likelihood that our test statistic is "at least as extreme as" that actually observed—*assuming H_0 were true*.

Previously, under a one-tailed test, we interpreted "our test statistic is *at least as extreme as* that actually observed" to mean the event $T \leq t = 1$.

Now that we're doing a two-tailed test, we'll instead interpret the same phrase to mean both the event $T \leq t = 1$ *and* the event that T is as far away on the other side of $\mathrm{E}\left[T \middle| H_0\right] = 3$. The second event is, specifically, $T \geq 5$. Altogether then, the p-value is given by

$$\begin{aligned} p &= \mathrm{P}\left(T \leq 1, T \geq 5 \middle| H_0\right) \\ &= \mathrm{P}\left(T = 0 \middle| H_0\right) + \mathrm{P}\left(T = 1 \middle| H_0\right) + \mathrm{P}\left(T = 5 \middle| H_0\right) \\ &= \begin{pmatrix} 5 \\ 0 \end{pmatrix} 0.6^0 0.4^5 + \begin{pmatrix} 5 \\ 1 \end{pmatrix} 0.6^1 0.4^4 + \begin{pmatrix} 5 \\ 5 \end{pmatrix} 0.6^1 0.4^4 = 0.1648. \end{aligned}$$

Since $p = 0.1648 \geq \alpha = 0.1$, we say that **we fail to reject H_0 at the $\alpha = 0.1$ significance level**.

Observe that previously, under the one-tailed test, we could reject H_0 at the $\alpha = 0.1$ significance level, because there $p = 0.08704$. Now, in contrast, under the two-tailed test, we fail to reject H_0 at the same significance level.

In general, all else equal, the p-value for an observed random sample is greater under a two-tailed test than under a one-tailed test. Thus, under a two-tailed test, we are less likely to reject H_0.

Example 1300 (Dr. Chee election). We change the alternative hypothesis:

$$H_0 : \mu = 0.3,$$
$$H_A : \mu \neq 0.3.$$

Say we observe the same random sample as before: $(x_1, x_2, \ldots, x_{100})$, in which 39 votes were in favour of Dr. Chee. So again our observed test statistic is $t = x_1 + x_2 + \cdots + x_{100} = 39$.

The difference now is how the p-value (of the observed sample) is calculated. In words, the p-value gives the likelihood that our test statistic is "at least as extreme as" that actually observed—*assuming* H_0 *were true*.

Previously, under a one-tailed test, we interpreted "our test statistic is *at least as extreme as* that actually observed" to mean the event $T \geq t = 39$.

Now that we're doing a two-tailed test, we'll instead interpret the same phrase to mean both the event $T \geq t = 39$ *and* the event that T is as far away on the other side of $\mathrm{E}\left[T \middle| H_0\right] = 30$. The second event is, specifically, $T \leq 21$. Altogether then, the p-value is given by

$$\begin{aligned} p = \mathrm{P}\left(T \leq 21, T \geq 39 \middle| H_0\right) = 1 - \mathrm{P}\left(22 \leq T \leq 38 \middle| H_0\right) \\ = 1 - \left[\mathrm{P}\left(T = 22 \middle| H_0\right) + \mathrm{P}\left(T = 23 \middle| H_0\right) + \cdots + \mathrm{P}\left(T = 38 \middle| H_0\right)\right] \\ = 1 - \left[\binom{100}{22} 0.3^{22} 0.7^{78} + \binom{100}{23} 0.3^{23} 0.7^{77} + \cdots + \binom{100}{38} 0.3^{38} 0.7^{62}\right] \approx 0.06281. \end{aligned}$$

Since $p = 0.06281 \geq \alpha = 0.05$, we say that **we fail to reject H_0 at the $\alpha = 0.05$ significance level**.

Again observe that previously, under the one-tailed test, we could reject H_0 at the $\alpha = 0.05$ significance level, because there $p = 0.03398$. Now, in contrast, under the two-tailed test, we fail to reject H_0 at the same significance level.

Exercise 513. We flip a coin 20 times and get 17 heads. Test, at the 5% significance level, whether the coin is biased.(Answer on p. **1983**.)

131.2. The Abuse of NHST (Optional)

NHST is popular because it gives a simplistic, formulaic cookbook procedure. Moreover, its conclusion appears to be binary: either we reject H_0 or we fail to reject H_0.

However, NHST is widely misunderstood, misinterpreted, and misused even within scientific communities. It has long been heavily criticised. In March 2016, the American Statistical Association even issued an official policy statement on how NHST should be used!

Here I discuss only the most important, commonly made error.

We may write the p-value as

$$p = \mathrm{P}\left(D|H_0\right),$$

where D stands for the observed data and H_0 stands for the null hypothesis. The p-value answers the following question:—*assuming H_0 were true*, what's the probability that we'd get data "at least as extreme" as those actually observed (D)?

Say we get a p-value of 0.03. We should then say simply that

- The small p-value **casts doubt on or provides evidence against** H_0.
- If the pre-selected significance level was $\alpha = 0.05$, then we may say that **we reject** H_0 **at the** 5% **significance level**.

However, instead of merely saying the above, some researchers may instead conclude that

$$H_0 \text{ is true with probability } 0.03.$$

Do you see the error here? The researcher has gone from the finding that $p = \mathrm{P}\left(D|H_0\right) = 0.03$ to the conclusion that $\mathrm{P}\left(H_0|D\right) = 0.03$. This is precisely the Conditional Probability Fallacy (CPF), which we discussed at length in subsection 116.1.

The error is the same as leaping from "*A lottery ticket buyer who doesn't cheat has a small probability q of winning*" to "*Jane bought a lottery ticket and won. Therefore, there is only probability q that she didn't cheat.*"

The p-value is **NOT** the probability that H_0 is true.[542] Instead, it is the probability that—assuming H_0 were true—we would have gotten data "at least as extreme" as those actually observed. This is an important difference. But it is also a subtle one, which is why even researchers get confused.

[542]Indeed, under the objectivist view, such a statement is nonsensical anyway, because H_0 is either true or not true; it makes no sense to talk probabilistically about whether H_0 is true.

131.3. Common Misinterpretations of the Margin of Error (Optional)

The **sampling error** or **margin of error** is often misinterpreted by laypersons (and journalists).

Example 1545. On the night of the 2016 Bukit Batok SMC By-Election, the Elections Department announced[543] that based on a sample count of 900 ballots,

- Dr. Chee had won 39% of the votes.
- These sample counts have a confidence level of 95%, with a ±4% **margin of error**.

What does the above gobbledygook mean? Let μ be the true proportion of votes won by Dr. Chee. Let $\bar{X}$ be the sample proportion and $\bar{x}$ be the observed sample proportion.

It's clear enough what the 39% means—they randomly counted 900 ballots and found (after accounting for any spoilt votes) that $\bar{x} = 39\%$ were in favour of Dr. Chee.

What's less clear is what the 95% confidence level and ±4% margin of error mean.

Here are three possible interpretations of what is meant. Only one is correct.

1. "With probability 0.95, $\mu \in (\bar{x} - 0.04, \bar{x} + 0.04) = (0.35, 0.43)$."

2. "With probability 0.95, $\bar{X} \in (\bar{x} - 0.04, \bar{x} + 0.04) = (0.35, 0.43)$."

Equivalently, suppose we repeatedly observe many random samples of size 900. Then we should find that in 0.95 of these observed random samples, the observed sample mean is between 0.35 and 0.43.

3. "With probability 0.95, $\bar{X} \in (\mu - 0.04, \mu + 0.04)$."

We have no idea what μ is. All we can say is that with probability 0.95, the sample mean $\bar{X}$ of votes for Dr. Chee is between $\mu - 0.04$ and $\mu + 0.04$.

Equivalently, suppose we repeatedly observe many random samples of size 900. Then we should find that in 0.95 of these observed random samples, the observed sample mean is between $\mu - 0.04$ and $\mu + 0.04$.

Take a moment to understand what each of the above interpretations say. Then decide which you think is the correct interpretation, before turning to the next page.

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Interpretation #1—"with probability 0.95, $\mu \in (\bar{x} - 0.04, \bar{x} + 0.04) = (0.35, 0.43)$"—is perhaps the one most commonly made by laypersons.[544] It makes two errors:

1. It is **nonsensical to speak probabilistically about the proportion μ of votes won** by Dr. Chee. μ is some fixed number. So either μ is in the interval $(0.35, 0.43)$, or it isn't. It makes no sense to speak probabilistically about whether μ is in that interval.
2. The **margin of error is applicable to the true proportion** μ and not to the observed sample proportion $\bar{x} = 0.39$.

Some "authorities" often attempt[545] to correct Interpretation #1 by offering Interpretation #2—"with probability 0.95, $\bar{X} \in (\bar{x} - 0.04, \bar{x} + 0.04) = (0.35, 0.43)$". However, Interpretation #2 is still wrong, because it still makes the second of the above two errors.

Unfortunately, the correct interpretation is also the one that says the least. It is Interpretation #3—"with probability 0.95, $\bar{X} \in (\mu - 0.04, \mu + 0.04)$".

This interpretation says merely that if we were somehow able to repeatedly observe random samples of size 900, then we'd find that 0.95 of the corresponding observed sample means will be in $(\mu - 0.04, \mu + 0.04)$. Which isn't saying much, because first of all, we have only one observed random sample; we do not get to repeatedly observe random samples. Secondly, this still doesn't tell us much about μ, which is what we're really interested in.

The correct interpretation (Interpretation #3) is the least interesting interpretation. Perhaps this explains why journalists often prefer to give an incorrect interpretation.

See section 147.8 (Appendices) for a discussion of where the Elections Department's ±4% margin of error comes from.

Journalists often try to explain what the confidence level and margin of error mean—they almost always get it wrong.

Example 1546. On the night of the 2016 Bukit Batok SMC By-Election, a website called Mothership.sg wrote:

"Based on the sample count of 100 votes,[546] it was revealed at 9.26pm that the SDP Sec-Gen received 39 percent of votes. In other words, Chee would score 35 per cent in the worst case scenario and 43 per cent in the best case scenario."

This is the most absurd misinterpretation of the margin of error I have ever seen.[547]

Let's see what the correct worst- and best-case scenarios are.

Suppose that in the observed random sample of 900 votes, exactly 39% or $0.39 \times 900 = 351$ were votes for Dr. Chee and the remaining 549 were for PAP Guy. Then,

- **Worst-case scenario**: The observed random sample of 900 votes happened to contain exactly all of the votes in favour of Dr. Chee. That is, Dr. Chee won only 351 votes and PAP Guy won the remaining $23,570 - 351 = 23,219$ votes. So the correct worst-case scenario is that Dr. Chee won $\approx 1.5\%$ of the votes.
- **Best-case scenario**: The observed random sample of 900 votes happened to contain exactly all of the votes in favour of PAP Guy. That is, PAP Guy won only 549 votes and Dr. Chee won the remaining $23570 - 549 = 23,021$ votes. So the correct best-case scenario is that Dr. Chee won $\approx 97.7\%$ of the votes.

These worst- and best-case scenarios are admittedly unlikely. Nonetheless, they are possible scenarios all the same. The journalist's purported worst- and best-case scenarios are completely wrong.

[546] By the way, even this basic fact was wrong. The sample count was not 100 votes. Instead, it was 900 votes, consisting of 100 votes from each of 9 polling stations.

Moreover, the Mothership.sg journalist failed to report the confidence level of 95%, either because he didn't know what it meant or because he didn't think it important. But it is important. It is pointless to inform the reader about the margin of error without also specifying the confidence level.

[547] You can find several misinterpretations of the margin of error collected in this academic paper. None is as absurdly bad as the error committed here.

131.4. Critical Region and Critical Value

Informally, the **critical region** is the set of values of the observed test statistic t for which we would reject the null hypothesis. The critical region is thus sometimes also called the **rejection region**.

And the **critical value(s)** is (are) the exact value(s) of the observed test statistic t at which we are *just* able to reject the null hypothesis.

Example 1300. (Dr. Chee election.) Say that as before, we have a one-tailed test where the two competing hypotheses are

$$H_0 : \mu = 0.3,$$
$$H_A : \mu > 0.3.$$

Say that as before, we choose $\alpha = 0.05$ as our **significance level**.

Say that as before, in our observed random sample of 100 votes, 39 are in favour of Dr. Chee, so that our observed test statistic is $t = 39$.

We calculated that the corresponding p-value is 0.03398 and so we were able to reject H_0 at the $\alpha = 0.05$ significance level.

We now calculate the **critical region** and the **critical value**. We can calculate that if $t = 38$, then the corresponding p-value is ≈ 0.053 (you should verify this for yourself). And so we would be unable to reject H_0.

We thus conclude that the critical value is 39, because this is the value of t at which we are *just* able to reject H_0.

And the critical region is the set $\{39, 40, 41, \ldots, 100\}$. These are the values at which we'd be able to reject H_0 at the $\alpha = 0.05$ significance level.

Same example as above, but now two-tailed:

Example 1300. (Dr. Chee election.)

Say that as before, we have a two-tailed test where the two competing hypotheses are

$$H_0 : \mu = 0.3,$$
$$H_A : \mu \neq 0.3.$$

The significance level is again $\alpha = 0.05$. Again, the observed random sample of 100 votes contains 39 in favour of Dr. Chee, so that our observed test statistic is $t = 39$.

We calculated that the corresponding p-value is 0.06281 and so we failed to reject H_0 at the $\alpha = 0.05$ significance level.

We calculate that if $t = 40$, then the corresponding p-value is ≈ 0.03745 (you should verify this for yourself). Thus, the critical value**s** are 20 and 40, because these are the values of t at which we are just able to reject H_0.

The critical region is the set $\{0, 1, \ldots, 20, 40, 41, \ldots, 100\}$. These are the values at which we'd be able to reject H_0 at the $\alpha = 0.05$ significance level.

Exercise 514. (Answer on p. **1984**.) We flip a coin 20 times. What are the critical region and critical value(s) in

(a) A test, at the 5% significance level, of whether the coin is biased towards heads.

(b) A test, at the 5% significance level, of whether the coin is biased.

131.5. Testing of a Population Mean (Small Sample, Normal Distribution, σ^2 Known)

Example 1547. The weight (in mg) of a grain of sand is $X \sim \mathrm{N}(\mu, 9)$. Our unknown parameter of interest is the true population mean μ (i.e. the true average weight of a grain of sand). Our "guess" is that $\mu = 5$. We thus write down two competing hypotheses:

$$H_0 : \mu = 5,$$
$$H_A : \mu \neq 5.$$

(Note that this is a two-sided test.)

We take a random sample of size 4—(X_1, X_2, X_3, X_4). Our test statistic is the sample mean $\bar{X} = (X_1 + X_2 + X_3 + X_4)/4$.

Our observed random sample is $(x_1, x_2, x_3, x_4) = (3, 9, 11, 7)$. That is, we randomly pick four grains of sand that happen to have weights 3, 9, 11, and 7 mg. Then the observed test statistic is

$$\bar{x} = \frac{3 + 9 + 11 + 7}{4} = 7.5.$$

The p-value is the probability that the test statistic $\bar{X}$ takes on values "at least as extreme as" our observed test statistic $\bar{x} = 7.5$, *assuming* $H_0 : \mu = 5$ *were true*. Note that *if* H_0 were true, *then* $\bar{X} \sim \mathrm{N}(\mu, \sigma^2/n) = \mathrm{N}(5, 9/4)$. Thus, the p-value is given by

$$p = \mathrm{P}\left(\bar{X} \geq 7.5, \bar{X} \leq 2.5 \middle| H_0\right) = \mathrm{P}\left(\bar{X} \geq 7.5 \middle| H_0\right) + \mathrm{P}\left(\bar{X} \leq 2.5 \middle| H_0\right)$$

$$= \mathrm{P}\left(Z \geq \frac{7.5 - 5}{\sqrt{9/4}}\right) + \mathrm{P}\left(Z \leq \frac{2.5 - 5}{\sqrt{9/4}}\right) \approx 0.04779 + 0.04779 = 0.09558.$$

Thus, we reject H_0 at the $\alpha = 0.1$ significance level. However, we would fail to reject H_0 at the $\alpha = 0.05$ significance level.

The table below summarises the tests to use for the population mean, in different circumstances. In this section, we learnt how to handle the first case (any sample size, normal distribution, σ^2 known). The following sections will deal with the other three cases.

Sample size	**Distribution**	σ^2	σ^2 **known**
Any	**Normal**	**Known**	Z-test: $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$.
Large	**Any**	**Known**	Z-test: $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$.
Large	**Any**	**Unknown**	Z-test: $\dfrac{\bar{X}-\mu}{s/\sqrt{n}} \sim \mathrm{N}(0,1)$.
Small	**Normal**	**Unknown**	t-test: $\dfrac{\bar{X}-\mu}{s/\sqrt{n}} \sim \mathrm{N}(0,1)$.[548]
Small	**Non-normal**	**Either**	Not in A-Levels.

Exercise 515. The Singapore daily high temperature (in °C) can be modelled by $X \sim \mathrm{N}(\mu, 8)$. Our unknown parameter of interest is the true population mean μ (i.e. the true average daily high temperature). Your friend guesses that $\mu = 34$. You gather the following data on daily high temperatures, of 10 randomly chosen days in 2015: $(35, 35, 31, 32, 33, 34, 31, 34, 35, 34)$. Test your friend's hypothesis, at the $\alpha = 0.05$ significance level. (Be sure to write down your null and alternative hypotheses.) (Answer on p. **1985**.)

131.6. Testing of a Population Mean (Large Sample, Any Distribution, σ^2 Known)

We'll recycle the same example from the previous section. Before, we knew that X was normally distributed. Now the big difference is that **we have absolutely no idea what distribution X comes from**!

To compensate, we require also that our random sample is "large enough", so that the CLT-approximation can be used.

Example 1548. The weight (in mg) of a grain of sand is $X \sim (\mu, 9)$. (This says simply that X is distributed with mean μ and variance 9.) Our unknown parameter of interest is the true population mean μ (i.e. the true average weight of a grain of sand). Again, we "guess" that $\mu = 5$. Again, we write down:

$$H_0 : \mu = 5,$$
$$H_A : \mu \neq 5.$$

(Note that this is, again, a two-sided test.)

This time, we'll take a random sample of size 100—$(X_1, X_2, \ldots, X_{100})$. Again, our test statistic is the sample mean $\bar{X} = (X_1 + X_2 + \cdots + X_{100}) / 100$.

Recall the magic of the CLT. Even if we have absolutely no idea what distribution X is drawn from, then provided n is sufficiently large, $\bar{X}$ is normally distributed. So here, since the sample is large ($n = 100 \geq 20$), by the CLT, we know that $\bar{X}$ has, approximately, the normal distribution $\mathrm{N}\left(\mu, \sigma^2/n\right)$. So, if H_0 were true, then we have, approximately, $\bar{X} \sim \mathrm{N}\left(\mu, \sigma^2/n\right) = \mathrm{N}(5, 9/100)$.

Say the observed test statistic we get is

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_{100}}{100} = 5.5.$$

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Again, the p-value is the probability that our test statistic $\bar{X}$ takes on values "at least as extreme as" our observed test statistic $\bar{x} = 5.6$, *assuming* $H_0 : \mu = 5$ *were true.* Thus, the p-value is given by

$$p = \mathrm{P}\left(\bar{X} \geq 5.6, \bar{X} \leq 4.4 \middle| H_0\right) = \mathrm{P}\left(\bar{X} \geq 5.6 \middle| H_0\right) + \mathrm{P}\left(\bar{X} \leq 4.4 \middle| H_0\right)$$

$$\overset{\text{CLT}}{\approx} \mathrm{P}\left(Z \geq \frac{5.6 - \mu}{\sigma/\sqrt{n}}\right) + \mathrm{P}\left(Z \leq \frac{4.4 - \mu}{\sigma/\sqrt{n}}\right) = \mathrm{P}\left(Z \geq \frac{5.6 - 5}{\sqrt{9/100}}\right) + \mathrm{P}\left(Z \leq \frac{4.4 - 5}{\sqrt{9/100}}\right)$$

$$= \mathrm{P}\left(Z \geq 2\right) + \mathrm{P}\left(Z \leq -2\right) \approx 0.0455.$$

Thus, we reject H_0 at the $\alpha = 0.05$ significance level.

Exercise 516. The Singapore daily high temperature (in °C) can be modelled by $X \sim (\mu, 8)$. Our unknown parameter of interest is the true population mean μ (i.e. the true average daily high temperature). Your friend guesses that $\mu = 34$. You gather the data on daily high temperatures, of 100 randomly chosen days in 2015 and find the observed sample average temperature to be 33.4 °C. Test your friend's hypothesis, at the $\alpha = 0.05$ significance level. (Be sure to write down your null and alternative hypotheses. Also, clearly state where you use the CLT.) (Answer on p. **1985**.)

131.7. Testing of a Population Mean (Large Sample, Any Distribution, σ^2 Unknown)

We'll recycle the same example from the previous section. Again, we have absolutely no idea what distribution X comes from. And again, the random sample is large enough, so that the CLT can be used.

But now, σ^2 **is unknown**. This turns out to be no big deal. We can simply replace σ^2 with the observed unbiased sample variance s^2, and do the same thing as before.

Example 1549. The weight (in mg) of a grain of sand is $X \sim \left(\mu, \sigma^2\right)$. (This says simply that X is distributed with mean μ and variance σ^2.) Our unknown parameter of interest is the true population mean μ (i.e. the true average weight of a grain of sand). Again, we "guess" that $\mu = 5$. Again, we write down

$$H_0 : \mu = 5,$$
$$H_A : \mu \neq 5.$$

(Note that this is, again, a two-sided test.)

Again, we take a random sample of size 100—$\left(X_1, X_2, \ldots, X_{100}\right)$. Again, our test statistic is the sample mean $\bar{X} = \left(X_1 + X_2 + \cdots + X_{100}\right)/100$.

Again, since the sample is large ($n = 100 \geq 20$), by the CLT, that $\bar{X}$ has, approximately, the normal distribution $\mathrm{N}\left(\mu, \sigma^2/n\right)$. So, if H_0 were true, then we have, approximately, $\bar{X} \sim \mathrm{N}\left(\mu, \sigma^2/n\right) = \mathrm{N}\left(5, \sigma^2/100\right)$. Since the sample variance S^2 is an unbiased estimator for σ^2, it is plausible that we also have, approximately, $\bar{X} \sim \mathrm{N}\left(\mu, \sigma^2/n\right) = \mathrm{N}\left(5, s^2/100\right)$, where s^2 is the observed sample variance.

Say the observed sample mean and observed sample variance we get are

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_{100}}{100} = 5.6 \quad \text{and} \quad s^2 = \frac{\sum_{i=1}^{100}\left(x_i - \bar{x}\right)^2}{n-1} = 8$$

(*Example continues on the next page ...*)

(... *Example continued from the previous page.*)

Again, the p-value is the probability that our test statistic $\bar{X}$ takes on values "at least as extreme as" our observed test statistic $\bar{x} = 5.6$, *assuming* $H_0 : \mu = 5$ *were true*. Thus, the p-value is given by

$$p = \mathrm{P}\left(\bar{X} \geq 5.6, \bar{X} \leq 4.4 \middle| H_0\right) = \mathrm{P}\left(\bar{X} \geq 5.6 \middle| H_0\right) + \mathrm{P}\left(\bar{X} \leq 4.4 \middle| H_0\right)$$

$$\stackrel{\text{CLT}}{\approx} \mathrm{P}\left(Z \geq \frac{5.6 - \mu}{s/\sqrt{n}}\right) + \mathrm{P}\left(Z \leq \frac{4.4 - \mu}{s/\sqrt{n}}\right) = \mathrm{P}\left(Z \geq \frac{5.6 - 5}{\sqrt{8/100}}\right) + \mathrm{P}\left(Z \leq \frac{4.4 - 5}{\sqrt{8/100}}\right)$$

$$\approx \mathrm{P}\left(Z \geq 2.1213\right) + \mathrm{P}\left(Z \leq -2.1213\right) \approx 0.03389.$$

Thus, we reject H_0 at the $\alpha = 0.05$ significance level.

Exercise 517. The Singapore daily high temperature (in °C) can be modelled by $X \sim \left(\mu, \sigma^2\right)$. Our unknown parameter of interest is the true population mean μ (i.e. the true average daily high temperature). Your friend guesses that $\mu = 34$. You gather the data on daily high temperatures, of 100 randomly chosen days in 2015. Your observed sample mean temperature is 33.4 °C and your observed sample variance is 11.2 °C^2. Test your friend's hypothesis, at the $\alpha = 0.05$ significance level. (Be sure to write down your null and alternative hypotheses. Also, clearly state where you use the CLT.) (Answer on p. **1986**.)

131.8. Formulation of Hypotheses

Example 1550. *We flip a coin* 100 *times. We get* 100 *heads. What can we say about the coin?*

This is an open-ended question, to which there can be many different answers. Here's the answer we're taught to give for H2 Maths:

Let μ be the probability that a coin-flip is heads. We formulate a pair of competing hypotheses:

$$H_0 : \mu = 0.5,$$
$$H_A : \mu \neq 0.5.$$

Our test statistic T is the number of heads (out of 100 coin-flips). Our observed test statistic t is 100. The corresponding p-value (note that this is a two-tailed test) is

$$\begin{aligned} \mathrm{P}(T \geq 100, T \leq 0 | H_0) &= \mathrm{P}(T = 0 | H_0) + \mathrm{P}(T = 100 | H_0) \\ &= \binom{100}{0} 0.5^0 0.5^{100} + \binom{100}{100} 0.5^{100} 0.5^0 \approx 1.578 \times 10^{-30}. \end{aligned}$$

The tiny p-value may be interpreted as casting on or providing evidence against H_0.

We note also that we can easily reject H_0 at any of the conventional significance levels ($\alpha = 0.1$, $\alpha = 0.05$, or $\alpha = 0.01$).

Exercise 518. (Answer on p. **1986**.) We observe the weights (in kg) of a random sample of 50 Singaporeans: $(x_1, x_2, \ldots, x_{50})$. We observe that $\sum x_i/50 = 68$ and $\sum x_i^2/50 = 5000$.

A friend claims that the average American is heavier than the average Singaporean. It is known that the average American weighs 75 kg. Is your friend correct? If you make any assumptions or approximations, make clear exactly where you do so. (Hint: Use Fact 244(a)).

132. Correlation and Linear Regression

132.1. Bivariate Data and Scatter Diagrams

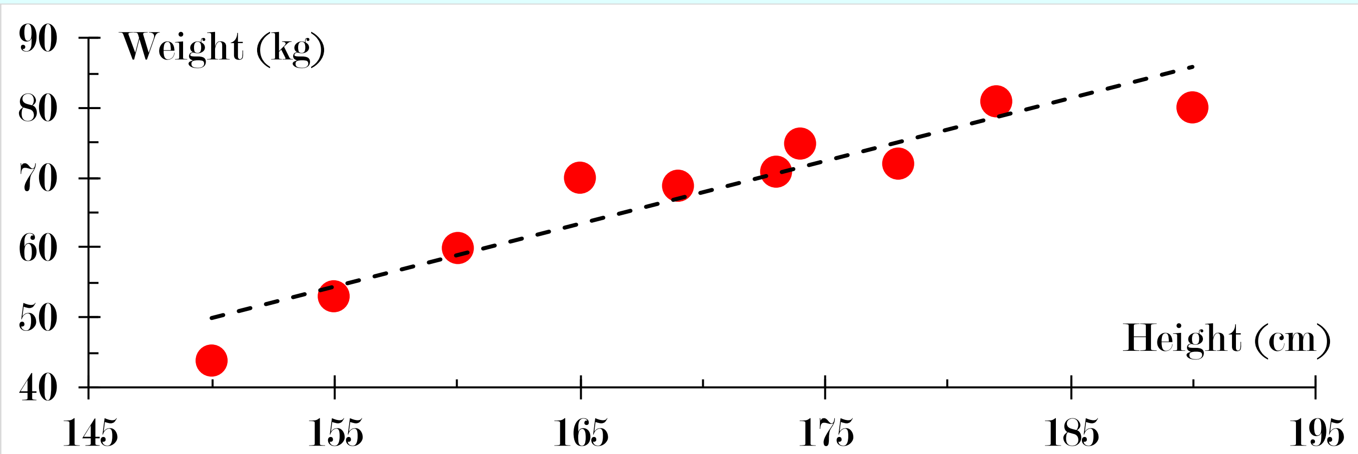
In this chapter, we'll be interested in the relationship between two sets of data.

Example 1551. We measure the heights and weights of 10 adult male Singaporeans. Their heights (in cm) and weights (in kg) are given in this table:

i	1	2	3	4	5	6	7	8	9	10
h_i (cm)	182	165	173	155	178	174	169	160	150	190
w_i (kg)	81	70	71	53	72	75	69	60	44	80

We call (h_i, w_i) **observation** i. So for example, observation 5 is $(178, 72)$ and observation 9 is $(150, 44)$.

We can plot a **scatter diagram** of these 10 persons' weights (vertical axis) against their heights (horizontal).



The black dotted line is called **a line of best fit**. Shortly (section 132.4), we'll learn how to construct this line of best fit.

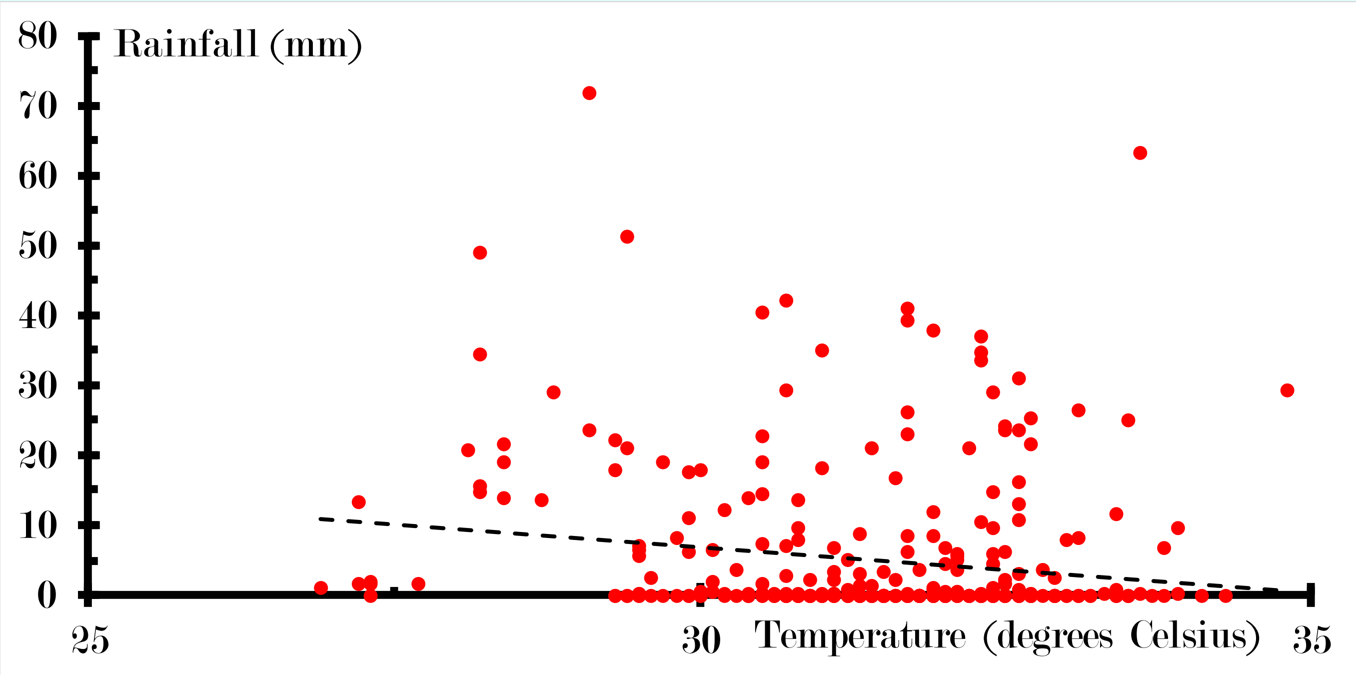
The more closely the data points in the above scatter diagram lie to a straight line, the more strongly **linearly correlated** are weight and height. So here with these particular data, the linear correlation between weight and height seems **strong**. In the next section, we'll learn about the **product moment correlation coefficient**, which is a way to precisely quantify the degree to which two sets of data are linearly correlated.

Because the line of best fit is upward-sloping, we can also say that the linear correlation is **positive**.

Example 1552. We have data from the Clementi weather station for the daily high temperature (in °C) and daily rainfall (in mm) on 361 days in 2015. (Strangely, data were missing for four days, namely Feb 10–13.)

i	1	2	3	4	...	361
t_i (°C)	27.3	29.5	31.1	32		30.2
p_i (mm)	0	0.2	0	0		12.4

We can again plot a **scatter diagram** of rainfall against temperature.



Again, the black dotted line is **a line of best fit**. The data points do not seem close to this line. Thus, it seems that the linear correlation between temperature and rainfall is **weak**.

The line of best fit is downward-sloping and so we say that the linear correlation is **negative**.

Exercise 519. (Answer on p. 1987.) The table below shows the prices charged (p) and the number of haircuts (q) given by 5 different barbers, during June 2016.

Draw a scatter diagram with price on the horizontal axis. Plot also what you think looks like a line of best fit.

i	1	2	3	4	5
p_i ($)	8	9	4	10	8
q_i	300	250	1000	400	400

132.2. Product Moment Correlation Coefficient (PMCC)

In the previous section, we used a scatter diagram to determine if there was a plausible linear relationship between two sets of data. This, though, was a very crude method.

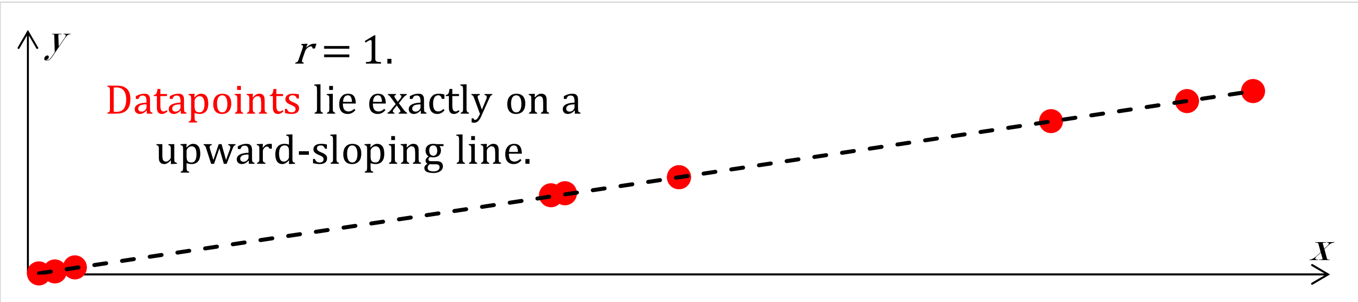
A more precise measure of the degree to which two sets of data are linearly correlated is called **the product moment correlation coefficient (PMCC)**. Formally:

Definition 261. Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two ordered sets of real numbers. The *product moment correlation coefficient (PMCC)* is the following real number:

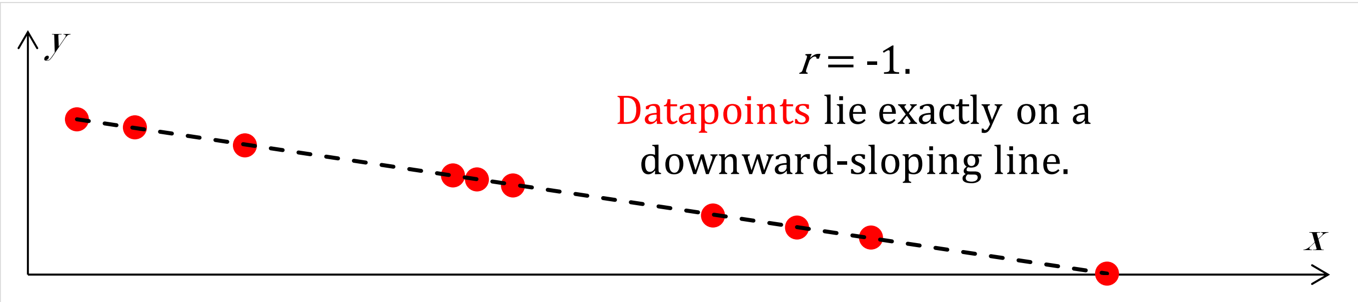
$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}.$$

Properties of the PMCC.

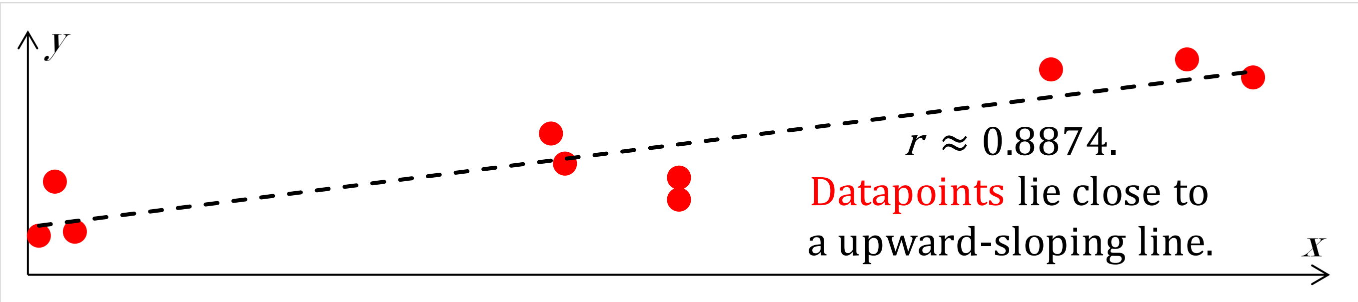
1. $-1 \le r \le 1$. (Surprisingly, this can be proven using vectors: Fact 302 (Appendices).)
2. We say the linear correlation is **positive** if $r > 0$ and **negative** if $r < 0$.
3. If $r = 1$, the linear correlation is positive and **perfect**.



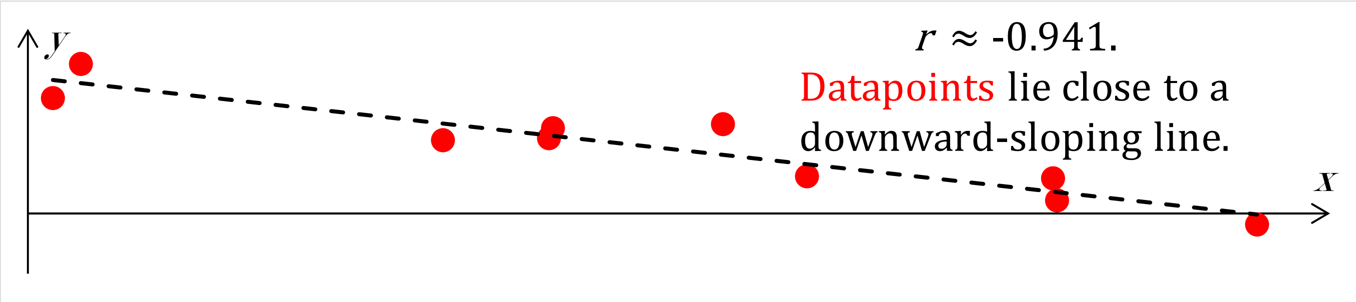
4. If $r = -1$, the linear correlation is negative and **perfect**.



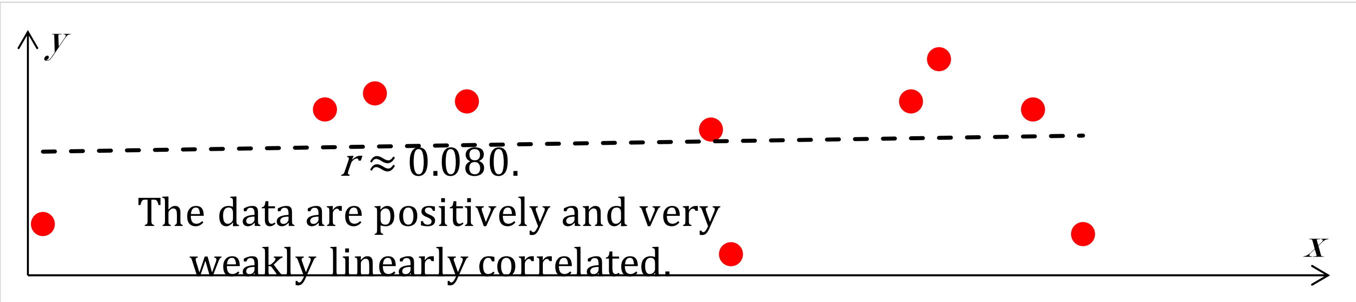
5. If r is close to 1, the linear correlation is **very strong**.

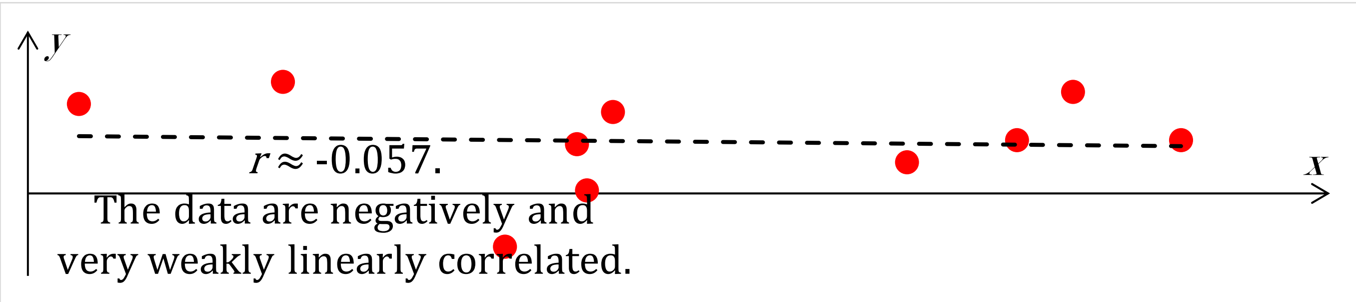


6. If r is close to -1, the linear correlation is **very strong**.

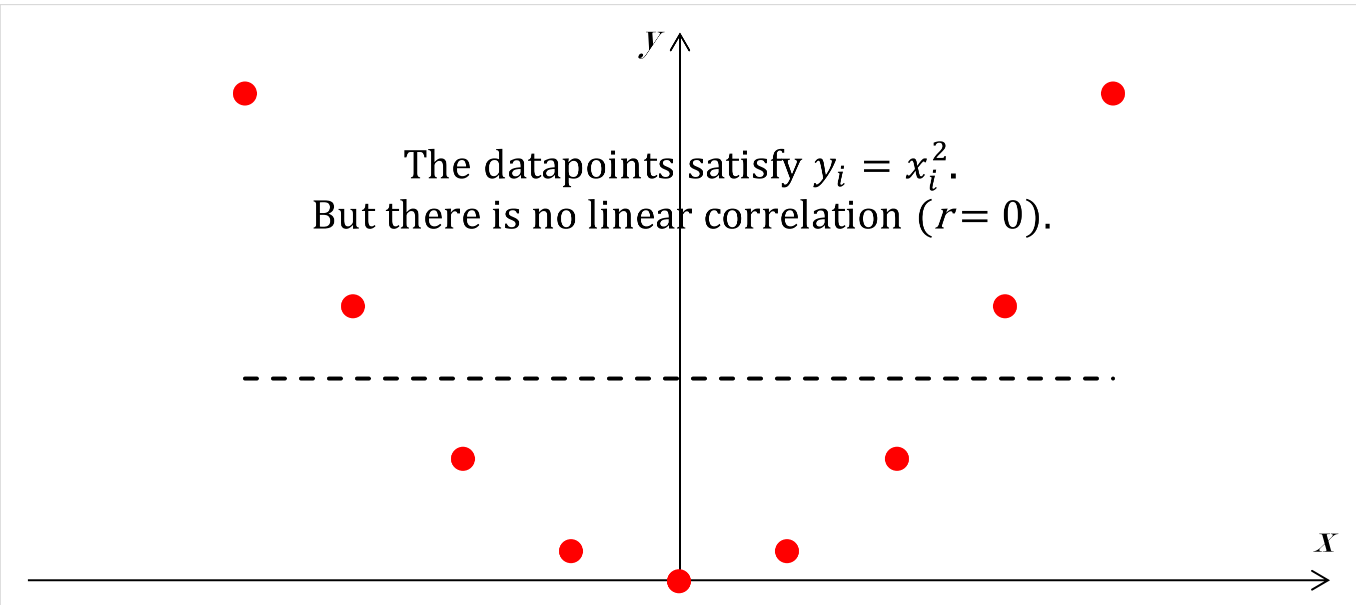


7. If r is close to 0, the linear correlation is **very weak**.



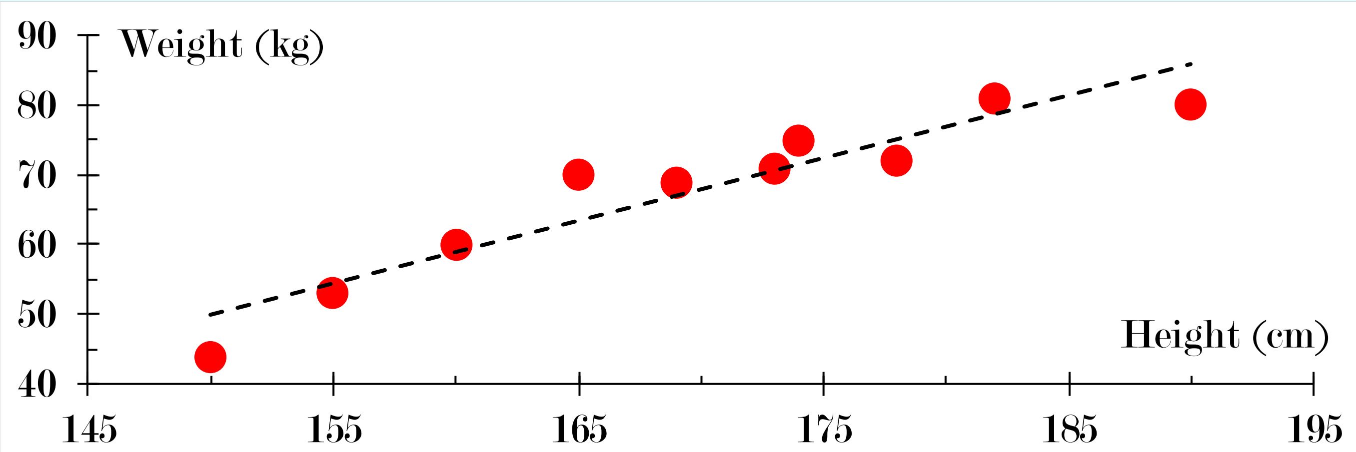


8. r is merely a measure of linear correlation and nothing else. Two variables may be very closely related but not linearly correlated. For example, data generated by the quadratic model $y_i = x_i^2$ may have a very low r.



Example 1551 (continued from above). This is the height and weight example revisited. For convenience, we reproduce the data and scatter diagram:

i	1	2	3	4	5	6	7	8	9	10
h_i (cm)	182	165	173	155	178	174	169	160	150	190
w_i (kg)	81	70	71	53	72	75	69	60	44	80



$$\bar{h} = \frac{182+165+173+155+178+174+169+160+150+190}{10} = 169.6,$$

$$\bar{w} = \frac{81+70+71+53+72+75+69+60+44+80}{10} = 67.5,$$

$$\sum_{i=1}^{n}\left(h_i - \bar{h}\right)\left(w_i - \bar{w}\right) = \left(182 - \bar{h}\right)\left(81 - \bar{w}\right) + \cdots + \left(190 - \bar{h}\right)\left(80 - \bar{w}\right) = 1237$$

$$\sqrt{\sum_{i=1}^{n}\left(h_i - \bar{h}\right)^2} = \sqrt{(182-169.6)^2 + \cdots + (190-169.6)^2} \approx 37.180640,$$

$$\sqrt{\sum_{i=1}^{n}\left(w_i - \bar{w}\right)^2} = \sqrt{(81-67.5)^2 + + \cdots + (80-67.5)^2} \approx 35.418922,$$

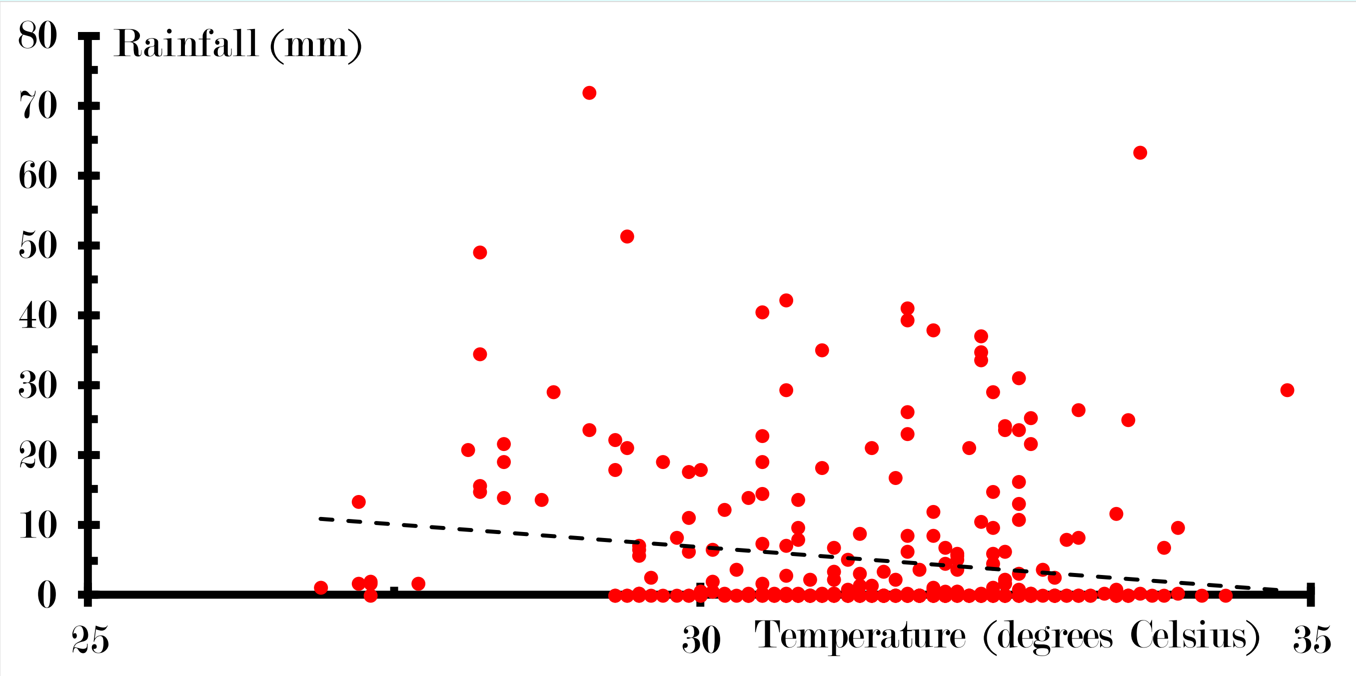
$$\implies r = \frac{\sum_{i=1}^{n}\left(h_i - \bar{h}\right)\left(w_i - \bar{w}\right)}{\sqrt{\sum_{i=1}^{n}\left(h_i - \bar{h}\right)^2}\sqrt{\sum_{i=1}^{n}\left(w_i - \bar{w}\right)^2}} \approx 0.9393.$$

As expected, $r > 0$ (the linear correlation is positive or, equivalently, the line of best fit is upward-sloping). Moreover, r is close to 1 (the linear correlation is very strong).

Example 1552 (continued from above). This is the temperature and rainfall example revisited. For convenience, we reproduce the data and scatter diagram:

i	1	2	3	4	...	361
t_i (°C)	27.3	29.5	31.1	32		30.2
p_i (mm)	0	0.2	0	0		12.4

We can again plot a **scatter diagram** of rainfall against temperature.



$$\bar{t} = \frac{27.3 + 29.5 + 31.1 + 32 + \cdots + 30.2}{361} \approx 31.5, \quad \bar{w} = \frac{0 + 0.2 + 0 + 0 + \cdots + 12.4}{361} \approx 5.0.$$

$$\implies r = \frac{\sum_{i=1}^{n} \left(t_i - \bar{t}\right)(w_i - \bar{w})}{\sqrt{\sum_{i=1}^{n} \left(t_i - \bar{t}\right)^2} \sqrt{\sum_{i=1}^{n} (w_i - \bar{w})^2}}$$

$$= \frac{(27.3 - 31.5)(0 - 5.0) + \cdots + (30.2 - 31.5)(12.4 - 5.0)}{\sqrt{(27.3 - 31.5)^2 + \cdots + (30.2 - 31.5)^2} \sqrt{(0 - 5.0)^2 + \cdots + (12.4 - 5.0)^2}}$$

$$\approx -0.1623.$$

As expected, $r < 0$ (the linear correlation is negative or, equivalently, the line of best fit is downward-sloping). Moreover, r is fairly close to 0 (the linear correlation is weak).

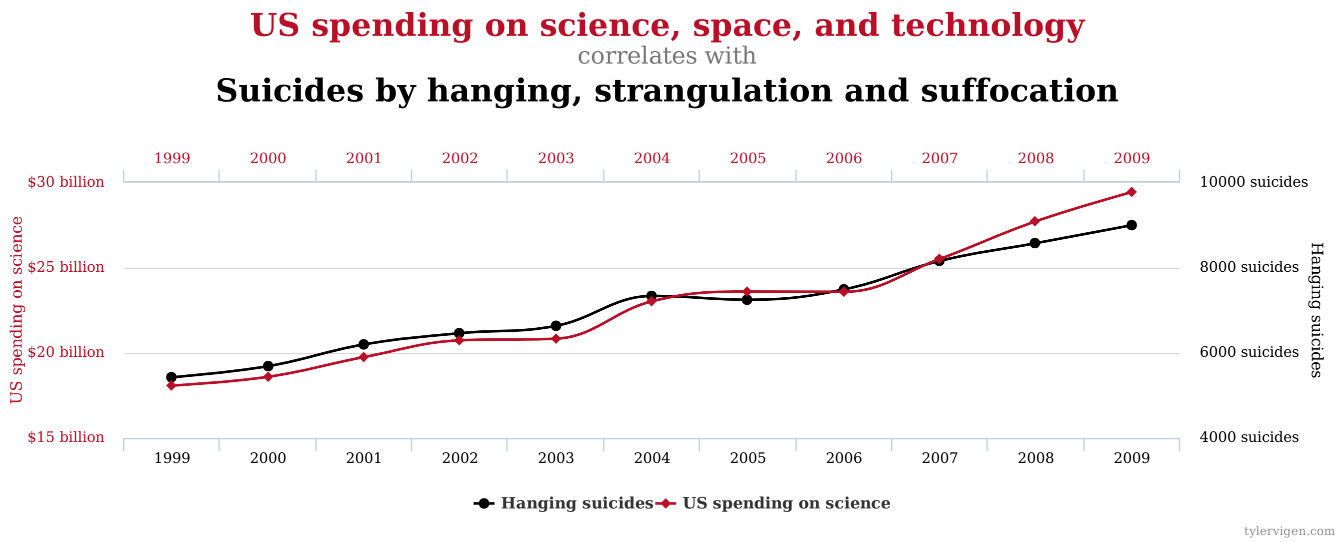
Exercise 520. Compute the PMCC between p and q, using the data below. (Answer on p. **1987**.)

i	1	2	3	4	5
p_i (\$)	8	9	4	10	8
q_i	300	250	1000	400	400

132.3. Correlation Does Not Imply Causation (Optional)

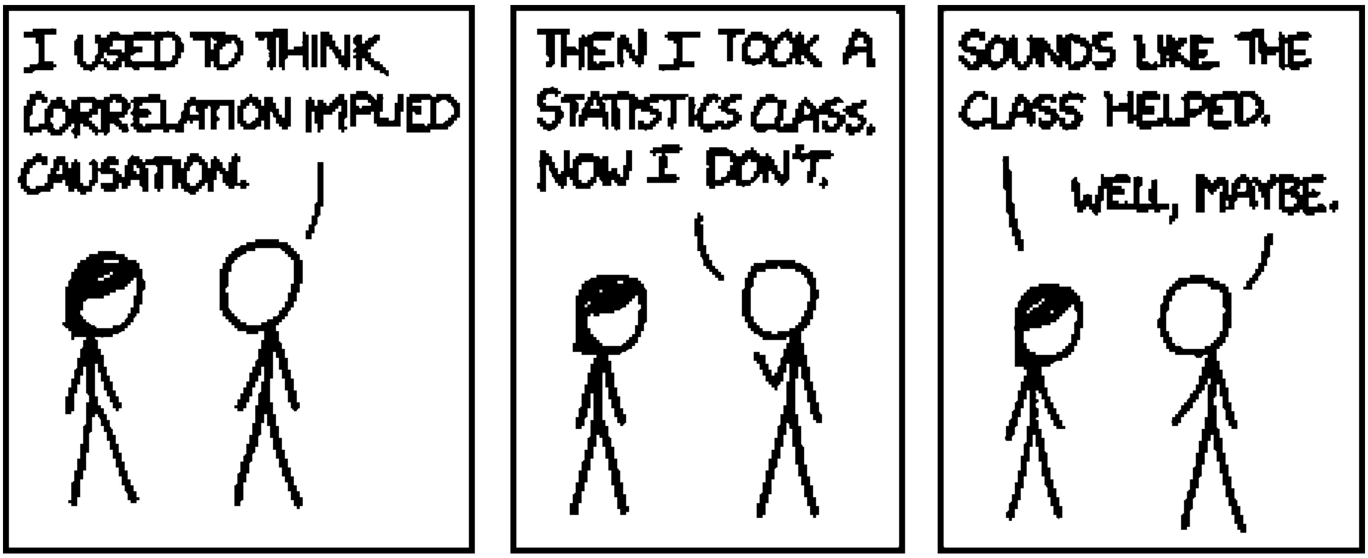
Correlation does not imply causation. This saying has now become a cliché. Doesn't make it any less true.

Below is an amusing but spurious correlation (source):



The PMCC is $r \approx 0.99789126$. So the two sets of data are almost perfectly linearly correlated. But of course, this doesn't mean that spending on science *causes* suicides or that suicides *cause* spending on science. More likely, the correlation is simply spurious.

A comic from xkcd:



Example 506 (continued from above). We suspect that the heights and weights of adult male Singaporeans are linearly correlated. We thus write down this linear model:

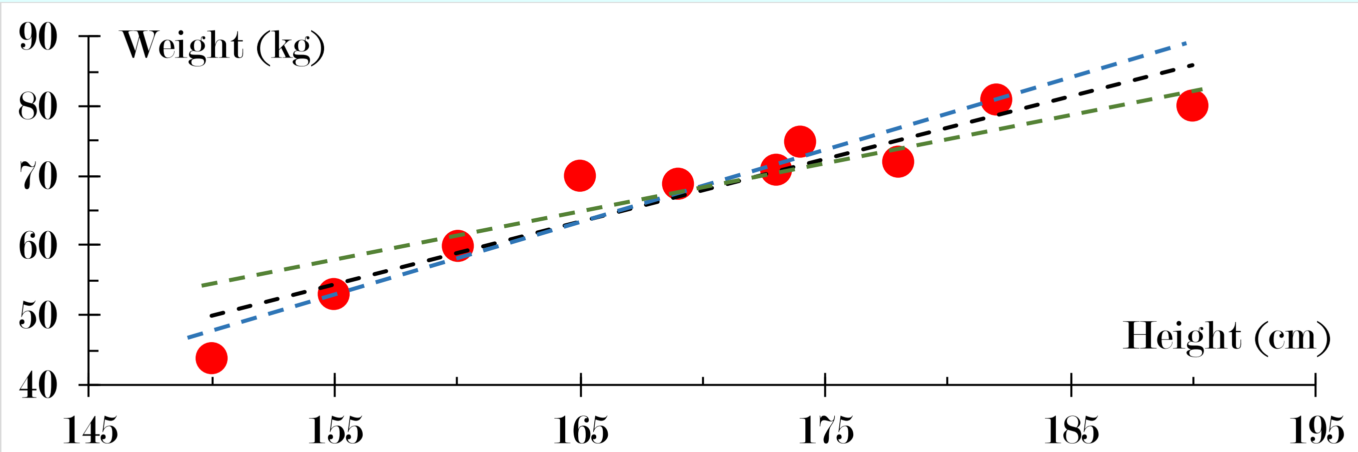
$$w = a + bh.$$

Recall the quote: "*All models are wrong, but some are useful.*" The model $w = a + bh$ is unlikely to be exactly correct. But hopefully it will be useful.

We treat a and b as unknown parameters (do you expect b to be positive or negative?). Our goal is to try to get estimates for a and b, from an observed random sample of height and weight data.

We recycle the data from earlier. These, along with the scatter diagram, are reproduced for convenience.

i	1	2	3	4	5	6	7	8	9	10
h_i (cm)	182	165	173	155	178	174	169	160	150	190
w_i (kg)	81	70	71	53	72	75	69	60	44	80



The basic idea of **linear regression** is this: *Find the line that "best fits" the given data.* Drawn in the figure above are three plausible candidates for the "line of best fit". But there can only be one line of best fit. Which is it?

At the end of the day, we'll choose black dotted line as "the" line of best fit. But why? This will be answered in the next section.

Example 1552 (continued from above). We suspect that daily rainfall and daily high temperatures for 2015 were linearly correlated. We thus write down this linear model:

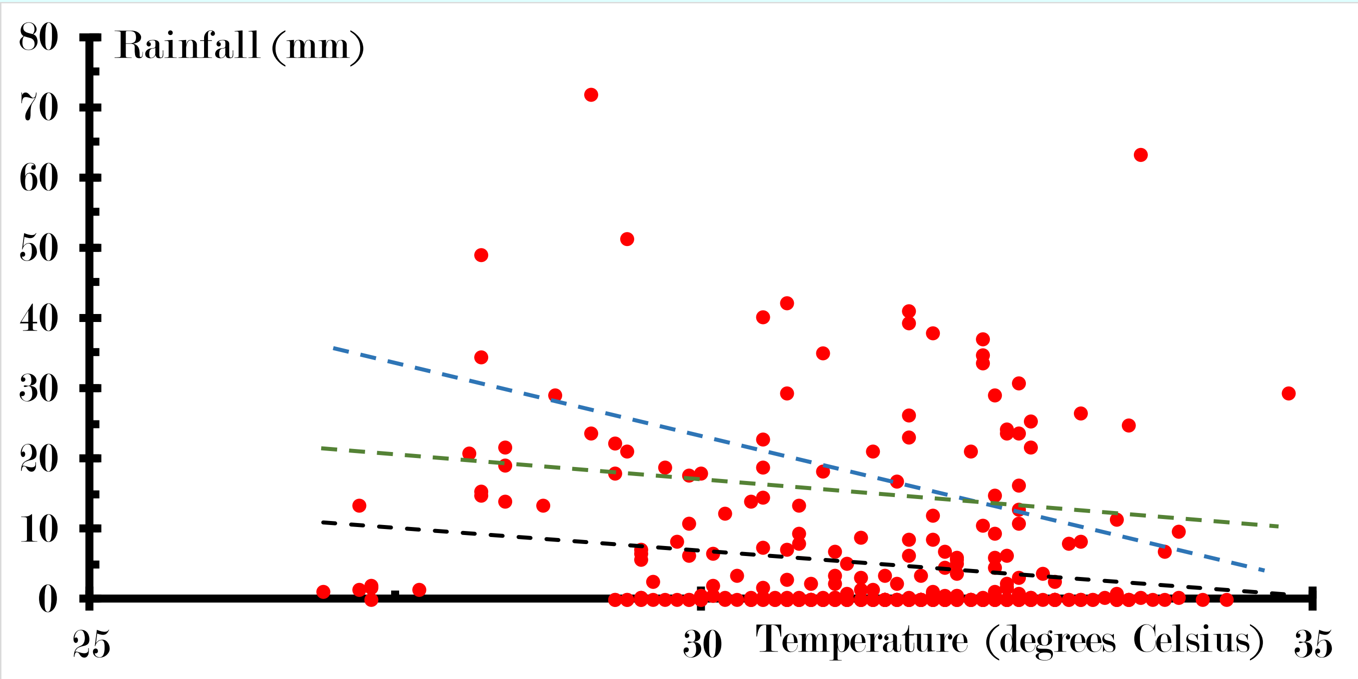
$$p = a + bt.$$

Again, our goal is to get estimates for the unknown parameters a and b (do you expect b to be positive or negative?).

We gather the following data (recycled from before):

i	1	2	3	4	...	361
t_i (°C)	27.3	29.5	31.1	32		30.2
p_i (mm)	0	0.2	0	0		12.4

We can again plot a **scatter diagram** of rainfall against temperature.



Again, drawn in the figure above are several plausible candidates for the "line of best fit". It turns out that the black dotted line will be "the" line of best fit.

132.5. Ordinary Least Squares (OLS)

There are different methods for determining "the" line of best fit. Each method will give a different line of best fit.

The method we'll learn in H2 Maths is the most basic and most standard method. It is called the method of **ordinary least squares (OLS)**.

Let's assume there is some true linear model, which may be written as $y = a + bx$. As always, we stick to the objectivist interpretation. The parameters a and b have some true, fixed values. However, they are unknown (and may forever be unknown).

Nonetheless, we'll try to do our best and get estimates for a and b. These estimates will be denoted $\hat{a}$ and $\hat{b}$. And our line of best fit will then be $y = \hat{a} + \hat{b}x$.

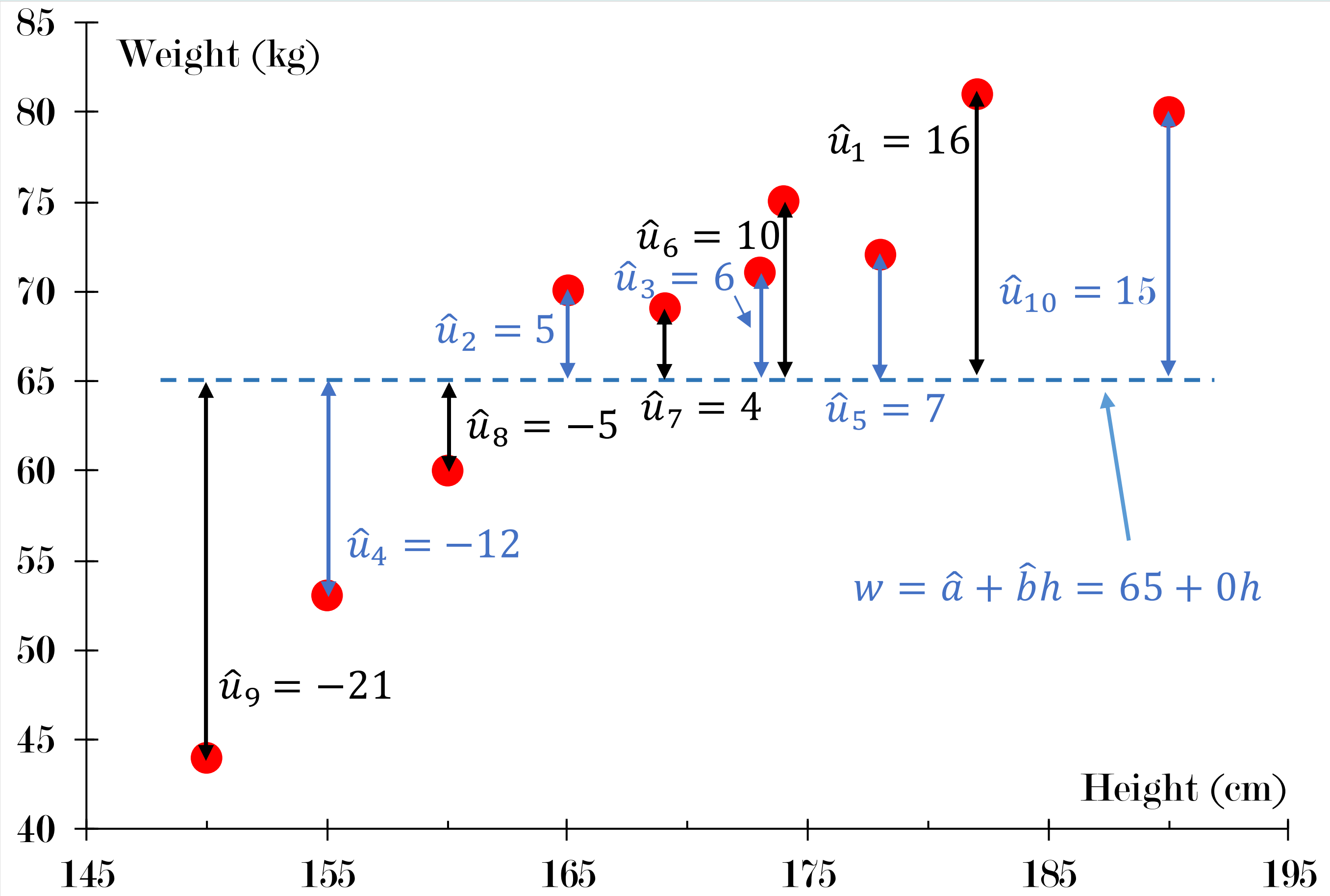
How do we find this line of best fit? Intuitively, this will be the line to which the data points are "as close as possible". But there are many ways to define the term "as close as possible". For example, we could try to minimise the sum of the distances between the points and the line. But we shall not do this.

Instead, we'll use the method of OLS:

1. Measure the vertical distance of each data point (x_i, y_i) from the line. This is called the **residual** and is denoted $\hat{u}_i$.
2. Our goal is to find the line $y = \hat{a} + \hat{b}x$ that minimises $\sum \hat{u}_i^2$—this quantity is called the Sum of Squared Residuals (SSR).

Example:

Example 1551 (height and weight example revisited). Our candidate line of best fit is $w = \hat{a} + \hat{b}h = 65 + 0h = 65$. This is a horizontal line, which simply "predicts" that everyone's weight is always 65 kg, regardless of their height. (This is a somewhat silly candidate line of best fit. Not surprisingly, this is not the actual line of best fit.)



i	1	2	3	4	5	6	7	8	9	10
h_i (cm)	182	165	173	155	178	174	169	160	150	190
w_i (kg)	81	70	71	53	72	75	69	60	44	80
$\hat{w}_i$ (kg)	65	65	65	65	65	65	65	65	65	65
$\hat{u}_i = w_i - \hat{w}_i$ (kg)	16	5	6	−12	7	10	4	−5	−21	15

The second last row of the above table gives, for each person with height h_i, the corresponding predicted weight $\hat{w}_i$ (as per our candidate line of best fit). The residual $\hat{u}_i$ (last row) is then defined as the vertical distance between the data point and the weight predicted by the candidate line of best fit.

The SSR is $\sum_{i=1}^{10} \hat{u}_i^2 = 16^2 + 5^2 + 6^2 + (-12)^2 + 7^2 + 10^2 + 4^2 + (-5)^2 + (-21)^2 + 15^2 = 1317$.

Can we do better than this? That is, can we find another candidate line of best fit whose SSR is smaller than 1317?

The following fact gives two formulae for $\hat{b}$, the gradient of the line of best fit. Formula (i) is printed in the List of Formulae (MF26) you get during exams, but formula (ii) is not.

Fact 248. *Let* $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *be two ordered sets of data. The OLS regression line of* y *on* x *is* $y - \bar{y} = \hat{b}(x - \bar{x})$, *where*

$$\textbf{(i)} \quad \hat{b} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

$$\textbf{(ii)} \quad \hat{b} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2}.$$

Moreover, the regression line can also be written in the form $y = \hat{a} + \hat{b}x$, *where* $\hat{b}$ *is as given above and* $\hat{a} = \bar{y} - \hat{b}\bar{x}$.

Proof. We want to find $\hat{a}$ and $\hat{b}$ such that the line $y = \hat{a} + \hat{b}x$ has the smallest SSR possible.

The residual $\hat{u}_i$ is defined as the vertical distance between (x_i, y_i) and the line $y = \hat{a} + \hat{b}x$. That is,

$$\hat{u}_i = y_i - y = y_i - \left(\hat{a} + \hat{b}x_i\right).$$

Thus, the SSR is $\sum \hat{u}_i^2 = \sum \left[y_i - \left(\hat{a} + \hat{b}x_i\right)\right]^2$.

We wish to minimise the SSR, by choosing appropriate values of $\hat{a}$ and $\hat{b}$. This involves the following pair of first order conditions:[549]

$$\frac{\partial}{\partial \hat{a}} \sum \hat{u}_i^2 = 0, \quad \frac{\partial}{\partial \hat{b}} \sum \hat{u}_i^2 = 0.$$

The remainder of the proof simply involves taking derivatives and doing the algebra, and is continued on p. 1744 (Appendices). □

Remark 177. Whenever we simply say **regression line** or **line of best fit**, it may safely be assumed that we are talking about the OLS regression line.

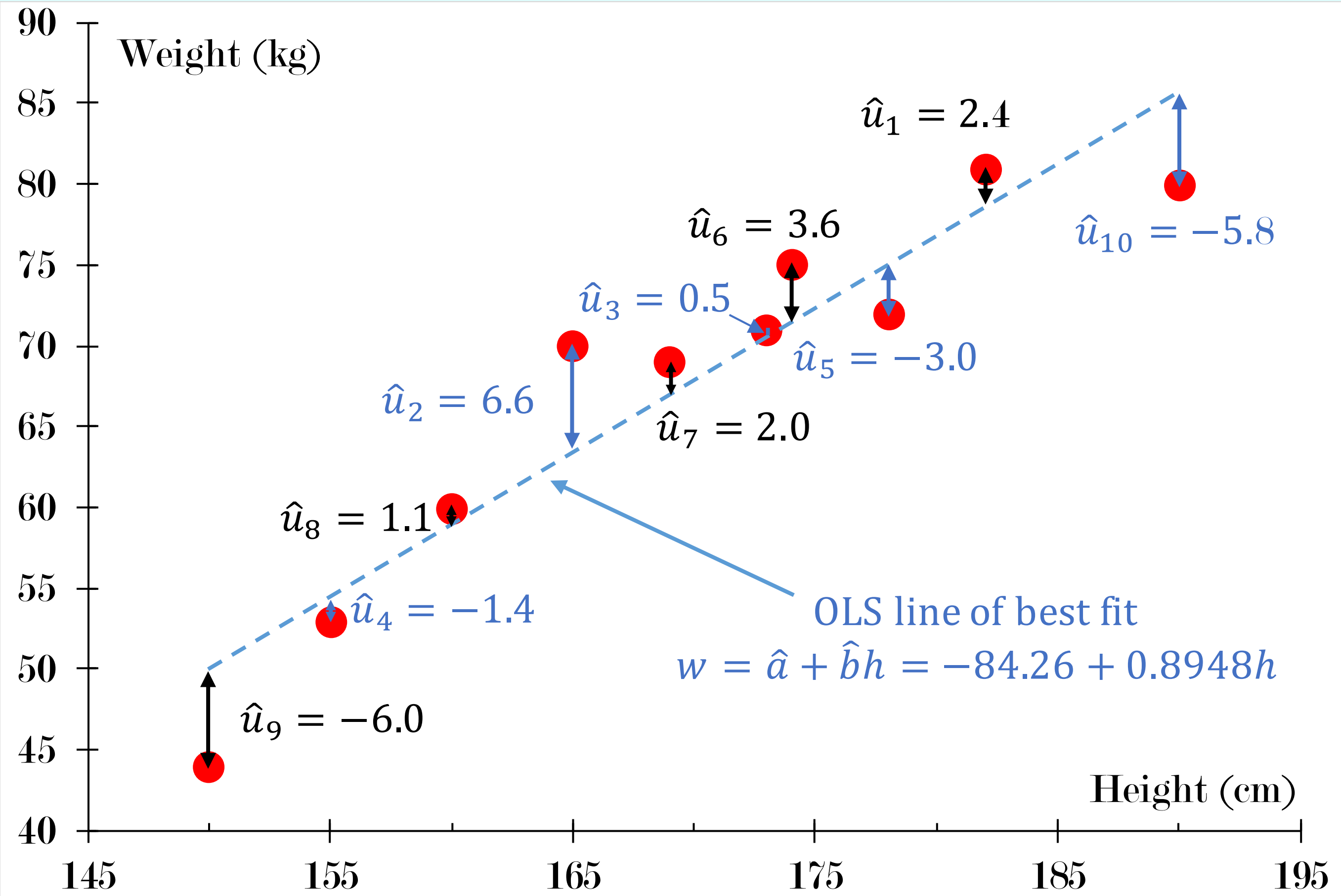
[549] There's a bit of hand-waving here.

Example 1551 (height and weight example revisited). We already calculated

$$\bar{h} = 169.6, \quad \bar{w} = 67.5, \quad \sum_{i=1}^{n}\left(h_i - \bar{h}\right)^2 = 1382.4, \quad \sum_{i=1}^{n}\left(h_i - \bar{h}\right)\left(w_i - \bar{w}\right) = 1237.$$

$$\text{So,} \quad \hat{b} = \frac{\sum_{i=1}^{n}\left(h_i - \bar{h}\right)\left(w_i - \bar{w}\right)}{\sum_{i=1}^{n}\left(h_i - \bar{h}\right)^2} = \frac{1237}{1382.4} \approx 0.8948.$$

Thus, the regression line is $w - 67.5 = 0.8948\left(h - 169.6\right)$ or $w = \hat{a} + \hat{b}h = -84.26 + 0.8948h$.



i	1	2	3	4	5	6	7	8	9	10
h_i (cm)	182	165	173	155	178	174	169	160	150	190
w_i (kg)	81	70	71	53	72	75	69	60	44	80
$\hat{w}_i$ (kg)	78.6	63.4	70.5	54.4	75.0	71.4	67.0	58.9	50.0	85.8
$\hat{u}_i = w_i - \hat{w}_i$ (kg)	2.4	6.6	0.5	−1.4	−3.0	3.6	2.0	1.1	−6.0	−5.8

The SSR for the actual line of best fit is $\sum_{i=1}^{10} \hat{u}_i^2 = 2.4^2 + \cdots + (-5.8)^2 \approx 147.6$. This is much better than the SSR of 1317 that we found for the previous candidate line of best fit, which was simply a horizontal line.

Exercise 521. (a) Find the regression line of q on p, using the data below. (b) Complete the table. (c) Draw the scatter diagram, including the regression line and the corresponding residuals. (d) Compute the SSR. (Answer on p. **1988**.)

i	1	2	3	4	5
p_i (\$)	8	9	4	10	8
q_i	300	250	1000	400	400
$\hat{q}_i$					
$\hat{u}_i = q_i - \hat{q}_i$					

Example 1553. We'll find the PMCC and the regression line for these data:

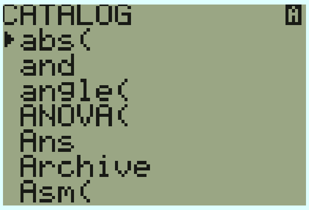
i	1	2	3	4	5
x_i	1	7	3	11	8
y_i	14	5	6	4	4

1. Press ON to turn on your calculator.
2. Press the blue 2ND button and then CATALOG (which corresponds to the 0 button). This brings up the CATALOG menu.
3. Using the down arrow key ⌄, scroll down until the cursor is on DiagnosticOn.
4. Press ENTER once. And press ENTER a second time. The TI84 now says "DONE", telling you that the Diagnostic option has been turned on.

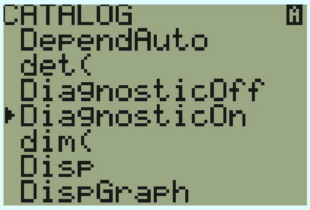
The above steps need only be performed once. Unless of course you've just reset your calculator (as is required before each exam). In which case you have to go through the above steps again.

Step 1.

Step 2.



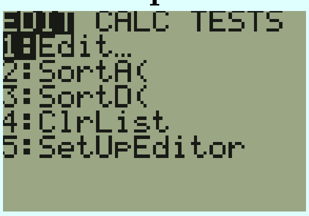
Step 3.



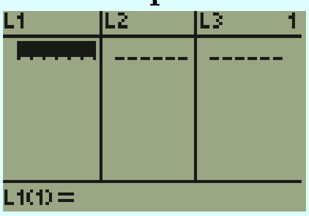
Step 4.

5. Press STAT to bring up the STAT menu.
6. Press 1 to select the "1:Edit" option.
7. The TI84 now prompts you to enter data under the column titled "L1". This is where you should enter the data for x, using the numeric pad and the ENTER key as is appropriate. (I omit from this step the exact buttons you should press.)
8. After entering the last entry, press the right arrow key ⟩ to go to column L2. So enter the data for y, again using the numeric pad and the ENTER key as is appropriate.

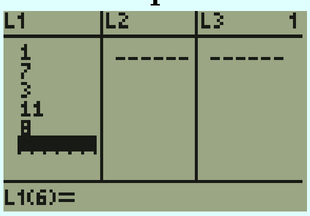
Step 5.



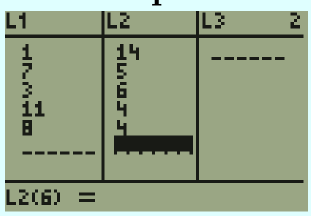
Step 6.



Step 7.



Step 8.



(*Example continues on the next page ...*)

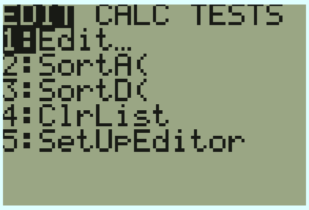
(*... Example continued from the previous page.*)

9. Now press STAT to again bring up the STAT menu.
10. Press the right arrow key ⟩ to go to the CALC submenu.
11. Press 4 to select the "4:LinReg(ax+b)" option.
12. To tell the TI84 to go ahead and do the calculations, simply press ENTER.

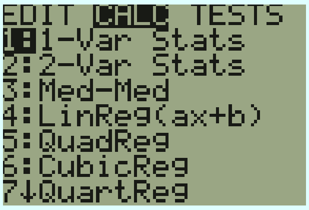
The TI84 tells you that the PMCC is $r = -.8147656398$. The equation of the regression line of y on x is $y = ax + b = -.859375x + 11.75625$.

(Be careful to note that the TI84 uses the symbol "a" for the coefficient for x, whereas in the List of Formulae (MF26), they use b instead. Don't get these mixed up!)

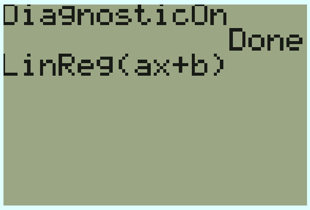
Step 9.



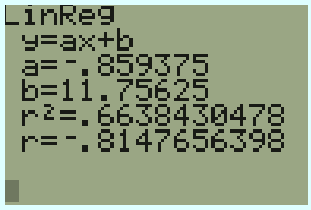
Step 10.



Step 11.



Step 12.



Exercise 522. Using your TI84, find the PMCC between q and p, and also find the regression line of q on p (see data below). Verify that your answer for this exercise is the same as those in the last two exercises. (Answer on p. **1989**.)

i	1	2	3	4	5
p_i (\$)	8	9	4	10	8
q_i	300	250	1000	400	400

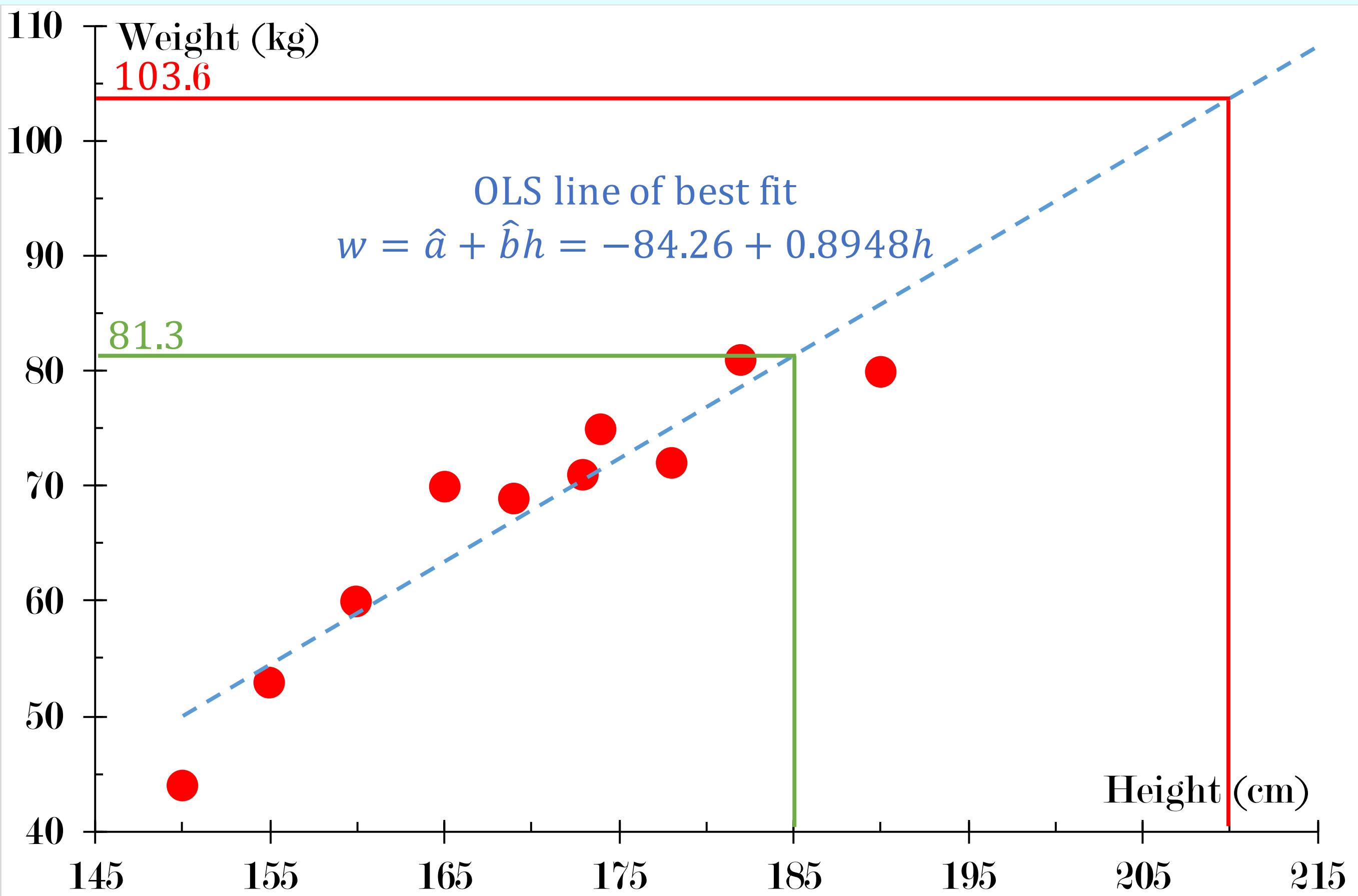
132.7. Interpolation and Extrapolation

Given any value of x, we call the corresponding $\hat{y} = \hat{b}(x - \bar{x}) + \bar{y}$ the **fitted value** or the **predicted value**. One use of the regression line is that it can help us **predict** (or "guess") the value of y, even for x for which we have no data.

Example 1551 (height and weight example revisited). Say we want to guess the weight of an adult male Singaporean who is 185 cm tall. Using our regression line, we predict that his weight is $\hat{w}_{h=185} = 0.8948 \times 185 - 84.26 \approx 81.3$ kg. This is called **interpolation**, because we are predicting the weight of a person whose height is between two of our observations.

Say instead we want to guess the weight of an adult male Singaporean who is 210 cm tall. Using our regression line, we predict that his weight is $\hat{w}_{h=210} = 0.8948 \times 210 - 84.26 \approx 103.6$ kg. This is called **extrapolation**, because we are predicting the weight of a person whose height is beyond on our rightmost observation.

i	1	2	3	4	5	6	7	8	9	10		
h_i (cm)	182	165	173	155	178	174	169	160	150	190	185	210
w_i (kg)	81	70	71	53	72	75	69	60	44	80	-	-
$\hat{w}_i$ (kg)	78.6	63.4	70.5	54.4	75.0	71.4	67.0	58.9	50.0	85.8	81.3	103.6



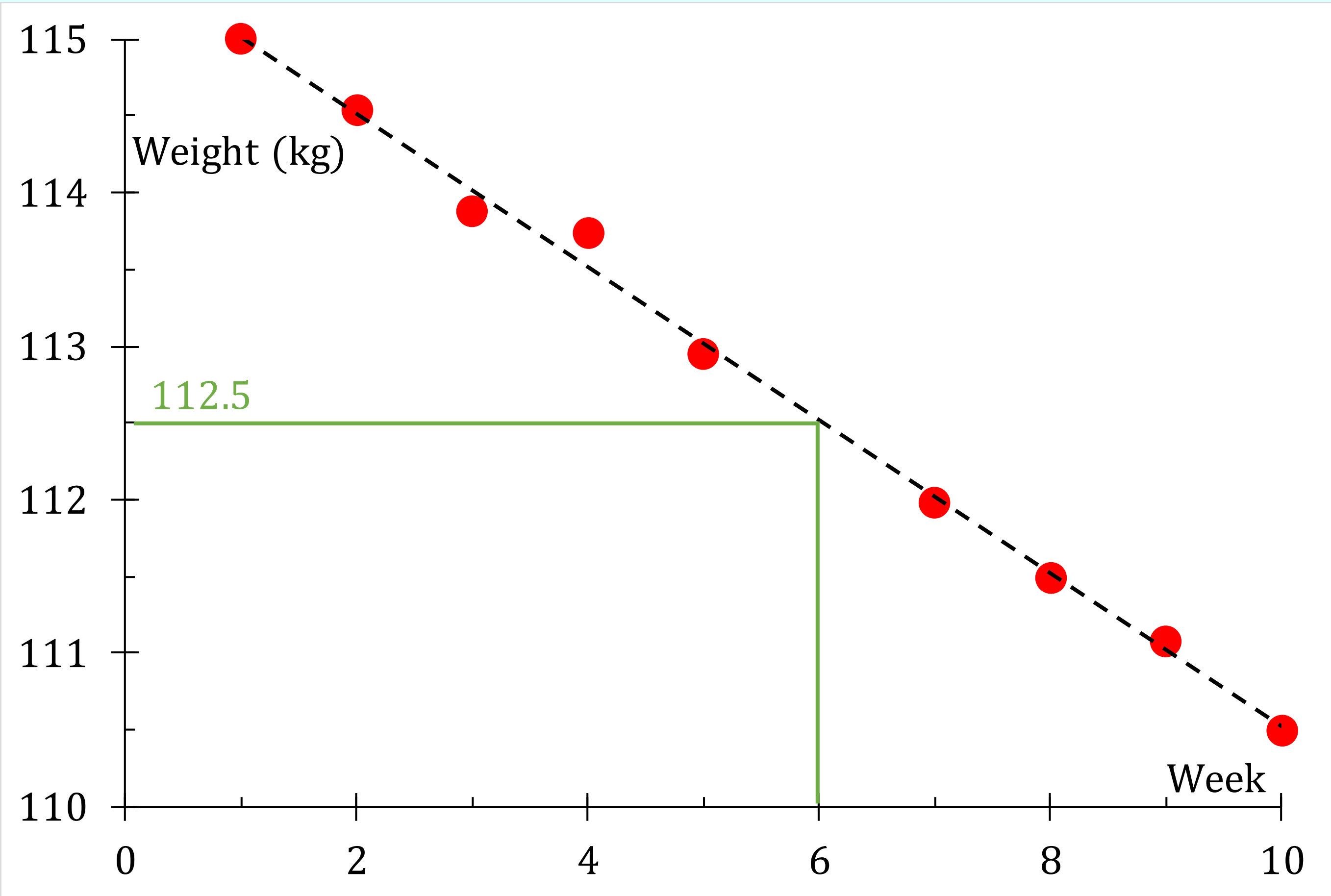
For the A-Level exams, you are supposed to mindlessly and formulaically say that "Extrapolation is less reliable than interpolation", because

> The former predicts what's *beyond* the known observations; the latter predicts what's *between* two known observations.

This, though, is not a very satisfying explanation for why extrapolation is "less reliable" than interpolation. It merely leads to another question: "Why should a prediction be more reliable if done between two known observations, than if done to the right of the right-most observation (or to the left of the left-most observation)?"

We won't give an adequate answer to this latter question. Instead, we'll simply give a bunch of examples to illustrate the dangers of extrapolation:

Example 1554. A man on a diet weighs 115 kg in Week #1. Here's a chart of his weight loss.

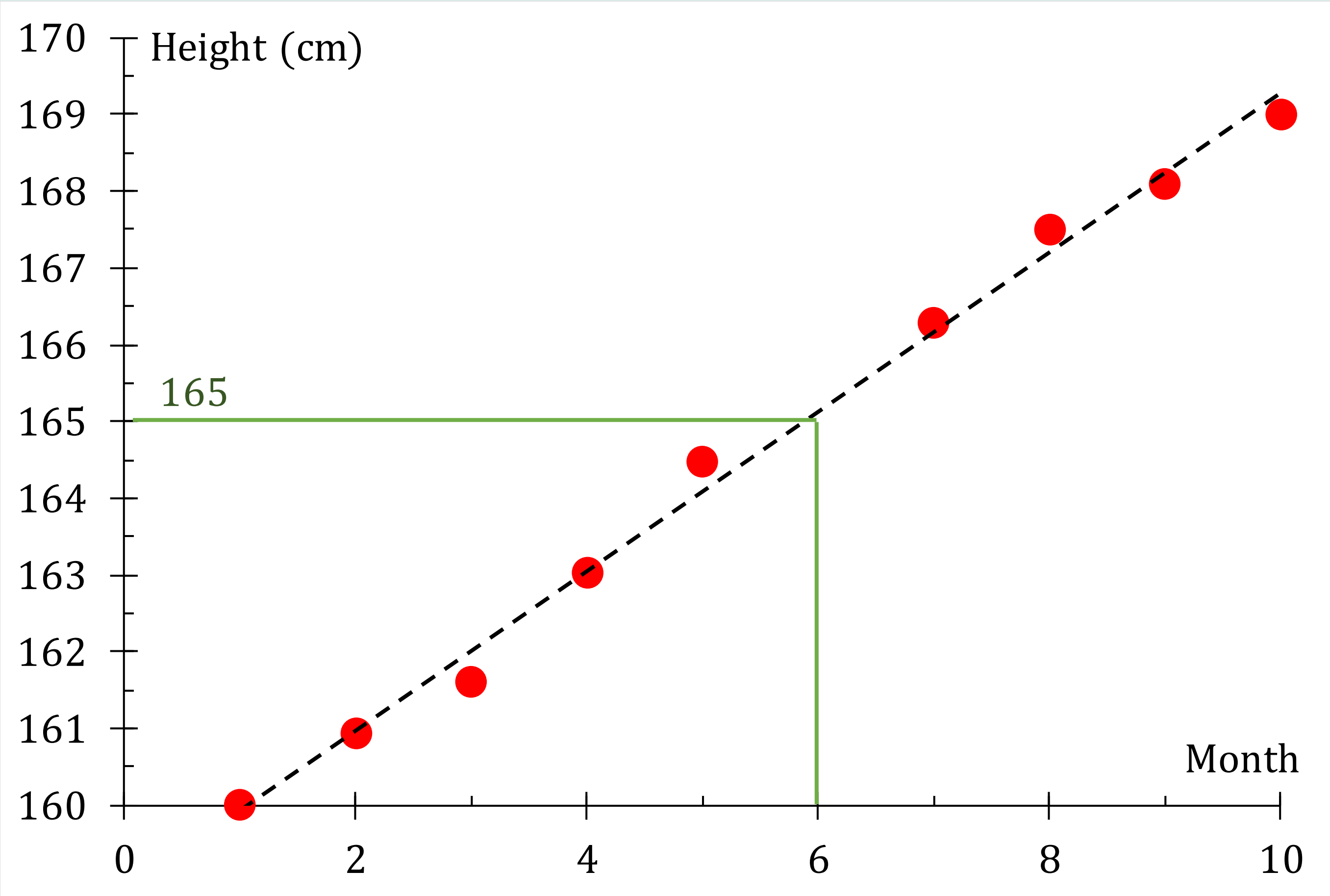


The OLS line of best fit suggests that he has been losing about 0.5 kg a week.

He forgot to record his weight on Week #6. By interpolation, we "predict" that his weight that week was 112.5 kg. This is probably a reliable guess.

By extrapolation, we predict that his weight on Week #201 will be 15 kg. This guess is obviously absurd. It requires that he keeps losing 0.5 kg a week for nearly 4 years.

Example 1555. A growing boy is 160 cm tall in Month #1. Here's a chart of his growth.



The OLS line of best fit suggests that he has been growing by about 1 cm a month.

He forgot to record his height in Month #6. By interpolation, we "predict" that his height that month was 165 cm. This is probably a reliable guess.

By extrapolation, we predict that his height in Month #101 will be 260 cm. This guess is obviously absurd. It requires that he keep growing by 1 cm a month for the 8-plus years.

Here are three colourful examples of the dangers of extrapolation from other contexts.

Example 1556. Russell's Chicken (*Problems of Philosophy*, 1912, Google Books link):

The man who has fed the chicken every day throughout its life at last wrings its neck instead, showing that more refined views as to the uniformity of nature would have been useful to the chicken. ... The mere fact that something has happened a certain number of times causes animals and men to expect that it will happen again. Thus our instincts certainly cause us to believe the sun will rise to-morrow, but we may be in no better a position than the chicken which unexpectedly has its neck wrung.

Example 1557. The Fermat numbers are

$$\begin{aligned}
F_0 &= 2^{2^0} + 1 = 3, \\
F_1 &= 2^{2^1} + 1 = 5, \\
F_2 &= 2^{2^2} + 1 = 17, \\
F_3 &= 2^{2^3} + 1 = 257, \\
F_4 &= 2^{2^4} + 1 = 65537.
\end{aligned}$$

Remarkably, the first five Fermat numbers are all prime. This observation led Fermat to conjecture (guess) in the 17th century that all Fermat numbers are prime. This was an act of extrapolation.

Unfortunately, Fermat's act of extrapolation was wrong. About a century later, Euler showed that $F_5 = 2^{2^5} + 1 = 4294967297 = 641 \times 6700417$ is composite (not prime).

Today, the Fermat numbers F_5, F_6, ..., F_{32} are all known to be composite. Indeed, it was shown in 1964 that F_{32} is composite. Over half a century later, it is not yet known if $F_{33} = 2^{2^{33}} + 1$ is prime or composite. F_{33} is an unimaginably huge number, with $2,585,827,973$ *digits*.

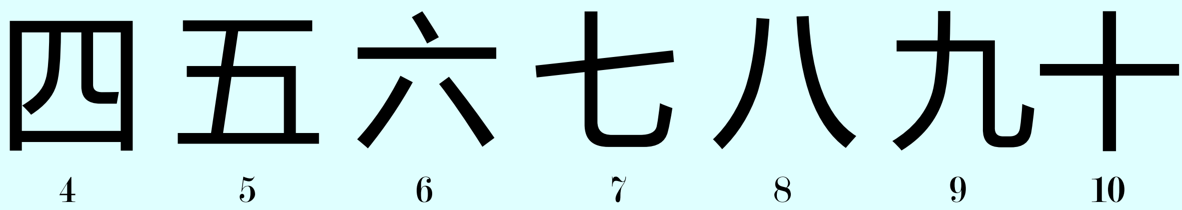
Example 1558. On Ah Beng's first day at school, he learns in Chinese class that the Chinese character for the number 1 is written as a single horizontal stroke.

On his second day at school, he learns that the Chinese character for the number 2 is written as two horizontal strokes.

On his third day at school, he learns that the Chinese character for the number 3 is written as three horizontal strokes.

After his third day at school, Ah Beng decides he'll skip at least the next few Chinese classes, because he thinks he knows how to write the Chinese characters for the numbers 4 and above. 4 simply consists of four horizontal strokes; 5 simply consists of five horizontal strokes; etc. Unfortunately, Ah Beng's act of extrapolation is wrong.

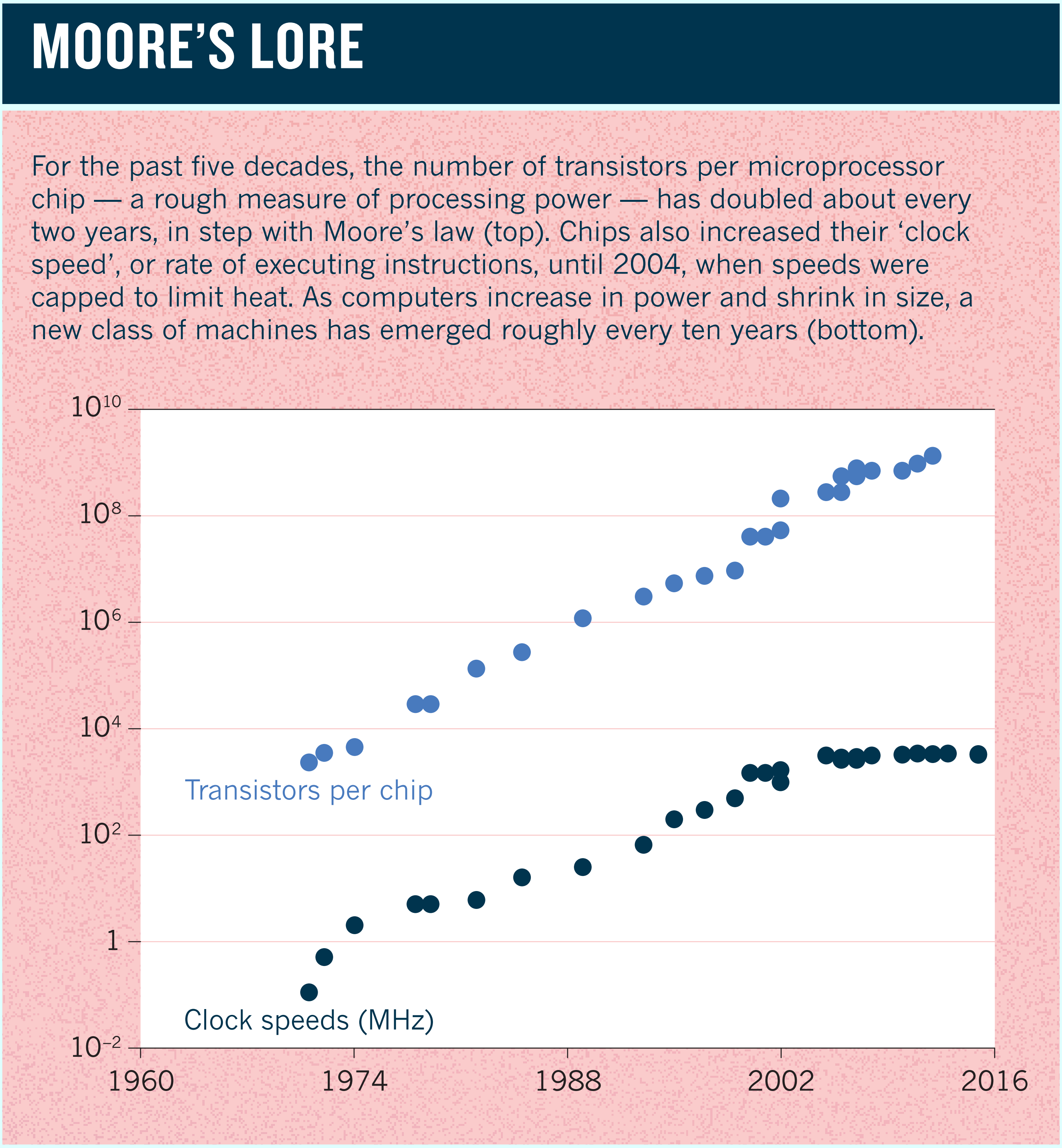
The characters for the numbers 4 through 10 look instead like this:



On the other hand, here are two historical examples of extrapolation that, to everyone's surprise, have held up remarkably well (at least to date).

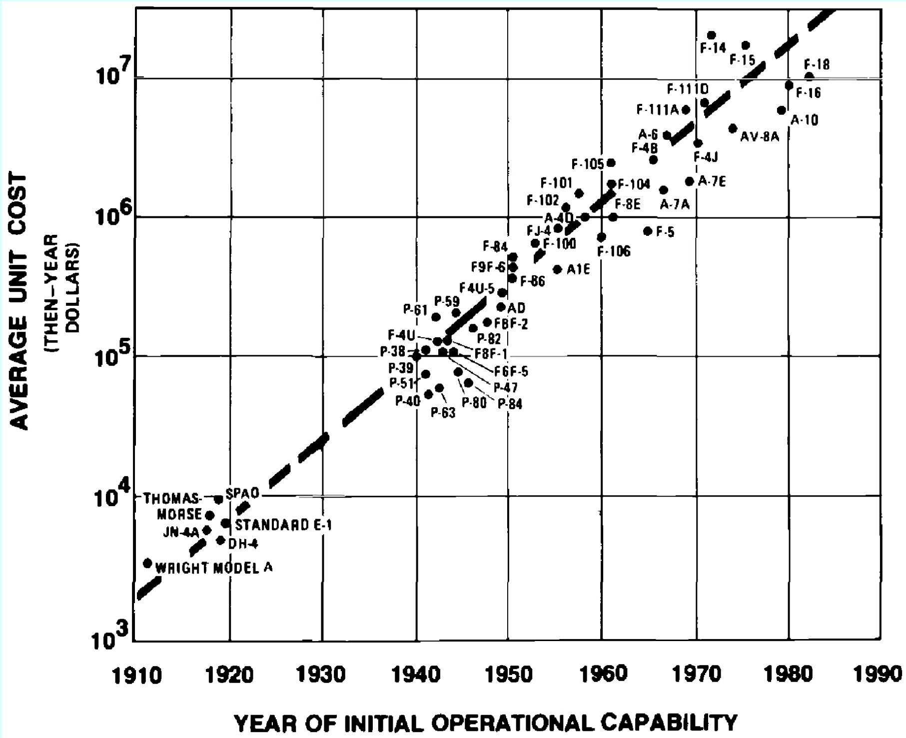
Example 1559. Moore's Law. In 1965, Gordon Moore observed that the number of components that could be crammed onto each integrated circuit *doubled every year*. He predicted that this rate of progress would continue at least through 1975.

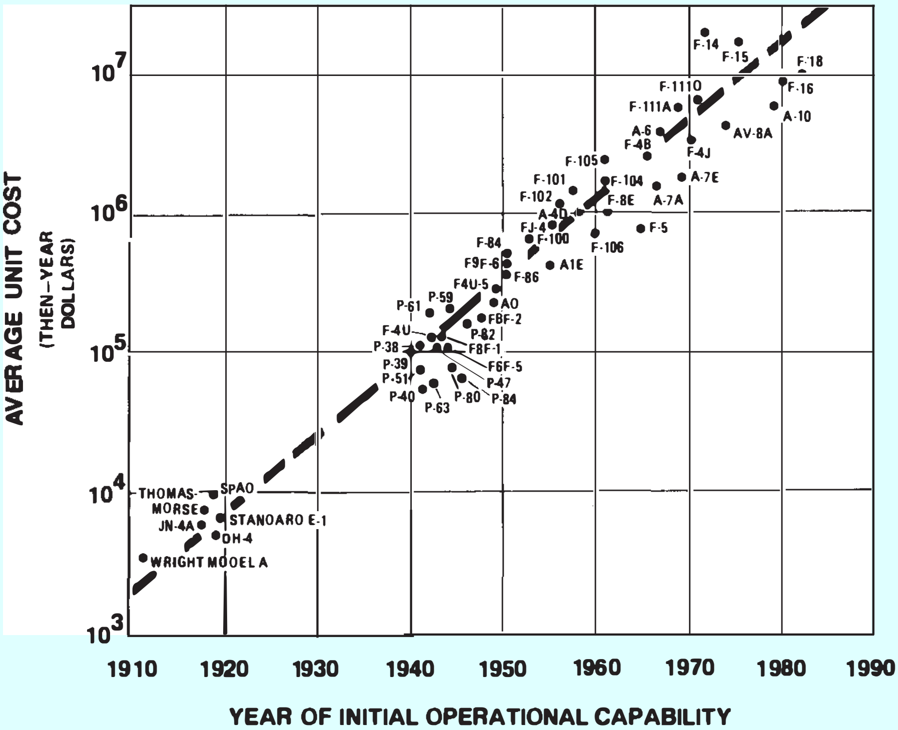
In 1975, he adjusted his prediction to a more modest rate of *doubling every two years*. Thus far, this latter prediction has held up remarkably well. The following from Nature:



Unfortunately, as stated in the same Nature article, it "has become increasingly obvious to everyone involved" that "Moore's law … is nearing its end".

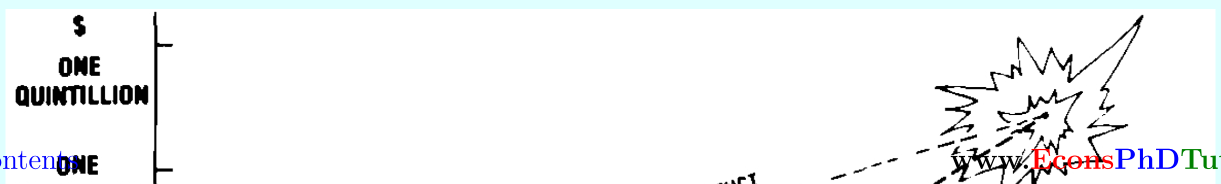
Example 1560. Augustine's Law. In 1983, Norman Augustine observed that the cost of a tactical aircraft grows four-fold every ten years. (Google Books.)





This is considerably quicker than the rate at which the annual US defense budget and US Gross National Product (GNP) grows. Extrapolating, he concluded:

- In 2054, the entire annual US defense budget will be spent on a single aircraft.
- Early in the 22nd century, the entire US GNP will be spent on a single aircraft.

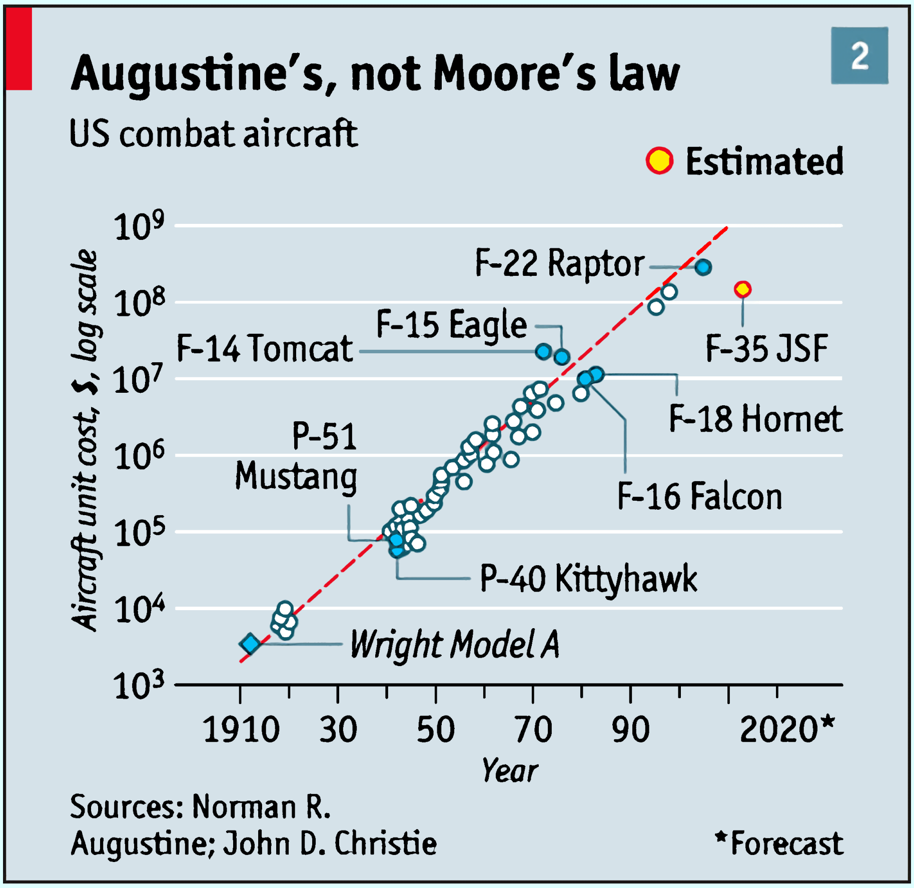


(... *Example continued from the previous page.*)

These seemingly absurd conclusions were written at least partly in jest.

Except so far they have been right on track. In a 2010 Economist article, Augustine was quoted as saying, "We are right on target. Unfortunately nothing has changed." That article also presented an updated version of Augustine's Law.

The latest F-35 fighter program is estimated to cost the US Department of Defense US$1.124 *trillion*. To be fair, that estimate is the cost of the entire program over its projected 60-year lifespan (through 2070)—this includes R&D, the purchase of over 2,000 F-35s, and operating costs. But still, US$1.124 *trillion* is a mind-blowing figure.[550]



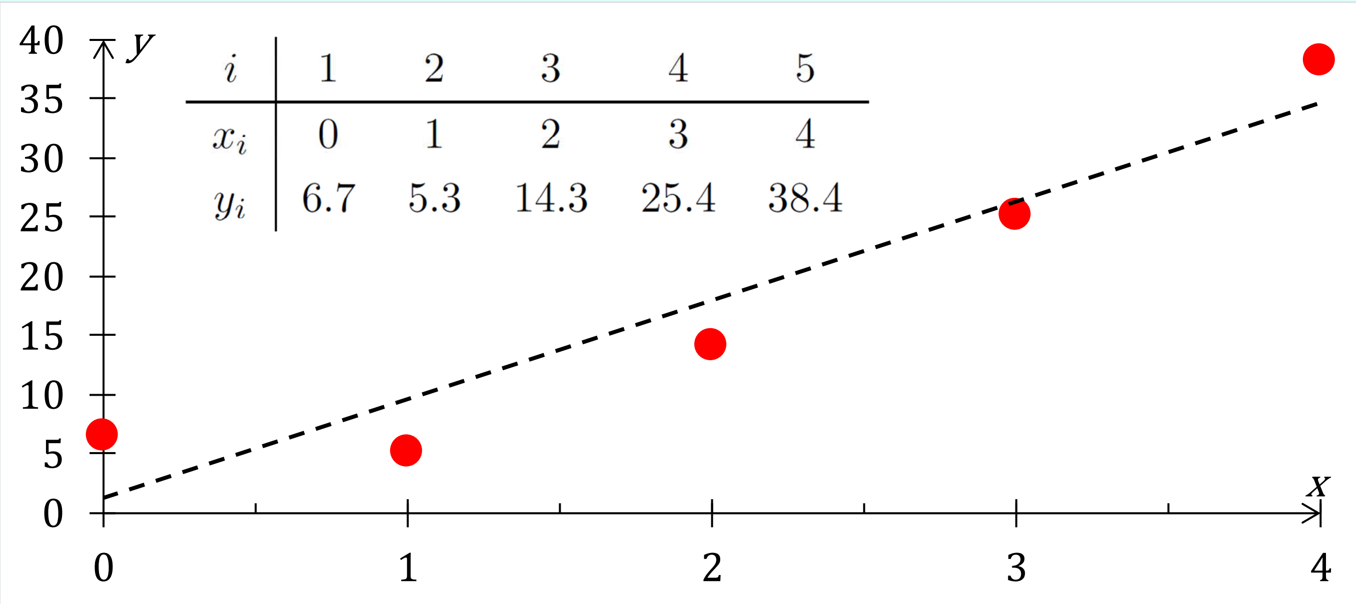
Exercise 523. Using the data below, "predict" how many haircuts were sold in June 2016 by (a) a barber who charged \$7 per haircut; and (b) a barber who charged \$200 per haircut. Which prediction is an act of interpolation and which is an act of extrapolation? Which prediction do you think is more reliable?(Answer on p. **1989**.)

i	1	2	3	4	5
p_i (\$)	8	9	4	10	8
q_i	300	250	1000	400	400

132.8. Transformations to Achieve Linearity

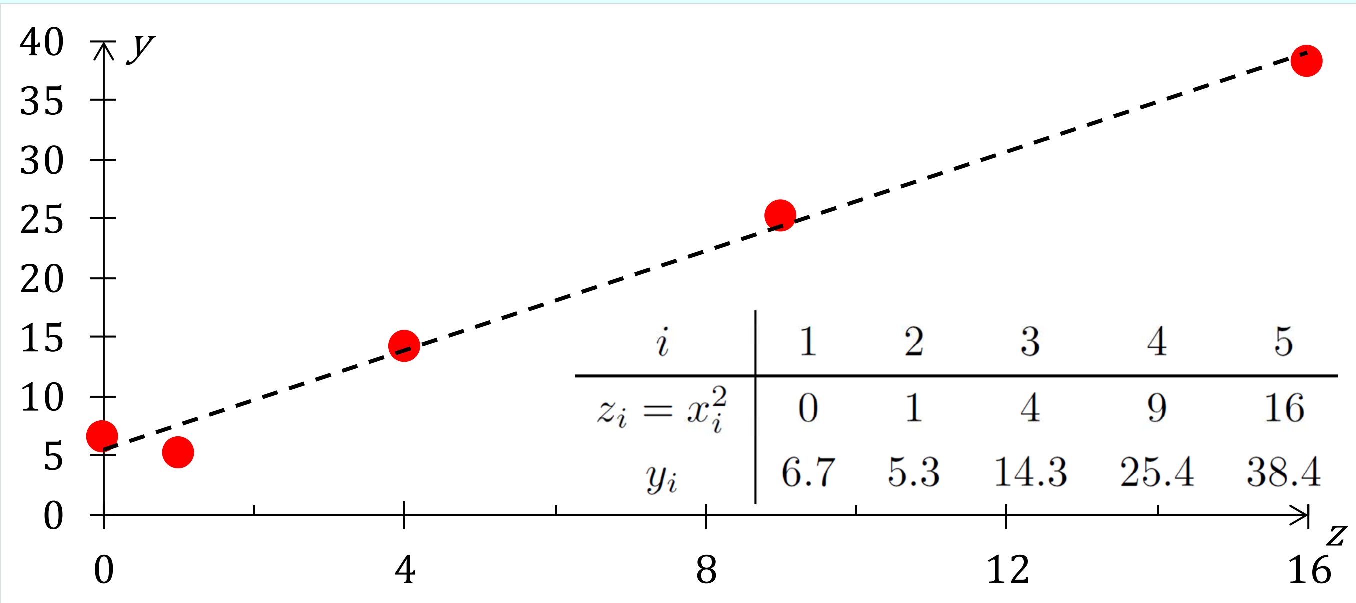
Two variables may have a relationship, but not a linear one. Here we consider cases where the relationship is quadratic, reciprocal, or logarithmic.

Example 1561. Quadratic. Consider the following data. There is a very strong, but not perfect degree of linear correlation between x and y ($r \approx 0.950$). The observations are very close to, but are not exactly on the OLS line of best fit.



i	1	2	3	4	5
x_i	0	1	2	3	4
y_i	6.7	5.3	14.3	25.4	38.4

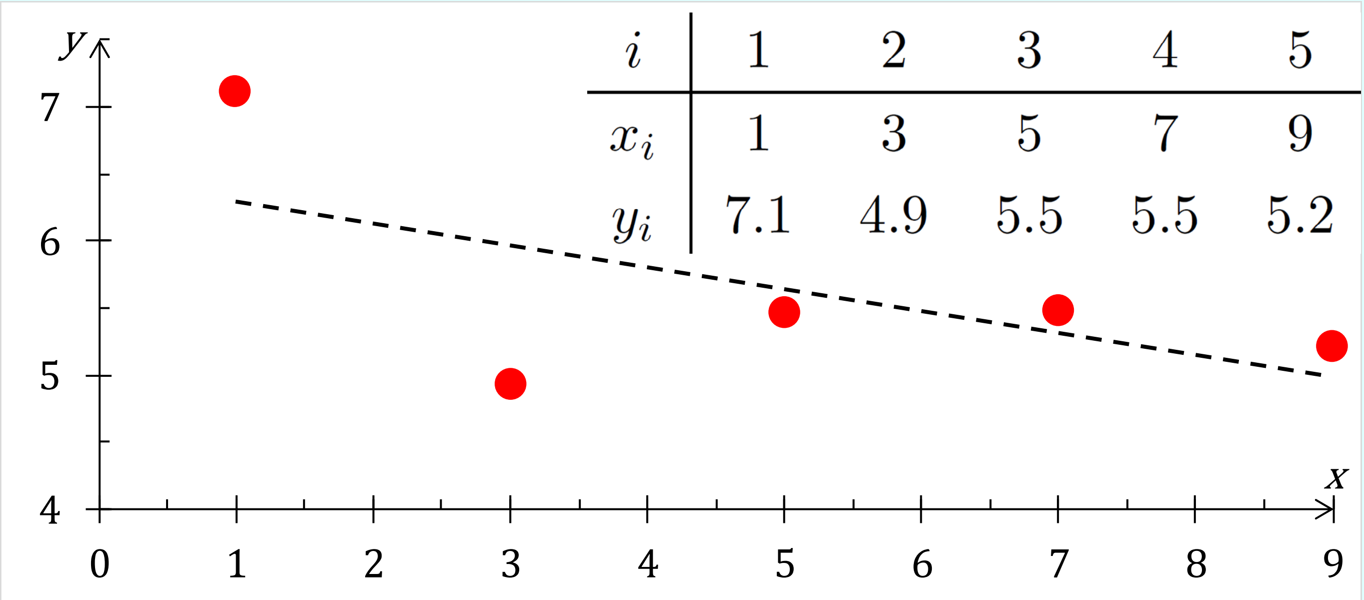
Perhaps we can do better by transforming the data. We'll do a quadratic transformation: let $z_i = x_i^2$. Then we have the following data.



i	1	2	3	4	5
$z_i = x_i^2$	0	1	4	9	16
y_i	6.7	5.3	14.3	25.4	38.4

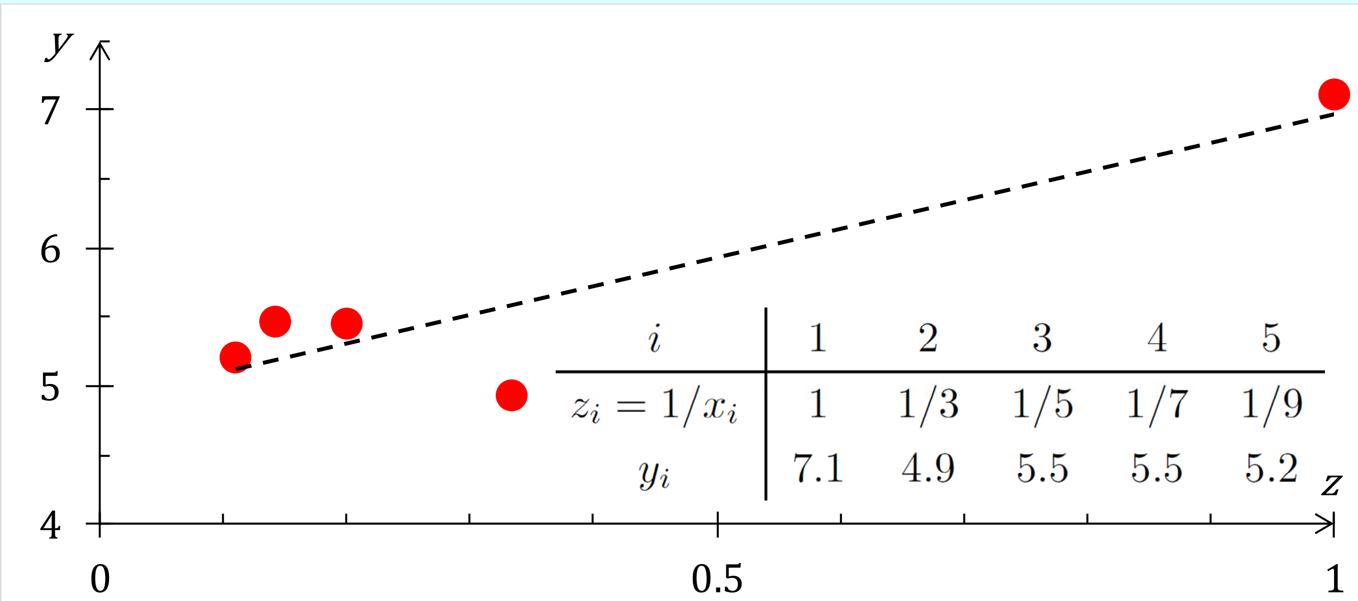
The degree of linear correlation between z and y is near perfect ($r \approx 0.995$). The observations also lie closer to the line of best fit than before.

Example 1562. Reciprocal. Consider the following data. There seems to be a moderate degree of linear correlation between x and y ($r \approx -0.603$). The observations are fairly close to the OLS line of best fit.



i	1	2	3	4	5
x_i	1	3	5	7	9
y_i	7.1	4.9	5.5	5.5	5.2

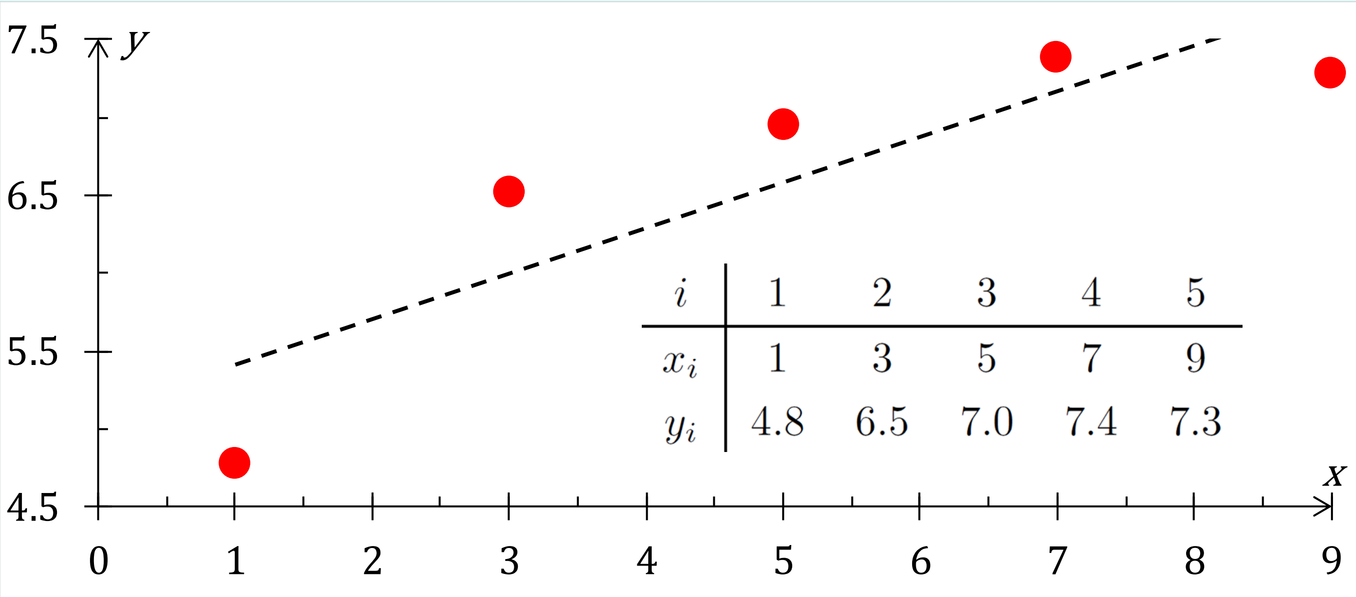
Perhaps we can do better by transforming the data. We'll do a reciprocal transformation: let $z_i = 1/x_i$. Then we have the following data and scatter diagram.



i	1	2	3	4	5
$z_i = 1/x_i$	1	1/3	1/5	1/7	1/9
y_i	7.1	4.9	5.5	5.5	5.2

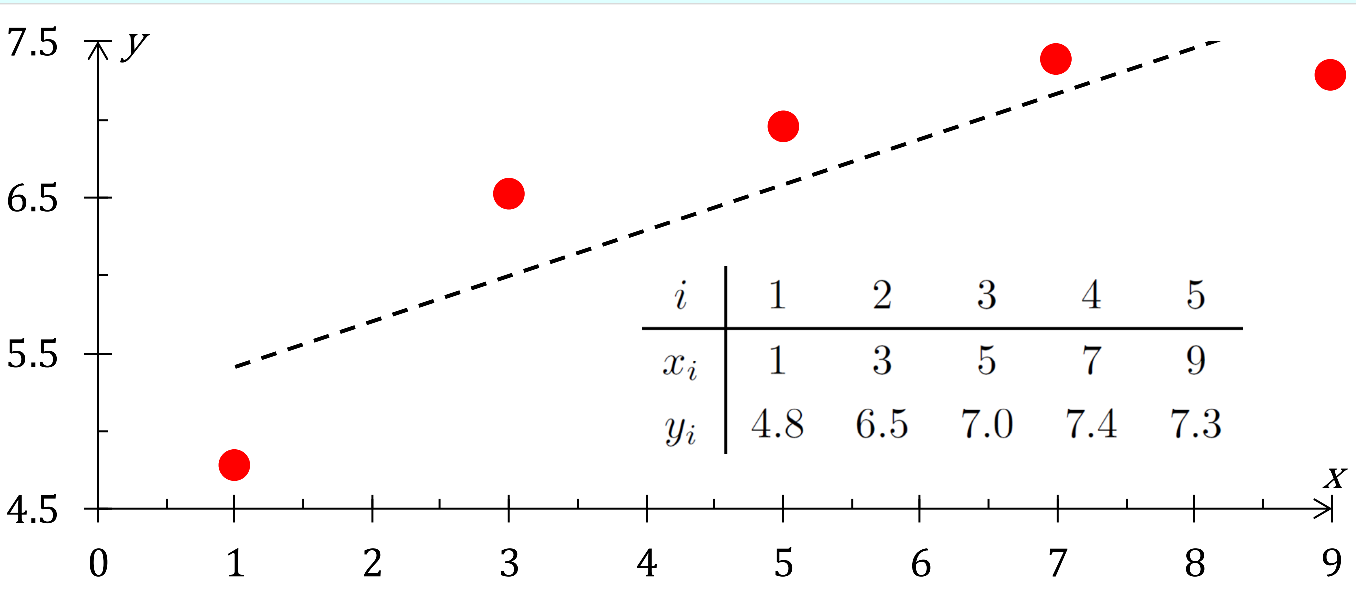
The degree of linear correlation between z and y is much stronger ($r \approx 0.899$). The observations also lie closer to the line of best fit.

Example 1563. Logarithmic. Consider the following data. There seems to be a fairly strong degree of linear correlation between x and y ($r \approx 0.873$). The observations are fairly close to, but are not exactly on the OLS line of best fit.



i	1	2	3	4	5
x_i	1	3	5	7	9
y_i	4.8	6.5	7.0	7.4	7.3

Perhaps we can do better by transforming the data. We'll do a reciprocal transformation: let $z_i = \ln x_i$. Then we have the following data and scatter diagram.



i	1	2	3	4	5
x_i	1	3	5	7	9
y_i	4.8	6.5	7.0	7.4	7.3

The degree of linear correlation between z and y is much stronger ($r \approx 0.978$). The observations also lie closer to the line of best fit.

Exercise 524. You are given the following data. (Answer on p. **1990**.)

i	1	2	3	4	5
x_i	1	2	3	4	5
y_i	10.59	10.54	27.30	33.84	56.6

(a) Plot the above data in a scatter diagram and find the PMCC.

(b) Apply an appropriate transformation to x. Plot the transformed data in a scatter diagram and find the PMCC.

132.9. The Higher the PMCC, the Better the Model?

There are no routine statistical questions, only questionable statistical routines.

— J.M. Hammersley[551]

It's much more interesting to live not knowing than to have answers which might be wrong.

— Richard Feynman (1981).

The A-Level examiners[552] want you to say, mindlessly and formulaically, that

All else equal, a model with a higher PMCC is better than a model with a lower PMCC.

Regurgitating the above sentence will earn you your full mark. But in fact, without the "all else equal" clause, **it is nonsense**. And since it is almost never true that "all else is equal", it is almost always nonsense.

In every introductory course or text on statistics, one is told that the PMCC is merely a relatively unimportant consideration, in deciding between models. Yet somehow, the A-Level examiners seem to consider the PMCC an all-important consideration.

Here's a quick example to illustrate.

Example 1564. (From the 2015 exam—see Exercise **793** below.) In an experiment the following information was gathered about air pressure P, measured in inches of mercury, at different heights above seA-Level h, measured in feet.

h	2000	5000	10000	15000	20000	25000	30000	35000	40000	45000
P	27.8	24.9	20.6	16.9	13.8	11.1	8.89	7.04	5.52	4.28

The exam first asks us to find the PMCCs between (a) h and P; (b) $\ln h$ and P; and (c) $\sqrt{h}$ and P. The answers are (a) $r_a \approx -0.980731$; (b) $r_b \approx -0.974800$; and (c) $r_c \approx -0.998638$.

The A-Level exam then says, "Using the most appropriate case ..., find the equation which best models air pressure at different heights." The "correct" answer is that (c) $P = a + b\sqrt{h}$ is the "most appropriate" model, simply because the PMCC there is the largest.

(*Example continues on the next page ...*)

[552]See 9740 N2015/II/10(iii), N2014/II/8(b)(ii), N2012/II/8(v), N2011/II/8(iii), N2010/II/10(iii), and N2008/II/8(i). These are given in this textbook as Exercises 793, 799, 814, 820, 830, and 842.

(... *Example continued from the previous page.*)

But this is utter nonsense. One does not conclude that one model is "more appropriate" than another simply because its PMCC is 0.018 larger. Small measurement errors or plain bad luck could easily explain these tiny differences in PMCCs.

Moreover, even if one model has $r = 0.9$ and another has $r = 0.4$, it does not automatically follow that the first model is "more appropriate" than the second. In deciding which statistical model to use, there are very many considerations, of which the PMCC is a relatively unimportant one.

In my view, the correct answer should have been this:

We have far too little information to make any conclusions.

Sadly, in the Singapore education system, what I consider to be the correct answer would not have gotten you any marks. Instead, one is taught that there must always be one single, simplistic, formulaic, definitive, "correct" answer. This is a convenient substitute for thinking.

As it turns out, the "most correct" linear model—based on the actual barometric formula (see subsection 147.10 (Appendices))—is actually the following:

$$\ln P = a + b \ln \left(1 + \frac{L}{T} h\right).$$

The constants $L = -0.0065$ kelvin per metre (Km^{-1}) and $T = 288.15$ kelvin (K) are, respectively, the standard temperature lapse rate (up to $11,000$ m above sea level) and the standard temperature (at sea level).

The PMCC for the above model is $r_d \approx 0.999998$, which is "better" than the cases examined above. (See this Google spreadsheet for the data and calculations.)

But again, **the PMCC is merely one relatively unimportant consideration**. Our conclusion that this last model is superior to the model $P = a + b\sqrt{h}$ is based not on the fact that r_d is 0.001 larger than r_c.

Instead, we are confident in this model because it was *derived* from physical theories. In contrast, the model $P = a + b\sqrt{h}$ (or indeed any of the other models suggested above) is completely arbitrary and has no theoretical justification. Hence, even if the model $P = a + b\sqrt{h}$ had a PMCC of 1, we'd still prefer this last model.

Part VII.
Ten-Year Series

Answers for Part VI (Probability and Statistics) 2016–19 questions will be written “soon”. ☺

As stated on p. 4 of your H2 Maths syllabus, your A-Level exam will consist of **two papers** of **3 hours each**.

- Paper 1 [total 100 points]: Pure Mathematics, 3 hours, 10–12 questions.
- Paper 2 [total 100 points]: 3 hours
  - Section A [total 40 points]: Pure Mathematics, 4–5 questions.
  - Section B [total 60 points]: Probability and Statistics, 6–8 questions.

Each paper contains 100 points. So, you have an average of 1.8 min = 108 s to spend on each point and each point is 0.5% of the maximum score of 200.

For more practice, try the TYS questions for H1 Maths (in my *H1 Maths Textbook*). They're very similar!

This part lists all the questions from the 2006–2017 A-Level exams, sorted into the six different parts and in reverse chronological order.

In the older exams, they had the habit of not distinctly numbering different parts within the same question as parts (i), (ii), etc. So I have sometimes taken the liberty of adding or modifying such numbers.

Many questions are **out-of-syllabus** and can be skipped.

(They're printed in grey).[553]

[553]Happily, the present 9758 syllabus (first examined in 2017) is considerably lighter than the previous 9740 syllabus (last examined in 2017), which was in turn lighter than the previous 9233 syllabus (last examined in 2008). Thus, many past-year questions printed here are no longer in the current 9758 syllabus and you can skip them. Answers have been provided anyway and you're perfectly welcome to try them.

The following appears on the cover page of each of your 9758 A-Level exam papers.

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

133. Past-Year Questions for Part I. Functions and Graphs

Exercise 525. (9758 N2019/I/3.) (Answer on p. 1991.)

A function is defined as $f(x) = 2x^3 - 6x^2 + 6x - 12$.

(i) Show that $f(x)$ can be written in the form $p\left\{(x+q)^3 + r\right\}$, where p, q and r are constants to be found. [2]

(ii) Hence, or otherwise, describe a sequence of transformations that transform the graph of $y = x^3$ onto the graph of $y = f(x)$. [3]

Exercise 526. (9758 N2019/I/4.) (Answer on p. 1991.)

(i) Sketch the graph of $y = |2^x - 10|$, giving the exact values of any points where the curve meets the axes. [3]

(ii) Without using a calculator, and showing all your working, find the exact interval, or intervals, for which $|2^x - 10| \le 6$. Give your answer in its simplest form. [3]

Exercise 527. (9758 N2019/I/5.) (Answer on p. 1992.)

The functions f and g are defined by

$$f(x) = e^{2x} - 4, \quad x \in \mathbb{R},$$
$$g(x) = x + 2, \quad x \in \mathbb{R}.$$

(i) Find $f^{-1}(x)$ and state its domain. [3]

(ii) Find the exact solution of $fg(x) = 5$, giving your answer in its simplest form. [3]

Exercise 528. (9758 N2019/I/7.) (Answer on p. 1992.)

A curve C has equation $y = xe^{-x}$.

(i) Find the equations of the tangents to C at the points where $x = 1$ and $x = -1$. [6]

(ii) Find the acute angle between these tangents. [2]

Exercise 529. (9758 N2019/II/2.) (Answer on p. 1992.)

(i) Sketch the graph of $y = \dfrac{2-x}{3x^2+5x-8}$. Give the equations of the asymptotes and the coordinates of the point(s) where the curve crosses either axis. [4]

(ii) Solve the inequality $\dfrac{2-x}{3x^2+5x-8} > 0$. [1]

(iii) Hence solve the inequality $\dfrac{x-2}{3x^2+5x-8} > 0$. [1]

Exercise 530. (9758 N2018/I/4.) (Answer on p. 1993.)

(i) Find the exact roots of the equation $\left|2x^2+3x-2\right|=2-x$. [4]

(ii) On the same axes, sketch the curves with equations $y=\left|2x^2+3x-2\right|$ and $y=2-x$. Hence solve exactly the inequality

$$\left|2x^2+3x-2\right|<2-x.$$ [4]

Exercise 531. (9758 N2018/I/5.) (Answer on p. 1993.)

Functions f and g are defined by

$$f:x\mapsto\frac{x+a}{x+b}\qquad\text{for } x\in\mathbb{R},\ x\neq -b,\ a\neq -1,$$
$$g:x\mapsto x\qquad\text{for } x\in\mathbb{R}.$$

It is given that $ff=g$.

Find the value of b.

Find $f^{-1}(x)$ in terms of x and a. [5]

Remark 178. This question contains an error:

Observe that $ff(-b)$ is undefined, while $g(-b)=-b$. So, $ff\neq g$.

Below is how I would've written this question (leaving the last two sentences unchanged):

> Let $a,b\in\mathbb{R}$ with $a\neq -1$. Define $f:\mathbb{R}\setminus\{-b\}\to\mathbb{R}$ by $f(x)=\dfrac{x+a}{x+b}$.
>
> It is given that $ff(x)=x$ for all $x\in\mathbb{R}\setminus\{-b\}$.
>
> Find the value of b.
>
> Find $f^{-1}(x)$ in terms of x and a.

(The function g in the original erroneous question was superfluous.)

The answer I provide is to the above rewritten question.

Exercise 532. (9758 N2017/I/2.) (Answer on p. 1994.)

(i) On the same axes, sketch the graphs of $y=\dfrac{1}{x-a}$ and $y=b|x-a|$, where a and b are positive constants. [2]

(ii) Hence, or otherwise, solve the inequality $\dfrac{1}{x-a}<b|x-a|$. [4]

Exercise 533. (9758 N2017/I/4.) (Answer on p. **1995**.)

A curve C has equation $y = \dfrac{4x+9}{x+2}$.

(i) Show that the gradient of C is negative for all points on C. [3]

(ii) By expressing the equation of C in the form $y = a + \dfrac{b}{x+2}$, where a and b are constants, write down the equations of the asymptotes of C. [3]

(iii) Describe a pair of transformations which transforms the graph of C on to the graph of $y = \dfrac{1}{x}$. [2]

Exercise 534. (9758 N2017/I/5.) (Answer on p. **1995**.)

When the polynomial $x^3 + ax^2 + bx + c$ is divided by $(x-1)$, $(x-2)$ and $(x-3)$, the remainders are 8, 12 and 25 respectively.

(i) Find the values of a, b and c. [4]

A curve has equation $y = f(x)$, where $f(x) = x^3 + ax^2 + bx + c$, with the values of a, b and c found in part **(i)**.

(ii) Show that the gradient of the curve is always positive. Hence explain why the equation $f(x) = 0$ has only one real root and find this root. [3]

(iii) Find the x-coordinates of the points where the tangent to the curve is parallel to the line $y = 2x - 3$. [3]

Exercise 535. (9758 N2017/II/1.) (Answer on p. **1996**.)

A curve C has parametric equations

$$x = \frac{3}{t}, \qquad y = 2t.$$

(i) The line $y = 2x$ cuts C at the points A and B. Find the exact length of AB. [3]

(ii) The tangent at the point $P\left(\dfrac{3}{p}, 2p\right)$ on C meets the x-axis at D and the y-axis at E. The point F is the midpoint of DE. Find a cartesian equation of the curve traced by F as p varies. [5]

Exercise 536. (9758 N2017/II/3.) (Answer on p. **1996**.)

(a) The curve $y = f(x)$ cuts the axes at $(a, 0)$ and $(0, b)$. It is given that $f^{-1}(x)$ exists. State, if it is possible to do so, the coordinates of the points where the following curves cut the axes.

(i) $y = f(2x)$.

(ii) $y = f(x - 1)$.

(iii) $y = f(2x - 1)$.

(iv) $y = f^{-1}(x)$. [4]

(b) The function g is defined by

$$g : x \mapsto 1 - \frac{1}{1 - x}, \qquad \text{where } x \in \mathbb{R},\ x \neq a.$$

(i) State the value of a and explain why this value has to be excluded from the domain of g. [2]

(ii) Find $g^2(x)$ and $g^{-1}(x)$, giving your answers in simplified form. [4]

(iii) Find the values of b such that $g^2(b) = g^{-1}(b)$. [2]

Exercise 537. (9740 N2016/I/1.) (Answer on p. **1996**.)

Express $\dfrac{4x^2 + 4x - 14}{x - 4} - (x + 3)$ as a single simplified fraction. [2]

Hence, without using a calculator, solve the inequality

$$\frac{4x^2 + 4x - 14}{x - 4} < x + 3. \qquad [3]$$

Exercise 538. (9740 N2016/I/3.) (Answer on p. **1997**.)

The curve $y = x^4$ is transformed onto the curve with equation $y = f(x)$. The turning point on $y = x^4$ corresponds to the point with coordinates (a, b) on $y = f(x)$. The curve $y = f(x)$ also passes through the point with coordinates $(0, c)$. Given that $f(x)$ has the form $k(x - l)^4 + m$ and that a, b and c are positive constants with $c > b$, express k, l and m in terms of a, b and c. [2]

By sketching the curve $y = f(x)$, or otherwise, sketch the curve $y = \dfrac{1}{f(x)}$. State, in terms of a, b and c, the coordinates of any points where $y = \dfrac{1}{f(x)}$ crosses the axes and of any turning points. [4]

Exercise 539. (9740 N2016/I/10.) (Answer on p. **1998**.)

(a) The function f is given by $f : x \mapsto 1 + \sqrt{x}$, for $x \in \mathbb{R}$, $x \geq 0$.

(i) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Show[554] that if $ff(x) = x$ then $x^3 - 4x^2 + 4x - 1 = 0$.

(iii) Hence find the value of x for which $ff(x) = x$.

(iv) Explain why this value of x satisfies the equation $f(x) = f^{-1}(x)$. [5]

(b) The function g, with domain the set of non-negative integers, is given by

$$g(n) = \begin{cases} 1, & \text{for } n = 0, \\ 2 + g\left(\dfrac{1}{2}n\right), & \text{for } n \text{ even}, \\ 1 + g(n-1), & \text{for } n \text{ odd}. \end{cases}$$

(i) Find $g(4)$, $g(7)$, and $g(12)$. [3]

(ii) Does g have an inverse? Justify your answer. [2]

Exercise 540. (9740 N2015/I/1.) (Answer on p. **1998**.)

A curve C has equation

$$y = \frac{a}{x^2} + bx + c,$$

where a, b and c are constants. It is given that C passes through the points with coordinates $(1.6, -2.4)$ and $(-0.7, 3.6)$, and that the gradient of C is 2 at the point where $x = 1$.

(i) Find the values of a, b and c, giving your answers correct to 3 decimal places. [4]

(ii) Find the x-coordinate of the point where C crosses the x-axis, giving your answer correct to 3 decimal places. [2]

(iii) One asymptote of C is the line with equation $x = 0$. Write down the equation of the other asymptote of C. [1]

Exercise 541. (9740 N2015/I/2.) (Answer on p. **1999**.)

(i) Sketch the curve with equation $y = \left|\dfrac{x+1}{1-x}\right|$, stating the equations of the asymptotes.

On the same diagram, sketch the line with equation $y = x + 2$. [3]

(iv) Solve the inequality $\left|\dfrac{x+1}{1-x}\right| < x + 2$. [3]

[554] Originally, parts **(a)(ii)**, **(a)(iii)**, and **(a)(iv)** here were combined into a single part **(a)(ii)** . But for great clarity, I've split the original **(a)(ii)** into three separate parts.

Exercise 542. (9740 N2015/I/5.) (Answer on p. 2000.)

(i) State a sequence of transformations that will transform the curve with equation $y = x^2$ on to the curve with equation $y = 0.25(x-3)^2$. [2]

A curve has equation $y = f(x)$ where

$$y = \begin{cases} 1, & \text{for } 0 \le x \le 1, \\ 0.25(x-3)^2, & \text{for } 1 < x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Sketch the curve for $-1 \le x \le 4$. [3]

(iii) On a separate diagram, sketch the curve with equation $y = 1 + f(0.5x)$, for $-1 \le x \le 4$. [2]

Exercise 543. (9740 N2015/II/3.) (Answer on p. 2001.)

(a) The function f is defined by

$$f : x \to \frac{1}{1-x^2}, \qquad x \in \mathbb{R},\ x > 1.$$

(i) Show that f has an inverse. [2]

(ii) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(b) The function g is defined by

$$g : x \to \frac{2+x}{1-x^2}, \qquad x \in \mathbb{R},\ x \ne \pm 1.$$

Find algebraically the range of g, giving your answer in terms of $\sqrt{3}$ as simply as possible. [5]

Exercise 544. (9740 N2014/I/1.) (Answer on p. 2001.)

The function f is defined by

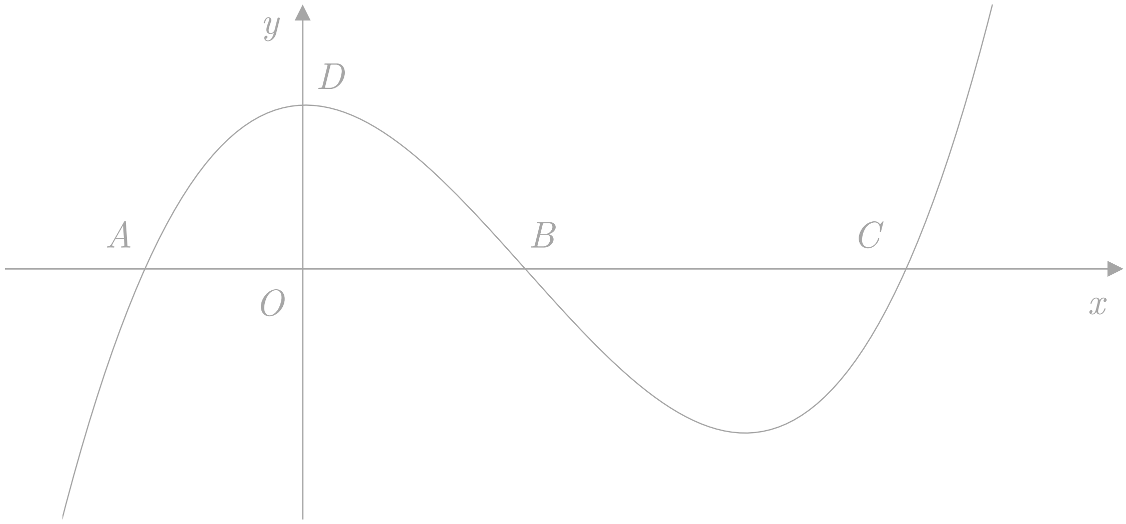
$$y = \frac{1}{1-x}, \qquad x \in \mathbb{R},\ x \ne 1,\ x \ne 0.$$

(i) Show that $f^2(x) = f^{-1}(x)$. [4]

(ii) Find $f^3(x)$ in simplified form. [1]

Exercise 545. (9740 N2014/I/4.) (Answer on p. **2001**.)

The diagram shows the curve $y = f(x)$. The curve crosses the x-axis at the points A, B and C, and has a maximum turning point at D where it crosses the y-axis. The coordinates of A, B, C and D are $(-a, 0)$, $(b, 0)$, $(c, 0)$ and $(0, d)$ respectively, where a, b, c and d are positive constants.



(i) Sketch the curve $y^2 = f(x)$, stating, in terms of a, b, c and d, the coordinates of any turning points and of the points where the curve crosses the x-axis. [4]

(ii) What can be said about the tangents to the curve $y^2 = f(x)$ at the points where it crosses the x-axis? [1]

Exercise 546. (9740 N2014/II/1.) (Answer on p. **2002**.)

A curve C has parametric equations $x = 3t^2$, $y = 6t$.

(i) Find the value of t at the point on C where the tangent has gradient 0.4. [3]

(ii) The tangent at the point $P\left(3p^2, 6p\right)$ on C meets the y-axis at the point D. Find the cartesian equation of the locus of the mid-point of PD as p varies. [4]

Remark 179. For **(ii)**, assume also that $p \neq 0$.[555]

Exercise 547. (9740 N2013/I/2.) (Answer on p. **2002**.)

It is given that

$$y = \frac{x^2 + x + 1}{x - 1}, \qquad x \in \mathbb{R}, x \neq 1.$$

Without using a calculator, find the set of values that y can take. [5]

[555] If $p = 0$, then $P = (0, 0)$ and the tangent at P is vertical, so that D could be any point on the y-axis.

Exercise 548. (9740 N2013/I/3.) (Answer on p. 2002.)

(i) Sketch the curve with equation

$$y = \frac{x+1}{2x-1},$$

stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [4]

(ii) Solve the inequality

$$\frac{x+1}{2x-1} < 1. \qquad [1]$$

Exercise 549. (9740 N2013/II/1.) (Answer on p. 2003.)

Functions f and g are defined by

$$f : x \mapsto \frac{2+x}{1-x}, \qquad x \in \mathbb{R}, x \neq 1,$$

$$g : x \mapsto 1 - 2x, \qquad x \in \mathbb{R}.$$

(i) Explain why the composite function fg does not exist. [2]

(ii) Find[556] an expression for $gf(x)$.

(iii) And hence, or otherwise, find $(gf)^{-1}(5)$. [4]

Exercise 550. (9740 N2012/I/1.) (Answer on p. 2004.)

A cinema sells tickets at three different prices, depending on the age of the customer. The age categories are under 16 years, between 16 and 65 years, and over 65 years. Three groups of people, A, B, and C, go to the cinema on the same day. The numbers in each category for each group, together with the total cost of the tickets for each group, are given in the following table.

Group	Under 16 years	Between 16 and 65 years	Over 65 years	Total cost
A	9	6	4	\$162.03
B	7	5	3	\$128.36
C	10	4	5	\$158.50

Write down and solve equations to find the cost of a ticket for each of the age categories. [4]

[556] Originally, parts **(ii)** and **(iii)** here were combined into a single part **(ii)**. But for great clarity, I've split the original **(ii)** into two separate parts.

Exercise 551. (9740 N2012/I/7.) (Answer on p. **2004**.)

A function f is said to be self-inverse if $f(x) = f^{-1}(x)$ for all x in the domain of f. The function g is defined by

$$g : x \mapsto \frac{x+k}{x-1}, \qquad x \in \mathbb{R}, x \neq 1.$$

where k is a constant, $k \neq -1$.

(i) Show that g is self-inverse. [2]

(ii) Given that $k > 0$, sketch the curve $y = g(x)$, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the x- and y-axes. [3]

(iii) State the equation of one line of symmetry of the curve in part **(ii)**, and describe fully a sequence of transformations which would transform the curve $y = 1/x$ onto this curve. [4]

Exercise 552. (9740 N2012/II/3.) (Answer on p. **2005**.)

It is given that $f(x) = x^3 + x^2 - 2x - 4$.

(i) Sketch the graph of $y = f(x)$. [1]

(ii) Find the integer solution of the equation $f(x) = 4$, and prove algebraically that there are no other real solutions. [3]

(iii) State the integer solution of the equation $(x+3)^3 + (x+3)^2 - 2(x+3) - 4 = 4$. [1]

(iv) Sketch the graph of $y = |f(x)|$. [1]

(v) Write down two different cubic equations which between them give the roots of the equation $|f(x)| = 4$. Hence find all the roots of this equation. [4]

Exercise 553. (9740 N2011/I/1.) (Answer on p. **2007**.)

Without using a calculator, solve the inequality

$$\frac{x^2+x+1}{x^2+x-2} < 0. \qquad [4]$$

Exercise 554. (9740 N2011/I/2.) (Answer on p. **2007**.)

It is given that $f(x) = ax^2 + bx + c$, where a, b, and c are constants.

(i) Given that the curve with equation $y = f(x)$ passes through the points with coordinates $(-1.5, 4.5)$, $(2.1, 3.2)$ and $(3.4, 4.1)$, find the values of a, b, and c. Give your answers correct to 3 decimal places. [3]

(ii) Find the set of values of x for which $f(x)$ is an increasing function.

Remark 180. In **(ii)**, following the practice of this textbook, I take "increasing" to mean *weakly* increasing (and not *strictly* increasing).

Exercise 555. (9740 N2011/II/3.) (Answer on p. **2007**.)

The function f is defined by

$$f : x \mapsto \ln(2x+1) + 3, \qquad x \in \mathbb{R},\ x > -\frac{1}{2}.$$

(i) Find $f^{-1}(x)$ and write down the domain and range of f^{-1}. [4]

(ii) Sketch on the same diagram the graphs of $y = f(x)$ and $y = f^{-1}(x)$ giving the equations of any asymptotes and the exact coordinates of any points where the curves cross the x- and y-axes. [4]

(iii) Explain why the x-coordinates of the points of intersection of the curves in part **(ii)** satisfy the equation $\ln(2x+1) = x - 3$, and find the values of these x-coordinates, correct to 4 significant figures. [3]

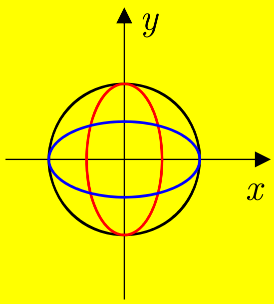
Exercise 556. (9740 N2010/I/5.) (Answer on p. **2008**.)

The curve with equation $y = x^3$ is transformed by a translation of 2 units in the positive x-direction, followed by a stretch with scale factor 0.5 parallel to the y-axis, followed by a translation of 6 units in the negative y-direction.

(i) Find the equation of the new curve in the form $y = f(x)$ and the exact coordinates of the points where this curve crosses the x- and y-axes. Sketch the new curve. [5]

(ii) On the same diagram, sketch the graph of $y = f^{-1}(x)$, stating the exact coordinates of the points where the graph crosses the x- and y-axes. [3]

Remark 181. In the above question's first sentence, the second step is a little ambiguous. Say we transform the black circle on the right "by a stretch with scale factor 0.5 parallel to the y-axis". Then do we get **(a)** the red ellipse; or **(b)** the blue ellipse? Perhaps this was clear in the mind of whoever that wrote this question, but it isn't to me and probably not to others either. In my answer, I shall assume **(a)** the stretch is outwards from the y-axis.



Exercise 557. (9740 N2010/II/4.) (Answer on p. **2009**.)

The function f is defined as follows.

$$f : x \mapsto \frac{1}{x^2 - 1}, \qquad \text{for } x \in \mathbb{R},\ x \neq -1,\ x \neq 1.$$

(i) Sketch the graph of $y = f(x)$. [1]

(ii) If the domain of f is further restricted to $x \geq k$, state with a reason the least value of k for which the function f^{-1} exists. [2]

In the rest of the question, the domain of f is $x \in \mathbb{R}$, $x \neq -1$, $x \neq 1$, as originally defined.

The function g is defined as follows.

$$g : x \mapsto \frac{1}{x - 3}, \qquad \text{for } x \in \mathbb{R},\ x \neq 2,\ x \neq 3,\ x \neq 4.$$

(iii) Show that $fg(x) = \dfrac{(x-3)^2}{(4-x)(x-2)}$. [2]

(iv) Solve the inequality $fg(x) > 0$. [3]

(v) Find the range of fg. [3]

Exercise 558. (9740 N2009/I/1.) (Answer on p. **2010**.)

(i) The first three terms of a sequence are given by $u_1 = 10$, $u_2 = 6$, $u_3 = 5$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. [4]

(ii) Find the set of values of n for which u_n is greater than 100. [2]

Exercise 559. (9740 N2009/I/6.) (Answer on p. **2011**.)

The curve C_1 has equation $y = \dfrac{x-2}{x+2}$. The curve C_2 has equation $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$.

(i) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

(ii) Show algebraically that the x-coordinates of the points of intersection of C_1 and C_2 satisfy the equation $2(x-2)^2 = (x+2)^2(6 - x^2)$. [2]

(iii) Use your calculator to find these x-coordinates. [2]

Exercise 560. (9740 N2009/II/3.) (Answer on p. **2012**.)

The function f is defined by

$$f : x \mapsto \frac{ax}{bx-a}, \qquad \text{for } x \in \mathbb{R},\ x \neq \frac{a}{b},$$

where a and b are non-zero constants.

(i) Find $f^{-1}(x)$. Hence or otherwise find $f^2(x)$ and state the range of f^2. [5]

(ii) The function g is defined by $g : x \mapsto \dfrac{1}{x}$ for all real non-zero x. State whether the composite function fg exists, justifying your answer. [2]

(ii) Solve the equation $f^{-1}(x) = x$. [3]

Exercise 561. (9740 N2008/I/9.) (Answer on p. **2012**.)

It is given that

$$f(x) = \frac{ax+b}{cx+d},$$

for non-zero constants a, b, c, and d.

(i) Given that $ad - bc \neq 0$, show by differentiation that the graph of $y = f(x)$ has no turning points. [3]

(ii) What can be said about the graph of $y = f(x)$ when $ad - bc = 0$? [2]

(iii) Deduce from part **(i)** that the graph of

$$y = \frac{3x-7}{2x+1}$$

has a positive gradient at all points of the graph. [1]

(iv) On separate diagrams, draw sketches of the graphs of

(a) $y = \dfrac{3x-7}{2x+1}$,

(b) $y^2 = \dfrac{3x-7}{2x+1}$,

including the coordinates of the points where the graphs cross the axes and the equations of any asymptotes. [5]

Exercise 562. (9233 N2008/I/14.) (Answer on p. 2014.)

Sketch, on separate diagrams, the curves

(i) $y = \dfrac{x}{x^2 - 1}$, stating the equations of the asymptotes, [4]

(ii) $y^2 = \dfrac{x}{x^2 - 1}$, making clear the form of the curve at the origin. [3]

(iii) Show that the x-coordinates of the points of intersection of the curves $y = \dfrac{x}{x^2 - 1}$ and $y = e^x$ satisfy the equation $x^2 = 1 + xe^{-x}$. [1]

(iv) Use the iterative formula $x_{n+1} = \sqrt{1 + x_n e^{-x_n}}$, together with a suitable initial value x_1, to find the positive root of this equation correct to 2 decimal places. [2]

Exercise 563. (9740 N2008/II/4.) (Answer on p. **2016**.)

The function f is defined by $f : x \mapsto (x - 4)^2 + 1$ for $x \in \mathbb{R}$, $x > 4$.

(i) Sketch the graph of $y = f(x)$. Your sketch should indicate the position of the graph in relation to the origin. [2]

(ii) Find $f^{-1}(x)$, stating the domain of f^{-1}. [3]

(iii) On the same diagram as in part **(i)**, sketch the graph of $y = f^{-1}(x)$. [1]

(iv) Write down the equation of the line in which the graph of $y = f(x)$ must be reflected in order to obtain the graph of $y = f^{-1}(x)$.[557]

(v) And hence find the exact solution of the equation $f(x) = f^{-1}(x)$. [5]

Remark 182. The writers of **(v)** probably made a mistake (more on this in the answer).

Exercise 564. (9740 N2007/I/1.) (Answer on p. **2018**.)

Show that $\dfrac{2x^2 - x - 19}{x^2 + 3x + 2} - 1 = \dfrac{x^2 - 4x - 21}{x^2 + 3x + 2}$. [1]

Hence, without using a calculator, solve the inequality

$$\frac{2x^2 - x - 19}{x^2 + 3x + 2} > 1. \qquad [4]$$

[557] Originally, parts **(iv)** and **(v)** here were combined into a single part **(iv)**. But for great clarity, I've split the original **(iv)** into two separate parts.

Exercise 565. (9740 N2007/I/2.) (Answer on p. **2018**.)

Functions f and g are defined by

$$f : x \mapsto \frac{1}{x-3} \qquad \text{for } x \in \mathbb{R},\ x \neq 3,$$

$$g : x \mapsto x^2 \qquad \text{for } x \in \mathbb{R}.$$

(i) Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist. [3]

(ii) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]

Exercise 566. (9740 N2007/I/5.) (Answer on p. **2019**.)

Show that the equation $y = \dfrac{2x+7}{x+2}$ can be written as $y = A + \dfrac{B}{x+2}$, where A and B are constants to be found. Hence state a sequence of transformations which transform the graph of $y = \dfrac{1}{x}$ to the graph of $y = \dfrac{2x+7}{x+2}$. [4]

Sketch the graph of $y = \dfrac{2x+7}{x+2}$, giving the equations of any asymptotes and the coordinates of any points of intersection with the x- and y-axes. [3]

Exercise 567. (9740 N2007/II/1.) (Answer on p. **2019**.)

Four friends buys three different kinds of fruits in the market. When they get home they cannot remember the individual prices per kilogram, but three of them can remember the total amount that they each paid. The weights of fruit and the total amounts paid are shown in the following table.

	Suresh	Fandi	Cindy	Lee Lian
Pineapples (kg)	1.15	1.20	2.15	1.30
Mangoes (kg)	0.60	0.45	0.90	0.25
Lychees (kg)	0.55	0.30	0.65	0.50
Total amount paid in $	8.28	6.84	13.05	

Assuming that, for each variety of fruit, the price per kilogram paid by each of the friends is the same, calculate the total amount that Lee Lian paid. [6]

Exercise 568. (9233 N2007/II/4.) (Answer on p. **2020**.)

The function f is defined by

$$f : x \mapsto \frac{4x+1}{x-3}, \qquad x \in \mathbb{R},\ x \neq 3.$$

(i) State the equations of the two asymptotes of the graph of $y = f(x)$. [2]

(ii) Sketch the graph of $y = f(x)$, showing its asymptotes and stating the coordinates of the points of intersection with the axes. [3]

(iii) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

Exercise 569. (9233 N2006/I/3.) (Answer on p. **2020**.)

Functions f and g are defined by

$$\begin{aligned} f &: x \mapsto 5x+3, \qquad && x > 0, \\ g &: x \mapsto \frac{3}{x}, && x > 0. \end{aligned}$$

(i) Find, in a similar form, fg, g^2 and g^{35}. [3]

[Note: g^2 denotes gg.]

(ii) Express h in terms of one or both f and g, where

$$h : x \mapsto 25x+18, \qquad x > 0.$$ [1]

Exercise 570. (9233 N2006/II/1.) (Answer on p. **2021**.)

Solve the inequality

$$\frac{x-9}{x^2-9} \leq 1.$$ [5]

Exercise 571. (9758 N2019/I/6.) (Answer on p. 2022.)

(i) By writing $\dfrac{1}{4r^2-1}$ in partial fractions, find an expression for $\displaystyle\sum_{r=1}^{n}\frac{1}{4r^2-1}$. [4]

(ii) Hence find the exact value of $\displaystyle\sum_{r=11}^{\infty}\frac{1}{4r^2-1}$. [2]

Exercise 572. (9758 N2019/I/8.) (Answer on p. 2022.)

(i) An arithmetic series has first term a and common difference $2a$ where $a \neq 0$. A geometric series has first term a and common ratio 2. The kth term of the geometric series is equal to the sum of the first 64 terms of the arithmetic series. Find the value of k. [3]

(ii) A geometric series has first term f and common ratio r where $f, r \in \mathbb{R}$ and $f \neq 0$. The sum of the first four terms of the series is 0. Find the possible values of f and r. Find also, in terms of f, the possible values of the sum of the first n terms of the series.[4]

(iii) The first term of an arithmetic series is negative. The sum of the first four terms of the series is 14 and the product of the first four terms of the series is 0. Find the 11th term of the series. [4]

Exercise 573. (9758 N2018/I/8.) (Answer on p. 2023.)

A sequence $u_1, u_2, u_3, \ldots$ is such that $u_{n+1} = 2u_n + An$, where A is a constant and $n \geq 1$.

(i) Given that $u_1 = 5$ and $u_2 = 15$, find A and u_3. [2]

Remark 183. The old (9740) syllabus[558] included "sequence generated by a simple recurrence relation of the form $x_{n+1} = f(x_n)$". This was explicitly removed from the present (9758) syllabus. So, such material can't be on the present 9758 exams, right? (Wrong.)

It is known that the nth term of this sequence is given by

$$u_n = a\left(2^n\right) + bn + c,$$

where a, b and c are constants.

(i) Find a, b and c. [4]

(ii) Find $\displaystyle\sum_{r=1}^{n} u_r$ in terms of n. (You need not simplify your answer.) [4]

[558] Last examined in 2017.

Exercise 574. (9758 N2018/I/11.) (Answer on p. 2023.)

Mr Wong is considering investing money in a savings plan. One plan, P, allows him to invest \$100 into the account on the first day of every month. At the end of each month the total in the account is increased by $a\%$.

(i) It is given that $a = 0.2$.

(a) Mr Wong invests \$100 on 1 January 2016. Write down how much this \$100 is worth at the end of 31 December 2016. [1]

(b) Mr Wong invests \$100 on the first day of each of the 12 months of 2016. Find the total amount in the account at the end of 31 December 2016. [3]

(c) Mr Wong continues to invest \$100 on the first day of each month. Find the month in which the total in the account will first exceed \$3000. Explain whether this occurs on the first or last day of the month. [5]

An alternative plan, Q, also allows him to invest \$100 on the first day of every month. Each \$100 invested earns a fixed bonus of \$$b$ at the end of every month for which it has been in the account. This bonus is added to the account. The accumulated bonuses themselves do not earn any further bonus.

(ii) **(a)** Find, in terms of b, how much \$100 invested on 1 January 2016 will be worth at the end of 31 December 2016. [1]

(b) Mr Wong invests \$100 on the first day of each of the 24 months in 2016 and 2017. Find the value of b such that the total value of all the investments, including bonuses, is worth \$2800 at the end of 31 December 2017. [3]

It is given instead that $a = 1$ for plan P.

(ii) Find the value of b for plan Q such that both plans give the same total value in the account at the end of the 60th month. [3]

Exercise 575. (9758 N2017/I/9.) (Answer on p. **2024**.)

(a) A sequence of numbers u_1, u_2, u_3, ... has a sum S_n where $S_n = \sum_{r=1}^{n} u_r$. It is given that $S_n = An^2 + Bn$, where A and B are non-zero constants.

(i) Find an expression for u_n in terms of A, B and n. Simplify your answer. [3]

(ii) It is also given that the tenth term is 48 and the seventeenth term is 90. Find A and B. [2]

(b) Show that $r^2(r+1)^2 - (r-1)^2 r^2 = kr^3$, where k is a constant to be determined. Use this result to find a simplified expression for $\sum_{r=1}^{n} r^3$. [4]

(c) D'Alembert's ratio test states that a series of the form $\sum_{r=0}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, and diverges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$. When $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=0}^{\infty} \frac{x^r}{r!}$ converges for all real values of x and state the sum to infinity of this series, in terms of x. [4]

Exercise 576. (9758 N2017/II/2.) (Answer on p. **2024**.)

An arithmetic progression has first term 3. The sum of the first 13 terms of the progression is 156.

(i) Find the common difference. [2]

A geometric progression has first term 3 and common ratio r. The sum of the first 13 terms of the progression is 156.

(ii) Show that $r^{13} - 52r + 51 = 0$. Show that the common ratio cannot be 1 even though $r = 1$ is a root of this equation. Find the possible values of the common ratio. [4]

(iii) It is given that the common ratio of the geometric progression is positive, and that the nth term of this geometric progression is more than 100 times the nth term of the arithmetic progression. Write down an inequality, and hence find the smallest possible value of n. [3]

Exercise 577. (9740 N2016/I/4.) (Answer on p. **2025**.)

An arithmetic series has first term a and common difference d, where a and d are non-zero. A geometric series has first term b and common ratio r, where b and r are non-zero. It is given that the 4th, 9th and 12th terms of the arithmetic series are equal to the 5th, 8th and 15th terms of the geometric series respectively.

(i) Show that r satisfies the equation $5r^{10} - 8r^3 + 3 = 0$. Given that $|r| < 1$, solve this equation, giving your answer correct to 2 decimal places. [4]

(ii) Using this value of r, find, in terms of b and n, the sum of the terms of the geometric series after, but not including, the nth term, simplifying your answer. [3]

Exercise 578. (9740 N2016/I/6.) (Answer on p. **2024**.)

(i) Prove by the method of mathematical induction that

$$\sum_{r=1}^{n} r\left(r^2+1\right) = \frac{1}{4}n\left(n+1\right)\left(n^2+n+2\right). \qquad [5]$$

(ii) A sequence u_0, u_1, u_2, … is given by

$$u_0 = 2 \qquad \text{and} \qquad u_n = u_{n-1} + n^3 + n \text{ for } n \geq 1.$$

Find u_1, u_2, and u_3. [2]

(iii) By considering $\sum_{r=1}^{n}\left(u_r - u_{r-1}\right)$, find a formula for u_n in terms of n. [3]

Exercise 579. (9740 N2015/I/8.) (Answer on p. **2026**.)

Two athletes are to run 20 km by running 50 laps around a circular track of length 400 m. They aim to complete the distance in between 1.5 hours and 1.75 hours inclusive.

(i) Athlete A runs the first lap in T seconds and each subsequent lap takes 2 seconds longer than the previous lap. Find the set of values of T which will enable A to complete the distance within the required time interval. [4]

(ii) Athlete B runs the first lap in t seconds and the time for each subsequent lap is 2% more than the time for the previous lap. Find the set of values of t which will enable B to complete the distance within the required time interval. [4]

(iii) Assuming each athlete completes the 20 km run in exactly 1.5 hours, find the difference in the athletes' times for their 50th laps, giving your answer to the nearest second. [3]

Exercise 580. (9740 N2015/II/4.) (Answer on p. **2026**.)

(a) Prove by the method of mathematical induction that

$$1\times3\times6 + 2\times4\times7 + 3\times5\times8 + \cdots + n\left(n+2\right)\left(n+5\right) = \frac{1}{12}n\left(n+1\right)\left(3n^2+31n+74\right). \qquad [6]$$

(b) **(i)** Show that $\dfrac{2}{4r^2+8r+3}$ can be expressed as $\dfrac{A}{2r+1} + \dfrac{B}{2r+3}$, where A and B are constants to be determined. [1]

The sum $\displaystyle\sum_{r=1}^{n} \frac{2}{4r^2+8r+3}$ is denoted by S_n.

(ii) Find an expression for S_n in terms of n. [4]

(iii) Find the smallest value of n for which S_n is within 10^{-3} of the sum to infinity. [3]

Exercise 581. (9740 N2014/I/6.) (Answer on p. 2027.)

(a) A sequence p_1, p_2, p_3, ... is given by

$$p_1 = 1 \text{ and } p_{n+1} = 4p_n - 7 \text{ for } n \geq 1.$$

(i) Use the method of mathematical induction to prove that

$$p_n = \frac{7 - 4^n}{3}. \qquad [5]$$

(ii) Find $\sum_{r=1}^{n} p_r$. [3]

(b) The sum S_n of the first n terms of a sequence u_1, u_2, u_3, ... is given by

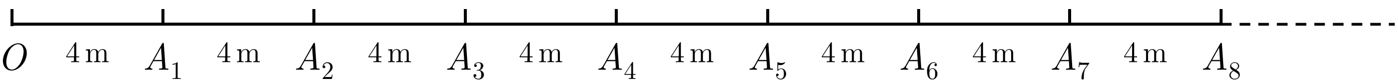
$$S_n = 1 - \frac{1}{(n+1)!}.$$

(i) Give a reason why the series $\sum u_r$ converges, and write down the value of the sum to infinity. [2]

(ii) Find a formula for u_n in simplified form. [2]

Exercise 582. (9740 N2014/II/3.) (Answer on p. 2028.)

In a training exercise, athletes run from a starting point O to and from a series of points A_1, A_2, A_3, ..., increasingly far away in a straight line. In the exercise, athletes start at O and run stage 1 from O to A_1 and back to O, then stage 2 from O to A_2 and back to O, and so on.

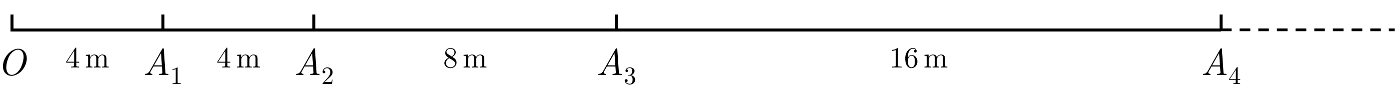


(i) In Version 1 of the exercise (above), the distances between adjacent points are all 4 m.

(a) Find the distance run by an athlete who completes the first 10 stages of Version 1 of the exercise. [2]

(b) Write down an expression for the distance run by an athlete who completes n stages of Version 1. Hence find the least number of stages that the athlete needs to complete to run at least 5 km. [4]

(ii) In Version 2 of the exercise (below), the distances between the points are such that $OA_1 = 4$ m, $A_1A_2 = 4$ m, $A_2A_3 = 8$ m and $A_nA_{n+1} = 2A_{n-1}A_n$. Write down an expression for the distance run by an athlete who completes n stages of Version 2. Hence find the distance from O, and the direction of travel, of the athlete after he has run exactly 10 km using Version 2. [5]



Exercise 583. (9740 N2013/I/7.) (Answer on p. 2029.)

A gardener is cutting off pieces of string from a long roll of string. The first piece he cuts off is 128 cm long and each successive piece is 2/3 as long as the preceding piece.

(i) The length of the nth piece of string cut off is p cm. Show that $\ln p = (An + B)\ln 2 + (Cn + D)\ln 3$, for constants A, B, C and D to be determined. [3]

(ii) Show that the total length of string cut off can never be greater than 384 cm. [2]

(iii) How many pieces must be cut off before the total length cut off is greater than 380 cm? You must show sufficient working to justify your answer. [4]

Remark 184. The wording of **(iii)** is a little ambiguous. Is the desired answer **(a)** the **maximum** number of pieces one can cut off **before** the total length cut off is greater than 380 cm? Or is it **(b)** the **minimum** number of pieces one can cut off **in order** for the total length cut off to be greater than 380 cm? (Of course, the latter is simply one more than the former.)

In my answer, I shall assume **(b)**.

Exercise 584. (9740 N2013/I/9.) (Answer on p. 2029.)

(i) Prove by the method of mathematical induction that

$$\sum_{r=1}^{n} r\left(2r^2+1\right) = \frac{1}{2}n(n+1)\left(n^2+n+1\right). \qquad [5]$$

(ii) It is given that $f(r) = 2r^3+3r^2+r+24$. Show that $f(r)-f(r-1) = ar^2$, for a constant a to be determined. Hence find a formula for $\sum_{r=1}^{n} r^2$, fully factorizing your answer. [5]

(iii) Find $\sum_{r=1}^{n} f(r)$. (You should not simplify your answer.) [3]

Exercise 585. (9740 N2012/I/3.) (Answer on p. 2030.)

A sequence u_1, u_2, u_3, … is given by

$$u_1 = 2 \qquad \text{and} \qquad u_{n+1} = \frac{3u_n - 1}{6} \text{ for } n \geq 1.$$

(i) Find the exact values of u_2 and u_3. [2]

(ii) It is given that $u_n \to l$ as $n \to \infty$. Showing your working, find the exact value of l. [2]

(iii) For this value of l, use the method of mathematical induction to prove that

$$u_n = \frac{14}{3}\left(\frac{1}{2}\right)^n + l. \qquad [4]$$

Exercise 586. (9740 N2012/II/4.) (Answer on p. 2031.)

On 1 January 2001 Mrs A put \$100 into a bank account, and on the first day of each subsequent month she put in \$10 more than in the previous month. Thus on 1 February she put \$110 into the account and on 1 March she put \$120 into the account, and so on. The account pays no interest.

(i) On what date did the value of Mrs A's account first become greater than \$5000? [5]

On 1 January 2001 Mr B put \$100 into a savings account, and on the first day of each subsequent month he put another \$100 into the account. The interest rate was 0.5% per month, so that on the last day of each month the amount in the account on that day was increased by 0.5%.

(ii) Use the formula for the sum of a geometric progression to find an expression for the value of Mr B's account on the last day of the nth month (where January 2001 was the 1st month, February 2001 was the 2nd month, and so on). Hence find in which month the value of Mr B's account first became greater than \$5000. [5]

(iii) Mr B wanted the value of his account to be \$5000 on 2 December 2003. What interest rate per month, applied from January 2001, would achieve this? [3]

Exercise 587. (9740 N2011/I/6.) (Answer on p. 2032.)

(i) Using the formulae for $\sin(A \pm B)$, prove that

$$\sin\left(r+\frac{1}{2}\right)\theta - \sin\left(r-\frac{1}{2}\right)\theta = 2\cos r\theta \sin\frac{1}{2}\theta. \quad [2]$$

Remark 185. Now assume also that θ is not an even integer multiple of π.[559]

(ii) Hence find a formula for $\sum_{r=1}^{n} \cos r\theta$ in terms of $\sin\left(n+\frac{1}{2}\right)\theta$ and $\sin\frac{1}{2}\theta$. [3]

(iii) Prove by the method of mathematical induction that

$$\sum_{r=1}^{n} \sin r\theta = \frac{\cos\frac{1}{2}\theta - \cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$$

for all positive integers n. [6]

[559] Otherwise some of the formulae that follow have 0 as denominators and are thus undefined.

Exercise 588. (9740 N2011/I/9.) (Answer on p. 2033.)

(i) A company is drilling for oil. Using machine A, the depth drilled on the first day is 256 metres. On each subsequent day, the depth drilled is 7 metres less than on the previous day. Drilling continues daily up to and including the day when a depth of less than 10 metres is drilled. What depth is drilled on the 10th day, and what is the total depth when drilling is completed? [6]

(ii) Using machine B, the depth drilled on the first day is also 256 metres. On each subsequent day, the depth drilled in $\frac{8}{9}$ of the depth drilled on the previous day. How many days does it take for the depth drilled to exceed 99% of the theoretical maximum total depth? [4]

Exercise 589. (9740 N2010/I/3.) (Answer on p. 2034.)

The sum S_n of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by

$$S_n = n(2n + c),$$

where c is a constant.

(i) Find u_n in terms of c and n. [3]

(ii) Find a recurrence relation of the form $u_{n+1} = f(u_n)$. [2]

Exercise 590. (9740 N2010/II/2.) (Answer on p. 2034.)

(i) Prove by mathematical induction that

$$\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7). \quad [5]$$

(ii) **(a)** Prove by the method of differences that

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}. \quad [4]$$

(b) Explain why $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$ is a convergent series, and state the value of the sum to infinity. [2]

Exercise 591. (9740 N2009/I/3.) (Answer on p. **2035**.)

(i) Show that

$$\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}=\frac{A}{n^3-n},$$

where A is a constant to be found. [2]

(ii) Hence find $\sum_{r=2}^{n}\frac{1}{r^3-r}$. (There is no need to express your answer as a single algebraic fraction.) [3]

(iii) Give a reason why the series $\sum_{r=2}^{n}\frac{1}{r^3-r}$ converges, and write down its value. [2]

Exercise 592. (9740 N2009/I/5.) (Answer on p. **2035**.)

(i) Use the method of mathematical induction to prove that

$$\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1). \quad [4]$$

(ii) Find $\sum_{r=n+1}^{2n}r^2$, giving your answer in fully factorized form. [4]

Exercise 593. (9740 N2009/I/8.) (Answer on p. **2036**.)

Two musical instruments, A and B, consist of metal bars of decreasing lengths.

(i) The first bar of instrument A has length 20 cm and the lengths of the bars form a geometric progression. The 25th bar has length 5 cm. Show that the total length of all the bars must be less than 357 cm, no matter how many bars there are. [4]

Instrument B consists of only 25 bars which are identical to the first 25 bars of instrument A.

(ii) Find the total length, L cm, of all the bars of instrument B and the length of the 13th bar. [3]

(iii) Unfortunately the manufacturer misunderstands the instructions and constructs instrument B wrongly, so that the lengths of the bars are in arithmetic progression with common difference d cm. If the total length of the 25 bars is still L cm and the length of the 25th bar is still 5 cm, find the value of d and the length of the longest bar. [4]

Exercise 594. (9740 N2008/I/2.) (Answer on p. **2036**.)

The nth term of a sequence is given by

$$u_n = n(2n+1),$$

for $n \geq 1$. The sum of the first n terms is denoted by S_n. Use the method of mathematical induction to show that

$$S_n = \frac{1}{6}n(n+1)(4n+5)$$

for all positive integers n. [5]

Exercise 595. (9740 N2008/I/10.) (Answer on p. **2036**.)

(i) A student saves \$10 on 1 January 2009. On the first day of each subsequent month she saves \$3 more than in the previous month, so that she saves \$13 on 1 February 2009, \$16 on 1 March 2009, and so on. On what date will she first have saved over \$2 000 in total? [5]

(ii) A second student puts \$10 on 1 January 2009 into a bank account which pays compound interest at a rate of 2% per month on the last day of each month. She puts a further \$10 into the account on the first day of each subsequent month.

(a) How much compound interest has her original \$10 earned at the end of 2 years? [2]

(b) How much in total is in the account at the end of 2 years? [3]

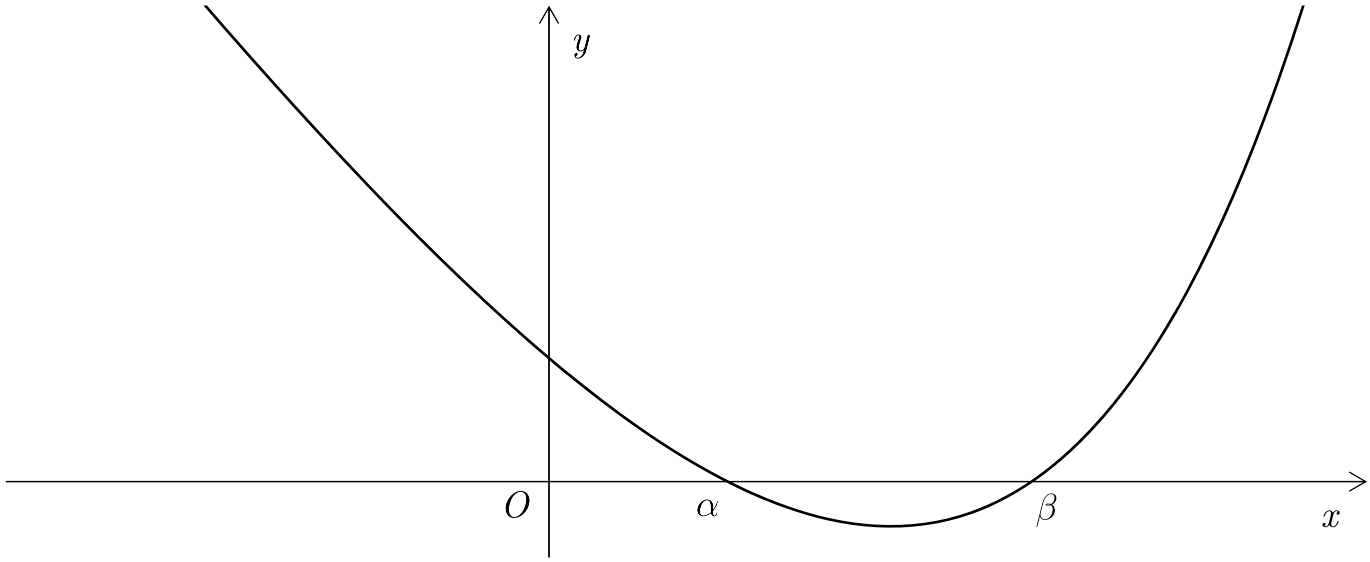
(c) After how many complete months will the total in the account first exceed \$2 000? [4]

Exercise 596. (9233 N2008/II/2.) (Answer on p. **2037**.)

An arithmetic progression and a geometric progression each have first term $\frac{1}{2}$.

The sum of their second terms is $\frac{1}{2}$ and the sum of their third terms is $\frac{1}{8}$. Given that the geometric progression is convergent, find its sum to infinity. [6]

Exercise 597. (9740 N2007/I/9.) (Answer on p. **2038**.)



The diagram shows the graph of $y = \mathrm{e}^x - 3x$. The two roots of the equation $\mathrm{e}^x - 3x = 0$ are denoted by α and β, where $\alpha < \beta$.

(i) Find the values of α and β, each correct to 3 decimal places. [2]

A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3}e^{x_n}, \qquad \text{for } n \geq 1.$$

(ii) Prove algebraically that, if the sequence converges, then it converges to either α or β. [2]

(iii) Use a calculator to determine the behaviour of the sequence for each of the cases $x_1 = 0$, $x_1 = 1$, $x_1 = 2$. [3]

(iv) By considering $x_{n+1} - x_n$, prove that

$$x_{n+1} < x_n \text{ if } \alpha < x_n < \beta,$$
$$x_{n+1} > x_n \text{ if } x_n < \alpha \text{ or } x_n > \beta. \qquad [2]$$

(v) State briefly how the results in part **(iv)** relate to the behaviours determined in **(iii)**. [2]

Exercise 598. (9740 N2007/I/10.) (Answer on p. **2038**.)

A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of the geometric series are equal to the first, fourth and sixth terms respectively of the arithmetic series.

(i) Show that $3r^2 - 5r + 2 = 0$. [4]

(ii) Deduce that the geometric series is convergent and find, in terms of a, the sum to infinity. [5]

(iii) The sum of the first n terms of the arithmetic series is denoted by S. Given that $a > 0$, find the set of possible values of n for which S exceeds $4a$. [5]

Exercise 599. (9740 N2007/II/2.) (Answer on p. 2039.)

A sequence u_1, u_2, u_3, … is such that $u_1 = 1$ and

$$u_{n+1} = u_n - \frac{2n+1}{n^2(n+1)^2}, \qquad \text{for all } n \geq 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \frac{1}{n^2}$. [4]

(ii) Hence find $\sum_{n=1}^{N} \frac{2n+1}{n^2(n+1)^2}$. [2]

(iii) Give a reason why the series in part **(ii)** is convergent and state the sum to infinity. [2]

(iv) Use your answer to part **(ii)** to find $\sum_{n=2}^{N} \frac{2n-1}{n^2(n+1)^2}$. [2]

Exercise 600. (9233 N2007/I/14.) (Answer on p. 2040.)

Use the method of mathematical induction to prove the following result.

$$\sum_{r=1}^{n} \sin rx = \frac{\cos\frac{1}{2}x - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}, \text{ where } 0 < x < 2\pi. \quad [6]$$

Exercise 601. (9233 N2007/II/1.) Find $\sum_{r=1}^{2n} 3^{r+2}$. [3] (Answer on p. 2040.)

Exercise 602. (9233 N2006/I/1.) (Answer on p. 2040.)

The sum S_n of the first n terms of a geometric progression is given by $S_n = 6 - \frac{2}{3^{n-1}}$. Find the first term and the common ratio. [4]

Exercise 603. (9233 N2006/I/11.) (Answer on p. 2040.)

(i) Prove by induction that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2. \quad [4]$$

(ii) Deduce that $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n+1)^2$. [1]

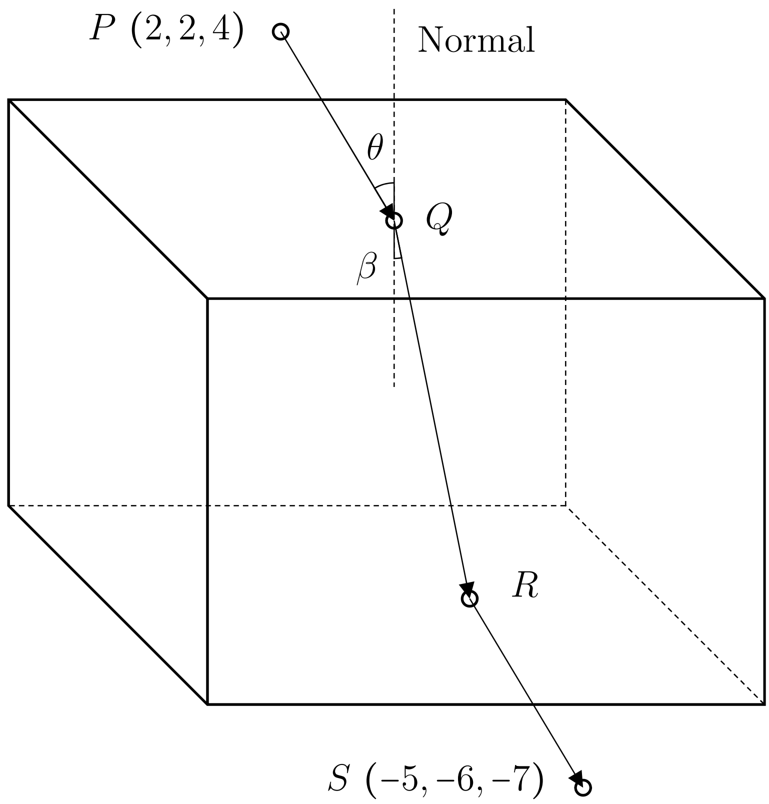
(iii) Hence or otherwise find

$$\sum_{r=1}^{n} (2r-1)^3,$$

simplifying your answer. [4]

Exercise 604. (9758 N2019/I/12.) (Answer on p. **2042**.)

A ray of light passes from air into a material made into a rectangular prism. The ray of light is sent in the direction $(-2,-3,-6)$ from a light source at the point P with coordinates $(2,2,4)$. The prism is placed so that the ray of light passes through the prism, entering at the point Q and emerging at the point R and is picked up by a sensor at point S with coordinates $(-5,-6,-7)$. The acute angle between PQ and the normal to the top of the prism at Q is θ and the acute angle between QR and the same normal is β (see diagram).



It is given that the top of the prism is a part of the plane $x+y+z=1$, and that the base of the prism is a part of the plane $x+y+z=-9$.

It is also given that the ray of light along PQ is parallel to the ray of light along RS so that P, Q, R and S lie in the same plane.

Remark 186. This question is unclear. In my answer, I make these two assumptions:

1. Q is on "the top of the prism" and hence part of the plane $x+y+z=1$;
2. R is on "the bottom top of the prism" and hence part of the plane $x+y+z=-9$.

I've drawn the above diagram to match my interpretation of the question.

(i) Find the exact coordinates of Q and R. [5]

(ii) Find the values of $\cos\theta$ and $\cos\beta$. [3]

(iii) Find the thickness of the prism measured in the direction of the normal at Q. [3]

Remark 187. I interpret **(iii)** as asking for the distance between the planes $x+y+z=1$ and $x+y+z=-9$.

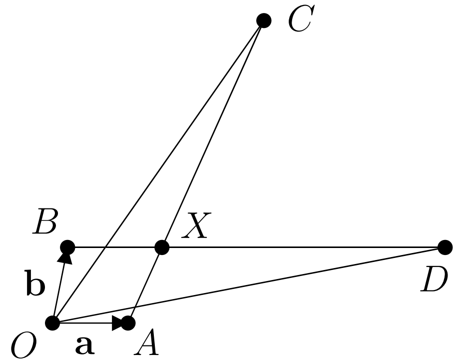
Snell's law states that $\sin\theta = k\sin\beta$, where k is a constant called the refractive index.

(iv) Find k for the material of this prism. [1]

(v) What can be said about the value of k for a material for which $\beta > \theta$? [1]

Exercise 605. (9758 N2019/II/5.) (Answer on p. 2042.)

With reference to the origin O, the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = 2\mathbf{a} + 4\mathbf{a}$ and $\overrightarrow{OD} = \mathbf{b} + 5\mathbf{a}$. The lines BD and AC cross at X (see diagram).



(i) Express $\overrightarrow{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

The point Y lies on CD and is such that the points O, X and Y are collinear.

(ii) Express $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and find the ratio $OX : OY$. [6]

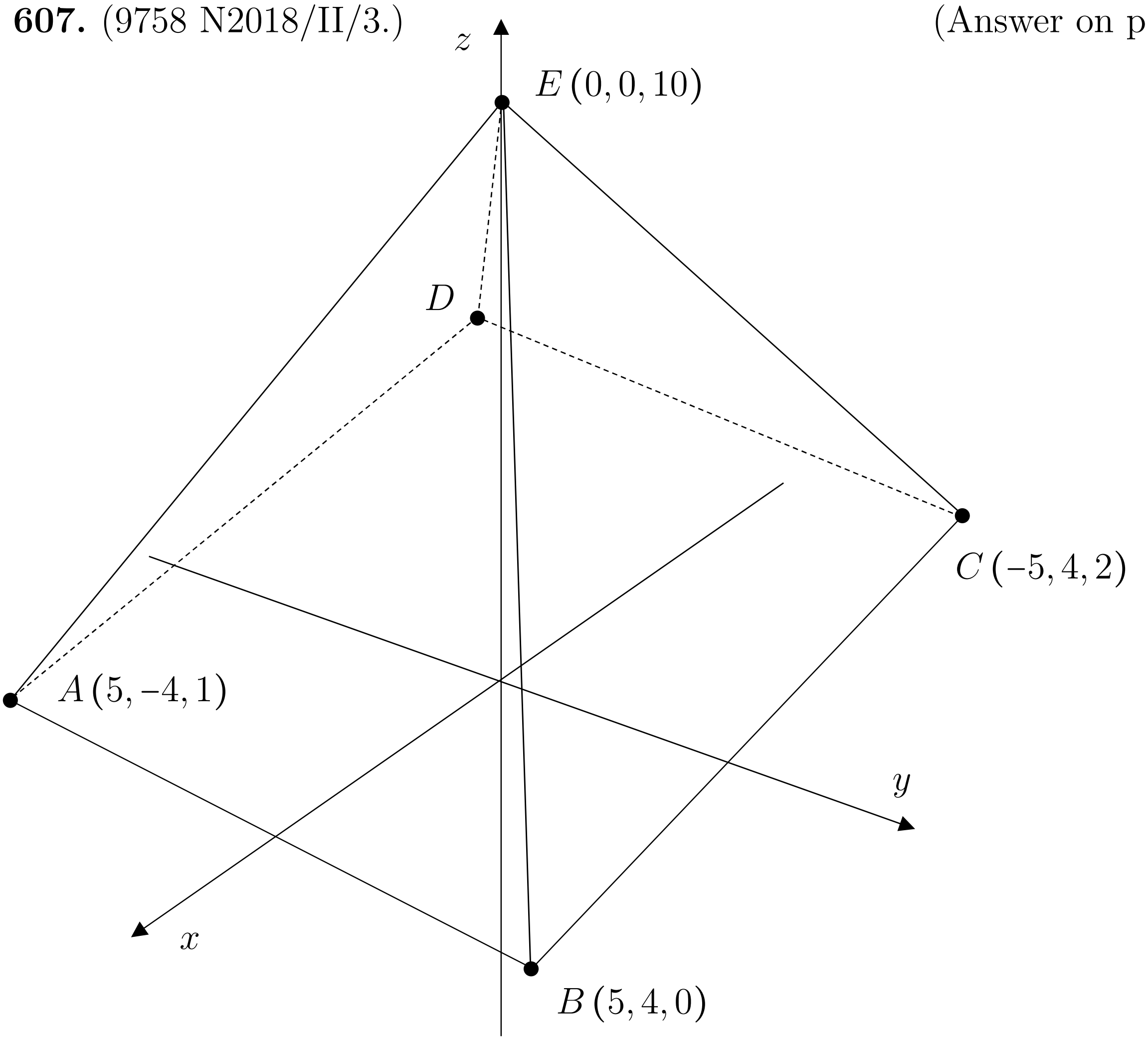
Exercise 606. (9758 N2018/I/6.) (Answer on p. 2043.)

Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \times 3\mathbf{b} = 2\mathbf{a} \times \mathbf{c}$.

(i) Show that $3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$, where λ is a constant. [2]

(ii) It is now given that $\mathbf{a}$ and $\mathbf{c}$ are unit vectors, that the modulus of $\mathbf{b}$ is 4 and that the angle between $\mathbf{b}$ and $\mathbf{c}$ is 60°. Using a suitable scalar product, find exactly the two possible values of λ. [5]

Exercise 607. (9758 N2018/II/3.) (Answer on p. 2043.)



(Exercise continues on the next page ...)

(Exercise continued from the previous page ...) An oblique pyramid has a plane base $ABCD$ in the shape of a parallelogram. The coordinates of A, B and C are $(5,-4,1)$, $(5,4,0)$ and $(-5,4,2)$ respectively. The apex of the pyramid is at $E\,(0,0,10)$ (see diagram).

(i) Find the coordinates of D. [1]

(ii) Find the cartesian equation of face BCE. [3]

Remark 188. Strangely enough, **(ii)** asks for the cartesian equation of the *face* BCE, which is a *triangle*. Given that the cartesian equation for a triangle is quite complicated (and certainly not taught in H2 Maths), they probably simply meant the plane BCE.

(iii) Find the angle between face BCE and the base of the pyramid. [3]

(iv) Find the shortest distance from the midpoint of edge AD to face BCE. [5]

Remark 189. Again, I believe what they meant to ask was for "the shortest distance from the midpoint of edge AD to *plane* BCE".

Exercise 608. (9758 N2017/I/6.) (Answer on p. 2044.)

(i) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter. [2]

(ii) Interpret geometrically the vector equation $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{n}$ is a constant unit vector and d is a constant scalar, stating what d represents. [3]

(iii) Given that $\mathbf{b} \cdot \mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} \cdot \mathbf{n} = d$ to find $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ and d. Interpret the solution geometrically. [3]

Remark 190. This question should have clearly stated if this was meant to be in the context of two- or three-dimensional space.[560] I shall assume the latter.

Exercise 609. (9758 N2017/I/10.) (Answer on p. 2044.)

Electrical engineers are installing electricity cables on a building site. Points (x, y, z) are defined relative to a main switching site at $(0,0,0)$, where units are metres. Cables are laid in straight lines and the widths of cables can be neglected.

An existing cable C starts at the main switching site and goes in the direction $(3,1,-2)$. A new cable is installed which passes through points $P\,(1,2,-1)$ and $Q\,(5,7,a)$.

(i) Find the value of a for which C and the new cable will meet. [4]

To ensure that the cables do not meet, the engineers use $a = -3$. The engineers wish to connect each of the points P and Q to a point R on C.

(ii) The engineers wish to reduce the length of cable required and believe in order to do this that angle PRQ should be $90°$. Show that this is not possible. [4]

(iii) The engineers discover that the ground between P and R is difficult to drill through and now decide to make the length of PR as small as possible. Find the coordinates of R in this case and the exact minimum length. [5]

[560] This is because in the context of two-dimensional space, the vector equation $\mathbf{r} \cdot \mathbf{n} = d$ describes a line.

Exercise 610. (9740 N2016/I/5.) (Answer on p. **2045**.)

The vectors $\mathbf{u}$ and $\mathbf{v}$ are given by $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = a\mathbf{i} + b\mathbf{k}$, where a and b are constants.

(i) Find $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ in terms of a and b. [2]

(ii) Given that the $\mathbf{i}$- and $\mathbf{k}$-components of the answer to part **(i)** are equal, express $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ in terms of a only. Hence find, in an exact form, the possible values of a for which $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ is a unit vector. [4]

(iii) Given instead that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$, find the numerical value of $|\mathbf{v}|$. [2]

Exercise 611. (9740 N2016/I/11.) (Answer on p. **2046**.)

The plane p has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} a \\ 4 \\ -2 \end{pmatrix}$, and the line l has equation $\mathbf{r} = \begin{pmatrix} a-1 \\ a \\ a+1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, where a is a constant and λ, μ and t are parameters.

(i) In the case where $a = 0$,

(a) show that l is perpendicular to p and find the values of λ, μ and t which give the coordinates of the point at which l and p intersect, [5]

(b) find the cartesian equations of the planes such that the perpendicular distance from each plane to p is 12. [5]

(ii) Find the value of a such that l and p do not meet in a unique point. [3]

Exercise 612. (9740 N2015/I/7.) (Answer on p. **2047**.)

Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OA, between O and A, such that $OC : CA = 3 : 2$. Point D lies on OB, between O and B, such that $OD : DB = 5 : 6$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(ii) Show that the vector equation of the line BC can be written as $\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$, where λ is a parameter. Find in a similar form the vector equation of the line AD in terms of a parameter μ. [3]

(iii) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines BC and AD meet and find the ratio $AE : ED$. [5]

Exercise 613. (9740 N2015/II/2.) (Answer on p. **2048**.)

The line L has equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$.

(i) Find the acute angle between L and the x-axis. [2]

The point P has position vector $2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

(ii) Find the points on L which are a distance of $\sqrt{33}$ from P. Hence or otherwise find the point on L which is closest to P. [5]

(iii) Find a cartesian equation of the plane that includes the line L and the point P. [3]

Exercise 614. (9740 N2014/I/3.) (Answer on p. **2048**.)

(i) Given that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, what can be deduced about the vectors $\mathbf{a}$ and $\mathbf{b}$? [2]

(ii) Find a unit vector $\mathbf{n}$ such that $\mathbf{n} \times (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \mathbf{0}$. [2]

(iii) Find the cosine of the acute angle between $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and the z-axis. [1]

Exercise 615. (9740 N2014/I/9.) (Answer on p. **2048**.)

Planes p and q are perpendicular. Plane p has equation $x + 2y - 3z = 12$. Plane q contains the line l with equation

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}.$$

The point A on l has coordinates $(1, -1, 3)$.

(i) Find a cartesian equation of q. [4]

(ii) Find a vector equation of the line m where p and q meet. [4]

(iii) B is a general point on m. Find an expression for the square of the distance AB. Hence, or otherwise, find the coordinates of the point on m which is nearest to A. [5]

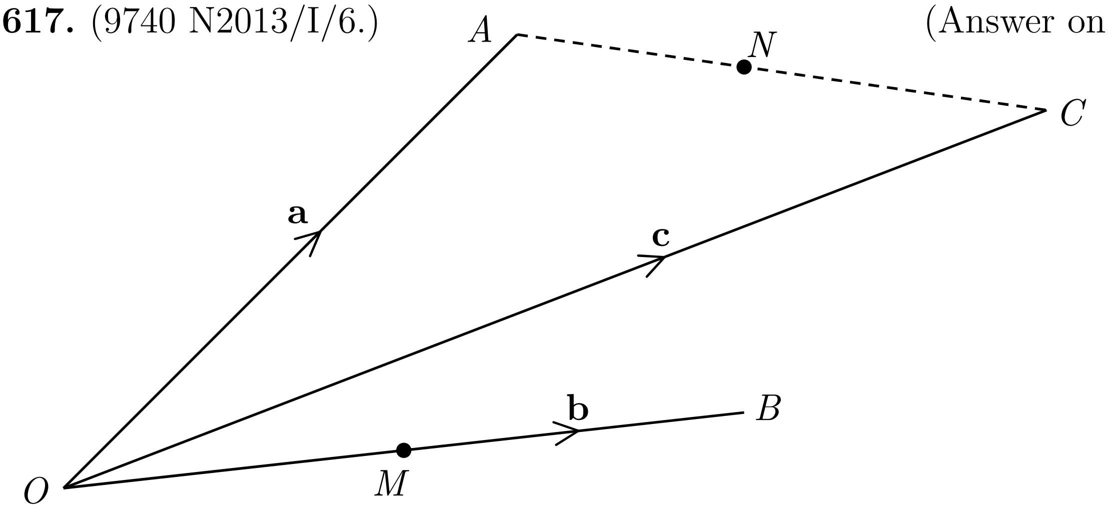
Exercise 616. (9740 N2013/I/1.) (Answer on p. **2048**.)

Planes p, q and r have equations $x - 2z = 4$, $2x - 2y + z = 6$ and $5x - 4y + \mu z = -9$ respectively, where μ is a constant.

(i) Given that $\mu = 3$, find the coordinates of the point of intersection of p, q and r. [2]

(ii) Given instead that $\mu = 0$, describe the relationship between p, q and r. [3]

Exercise 617. (9740 N2013/I/6.) (Answer on p. 2049.)



The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$ (see diagram).

(i) Explain why $\mathbf{c}$ can be expressed as $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for constants λ and μ. [1]

The point N is on AC such that $AN : NC = 3 : 4$.

(ii) Write down the position vector of N in terms of $\mathbf{a}$ and $\mathbf{c}$. [1]

(iii) It is given that the area of triangle ONC is equal to the area of triangle OMC, where M is the mid-point of OB. By finding the areas of these triangles in terms of $\mathbf{a}$ and $\mathbf{b}$, find λ in terms of μ in the case where λ and μ are both positive. [5]

Exercise 618. (9740 N2013/II/4.) (Answer on p. 2049.)

The planes p_1 and p_2 have equations $\mathbf{r} \cdot (2, -2, 1) = 1$ and $\mathbf{r} \cdot (-6, 3, 2) = -1$ respectively, and meet in the line l.

(i) Find the acute angle between p_1 and p_2. [3]

(ii) Find a vector equation for l. [4]

(iii) The point $A\,(4, 3, c)$ is equidistant from the planes $p_1$and p_2. Calculate the two possible values of c. [6]

Exercise 619. (9740 N2012/I/5.) (Answer on p. 2050.)

Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$. The point C has position vector $\mathbf{c}$ given by $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, where λ and μ are positive constants.

(i) Given that the area of triangle OAC is $\sqrt{126}$, find μ. [4]

(ii) Given instead that $\mu = 4$ and that $OC = 5\sqrt{3}$, find the possible coordinates of C. [4]

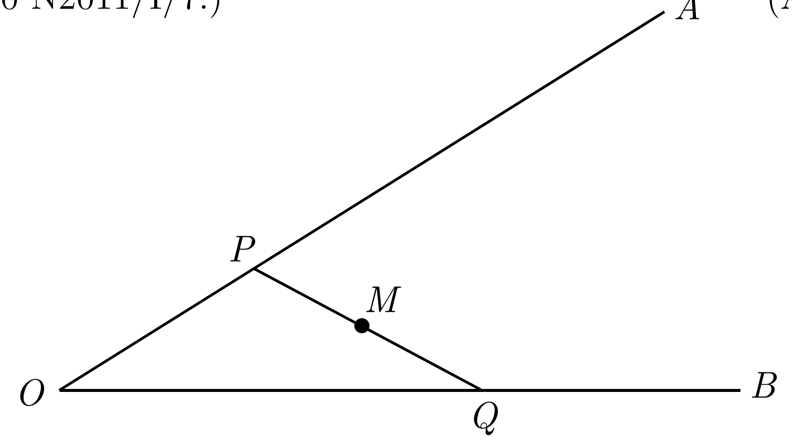
Exercise 620. (9740 N2012/I/9.) (Answer on p. **2050**.)

(i) Find a vector equation of the line through the points A and B with position vectors $7\mathbf{i}+8\mathbf{j}+9\mathbf{k}$ and $-\mathbf{i}-8\mathbf{j}+\mathbf{k}$ respectively. [3]

(ii) The perpendicular to this line from the point C with position vector $\mathbf{i}+8\mathbf{j}+3\mathbf{k}$ meets the line at the point N. Find the position vector of N and the ratio $AN : NB$. [5]

(iii) Find a cartesian equation of the line which is a reflection of the line AC in the line AB. [4]

Exercise 621. (9740 N2011/I/7.) (Answer on p. **2051**.)



Referred to the origin O, the points A and B are such that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. The point P on OA is such that $OP : PA = 1 : 2$, and the point Q on OB is such that $OQ : QB = 3 : 2$. The mid-point of PQ is M (see diagram).

(i) Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and show that the area of triangle OMP can be written as $k\,|\mathbf{a}\times\mathbf{b}|$, where k is a constant to be found. [6]

(ii) The vectors $\mathbf{a}$ and $\mathbf{b}$ are now given by $\mathbf{a}=2p\mathbf{i}-6p\mathbf{j}+3p\mathbf{k}$ and $\mathbf{b}=\mathbf{i}+\mathbf{j}-2\mathbf{k}$, where p is a positive constant. Given that $\mathbf{a}$ is a unit vector,

(a) find the exact value of p, [2]

(b) give a geometrical interpretation of $|\mathbf{a}\cdot\mathbf{b}|$, [1]

(c) evaluate $\mathbf{a}\times\mathbf{b}$. [2]

Exercise 622. (9740 N2011/I/11.) (Answer on p. **2051**.)

The plane p passes through the points with coordinates $(4,-1,-3)$, $(-2,-5,2)$ and $(4,-3,-2)$.

(i) Find a cartesian equation of p. [4]

The lines l_1 and l_2 have, respectively, equations

$$\frac{x-1}{2}=\frac{y-2}{-4}=\frac{z+3}{1} \text{ and } \frac{x+2}{1}=\frac{y-1}{5}=\frac{z-3}{k},$$

where k is a constant. It is given that l_1 and l_2 intersect.

(ii) Find the value of k. [4]

(iii) Show that $l_1$lies in p and find the coordinates of the point at which l_2 intersects p. [4]

(iv) Find the acute angle between l_2 and p. [3]

Exercise 623. (9740 N2010/I/1.) (Answer on p. **2052**.)

The position vectors **a** and **b** are given by

$$\mathbf{a} = 2p\mathbf{i} + 3p\mathbf{j} + 6p\mathbf{k} \text{ and } \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k},$$

where $p > 0$. It is given that $|\mathbf{a}| = |\mathbf{b}|$.

(i) Find the exact value of p. [2]

(ii) Show that $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) = 0$. [3]

Exercise 624. (9740 N2010/I/10.) (Answer on p. **2052**.)

The line l and plane p have, respectively, equations

$$\frac{x-10}{-3} = \frac{y+1}{6} = \frac{z+3}{9} \text{ and } x - 2y - 3z = 0.$$

(i) Show that l is perpendicular to p. [2]

(ii) Find the coordinates of the point of intersection of l and p. [4]

(iii) Show that the point A with coordinates $(-2, 23, 33)$ lies on l. Find the coordinates of the point B which is the mirror image of A in p. [3]

(iv) Find the area of triangle OAB, where O is the origin, giving your answer to the nearest whole number. [3]

Exercise 625. (9740 N2009/I/10.) (Answer on p. **2052**.)

The planes p_1 and p_2 have equations $\mathbf{r} \cdot (2, 1, 3) = 1$ and $\mathbf{r} \cdot (-1, 2, 1) = 2$ respectively, and meet in a line l.

(i) Find the acute angle between p_1 and p_2. [3]

(ii) Find a vector equation of l. [4]

(iii) The plane p_3 has equation $2x + y + 3z - 1 + k(-x + 2y + z - 2) = 0$. Explain why l lies in p_3 for any constant k. Hence, or otherwise, find a cartesian equation of the plane in which both l and the point $(2, 3, 4)$ lie. [5]

Exercise 626. (9740 N2009/II/2.) (Answer on p. **2053**.)

Relative to the origin O, two points A and B have position vectors given by $\mathbf{a} = 14\mathbf{i}+14\mathbf{j}+14\mathbf{k}$ and $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$ respectively.

(i) The point P divides the line AB in the ratio $2:1$. Find the coordinates of P. [2]

(ii) Show that AB and OP are perpendicular. [2]

(iii) The vector **c** is a unit vector in the direction of $\overrightarrow{OP}$. Write **c** as a column vector, and give the geometrical meaning of $|\mathbf{a} \cdot \mathbf{c}|$. [2]

(iv) Find $\mathbf{a} \times \mathbf{p}$, where **p** is the vector $\overrightarrow{OP}$, and give the geometrical meaning of $|\mathbf{a} \times \mathbf{p}|$. Hence write down the area of the triangle OAP. [4]

Exercise 627. (9740 N2008/I/3.) (Answer on p. 2053.)

Points O, A, B are such that $\overrightarrow{OA} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\overrightarrow{OB} = 5\mathbf{i} - \mathbf{j}$, and the point P is such that $OAPB$ is a parallelogram.

(i) Find $\overrightarrow{OP}$. [1]

(ii) Find the size of angle AOB. [3]

(iii) Find the exact area of the parallelogram $OAPB$. [2]

Exercise 628. (9740 N2008/I/11.) (Answer on p. 2053.)

The equations of three planes p_1, p_2, p_3 are

$$\begin{aligned} 2x - 5y + 3z &= 3, \\ 3x + 2y - 5z &= -5, \\ 5x + \lambda y + 17z &= \mu, \end{aligned}$$

respectively, where λ and $\mu$are constants. When $\lambda = -20.9$ and $\mu = 16.6$, find the coordinates of the point at which these planes meet. [2]

The planes p_1 and p_2 intersect in a line l.

(i) Find a vector equation of l. [4]

(ii) Given that all three planes meet in the line l, find λ and μ. [3]

(iii) Given instead that the three planes have no points in common, what can be said about the values of λ and μ? [2]

(iv) Find the cartesian equation of the plane which contains l and the point $(1, -1, 3)$. [4]

Exercise 629. (9233 N2008/I/11.) (Answer on p. 2054.)

The cartesian equations of two lines are

$$\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1} \text{ and } \frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}.$$

(i) Show that the lines intersect and state the point of intersection. [5]

(ii) Find the acute angle between the lines. [4]

Exercise 630. (9740 N2007/I/6.) (Answer on p. 2054.)

Referred to the origin O, the position vectors of the points A and B are $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ respectively.

(i) Show that OA is perpendicular to OB. [2]

(ii) Find the position vector of the point M on the line segment AB such that $AM : MB = 1 : 2$. [3]

(iii) The point C has position vector $-4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Use a vector product to find the exact area of triangle OAC. [4]

Exercise 631. (9740 N2007/I/8.) (Answer on p. **2054**.)

The line l passes through the points A and B with coordinates $(1,2,4)$ and $(-2,3,1)$ respectively. The plane p has equation $3x-y+2z=17$. Find

(i) the coordinates of the point of intersection of l and p, [5]

(ii) the acute angle between l and p, [3]

(iii) the perpendicular distance from A to p. [3]

Exercise 632. (9233 N2007/I/7.) (Answer on p. **2054**.)

The point P is the foot of the perpendicular from the point $A\,(1,3,-2)$ to the line given by

$$\frac{x+3}{2}=\frac{y-8}{2}=\frac{z-3}{3}.$$

Find the coordinates of P, and hence find the length of AP. [7]

Exercise 633. (9233 N2007/II/2.) (Answer on p. **2054**.)

Referred to an origin O the position vectors of two points A and B are $4\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $\mathbf{i}-3\mathbf{j}+4\mathbf{k}$ respectively. Two other points, C and D, are given by $\overrightarrow{OC}=0.25\overrightarrow{OA}$ and $\overrightarrow{OD}=0.75\overrightarrow{OB}$.

(i) Find a vector equation for the line AD. [2]

(ii) Find the position vector of the point of intersection of AD and BC. [5]

Exercise 634. (9233 N2006/I/14.) (Answer on p. **2055**.)

The points A, B, C and D have position vectors $\mathbf{i}-2\mathbf{j}+5\mathbf{k}$, $\mathbf{i}+3\mathbf{j}$, $10\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $-2\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ respectively, with respect to an origin O. The point P on AB is such that $AP:PB=\lambda:1-\lambda$ and the point Q on CD is such that $CQ:QD=\mu:1-\mu$. Find $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of λ and μ respectively. [3]

Given that PQ is perpendicular to both AB and CD,

(i) show that $\overrightarrow{PQ}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, [7]

(ii) Find the area of triangle ABQ. [2]

136. Past-Year Questions for Part IV. Complex Numbers

Exercise 635. (9758 N2019/I/1.) (Answer on p. **2056**.)

The function f is defined by $f(z) = az^3 + bz^2 + cz + d$, where a, b, c and d are real numbers. Given that $2 + \mathrm{i}$ and -3 are roots of $f(z) = 0$, find b, c and d in terms of a. [4]

Exercise 636. (9758 N2019/I/9.) (Answer on p. **2056**.)

(i) The complex number w can be expressed as $\cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $w + \dfrac{1}{w}$ is a real number. [2]

(b) Show that $\dfrac{w-1}{w+1}$ can be expressed as $k\tan\frac{1}{2}\theta$, where k is a complex number to be found. [4]

(ii) The complex number z has modulus 1. Find the modulus of the complex number $\dfrac{z-3\mathrm{i}}{1+3\mathrm{i}z}$. [5]

Exercise 637. (9758 N2018/II/2.) (Answer on p. **2057**.)

(i) One of the roots of the equation $4x^4 - 20x^3 + sx^2 - 56x + t = 0$, where s and t are real, is $2 - 3\mathrm{i}$. Find the other roots of the equation and the values of s and t. [5]

(ii) The complex number w is such that $w^3 = 27$.

(a) Given that one possible value of w is 3, use **a non-calculator method** to find the other possible values of w. Give your answers in the form $a + \mathrm{i}b$, where a and b are exact values. [3]

(b) Write these values of w in modulus-argument form and represent them on an Argand diagram. [2]

(c) Find the sum and the product of all the possible values of w, simplifying your answers. [2]

Exercise 638. (9758 N2017/I/8.) (Answer on p. **2058**.)

Do not use a calculator in answering this question.

(a) Find the roots of the equation $z^2(1-\mathrm{i}) - 2z + (5+5\mathrm{i}) = 0$, giving your answers in cartesian form $a + \mathrm{i}b$. [3]

(b) **(i)** Given that $\omega = 1 - \mathrm{i}$, find ω^2, ω^3 and ω^4 in cartesian form. Given also that

$$\omega^4 + p\omega^3 + 39\omega^2 + q\omega + 58 = 0,$$

where p and q are real, find p and q. [4]

(ii) Using the values of p and q in part **(b)(i)**, express $\omega^4 + p\omega^3 + 39\omega^2 + q\omega + 58$ as the product of two quadratic factors. [3]

Exercise 639. (9740 N2016/I/7.) (Answer on p. **2058**.)

Do not use a calculator in answering this question.

(a) Verify that $-1+5\mathrm{i}$ is a root of the equation $w^2+(-1-8\mathrm{i})\,w+(-17+7\mathrm{i})=0$. Hence, or otherwise, find the second root of the equation in cartesian form, $p+\mathrm{i}q$, showing your working. [5]

(b) The equation $z^3-5z^2+16z+k=0$, where k is a real constant, has a root $z=1+a\mathrm{i}$, where a is a positive real constant. Find the values of a and k, showing your working. [5]

Exercise 640. (9740 N2016/II/4.) (Answer on p. **2059**.)

(a) Two loci in the Argand diagram are given by the equations

$$|z-3-\mathrm{i}|=1 \qquad \text{and} \qquad \arg z=\alpha, \text{ where } \tan\alpha=0.4.$$

The complex numbers $z_1$and z_2, where $|z_1|<|z_2|$, correspond to the points of intersection of these loci.

(i) Draw an Argand diagram to show both loci, and mark the points represented by z_1 and z_2. [2]

(ii) Find the two values of z which represent points on $|z-3-\mathrm{i}|=1$ such that $|z-z_1|=|z-z_2|$. [4]

(b) **(i)** The complex number $2-2\mathrm{i}$ is denoted by w. By writing w in polar form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$, find exactly all the cube roots of w in polar form. [3]

(ii) Find the smallest positive whole number value of n such that $\arg\left(w^*w^n\right)=\frac{1}{2}\pi$.[3]

Exercise 641. (9740 N2015/I/9.) (Answer on p. **2059**.)

(a) The complex number w is such that $w=a+\mathrm{i}b$, where a and b are non-zero real numbers. The complex conjugate of w is denoted by w^*. Given that $\dfrac{w^2}{w^*}$ is purely imaginary, find the possible values of w in terms of a. [5]

(b) The complex number z is such that $z^5=-32\mathrm{i}$.

(i) Find the modulus and argument of each of the possible values of z. [4]

(ii) Two of these values are z_1 and z_2, where $0.5\pi<\arg z_1<\pi$ and $-\pi<\arg z_2<-0.5\pi$. Find the exact value of $\arg\left(z_1-z_2\right)$ in terms of π and show that $|z_1-z_2|=4\sin\left(0.2\pi\right)$. [4]

Exercise 642. (9740 N2014/I/5.) (Answer on p. **2060**.)

It is given that $z = 1 + 2\mathrm{i}$.

(i) Without using a calculator, find the values of z^2 and $\dfrac{1}{z^3}$ in cartesian form $x + \mathrm{i}y$, showing your working. [4]

(ii) The real numbers p and q are such that $pz^2 + \dfrac{q}{z^3}$ is real. Find, in terms of p, the value of q and the value of $pz^2 + \dfrac{q}{z^3}$. [3]

Exercise 643. (9740 N2014/II/4.) (Answer on p. **2060**.)

(a) The complex number z satisfies $|z + 5 - \mathrm{i}| = 4$.

(i) On an Argand diagram show the locus of z. [2]

(ii) The complex number z also satisfies $|z - 6\mathrm{i}| = |z + 10 + 4\mathrm{i}|$. Find exactly the possible values of z, giving your answers in the form $x + \mathrm{i}y$. [4]

(b) It is given that $w = \sqrt{3} - \mathrm{i}$.

(i) Without using a calculator, find an exact expression for w^6. Give your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [3]

(ii) Without using a calculator, find the three smallest positive whole number values of n for which $\dfrac{w^n}{w^*}$ is a real number. [4]

Exercise 644. (9740 N2013/I/4.) (Answer on p. **2061**.)

The complex number w is given by $1 + 2\mathrm{i}$.

(i) Find w^3 in the form $x + \mathrm{i}y$, showing your working. [2]

(ii) Given that w is a root of the equation $az^3 + 5z^2 + 17z + b = 0$, find the values of the real numbers a and b. [3]

(iii) Using these values of a and b, find all the roots of this equation in exact form. [3]

Exercise 645. (9740 N2013/I/8.) (Answer on p. **2062**.)

The complex number z is given by $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta \le 0.5\pi$.

(i) Given that $w = \left(1 - \mathrm{i}\sqrt{3}\right)z$, find $|w|$ in terms of r and $\arg w$ in terms of θ. [2]

(ii) Given that r has a fixed value, draw an Argand diagram to show the locus of z as θ varies. On the same diagram, show the corresponding locus of w. You should identify the modulus and argument of the endpoints of each locus. [4]

(iii) Given that $\arg \dfrac{z^{10}}{w^2} = \pi$, find θ. [3]

Exercise 646. (9740 N2012/I/6.) (Answer on p. **2063**.)

Do not use a calculator in answering this question.

The complex number z is given by $z = 1 + \mathrm{i}c$, where c is a non-zero real number.

(i) Find z^3 in the form $x + \mathrm{i}y$. [2]

(ii) Given that z^3 is real, find the possible values of z. [2]

(iii) For the value of z found in part (ii) for which $c < 0$, find the smallest positive integer n such that $|z^n| > 1000$. State the modulus and argument of z^n when n takes this value. [4]

Exercise 647. (9740 N2012/II/2.) (Answer on p. **2063**.)

The complex number z satisfies the equation $|z - (7 - 3\mathrm{i})| = 4$.

(i) Sketch an Argand diagram to illustrate this equation. [2]

(ii) Given that $|z|$ is as small as possible,

(a) find the exact value of $|z|$, [2]

(b) hence find an exact expression for z, in the form $x + \mathrm{i}y$. [2]

(iii) It is given instead that $-\pi < \arg z \le \pi$ and that $|\arg z|$ is as large as possible. Find the value of $\arg z$ in radians, correct to 4 significant figures. [3]

Exercise 648. (9740 N2011/I/10.) (Answer on p. **2064**.)

Do not use a graphic calculator in answering this question.

(i) The roots of the equation $z^2 = -8\mathrm{i}$ are z_1 and z_2. Find z_1 and z_2 in cartesian form $x + \mathrm{i}y$, showing your working. [4]

(ii) Hence, or otherwise, find in cartesian form the roots w_1 and w_2 of the equation $w^2 + 4w + (4 + 2\mathrm{i}) = 0$. [3]

(iii) Using a single Argand diagram, sketch the loci

(a) $|z - z_1| = |z - z_2|$, [1]

(b) $|z - w_1| = |z - w_1|$, [1]

(iv) Give a reason why there are no points which lie on both of these loci. [1]

Exercise 649. (9740 N2011/II/1.) (Answer on p. **2065**.)

The complex number z satisfies $|z - 2 - 5\mathrm{i}| \le 3$.

(i) On an Argand diagram, sketch the region in which the point representing z can lie. [3]

(ii) Find exactly the maximum and minimum possible values of $|z|$. [2]

(iii) It is given that $0 \le \arg z \le \dfrac{\pi}{4}$. With this extra information, find the maximum value of $|z - 6 - \mathrm{i}|$. Label the point(s) that correspond to this maximum value on your diagram with the letter P. [3]

Exercise 650. (9740 N2010/I/8.) (Answer on p. 2066.)

The complex numbers z_1 and z_2 are given by $1 + \mathrm{i}\sqrt{3}$ and $-1 - \mathrm{i}$ respectively.

(i) Express each of z_1 and z_2 in polar form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give r and θ in exact form. [2]

(ii) Find the complex conjugate of $\frac{z_1}{z_2}$ in exact polar form. [3]

(iii) On a single Argand diagram, sketch the loci

(a) $|z - z_1| = 2$,

(b) $\arg(z - z_2) = \frac{\pi}{4}$. [4]

(iv) Find where the locus $|z - z_1| = 2$ meets the positive real axis. [2]

Exercise 651. (9740 N2010/II/1.) (Answer on p. 2067.)

(i) Solve the equation $x^2 - 6x + 34 = 0$. [2]

(ii) One root of the equation $x^4 + 4x^3 + x^2 + ax + b = 0$, where a and b are real, is $x = -2 + \mathrm{i}$. Find the values of a and b and the other roots. [5]

Exercise 652. (9740 N2009/I/9.) (Answer on p. 2068.)

(i) Solve the equation

$$z^7 - (1 + \mathrm{i}) = 0,$$

giving the roots in the form $r\mathrm{e}^{\mathrm{i}\alpha}$, where $r > 0$ and $-\pi < \alpha \leq \pi$. [5]

(ii) Show the roots on an Argand diagram. [2]

(iii) The roots represented by z_1 and z_2 are such that $0 < \arg z_1 < \arg z_2 < \frac{\pi}{2}$. Explain why the locus of all points z such that $|z - z_1| = |z - z_2|$ passes through the origin. Draw this locus on your Argand diagram and find its exact cartesian equation. [5]

Exercise 653. (9740 N2008/I/8.) (Answer on p. 2069.)

A graphic calculator is not to be used in answering this question.

(i) It is given that $z_1 = 1 + \sqrt{3}\mathrm{i}$. Find the value of z_1^3, showing clearly how you obtain your answer. [3]

(ii) Given that $1 + \sqrt{3}\mathrm{i}$ is a root of the equation $2z^3 + az^2 + bz + 4 = 0$, find the values of the real numbers a and b. [4]

(iii) For these values a and b, solve the equation in part **(ii)**, and show all the roots on an Argand diagram. [4]

Exercise 654. (9740 N2008/II/3.) (Answer on p. 2070.)

(a) The complex number w has modulus r and argument θ, where $0 < \theta < \frac{\pi}{2}$, and w^* denotes the conjugate of w. State the modulus and argument of p, where $p = \frac{w}{w^*}$. [2]

Given that p^5 is real and positive, find the possible values of θ. [2]

(b) The complex number z satisfies the relations $|z| \leq 6$ and $|z| = |z - 8 - 6\mathrm{i}|$.

- **(i)** Illustrate both of these relations on a single Argand diagram. [3]
- **(ii)** Find the greatest and least possible values of $\arg z$, giving your answers in radians correct to 3 decimal places. [4]

Exercise 655. (9233 N2008/I/9.) (Answer on p. 2071.)

In an Argand diagram, the point P represents the complex number z. Clearly labelling any relevant points, draw three separate diagrams to show the locus of P in each of the following cases.

- **(i)** $|z + 2\mathrm{i}| = 2$. [2]
- **(ii)** $|z - 2 - \mathrm{i}| = |z - \mathrm{i}|$. [2]
- **(iii)** $\frac{\pi}{6} \leq \arg(z + 1 - 3\mathrm{i}) \leq \frac{\pi}{3}$. [4]

Exercise 656. (9233 N2008/II/3.) (Answer on p. 2073.)

- **(i)** Verify that $w = 1 - \mathrm{i}$ satisfies the equation $w^2 = -2\mathrm{i}$ and write down the other root of this equation. [3]
- **(ii)** Use the quadratic formula to solve the equation $z^2 - (3 + 5\mathrm{i})z - 4(1 - 2\mathrm{i}) = 0$. [4]

Exercise 657. (9740 N2007/I/3.) (Answer on p. 2073.)

(a) Sketch, on an Argand diagram, the locus of points representing the complex number z such that $|z + 2 - 3\mathrm{i}| = \sqrt{13}$. [3]

(b) The complex number w is such that $ww^* + 2w = 3 + 4\mathrm{i}$, where w^* is the complex conjugate of w. Find w in the form $a + \mathrm{i}b$, where a and b are real. [4]

Exercise 658. (9740 N2007/I/7.) (Answer on p. 2074.)

The polynomial $P(z)$ has real coefficients. The equation $P(z) = 0$ has a root $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 < \theta < \pi$.

- **(i)** Write down a second root in terms of r and θ, and hence show that a quadratic factor of $P(z)$ is $z^2 - 2rz\cos\theta + r^2$. [3]
- **(ii)** Solve the equation $z^6 = -64$, expressing the solutions in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]
- **(iii)** Hence, or otherwise, express $z^6 + 64$ as the product of three quadratic factors with real coefficients, giving each factor in non-trigonometrical form. [3]

Exercise 659. (9233 N2007/I/9.) (Answer on p. **2075**.)

(i) The equation $az^4 + bz^3 + cz^2 + dz + e = 0$ has a root $z = k\mathrm{i}$, where k is real and non-zero. Given that the coefficients a, b, c, d and e are real, show that $ad^2 + b^2e = bcd$. [5]

(ii) Verify that this condition is satisfied for the equation $z^4 + 3z^3 + 13z^2 + 27z + 36 = 0$ and hence find two roots of this equation which are of the form $z = k\mathrm{i}$, where k is real. [3]

Exercise 660. (9233 N2007/II/5.) (Answer on p. **2075**.)

(i) Illustrate, on an Argand diagram, the locus of a point P representing the complex number z, where $\arg(z - 2\mathrm{i}) = \frac{\pi}{3}$. [3]

(ii) Illustrate, using the same Argand diagram, the locus of a point Q representing the complex number z, where $|z - 4| = |z + 2|$. [2]

(iii) Hence find the exact value of z such that $\arg(z - 2\mathrm{i}) = \frac{\pi}{3}$ and $|z - 4| = |z + 2|$, giving your answer in the form $a + \mathrm{i}b$. [3]

(iv) Show that, in this case, $zz^* = 8 + 4\sqrt{3}$. [2]

Exercise 661. (9233 N2006/I/5.) (Answer on p. **2076**.)

The complex number z satisfies $|z + 4 - 4\mathrm{i}| = 3$.

(i) Describe, with the aid of a sketch, the locus of the point which represents z in an Argand diagram. [3]

(ii) Find the least possible value of $|z - \mathrm{i}|$. [2]

Exercise 662. (9233 N2006/I/6.) (Answer on p. **2077**.)

(i) Show that the equation $z^4 - 2z^3 + 6z^2 - 8z + 8 = 0$ has a root of the form $k\mathrm{i}$ where k is real. [3]

(ii) Hence solve the equation $z^4 - 2z^3 + 6z^2 - 8z + 8 = 0$. [3]

137. Past-Year Questions for Part V. Calculus

Exercise 663. (9758 N2019/I/2.) (Answer on p. **2079**.)

The curve C has equation $y = x^3 + x - 1$.

(i) C crosses the x-axis at the point with coordinates $(a, 0)$. Find the value of a correct to 3 decimal places. [1]

(ii) You are given that $b > a$.

The region P is bounded by C, the x-axis and the lines $x = -1$ and $x = 0$. The region Q is bounded by C, the line $x = b$ and the part of the x-axis between $x = a$ and $x = b$. Given that the area of Q is 2 times the area of P, find the value of b correct to 3 decimal places. [4]

Exercise 664. (9758 N2019/I/10.) (Answer on p. **2080**.)

A curve C has parametric equations

$$x = a\left(2\cos\theta - \cos 2\theta\right),$$
$$y = a\left(2\sin\theta - \sin 2\theta\right),$$

for $0 \le \theta \le 2\pi$.

(i) Sketch C and state the Cartesian equation of its line of symmetry. [2]

(ii) Find the values of θ at the points where C meets the x-axis. [2]

(iii) Show that the area enclosed by the x-axis, and the part of C above the x-axis, is given by

$$\int_{\theta_1}^{\theta_2} a^2\left(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta\right)\,\mathrm{d}\theta,$$

where θ_1 and θ_2 should be stated. [3]

(iv) Hence find, in terms of a, the exact total area enclosed by C. [5]

Remark 191. Question **(iii)** was terrible. A proper solution of **(iii)** requires knowledge of multivariate (or vector) calculus and, in particular, a result known as Green's Theorem.

It is possible to arrive at the "correct answer" by smoking your way through with arbitrary guesswork and having a lot of luck. (More on this in the answer.) This is apparently what counts for maths education in Singapore: Blindly grope your way through and hopefully arrive at the "correct answer", even if you have no idea what you're doing.

By the way, this question also looks a lot like it was simply copied from the website Brilliant.org (but with the letters r and t changed to a and θ):[561]

A **cardioid** is a curve traced by a fixed point on the perimeter of a circle of radius r which is rolling around another circle of radius r. It is parameterized by the equations

$$x = r(2\cos t - \cos 2t)$$
$$y = r(2\sin t - \sin 2t),$$

where $0 \le t \le 2\pi$. What is the area inside the cardioid?

[561] Archived on Wayback Machine since at least 2017-09-19.

Exercise 665. (9758 N2019/I/11.) (Answer on p. **2082**.)

Scientists are investigating how the temperature of water changes in various environments.

(i) The scientists begin by investigating how hot water cools.

The water is heated in a container and then placed in a room which is kept at a constant temperature of 16 °C. The temperature of the water t minutes after it is placed in the room θ °C. This temperature decreases at a rate proportional to the difference between the temperature of the water and the temperature of the room. The temperature of the water falls from a value of 80 °C to 32 °C in the first 30 minutes.

(a) Write down a differential equation for this situation. Solve this differential equation to get θ as an exact function of t. [6]

(b) Find the temperature of the water 45 minutes after it is placed in the room. [1]

(ii) The scientists then model the thickness of ice on a pond.

In winter the surface of the water in the pond freezes. Once the thickness of the ice reaches 3 cm, it is safe to skate on the ice. The thickness of the ice is T cm, t minutes after the water starts to freeze. The freezing of the water is modelled by a differential equation in which the rate of change of the thickness of the ice is inversely proportional to its thickness. It is given that $T = 0$ when $t = 0$. After 60 minutes, the ice is 1 cm thick.

Find the time from when freezing commences until the ice is first safe to skate on. [6]

Exercise 666. (9758 N2019/II/1.) (Answer on p. **2082**.)

You are given that $I = \int x(1-x)^{\frac{1}{2}}\,\mathrm{d}x$.

(i) Use integration by parts to find an expression for I. [2]

(ii) Use the substitution $u^2 = 1 - x$ to find another expression for I. [2]

(iii) Show algebraically that your answers to parts **(i)** and **(ii)** differ by a constant. [2]

Exercise 667. (9758 N2019/II/3.) (Answer on p. **2083**.)

A solid cylinder has radius r cm, height h cm and total surface area 900 cm^2. Find the exact value of the maximum possible volume of the cylinder. Find also the ratio $r : h$ that gives this maximum volume. [7]

Exercise 668. (9758 N2019/II/4.) (Answer on p. **2083**.)

(i) Given that $f(x) = \sec 2x$, find $f'(x)$ and $f''(x)$. Hence, or otherwise, find the Maclaurin series for $f(x)$, up to and including the term in x^2. [5]

(ii) Use your series from part **(i)** to estimate $\int_0^{0.02} \sec 2x \, \mathrm{d}x$, correct to 5 decimal places. [2]

(iii) Use your calculator to find $\int_0^{0.02} \sec 2x \, \mathrm{d}x$, correct to 5 decimal places. [1]

(iv) Comparing your answers to parts **(ii)** and **(iii)**, and with reference to the value of x, comment on the accuracy of your approximations. [2]

(v) Explain why a Maclaurin series for $g(x) = \mathrm{cosec} 2x$ cannot be found. [1]

Exercise 669. (9758 N2018/I/1.) (Answer on p. **2083**.)

(i) Given that $y = \dfrac{\ln x}{x}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. [2]

(ii) Hence, or otherwise, find the exact value of $\int_1^{\mathrm{e}} \dfrac{\ln x}{x^2} \, \mathrm{d}x$, showing your working. [4]

Exercise 670. (9758 N2018/I/2.) (Answer on p. **2084**.)

Do not use a calculator in answering this question.

A curve has equation $y = \dfrac{3}{x}$ and a line has equation $y + 2x = 7$. The curve and the line intersect at the points A and B.

(i) Find the x-coordinates of A and B. [2]

(ii) Find the exact volume generated when the area bounded by the curve and the line is rotated about the x-axis through 360°. [4]

Exercise 671. (9758 N2018/I/3.) (Answer on p. **2084**.)

(i) It is given that $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y - 6$. Using the substitution $y = ux^2$, show that the differential equation can be transformed to $\dfrac{\mathrm{d}u}{\mathrm{d}x} = f(x)$, where the function $f(x)$ is to be found. [3]

(ii) Hence, given that $y = 2$ when $x = 1$, solve the differential equation $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y - 6$, to find y in terms of x. [4]

Exercise 672. (9758 N2018/I/7.) (Answer on p. **2085**.)

A curve C has equation $\dfrac{x^2 - 4y^2}{x^2 + xy^2} = \dfrac{1}{2}$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{2x - y^2}{2xy + 16y}$. [3]

The points P and Q on C each have x-coordinate 1. The tangents to C at P and Q meet at the point N.

(ii) Find the exact coordinates of N. [6]

Exercise 673. (9758 N2018/I/9.) (Answer on p. **2085**.)

A curve C has parametric equations

$$x = 2\theta - \sin 2\theta, \quad y = 2\sin^2\theta,$$

for $0 \le \theta \le \pi$.

(i) Show that $\dfrac{dy}{dx} = \cot\theta$. [4]

(ii) The normal to the curve at the point where $\theta = \alpha$ meets the x-axis at the point A. Show that the x-coordinate of A is $k\alpha$, where k is a constant to be found. [4]

(iii) Do not use a calculator in answering this part.

The distance between two points along a curve is the arc-length. Scientists and engineers need to use arc-length in applications such as finding the work done in moving an object along the path described by a curve or the length of cabling used on a suspension bridge.

The arc-length between two points on C, where $\theta = \beta$ and $\theta = \gamma$, is given by the formula

$$\int_\beta^\gamma \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Find the total length of C. [5]

Remark 192. The H2 Further Mathematics (9649) syllabus includes "arc length of curves defined in cartesian or parametric form". Your H2 Maths (9758) does not. So, such material can't be on your H2 Maths (9758) exams, right? (Wrong.)

Exercise 674. (9758 N2018/I/10.) (Answer on p. 2086.)

An electrical circuit comprises a power source of V volts in series with a resistance of R ohms, a capacitance of C farads and an inductance of L henries. The current in the circuit, t seconds after turning on the power, is I amps and the charge on the capacitor is q coulombs. The circuit can be used by scientists to investigate resonance, to model heavily damped motion and to tune into radio stations on a stereo tuner. It is given that R, C and L are constants, and that $I = 0$ when $t = 0$.

A differential equation for the circuit is $L\frac{\mathrm{d}I}{\mathrm{d}t} + RI + \frac{q}{C} = V$, where $I = \frac{\mathrm{d}q}{\mathrm{d}t}$.

(i) Show that, under certain conditions on V which should be stated,

$$L\frac{\mathrm{d}^2I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} = 0. \qquad [2]$$

It is now given that the differential equation in part **(i)** holds for the rest of the question.

(i) Given that $I = Ate^{-\frac{Rt}{2L}}$ is a solution of the differential equation, where A is a positive constant, show that $C = \frac{4L}{R^2}$. [5]

(ii) In a particular circuit, $R = 4$, $L = 3$ and $C = 0.75$. Find the maximum value of I in terms of A, showing that this value is a maximum. [4]

(iii) Sketch the graph of I against t. [2]

Exercise 675. (9758 N2018/II/1.) (Answer on p. 2087.)

The curve $y = f(x)$ passes through the point $(0, 69)$ and has gradient given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{1}{3}y - 15\right)^{\frac{1}{3}}.$$

(i) Find $f(x)$. [4]

(ii) Find the coordinates of the point on the curve where the gradient is 4. [2]

Exercise 676. (9758 N2018/II/4.) (Answer on p. 2087.)

In this question you may use expansions from the List of Formulae (MF26).

(i) Find the Maclaurin expansion of $\ln(\cos 2x)$ in ascending powers of x, up to and including the term in x^6. State any value(s) of x in the domain $0 \le x \le \frac{1}{4}\pi$ for which the expansion is **not** valid. [6]

(ii) Use your expansion from part **(i)** and integration to find an approximate expression for $\int \frac{\ln(\cos 2x)}{x^2}\,\mathrm{d}x$. Hence find an approximate value for $\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,\mathrm{d}x$, giving your answer to 4 decimal places. [3]

(iii) Use your graphing calculator to find a second approximate value for $\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,\mathrm{d}x$, giving your answer to 4 decimal places. [1]

Exercise 677. (9758 N2017/I/1.) (Answer on p. 2088.)

Using standard series from the List of Formulae (MF26), expand $e^{2x} \ln(1 + ax)$ as far as the term in x^3, where a is a non-zero constant. Hence find the value of a for which there is no term in x^2. [4]

Exercise 678. (9758 N2017/I/3.) (Answer on p. 2088.)

Do not use a calculator in answering this question.

A curve C has equation $y^2 - 2xy + 5x^2 - 10 = 0$.

(i) Find the exact x-coordinates of the stationary points of C. [4]

(ii) For the stationary point with $x > 0$, determine whether it is a maximum or minimum. [3]

Exercise 679. (9758 N2017/I/7.) (Answer on p. 2089.)

It is given that $f(x) = \sin 2mx + \sin 2nx$, where m and n are positive integers and $m \neq n$.

(i) Find $\int \sin 2mx \sin 2nx \, dx$. [3]

(ii) Find $\int_0^\pi (f(x))^2 \, dx$. [5]

Exercise 680. (9758 N2017/I/11.) (Answer on p. 2089.)

Sir Isaac Newton was a famous scientist renowned for his work on the laws of motion. One law states that, for an object falling vertically in a vacuum, the rate of change of velocity, $v\,\mathrm{m\,s^{-1}}$, with respect to time, t seconds, is a constant, c.

(i) (a) Write down a differential equation relating v, t and c. [1]

(b) Initially the velocity of the object is $4\,\mathrm{m\,s^{-1}}$ and, after a further $2.5\,\mathrm{s}$, the velocity of the object is $29\,\mathrm{m\,s^{-1}}$. Find v in terms of t and state the value of c. [3]

For an object falling vertically through the atmosphere, the rate of change of velocity is less than that for an object falling in a vacuum. The new rate of change of v is modelled as the difference between the value of c found in part **(i)(b)** and an amount proportional to the velocity v, with a constant of proportionality k.

(ii) Given that in this case the initial velocity is zero, find v in terms of t and k. [5]

For an object falling through the atmosphere, the 'terminal velocity' is the value approached by the velocity after a long time.

(iii) A falling object has initial velocity zero and terminal velocity $40\,\mathrm{m\,s^{-1}}$. Find how long it takes the object to reach 90% of its terminal velocity. [4]

Exercise 681. (9758 N2017/II/4.) (Answer on p. **2090**.)

(a) A flat novelty plate for serving food on is made in the shape of the region enclosed by the curve $y = x^2 - 6x + 5$ and the line $2y = x - 1$. Find the area of the plate. [4]

(b) A curved container has a flat circular top. The shape of the container is formed by rotating the part of the curve $x = \dfrac{\sqrt{y}}{a - y^2}$, where a is a constant greater than 1, between the points $(0, 0)$ and $\left(\dfrac{1}{a-1}, 1\right)$ through 2π radians about the y-axis.

(i) Find the volume of the container, giving your answer as a single fraction in terms of a and π. [4]

(ii) Another curved container with a flat circular top is formed in the same way from the curve $x = \dfrac{\sqrt{y}}{b - y^2}$ and the points $(0, 0)$ and $\left(\dfrac{1}{b-1}, 1\right)$. It has a volume that is four times as great as the container in part **(i)**. Find an expression for b in terms of a. [3]

Remark 193. In **(b)(ii)**, assume also that $b > 1$.

Exercise 682. (9740 N2016/I/2.) (Answer on p. **2091**.)

(i) Use your calculator to find the gradient of the curve $y = 2^{\cos x}$ at the points where $x = 0$ and $x = \dfrac{1}{2}\pi$. [2]

(ii) Find the equations of the tangents to this curve at the points where $x = 0$ and $x = \dfrac{1}{2}\pi$ and find the coordinates of the point where these tangents meet. [3]

Exercise 683. (9740 N2016/I/8.) (Answer on p. **2091**.)

It is given that $y = f(x)$, where $f(x) = \tan(ax + b)$ for constants a and b.

(i) Show that $f'(x) = a + ay^2$. Use this result to find $f''(x)$ and $f'''(x)$ in terms of a and y. [5]

(ii) In the case where $b = \dfrac{1}{4}\pi$, use your results from part **(i)** to find the Maclaurin series for $f(x)$ in terms of a, up to and including the term in x^3. [3]

(iii) Find the first two non-zero terms in the Maclaurin series for $\tan 2x$. [3]

Exercise 684. (9740 N2016/I/9.) (Answer on p. **2092**.)

A stone is held on the surface of a pond and released. The stone falls vertically through the water and the distance, x metres, that the stone has fallen in time t seconds is measured.

It is given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

(i) The motion of the stone is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 10.$$

(a) By substituting $y = \frac{dx}{dt}$, show that the differential equation can be written as

$$\frac{dy}{dt} = 10 - 2y. \qquad [1]$$

(b) Find y in terms of t and hence find x in terms of t. [6]

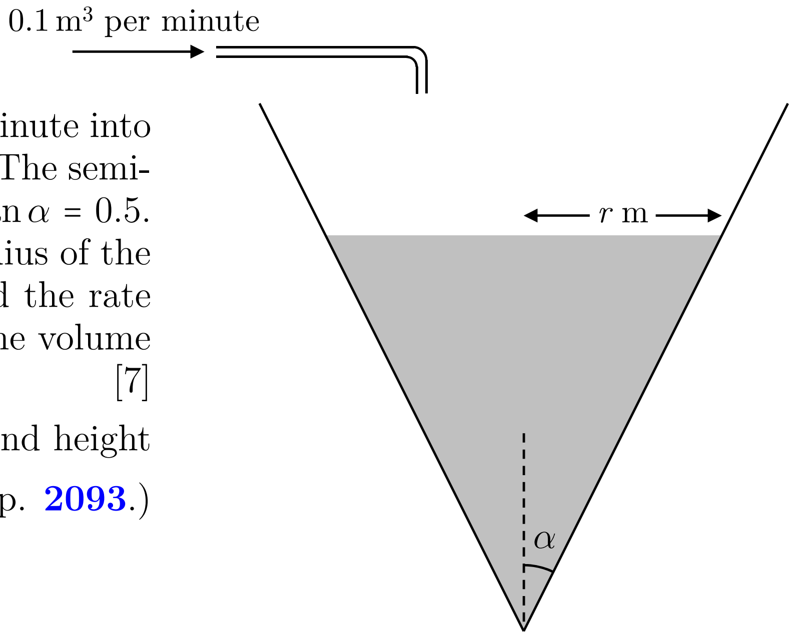
(ii) A second model for the motion of the stone is suggested, given by the differential equation

$$\frac{d^2x}{dt^2} = 10 - 5\sin\frac{1}{2}t.$$

Find x in terms of t for this model. [3]

(iii) The pond is 5 metres deep. For each of these models, find the time the stone takes to reach the bottom of the pond, giving your answers correct to 2 decimal places. [2]

Exercise 685. (9740 N2016/II/1.)



Water is poured at a rate of $0.1\,\mathrm{m}^3$ per minute into a container in the form of an open cone. The semi-vertical angle of the cone is α, where $\tan\alpha = 0.5$. At time t minutes after the start, the radius of the water surface is r m (see diagram). Find the rate of increase of the depth of water when the volume of water in the container is $3\,\mathrm{m}^3$. [7]

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.] (Answer on p. **2093**.)

Exercise 686. (9740 N2016/II/2.) (Answer on p. **2093**.)

(a) **(i)** Find $\int x^2 \cos nx \, dx$, where n is a positive integer. [3]

(ii) Hence find $\int_{\pi}^{2\pi} x^2 \cos nx \, dx$, giving your answers in the form $a\frac{\pi}{n^2}$, where the possible values of a are to be determined. [2]

(b) The region bounded by the curve $y = \frac{x\sqrt{x}}{9 - x^2}$, the x-axis and the lines $x = 0$ and $x = 2$ is rotated through 2π radians about the x-axis. Use the substitution $u = 9 - x^2$ to find the exact volume of the solid obtained, simplifying your answer. [5]

Exercise 687. (9740 N2016/II/3.) (Answer on p. **2094**.)

A curve D has parametric equations

$$x = t - \cos t, \qquad y = 1 - \cos t, \qquad \text{for } 0 \le t \le 2\pi.$$

(i) Sketch the graph of D. Give in exact form the coordinates of the points where D meets the x-axis, and also give in exact form the coordinates of the maximum point on the curve. [4]

(ii) Find, in terms of a, the area under D for $0 \le t \le a$, where a is a positive constant less than 2π. [3]

The normal to D at the point where $t = \frac{1}{2}\pi$ cuts the x-axis at E and the y-axis at F.

(iii) Find the exact area of triangle OEF, where O is the origin. [4]

Exercise 688. (9740 N2015/I/3.) (Answer on p. **2095**.)

(i) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$$

is $\int_0^1 f(x)\, dx$. [2]

(ii) Hence evaluate $\lim_{n\to\infty} \frac{1}{n}\left(\frac{\sqrt[3]{1} + \sqrt[3]{2} + \cdots + \sqrt[3]{n}}{\sqrt[3]{n}}\right)$. [3]

Exercise 689. (9740 N2015/I/4.) (Answer on p. 2096.)

A piece of wire of fixed length d m is cut into two parts. One part is bent into the shape of a rectangle with sides of length x m and y m. The other part is bent into the shape of a semicircle, including its diameter. The radius of the semicircle is x m. Show that the maximum value of the total area of the two shapes can be expressed as kd^2 m^2, where k is a constant to be found. [6]

Exercise 690. (9740 N2015/I/6.) (Answer on p. 2096.)

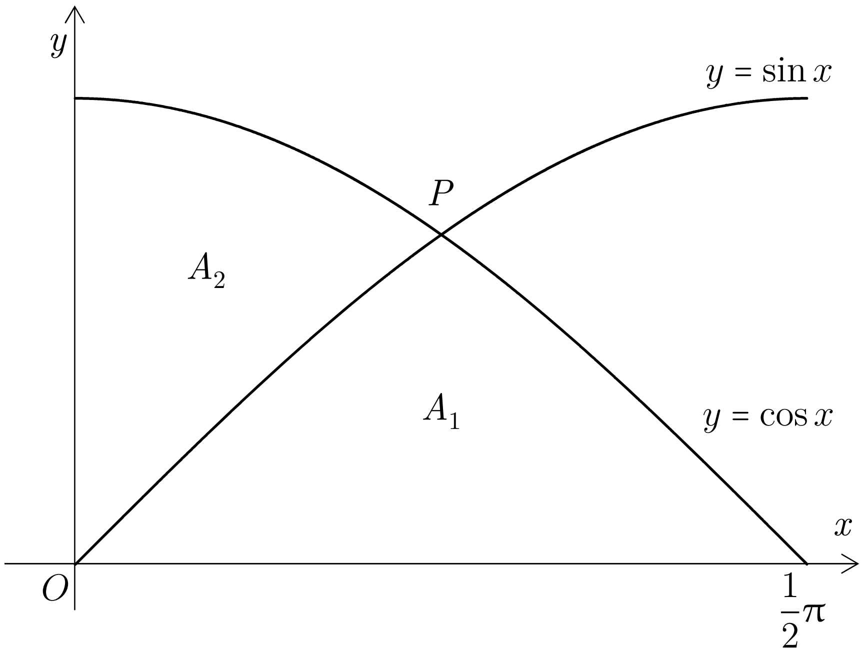
(i) Write down the first three non-zero terms in the Maclaurin series for $\ln(1+2x)$, where $-\frac{1}{2} < x \leq \frac{1}{2}$, simplifying the coefficients. [2]

(ii) It is given that the three terms found in part **(i)** are equal to the first three terms in the series expansion of $ax(1+bx)^c$ for small x. Find the exact values of the constants a, b and c and use these values to find the coefficient of x^4 in the expansion of $ax(1+bx)^c$, giving your answer as a simplified rational number. [6]

Exercise 691. (9740 N2015/I/10.) (Answer on p. 2097.)

With origin O, the curves with equations $y = \sin x$ and $y = \cos x$, where $0 \leq x \leq \frac{1}{2}\pi$, meet at the point P with coordinates $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$.

The area of the region bounded by the curves and the x-axis is A_1 and the area of the region bounded by the curves and the y-axis is A_2 (see diagram).



(i) Show that $\dfrac{A_1}{A_2} = \sqrt{2}$. [4]

(ii) The region bounded by $y = \sin x$ between O and P, the line $y = \frac{1}{2}\sqrt{2}$ and the y-axis is rotated about the y-axis through 360°. Show that the volume of the solid formed is given by

$$\pi \int_0^{\frac{1}{2}\sqrt{2}} \left(\sin^{-1} y\right)^2 \, dy. \qquad [2]$$

(iii) Show that the substitution $y = \sin u$ transforms the integral in **(ii)** to $\pi \int_a^b u^2 \cos u \, du$, for limits a and b to be determined. Hence find the exact volume. [6]

Exercise 692. (9740 N2015/I/11.) (Answer on p. 2098.)

A curve C has parametric equations

$$x = \sin^3\theta, \qquad y = 3\sin^2\theta\cos\theta, \qquad \text{for } 0 \le \theta \le \frac{1}{2}\pi.$$

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cot\theta - \tan\theta$. [3]

(ii) Show that C has a turning point when $\tan\theta = \sqrt{k}$, where k is an integer to be determined. Find, in non-trigonometric form, the exact coordinates of the turning point and explain why it is a maximum. [6]

(iii) Show that the area of the region bounded by C and the x-axis is given by

$$\int_0^{\frac{1}{2}\pi} 9\sin^4\theta\cos^2\theta\,\mathrm{d}\theta.$$

Use your calculator to find the area, giving your answer correct to 3 decimal places. [3]

The line with equation $y = ax$, where a is a positive constant, meets C at the origin and at the point P.

(iv) Show that $\tan\theta = \dfrac{3}{a}$ at P. Find the exact value of a such that the line passes through the maximum point of C. [3]

Exercise 693. (9740 N2015/II/1.) (Answer on p. 2099.)

As a tree grows, the rate of increase of its height, h m, with respect to time, t years after planting, is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{10}\sqrt{16 - \frac{1}{2}h}.$$

The tree is planted as a seedling of negligible height, so that $h = 0$ when $t = 0$.

(i) State the maximum height of the tree, according to this model. [1]

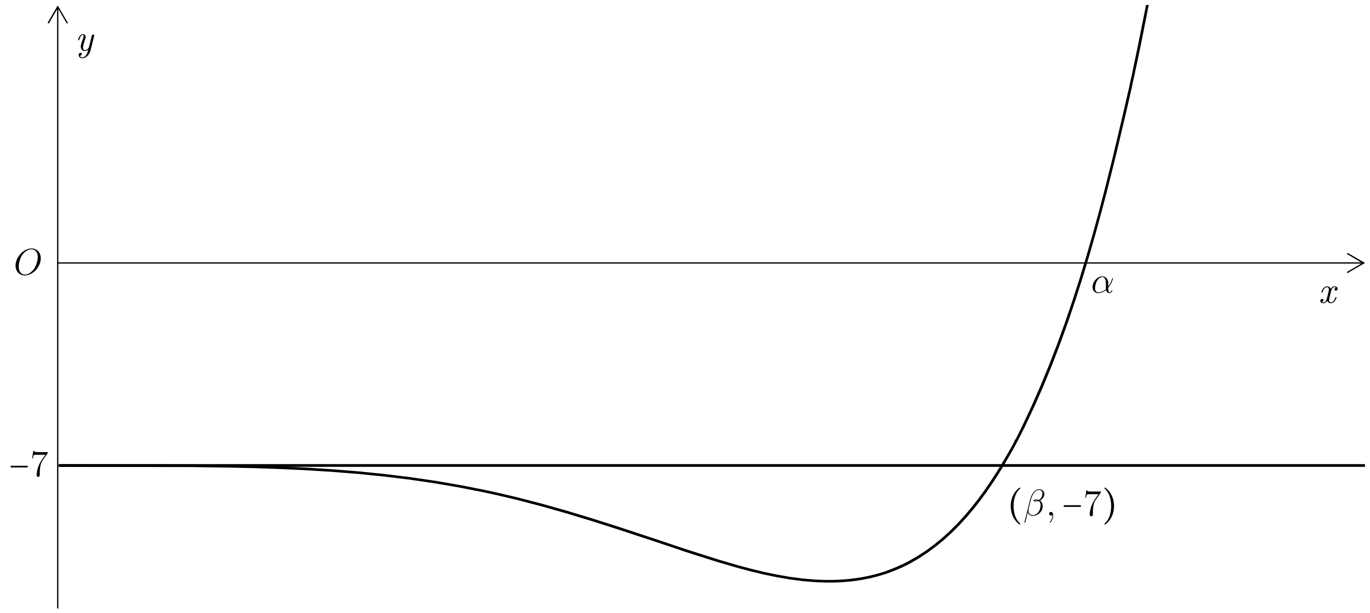
(ii) Find an expression for t in terms of h, and hence find the time the tree takes to reach half its maximum height. [5]

Exercise 694. (9740 N2014/I/2.) (Answer on p. 2099.)

The curve C has equation $x^2y + xy^2 + 54 = 0$. Without using a calculator, find the coordinates of the point on C at which the gradient is -1, showing that there is only one such point. [6]

Exercise 695. (9740 N2014/I/7.) (Answer on p. 2100.)

It is given that $f(x) = x^6 - 3x^4 - 7$. The diagram shows the curve with equation $y = f(x)$ and the line with equation $y = -7$, for $x \geq 0$. The curve crosses the positive x-axis at $x = \alpha$, and the curve and the line meet where $x = 0$ and $x = \beta$.



(i) Find the value of α, giving your answer correct to 3 decimal places, and find the exact value of β. [2]

(ii) Evaluate $\int_\beta^\alpha f(x)\,\mathrm{d}x$, giving your answer correct to 3 decimal places. [2]

(iii) Find, in terms of $\sqrt{3}$, the area of the finite region bounded by the curve and the line, for $x \geq 0$. [3]

(iv) Show that $f(x) = f(-x)$. What can be said about the six roots of the equation $f(x) = 0$? [4]

Exercise 696. (9740 N2014/I/8.) (Answer on p. 2101.)

It is given that $f(x) = \dfrac{1}{\sqrt{9 - x^2}}$, where $-3 < x < 3$.

(i) Write down $\int f(x)\,\mathrm{d}x$. [1]

(ii) Find the binomial expansion for $f(x)$, up to and including the term in x^6. Give the coefficients as exact fractions in their simplest form. [4]

Remark 194. For **(ii)**, replace *binomial expansion* (no longer on the 9758 syllabus) with *Maclaurin expansion*.

(iii) Hence, or otherwise, find the first four non-zero terms of the Maclaurin series for $\sin^{-1}\dfrac{x}{3}$. Give the coefficients as exact fractions in their simplest form. [4]

Exercise 697. (9740 N2014/I/10.) (Answer on p. **2102**.)

The mass, x grams, of a certain substance present in a chemical reaction at time t minutes satisfies the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(1 + x - x^2\right),$$

where $0 \le x \le \frac{1}{2}$ and k is a constant. It is given that $x = \frac{1}{2}$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{4}$ when $t = 0$.

(i) Show that $k = -\frac{1}{5}$. [1]

(ii) By first expressing $1 + x - x^2$ in completed square form, find t in terms of x. [5]

(iii) Hence find

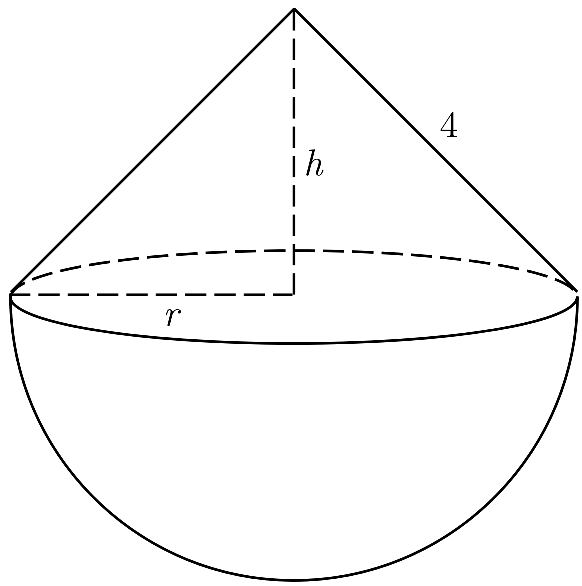
- **(a)** the exact time taken for the mass of the substance present in the chemical reaction to become half of its initial value, [1]
- **(b)** the time taken for there to be none of the substance present in the chemical reaction, giving your answer correct to 3 decimal places. [1]

(iv) Express the solution of the differential equation in the form $x = f(t)$ and sketch the part of the curve with this equation which is relevant in this context. [5]

Exercise 698. (9740 N2014/I/11.) (Answer on p. **2103**.)

[It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]

A toy manufacturer makes a toy which consists of a hemisphere of radius r cm joined to a circular cone of base radius r cm and height h cm (see diagram). The manufacturer determines that the length of the slant edge of the cone must be 4 cm and that the total volume of the toy, V cm^3, should be as large as possible.



(i) Find a formula for V in terms of r. Given that $r = r_1$ is the value of r which gives the maximum value of V, show that r_1 satisfies the equation $45r^4 - 768r^2 + 1024 = 0$. [6]

(ii) Find the two solutions to the equation in part **(i)** for which $r > 0$, giving your answers correct to 3 decimal places. [2]

(iii) Show that one of the solutions found in part **(ii)** does not give a stationary value of V. Hence write down the value of r_1 and find the corresponding value of h. [3]

(iv) Sketch the graph showing the volume of the toy as the radius of the hemisphere varies. [3]

Exercise 699. (9740 N2014/II/2.) (Answer on p. **2104**.)

Using partial fractions, find

$$\int_0^2 \frac{9x^2+x-13}{(2x-5)(x^2+9)}\,\mathrm{d}x.$$

Give your answer in the form $a\ln b + c\tan^{-1} d$, where a, b, c and d are rational numbers to be determined. [9]

Exercise 700. (9740 N2013/I/5.) (Answer on p. **2104**.)

It is given that

$$f(x) = \begin{cases} \sqrt{1-\dfrac{x^2}{a^2}} & \text{for } -a \le x \le a, \\ 0 & \text{for } a < x < 2a, \end{cases}$$

and that $f(x+3a) = f(x)$ for all real values of x, where a is a real constant.

(i) Sketch the graph of $y = f(x)$ for $-4a \le x \le 6a$. [3]

(ii) Use the substitution $x = a\sin\theta$ to find the exact value of $\int_{\frac{1}{2}a}^{\frac{\sqrt{3}}{2}a} f(x)\,\mathrm{d}x$ in terms of a and π. [5]

Exercise 701. (9740 N2013/I/10.) (Answer on p. **2105**.)

The variables x, y and z are connected by the following differential equations.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3-2z \qquad \text{(A)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = z \qquad \text{(B)}$$

(i) Given that $z < \frac{3}{2}$, solve equation (A) to find z in terms of x. [4]

(ii) Hence find y in terms of x. [2]

(iii) Use the result in part **(ii)** to show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = a\frac{\mathrm{d}y}{\mathrm{d}x} + b,$$

for constants a and b to be determined. [3]

(iv) The result in part **(ii)** represents a family of curves. Some members of the family are straight lines. Write down the equations of two of these lines. On a single diagram, sketch one of your lines together with a non-linear member of the family of curves that has your line as an asymptote. [4]

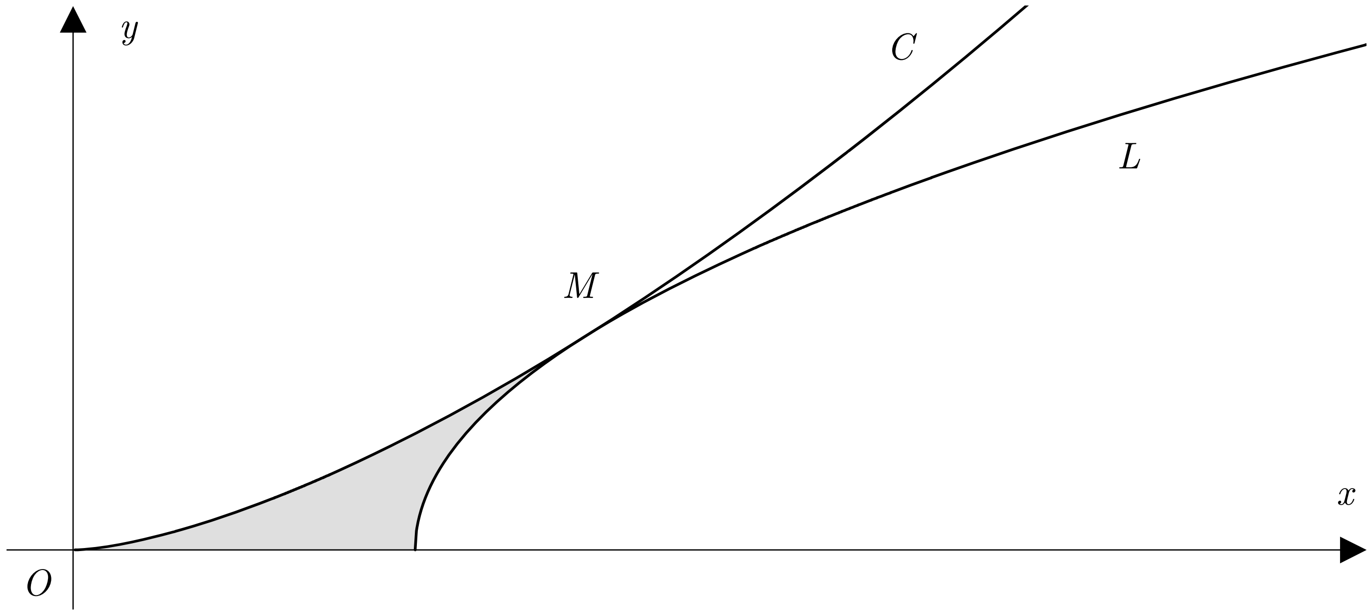
Exercise 702. (9740 N2013/I/11.) (Answer on p. **2106**.)

A curve C has parametric equations

$$x = 3t^2, \qquad y = 2t^3.$$

(i) Find the equation of the tangent to C at the point with parameter t. [3]

(ii) Points P and Q on C have parameters p and q respectively. The tangent at P meets the tangent at Q at the point R. Show that the x-coordinate of R is $p^2 + pq + q^2$, and find the y-coordinate of R in terms of p and q. Given that $pq = -1$, show that R lies on the curve with equation $x = y^2 + 1$. [5]



A curve L has equation $x = y^2 + 1$. The diagram shows the parts of C and L for which $y \geq 0$. The curves C and L touch at the point M.

(iii) Show that $4t^6 - 3t^2 + 1 = 0$ at M. Hence, or otherwise, find the exact coordinates of M. [3]

(iv) Find the exact value of the area of the shaded region bounded by C and L for which $y \geq 0$. [6]

Exercise 703. (9740 N2013/II/2.) (Answer on p. **2107**.)

Fig. 1 shows a piece of card, ABC, in the form of an equilateral triangle of side a. A kite shape is cut from each corner, to give the shape shown in Fig. 2. The remaining card shown in Fig. 2 is folded along the dotted lines, to form the open triangular prism of height x shown in Fig. 3.

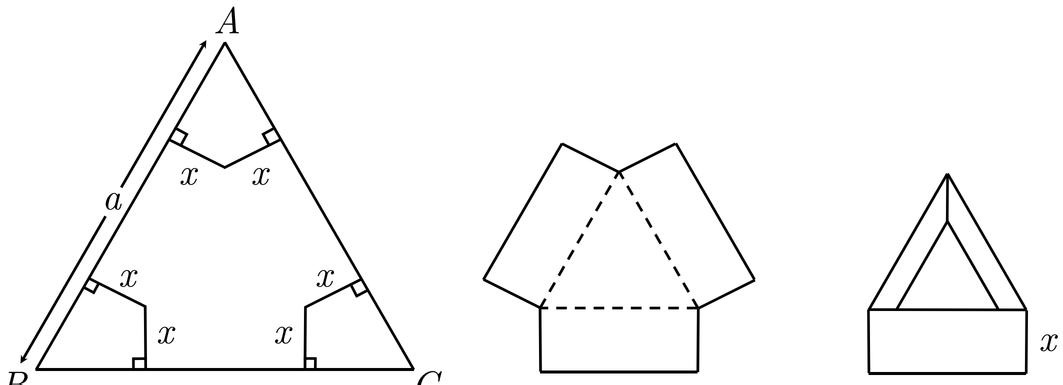


Fig. 1 **Fig. 2** **Fig. 3**

(i) Show that the volume V of the prism is given by $V = \frac{1}{4}x\sqrt{3}\left(a - 2x\sqrt{3}\right)^2$. [3]

(ii) Use differentiation to find, in terms of a, the maximum value of V, proving that it is a maximum. [6]

Remark 195. For **(ii)**, assume also that a is a fixed constant. Otherwise, V has no maximum because we can simply let both a and x grow without bound.

Exercise 704. (9740 N2013/II/3.) (Answer on p. **2108**.)

(i) Given that $f(x) = \ln(1 + 2\sin x)$, find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$. Hence write down the first three non-zero terms in the Maclaurin series for $f(x)$. [7]

(ii) The first two non-zero terms in the Maclaurin series for $f(x)$ are equal to the first two non-zero terms in the series expansion of $e^{ax}\sin nx$. Using appropriate expansions from the List of Formulae (MF15), find the constants a and n. Hence find the third non-zero term of the series expansion of $e^{ax}\sin nx$ for these values of a and n. [5]

Exercise 705. (9740 N2012/I/2.) (Answer on p. **2108**.)

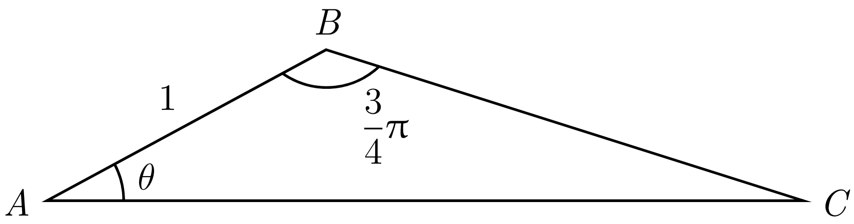
(i) Find $\displaystyle\int \frac{x^3}{1+x^4}\,dx$. [2]

(ii) Use the substitution $u = x^2$ to find $\displaystyle\int \frac{x}{1+x^4}\,dx$. [3]

(iii) Evaluate $\displaystyle\int_0^1 \left(\frac{x}{1+x^4}\right)^2 dx$, giving the answer correct to 3 decimal places. [1]

Exercise 706. (9740 N2012/I/4.) (Answer on p. **2109**.)

In the triangle ABC, $AB = 1$, angle $BAC = \theta$ radians and angle $ABC = \frac{3}{4}\pi$ radians (see diagram).



(i) Show that $AC = \dfrac{1}{\cos\theta - \sin\theta}$. [4]

(ii) Given that θ is a sufficiently small angle, show that

$$AC \approx 1 + a\theta + b\theta^2,$$

for constants a and b to be determined. [4]

Exercise 707. (9740 N2012/I/8.) (Answer on p. **2109**.)

The curve C has equation

$$x - y = (x + y)^2.$$

It is given that C has only one turning point.

(i) Show that $1 + \dfrac{dy}{dx} = \dfrac{2}{2x + 2y + 1}$. [4]

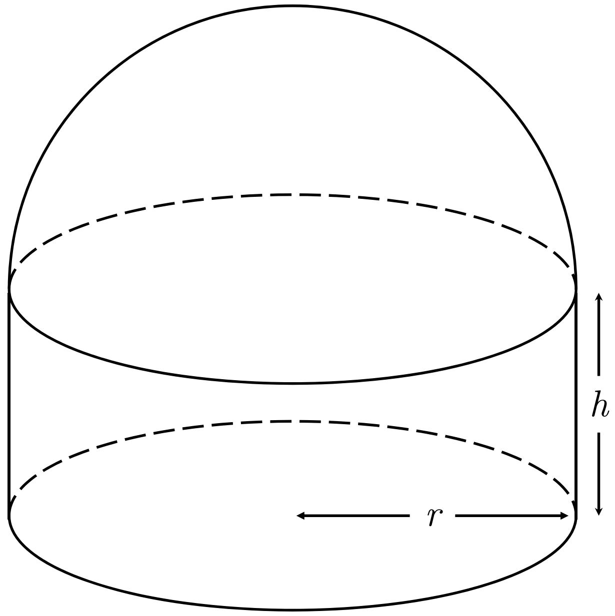
(ii) Hence, or otherwise, show that $\dfrac{d^2y}{dx^2} = -\left(1 + \dfrac{dy}{dx}\right)^3$. [3]

(iii) Hence, state, with a reason, whether the turning point is a maximum or a minimum. [2]

Exercise 708. (9740 N2012/I/10.) (Answer on p. 2110.)

[It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

A model of a concert hall is made up of three parts. The roof is modelled by the curved surface of a hemisphere of radius r cm. The walls are modelled by the curved surface of a cylinder of radius r cm and height h cm. The floor is modelled by a circular disc of radius r cm. The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.



(i) It is given that the volume of the model is a fixed value $k\,\text{cm}^3$, and the external surface area is a minimum. Use differentiation to find the values of r and h in terms of k. Simplify your answers. [7]

(ii) It is given instead that the volume of the model is $200\,\text{cm}^3$ and its external surface area is $180\,\text{cm}^2$. Show that there are two possible values of r. Given also that $r < h$, find the value of r and the value of h. [5]

Exercise 709. (9740 N2012/I/11.) (Answer on p. 2111.)

A curve C has parametric equations

$$x = \theta - \sin\theta, \qquad y = 1 - \cos\theta,$$

where $0 \le \theta \le 2\pi$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \cot\frac{1}{2}\theta$ and find the gradient of C at the point where $\theta = \pi$. What can be said about the tangents to C as $\theta \to 0$ and $\theta \to 2\pi$? [5]

(ii) Sketch C, showing clearly the features of the curve at the points where $\theta = 0$, π and 2π. [3]

(iii) Without using a calculator, find the exact area of the region bounded by C and the x-axis. [5]

(iv) A point P on C has parameter p, where $0 < p < \frac{1}{2}\pi$. Show that the normal to C at P crosses the x-axis at the point with coordinates $(p, 0)$. [3]

Exercise 710. (9740 N2012/II/1.) (Answer on p. **2112**.)

(a) Find the general solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 16 - 9x^2$, giving your answer in the form $y = f(x)$. [3]

(b) Given that u and t are related by $\frac{\mathrm{d}u}{\mathrm{d}t} = 16 - 9u^2$, and that $u = 1$ when $t = 0$, find t in terms of u, simplifying your answer. [5]

Exercise 711. (9740 N2011/I/3.) (Answer on p. **2112**.)

The parametric equations of a curve are $x = t^2$, $y = \frac{2}{t}$.

(i) Find the equation of the tangent to the curve at the point $\left(p^2, \frac{2}{p}\right)$, simplifying your answer. [2]

(ii) Hence find the coordinates of the points Q and R where this tangent meets the x- and y-axes respectively. [2]

(iii) Find a cartesian equation of ~~the locus of~~ the mid-point of QR as p varies.[562] [3]

Exercise 712. (9740 N2011/I/4.) (Answer on p. **2112**.)

(i) Use the first three non-zero terms of the Maclaurin series for $\cos x$ to find the Maclaurin series for $g(x)$, where $g(x) = \cos^6 x$, up to and including the term in x^4. [3]

(ii) **(a)** Use your answer to part **(i)** to give an approximation for $\int_0^a g(x)\,\mathrm{d}x$ in terms of a, and evaluate this approximation in the case where $a = \frac{\pi}{4}$. [3]

(b) Use your calculator to find an accurate value for $\int_0^{\frac{1}{4}\pi} g(x)\,\mathrm{d}x$. Why is the approximation in part **(ii)** **(a)** not very good?

Exercise 713. (9740 N2011/I/5.) (Answer on p. **2113**.)

It is given that $f(x) = 2 - x$.

(i) On separate diagrams, sketch the graphs of $y = f(|x|)$ and $y = |f(x)|$, giving the coordinates of any points where the graphs meet the x- and y-axes. You should label the graphs clearly. [3]

(ii) State the set of values of x for which $f(|x|) = |f(x)|$. [1]

(iii) Find the exact value of the constant a for which $\int_{-1}^{1} f(|x|)\,\mathrm{d}x = \int_{1}^{a} |f(x)|\,\mathrm{d}x$. [3]

[562] The word **locus** has been removed from your H2 Maths syllabus. But here this word doesn't really mean much. We'll just delete these three words without changing the question's meaning.

Exercise 714. (9740 N2011/I/8.) (Answer on p. **2114**.)

(i) Find $\int \left(100 - v^2\right)^{-1} \mathrm{d}v$.

(ii) A stone is dropped from a stationary balloon. It leaves the balloon with zero speed, and t seconds later its speed v metres per second satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.1v^2.$$

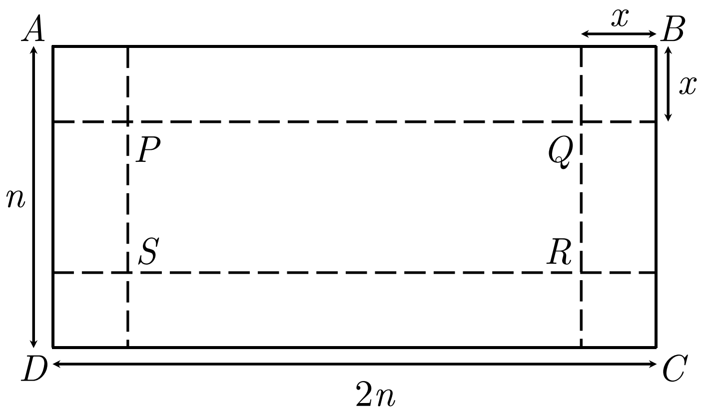
(a) Find t in terms of v. Hence find the exact time the stone takes to reach a speed of 5 metres per second. [5]

(b) Find the speed of the stone after 1 second. [3]

(c) What happens to the speed of the stone for large values of t? [2]

Exercise 715. (9740 N2011/II/2.) (Answer on p. **2114**.)

The diagram shows a rectangular piece of cardboard $ABCD$ of sides n metres and $2n$ metres, where n is a positive constant. A square of side x metres is removed from each corner of $ABCD$. The remaining shape is now folded along PQ, QR, RS and SP to form an open rectangular box of height x metres.



(i) Show that the volume V cubic metres of the box is given by $V = 2n^2x - 6nx^2 + 4x^3$. [3]

(ii) Without using a calculator, find in surd form the value of x that gives a stationary value of V, and explain why there is only one answer. [6]

Exercise 716. (9740 N2011/II/4.) (Answer on p. **2115**.)

(a) **(i)** Obtain a formula for $\int_0^n x^2 \mathrm{e}^{-2x} \, \mathrm{d}x$ in terms of n, where $n > 0$. [5]

(ii) Hence evaluate $\int_0^\infty x^2 \mathrm{e}^{-2x} \, \mathrm{d}x$. [1]

[You may assume that $n\mathrm{e}^{-2n}$ and $n^2\mathrm{e}^{-2n} \to 0$ as $n \to \infty$.]

(a) The region bounded by the curve $y = \dfrac{4x}{x^2+1}$, the x-axis and the lines $x = 0$ and $x = 1$ is rotated through 2π radians about the x-axis. Use the substitution $x = \tan\theta$ to show that the volume of the solid obtained is given by $16\pi \int_0^{\pi/4} \sin^2\theta \, \mathrm{d}\theta$, and evaluate this integral exactly. [6]

Exercise 717. (9740 N2010/I/2.) (Answer on p. **2115**.)

(i) Find the first three terms of the Maclaurin series for $e^x(1+\sin 2x)$. [You may use standard results given in the List of Formulae (MF15).] [3]

(ii) It is given that the first two terms of this series are equal to the first two terms in the series expansion, in ascending powers of x, of $\left(1+\frac{4}{3}x\right)^n$. Find n and show that the third terms in each of these series are equal. [3]

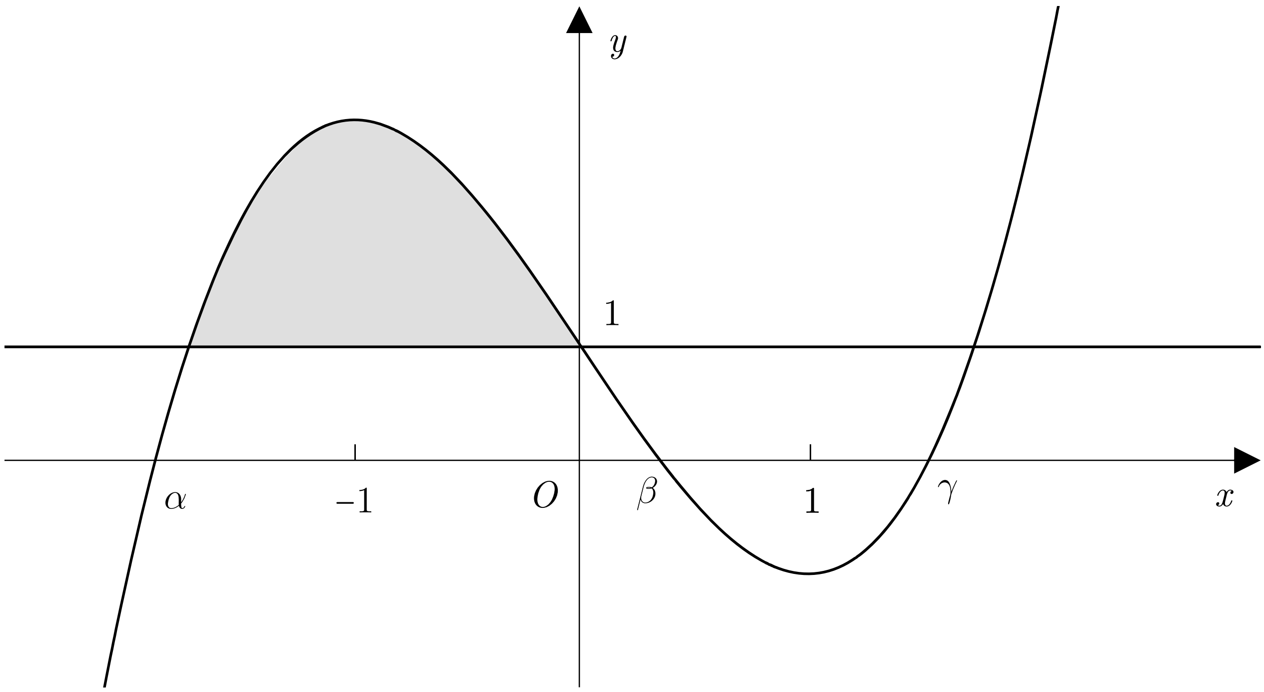
Exercise 718. (9740 N2010/I/4.) (Answer on p. **2115**.)

(i) Given that $x^2-y^2+2xy+4=0$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

(ii) For the curve with equation $x^2-y^2+2xy+4=0$, find the coordinates of each point at which the tangent is parallel to the x-axis. [4]

Exercise 719. (9740 N2010/I/6.) (Answer on p. **2116**.)

The diagram shows the curve with equation $y=x^3-3x+1$ and the line with equation $y=1$. The curve crosses the x-axis at $x=\alpha$, $x=\beta$ and $x=\gamma$ and has turning points $x=-1$ and $x=1$.



(i) Find the values of β and γ, giving your answers correct to 3 decimal places. [2]

(ii) Find the area of the region bounded by the curve and the x-axis between $x=\beta$ and $x=\gamma$. [2]

(iii) Use a non-calculator method to find the shaded area. [4]

(iv) Find the set of values of k for which the equation $x^3-3x+1=k$ has three real distinct roots. [2]

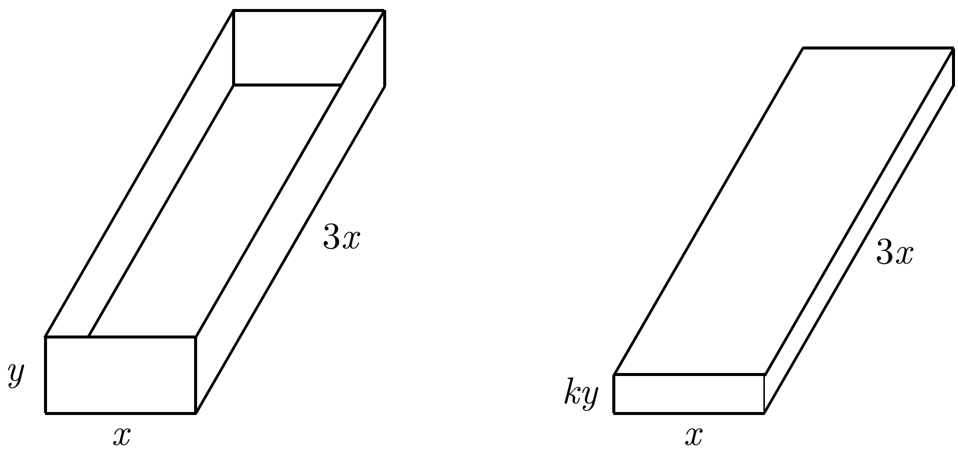
Exercise 720. (9740 N2010/I/7.) (Answer on p. **2116**.)

(i) A bottle containing liquid is taken from a refrigerator and placed in a room where the temperature is a constant 20 °C. As the liquid warms up, the rate of increase of its temperature θ °C after time t minutes is proportional to the temperature difference $(20 - \theta)$ °C. Initially the temperature of the liquid is 10 °C and the rate of increase of the temperature is 1 °C per minute. By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-\frac{1}{10}t}$. [7]

(ii) Find the time it takes the liquid to reach a temperature of 15 °C, and state what happens to θ for large values of t. Sketch a graph of θ against t. [4]

Exercise 721. (9740 N2010/I/9.) (Answer on p. **2117**.)

A company requires a box made of cardboard of negligible thickness to hold 300 cm^3 of powder when full. The length of the box is $3x$ cm, the width is x cm and the height is y cm. The lid has depth ky cm, where $0 < k \leq 1$ (see diagram).



Box   Lid

(i) Use differentiation to find, in terms of k, the value of x which gives a minimum total external surface area of the box and the lid. [6]

Remark 196. I interpret "total external surface area" as referring to that when the box and lid are kept separate, as depicted. Also, I interpret the box's *external* surface area consists of five rectangles and likewise for the lid. Further, I assume k is constant.

(ii) Find also the ratio of the height to the width, $\frac{y}{x}$, in this case, simplifying your answer. [2]

(iii) Find the values between which $\frac{y}{x}$ must lie. [2]

Remark 197. I interpret **(iii)** to mean, "Find the values between which the ratio you found in **(ii)** must lie."

(iv) Find the value of k for which the box has square ends. [2]

Remark 198. I interpret **(iv)** to mean, "Given the ratio you found in **(ii)**, find the value of k for which $x = y$."

Exercise 722. (9740 N2010/I/11.) (Answer on p. **2118**.)

A curve C has parametric equations

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}.$$

(i) The point P on the curve has parameter p. Show that the equation of the tangent at P is
$$\left(p^2 + 1\right) x - \left(p^2 - 1\right) y = 4p.$$ [4]

(ii) The tangent at P meets the line $y = x$ at the point A and the line $y = -x$ at the point B. Show that the area of triangle OAB is independent of p, where O is the origin. [4]

(iii) Find a cartesian equation of C. Sketch C, giving the coordinates of any points where C crosses the x- and y-axes and the equations of any asymptotes. [4]

Exercise 723. (9740 N2010/II/3.) (Answer on p. **2119**.)

(i) Given that $y = x\sqrt{x+2}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$, expressing your answer as a single algebraic fraction.

Hence, show that there is only one value of x for which the curve $y = x\sqrt{x+2}$ has a turning point, and state this value. [5]

(ii) A curve has equation $y^2 = x^2 (x + 2)$.

(a) Find exactly the possible values of the gradient at the point where $x = 0$. [2]

(b) Sketch the curve $y^2 = x^2 (x + 2)$. [2]

(iii) On a separate diagram sketch the graph of $y = f'(x)$, where $f(x) = x\sqrt{x+2}$. State the equations of any asymptotes. [2]

Exercise 724. (9740 N2009/I/2.) (Answer on p. **2119**.)

Find the exact value of p such that

$$\int_0^1 \frac{1}{4 - x^2}\,\mathrm{d}x = \int_0^{\frac{1}{2p}} \frac{1}{\sqrt{1 - p^2 x^2}}\,\mathrm{d}x.$$ [5]

Exercise 725. (9740 N2009/I/4.) (Answer on p. **2120**.)

It is given that

$$f(x) = \begin{cases} 7 - x^2 & \text{for } 0 < x \le 2, \\ 2x - 1 & \text{for } 2 < x \le 4, \end{cases}$$

and that $f(x) = f(x + 4)$ for all real values of x.

(i) Evaluate $f(27) + f(45)$. [2]

(ii) Sketch the graph of $y = f(x)$ for $-7 \le x \le 10$. [3]

(iii) Find $\int_{-4}^{3} f(x)\,\mathrm{d}x$. [3]

Exercise 726. (9740 N2009/I/7.) (Answer on p. **2120**.)

(i) Given that $f(x) = e^{\cos x}$, find $f(0)$, $f'(0)$ and $f''(0)$. Hence write down the first two non-zero terms in the Maclaurin series for $f(x)$. Give the coefficients in terms of e. [5]

(ii) Given that the first two non-zero terms in the Maclaurin series for $f(x)$ are equal to the first two non-zero terms in the series expansion of $\frac{1}{a+bx^2}$, where a and b are constants, find a and b in terms of e. [4]

Exercise 727. (9740 N2009/I/11.) (Answer on p. **2121**.)

The curve C has equation $y = f(x)$, where $f(x) = xe^{-x^2}$.

(i) Sketch the curve C. [2]

(ii) Find the exact coordinates of the turning points on the curve. [4]

(iii) Use the substitution $u = x^2$ to find $\int_0^n f(x)\,dx$, for $n > 0$. Hence find the area of the region between the curve and the positive x-axis. [4]

(iv) Find the exact value of $\int_{-2}^{2} |f(x)|\,dx$. [2]

(v) Find the volume of revolution when the region bounded by the curve, the lines $x = 0$, $x = 1$ and the x-axis is rotated completely about the x-axis. Give your answer correct to 3 significant figures. [2]

Exercise 728. (9740 N2009/II/1.) (Answer on p. **2122**.)

The curve C has parametric equations

$$x = t^2 + 4t, \qquad y = t^3 + t^2.$$

(i) Sketch the curve for $-2 \le t \le 1$. [1]

The tangent to the curve at the point P where $t = 2$ is denoted by l.

(ii) Find the cartesian equation of l. [3]

(iii) The tangent l meets C again at the point Q. Use a non-calculator method to find the coordinates of Q. [4]

Exercise 729. (9740 N2009/II/4.) (Answer on p. 2123.)

Two scientists are investigating the change of a certain population of size n thousand at time t years.

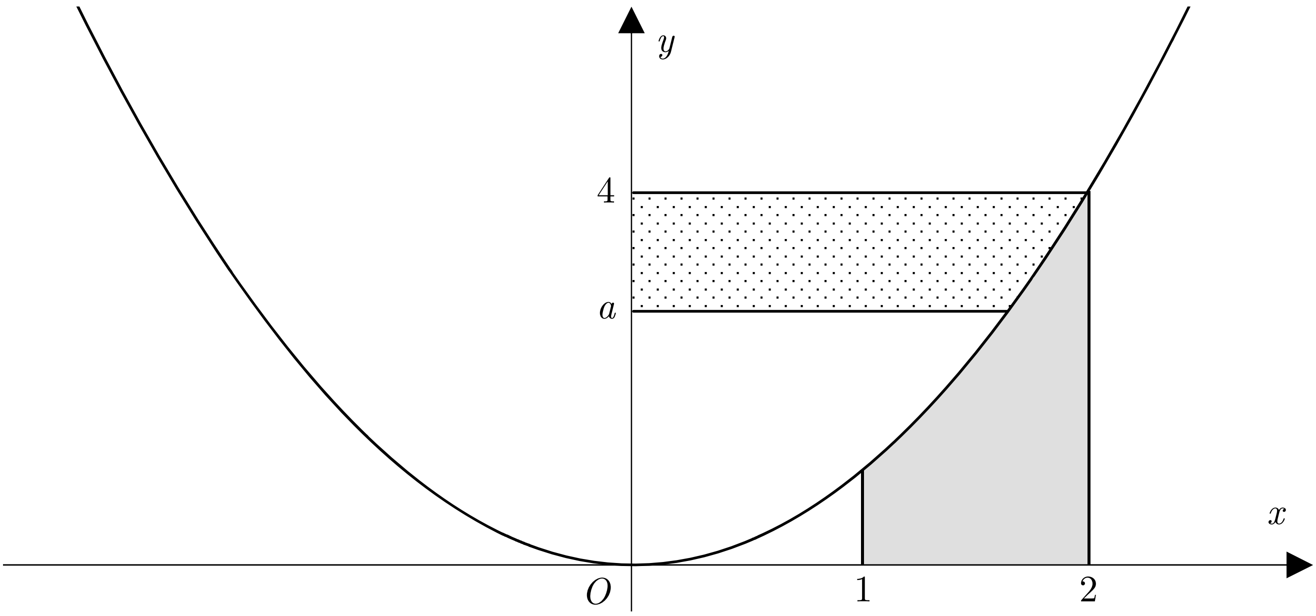
(i) One scientist suggests that n and t are related by the differential equation $\frac{\mathrm{d}^2 n}{\mathrm{d}t^2} = 10 - 6t$.

Find the general solution of this differential equation. Sketch three members of the family of solution curves, given that $n = 100$ when $t = 0$. [5]

(ii) The other scientist suggests that n and t are related by the differential equation $\frac{\mathrm{d}n}{\mathrm{d}t} = 3 - 0.02n$. Find n in terms of t, given again that $n = 100$ when $t = 0$. Explain in simple terms what will eventually happen to the population using this model. [7]

Exercise 730. (9740 N2008/I/1.) (Answer on p. 2123.)

The diagram shows the curve with equation $y = x^2$. The area of the region bounded by the curve, the lines $x = 1$, $x = 2$ and the x-axis is equal to the area of the region bounded by the curve, the lines $y = a$, $y = 4$ and the y-axis , where $a < 4$. Find the value of a. [4]



Exercise 731. (9740 N2008/I/4.) (Answer on p. 2124.)

(i) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x}{x^2 + 1}.$$ [2].

(ii) Find the particular solution of the differential equation for which $y = 2$ when $x = 0$.[1]

(iii) What can you say about the gradient of every solution curve as $x \to \pm\infty$? [1]

(iv) Sketch, on a single diagram, the graph of the solution found in part **(ii)**, together with 2 other members of the family of solution curves. [3]

Exercise 732. (9740 N2008/I/5.) (Answer on p. **2124**.)

(i) Find the exact value of $\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9x^2}\,\mathrm{d}x$. [3]

(ii) Find, in terms of n and e, $\int_1^{\mathrm{e}} x^n \ln x \,\mathrm{d}x$, where $n \neq -1$. [4]

Exercise 733. (9740 N2008/I/6.) (Answer on p. **2125**.)

(a) In the triangle ABC, $AB = 1$, $BC = 3$ and angle $ABC = \theta$ radian. Given that θ is a sufficiently small angle, show that

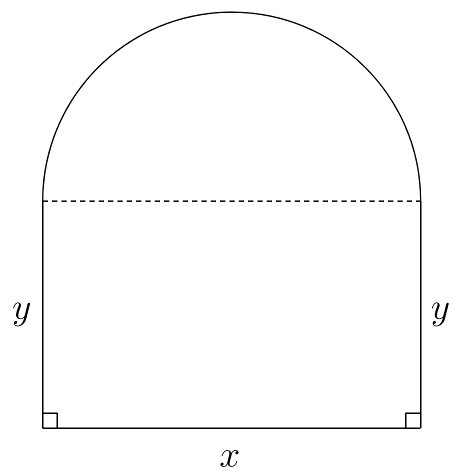
$$AC \approx \sqrt{4 + 3\theta^2} \approx a + b\theta^2,$$

for constants a and b to be determined. [5]

(b) Given that $f(x) = \tan\left(2x + \frac{\pi}{4}\right)$, find $f(0)$, $f'(0)$ and $f''(0)$. Hence find the first 3 terms in the Maclaurin series of $f(x)$. [5]

Exercise 734. (9740 N2008/I/7.) (Answer on p. **2125**.)

A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle x m by y m together with a semicircle of diameter x m, as shown in the diagram. A low wall will be built around the flower-bed. The time needed to build the wall will be 3 hours per metre for the straight parts and 9 hours per metre for the semicircular part. Given that a total time of 180 hours is taken to build the wall, find, using differentiation, the values of x and y which give a flower-bed of maximum area. [10]



Exercise 735. (9740 N2008/II/1.) Let $f(x) = \mathrm{e}^x \sin x$. (Answer on p. **2126**.)

(i) Sketch the graph for $y = f(x)$ for $-3 \leq x \leq 3$. [2]

(ii) Find the series expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [3]

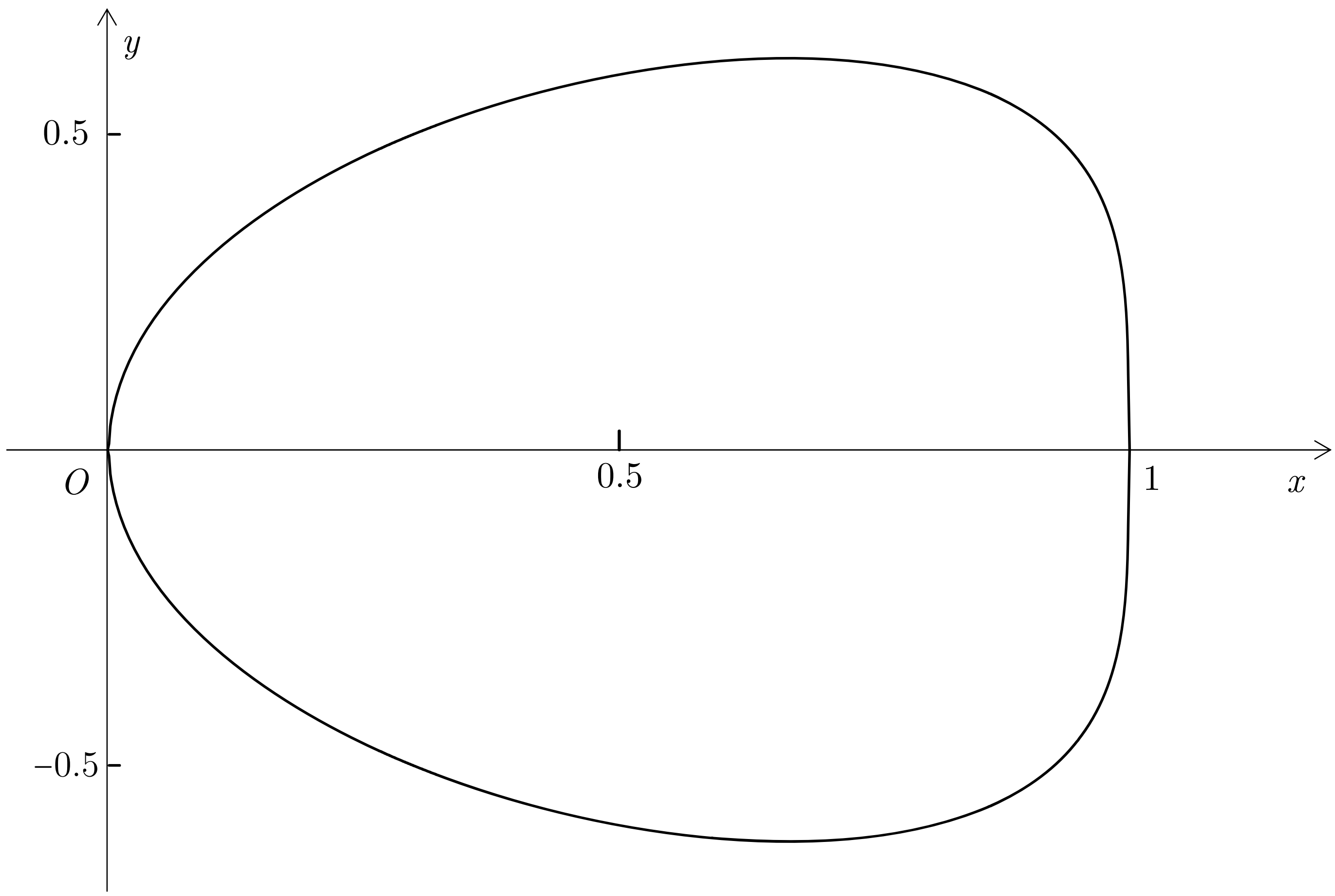
Denote the answer to part **(ii)** by $g(x)$.

(iii) On the same diagram as in part **(i)**, sketch the graph of $y = g(x)$. Label the two graphs clearly. [1]

(iv) Find, for $-3 \leq x \leq 3$, the set of values of x for which the value of $g(x)$ is within ± 0.5 of the value of $f(x)$. [3]

Exercise 736. (9740 N2008/II/2.) (Answer on p. **2126**.)

The diagram shows the curve C with equation $y^2 = x\sqrt{1-x}$. The region enclosed by C is denoted by R.



(i) Write down an integral that gives the area of R, and evaluate this integral numerically. [3]

(ii) The part of R above the x-axis is rotated through 2π radians about the x-axis. By using the substitution $u = 1 - x$, or otherwise, find the exact value of the volume obtained. [3]

(iii) Find the exact x-coordinate of the maximum point of C. [3]

Exercise 737. (9233 N2008/I/2.) (Answer on p. **2126**.)

Find the constants a and b such that, when x is small,

$$\frac{\cos 2x}{\sqrt{1+x^2}} \approx a + bx^2. \qquad [4]$$

Exercise 738. (9233 N2008/I/3.) Show that (Answer on p. **2127**.)

$$\int_0^1 x\mathrm{e}^{-2x}\,\mathrm{d}x = \frac{1}{4} - \frac{3}{4}\mathrm{e}^{-2}. \qquad [5]$$

Exercise 739. (9233 N2008/I/4.) (Answer on p. **2127**.)

Use the substitution $t = \ln x$ to find the value of

$$\int_{\mathrm{e}}^{\mathrm{e}^3} \frac{1}{x(\ln x)^2}\,\mathrm{d}x. \qquad [6]$$

Exercise 740. (9233 N2008/I/6.) (Answer on p. **2127**.)

(i) Given that $0 < a < b$, sketch the graph of $y = |x-a|$ for $-b \le x \le b$. [3]

(ii) Find $\int_{-b}^{b} |x-a|\,\mathrm{d}x$. [2]

Exercise 741. (9233 N2008/I/8.) (Answer on p. **2127**.)

Find the exact value of a for which

$$\int_a^{\infty} \frac{1}{4+x^2}\,\mathrm{d}x = \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x. \qquad [5]$$

Exercise 742. (9233 N2008/I/10.) (Answer on p. **2128**.)

(i) Prove that the substitution $y = xz$ reduces the differential equation $xy\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$ to

$$xz\frac{\mathrm{d}z}{\mathrm{d}x} = 1. \qquad [3]$$

(ii) Hence find the solution of the differential equation $xy\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$ for which $y = 6$ when $x = 2$. [5]

Exercise 743. (9233 N2008/I/13.) (Answer on p. **2128**.)

A curve is defined by the parametric equations

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad \text{for } 0 < t < \frac{1}{4}\pi.$$

(i) Show that the equation of the normal to the curve at the point $P\left(\cos^3 t, \sin^3 t\right)$ is

$$x\cos t - y\sin t = \cos^4 t - \sin^4 t. \qquad [5]$$

(ii) Prove the identity $\cos^4 t - \sin^4 t \equiv \cos 2t$. [2]

(iii) The normal at P meets the x-axis at A and the y-axis at B. Show that the length of AB can be expressed in the form $k\cot 2t$, where k is a constant to be found. [5]

Exercise 744. (9233 N2008/I/14 EITHER.) (Answer on p. 2129.)

It is required to prove the statement:

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

(i) Use mathematical induction to prove the statement for all positive integers n. [6]

(ii) By considering the expression obtained by integrating each term on the left hand side, prove the statement without using mathematical induction. [6]

Exercise 745. (9233 N2008/II/1.) (Answer on p. 2129.)

Use the formulae for $\cos(A+B)$ and $\cos(A-B)$, with $A = 5x$ and $B = x$, to show that $2\sin 5x \sin x$ can be written as $\cos px - \cos qx$, where p and q are positive integers. [2]

Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} \sin 5x \sin x \, dx. \qquad [3]$$

Exercise 746. (9233 N2008/II/5.) (Answer on p. 2129.)

(i) Show that the derivative of the function

$$\ln(1+x) - \frac{2x}{x+2}$$

is never negative. [5]

(ii) Hence show that $\ln(1+x) \geq \dfrac{2x}{x+2}$ when $x \geq 0$. [3]

Remark 199. In **(i)**, it is claimed that "$\ln(1+x) - \dfrac{2x}{x+2}$" is a function. But it is not. It is simply an expression.

As repeatedly stressed in Ch. 17, to specify a function, we must state its domain, codomain, and mapping rule. It turns out that the failure to do so here has important consequences (see footnote in answer).

Exercise 747. (9740 N2007/I/4.) (Answer on p. 2130.)

The current I in an electric circuit at time t satisfies the differential equation

$$4\frac{dI}{dt} = 2 - 3I.$$

(i) Find I in terms of t, given that $I = 2$ when $t = 0$. [6]

(ii) State what happens to the current in this circuit for large values of t. [1]

Exercise 748. (9740 N2007/I/11.) (Answer on p. 2130.)

A curve has parametric equations

$$x = \cos^2 t, \qquad y = \sin^3 t, \qquad \text{for } 0 \le t \le \frac{1}{2}\pi.$$

(i) Sketch the curve. [2]

(ii) The tangent to the curve at the point $\left(\cos^2\theta, \sin^3\theta\right)$, where $0 < \theta < \frac{1}{2}\pi$, meets the x- and y-axes at Q and R respectively. The origin is denoted by O. Show that the area of $\triangle OQR$ is

$$\frac{1}{12}\sin\theta\left(3\cos^2\theta + 2\sin^2\theta\right)^2. \qquad [6]$$

(iii) Show that the area under the curve for $0 \le t \le 0.5\pi$ is $2\int_0^{\frac{1}{2}\pi} \cos t \sin^4 t \, dt$, and use the substitution $\sin t = u$ to find this area. [5]

Exercise 749. (9740 N2007/II/3.) (Answer on p. 2131.)

(i) By successively differentiating $(1+x)^n$, find Maclaurin series for $(1+x)^n$, up to and including the term in x^3. [4]

(ii) Obtain the expansion of $(4-x)^{\frac{3}{2}}\left(1+2x^2\right)^{\frac{3}{2}}$ up to and including the term in x^3. [5]

(iii) Find the set of values of x for which the expansion in part **(ii)** is valid. [2]

Exercise 750. (9740 N2007/II/4.) (Answer on p. 2131.)

(i) Find the exact value of $\int_0^{\frac{5}{3}\pi} \sin^2 x \, dx$. Hence find the exact value of $\int_0^{\frac{5}{3}\pi} \cos^2 x \, dx$. [6]

(ii) The region R is bounded by the curve $y = x^2 \sin x$, the line $x = \frac{1}{2}\pi$ and the part of the x-axis between 0 and $\frac{1}{2}\pi$. Find

(a) the exact area of R, [5]

(b) the numerical value of the volume of revolution formed when R is rotated completely about the x-axis, giving your answer correct to 3 decimal places. [2]

Exercise 751. (9233 N2007/I/2.) (Answer on p. 2131.)

Find the first negative coefficient in the expansion of $(4+3x)^{\frac{5}{2}}$ in a series of ascending powers of x, where $|x| < \frac{4}{3}$. Give your answer as a fraction in its lowest terms. [3]

Exercise 752. (9233 N2007/I/3.) (Answer on p. **2132**.)

The region bounded by the curve $y = \dfrac{1}{\sqrt{1+4x^2}}$, the x-axis and the lines $x = \dfrac{1}{2}$ and $x = \dfrac{1}{2}\sqrt{3}$ is rotated through 4 right angles about the x-axis to form a solid of revolution of volume V. Find the exact value of V, giving your answer in the form $k\pi^2$. [5]

Exercise 753. (9233 N2007/I/8.) (Answer on p. **2132**.)

(i) Use the substitution $t = \sin u$ to show that

$$\int \frac{\left(\sin^{-1} t\right)\cos\left[\left(\sin^{-1} t\right)^2\right]}{\sqrt{1-t^2}}\,\mathrm{d}t$$

simplifies to $\int u \cos u^2 \,\mathrm{d}u$. [3]

Remark 200. For **(i)**, let us specify also that $u \in (-\pi/2, \pi/2)$ so that **(a)** $t = \sin u \in (-1, 1)$; **(b)** the integrand is well defined for all t; and **(c)** $\cos u \in (0, 1)$.

(i) Hence evaluate $\displaystyle\int_0^1 \frac{\left(\sin^{-1} t\right)\cos\left[\left(\sin^{-1} t\right)^2\right]}{\sqrt{1-t^2}}\,\mathrm{d}t$. [4]

Exercise 754. (9233 N2007/I/10.) (Answer on p. **2132**.)

(i) By sketching the graphs of $y = \cos x$ and $y = \sin x$, or otherwise, solve the inequality

$$\cos x > \sin x$$

for $0 \le x \le 2\pi$. [3]

(ii) Evaluate $\displaystyle\int_0^{2\pi} |\cos x - \sin x| \,\mathrm{d}x$. [5]

Exercise 755. (9233 N2007/I/11.) (Answer on p. **2132**.)

Use partial fractions to evaluate

$$\int_1^4 \frac{5x+4}{(x-5)\left(x^2+4\right)}\,\mathrm{d}x,$$

giving your answer in the form $-\ln a$, where a is a positive integer. [9]

Exercise 756. (9233 N2007/I/13.) (Answer on p. **2133**.)

In this question, the result $\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$ may be quoted without proof.

Given that $y = \ln(\sec x)$, show that

(i) $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\frac{\mathrm{d}y}{\mathrm{d}x}$, [3]

(ii) the value of $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$ when $x = 0$ is 2. [4]

(iii) Write down the Maclaurin series for $\ln(\sec x)$ up to and including the term in x^4. [2]

(iv) By substituting $x = \frac{1}{4}\pi$, show that $\ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1\,536}$. [3]

Exercise 757. (9233 N2007/I/14.) (Answer on p. **2133**.)

A family of curves is given by $x^2 - y^2 = Ax$, where A is an arbitrary constant.

(i) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2+y^2}{2xy}.$$ [4]

Remark 201. For **(i)**, add the assumption that $x, y \neq 0$ (otherwise $\mathrm{d}y/\mathrm{d}x$ is undefined).

A second, related family of curves is given by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy}{x^2+y^2}.$$

(ii) By substituting $y = vx$, where v is a function of x, show that, for the second family of curves,

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{3v+v^3}{1+v^2}.$$ [4]

(iii) Hence show that the second family of curves is given by

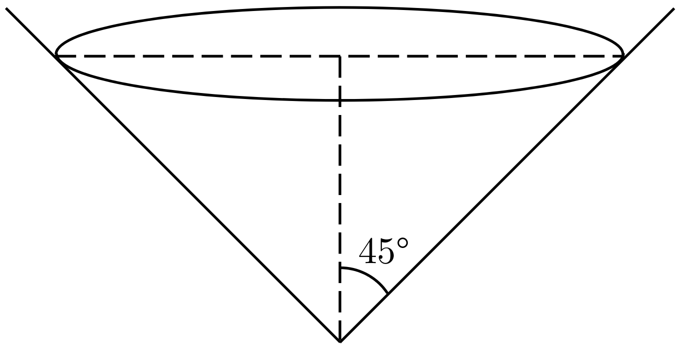
$$3x^2y + y^3 = C,$$

where C is an arbitrary constant. [4]

Remark 202. For **(ii)** and **(iii)**, add the assumption that at least one of x or y is non-zero (otherwise $\mathrm{d}y/\mathrm{d}x$ is undefined).

Exercise 758. (9233 N2006/I/7.) (Answer on p. **2133**.)

A hollow cone of semi-vertical angle 45° is held with its axis vertical and vertex downwards (see diagram). At the beginning of an experiment, it is filled with 390 cm^3 of liquid. The liquid runs out through a small hole at the vertex at a constant rate of $2\,\text{cm}^3\,\text{s}^{-1}$. Find the rate at which the depth of the liquid is decreasing 3 minutes after the start of the experiment. [6]



Remark 203. A cone's volume is Height × Base area/3.

Exercise 759. (9233 N2006/I/8.) (Answer on p. **2134**.)

Find the coordinates of the points on the curve

$$3x^2 + xy + y^2 = 33$$

at which the tangent is parallel to the x-axis. [7]

Exercise 760. (9233 N2006/I/9.) (Answer on p. **2134**.)

(i) Use the derivative of $\cos\theta$ to show that $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\sec\theta) = \sec\theta\tan\theta$. [2]

(ii) Use the substitution $x = \sec\theta - 1$ to find the exact value of

$$\int_{\sqrt{2}-1}^{1} \frac{1}{(x+1)\sqrt{x^2+2x}}\,\mathrm{d}x.$$ [6]

Remark 204. For **(ii)**, take $\theta \in (0, \pi/2)$ (this has the implication that $\tan\theta > 0$).

Exercise 761. (9233 N2006/I/12.) (Answer on p. **2134**.)

(i) Express

$$f(x) = \frac{1 + x - 2x^2}{(2-x)(1+x^2)}$$

in partial fractions. [4]

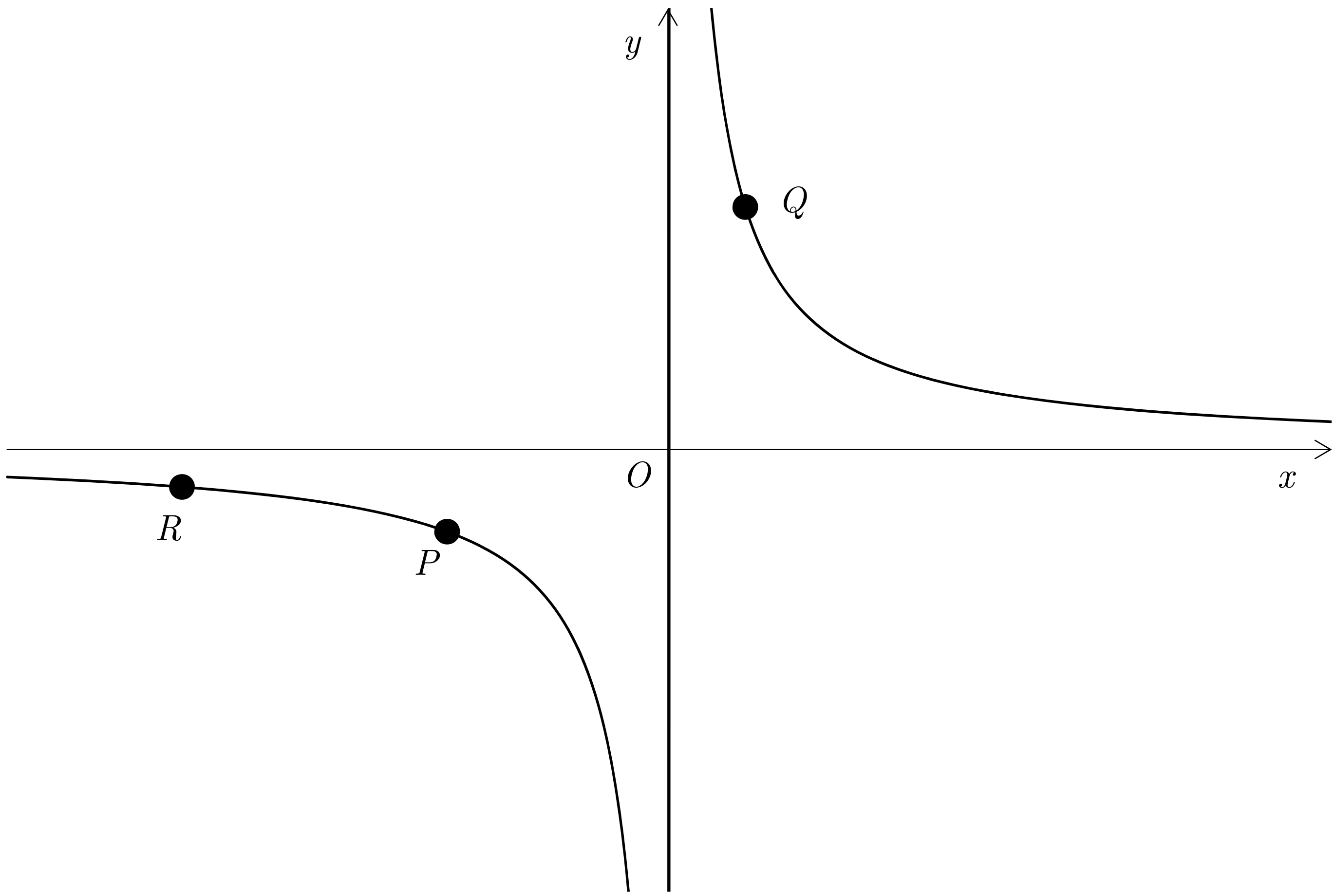
(ii) Expand $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

(iii) State the set of values of x for which the expansion is valid. [1]

Exercise 762. (9233 N2006/I/14.) (Answer on p. **2135**.)

A curve has parametric equations $x = ct$, $y = \frac{c}{t}$, where c is a positive constant.

Three points $P\left(cp, \frac{c}{p}\right)$, $Q\left(cq, \frac{c}{q}\right)$, $R\left(cr, \frac{c}{r}\right)$ on the curve are shown in the diagram.



(i) Prove that the gradient of QR is $-\frac{1}{qr}$. [2]

(ii) Given that the line through P perpendicular to QR meets the curve at $V\left(cv, \frac{c}{v}\right)$, find v in terms of p, q and r. [2]

(iii) Find the gradient of the normal at P. [3]

(iv) The normal at P meets the curve again at $S\left(cs, \frac{c}{s}\right)$. Show that $s = -\frac{1}{p^3}$. [2]

(v) Given that angle QPR is $90°$, prove that QR is parallel to the normal at P. [3]

Exercise 763. (9233 N2006/II/2.) (Answer on p. **2135**.)

(i) Given that $z = \frac{x}{(x^2+32)^{\frac{1}{2}}}$, show that $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{32}{(x^2+32)^{\frac{3}{2}}}$. [3]

(ii) Find the exact value of the area of the region bounded by the curve $y = \frac{1}{(x^2+32)^{\frac{3}{2}}}$, the x-axis and the lines $x = 2$ and $x = 7$. [3]

(Sorry *hor*, I haven't written the answers for 2016–19 questions yet.)

Exercise 764. (9758 N2019/II/6.) (Answer on p. **1451**.)

In a certain country, there are 100 professional football clubs, arranged in 4 divisions. There are 22 clubs in Division One, 24 in Division Two, 26 in Division Three and 28 in Division Four.

(i) Alice wishes to find out about approaches to training by clubs in Division One, so she sends a questionnaire to the 22 clubs in Division One. Explain whether these 22 clubs form a sample or a population. [1]

(ii) Dillip wishes to investigate the facilities for supporters at the football clubs, but does not want to obtain the detailed information necessary from all 100 clubs. Explain how he should carry out his investigation, and why he should do the investigation in this way. [2]

(iii) Find the number of different possible samples of 20 football clubs, with 5 clubs chosen from each division. [3]

A764 (9758 N2019/II/6).

Exercise 765. (9758 N2019/II/7.) (Answer on p. **1451**.)

A company produces drinking mugs. It is known that, on average, 8% of the mugs are faulty. Each day the quality manager collects 50 of the mugs at random and checks them; the number of faulty mugs found is the random variable F.

(i) State, in the context of the question, two assumptions needed to model F by a binomial distribution. [2]

You are now given that F can be modelled by a binomial distribution.

(ii) Find the probability that, on a randomly chosen day, at least 7 faulty mugs are found. [2]

(iii) The number of faulty mugs produced each day is independent of other days. Find the probability that, in a randomly chosen working week of 5 days, at least 7 faulty mugs are found on no more than 2 days. [2]

The company also makes saucers. The number of faulty saucers also follows a binomial distribution. The probability that a saucer is faulty is p. Faults on saucers are independent of faults on mugs.

(iv) Write down an expression in terms of p for the probability that, in a random sample of 10 saucers, exactly 2 are faulty. [1]

The mugs and saucers are sold in sets of 2 randomly chosen mugs and 2 randomly chosen saucers. The probability that a set contains at most 1 faulty item is 0.97.

(v) Write down an equation satisfied by p. Hence find the value of p. [4]

A765 (9758 N2019/II/7).

Exercise 766. (9758 N2019/II/8.) (Answer on p. **1452**.)

Gerri collects characters given away in packets of breakfast cereal. There are four different characters: Horse, Rider, Dog and Bird. Each character is made in four different colours: Orange, Yellow, Green and White. Gerri has collected 56 items; the numbers of each character and colour are shown in the table.

	Orange	Yellow	Green	White
Horse	1	1	3	4
Rider	1	1	7	5
Dog	3	7	1	6
Bird	4	5	6	1

(i) Gerri puts all the items in a bag and chooses one item at random.

(a) Find the probability that this item is either a Horse or a Rider. [1]

(b) Find the probability that this item is either a Dog or a Bird but the item is not White. [1]

(ii) Gerri now puts the item back in the bag and chooses two items at random.

(a) Find the probability that both of the items are Horses, but neither of the items is Orange. [1]

(b) Find the probability that Gerri's two items include exaclty one Dog and exactly one item that is Yellow. [3]

(iii) Gerri has two favourites among the 16 possible colour/character combinations. The probability of choosing these two at random from the 56 items is $\frac{1}{77}$. Write down all the possibilities for Gerri's two favourite colour/character combinations. [3]

A766 (9758 N2019/II/8).

Exercise 767. (9758 N2019/II/9.) (Answer on p. **1452**.)

A company produces resistors rated at 750 ohms for use in electronic circuits. The production manager wishes to test whether the mean resistance of these resistors is in fact 750 ohms. He knows that the resistances are normally distributed with variance 100 ohms^2.

(i) Explain whether the manager should carry out a 1-tail test or a 2-tail test. State hypotheses for the test, defining any symbols you use. [2]

The production manager takes a random sample of 8 of these resistors. He finds that the resistances, in ohms, are as follows.

742 771 768 738 769 752 742 766

(ii) Find the mean of the sample of 8 resistors. Carry out the test, at 5% level of significance, for the production manager. Give your conclusion in context. [5]

The company also produces resistors rated at 1250 ohms. Nothing is known about the distribution of the resistances of these resistors.

(iii) Describe how, and why, a test of the mean resistance of the 1250 ohms resistors would need to differ from that for the 750 ohms resistors. [2]

A767 (9758 N2019/II/9).

Exercise 768. (9758 N2019/II/10.) (Answer on p. **1453**.)

Abi and Bhani find the fuel consumption for a car driven at different constant speeds. The table shows the fuel consumption, y kilometres per litre, for different constant speeds, x kilometres per hour.

(i) Abi decides to model the data using the line $y = 35 - \frac{1}{3}x$.

(a) On the grid opposite [*omitted from this textbook*]

- draw a scatter diagram of the data,
- draw the line $y = 35 - \frac{1}{3}x$. [2]

(b) For a line of best fit $y = f(x)$, the residual for a point (a, b) plotted on the scatter diagram is the vertical distance between $(a, f(a))$ and (a, b). Mark the residual for each point on your diagram. [1]

(c) Calculate the sum of the squares of the residuals for Abi's line. [1]

(d) Explain why, in general, the sum of the squares of the residuals rather than the sum of the residuals is used. [1]

Bhani models the same data using a straight line passing through the points $(40, 22)$ and $(55, 17)$. The sum of the squares of the residuals for Bhani's line is 1.

(ii) State, with a reason, which of the two models, Abi's or Bhani's, gives a better fit. [1]

(iii) State the coordinates of the point that the least squares regression line must pass through. [1]

(iv) Use your calculator to find the equation of the least squares regression line of y on x. State the value of the product moment correlation coefficient. [3]

(v) Use the equation of the regression line to estimate the fuel consumption when the speed is 30 kilometres per hour. Explain whether you would expect this value to be reliable. [2]

(vi) Cerie performs a similar experiment on a different car. She finds that the sum of the squares of the residuals for her line is 0. What can you deduce about the data points in Cerie's experiment? [1]

A768 (9758 N2019/II/10).

Exercise 769. (9758 N2019/II/11.) (Answer on p. 1454.)

In this question you should state clearly all the distributions that you use, together with the values of the appropriate parameters.

Arif is making models of hydrocarbon molecules. Hydrocarbons are chemical compounds made from carbon atoms and hydrogen atoms. Arif has a bag containing a large number of white balls to represent the carbon atoms, and a bag containing a large number of black balls to represent the hydrogen atoms.

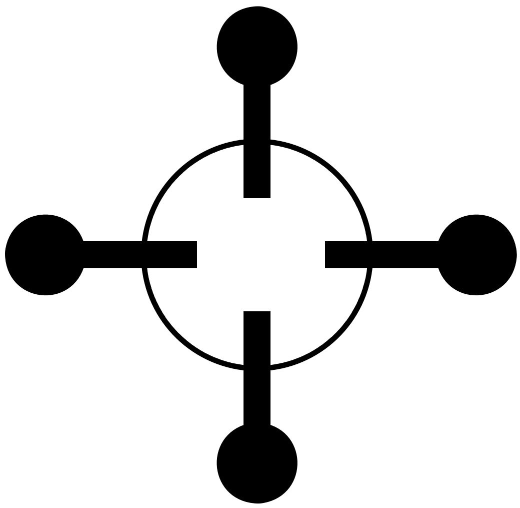
The masses of the white balls have the distribution $\mathrm{N}\left(110, 4^2\right)$ and the masses of the black balls have the distribution $\mathrm{N}\left(55, 2^2\right)$. The units for mass are grams.

(i) Find the probability that the total mass of 4 randomly chosen white balls is more than 425 grams. [2]

(ii) Find the probability that the total mass of a randomly chosen white ball and a randomly chosen black ball is between 161 and 175 grams. [2]

(iii) The probability that 2 randomly chosen white balls and 3 randomly chosen black balls have total mass less than M grams is 0.271. Find the value of M. [4]

Arif also has a bag containing a large number of connecting rods to fix the balls together. The masses of the connecting rods, in grams, have the distribution $\mathrm{N}\left(20, 0.9^2\right)$. In order to make models of methane (a hydrocarbon), Arif has to drill 1 hole in each black ball, and 4 holes in each white ball, for the connecting rods to fit in. This reduces the mass of each black ball by 10% and reduces the mass of each white ball by 30%.



A methane molecule consists of 1 carbon atom and 4 hydrogen atoms. Arif makes a model of a methane molecule using 4 black balls, 1 white ball and 4 connecting rods (see diagram). The balls and connecting rods are all chosen at random.

(iv) Find the probability that the mass of Arif's model is more than 350 grams. [4]

A769 (9758 N2019/II/11).

Exercise 770. (9758 N2018/II/5.) (Answer on p. 1455.)

The manufacturer of a certain type of fan used for cooling electronic devices claims that the mean time to failure (MTTF) is 65 000 hours. The quality control manager suspects that the MTTF is actually less than 65 000 hours and decides to carry out a hypothesis test on a sample of these fans. (An accelerated testing procedure is used to find the MTTF.)

(i) Explain why the manager should take a sample of at least 30 fans, and state how these fans should be chosen. [2]

(ii) State suitable hypotheses for the test, defining any symbols that you use. [2]

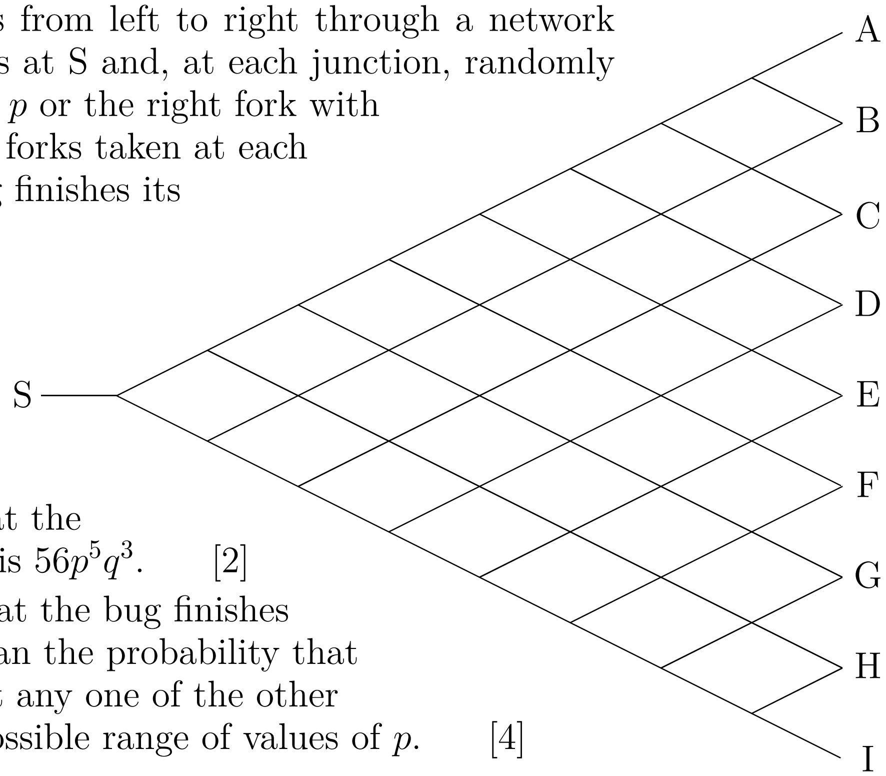
The quality control manager takes a suitable sample of 43 fans, and finds that they have an MTTF of 64 230 hours.

(iii) Given that the manager concludes that there is no reason to reject the null hypothesis at the 5% level of significance, find the range of possible values of the variance used in calculating the test statistic. [3]

A770 (9758 N2018/II/5).

Exercise 771. (9758 N2018/II/6.) (Answer on p. 1455.)

In a computer game, a bug moves from left to right through a network of connected paths. The bug starts at S and, at each junction, randomly takes the left fork with probability p or the right fork with probability q, where $q = 1 - p$. The forks taken at each junction are independent. The bug finishes its journey at one of the 9 endpoints labelled A–I (see diagram).



(i) Show that the probability that the bug finishes its journey at D is $56p^5q^3$. [2]

(ii) Given that the probability that the bug finishes its journey at D is greater than the probability that the bug finishes its journey at any one of the other endpoints, find exactly the possible range of values of p. [4]

In another version of the game, the probability that, at each junction, the bug takes the left fork is $0.9p$, the probability that the bug takes the right fork is $0.9q$ and the probability that the bug is swallowed up by a 'black hole' is 0.1.

(iii) Find the probability that, in this version of the game, the bug reaches one of the endpoints A–I, without being swallowed up by a black hole. [1]

A771 (9758 N2018/II/6).

Exercise 772. (9758 N2018/II/7.) (Answer on p. **1456**.)

The events A, B and C are such that $\mathrm{P}(A) = a$, $\mathrm{P}(B) = b$ and $\mathrm{P}(C) = c$. A and B are independent events. A and C are mutually exclusive events.

(i) Find an expression for $\mathrm{P}(A' \cap B')$ and hence prove that A' and B' are independent events. [2]

(ii) Find an expression for $\mathrm{P}(A' \cap C')$. Draw a Venn diagram to illustrate the case when A' and C' are also mutually exclusive events. (You should not show event B on your diagram.) [2]

You are now given that A' and C' are **not** mutually exclusive, $\mathrm{P}(A) = \frac{2}{5}$, $\mathrm{P}(B \cap C) = \frac{1}{5}$ and $\mathrm{P}(A' \cap B' \cap C') = \frac{1}{10}$.

(iii) Find exactly the maximum and minimum possible values of $\mathrm{P}(A \cap B)$. [4]

A772 (9758 N2018/II/7).

Exercise 773. (9758 N2018/II/8.) (Answer on p. **1456**.)

A bag contains $(n+5)$ numbered balls. Two of the balls are numbered 3, three of the balls are numbered 4 and n of the balls are numbered 5. Two balls are taken, at random and without replacement, from the bag. The random variable S is the sum of the numbers on the two balls taken.

(i) Determine the probability distribution of S. [4]

(ii) For the case where $n = 1$, find $\mathrm{P}(S = 10)$ and explain this result. [1]

(iii) Show that $\mathrm{e}(S) = \dfrac{10n+36}{n+5}$ and $\mathrm{Var}(S) = \dfrac{g(n)}{(n+5)^2(n+4)}$ where $g(n)$ is a quadratic polynomial to be determined. [6]

A773 (9758 N2018/II/8).

Exercise 774. (9758 N2018/II/9.) (Answer on p. **1457**.)

Many electronic devices need a fan to keep them cool. In order to maximise the lifetime of such fans, the speed they run at is reduced when conditions allow. Running a fan at a lower speed reduces the power required. The following table gives details, for a particular type of fan, of the power required (P watts) at different fan speeds (R revolutions per minute).

Fan speed (R)	3600	4500	5400	6300	7200	8100	9000	9900
Power (P)	0.22	0.34	0.52	0.78	1.06	1.48	2.04	2.64

(i) Draw a scatter diagram of these data. Use your diagram to explain whether the relationship between P and R is likely to be well modelled by an equation of the form $P = aR + b$, where a and b are constants. [2]

(ii) By calculating the relevant product moment correlation coefficients, determine whether the relationship between P and R is modelled better by $P = aR + b$ or by $P = aR^2 + b$. Explain how you decide which model is better, and state the equation in this case.[5]

(iii) Use your equation to estimate the speed of the fan when the power is 0.9 watts. Explain whether your estimate is reliable. [2]

(iv) Use your equation to estimate the power used when the speed of the fan is 3300 revolutions per minute. Explain whether your estimate is reliable. [2]

(v) Re-write your equation from part **(ii)** so that it can be used when the speed of the fan, R, is given in revolutions per second. [1]

A774 (9758 N2018/II/9).

Exercise 775. (9758 N2018/II/10.) (Answer on p. **1458**.)

In this question you should state the parameters of any distributions that you use.

A manufacturer produces specialist light bulbs. The masses in grams of one type of light bulb have the normal distribution $\mathrm{N}\left(50, 1.5^2\right)$.

(i) Sketch the distribution for masses between 40 grams and 60 grams. [2]

(ii) Find the probability that the mass of a randomly chosen bulb is less than 50.4 grams. [1]

Each light bulb is packed into a randomly chosen box. The masses of the empty boxes have the distribution $\mathrm{N}\left(75, 2^2\right)$.

(iii) Find the probability that the total mass of 4 randomly chosen empty boxes is more than 297 grams. [2]

(iv) Find the probability that the total mass of a randomly chosen light bulb and a randomly chosen box is between 124.9 and 125.7 grams. [3]

In order to protect the bulbs in transit each bulb is surrounded by padding before being packed in a box. The mass of the padding is modelled as 30% of the mass of the bulb.

(v) The probability that the total mass of a box containing a bulb and padding more than k grams is 0.9. Find k. [4]

(vi) Find the probability that the total mass of 4 randomly chosen boxes, each containing a bulb and padding, is more than 565 grams. [3]

A775 (9758 N2018/II/10).

Exercise 776. (9758 N2017/II/5.) (Answer on p. **2136**.)

A bag contains 6 red counters and 3 yellow counters. In a game, Lee removes counters at random from the bag, one at a time, until he has taken out 2 red counters. The total number of counters Lee removes from the bag is denoted by T.

(i) Find $\mathrm{P}(T = t)$ for all possible values of t. [3]

(ii) Find $\mathrm{E}(T)$ and $\mathrm{Var}(T)$. [2]

Lee plays this game 15 times.

3. Find the probability that Lee has to take at least 4 counters out of the bag in at least 5 of his 15 games. [2]

Exercise 777. (9758 N2017/II/6.) (Answer on p. **2136**.)

A children's game is played with 20 cards, consisting of 5 sets of 4 cards. Each set consists of a father, mother, daughter and son from the same family. The family names are Red, Blue, Green, Yellow and Orange. So, for example, the Red family cards are father Red, mother Red, daughter Red and son Red.

The 20 cards are arranged in a row.

(i) In how many different ways can the 20 cards be arranged so that the 4 cards in each family set are next to each other? [2]

(ii) In how many different ways can the cards be arranged so that all five father cards are next to each other, all four Red family cards are next to each other and all four Blue family cards are next to each other? [3]

The cards are now arranged at random in a circle.

(iii) Find the probability that no two father cards are next to each other. [4]

Exercise 778. (9758 N2017/II/7.) (Answer on p. **2136**.)

The production manager of a food manufacturing company wishes to take a random sample of a certain type of biscuit bar from the thousands produced one day at his factory, for quality control purposes. He wishes to check that the mean mass of the bars is 32 grams, as stated on the packets.

(i) State what it means for a sample to be random in this context. [1]

The masses, x grams, of a random sample of 40 biscuit bars are summarised as follows.

$$n = 40, \qquad \sum (x - 32) = -7.7, \qquad \sum (x - 32)^2 = 11.05.$$

(ii) Calculate unbiased estimates of the population mean and variance of the mass of biscuit bars. [2]

(iii) Test, at the 1% level of significance, the claim that the mean mass of biscuit bars is 32 grams. You should state your hypotheses and define any symbols you use. [5]

(iv) Explain why there is no need for the production manager to know anything about the population distribution of the masses of the biscuit bars. [2]

Exercise 779. (9758 N2017/II/8.) (Answer on p. **2136**.)

(a) Draw separate scatter diagrams, each with 8 points, all in the first quadrant, which represent the situation where the produce moment correlation coefficient between variables x and y is

(i) -1,

(ii) 0,

(iii) between 0.5 and 0.9. [3]

(b) An investigation into the effect of a fertiliser on yields of corn found that the amount of fertiliser applied, x, resulted in the average yields of corn, y, given below, where x and y are measured in suitable units.

x	0	40	80	120	160	200
y	70	104	118	119	126	129

(i) Draw a scatter diagram for these values. State which of the following equations, where a and b are positive constants, provides the most accurate model of the relationship between x and y.

(A) $y = ax^2 + b$. (B) $y = \dfrac{a}{x^2} + b$.

(C) $y = a \ln 2x + b$. (D) $y = a\sqrt{x} + b$. [2]

(ii) Using the model you chose in part **(i)**, write down the equation for the relationship between x and y, giving the numerical values of the coefficients. State the product moment correlation coefficient for this model. [3]

(iii) Give two reasons why it would be reasonable to use your model to estimate the value of y when $x = 189$. [2]

Exercise 780. (9758 N2017/II/9.) (Answer on p. **2136**.)

On average 8% of a certain brand of kitchen lights are faulty. The lights are sold in boxes of 12.

(i) State, in context, two assumptions needed for the number of faulty lights in a box to be well modelled by a binomial distribution. [2]

Assume now that the number of faulty lights in a box has a binomial distribution.

(ii) Find the probability that a box of 12 of these kitchen lights contains at least 1 faulty light. [1]

The boxes are packed into cartons. Each carton contains 20 boxes.

(iii) Find the probability that each box in one randomly selected carton contains at least one faulty light. [1]

(iv) Find the probability that there are at least 20 faulty lights in a randomly selected carton. [2]

(v) Explain why the answer to part **(iv)** is greater than the answer to part **(iii)**. [1]

The manufacturer introduces a quick test to check if lights are faulty. Lights identified as faulty are discarded. If a light is faulty there is a 95% chance that the quick test will correctly identify the light as faulty. If the light is not faulty, there is a 6% chance that the quick test will incorrectly identify the light as faulty.

(vi) Find the probability that a light identified as faulty by the quick test is **not** faulty.[3]

(vii) Find the probability that the quick test correctly identifies lights as faulty or not faulty. [1]

(viii) Discuss briefly whether the quick test is worthwhile. [1]

Exercise 781. (9758 N2017/II/10.) (Answer on p. **2136**.) A small component for a machine is made from two metal spheres joined by a short metal bar. The masses in grams of the spheres have the distribution $\mathrm{N}\left(20, 0.5^2\right)$.

(i) Find the probability that the mass of a randomly selected sphere is more than 20.2 grams. [1]

In order to protect them from rusting, the spheres are given a coating which increases the mass of each sphere by 10%.

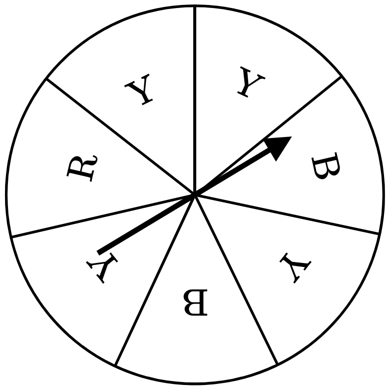
(ii) Find the probability that the mass of a coated sphere is between 21.5 and 22.45 grams. State the distribution you use and its parameters. [3]

(iii) The masses of the metal bars are normally distributed such that 60% of them have a mass greater than 12.2 grams and 25% of them have a mass less than 12 grams. Find the mean and standard deviation of the masses of metal bars. [4]

(iv) The probability that the total mass of a component, consisting of two randomly chosen coated spheres and one randomly chosen bar, is more than k grams is 0.75. Find k, stating the parameters of any distribution you use. [4]

Exercise 782. (9740 N2016/II/5.) (Answer on p. **2136**.)

In a game of chance, a player has to spin a fair spinner. The spinner has 7 sections and an arrow which has an equal chance of coming to rest over any of the 7 sections. The spinner has 1 section labelled **R**, 2 sections labelled **B** and 4 sections labelled **Y** (see diagram).



The player then has to throw one of three fair six-sided dice, coloured red, blue or yellow. If the spinner comes to rest over **R** the red die is thrown, if the spinner comes to rest over **B** the blue die is thrown and if the spinner comes to rest over **Y** the yellow die is thrown. The yellow die has one face with $*$ on it, the blue die has two faces with $*$ on it and the red die has three faces with $*$ on it. The player wins the game if the die thrown comes to rest with a face showing $*$ uppermost.

(i) Find the probability that a player wins a game. [2]

(ii) Given that a player wins a game, find the probability that the spinner came to rest over **B**. [1]

(iii) Find the probability that a player wins 3 consecutive games, each time throwing a die of a different colour. [2]

Exercise 783. (9740 N2016/II/6.) (Answer on p. 2136.) The number of employees of a company, classified by department and gender, is shown below.

	Production	Development	Administration	Finance
Male	2345	1013	237	344
Female	867	679	591	523

(i) The directors wish to survey a sample of 100 of the employees. This sample is to be a stratified sample, based on department and gender.

(a) How many males should be in the sample? [1]

(b) How many females from the Development department should be in the sample? [1]

The Managing Director knows that, some years ago, the mean age of employees was 37 years. He believes that the mean age of employees now is less than 37 years.

(ii) State why the stratified sample from part **(i)** should not be used for a hypothesis test of the Managing Director's belief. [1]

The Company Secretary obtains a suitable sample of 80 employees in order to carry out a hypothesis test of the Managing Director's belief that the mean age of the employees now is less than 37 years. You are given that the population variance of the ages is 140 years^2.

(iii) Write down appropriate hypotheses to test the Managing Director's belief. You are given that the result of the test, using a 5% significance level, is that the Managing Director's belief should be accepted. Determine the set of possible values of the mean age of the sample of employees. [4]

(iv) You are given instead that the mean age of the sample of employees is 35.2 years, and that the result of a test at the $\alpha\%$ significance level is that the Managing Director's belief should not be accepted. Find the set of possible values of α. [3]

Exercise 784. (9740 N2016/II/7.) (Answer on p. 2136.)

The management board of a company consists of 6 men and 4 women. A chairperson, a secretary and a treasurer are chosen from the 10 members of the board. Find the number of ways the chairperson, the secretary and the treasurer can be chosen so that

(i) they are all women, [1]

(ii) at least one is a woman and at least one is a man. [3]

The 10 members of the board sit at random around a table. Find the probability that

(iii) the chairperson, the secretary and the treasurer sit in three adjacent places, [3]

(iv) the chairperson, the secretary and the treasurer are all separated from each other by at least one other person. [3]

Exercise 785. (9740 N2016/II/8.) (Answer on p. 2136.)

A website about electric motors gives information about the percentage efficiency of motors depending on their power, measured in horsepower. Xian has copied the following table for a particular type of electric motor, but he has copied one of the efficiency values wrongly.

Power, x	1	1.5	2	3	5	7.5	10	20	30	40	50
Efficiency, y	72.5	82.5	84.0	87.4	87.5	88.5	89.5	90.2	91.0	91.7	92.4

(i) Plot a scatter diagram on graph paper for these values, labelling the axes, using a scale of 2 cm to represent 10% efficiency on the y-axis and an appropriate scale for the x-axis. On your diagram, circle the point that Xian has copied wrongly. [2]

For parts **(ii)**, **(iii)** and **(iv)** of this question you should **exclude** the point for which Xian has copied the efficiency value wrongly.

(ii) Explain from your scatter diagram why the relationship between x and y should not be modelled by an equation of the form $y = ax + b$. [1]

(iii) Suppose that the relationship between x and y is modelled by an equation of the form $y = \frac{c}{x} + d$, where c and d are constants. State with a reason whether each of c and d is positive or negative. [2]

(iv) Find the product moment correlation coefficient and the constants c and d for the model in part **(iii)**. [3]

(v) Use the model $y = \frac{c}{x} + d$, with the values of c and d found in part (iv), to estimate the efficiency value (y) that Xian has copied wrongly. Give two reasons why you would expect this estimate to be reliable. [3]

Exercise 786. (9740 N2016/II/9.) (Answer on p. 2136.)

(a) The random variable X has distribution $\mathrm{N}\left(15, a^2\right)$ and $\mathrm{P}(10 < X < 20) = 0.5$. Find the value of a. [2]

(b) The random variable Y has distribution $\mathrm{B}(4, p)$ and $\mathrm{P}(Y = 1) + \mathrm{P}(Y = 2) = 0.5$. Show that $4p^4 - 12p^2 + 8p = 1$ and hence find the possible values of p. [4]

(c) On a television quiz show contestants have to select the right answer from one of three alternatives. George decides to do this entirely by guesswork. Use a suitable approximation, which should be stated, to find the probability that George guesses at least 30 questions right out of 100. [4]

Exercise 787. (9740 N2016/II/10.) (Answer on p. **2136**.)

Mia owns a field. Various types of weed are found in Mia's field.

(i) State, in this context, two conditions that must be met for the numbers of a particular type of weed in Mia's field to be well modelled by a Poisson distribution. [2]

For the remainder of this question assume that these conditions are met.

There is an average of 1.5 dandelion plants (a type of weed) per m^2 in Mia's field.

(ii) Find the probability that in $1\,\text{m}^2$ of Mia's field there are at least 2 dandelion plants. [2]

(iii) Find the probability that in $4\,\text{m}^2$ of Mia's field there are at most 3 dandelion plants. [2]

(iv) Use a suitable approximation, which should be stated, to find the probability that the number of dandelion plants in an $80\,\text{m}^2$ area of Mia's field is between 110 and 140 inclusive. [4]

The distribution of daisies (another type of weed) per m^2 in Mia's field can be modelled by $\text{Po}(\lambda)$. The probability that the number of daisies in a $1\,\text{m}^2$ area of the field is less than or equal to 2 is the same as the probability that the number of daisies in a $2\,\text{m}^2$ area of the field is more than 2.

(v) Write down an equation in λ and solve it to find λ. [4]

Exercise 788. (9740 N2015/II/5.) (Answer on p. **2136**.)

The manager of a busy supermarket wishes to conduct a survey of the opinions of customers of different ages about different types of cola drink.

(i) Give a reason why the manager would not be able to use stratified sampling. [1]

(ii) Explain briefly how the manager could carry out a survey using quota sampling. [2]

(iii) Give one reason why quota sampling would not necessarily provide a sample which is representative of the customers of the supermarket. [1]

Exercise 789. (9740 N2015/II/6.) (Answer on p. **2136**.)

'Droppers' are small sweets that are made in a variety of colours. Droppers are sold in packets and the colours of the sweets in a packet are independent of each other. On average, 25% of Droppers are red.

(i) A small packet of Droppers contains 10 sweets. Find the probability that there are at least 4 red sweets in a small packet. [2]

A large packet of Droppers contains 100 sweets.

(ii) Use a suitable approximation, which should be stated, to find the probability that a large packet contains at least 30 red sweets. [3]

(iii) Yip buys 15 large packets of Droppers. Find the probability that no more than 3 of these packets contain at least 30 red sweets. [2]

Exercise 790. (9740 N2015/II/7.) (Answer on p. **2137**.)

The average number of errors per page for a certain daily newspaper is being investigated.

(i) State, in context, two assumptions that need to be made for the number of errors per page to be well modelled by a Poisson distribution. [2]

Assume that the number of errors per page has the distribution Po (1.3).

(ii) Find the probability that, on one day, there are more than 10 errors altogether on the first 6 pages. [3]

(iii) The probability that there are fewer than 2 errors altogether on the first n pages of the newspaper is less than 0.05. Write down an inequality in terms of n to represent this information, and hence find the least possible value of n. [2]

Exercise 791. (9740 N2015/II/8.) (Answer on p. **2137**.)

A market stall sells pineapples which have masses that are normally distributed. The stall owner claims the mean mass of the pineapples is at least 0.9 kg. Nur buys a random selection of 8 pineapples from the stall. The 8 pineapples have masses, in kg, as follows.

0.80 1.000 0.82 0.85 0.93 0.96 0.81 0.89

(i) Find unbiased estimates of the population mean and variance of the mass of pineapples.

(ii) Test the stall owner's claim at the 10% level of significance. [7]

Exercise 792. (9740 N2015/II/9.) (Answer on p. **2137**.)

For events A, B and C it is given that $\mathrm{P}(A) = 0.45$, $\mathrm{P}(B) = 0.4$, $\mathrm{P}(C) = 0.3$ and $\mathrm{P}(A \cap B \cap C) = 0.1$. It is also given that events A and B are independent, and that events A and C are independent.

(i) Find $\mathrm{P}(B|A)$. [1]

(ii) Given also that events B and C are independent, find $\mathrm{P}(A' \cap B' \cap C')$. [3]

(iii) Given instead that events B and C are **not** independent, find the greatest and least possible values of $\mathrm{P}(A' \cap B' \cap C')$. [4]

Exercise 793. (9740 N2015/II/10.) (Answer on p. 2138.)

In an experiment the following information was gathered about air pressure P, measured in inches of mercury, at different heights above sea-level h, measured in feet.

h	2 000	5 000	10 000	15 000	20 000	25 000	30 000	35 000	40 000	45 000
P	27.8	24.9	20.6	16.9	13.8	11.1	8.89	7.04	5.52	4.28

(i) Draw a scatter diagram for these values, labelling the axes. [1]

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between

(a) h and P,

(b) $\ln h$ and P,

(c) $\sqrt{h}$ and P. [3]

(iii) Using the most appropriate case from part **(ii)**, find the equation which best models air pressure at different heights. [3]

(iv) Given that 1 metre = 3.28 feet, re-write your equation from part **(iii)** so that it can be used to estimate the air pressure when the height is given in metres. [2]

Exercise 794. (9740 N2015/II/11.) (Answer on p. 2139.)

This question is about arrangements of all eight letters in the word CABBAGES.

(i) Find the number of different arrangements of the eight letters that can be made. [2]

(ii) Write down the number of these arrangements in which the letters are **not** in alphabetical order. [1]

(iii) Find the number of different arrangements that can be made with both the A's together and both the B's together. [2]

(iv) Find the number of different arrangements that can be made with no two adjacent letters the same. [4]

Exercise 795. (9740 N2015/II/12.) (Answer on p. 2139.)

In this question you should state clearly the values of the parameters of any normal distribution you use.

The masses in grams of apples have the distribution $\mathrm{N}\left(300, 20^2\right)$ and the masses in grams of pears have the distribution $\mathrm{N}\left(200, 15^2\right)$. A certain recipe requires 5 apples and 8 pears.

(i) Find the probability that the total mass of 5 randomly chosen apples is more than 1600 grams. [2]

(ii) Find the probability that the total mass of 5 randomly chosen apples is more than the total mass of 8 randomly chosen pears. [3]

The recipe requires the apples and pears to be prepared by peeling them and removing the cores. This process reduces the mass of each apple by 15% and the mass of each pear by 10%.

(iii) Find the probability that the total mass, after preparation, of 5 randomly chosen apples and 8 randomly chosen pears is less than 2750 grams. [4]

Exercise 796. (9740 N2014/II/5.) (Answer on p. **2140**.)

An Internet retailer has compiled a list of 10 000 regular customers and wishes to carry out a survey of customer opinions involving 5% of its customers.

(i) Describe how the marketing manager could choose customers for this survey using systematic sampling. [2]

(ii) Give one advantage and one disadvantage of systematic sampling in this context. [2]

Exercise 797. (9740 N2014/II/6.) (Answer on p. **2140**.)

A team in a particular sport consists of 1 goalkeeper, 4 defenders, 2 midfielders and 4 attackers. A certain club has 3 goalkeepers, 8 defenders, 5 midfielders and 6 attackers.

(i) How many different teams can be formed by the club? [2]

One of the midfielders in the club is the brother of one of the attackers in the club.

(ii) How many different teams can be formed which include exactly one of the two brothers? [3]

The two brothers leave the club. The club manager decides that one of the remaining midfielders can play either as a midfielder or as a defender.

(iii) How many different teams can now be formed by the club? [3]

Exercise 798. (9740 N2014/II/7.) (Answer on p. **2140**.)

Yan is carrying out an experiment with a fair 6-sided die and a biased 6-sided die, each numbered from 1 to 6.

(i) Yan rolls the fair die 10 times. Find the probability that it shows a 6 exactly thrice. [1]

(ii) Yan now rolls the fair die 60 times. Use a suitable approximate distribution, which should be stated, to find the probability that the die shows a 6 between 5 and 8 times, inclusive. [3]

The probability that the biased die shows a 6 is $\dfrac{1}{15}$.

(iii) Yan rolls the biased die 60 times. Use a suitable approximate distribution, which should be stated, to find the probability that the biased die shows a 6 between 5 and 8 times, inclusive. [3]

Exercise 799. (9740 N2014/II/8.) (Answer on p. 2141.)

(a) Sketch a scatter diagram that might be expected when x and y related approximately by $y = px^2 + t$ in each of the cases **(i)** and **(ii)** below. In each case your diagram should include 6 points, approximately equally spaced with respect to x, and with all x-values positive.

(i) p and t are both positive.

(ii) p is negative and t is positive. [2]

(b) The age in months (m) and prices in dollars (P) of a random sample of ten used cars of a certain model are given in the table.

m	11	20	28	36	40	47	58	62	68	75
P	112 800	102 600	76 500	72 000	72 000	69 000	65 800	57 000	50 600	47 600

It is thought that the price after m months can be modelled by one of the formulae

$$P = am + b, \qquad P = c \ln m + d,$$

where a, b, c and d are constants.

(i) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(A) m and P; and

(B) $\ln m$ and P. [2]

(ii) Explain which of $P = am + b$ and $P = c \ln m + d$ is the better model and find the equation of a suitable regression line for this model. [3]

(iii) Use the equation of your regression line to estimate the price of a car that is 50 months old. [1]

Exercise 800. (9740 N2014/II/9.) (Answer on p. 2141.)

The number of minutes that the 0815 bus arrives late at my local bus stop has a normal distribution; the mean number of minutes the bus is late has been 4.3. A new company takes over the service, claiming that punctuality will be improved. After the new company takes over, a random sample of 10 days is taken and the number of minutes that the bus is late is recorded. The sample mean is $\bar{t}$ minutes and the sample variance is k^2 minutes^2. A test is to be carried out at the 10% level of significance to determine whether the mean number of minutes late has been reduced.

(i) State appropriate hypotheses for the test, defining any symbols that you use. [2]

(ii) Given that $k^2 = 3.2$, find the set of values of $\bar{t}$ for which the result of the test would be that the null hypothesis is not rejected. [4]

(iii) Given instead that $\bar{t} = 4.0$, find the set of values of k^2 for which the result of the test would be to reject the null hypothesis. [3]

Exercise 801. (9740 N2014/II/10.) (Answer on p. **2142**.)

A game has three sets of ten symbols, and one symbol from each set is randomly chosen to be displayed on each turn. The symbols are as follows.

Set 1	+	+	+	+	×	×	×	◯	◯	⋆
Set 2	+	+	+	×	◯	◯	◯	◯	⋆	⋆
Set 3	+	+	×	×	×	×	◯	◯	◯	⋆

For example, if a + symbol is chosen from set 1, a ◯ symbol is chosen from set 2 and a ⋆ symbol is chosen from set 3, the display would be +◯⋆.

(i) Find the probability that, on one turn,

(a) ⋆ ⋆ ⋆ is displayed, [1]

(b) at least one ⋆ symbol is displayed, [2]

(c) two ×symbols and one + symbol are displayed in any order. [3]

(ii) Given that exactly one of the symbols displayed is ⋆, find the probability that the other two symbols are + and ◯. [4]

Exercise 802. (9740 N2014/II/11.) (Answer on p. **2142**.)

An art dealers sells both original paintings and prints. (Prints are copies of paintings.) It is to be assumed that his sales of originals per week can be modelled by the distribution $\text{Po}(2)$ and his sales of prints per week can be modelled by the independent distribution $\text{Po}(11)$.

(i) Find the probability that, in a randomly chosen week,

(a) the art dealer sells more than 8 prints, [2]

(b) the art dealer sells a total of fewer than 15 prints and originals combined. [2]

(ii) The probability that the art dealer sells fewer than 3 originals in a period of n weeks is less than 0.01. Express this information as an inequality in n, and hence find the smallest possible integer value of n. [5]

(iii) Using a suitable approximation, which should be stated, find the probability that the art dealer sells more than 550 prints in a year (52 weeks). [3]

(iv) Give two reasons in context why the assumptions made at the start of this question may not be valid. [2]

Exercise 803. (9740 N2013/II/5.) (Answer on p. **2143**.)

A large multi-national company has 100 000 employees based in several different countries. To celebrate the 90th anniversary of the founding of the company, the Chief Executive wishes to invite a representative sample of 90 employees to a party, to be held at the company's Headquarters in Singapore.

(i) Explain how random sampling could be carried out to choose the 90 employees. Explain briefly why this may not provide the representative sample that the Chief Executive wants. [2]

(ii) Name a more appropriate sampling method, and explain how it can be carried out to provide the representative sample that the Chief Executive wants. [2]

Exercise 804. (9740 N2013/II/6.) (Answer on p. **2143**.)

The continuous random variable Y has the distribution $\mathrm{N}\left(\mu, \sigma^2\right)$. It is known that $\mathrm{P}(Y < 2a) = 0.95$ and $\mathrm{P}(Y < a) = 0.25$. Express μ in the form ka, where k is a constant to be determined. [4]

Exercise 805. (9740 N2013/II/7.) (Answer on p. **2143**.)

On average one in 20 packets of a breakfast cereal contains a free gift. Jack buys n packets from a supermarket. The number of these packets containing a free gift is the random variable F.

(i) State, in context, two assumptions needed for F to be well modelled by a binomial distribution. [2]

Assume now that F has a binomial distribution.

(ii) Given that $n = 20$, find $\mathrm{P}(F = 1)$. [1]

(iii) Given instead that $n = 60$, use a suitable approximation to find the probability that F is at least 5. State the parameter(s) of the distribution that you use. [3]

Exercise 806. (9740 N2013/II/8.) (Answer on p. **2143**.) For events A and B it is given that $\mathrm{P}(A) = 0.7$, $\mathrm{P}(B|A') = 0.8$ and $\mathrm{P}(A|B') = 0.88$. Find

(i) $\mathrm{P}(B \cap A')$, [1]

(ii) $\mathrm{P}(A' \cap B')$, [2]

(iii) $\mathrm{P}(A \cap B)$. [3]

Exercise 807. (9740 N2013/II/9.) (Answer on p. **2143**.)

A motoring magazine editor believes that the figures quoted by car manufacturers for distances travelled per litre of fuel are too high. He carries out a survey into this by asking for information from readers. For a certain model of car, 8 readers reply with the following data, measured in km per litre.

14.0	12.5	11.0	11.0	12.5	12.6	15.6	13.2

(i) Calculate unbiased estimates of the population mean and variance. [2]

The manufacturer claims that this model of car will travel 13.8 km per litre on average. It is given that the distances travelled per litre for cars of this model are normally distributed.

(ii) Stating a necessary assumption, carry out a t-test of the magazine editor's belief at the 5% significance level. [5]

Exercise 808. (9740 N2013/II/10.) (Answer on p. **2144**.)

(i) Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A), (B) and (C) below. In each case your diagram should include 6 points, approximately equally spaced with respect to x, and with all x- and y-values positive. The letters a, b, c, d, e and f represent constants.

(A) $y = a + bx^2$, where a is positive and b is negative,

(B) $y = c + d \ln x$, where c is positive and d is negative,

(C) $y = e + \dfrac{f}{x}$, where e is positive and f is negative. [3]

A motoring website gives the following information about the distance travelled, y km, by a certain type of car at different speeds, $x\,\mathrm{km\,h^{-1}}$, on a fixed amount of fuel.

Speed, x	88	96	104	112	120	128
Distance, y	148	147	144	138	126	107

(ii) Draw the scatter diagram for these values, labelling the axes. [1]

(iii) Explain which of the three cases in part **(i)** is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. [2]

(iv) It is required to estimate the distance travelled at a speed of $110\,\mathrm{km\,h^{-1}}$. Use the case that you identified in part **(iii)** to find the equation of a suitable regression line, and use your equation to find the required estimate. [3]

Exercise 809. (9740 N2013/II/11.) (Answer on p. **2145**.)

A machine is used to generate codes consisting of three letters followed by two digits. Each of the three letters generated is equally likely to be any of the twenty-six letters of the alphabet A–Z. Each of the two digits generated is equally likely to be any of the nine digits 1–9. The digit 0 is not used. Find the probability that a randomly chosen code has

(i) three different letters and two different digits, [2]

(ii) the second digit higher than the first digit, [2]

(iii) exactly two letters the same or two digits the same, but not both, [4]

(iv) exactly one vowel (A, E, I, O or U) and exactly one even digit. [4]

Exercise 810. (9740 N2013/II/12.) (Answer on p. **2145**.)

A company has two departments and each department records the number of employees absent through illness each day. Over a long period of time it is found that the average numbers absent on a day are 1.2 for the Administration Department and 2.7 for the Manufacturing Department.

(i) State, in this context, two conditions that must be met for the numbers of absences to be well modelled by Poisson distributions. Explain why each of your two conditions may not be met. [3]

For the remainder of this question assume that these conditions are met. You should assume also that absences in the two departments are independent of each other.

(ii) Find the smallest number of days for which the probability that no employee is absent through illness from the Administration Department is less than 0.01. [2]

Each employee absent on a day represents one 'day of absence'. So, one employee absent for 3 days contributes 3 days of absence, and 5 employees absent on 1 day contribute 5 days of absence.

(iii) Find the probability that, in a 5-day period, the total number of days of absence in the two departments is more than 20. [3]

(iv) Use a suitable approximation, which should be stated together with its parameter(s), to find the probability that, in a 60-day period, the total number of days of absence in the two departments is between 200 and 250 inclusive. [4]

Exercise 811. (9740 N2012/II/5.) (Answer on p. **2146**.)

The probability that a hospital patient has a particular disease is 0.001. A test for the disease has probability p of giving a positive result when the patient has the disease, and equal probability p of giving a negative result when the patient does not have the disease. A patient is given the test.

(i) Given that $p = 0.995$, find the probability that

(a) the result of the test is positive, [2]

(b) the patient has the disease given that the result of the test is positive. [2]

(ii) It is given instead that there is a probability of 0.75 that the patient has the disease given that the result of the test is positive. Find the value of p, giving your answer correct to 6 decimal places. [3]

Exercise 812. (9740 N2012/II/6.) (Answer on p. **2146**.)

On a remote island a zoologist measures the tail lengths of a random sample of 20 squirrels. In a species of squirrel known to her, the tail lengths have mean 14.0 cm. She carries out a test, at the 5% significance level of whether squirrels on the island have the same mean tail length as the species known to her. She assumes that the tail lengths of squirrels on the island are normally distributed with standard deviation 3.8 cm.

(i) State appropriate hypotheses for the test. [1]

The sample mean tail length is denoted by $\bar{x}$ cm.

(ii) Use an algebraic method to calculate the set of values of $\bar{x}$ for which the null hypothesis would not be rejected. (Answers obtained by trial and improvement from a calculator will obtain no marks.) [3]

(iii) State the conclusion of the test in the case where $\bar{x} = 15.8$. [2]

Exercise 813. (9740 N2012/II/7.) (Answer on p. **2146**.)

A group of fifteen people consists of one pair of sisters, one set of three brothers and ten other people. The fifteen people are arranged randomly in a line.

(i) Find the probability that the sisters are next to each other. [2]

(ii) Find the probability that the brothers are *not* all next to each other. [2]

(iii) Find the probability that the sisters are next to each other and the brothers *are* all next to each other. [2]

(iv) Find the probability that *either* the sisters are next to each other or the brothers *are* all next to each other *or* both. [2]

Instead the fifteen people are arranged randomly in a circle.

(v) Find the probability that the sisters are next to each other. [1]

Exercise 814. (9740 N2012/II/8.) (Answer on p. 2147.)

Amy is revising for a mathematics examination and takes a different practice paper each week. Her marks, $y\%$ in week x, are as follows.

Week x	1	2	3	4	5	6
Percentage mark y	38	63	67	75	71	82

(i) Draw a scatter diagram showing these marks. [1]

(ii) Suggest a possible reason why one of the marks does not seem to follow the trend. [1]

(iii) It is desired to predict Amy's marks on future papers. Explain why, in this context, neither a linear nor a quadratic model is likely to be appropriate. [2]

It is decided to fit a model of the form $\ln(L-y) = a + bx$, where L is a suitable constant. The product moment correlation coefficient between x and $\ln(L-y)$ is denoted by r. The following table gives values of r for some possible values of L.

L	91	92	93
r		$-0.929\,944$	$-0.929\,918$

(iv) Calculate the value of r for $L = 91$, giving your answer correct to 6 decimal places. [1]

(v) Use the table and your answer to part **(iv)** to suggest with a reason which of 91, 92 or 93 is the most appropriate value for L. [1]

(vi) Using the value for L, calculate the values of a and b, and use them to predict the week in which Amy will obtain her first mark of at least 90%. [4]

(vii) Give an interpretation, in context, of the value of L. [1]

Exercise 815. (9740 N2012/II/9.) (Answer on p. 2148.)

In an opinion poll before an election, a sample of 30 voters is obtained.

(i) The number of voters in the sample who support the Alliance Party is denoted by A. State, in context, what must be assumed for A to be well modelled by a binomial distribution. [2]

Assume now that A has the distribution $\mathrm{B}(30, p)$.

(ii) Given that $p = 0.15$, find $\mathrm{P}(A = 3 \text{ or } 4)$. [2]

(iii) Given instead that $p = 0.55$, explain whether it is possible to approximate the distribution of A with

(a) a normal distribution,

(b) a Poisson distribution. [3]

(iv) For an unknown value of p it is given that $\mathrm{P}(A = 15) = 0.06864$ correct to 5 decimal places. Show that p satisfies an equation of the form $p(1-p) = k$, where k is a constant to be determined. Hence find the value of p to a suitable degree of accuracy, given that $p < 0.5$. [5]

Exercise 816. (9740 N2012/II/10.) (Answer on p. **2148**.)

Gold coins are found scattered throughout an archaeological site.

(i) State two conditions needed for the number of gold coins found in a randomly chosen region of area 1 square metre to be well modelled by a Poisson distribution. [2]

Assume that the number of gold coins in 1 square metre has the distribution $\text{Po}(0.8)$.

(ii) Find the probability that in 1 square metre there are at least 3 gold coins. [1]

(iii) It is given that the probability that 1 gold coin is found in x square metres is 0.2. Write down an equation for x, and solve it numerically given that $x < 1$. [2]

(iv) Use a suitable approximation to find the probability that in 100 square metres there are at least 90 gold coins. State the parameter(s) of the distribution that you use. [3]

Pottery shards are also found scattered throughout the site. The number of pottery shards in 1 square metre is an independent random variable with the distribution $\text{Po}(3)$. Use suitable approximations, whose parameters should be stated, to find

(v) the probability that in 50 square metres the total number of gold coins and pottery shards is at least 200, [4]

(vi) the probability that in 50 square metres there are at least 3 times as many pottery shards as gold coins. [3]

Exercise 817. (9740 N2011/II/5.) (Answer on p. **2149**.)

The continuous random variable X has the distribution $\text{N}\left(\mu, \sigma^2\right)$. It is known that $\text{P}(X < 40.0) = 0.05$ and $\text{P}(X < 70.0) = 0.975$. Calculate the values of $\mu$and σ. [4]

Exercise 818. (9740 N2011/II/6.) (Answer on p. **2149**.)

It is desired to interview residents of a city suburb about the types of shop to be opened in a new shopping mall. In particular it is necessary to interview a representative range of ages.

(i) Explain how a quota sample might be carried out in this context. [2]

(ii) Explain a disadvantage of quota sampling in the context of your answer to part **(i)**. [1]

(iii) State the name of a method of sampling that would not have this disadvantage, and explain whether it would be realistic to use this method in this context. [2]

Exercise 819. (9740 N2011/II/7.) (Answer on p. **2149**.)

When I try to contact (by telephone) any of my friends in the evening, I know that on average the probability that I succeed is 0.7. On one evening I attempt to contact a fixed number, n, of different friends. If I do not succeed with a particular friend, I do not attempt to contact that friend again that evening. The number of friends whom I succeed in contacting is the random variable R.

(i) State, in the context of this question, two assumptions needed to model R by a binomial distribution. [2]

(ii) Explain why one of the assumptions stated in part **(i)** may not hold in this context. [1]

Assume now that these assumptions do in fact hold.

(iii) Given that $n = 8$, find the probability that R is at least 6. [1]

(iv) Given that $n = 40$, use an appropriate approximation to find $P(R < 25)$. State the parameters of the distribution you use. [4]

Exercise 820. (9740 N2011/II/8.) (Answer on p. **2150**.)

(i) Sketch a scatter diagram that might be expected for the case when x and y are related approximately by $y = a+bx^2$, where a is positive and b is negative. Your diagram should include 5 points, approximately equally spaced with respect to x, and with all x- and y-values positive. [1]

The table gives the values of seven observations of bivariate data, x and y.

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y	18.8	16.9	14.5	11.7	8.6	4.9	0.8

(ii) Calculate the value of the product moment correlation coefficient, and explain why its value does not necessarily mean that the best model for the relationship between x and y is $y = c + dx$. [2]

(iii) Explain how to use the values obtained by calculating product moment correlation coefficients to decide, for this data, whether $y = a + bx^2$ or $y = c + dx$ is the better model. [1]

(iv) It is desired to use the data in the table to estimate the value of y for which $x = 3.2$. Find the equation of the least-squares regression line of y on x^2. Use your equation to calculate the desired estimate. [3]

Exercise 821. (9740 N2011/II/9.) (Answer on p. **2150**.)

Camera lenses are made by two companies, A and B. 60% of all lenses are made by A and the remaining 40% by B. 5% of the lenses made by A are faulty. 7% of the lenses made by B are faulty. *(Author's remark: Assume that there are infinitely many lenses.)*

(i) One lens is selected at random. Find the probability that

(a) it is faulty, [2]

(b) it was made by A, given that it is faulty. [1]

(ii) Two lenses are selected at random. Find the probability that

(a) exactly one of them is faulty, [2]

(b) both were made by A, given that exactly one is faulty. [3]

Exercise 822. (9740 N2011/II/10.) (Answer on p. **2150**.)

In a factory, the time in minutes for an employee to install an electronic component is a normally distributed continuous random variable T. The standard deviation of T is 5.0 and under ordinary conditions the expected value of T is 38.0. After background music is introduced into the factory, a sample of n components is taken and the mean time taken for randomly chosen employees to install them is found to be $\bar{t}$ minutes. A test is carried out, at the 5% significance level, to determine whether the mean time taken to install a component has been reduced.

(i) State appropriate hypotheses for the test, defining any symbols you use. [2]

(ii) Given that $n = 50$, state the set of values of $\bar{t}$ for which the result of the test would be to reject the null hypothesis. [3]

(iii) It is given instead that $\bar{t} = 37.1$ and the result of the test is that the null hypothesis is not rejected. Obtain an inequality involving n, and hence find the set of values that n can take. [4]

Exercise 823. (9740 N2011/II/11.) (Answer on p. **2151**.)

A committee of 10 people is chosen at random from a group consisting of 18 women and 12 men. The number of women on the committee is denoted by R.

(i) Find the probability that $R = 4$. [3]

(ii) The most probable number of women on the committee is denoted by r. By using the fact that $\mathrm{P}(R = r) > \mathrm{P}(R = r + 1)$, show that r satisfies the inequality

$$(r+1)!\,(17-r)!\,(9-r)!\,(r+3)! > r!\,(18-r)!\,(10-r)!\,(r+2)!$$

and use this inequality to find the value of r. [5]

Exercise 824. (9740 N2011/II/12.) (Answer on p. **2152**.)

The number of people joining an airport check-in queue in a period of 1 minute is a random variable with the distribution Po (1.2).

(i) Find the probability that, in a period of 4 minutes, at least 8 people join the queue. [1]

(ii) The probability that no more than 1 person joins the queue in a period of t seconds is 0.7. Find an equation for t. Hence find the value of t, giving your answer correct to the nearest whole number. [4]

(iii) The number of people leaving the same queue in a period of 1 minute is a random variable with the distribution Po (1.8). At 0930 on a certain morning there are 35 people in the queue. Use appropriate approximations to find the probability that by 0945 there are at least 24 people in the queue, stating the parameters of any distributions that you use. (You may assume that the queue does not become empty during this period.) [5]

(iv) Explain why a Poisson model would probably not be valid if applied to a time period of several hours. [1]

Exercise 825. (9740 N2010/II/5.) (Answer on p. **2152**.)

At an international athletics competition, it is desired to sample 1% of the spectators to find their opinions of the catering facilities.

(i) Give a reason why it would be difficult to use a stratified sample. [1]

(ii) Explain how a systematic sample could be carried out. [2]

Exercise 826. (9740 N2010/II/6.) (Answer on p. **2152**.)

The time required by an employee to complete a task is a normally distributed random variable. Over a long period it is known that the mean time required is 42.0 minutes. Background music is introduced in the workplace, and afterwards the time required, t minutes, is measured for a random sample of 11 employees. The results are summarised as follows.

$$n = 11, \qquad \sum t = 454.3, \qquad \sum t^2 = 18\,779.43.$$

(i) Find unbiased estimates of the population mean and variance.

(ii) Test, at the 10% significance level, whether there has been a change in the mean time required by an employee to complete the task. [7]

Exercise 827. (9740 N2010/II/7.) (Answer on p. **2153**.)

For events A and B it is given that $\mathrm{P}(A) = 0.7$, $\mathrm{P}(B) = 0.6$ and $\mathrm{P}(A|B') = 0.8$. Find

(i) $\mathrm{P}(A \cap B')$, [2]

(ii) $\mathrm{P}(A \cup B)$, [2]

(iii) $\mathrm{P}(B'|A)$. [2]

For a third event C, it is given that $\mathrm{P}(C) = 0.5$ and that A and C are independent.

(iv) Find $\mathrm{P}(A' \cap C)$. [2]

(v) Hence state an inequality satisfied by $\mathrm{P}(A' \cap B \cap C)$.[563] [1]

Exercise 828. (9740 N2010/II/8.) (Answer on p. **2153**.)

The digits 1, 2, 3, 4 and 5 are arranged randomly to form a five-digit number. No digit is repeated. Find the probability that

(i) the number is greater than 30 000, [1]

(ii) the last two digits are both even, [2]

(iii) the number is greater than 30 000 and odd. [4]

Exercise 829. (9740 N2010/II/9.) (Answer on p. **2153**.)

In this question you should state clearly the values of the parameters of any normal distribution you use.

Over a three-month period Ken makes X minutes of peak-rate telephone calls and Y minutes of cheap-rate calls. X and Y are independent random variables with the distributions $\mathrm{N}\left(180, 30^2\right)$ and $\mathrm{N}\left(400, 60^2\right)$ respectively.

(i) Find the probability that, over a three-month period, the number of minutes of cheap-rate calls made by Ken is more than twice the number of minutes of peak-rate calls. [4]

Peak-rate calls cost \$0.12 per minute and cheap-rate calls cost \$0.05 per minute.

(ii) Find the probability that, over a three-month period, the total cost of Ken's calls is greater than \$45. [3]

(iii) Find the probability that the total cost of Ken's peak-rate calls over two independent three-month periods is greater than \$45. [3]

[563] This question is terribly vague. A trivial but perfectly correct answer would be $\mathrm{P}(A' \cap B \cap C) \geq 0$, but I suspect that any smart aleck who wrote this didn't get the mark.

Exercise 830. (9740 N2010/II/10.) (Answer on p. **2153**.)

A car is placed in a wind tunnel and the drag force F for different wind speeds v, in appropriate units, is recorded. The results are shown in the table.

v	0	4	8	12	16	20	24	28	32	36
F	0	2.5	5.1	8.8	11.2	13.6	17.6	22.0	27.8	33.9

(i) Draw the scatter diagram for these values, labelling the axes clearly. [2]

It is thought that the drag force F can be modelled by one of the formulae

$$F = a + bv \qquad \text{or} \qquad F = c + dv^2$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) v and F,

(b) v^2 and F. [2]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $F = a + bv$ or $F = c + dv^2$ is the better model. [1]

(iv) It is required to estimate the value of v for which $F = 26.0$. Find the equation of a suitable regression line, and use it to find the required estimate. Explain why neither the model $F = a + bv$ nor the model $F = c + dv^2$ should be used.[564] [4]

Exercise 831. (9740 N2010/II/11.) (Answer on p. **2154**.)

In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters.

The number of telephone calls received by a call centre in one minute is a random variable with distribution Po (3).

(i) Find the probability that exactly 8 calls are received in a randomly chosen period of 4 minutes. [2]

(ii) Find the length of time, to the nearest second, for which the probability that no calls are received is 0.2. [3]

(iii) Use a suitable approximation to find the probability that, on a randomly chosen working day of 12 hours, more than 2200 calls are received. [4]

A working day of 12 hours on which more than 2200 calls are received is said to be 'busy'.

(iv) Find the probability that, in six randomly chosen working days, exactly two are busy. [2]

(v) Use a suitable approximation to find the probability that, in 30 randomly chosen working days of 12 hours, fewer than 10 are busy. [4]

[564] I have changed the wording of this sentence slightly.

Exercise 832. (9740 N2009/II/5.) (Answer on p. **2155**.)

A cinema manager wishes to take a survey of opinions of cinema-goers. Describe how a quota sample of size 100 might be obtained, and state one disadvantage of quota sampling. [3]

Exercise 833. (9740 N2009/II/6.) (Answer on p. **2155**.)

The table gives the world record time, in seconds above 3 minutes 30 seconds, for running 1 mile as at 1st January in various years.

Year, x	1930	1940	1950	1960	1970	1980	1990	2000
Time, t	40.4	36.4	31.3	24.5	21.1	19.0	16.3	13.1

(i) Draw a scatter diagram to illustrate the data. [2]

(ii) Comment on whether a linear model would be appropriate, referring both to the scatter diagram and the context of the question. [2]

(iii) Explain why in this context a quadratic model would probably not be appropriate for long-term predictions. [1]

(iv) Fit a model of the form $\ln t = a + bx$ to the data and use it to predict the world record time as at 1st January 2010. Comment on the reliability of your prediction. [3]

Exercise 834. (9740 N2009/II/7.) (Answer on p. **2155**.)

A company buys $p\%$ of its electronic components from supplier A and the remaining $(100 - p)\%$ from supplier B. The probability that a randomly chosen component supplied by A is faulty is 0.05. The probability that a randomly chosen component supplied by B is faulty is 0.03.

(i) Given that $p = 25$, find the probability that a randomly chosen component is faulty. [2]

(ii) For a general value of p, the probability that a randomly chosen component that is faulty was supplied by A is denoted by $f(p)$. Show that $f(p) = \dfrac{0.05p}{0.02p + 3}$. Prove by differentiation that f is an increasing function for $0 \le p \le 100$, and explain what this statement means in the context of the question. [6]

Exercise 835. (9740 N2009/II/8.) (Answer on p. **2156**.)

Find the number of ways in which the letters of the word ELEVATED can be arranged if

(i) there are no restrictions, [1]

(ii) T and D must not be next to one another, [2]

(iii) consonants (L, V, T, D) and vowels (E, A) must alternate, [3]

(iv) between any two Es there must be at least 2 other letters. [3]

Exercise 836. (9740 N2009/II/9.) (Answer on p. **2156**.)

The thickness in cm of a mechanics textbook is a random variable with the distribution $\mathrm{N}\left(2.5, 0.1^2\right)$.

(i) The mean thickness of n randomly chosen mechanics textbooks is denoted by $\bar{M}$ cm. Given that $P\left(\bar{M} > 2.53\right) = 0.0668$, find the value of n. [3]

The thickness in cm of a statistics textbook is a random variable with the distribution $\mathrm{N}\left(2.0, 0.08^2\right)$.

(ii) Calculate the probability that 21 mechanics textbooks and 24 statistics textbooks will fit into a bookshelf of length 1 m. State clearly the mean and variance of any normal distribution you use in your calculation. [3]

(iii) Calculate the probability that the total thickness of 4 statistics textbooks is less than three times the thickness of 1 mechanics textbook. State clearly the mean and variance of any normal distribution you use in your calculation. [3]

(iv) State an assumption needed for your calculation in parts **(ii)** and **(iii)**. [1]

Exercise 837. (9740 N2009/II/10.) (Answer on p. **2156**.)

A company supplies sugar in small packets. The mass of sugar in one packet is denoted by X grams. The masses of a random sample of 9 packets are summarised by

$$\sum x = 86.4, \quad \sum x^2 = 835.92.$$

(i) Calculate unbiased estimates of the mean and variance of X. [2]

The mean mass of sugar in a packet is claimed to be 10 grams. The company directors want to know whether the sample indicates that this claim is incorrect.

(ii) Stating a necessary assumption, carry out a t-test at the 5% significance level. Explain why the Central Limit Theorem does not apply in this context. [7]

(iii) Suppose now that the population variance of X is known, and that the assumption made in part **(ii)** is still valid. What change would there be in carrying out the test? [1]

Exercise 838. (9740 N2009/II/11.) (Answer on p. **2157**.)

A fixed number, n, of cars is observed and the number of those cars that are red is denoted by R.

(i) State, in context, two assumptions needed for R to be well modelled by a binomial distribution. [2]

Assume now that R has the distribution $\mathrm{B}(n, p)$. [2]

(ii) Given that $n = 20$ and $p = 0.15$, find $\mathrm{P}(4 \leq R < 8)$. [2]

(iii) Given that $n = 240$ and $p = 0.3$, find $\mathrm{P}(R < 60)$ using a suitable approximation, which should be clearly stated.

(iv) Given that $n = 240$ and $p = 0.02$, find $\mathrm{P}(R = 3)$ using a suitable approximation, giving your answer correct to 4 decimal places and explaining why the approximation is appropriate in this case. [3]

(v) Given that $n = 20$ and $\mathrm{P}(R = 0 \text{ or } 1) = 0.2$, write down an equation for the value of p, and find this value numerically. [2]

Exercise 839. (9740 N2008/II/5.) (Answer on p. **2157**.)

A school has 950 pupils.

(i) A sample of 50 pupils is to be chosen to take part in a survey. Describe how the sample could be chosen using systematic sampling. [2]

The purpose of the survey is to investigate pupils' opinions about the sports facilities available at the school.

(ii) Give a reason why a stratified sample might be preferable in this context. [2]

Exercise 840. (9740 N2008/II/6.) (Answer on p. **2158**.)

In mineral water from a certain source, the mass of calcium, X mg, in a one-litre bottle is a normally distributed random variable with mean μ. Based on observations over a long period, it is known that $\mu = 78$. Following a period of extreme weather, 15 randomly chosen bottles of the water were analysed. The masses of calcium in the bottles are summarised by

$$\sum x = 1\,026.0, \qquad \sum x^2 = 77\,265.90.$$

Test, at the 5% significance level, whether the mean mass of calcium in a bottle has changed. [6]

Exercise 841. (9740 N2008/II/7.) (Answer on p. 2158.)

A computer game simulates a tennis match between two players, A and B. The match consists of at most three sets. Each set is won by either A or B, and the match is won by the first player to win two sets.

The simulation uses the following rules.

- The probability that A wins the first set is 0.6.
- For each set after the first, the conditional probability that A wins that set, given that A won the preceding set, is 0.7.
- For each set after the first, the conditional probability that B wins that set, given that B won the preceding set, is 0.8.

Calculate the probability that

(i) A wins the second set, [2]

(ii) A wins the match, [3]

(iii) B won the first set, given that A wins the match. [3]

Exercise 842. (9740 N2008/II/8.) (Answer on p. 2158.)

A certain metal discolours when exposed to air. To protect the metal against discolouring, it is treated with a chemical. In an experiment, different quantities, x ml, of the chemical were applied to standard samples of the metal, and the times, t hours, for the metal to discolour were measured. The results are given in the table.

x	1.2	2.0	2.7	3.8	4.8	5.6	6.9
t	2.2	4.5	5.8	7.3	7.6	9.0	9.9

(i) Calculate the product moment correlation coefficient between x and t, and explain whether your answer suggests that a linear model is appropriate. [3]

(ii) Draw a scatter diagram for the data. [1]

One of the values t appears to be incorrect.

(iii) Indicate the corresponding point on your diagram by labelling it P, and explain why the scatter diagram for the remaining points may be consistent with a model of the form $t = a + b\ln x$. [2]

(iv) Omitting P, calculate least square estimates of a and b for the model $t = a + b\ln x$. [2]

(v) Estimate the value of t at the value of x corresponding to P. [1]

(vi) Comment on the use of the model in part **(iv)** in predicting the value of t when $x = 8.0$. [1]

Exercise 843. (9740 N2008/II/9.) (Answer on p. **2159**.)

A shop sells two types of piano, 'grand' and 'upright'. The mean number of grand pianos sold in a week is 1.8.

(i) Use a Poisson distribution to find the probability that in a given week at least 4 grand pianos are sold. [2]

The mean number of upright pianos sold in a week is 2.6. The sales of the two types of piano is independent.

(ii) Use a Poisson distribution to find the probability that in a given week the total number of pianos sold is exactly 4. [2]

(iii) Use a normal approximation to the Poisson distribution to find the probability that the number of grand pianos sold in a year of 50 weeks is less than 80. [4]

(iv) Explain why the Poisson distribution may not be a good model for the number of grand pianos sold in a year. [2]

Exercise 844. (9740 N2008/II/10.) (Answer on p. **2159**.)

A group of diplomats is to be chosen to represent three islands, K, L and M. The group is to consist of 8 diplomats and is chosen from a set of 12 diplomats consisting of 3 from K, 4 from L and 5 from M. Find the number of ways in which the group can be chosen if it includes

(i) 2 diplomats from K, 3 from L and 3 from M, [2]

(ii) diplomats from L and M only, [2]

(iii) at least 4 diplomats from M, [2]

(iv) at least 1 diplomat from each island. [4]

Exercise 845. (9740 N2008/II/11.) (Answer on p. **2160**.)

The random variable X has the distribution $\mathrm{N}\left(50, 8^2\right)$. Given that X_1 and X_2 are two independent observations of X, find

1. $\mathrm{P}\left(X_1 + X_2 > 120\right)$, [2]
2. $\mathrm{P}\left(X_1 > X_2 + 15\right)$. [3]

The random variable Y is related to X by the formula $Y = aX + b$, where a and b are constants with $a > 0$.

3. Given that $\mathrm{P}(Y < 74) = \mathrm{P}(Y > 146) = 0.0668$, find the values of $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$, and hence find the values of a and b. [7]

Exercise 846. (9233 N2008/I/1.) (Answer on p. **2160**.)

On a bookshelf there are 15 different books; 6 have red covers, 5 have blue covers and 4 have green covers. All the red books are to be kept together, all the blue books are to be kept together and all the green books are to be kept together. In how many ways can the 15 books be arranged on the bookshelf? [3]

Exercise 847. (9233 N2008/II/23.) (Answer on p. **2160**.)

The events A, B and C are such that $\mathrm{P}(A) = 0.2$, $\mathrm{P}(C) = 0.4$, $\mathrm{P}(A \cup B) = 0.4$ and $\mathrm{P}(B \cap C) = 0.1$. Given that A and B are independent, find $\mathrm{P}(B)$ and show that B and C are also independent. [4]

Exercise 848. (9233 N2008/II/26.) (Answer on p. **2160**.)

The number of times that an office photocopying machine breaks down in a week follows a Poisson distribution with mean 3. Find the probability that

(i) the machine will break down more than twice in a given week, [2]

(ii) the machine will break down at most three times in a period of four weeks. [3]

(iii) Use a suitable approximation to find the probability that the machine will break down more than 50 times in a period of 16 weeks. [4]

Exercise 849. (9233 N2008/II/27.) (Answer on p. **2160**.)

The masses of a certain type of electronic component produced by a machine are normally distributed with mean 32.40 g. The machine is adjusted and a sample of 80 components is now taken and is found to have a mean mass 32.00 g. The unbiased estimate of the population variance, calculated from this sample, is $2.892\,\mathrm{g}^2$.

(i) Test at the 5% significance level whether this indicates a change in the mean. [5]

(ii) Explain what you understand by the phrase 'at the 5% significance' in the context of this question. [2]

(iii) Find the least level of significance at which this sample would indicate a decrease in the population mean. [3]

Exercise 850. (9233 N2008/II/29.) (Answer on p. **2161**.)

Mr Sim and Mr Lee work in the same office and are expected to arrive by 9 a.m. each day. Both men drive to work.

(i) The time taken for Mr Sim's journey follows a normal distribution with mean 50 minutes and standard deviation 4 minutes. Given that he regularly leaves home at 8.05 a.m., find the probability that he will be late no more than once in a working week of 5 days. [5]

(ii) Mr Lee's journey time follows a normal distribution with mean 40 minutes and standard deviation 5 minutes. Mr Lee leaves home at 8.10 a.m. each day. Find the probability that Mr Sim will arrive at work before Mr Lee on any particular day. [5]

(iii) Find the probability that in a working week of 5 days, Mr Sim arrives at work before Mr Lee on at least 3 days. [2]

Exercise 851. (9233 N2008/II/30.) (Answer on p. **2161**.)

(i) The masses of valves produced by a machine are normally distributed with mean μ and standard deviation σ. 12% of the valves have mass less than 86.50 g and 20% have mass more than 92.25 g. Find μ and σ. [4]

(ii) The setting of the machine is adjusted so that the mean mass of the valves produced is unchanged, but the standard deviation is reduced. Given that 80% of the valves now have a mass within 2 g of the mean, find the new standard deviation. [3]

(iii) After the machine has been adjusted, a random sample of n valves is taken. Find the smallest value of n such that the probability that the sample mean exceeds μ by at least 0.50 g is at most 0.1. [5]

Exercise 852. (9740 N2007/II/5.) (Answer on p. **2161**.)

(i) Give a real-life example of a situation in which quota sampling could be used. Explain why quota sampling would be appropriate in this situation, and describe briefly any disadvantage that quota sampling has. [4]

(ii) Explain briefly whether it would be possible to use stratified sampling in the situation you have described in part **(i)**. [1]

Exercise 853. (9740 N2007/II/6.) (Answer on p. **2162**.)

In a large population, 24% have a particular gene A, and 0.3% have gene B. Find the probability that, in a random sample of 10 people from the population, at most 4 have gene A. [2]

A random sample of 1000 people is taken from the population. Using appropriate approximations, find

(i) the probability that between 230 and 260 inclusive have gene A, [3]

(ii) the probability that at least 2 but fewer than 5 have gene B. [2]

Exercise 854. (9740 N2007/II/7.) (Answer on p. **2162**.)

A large number of students in a college have completed a geography project. The time, x hours, taken by a student to complete the project is noted for a random sample of 150 students. The results are summarised by

$$\sum x = 4626, \quad \sum x^2 = 147691.$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 5% significance level, whether the population mean time for a student to complete the project exceeds 30 hours. [4]

(iii) State giving a valid reason, whether any assumptions about the population are needed in order for the test to be valid. [1]

Exercise 855. (9740 N2007/II/8.) (Answer on p. **2162**.)

Chickens and turkeys are sold by weight. The masses, in kg, of chickens and turkeys are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean Mass	Standard Deviation
Chickens	2.2	0.5
Turkeys	10.5	2.1

Chickens are sold at \$3 per kg and turkeys at \$5 per kg.

(i) Find the probability that a randomly chosen chicken has a selling price exceeding \$7. [2]

(ii) Find the probability of the event that both a randomly chosen chicken has a selling price exceeding \$7 and a randomly chosen turkey has a selling price exceeding \$55. [3]

(iii) Find the probability that the total selling price of a randomly chosen chicken and a randomly chosen turkey is more than \$62. [4]

(iv) Explain why the answer to part **(iii)** is greater than the answer to part **(ii)**. [1]

Exercise 856. (9740 N2007/II/9.) (Answer on p. **2163**.)

A group of 12 people consists of 6 married couples.

(i) The group stand in a line.

- **(a)** Find the number of different possible orders. [1]
- **(b)** Find the number of different possible orders in which each man stands next to his wife. [3]

(ii) The group stand in a circle.

- **(a)** Find the number of different possible arrangements. [1]
- **(b)** Find the number of different possible arrangements if men and women alternate. [2]
- **(c)** Find the number of different possible arrangements if each man stands next to his wife and men and women alternate. [2]

Exercise 857. (9740 N2007/II/10.) (Answer on p. **2163**.)

A player throws three darts at a target. The probability that he is successful in hitting the target with his first throw is $\frac{1}{8}$. For each of his second and third throws, the probability of success is

- twice the probability of success on the preceding throw if that throw was successful,
- the same as the probability of success on the preceding throw if that throw was unsuccessful.

Construct a probability tree showing this information. [3]

Find

(i) the probability that all three throws are successful, [2]

(ii) the probability that at least two throws are successful, [2]

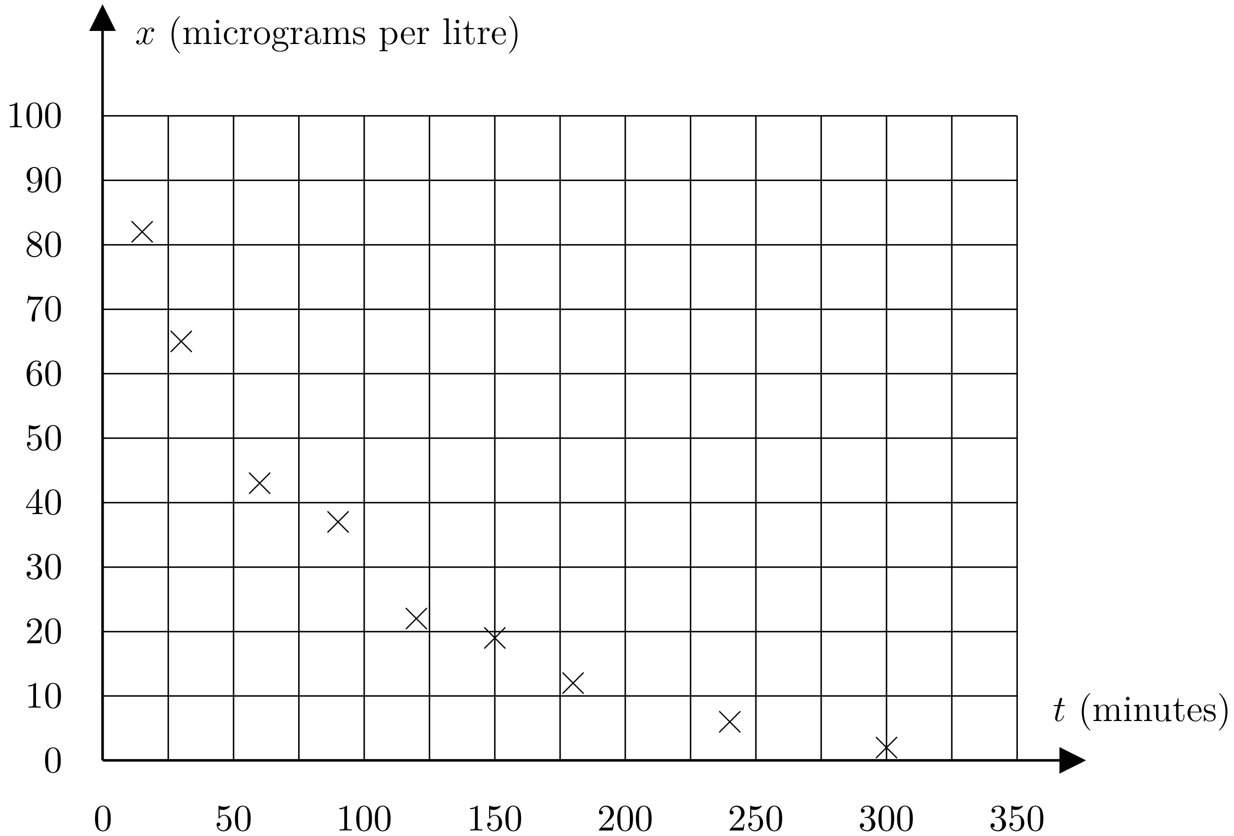
(iii) the probability that the third throw is successful given that exactly two of the three throws are successful. [4]

Exercise 858. (9740 N2007/II/11.) (Answer on p. **2163**.)

Research is being carried out into how the concentration of a drug in the bloodstream varies with time, measured from when the drug is given. Observations at successive times give the data shown in the following table.

Time (t minutes)	15	30	60	90	120	150	180	240	300
Concentration (x micrograms per litre)	82	65	43	37	22	19	12	6	2

It is given that the value of the product moment correlation coefficient for this data is −0.912, correct to 3 decimal places. The scatter diagram for the data is shown below.



(i) Calculate the equation of the regression line of x on t. [2]

(ii) Calculate the corresponding estimated value of x when $t = 300$, and comment on the suitability of the linear model. [2]

The variable y is defined by $y = \ln x$. For the variables y and t,

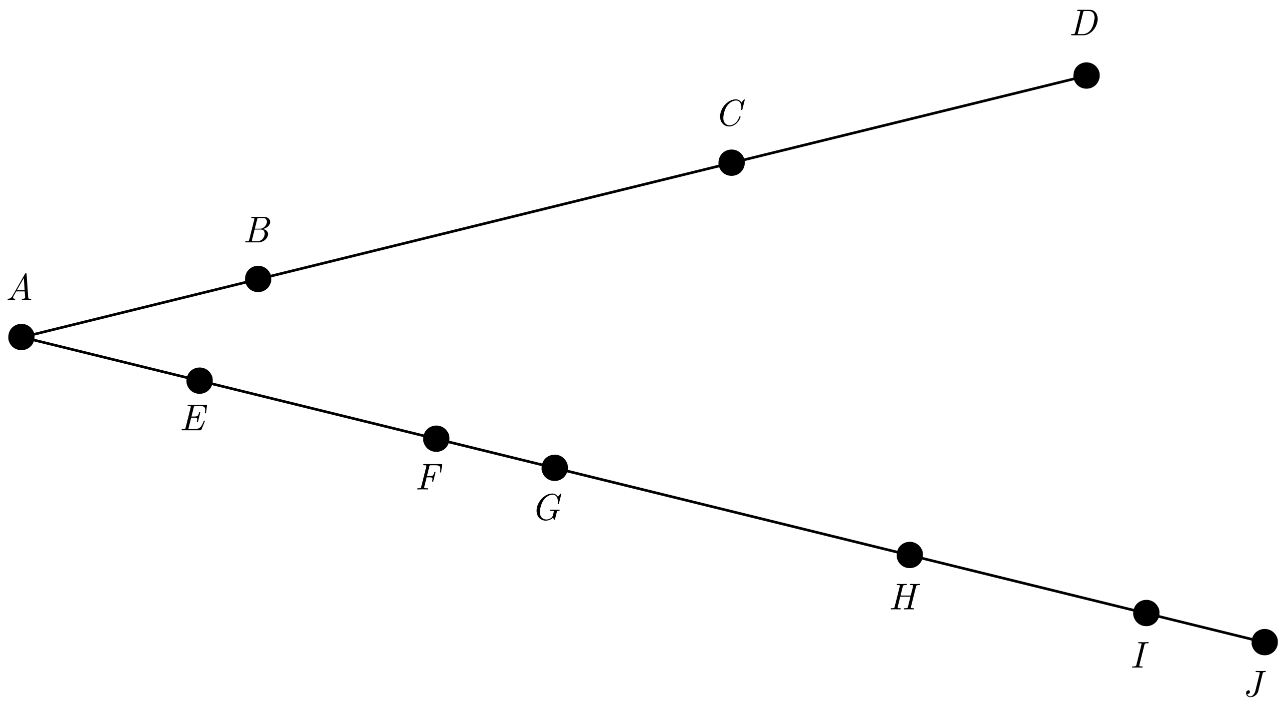
(iii) calculate the product moment correlation coefficient and comment on its value, [2]

(iv) calculate the equation of the appropriate regression line. [3]

(v) Use a regression line to give the best estimate that you can of the time when the drug concentration is 15 micrograms per litre. [2]

Exercise 859. (9233 N2007/I/4.) (Answer on p. **2164**.)

The diagram shows two straight lines, $ABCD$ and $AEFGHIJ$, which intersect at A. Triangles are to be drawn using three of the points A, B, C, D, E, F, G, H, I, J as vertices.



(i) How many different triangles can be drawn which have the point A as one of the vertices? [1]

(ii) How many different triangles in total can be drawn? [4]

Exercise 860. (9233 N2007/II/23.) (Answer on p. **2164**.)

(i) A random sample of size 100 is taken from a population with mean 30 and standard deviation 5. Find an approximate value for the probability that the sample mean lies between 29.2 and 30.8. [6]

(ii) Giving a reason, state whether it is necessary to make any assumptions about the distribution of the population. [1]

Exercise 861. (9233 N2007/II/25.) (Answer on p. **2164**.)

The numbers of men and women studying Chemistry, Physics and Biology at a college are given in the following table.

	Chemistry	Physics	Biology
Men	12	16	32
Women	8	12	20

One of these students is chosen at random by a researcher. Events M, W, C and B are defined as follows.

M: the student chosen is a man

W: the student chosen is a woman.

C: the student chosen is studying Chemistry

B: the student chosen is studying Biology

Find

(i) $\mathrm{P}(W|B)$. [1]

(ii) $\mathrm{P}(B|W)$. [1]

(iii) $\mathrm{P}(B \cup W)$. [2]

State, with a reason in each case, whether W and B are independent, and whether M and C are mutually exclusive. [4]

Exercise 862. (9233 N2007/II/26.) (Answer on p. **2164**.)

At a fire station, each call-out is classified as either genuine or false. Call-outs occur at random times. On average, there are two genuine call-outs in a week, and one false call-out in a two-week period.

(i) Calculate the probability that there are fewer than 6 genuine call-outs in a randomly chosen two-week period. [?]

(ii) Using a suitable approximation, calculate the probability that the total number of call-outs in a randomly chosen six-week period exceeds 19. [?]

Exercise 863. (9233 N2007/II/27.) (Answer on p. **2165**.)

An oil mixture is produced by mixing L litres of light oil with H litres of heavy oil. The random variables L and H are independent normal variables. The expected value of L is 5 and its standard deviation is 0.1. The expected value of H is 3 and its standard deviation is 0.05.

(i) Find the probability that the volume of the mixture lies between 7.9 litres and 8.2 litres. [6]

The density of light oil is 0.74 kilograms per litre, and the density of heavy oil is 0.86 kilograms per litre.

(ii) Find the probability that the mass of the mixture lies between 6.1 kg and 6.2 kg. [6]

[Density is defined by Density = $\dfrac{\text{Mass}}{\text{Volume}}$.]

Exercise 864. (9233 N2006/I/4.) (Answer on p. **2165**.)

A box contains 8 balls, of which 3 are identical (and so are indistinguishable from one another) and the other 5 are different from each other.[565] 3 balls are to be picked out of the box; the order in which they are picked out does not matter. Find the number of different possible selections of 3 balls. [4]

Exercise 865. (9233 N2006/II/23.) (Answer on p. **2165**.)

Two fair dice, one red and the other green, are thrown.

A is the event: The score on the red die is divisible by 3.

B is the event: The sum of two scores is 9.

(i) Justifying your conclusion, determine whether A and B are independent. [3]

(ii) Find $\mathrm{P}(A \cup B)$. [2]

Exercise 866. (9233 N2006/II/25.) (Answer on p. **2165**.)

The mass of vegetables in a randomly chosen bag has a normal distribution. The mass of the contents of a bag is supposed to be 10 kg. A random sample of 80 bags is taken and the mass of the contents of each bag, x grams, is measured. The data are summarised by

$$\sum (x - 10\,000) = -2\,510, \quad \sum (x - 10\,000)^2 = 2\,010\,203.$$

(i) Test, at the 5% significance level, whether the mean mass of the contents of a bag is less than 10 kg. [7]

(ii) Explain, in the context of the question, the meaning of 'at the 5% significance level'. [1]

Exercise 867. (9233 N2006/II/26.) (Answer on p. **2165**.)

In a weather model, severe floods are assumed to occur at random intervals, but at an average rate of 2 per 100 years.

(i) Using this model, find the probability that, in a randomly chosen 200-year period, there is exactly one severe flood in the first 100 years and exactly one severe flood in the second 100 years. [3]

(ii) Using the same model, and a suitable approximation, find the probability that there are more than 25 severe floods in 1000 years. [5]

[565] Assume also that each of the latter 5 balls is different from each of the first 3.

Exercise 868. (9233 N2006/II/28.) (Answer on p. 2166.)

Observations are made of the speeds of cars on a particular stretch of road during daylight hours. It is found that, on average, 1 in 80 cars is travelling at a speed exceeding $125\,\mathrm{km\,h^{-1}}$, and 1 in 10 is travelling at a speed less than $40\,\mathrm{km\,h^{-1}}$.

(i) Assuming a normal distribution, find the mean and the standard deviation of this distribution. [4]

(ii) A random sample of 10 cars is to be taken. Find the probability that at least 7 will be travelling at a speed in excess of $40\,\mathrm{km\,h^{-1}}$. [3]

(iii) A random sample of 100 cars is to be taken. Using a suitable approximation, find the probability that at most 8 cars will be travelling at a speed less than $40\,\mathrm{km\,h^{-1}}$. [3]

139. All Past-Year Questions, Listed and Categorised

There are in total 200 points, so each point accounts for 0.5% of your final A-Level grade. Below I list points for curveball questions[566] in red.

The clickable four-digit numbers are the page numbers for the Questions and Answers.

139.1. 2019 (9758)

[566]See Preface/Rant—p. xlix.

139.3. 2017 (9758)

There are in total 200 points, so each point accounts for 0.5% of your final A-Level grade. Below I list points for curveball questions[567] in red.

The clickable four-digit numbers are the page numbers for the Questions and Answers.

[567]See Preface/Rant—p. xlix.

Paper I: Pure Mathematics [100]					
	Q	**A**	**Part**	**Topics**	**Points**
1	1422	2088	Calc.	Maclaurin	$4 = \mathbf{4}$
2	1374	1994	F&G	graphs, absolute value	$2 + 4 = \mathbf{6}$
3	1422	2088	Calc.	differentiation, stationary points, turning points, maximum, minimum	$4 + 3 = \mathbf{7}$
4	1375	1995	F&G	conic sections, differentiation, asymptotes, transformations	$3 + 3 + 2 = \mathbf{8}$
5	1375	1995	F&G	factorisation, Remainder Theorem, differentiation, quadratic	$4 + 3 + 3 = \mathbf{10}$
6	1402	2044	Vectors	vector equations, lines, planes	$2 + 3 + 3 = \mathbf{8}$
7	1422	2089	Calc.	integration, trigonometry	$3 + 5 = \mathbf{8}$
8	1410	2058	Complex	quadratic, factorisation	$3 + 4 + 3 = \mathbf{10}$
9	1390	2024	S&S	summation, limits, Maclaurin	$3 + 2 + 4 + 4 = \mathbf{13}$
10	1402	2044	Vectors	vector equations, lines, scalar product, quadratic	$4 + 4 + 5 = \mathbf{13}$
11	1422	2089	Calc.	differential equations	$1 + 3 + 5 + 4 = \mathbf{13}$
Paper II, Section A: Pure Mathematics [40]					
1	1375	1996	F&G	parametric, differentiation, equations, points	$3 + 5 = \mathbf{8}$
2	1390	2024	S&S	arithmetic progression, geometric progression	$2 + 4 + 3 = \mathbf{9}$
3	1376	1996	F&G	inverse, composite functions transformations, conic sections	$4 + 2 + 4 + 2 = \mathbf{12}$
4	1423	2090	Calc.	graph, quadratic, integration, volume	$4 + 4 + 3 = \mathbf{11}$
Section B: Probability and Statistics [60]					
5	1459	2136	P&S		$3 + 2 + 2 = \mathbf{7}$
6	1459	2136	P&S		$2 + 3 + 4 = \mathbf{9}$
7	1459	2136	P&S		$1 + 2 + 5 + 2 = \mathbf{10}$
8	1460	2136	P&S		$3 + 2 + 3 + 2 = \mathbf{10}$
9	1461	2136	P&S		$2 + 1 + 1 + 2 + 1 + 3 + 1 + 1 = \mathbf{12}$
10	1461	2136	P&S		$1 + 3 + 4 + 4 = \mathbf{12}$

139.4. 2016 (9740)

Paper I: Pure Mathematics [100]					
	Q	A	Part	Topics	Points
1	1376	1996	F&G	inequalities	$2+3=\mathbf{5}$
2	1423	2091	Calc.	differentiation, calculator	$2+3=\mathbf{5}$
3	1376	1997			$2+4=\mathbf{6}$
4	1390	2025			$4+3=\mathbf{7}$
5	1403	2045			$2+4+2=\mathbf{8}$
6	1391	2024			$5+2+3=\mathbf{10}$
7	1411	2058			$5+5=\mathbf{10}$
8	1423	2091			$5+3+3=\mathbf{11}$
9	1424	2092			$1+6+3+2=\mathbf{12}$
10	1377	1998			$3+5+3+2=\mathbf{13}$
11	1403	2046			$5+5+3=\mathbf{13}$
Paper II, Section A: Pure Mathematics [40]					
1	1424	2093			$7=\mathbf{7}$
2	1425	2093			$3+2+5=\mathbf{10}$
3	1425	2094			$4+3+4=\mathbf{11}$
4	1411	2059			$2+4+3+3=\mathbf{12}$
Section B: Probability and Statistics [60]					
5	1462	2136	P&S		$2+1+2=\mathbf{5}$
6	1463	2136	P&S		$1+1+1+4+3=\mathbf{10}$
7	1463	2136	P&S		$1+3+3+3=\mathbf{10}$
8	1464	2136	P&S		$2+1+2+3+3=\mathbf{11}$
9	1464	2136	P&S		$2+4+4=\mathbf{10}$
10	1465	2136	P&S		$2+2+2+4+4=\mathbf{14}$

139.5. 2015 (9740)

	Q	A	Part	Topics	Points
Paper I: Pure Mathematics [100]					
1	1377	1998			$4+2+1=\mathbf{7}$
2	1377	1999			$3+3=\mathbf{6}$
3	1425	2095			$2+3=\mathbf{5}$
4	1426	2096			$6=\mathbf{6}$
5	1378	2000			$2+3+2=\mathbf{7}$
6	1426	2096			$2+6=\mathbf{8}$
7	1403	2047			$2+3+5=\mathbf{10}$
8	1391	2026			$4+4+3=\mathbf{11}$
9	1411	2059			$5+4+4=\mathbf{13}$
10	1426	2097			$4+2+6=\mathbf{12}$
11	1427	2098			$3+6+3+3=\mathbf{15}$
Paper II, Section A: Pure Mathematics [40]					
1	1427	2099			$1+5=\mathbf{6}$
2	1404	2048			$2+5+3=\mathbf{10}$
3	1378	2001			$2+3+5=\mathbf{10}$
4	1391	2026			$6+1+4+3=\mathbf{14}$
Section B: Probability and Statistics [60]					
5	1465	2136	P&S		$1+2+1=\mathbf{4}$
6	1465	2136	P&S		$2+3+2=\mathbf{7}$
7	1466	2137	P&S		$2+3+2=\mathbf{7}$
8	1466	2137	P&S		$7=\mathbf{7}$
9	1466	2137	P&S		$1+3+4=\mathbf{8}$
10	1467	2138	P&S		$1+3+3+2=\mathbf{9}$
11	1467	2139	P&S		$2+1+2+4=\mathbf{9}$
12	1467	2139	P&S		$2+3+4=\mathbf{9}$

139.6. 2014 (9740)

	Q	A	Part	Topics	Points
Paper I: Pure Mathematics [100]					
1	1378	2001			$4+1=\mathbf{5}$
2	1427	2099			$6=\mathbf{6}$
3	1404	2048			$2+2+1=\mathbf{5}$
4	1379	2001			$4+1=\mathbf{5}$
5	1412	2060			$4+3=\mathbf{7}$
6	1392	2027			$5+3+2+2=\mathbf{12}$
7	1428	2100			$2+2+3+4=\mathbf{11}$
8	1428	2101			$1+4+4=\mathbf{9}$
9	1404	2048			$4+4+5=\mathbf{13}$
10	1429	2102			$1+5+1+1+5=\mathbf{13}$
11	1429	2103			$6+2+3+3=\mathbf{14}$
Paper II, Section A: Pure Mathematics [40]					
1	1379	2002			$3+4=\mathbf{7}$
2	1430	2104			$9=\mathbf{9}$
3	1392	2028			$2+4+5=\mathbf{11}$
4	1412	2060			$2+4+3+4=\mathbf{13}$
Section B: Probability and Statistics [60]					
5	1468	2140	P&S		$2+2=\mathbf{4}$
6	1468	2140	P&S		$2+3+3=\mathbf{8}$
7	1468	2140	P&S		$1+3+3=\mathbf{7}$
8	1469	2141	P&S		$2+2+3+1=\mathbf{8}$
9	1469	2141	P&S		$2+4+3=\mathbf{9}$
10	1470	2142	P&S		$1+2+3+4=\mathbf{10}$
11	1470	2142	P&S		$2+2+5+3+2=\mathbf{14}$

139.7. 2013 (9740)

Paper I: Pure Mathematics [100]					
	Q	**A**	**Part**	**Topics**	**Points**
1	1404	2048			$2+3=\mathbf{5}$
2	1379	2002			$5=\mathbf{5}$
3	1380	2002			$4+2=\mathbf{6}$
4	1412	2061			$2+3+3=\mathbf{8}$
5	1430	2104			$3+5=\mathbf{8}$
6	1405	2049			$1+1+5=\mathbf{7}$
7	1393	2029			$3+2+4=\mathbf{9}$
8	1412	2062			$2+4+3=\mathbf{9}$
9	1393	2029			$5+5+3=\mathbf{13}$
10	1430	2105			$4+2+3+4=\mathbf{13}$
11	1431	2106			$3+5+3+6=\mathbf{17}$
Paper II, Section A: Pure Mathematics [40]					
1	1380	2003			$2+4=\mathbf{6}$
2	1432	2107			$3+6=\mathbf{9}$
3	1432	2108			$7+5=\mathbf{12}$
4	1405	2049			$3+4+6=\mathbf{13}$
Section B: Probability and Statistics [60]					
5	1470	2143	P&S		$2+2=\mathbf{4}$
6	1471	2143	P&S		$4=\mathbf{4}$
7	1471	2143	P&S		$2+1+3=\mathbf{6}$
8	1471	2143	P&S		$1+2+3=\mathbf{6}$
9	1471	2143	P&S		$2+5=\mathbf{7}$
10	1472	2144	P&S		$3+1+2+3=\mathbf{9}$
11	1472	2145	P&S		$2+2+4+4=\mathbf{12}$
12	1473	2145	P&S		$3+2+3+4=\mathbf{12}$

139.8. 2012 (9740)

	Q	A	Part	Topics	Points
Paper I: Pure Mathematics [100]					
1	1380	2004			$4 = \mathbf{4}$
2	1432	2108			$2+3+1 = \mathbf{6}$
3	1393	2030			$2+2+4 = \mathbf{8}$
4	1433	2109			$4+4 = \mathbf{8}$
5	1405	2050			$4+4 = \mathbf{8}$
6	1413	2063			$2+2+4 = \mathbf{8}$
7	1381	2004			$2+3+4 = \mathbf{9}$
8	1433	2109			$4+3+2 = \mathbf{9}$
9	1406	2050			$3+5+4 = \mathbf{12}$
10	1434	2110			$7+5 = \mathbf{12}$
11	1434	2111			$5+3+5+3 = \mathbf{16}$
Paper II, Section A: Pure Mathematics [40]					
1	1435	2112			$3+5 = \mathbf{8}$
2	1413	2063			$2+2+2+3 = \mathbf{9}$
3	1381	2005			$1+3+1+1+4 = \mathbf{10}$
4	1394	2031			$5+5+3 = \mathbf{13}$
Section B: Probability and Statistics [60]					
5	1473	2146	P&S		$2+2+3 = \mathbf{7}$
6	1474	2146	P&S		$1+3+2 = \mathbf{6}$
7	1474	2146	P&S		$2+2+2+2+1 = \mathbf{9}$
8	1475	2147	P&S		$1+1+2+1+1+4+1 = \mathbf{11}$
9	1475	2148	P&S		$2+2+3+5 = \mathbf{12}$
10	1476	2148	P&S		$2+1+2+3+4+3 = \mathbf{15}$

Paper I: Pure Mathematics [100]					
	Q	A	Part	Topics	Points
1	1381	2007			$4 = \mathbf{4}$
2	1381	2007			$3 + 2 = \mathbf{5}$
3	1435	2112			$2 + 2 + 3 = \mathbf{7}$
4	1435	2112			$3 + 3 + 2 = \mathbf{8}$
5	1435	2113			$3 + 1 + 3 = \mathbf{7}$
6	1394	2032			$2 + 3 + 6 = \mathbf{11}$
7	1406	2051			$6 + 2 + 1 + 2 = \mathbf{11}$
8	1436	2114			$2 + 5 + 3 + 2 = \mathbf{12}$
9	1395	2033			$6 + 4 = \mathbf{10}$
10	1413	2064			$4 + 3 + 1 + 1 + 1 = \mathbf{10}$
11	1406	2051			$4 + 4 + 4 + 3 = \mathbf{15}$
Paper II, Section A: Pure Mathematics [40]					
1	1413	2065			$3 + 2 + 3 = \mathbf{8}$
2	1436	2114			$3 + 6 = \mathbf{9}$
3	1382	2007			$4 + 4 + 3 = \mathbf{11}$
4	1436	2115			$5 + 1 + 6 = \mathbf{12}$
Section B: Probability and Statistics [60]					
5	1476	2149	P&S		$4 = \mathbf{4}$
6	1476	2149	P&S		$2 + 1 + 2 = \mathbf{5}$
7	1477	2149	P&S		$2 + 1 + 1 + 4 = \mathbf{8}$
8	1477	2150	P&S		$1 + 2 + 1 + 3 = \mathbf{7}$
9	1478	2150	P&S		$2 + 1 + 2 + 3 = \mathbf{8}$
10	1478	2150	P&S		$2 + 3 + 4 = \mathbf{9}$
11	1478	2151	P&S		$3 + 5 = \mathbf{8}$
12	1479	2152	P&S		$1 + 4 + 5 + 1 = \mathbf{11}$

139.10. 2010 (9740)

Paper I: Pure Mathematics [100]					
	Q	A	Part	Topics	Points
1	1407	2052			$2+3=\mathbf{5}$
2	1437	2115			$3+3=\mathbf{6}$
3	1395	2034			$3+2=\mathbf{5}$
4	1437	2115			$4+4=\mathbf{8}$
5	1382	2008			$5+3=\mathbf{8}$
6	1437	2116			$2+2+4+2=\mathbf{10}$
7	1438	2116			$7+4=\mathbf{11}$
8	1414	2066			$2+3+4+2=\mathbf{11}$
9	1438	2117			$6+2+2+2=\mathbf{12}$
10	1407	2052			$2+4+3+3=\mathbf{12}$
11	1439	2118			$4+4+4=\mathbf{12}$
Paper II, Section A: Pure Mathematics [40]					
1	1414	2067			$2+5=\mathbf{7}$
2	1395	2034			$5+4+2=\mathbf{11}$
3	1439	2119			$5+2+2+2=\mathbf{11}$
4	1383	2009			$1+2+2+3+3=\mathbf{11}$
Section B: Probability and Statistics [60]					
5	1479	2152	P&S		$1+2=\mathbf{3}$
6	1479	2152	P&S		$7=\mathbf{7}$
7	1480	2153	P&S		$2+2+2+2+1=\mathbf{9}$
8	1480	2153	P&S		$1+2+4=\mathbf{7}$
9	1480	2153	P&S		$4+3+3=\mathbf{10}$
10	1481	2153	P&S		$2+2+1+4=\mathbf{9}$
11	1481	2154	P&S		$2+3+4+2+4=\mathbf{15}$

139.11. 2009 (9740)

	Q	A	Part	Topics	Points
Paper I: Pure Mathematics [100]					
1	1383	2010			$4+2=\mathbf{6}$
2	1439	2119			$5=\mathbf{5}$
3	1396	2035			$2+3+2=\mathbf{7}$
4	1439	2120			$2+3+3=\mathbf{8}$
5	1396	2035			$4+4=\mathbf{8}$
6	1383	2011			$4+2+2=\mathbf{8}$
7	1440	2120			$5+4=\mathbf{9}$
8	1396	2036			$4+3+4=\mathbf{11}$
9	1414	2068			$5+2+5=\mathbf{12}$
10	1407	2052			$3+4+5=\mathbf{12}$
11	1440	2121			$2+4+4+2+2=\mathbf{14}$
Paper II, Section A: Pure Mathematics [40]					
1	1440	2122			$1+3+4=\mathbf{8}$
2	1407	2053			$2+2+2+4=\mathbf{10}$
3	1384	2012			$5+2+3=\mathbf{10}$
4	1441	2123			$5+7=\mathbf{12}$
Section B: Probability and Statistics [60]					
5	1482	2155	P&S		$3=\mathbf{3}$
6	1482	2155	P&S		$2+2+1+3=\mathbf{8}$
7	1482	2155	P&S		$2+6=\mathbf{8}$
8	1482	2156	P&S		$1+2+3+3=\mathbf{9}$
9	1483	2156	P&S		$3+3+3+1=\mathbf{10}$
10	1483	2156	P&S		$2+7+1=\mathbf{10}$
11	1484	2157	P&S		$2+2+3+3+2=\mathbf{12}$

139.12. 2008 (9740)

	Q	A	Part	Topics	Points
Paper I: Pure Mathematics [100]					
1	1441	2123			$4 =$ **4**
2	1397	2036			$5 =$ **5**
3	1408	2053			$1 + 3 + 2 =$ **6**
4	1441	2124			$2 + 1 + 1 + 3 =$ **7**
5	1442	2124			$3 + 4 =$ **7**
6	1442	2125			$5 + 5 =$ **10**
7	1442	2125			$10 =$ **10**
8	1414	2069			$3 + 4 + 4 =$ **11**
9	1384	2012			$3 + 2 + 1 + 5 =$ **11**
10	1397	2036			$5 + 2 + 3 + 4 =$ **14**
11	1408	2053			$2 + 4 + 3 + 2 + 4 =$ **15**
Paper II, Section A: Pure Mathematics [40]					
1	1442	2126			$2 + 3 + 1 + 3 =$ **9**
2	1443	2126			$3 + 3 + 3 =$ **9**
3	1415	2070			$2 + 2 + 3 + 4 =$ **11**
4	1385	2016			$2 + 3 + 1 + 5 =$ **11**
Section B: Probability and Statistics [60]					
5	1484	2157	P&S		$2 + 2 =$ **4**
6	1484	2158	P&S		$6 =$ **6**
7	1485	2158	P&S		$2 + 3 + 3 =$ **8**
8	1485	2158	P&S		$3 + 1 + 2 + 2 + 1 + 1 =$ **10**
9	1486	2159	P&S		$2 + 2 + 4 + 2 =$ **10**
10	1486	2159	P&S		$2 + 2 + 2 + 4 =$ **10**
11	1486	2160	P&S		$2 + 3 + 7 =$ **12**

139.13. 2007 (9740)

Paper I: Pure Mathematics [100]					
	Q	**A**	**Part**	**Topics**	**Points**
1	1385	2018			$1+4=\mathbf{5}$
2	1386	2018			$3+3=\mathbf{6}$
3	1415	2073			$3+4=\mathbf{7}$
4	1445	2130			$6+1=\mathbf{7}$
5	1386	2019			$4+3=\mathbf{7}$
6	1408	2054			$2+3+4=\mathbf{9}$
7	1415	2074			$3+4+3=\mathbf{10}$
8	1409	2054			$5+3+3=\mathbf{11}$
9	1398	2038			$2+2+3+2+2=\mathbf{11}$
10	1398	2038			$4+5+5=\mathbf{14}$
11	1446	2130			$2+6+5=\mathbf{13}$
Paper II, Section A: Pure Mathematics [40]					
1	1386	2019			$6=\mathbf{6}$
2	1399	2039			$4+2+2+2=\mathbf{10}$
3	1446	2131			$4+5+2=\mathbf{11}$
4	1446	2131			$6+5+2=\mathbf{13}$
Section B: Probability and Statistics [60]					
5	1488	2161	P&S		$4+1=\mathbf{5}$
6	1488	2162	P&S		$2+3+2=\mathbf{7}$
7	1488	2162	P&S		$2+4+1=\mathbf{7}$
8	1489	2162	P&S		$2+3+4+1=\mathbf{10}$
9	1489	2163	P&S		$1+3+1+2+2=\mathbf{9}$
10	1490	2163	P&S		$3+2+2+4=\mathbf{11}$
11	1491	2163	P&S		$2+2+2+3+2=\mathbf{11}$

The 9233 syllabus was significantly heftier. In particular, in addition to "Pure Mathematics" and "Probability and Statistics", there was also "Particle Mechanics". This textbook has omitted the questions on Particle Mechanics and also any other questions that would also be out of the 9740 syllabus. (This explains why there seem to be some missing questions.)

The format of the papers was also somewhat more complicated. The last question of Paper 1 was an either-or question (i.e. examinees had a choice of doing one of two questions given). Paper 2 contained four sections, of which only Section A (Pure Mathematics) was mandatory and examinees had to choose to do one of Sections B, C, or D. And again, the last question of each of these four sections was an either-or question.

Note also that Permutations and Combinations fell under "Pure Mathematics".

Paper 1: Pure Mathematics					
	Q	**A**	**Part**	**Topics**	**Points**
1	1486	2160			3 = **3**
2	1443	2126			4 + 5 = **9**
3	1443	2127			5 = **5**
4	1444	2127			4 = **4**
6	1444	2127			3 + 2 = **5**
8	1444	2127			5 = **5**
9	1415	2071			2 + 2 + 4 = **8**
10	1444	2128			3 + 5 = **8**
11	1408	2054			5 + 4 = **9**
13	1444	2128			5 + 2 + 5 = **12**
14	1445	2129			6 + 6 = **12**
14	1385	2014			4 + 3 + 1 + 2 = **10**
Paper 2, Section A: Pure Mathematics					
1	1445	2129			3 = **3**
2	1397	2037			6 = **6**
3	1415	2073			3 + 4 = **7**
5	746	2129			5 + 3 = **8**
Paper 2, Sections B–D: Probability and Statistics					
23	1487	2160	P&S		4 = **4**
26	1487	2160	P&S		2 + 3 + 4 = **9**
27	1487	2160	P&S		5 + 2 + 3 = **10**
29	1487	2161	P&S		5 + 5 + 2 = **12**
30	1488	2161	P&S		4 + 3 + 5 = **12**

139.15. 2007 (9233)

Paper 1: Pure Mathematics					
	Q	**A**	**Part**	**Topics**	**Points**
2	1446	2131			$3 = \mathbf{3}$
3	1447	2132			$5 = \mathbf{5}$
4	1492	2164			$1 + 4 = \mathbf{5}$
7	1409	2054			$7 = \mathbf{7}$
8	1447	2132			$3 + 4 = \mathbf{7}$
9	1416	2075			$5 + 3 = \mathbf{8}$
10	1447	2132			$3 + 5 = \mathbf{8}$
11	1447	2132			$9 = \mathbf{9}$
13	1448	2133			$3 + 4 + 2 + 3 = \mathbf{12}$
14	1399	2040			$6 = \mathbf{6}$
14	1448	2133			$4 + 4 + 4 = \mathbf{12}$
Paper 2, Section A: Pure Mathematics					
1	1399	2040			$3 = \mathbf{3}$
2	1409	2054			$2 + 5 = \mathbf{7}$
4	1387	2020			$2 + 3 + 3 = \mathbf{8}$
5	1416	2075			$3 + 2 + 3 + 2 = \mathbf{10}$
Paper 2, Sections B–D: Probability and Statistics					
23	1492	2164	P&S		$6 + 1 = \mathbf{7}$
25	1493	2164	P&S		$1 + 1 + 2 + 4 = \mathbf{8}$
26	1493	2164	P&S		$? = \mathbf{?}$
27	1493	2165	P&S		$6 + 6 = \mathbf{12}$

139.16. 2006 (9233)

Paper 1: Pure Mathematics					
	Q	**A**	**Part**	**Topics**	**Points**
1	1399	2040			$4 = \mathbf{4}$
3	1387	2020			$3 + 1 = \mathbf{4}$
4	1494	2165			$4 = \mathbf{4}$
5	1416	2076			$3 + 2 = \mathbf{5}$
6	1416	2077			$3 + 3 = \mathbf{6}$
7	1449	2133			$6 = \mathbf{6}$
8	1449	2134			$7 = \mathbf{7}$
9	1449	2134			$2 + 6 = \mathbf{8}$
11	1399	2040			$4 + 1 + 4 = \mathbf{9}$
14	1450	2135			$2 + 2 + 3 + 2 + 3 = \mathbf{12}$
14	1409	2055			$7 + 2 = \mathbf{9}$
Paper 2, Section A: Pure Mathematics					
1	1387	2021			$5 = \mathbf{5}$
2	1450	2135			$3 + 3 = \mathbf{6}$
Paper 2, Sections B–D: Probability and Statistics					
23	1494	2165	P&S		$3 + 2 = \mathbf{5}$
25	1494	2165	P&S		$7 + 1 = \mathbf{8}$
26	1494	2165	P&S		$3 + 5 = \mathbf{8}$
28	1495	2166	P&S		$4 + 3 + 3 = \mathbf{10}$

140. H1 Maths Questions (2016–19)

140.1. H1 Maths 2019 Questions

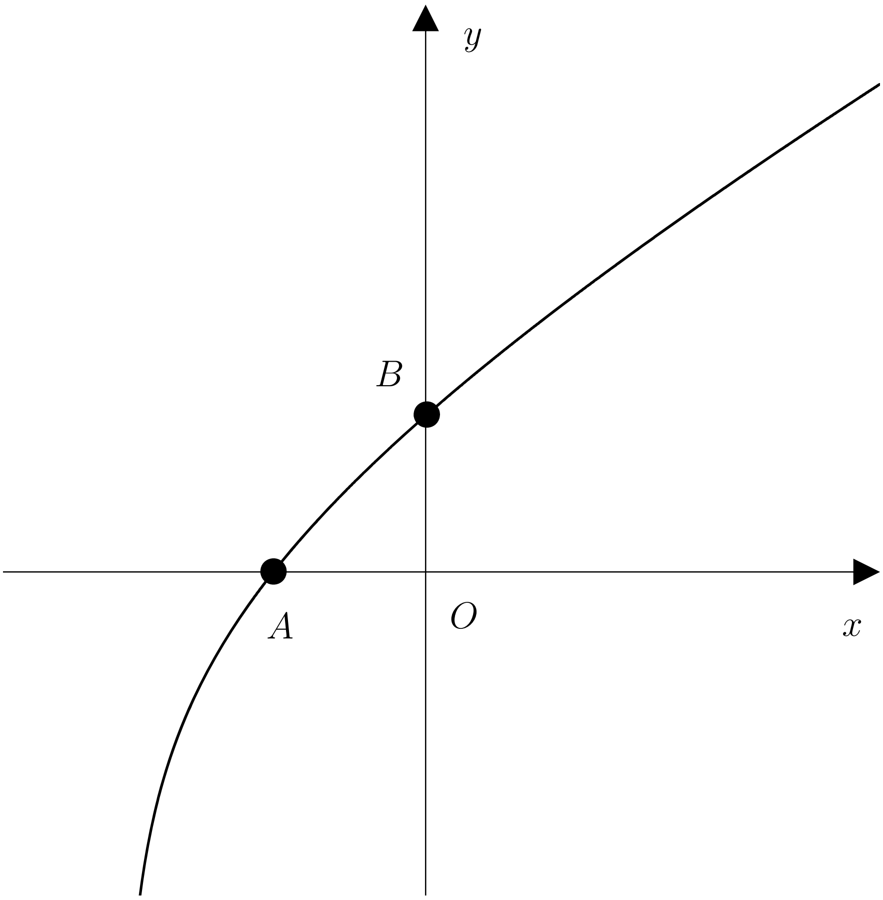
Answers for Probability and Statistics questions to be written.

Exercise 869. (8865 N2019/1.) (Answer on p. 1518.)

An online company sells three styles of television: Style A, Style B and Style C. Last month, the company sold 22 Style A, 16 Style B and 8 Style C televisions and the total income from these sales was \$96 480. The income from the Style A sales was \$3 120 more than the income from Style B sales. This month, the company has sold two more of each of the three styles of television than last month and the total income has increased by \$13 260. The prices instead of the televisions have not changed.

Find the price of a Style B television. [5]

Exercise 870. (8865 N2019/2.) (Answer on p. 1518.)



The diagram shows a sketch of the curve with equation $y = x + \ln(3x + 4)$. The points A and B are where the curve cuts the x-axis and the y-axis respectively.

(i) Find the coordinates of A, correct to 3 significant figures. [1]

(ii) Find the equation of the asymptote to the curve. [1]

(iii) Without using a calculator, find the equation of the tangent to the curve at the point B, giving your answer in the form $y = mx + c$, where m and c are exact constants. [5]

Exercise 871. (8865 N2019/3.) (Answer on p. 1518.)

(i) Without using a calculator, solve the simultaneous equations

$$y = 2x^2 + 6x - 3,$$
$$y = 11x + 9. \qquad [4]$$

(ii) Hence, or otherwise, solve the inequality

$$2x^2 + 6x - 3 \le 11x + 9. \qquad [2]$$

Exercise 872. (8865 N2019/4.) (Answer on p. 1519.)

(i) Differentiate $\dfrac{1}{(2-3x)^4}$ with respect to x. [2]

(ii) Differentiate $\left(2\sqrt{x} - \dfrac{3}{\sqrt{x}}\right)^2$ with respect to x, giving your answer in the form $p + \dfrac{q}{x^2}$, where p and q are constants to be determined. [3]

(iii) Without using a calculator, find the exact value of $\displaystyle\int_0^1 \left(x^2 + 2 - \mathrm{e}^{-2x}\right) \mathrm{d}x$. [4]

Exercise 873. (8865 N2019/5.) (Answer on p. 1519.)

Mr Tan has just set up a company to manufacture and sell computers. He will monitor the profit from the sale of his computers over a period of six years, before he decides whether to continue with his business.

Mr Tan is a mathematician and he wants to model his total profit, P thousand dollars, at time t years. He believes that

$$P = k \times 1.5^t - 0.6,$$

where k is a positive constant.

(i) Explain why $k = 0.6$. [1]

(ii) Sketch the graph of P against t for $0 \le t \le 6$. [1]

(iii) Find the value of P when $t = 6$. [1]

(iv) Find the rate at which P is increasing when $t = 3$. [1]

Mr Tan is also interested in modelling his costs, C thousand dollars per year, at any point in time. The model he uses is

$$C = t^3 - 10t^2 + 25t + 10, \qquad \text{for } 0 \le t \le 6.$$

(v) Use differentiation to find the values of t which give stationary points on the graph of C against t. For each point, justify whether it is a minimum or a maximum. [5]

(vi) Sketch the graph of C against t, stating the coordinates of any intersections with the coordinate axes. [2]

(vii) Use your calculator to find the value of $\displaystyle\int_0^6 \left(t^3 - 10t^2 + 25t + 10\right) \mathrm{d}t$. In the context of the question, what does this value represent? [2]

Exercise 874. (8865 N2019/6.) (Answer on p. 1515.)

(i) Find the number of different arrangements of the 9 letters of the word CHEMISTRY in which the two vowels (E and I) are next to each other. [2]

(ii) Find the number of ways in which 4 letters can be selected from the 9 letters of the word CHEMISTRY, given that at least one vowel must be included. [3]

A874 (8865 N2019/6). XXX

Exercise 875. (8865 N2019/7.) (Answer on p. 1515.)

In a large insurance company, the most successful members of the sales team are rewarded with an annual bonus. In 2016 exactly 32% of the members of the sales team received the annual bonus.

Eight members of the sales team were randomly selected.

(i) (a) Find the probability that exactly four of them received the bonus in 2016. [1]

(b) Find the probability that at least five of them received the bonus in 2016. [2]

(ii) For random samples of size eight, find the mean and variance of the number of members of the sales team who received the bonus in 2016. [2]

A875 (8865 N2019/7). XXX

Exercise 876. (8865 N2019/8.) (Answer on p. 1515.)

Two events A and B are such that $\mathrm{P}(A) = 0.6$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A|B) = 0.1$.

(i) Find the probability that both A and B occur. [2]

(ii) Describe in words what is meant by $\mathrm{P}(A' \cap B)$. [2]

(iii) Find $\mathrm{P}(A' \cup B)$. [2]

A876 (8865 N2019/8). XXX

Exercise 877. (8865 N2019/9.) (Answer on p. 1515.)

A large number of students at a college take two tests in science: a written test marked out of 30 and a practical test marked out of 5. The table shows the marks, x, for the written test and the marks, y, for the practical test, for a random sample of 9 students.

x	16	20	25	19	17	30	21	24	25
y	3.1	3.6	3.9	3.4	2.9	4.8	3.9	3.8	3.7

(i) Give a sketch of the scatter diagram for the data, as shown on your calculator. [2]

(ii) Find the product moment correlation coefficient and comment on its value in the context of the data. [2]

(iii) Find the equation of the regression line of y on x, giving your answer in the form $y = ax + b$, with the values of a and b correct to 3 significant figures. Sketch this line on your scatter diagram. [2]

(iv) Use the equation of your regression line to calculate an estimate for the mark obtained in the practical test by a student who obtained 22 marks in the written test. Comment on the reliability of your estimate. [2]

A877 (8865 N2019/9). XXX

Exercise 878. (8865 N2019/10.) (Answer on p. 1516.)

Mia's sock drawer contains 10 red socks, 8 pink socks and 7 blue socks. She chooses 2 socks at random from the drawer.

(i) Draw a tree diagram to represent this situation, showing all possible outcomes. [2]

(ii) Find the probability that both socks are red. [1]

(iii) Find the probability that the 2 socks are of different colours. [3]

Mia now chooses a third sock from the drawer, without replacing the other 2 socks.

(iv) Find the probability that Mia now has at least 2 red socks. [4]

A878 (8865 N2019/10). XXX

Exercise 879. (8865 N2019/11.) (Answer on p. 1516.)

The times taken, in minutes, for runners from two large athletics clubs, the Arrows and the Beavers, to run 5 000 metres have independent normal distributions with means and standard deviations as shown in the following table.

	Mean	Standard Deviation
Arrows	14.8	0.55
Beavers	15.2	0.65

(i) Find the probability that the time taken to run 5 000 metres by a randomly chosen runner from the Arrows is more than 15.0 minutes. [1]

(ii) Find the probability that the sum of the times taken by 2 randomly chosen runners from the Arrows and 3 randomly chosen runners from the Beavers is less than 75 minutes. [4]

In a competition, teams of 6 runners enter a 5 000-metre race. The score for each team is the sum of the times taken, in minutes, by the 6 runners to complete the race. The team with the lowest score wins the competition. The Arrows and the Beavers each enter a team of 6 randomly chosen runners from their clubs.

(iii) Find the probability that the score of the Arrows team is within ±5 of the score of the Beavers team. [4]

Any team that scores fewer than 90 wins a medal.

(iv) Find the probability that neither the Arrows nor the Beavers win a medal. [4]

A879 (8865 N2019/11). XXX

Exercise 880. (8865 N2019/12.) (Answer on p. 1517.)

A baker states that a certain a small cake that he produces has a mean mass of 100 grams. A food inspector wishes to investigate whether the mean mass of these cakes is actually less than 100 grams. The food inspector selects a random sample of 60 cakes of this type. The masses, x, in grams, are summarised by

$$\sum x = 5\,928, \qquad \sum x^2 = 587\,000.$$

(i) Find unbiased estimates of the population mean and variance. [3]

(ii) Determine the conclusion the food inspector should reach if she carries out a test at the 5% significance level. [5]

The baker still states that the mean mass of these cakes is 100 grams. Using a 5% significance level, a test is carried out on a new random sample of 60 cakes of this type with the null hypothesis $\mu = 100$ and the alternative hypothesis $\mu \neq 100$, where μgrams is the

population mean mass. The test indicates that there is sufficient evidence to reject the baker's statement.

(iii) Find the set of values within which the mean mass of this sample must lie. [5]

A880 (8865 N2019/12). XXX

140.2. H1 Maths 2019 Answers

A869 (8865 N2019/1). Let A, B, and C be the price (\$) of a Style A, Style B, and Style C television, respectively. Then the information we were given may be written as

$$22A + 16B + 8C \overset{1}{=} 96\,480, \qquad 22A \overset{2}{=} 16B + 3\,120, \qquad 2(A+B+C) \overset{3}{=} 13\,260.$$

Rearrange $\overset{2}{=}$ to get $22A - 16B \overset{4}{=} 3\,120$.

Take $\overset{1}{=} + \frac{1}{2} \times \overset{4}{=} -4 \times \overset{3}{=}$ to get

$$22A + 16B + 8C + \frac{1}{2} \times (22A - 16B) - 4 \times 2(A+B+C) = 96\,480 + \frac{1}{2} \times 3\,120 - 4 \times 13\,260,$$

$$\iff \qquad 25A = 45\,000 \qquad \text{or} \qquad A \overset{5}{=} 1\,800.$$

Plug $\overset{5}{=}$ into $\overset{4}{=}$ to get $22 \times 1\,800 - 16B = 3\,120$ or $B = (22 \times 1\,800 - 3\,120)/16 = 2\,280$.

The price of a Style B television is \$2 280.

A870 (8865 N2019/2)(i) You're supposed to just mindlessly use your calculator:

$$A \approx (-0.677, 0).$$

(ii) $3x + 4 = 0$ or $x = -4/3$.

(iii) $B = (0, \ln 4)$. Compute $\frac{dy}{dx} = 1 + \frac{3}{3x+4}$. So, $\left.\frac{dy}{dx}\right|_{x=0} = \frac{7}{4}$. Hence, the requested tangent line is

$$y - \ln 4 = \frac{7}{4}(x - 0) \qquad \text{or} \qquad y = \frac{7}{4}x + \ln 4.$$

A871 (8865 N2019/3)(i) Plug the second equation into the first and rearrange:

$$11x + 9 = 2x^2 + 6x - 3 \qquad \iff \qquad 0 = 2x^2 - 5x - 12.$$

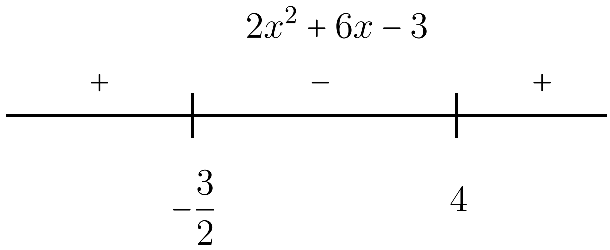
By the quadratic formula,

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-12)}}{2(2)} = \frac{5 \pm \sqrt{25 + 96}}{4} = \frac{5 \pm 11}{4} = 4, -\frac{3}{2}.$$

Plugging these values of x into the second given equation, we get $y = 53, -\frac{15}{2}$. Hence, the solutions to the pair of given simultaneous equations are

$$(4, 53) \qquad \text{and} \qquad \left(-\frac{3}{2}, -\frac{15}{2}\right).$$

(ii) In the quadratic polynomial $2x^2 - 5x - 12$, the coefficient on x^2 is $2 > 0$. Hence, the graph of the quadratic equation $y = 2x^2 - 5x - 12$ is ∪-shaped.



So, $2x^2+6x-3 \leq 11x+9 \iff 0 \geq 2x^2-5x-12 \iff x \in \left[-\frac{3}{2}, 4\right]$.

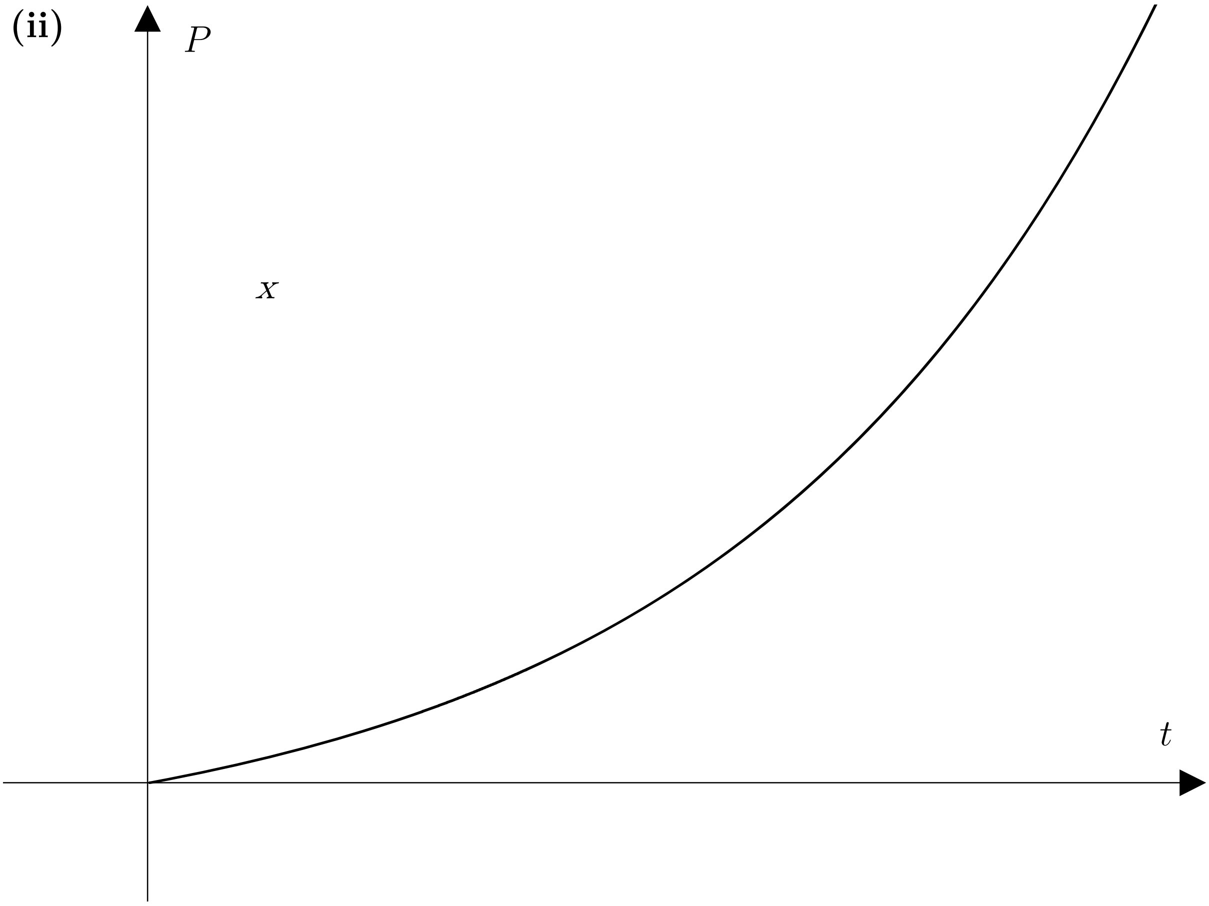
A872 (8865 N2019/4)(i) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{(2-3x)^4} = -4\frac{-3}{(2-3x)^5} = \frac{12}{(2-3x)^5}$.

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\left(2\sqrt{x}-\frac{3}{\sqrt{x}}\right)^2 = 2\left(2\sqrt{x}-\frac{3}{\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}+\frac{3}{2x^{3/2}}\right) = \frac{1}{\sqrt{x}}\frac{1}{x^{3/2}}(2x-3)(2x+3) = \frac{1}{x^2}\left(4x^2-9\right) = 4-\frac{9}{x^2}$.

(iii) $\int_0^1 \left(x^2+2-\mathrm{e}^{-2x}\right)\mathrm{d}x = \left[\frac{x^2}{3}+2x+\frac{1}{2}\mathrm{e}^{2x}\right]_0^1 = \left(\frac{1}{3}+2+\frac{1}{2}\mathrm{e}^2\right)-\left(0+0+\frac{1}{2}\right) = \frac{11}{6}+\frac{1}{2}\mathrm{e}^2$.

A873 (8865 N2019/5)(i) At $t=0$, $P=0=k\times 1.5^t-0.6 \iff k=0.6$.

(ii)



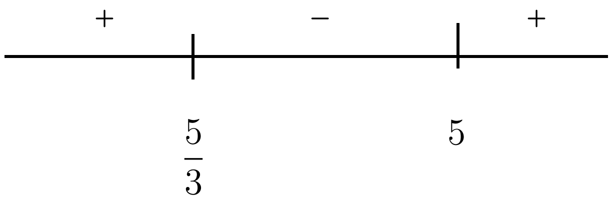
(iii) $P(6) = 0.6\times 1.5^{0.6}-0.6 \approx 6.23$.

(iv) $\frac{\mathrm{d}P}{\mathrm{d}t} = 6(\ln 1.5)1.5^t$. So, $\left.\frac{\mathrm{d}P}{\mathrm{d}t}\right|_{t=3} = 6(\ln 1.5)1.5^3 \approx 8.21$.

(v) $\frac{\mathrm{d}C}{\mathrm{d}t} = 3t^2-20t+25 \iff \left.\frac{\mathrm{d}C}{\mathrm{d}t}\right|_{t=\bar{t}} = 0 \iff \bar{t} = \frac{20\pm\sqrt{(-20)^2-4(3)(25)}}{2(3)} = \frac{20\pm\sqrt{100}}{6} =$

$\dfrac{20 \pm 10}{6} = 5, \dfrac{5}{3}$.

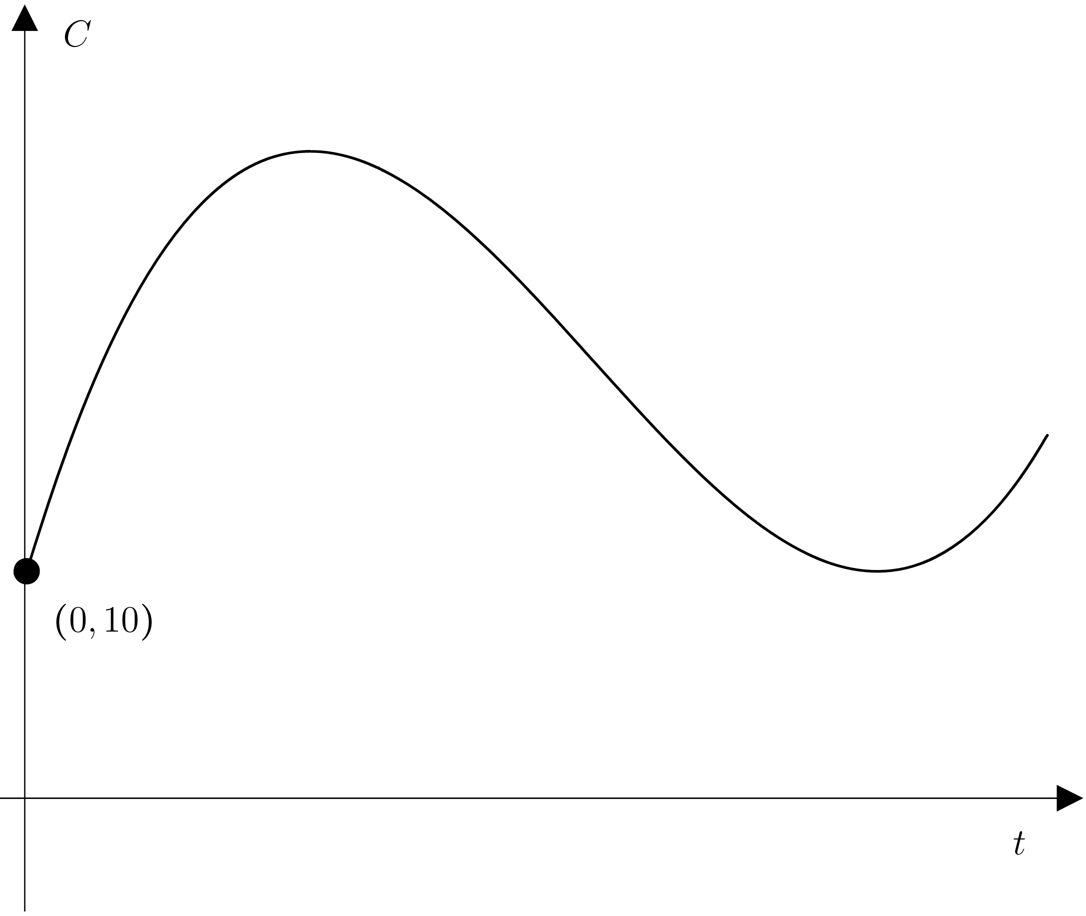
The coefficient on t^2 is $3 > 0$, so the graph is ∪-shaped. Sign diagram for $\dfrac{dC}{dt}$:



Since $\dfrac{dC}{dt} > 0$ to the left of $\dfrac{5}{3}$ and $\dfrac{dC}{dt} < 0$ to the right of $\dfrac{5}{3}$, the point $\dfrac{5}{3}$ is a strict local minimum.

Since $\dfrac{dC}{dt} < 0$ to the left of 5 and $\dfrac{dC}{dt} > 0$ to the right of 5, the point 5 is a strict local maximum.

(vi)



(vii) 114. Total costs over the first six years.

140.3. H1 Maths 2018 Questions

Answers for Probability and Statistics questions to be written.

Exercise 881. (8865 N2018/1.) (Answer on p. 1526.)

Find the values of the real constant k for which the equation $3kx^2 - (k-6)x - 2 = 0$ has two distinct roots. [4]

Exercise 882. (8865 N2018/2.) (Answer on p. 1526.)

(i) Differentiate $3\ln\left(4x^3+2\right)$. [2]

(ii) Integrate $\dfrac{3x^2-2}{\sqrt{x}}$. [3]

Exercise 883. (8865 N2018/3.) (Answer on p. 1526.)

Three large companies P, Q and R recruit graduates. The companies take part in a scheme to train the graduates as accountants, financial advisers or consultants. The training cost for each accountant is the same for each company, as are the training costs for each financial adviser and each consultant.

Last year, Company P paid \$294 100 to train 5 accountants, 12 financial advisers and 8 consultants.

Company Q paid \$270 100 to train 9 accountants, 7 financial advisers and 3 consultants.

Company R paid \$122 700 to train 3 accountants and 6 financial advisers.

(i) Express this information as 3 linear equations and hence find the cost of training one financial adviser last year. [5]

Company S decides to join the scheme this year. The company will train an equal number of accountants and financial advisers but no consultants. Training costs have increased by 10%. The company has \$200 000 to spend on this training.

(ii) What is the greatest possible number of accountants that company S will be able to train this year? [3]

Exercise 884. (8865 N2018/4.) (Answer on p. 1526.)

A curve C has equation $y = x + \mathrm{e}^{1-2x}$.

(i) Find the exact coordinates of the turning point of C. [3]

(ii) Sketch the graph of C, stating the coordinates of any points of intersection with the axes. [2]

(iii) Find the equation of the tangent to C at the point where $x = 2$, giving your answer in the form $y = mx + c$, with m and c correct to 3 significant figures. [3]

(iv) Find $\displaystyle\int y\,\mathrm{d}x$ and hence find the area between C, the x-axis and the lines $x = 0$ and $x = 1$, giving your answer in terms of e. [3]

Exercise 885. (8865 N2018/5.) (Answer on p. 1527.)

A company buys televisions from a manufacturer and stores them in a warehouse before selling them in it retail outlets. Space in the warehouse is limited. On every occasion throughout the year when the stock in the warehouse is low, the manager places an order for x televisions, where x is a fixed positive integer. He estimates his total cost for the year using a model with three components:

- the purchase from the manufacturer,
- the storage cost,
- the ordering cost.

Next year, the manager estimates that the company will need to buy 1 200 televisions. The purchase order will be \$200 each, the storage cost will be \$$6x$ for the year and the ordering cost will be \$50 per order.

(i) Show that the estimated total cost for the year, \$$C$, is given by

$$C = 6x + 240\,000 + \frac{60\,000}{x}.$$ [1]

(ii) The manager wishes to determine the value of x that will minimise C. Use differentiation to find the minimum value of C, justifying that this value is a minimum. [5]

A new manager takes over at the company and he wants to maximise the profits on sales of televisions. He uses a new model in which the total cost per year is constant and equal to \$240 000.

(iii) Is this a reasonable model? Justify your answer. [1]

Market research suggests that if televisions at \$300 each, then 1 200 will be sold per year. If they are priced at \$700 each, then none will be sold. Between these values, the graph of the number of sales against the selling price will be a straight line.

Assume that the total cost per year is \$240 000, the profit per year is \$$P$, and the selling price is \$$S$.

(iv) Show that $P = 2\,100S - 3S^2 - 240\,000$. [3]

(v) Find the value of S that leads to the maximum profit. [2]

Exercise 886. (8865 N2018/6.) (Answer on p. 1522.)

In a large population, the proportion of people who have blood group A is 40%. A sample of 6 people is randomly chosen from the population.

(i) Find the probability that exactly 2 of these people have blood group A. [1]

(ii) Find the probability that at least 4 of these people do **not** have blood group A. [3]

A886 (8865 N2018/6). XXX

Exercise 887. (8865 N2018/7.) (Answer on p. 1523.)

A music producer is forming a new band consisting of 3 guitarists, 1 drummer and 2 vocalists. The members of the band will be selected from the 8 guitarists, 4 drummers and 6 vocalists who have passed the first stage of the auditions.

(i) How many different selections of the band can the producer make? [2]

Three of the guitarists are from the same family.

(ii) Given that at most one of these three can be in the band, how many different selections of guitarists can the producer make? [2]

(iii) Given instead that the producer selects the guitarists at random, find the probability that the band will contain none of these three. [2]

A887 (8865 N2018/7). XXX

Exercise 888. (8865 N2018/8.) (Answer on p. 1523.)

Two events A and B are such that $\mathrm{P}(A) = p$, $\mathrm{P}(B) = 2p$, $\mathrm{P}(A|B) = 0.3$ and $\mathrm{P}(A \cup B) = 0.8$.

(i) Find the value of p. [4]

(ii) Explain what is meant by $\mathrm{P}(A' \cap B)$ and find its value. [3]

A888 (8865 N2018/8). XXX

Exercise 889. (8865 N2018/9.) (Answer on p. 1523.)

At a certain college, the numbers of male and female studying Art, Business and Statistics are as shown in the following table. Each student studies exactly one of these three subjects.

	Art	Business	Statistics	Total
Male	54	72	34	160
Female	36	40	64	140
Total	90	112	98	300

A student is chosen at random from these 300 students. Let

A be the event that the student studies Art,

B be the event that the student studies Business,

F be the event that the student is female,

M be the event that the student is male.

(i) Find $\mathrm{P}(B)$. [1]

(ii) Find $\mathrm{P}(F \cup B)$. [2]

(iii) Find $\mathrm{P}(M \cap A)$. [1]

(iv) Determine whether the events M and A are independent. [2]

On another occasion, 3 of these students are chosen at random, without replacement.

(v) Find the probability that exactly 2 are studying Art, giving your answer correct to 3 decimal places. [3]

A889 (8865 N2018/9). XXX

Exercise 890. (8865 N2018/10.) (Answer on p. 1524.)

The Harriers is an athletics club with a large number of members. At the beginning of each season, the athletes are timed running 100 metres and running 10 000 metres. For a random sample of 8 athletes, the times, x seconds, to run 100 metres and the times, y minutes, to run 10 000 metres, are shown in the following table.

Athlete	A	B	C	D	E	F	G	H
x	12.6	11.9	10.8	12.2	14.1	15.6	14.8	11.1
y	37.4	40.6	42.2	39.2	34.1	33.1	32.6	42.2

(i) Give a sketch of the scatter diagram for the data, as shown on your calculator. [2]

(ii) Find the product moment correlation coefficient and comment on its value in the context of the data. [3]

(iii) Find the equation of the regression line of y on x, in the form $y = mx + c$, giving the values of m and c correct to 2 decimal places. Sketch this line on your scatter diagram. [2]

(iv) Calculate an estimate of the time taken to run 10 000 metres by an athlete who runs 100 metres in 13.1 seconds. [1]

(v) Give a reason why you would expect this estimate to be reliable. [1]

A890 (8865 N2018/10). XXX

Exercise 891. (8865 N2018/11.) (Answer on p. 1524.)

A website states that the mean length of adults in a particular species of fish is 30 cm. A scientist claims that the true mean length is greater than 30 cm. To test this claim, a random sample of 100 adult fish of this species is captured from a lake. The lengths of the fish, x cm, are summarised by

$$\sum(x-30) = 15 \qquad \text{and} \qquad \sum(x-30)^2 = 82.$$

(i) Find unbiased estimates of the population mean and variance. [3]

(ii) Test at the 2.5% significance level whether the scientist's claim is supported by the data. [4]

(iii) State, with a reason, whether it is necessary to assume that the lengths of these fish are distributed normally for the test to be valid. [1]

A new random sample of 100 adult fish of this species is captured from a different lake. The mean length of the fish in this sample is m cm and the population variance is $0.9\,\text{cm}^2$. A test at the 10% significance level supports the scientist's claim that the mean length of the fish is greater than 30 cm.

(iv) Find the range of possible values of m. [3]

A891 (8865 N2018/11). XXX

Exercise 892. (8865 N2018/12.) (Answer on p. 1525.)

A company produces two electrical components, Type A and Type B. The masses of Type A components are normally distributed with mean 250 grams and standard deviation 3 grams. The masses of Type B components are normally distributed with mean 240 grams and standard deviation 4 grams. The masses of the electrical components are independent of each other.

(i) Find the probability that the mass of a randomly chosen Type A component is more than 2% above the mean mass. [2]

(ii) Find the probability that the mass of a randomly chosen Type A component is greater than the mass of a randomly chosen Type B component. [3]

(iii) Find the probability that the total mass of 6 randomly chosen Type A components and 3 randomly chosen Type B components is between 2 190 grams and 2 230 grams. [4]

The cost of producing a Type A component is \$0.02 per gram.

The cost of producing a Type B component is \$0.03 per gram.

(iv) Find the probability that the cost of producing 10 Type B components exceeds the cost of producing 10 Type A components by more than \$22.50. [5]

A892 (8865 N2018/12). XXX

140.4. H1 Maths 2018 Answers

A881 (8865 N2018/1). The equation has two distinct roots if and only if the discriminant is positive:

$$(k-6)^2 - 4(3k)(-2) = k^2 + 36 - 12k + 24k = k^2 + 12k + 36 \overset{1}{>} 0.$$

The coefficient on k^2 is $1 > 0$, so that the graph of $y = k^2 + 12k + 36 = (k+6)^2$ is ∪-shaped and just touches the horizontal axis at $k = -6$. Hence, $\overset{1}{>}$ holds if and only $k \neq -6$.

A882 (8865 N2018/2)(i) $\dfrac{\mathrm{d}}{\mathrm{d}x} 3\ln\left(4x^3+2\right) = \dfrac{3 \cdot 12x^2}{4x^3+2} = \dfrac{36x^2}{4x^3+2}$.

(ii) $\displaystyle\int \frac{3x^2-2}{\sqrt{x}}\,\mathrm{d}x = \int 3x^{3/2} - 2x^{-1/2}\,\mathrm{d}x = 3\frac{2}{5}x^{5/2} - \frac{2}{1/2}x^{1/2} + C = \frac{6}{5}x^{5/2} - 4x^{1/2} + C.$

A883 (8865 N2018/3)(i) Let a, f, and c be last year's cost (\$) of training one accountant, financial adviser, and consultant, respectively. So, the given information may be written as

$$5a + 12f + 8c \overset{1}{=} 294\,100, \qquad 9a + 7f + 3c \overset{2}{=} 270\,100, \qquad 3a + 6f \overset{3}{=} 122\,700.$$

Take $3\times \overset{1}{=} -8\times \overset{2}{=} +19\times \overset{3}{=}$ to get

$$3(5a + 12f + 8c) - 8(9a + 7f + 3c) + 19(3a + 6f) = 3 \times 294\,100 - 8 \times 270\,100 + 19 \times 122\,700$$

or, $$94f = 1\,052\,800 \qquad \text{or} \qquad f = 11\,200.$$

Hence, the cost of training one financial adviser last year was \$11 200.

(ii) From $\overset{3}{=}$, $a + f = (122\,700 - 3f)/3 = 40\,900 - 11\,200 = 29\,700$.

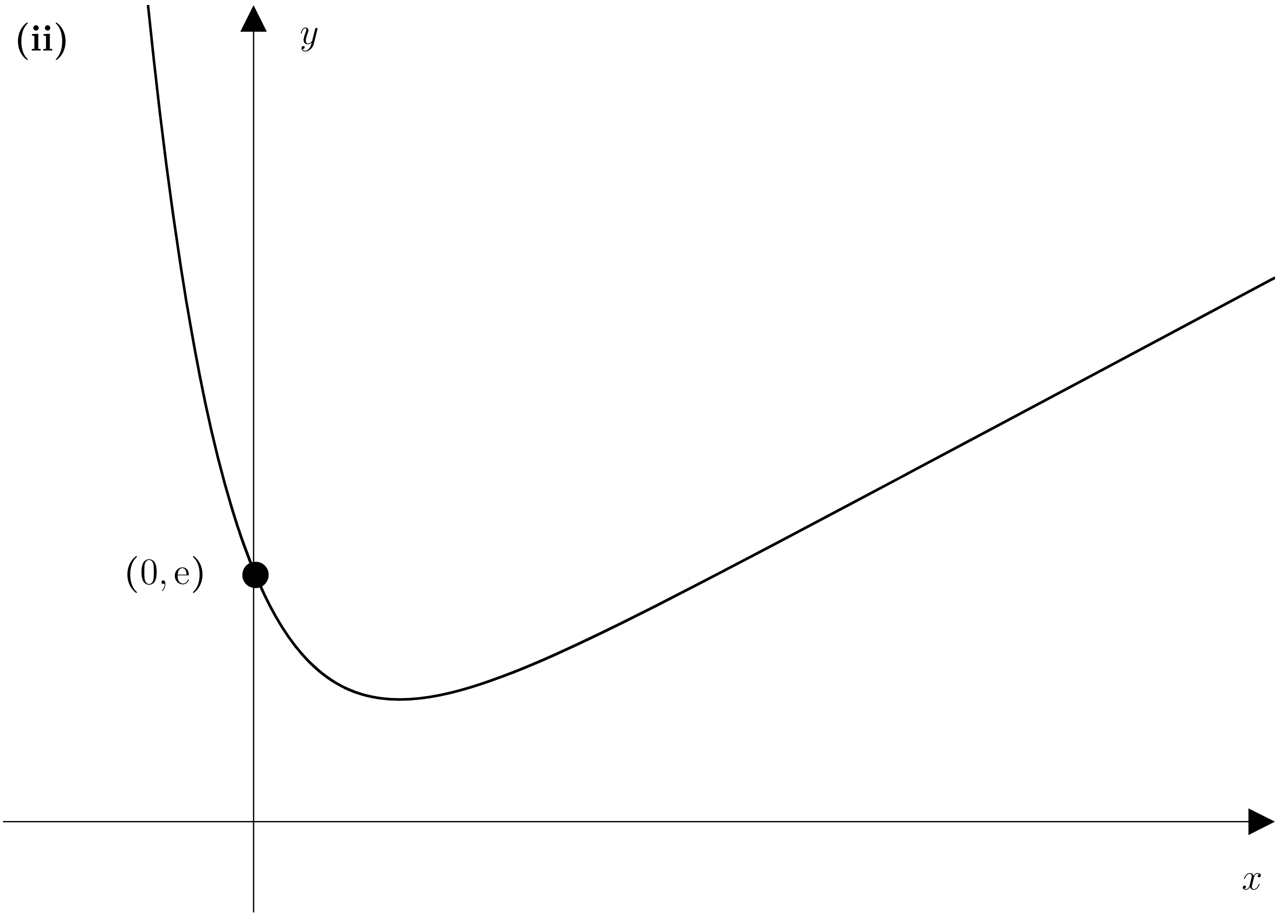
Hence, the answer is the greatest integer smaller than or equal to 200 000/29 700, which is 6.

A884 (8865 N2018/4)(i) Compute $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - 2\mathrm{e}^{1-2x}$.

Any stationary points are given by $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\bar{x}} = 0$ or $1 - 2\mathrm{e}^{1-2\bar{x}} = 0$ or $\ln 0.5 = 1 - 2\bar{x}$ or $\bar{x} = 0.5 - 0.5\ln 0.5 = 0.5(1 + \ln 2)$.

Observe that $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$ for $x < \bar{x}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ for $x > \bar{x}$, so that $\bar{x}$ is a strict extremum and hence, also a turning point.

(ii)



(iii) At $x = 2$, $y = 2 + \mathrm{e}^{1-2\cdot 2} = 2 + \mathrm{e}^{-3}$. Compute $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2} = 1 - 2\mathrm{e}^{1-2\cdot 2} = 1 - 2\mathrm{e}^{-3}$. Hence, the requested tangent line is $y - \left(2 + \mathrm{e}^{-3}\right) = \left(1 - 2\mathrm{e}^{-3}\right)(x - 2)$ or

$$y = \left(1 - 2\mathrm{e}^{-3}\right)(x - 2) + 2 + \mathrm{e}^{-3} = \left(1 - 2\mathrm{e}^{-3}\right)x + 2 + \mathrm{e}^{-3} - 2\left(1 - 2\mathrm{e}^{-3}\right) = \left(1 - 2\mathrm{e}^{-3}\right)x + 5\mathrm{e}^{-3} \approx 0.900x + 0.249.$$

(iv) $\int y\,\mathrm{d}x = \int x + \mathrm{e}^{1-2x}\,\mathrm{d}x = \frac{x^2}{2} - \frac{1}{2}\mathrm{e}^{1-2x} + D$ (where D is the constant of integration).

The requested area is $\int_0^1 x+\mathrm{e}^{1-2x}\,\mathrm{d}x = \left[\frac{x^2}{2} - \frac{1}{2}\mathrm{e}^{1-2x}\right]_0^1 = \left(\frac{1}{2} - \frac{1}{2}\mathrm{e}^{-1}\right) - \left(0 - \frac{1}{2}\mathrm{e}^{1}\right) = \frac{1}{2}\left(1 - \mathrm{e}^{-1} + \mathrm{e}\right)$.

A885 (8865 N2018/5)(i) $C = 6x + 1\,200 \times 200 + \frac{50 \times 1\,200}{x} = 6x + 240\,000 + \frac{60\,000}{x}$. (The number of orders made is $1\,200/x$.)

(ii) $\frac{\mathrm{d}C}{\mathrm{d}x} = 6 - \frac{60\,000}{x^2}$. The stationary points are given by $\left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=\bar{x}} = 0$ or $6 - \frac{60\,000}{\bar{x}^2} = 0$ or $\bar{x} = \pm 100$ (reject negative value).

Observe that $\frac{\mathrm{d}C}{\mathrm{d}x} > 0$ for all $x \in [0, \bar{x})$ and $\frac{\mathrm{d}C}{\mathrm{d}x} < 0$ for all $x \in (\bar{x}, \infty)$. Hence, $\bar{x}$ is a strict global minimum.

(iii) My answer would be, "Maybe, it depends," but this would have gotten you zero marks.

The "correct" answer that would've gotten you one mark was probably something like, "No, this is not a reasonable model. The total cost is not constant but varies depending on such factors as the number of televisions bought and orders made."

(iv) Let Q be the number sold. Then $Q = 2\,100 - 3S$ ($S \geq 0$). Hence, profit is

$$P = QS - \text{Total Cost} = (2\,100 - 3S)\,S - 240\,000 = 2\,100S - 3S^2 - 240\,000.$$

(v) Compute $\dfrac{\mathrm{d}P}{\mathrm{d}S} = 2\,100 - 6S$. The stationary points are given by $\left.\dfrac{\mathrm{d}P}{\mathrm{d}S}\right|_{S=\bar{S}} = 0$ or $\bar{S} = 350$. Observe that $\dfrac{\mathrm{d}P}{\mathrm{d}S} < 0$ for all $S \in \left[0, \bar{S}\right)$ and $\dfrac{\mathrm{d}P}{\mathrm{d}S} > 0$ for all $S \in \left(\bar{S}, \infty\right)$. Hence, $\bar{S}$ is a strict global maximum.

Answers for Probability and Statistics questions to be written.

Exercise 893. (8865 N2017/1.) (Answer on p. 1534.)

Find algebraically the set of values of k for which

$$x^2 + (k-4)x - (k-7) > 0$$

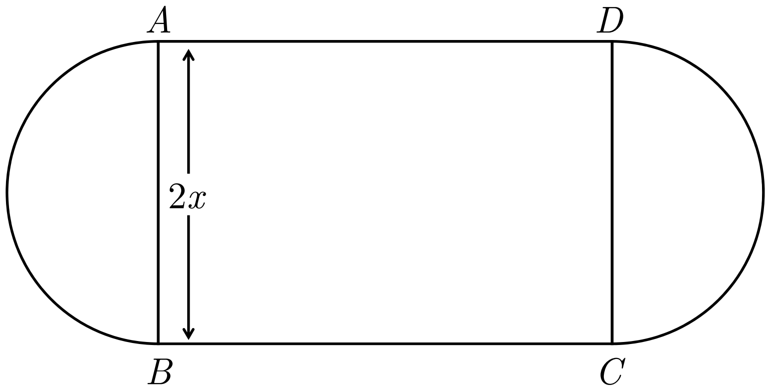
for all real values of x. [4]

Exercise 894. (8865 N2017/2.) (Answer on p. 1534.)

(i) Differentiate $\dfrac{1}{\sqrt{5x-2}}$ with respect to x. [2]

(ii) Find $\displaystyle\int \frac{(2x^2-1)^2}{x^2}\,\mathrm{d}x$. [4]

Exercise 895. (8865 N2017/3.) (Answer on p. 1534.)



The diagram shows a sign in the shape of a rectangle $ABCD$ with two semicircles, one attached to AB and one attached to CD. The length of AB is $2x$ cm and the total perimeter of the sign is 10 cm.

(i) Show that the area of the sign is $x(10-\pi x)$ cm^2. [3]

The area of the sign is to be as large as possible.

(ii) Use a non-calculator method to find the maximum value of this area, giving your answer in terms of π. Justify that this is the maximum value. [4]

Exercise 896. (8865 N2017/4.) (Answer on p. 1534.)

The equation of a curve is $y = \ln(4x-5)$.

(i) Sketch the curve, stating the equations of any asymptotes. [2]

(ii) Find the equation of the tangent to the curve at the point where $x = 2.5$, giving your answer in the form $ax + by = c$, where a and b are integers and c is in exact form. [4]

This tangent meets the x-axis at P and the y-axis at Q.

(iii) Find the length of PQ, giving your answer correct to 3 decimal places. [4]

Exercise 897. (8865 N2017/5.) (Answer on p. **1535**.)

A company produces three types of sports shoes: Supers, Runners and Walkers. The manufacturing cost of a pair of Supers is twice the manufacturing cost of a pair of Walkers. The total manufacturing cost of 10 pairs of Runners is \$50 more than the total manufacturing cost of 7 pairs of Supers. The total manufacturing cost of 2 pairs of Supers, 6 pairs of Runners and 4 pairs of Walkers is \$481.

(i) By writing down three linear equations, find the manufacturing cost of a pair of Runners. [5]

An economist advises the company on how to increase their profit.

[Profit = selling price − manufacturing cost]

In a simple model, the economist suggests that the selling price of all sports shoes should be \$80 a pair.

(ii) Find the profit from the sale of 100 pairs of each of Supers, Runners and Walkers. [2]

The company is trialling a new type of sports shoes, Extremes. The economist predicts that the profit \$$P$ will be related to the manufacturing cost (\$$x$) by the equation

$$P = 7\sqrt{x} - 0.9x.$$

(iii) Sketch the graph of P against x, stating the coordinates of the intersections with the x-axis. [2]

(iv) Use your calculator to estimate the maximum value of P. State also the value of x for which this maximum value occurs. [2]

(v) Given that the manufacturing cost of a pair of Extremes is \$55, find the selling price. [1]

(vi) If the manufacturing cost of a pair of Extremes increased to \$65, would you advise the company to produce Extremes? Justify your answer. [1]

Remark 205. For **(v)** and **(vi)**, use the economist's prediction.

Exercise 898. (8865 N2017/6.) (Answer on p. **1536**.)

As part of an assessment of the health of people in a particular country, the heights of a large number of adult males have been recorded. The results show that 20% of them have a height less than 1.6 mand 30% of them have a height greater than 1.75 m. Assuming that the heights of adult males are normally distributed, find the mean and variance of the distribution. [4]

Exercise 899. (8865 N2017/7.) (Answer on p. **1536**.)

Printers in a busy office produce large numbers of documents each week. The ink cartridges used in the printers often need replacing. The probability that an ink cartridge will last for one week or more is 0.7, independently of all other cartridges. The cartridges are supplied in boxes of 8. A box is selected at random.

(i) Find the probability that exactly 5 of the cartridges in the box will last for one week or more. [1]

(ii) Find the probability that at least half of the cartridges will last for less than one week. [2]

The office has 6 boxes of ink cartridges in stock.

(iii) Find the probability that, for at most 2 of the boxes, at least half of the cartridges will last less than one week. [2]

Exercise 900. (8865 N2017/8.) (Answer on p. **1536**.)

A code consists of 6 characters. The first 3 characters of the code consist of 3 digits chosen from $\{1, 2, 3, 4, 5, 6\}$. The last 3 characters of the code consist of 3 letters chosen from {A, B, C, D, E, F, G, H}.

(i) How many codes can be formed if repetitions are not allowed? [1]

Now suppose that repetitions are allowed.

(ii) Find the probability that a code chosen at random

(a) contains the digit 5 exactly once and the letter H exactly twice, [3]

(b) has 2 as its first character or H as its sixth character, but not both. [3]

Exercise 901. (8865 N2017/9.) (Answer on p. **1536**.)

A computer manufacturing company employs a large number of workers on the production line. The owner encourages his employees to stay with the company by giving them increases in their earnings to reward their length of service. The weekly earnings, y hundred dollars, of a random sample of 8 employees from the production line who have been with the company for x years are given in the following table.

Employee	A	B	C	D	E	F	G	H
x	14	16	11	24	36	28	22	40
y	4.9	5.5	5.2	6.5	9.7	7.5	6.2	9.8

(i) Give a sketch of the scatter diagram of the data. [2]

(ii) Find the product moment correlation coefficient and comment on its value in the context of the data. [2]

(iii) Find the equation of the regression line of y on x in the form $y = ax + b$, giving the values of a and b correct to 3 significant figures. Sketch this line on your scatter diagram. [2]

Sue has been employed by the company for 2 years and she earns 190 dollars per week.

(iv) Use the equation of your regression line to calculate an estimate of the weekly earnings for employees on the production line who have been with the company for 2 years. [1]

Sue concludes that she should be earning more.

(v) Give **two** reasons why her conclusion might not be justified. [2]

Exercise 902. (8865 N2017/10.) (Answer on p. **1536**.)

Bottles of a certain type of juice are said to contain 0.6 litres. A random sample of 50 bottles is taken and the volumes of juice (in litres) in the bottles are measured. The unbiased estimates for the population mean and variance are 0.568 and 0.01528. The population mean volume is denoted by μ. The null hypothesis $\mu = 0.6$ is to be tested against the alternative hypothesis $\mu < 0.6$.

(i) Find the p-value of the test and state the meaning of this p-value in this context. [2]

(ii) State, giving a reason, whether it is necessary to assume a normal distribution for this test to be valid. [1]

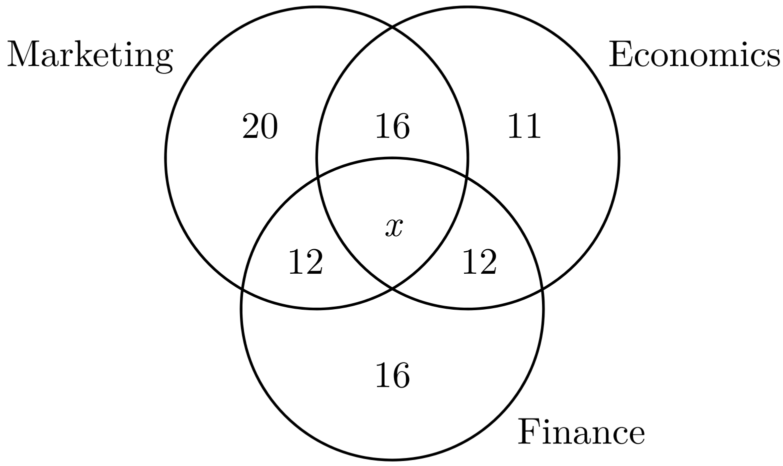
A second random sample of bottles of this juice is taken. The sample size is 110 and the volumes, y, are summarised by

$$\sum y = 70.4, \qquad \sum y^2 = 49.42.$$

(iii) Find unbiased estimates for the population mean and variance using this second sample. [3]

(iv) Using this second sample, test at the 5% significance level, whether there is evidence that the population mean volume of juice differs from 0.6 litres. [4]

Exercise 903. (8865 N2017/11.) (Answer on p. **1536**.)



Marketing, Economics and Finance are three subjects offered at a business college. The numbers of students studying different combinations of these subjects are shown in the above Venn diagram. Every student studies at least one of these subjects. The number who study all three subjects is x. One of the students is chosen at random.

- M is the event that the student studies Marketing.
- E is the event that the student studies Economics.
- F is the event that the student studies Finance.

(i) Write down expressions for $\mathrm{P}(M)$ and $\mathrm{P}(E)$ in terms of x. [2]

(ii) Given that events M and E are independent, find the value of x. [3]

(iii) Find $\mathrm{P}(M \cup F')$. [1]

(iv) Explain, in the context of this question, what is meant by $\mathrm{P}(F|M)$, and find its value. [3]

Three students are chosen at random, without replacement.

(v) Find the probability that each studies exactly two of these three subjects. [3]

Exercise 904. (8865 N2017/12.) (Answer on p. **1536**.)

There are bus and train services between the towns of Ayton and Beeton. The journey times, in minutes, by bus and by train have independent normal distributions. The means and standard deviations of these distributions are shown in the following table.

	Mean	Standard deviation
Bus	45	4
Train	42	3

(i) Find the probability that a randomly chosen bus journey takes less than 48 minutes. [1]

(ii) Find the probability that two randomly chosen bus journeys each take more than 48 minutes. [2]

(iii) The probability that the total time for two randomly chosen bus journeys is more than 96 minutes is denoted by p. Without calculating its value, explain why p will be greater than your answer to part **(ii)**. [1]

Lan lives in Ayton and works in Beeton. Three days a week he travels from home to work by bus and two days a week he travels from home to work by train.

(iv) Find the probability that for 3 randomly chosen bus journeys and 2 randomly chosen train journeys, Lan's total journey time is more than 210 minutes. [4]

Journeys are charged by the time taken. For bus journeys the charge is \$0.12 per minute and for train journeys the charge is \$0.15 per minute.

Let B represent the cost of one journey from Ayton to Beeton by bus.

Let T represent the cost of one journey from Ayton to Beeton by train.

(v) Find $\mathrm{P}(3B - 2T < 3)$ and explain, in the context of this question, what your answer represents. [5]

A893 (8865 N2017/1). The coefficient on x^2 is $1 > 0$, so the given quadratic is a ∪-shaped curve.

Hence, the given inequality holds for all real x if and only if the determinant is negative:

$$0 > (k-4)^2 - 4(1)[-(k-7)] = k^2 - 4k - 12 = (k-6)(k+2).$$

This latter inequality in turn holds if and only if $k \in (-2, 6)$.

A894 (8865 N2017/2)(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{\sqrt{5x-2}} = \dfrac{5}{2(5x-2)^{3/2}}$.

(ii) $\displaystyle\int \frac{(2x^2-1)^2}{x^2}\,\mathrm{d}x = \int \frac{4x^4-4x^2+1}{x^2}\,\mathrm{d}x = \int 4x^2 - 4 + \frac{1}{x^2}\,\mathrm{d}x = \frac{4}{3}x^3 - 4x - \frac{1}{x} + C.$

A895 (8865 N2017/3)(i) The total perimeter of the sign is $10 = 2|AD| + \pi(2x)$. Hence, $|AD| = 5 - \pi x$.

The area of the rectangle $ABCD$ is $2x(5-\pi x) = 10x - 2\pi x^2$.

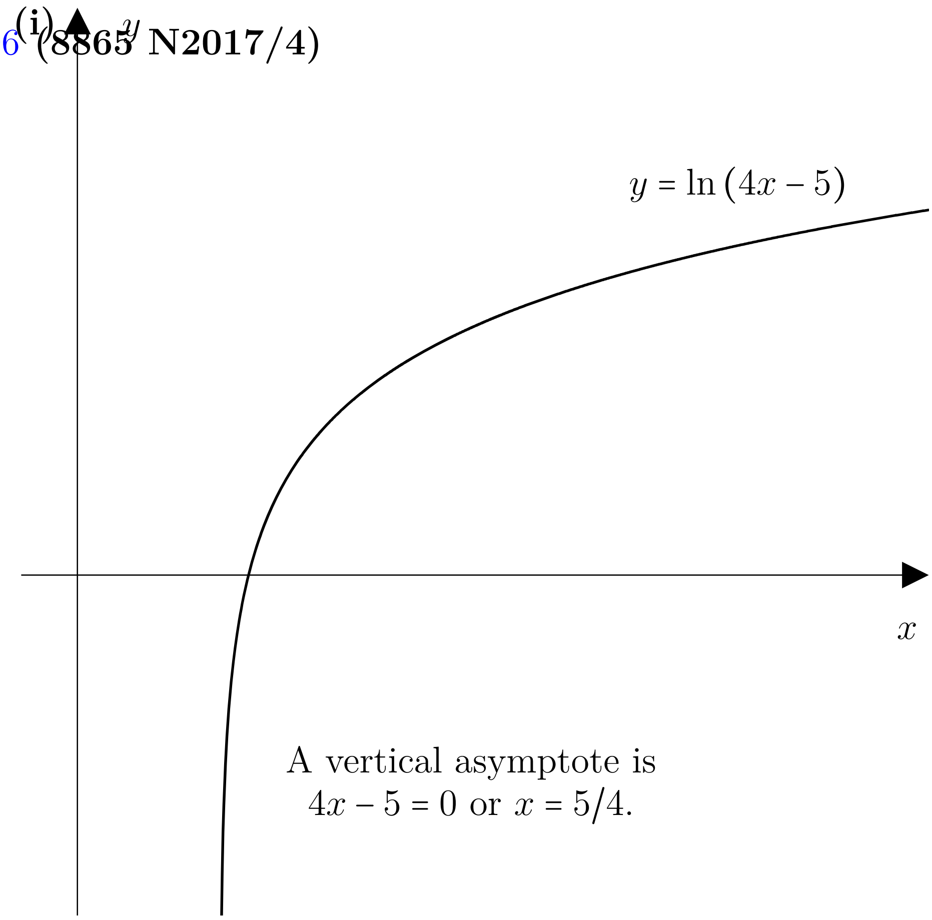
The area of the two semicircles is πx^2.

Hence, the area of the sign is $y = 10x - 2\pi x^2 + \pi x^2 = 10x - \pi x^2 = x(10-\pi x)$.

(ii) Compute $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 10 - 2\pi x$. The stationary points are given by $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\bar{x}} = 0$ or $10 - 2\pi\bar{x} = 0$ or $\bar{x} = \dfrac{5}{\pi}$.

Observe that $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ for all $x \in [0, \bar{x})$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ for all $x \in (\bar{x}, \infty)$. Hence, $\bar{x}$ is a strict global maximum.

A896 (8865 N2017/4)(i)



(ii) At $x = 2.5$, $y = \ln 5$. Compute $\left.\frac{dy}{dx}\right|_{x=2.5} = \left.\frac{4}{4x-5}\right|_{x=2.5} = \frac{4}{5}$. Hence, the requested tangent line is

$$y - \ln 5 = \frac{4}{5}(x - 2.5) \qquad \text{or} \qquad y - \frac{4}{5}x \overset{1}{=} \ln 5 - 2.$$

(iii) Plug $x = 0$ and $y = 0$ into $\overset{1}{=}$ to get, respectively, $y = \ln 5 - 2$ and $x = 2.5 - \frac{5}{4}\ln 5$. So,

$$|PQ| = \sqrt{(\ln 5 - 2)^2 + \left(2.5 - \frac{5}{4}\ln 5\right)^2} \approx 0.625.$$

A897 (8865 N2017/5)(i) Let s, r, and w be the manufacturing cost of a pair of Supers, Runners, and Walkers, respectively. The given information may be written as

$$s \overset{1}{=} 2w, \qquad 10r \overset{2}{=} 7s + 50, \qquad 2s + 6r + 4w \overset{3}{=} 481.$$

Plug $\overset{1}{=}$ into $\overset{2}{=}$ to get $10r = 14w + 50$ or $5r - 7w \overset{4}{=} 25$.

Plug $\overset{1}{=}$ into $\overset{3}{=}$ to get $6r + 8w \overset{5}{=} 481$.

Now take $8 \times \overset{4}{=} + 7 \times \overset{5}{=}$ to get $82r = 3\,567$ or $r \overset{6}{=} 43.5$.

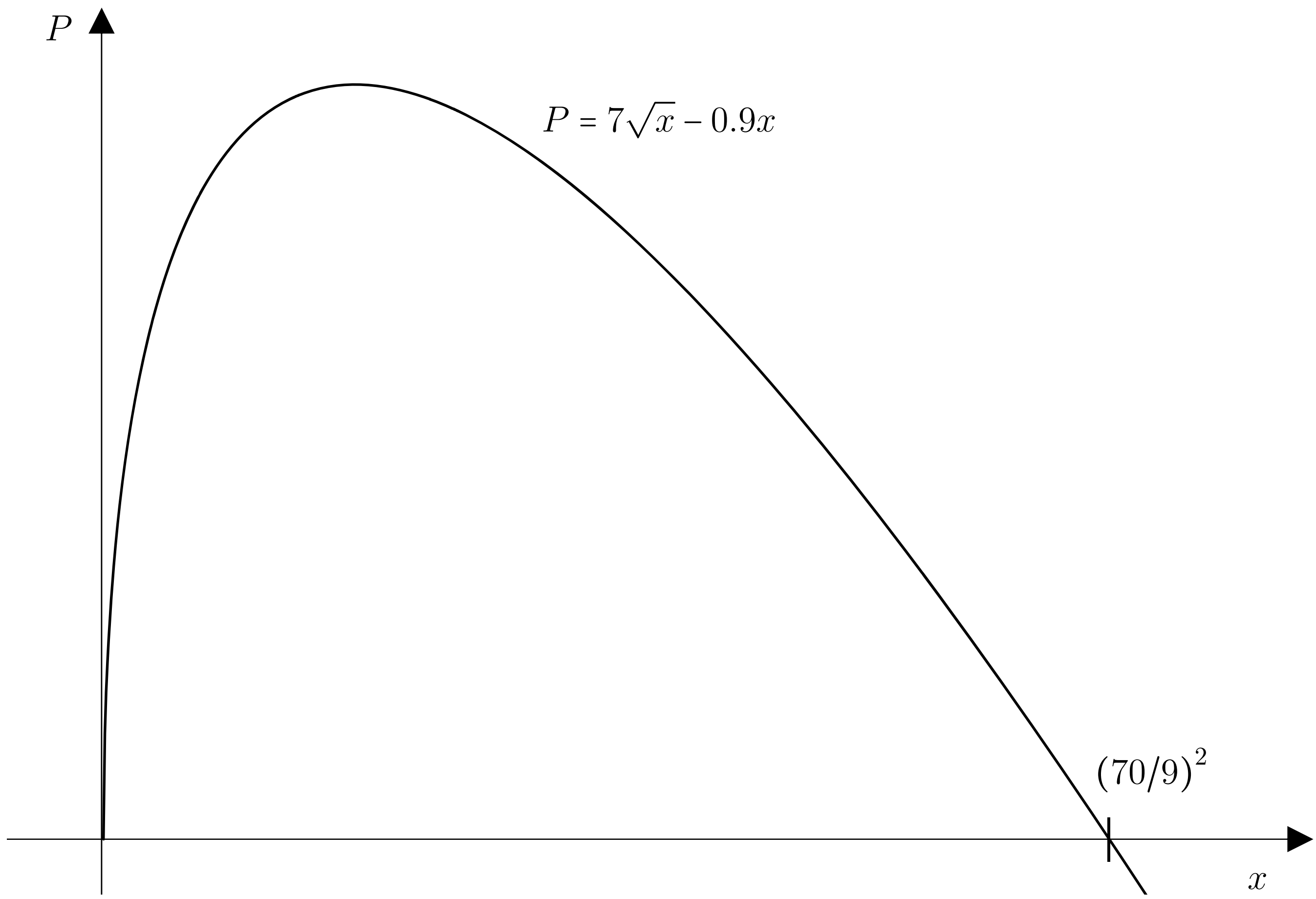
The manufacturing cost of a pair of Runners is \$43.50.

(ii) Plug $\overset{6}{=}$ into $\overset{2}{=}$ to get $435 = 7s + 50$ or $s \overset{7}{=} 55$.

Plug $\overset{7}{=}$ into $\overset{1}{=}$ to get $55 = 2w$ or $w = 27.5$.

So, the answer is $100\left[3 \times \$80 - \$\left(s + r + w\right)\right] = \$11\,400$.

(iii) $7\sqrt{x} - 0.9x = 0 \iff 7 - 0.9\sqrt{x} = 0 \iff 70/9 = \sqrt{x} \iff x = (70/9)^2$.



(iv) $P \approx 13.6$ at $x \approx 15.1$.

(v) $P(55) = 7\sqrt{55} - 0.9(55) \approx 2.41$. Hence, using the economist's model, the selling price is approximately $\$55 + \$2.41 = \$55.71$.

(vi) No, $65 > (70/9)^2$ and profit would be negative (according to the economist's model).

A898 (8865 N2017/6). XXX

A899 (8865 N2017/7). XXX

A900 (8865 N2017/8). XXX

A901 (8865 N2017/9). XXX

A902 (8865 N2017/10). XXX

A903 (8865 N2017/11). XXX

A904 (8865 N2017/12). XXX

140.7. H1 Maths 2016 Questions

Answers for Probability and Statistics questions to be written.

Exercise 905. (8864 N2016/1.) (Answer on p. **1541**.)

Differentiate

(i) $2\ln\left(3x^2+4\right)$, [2]

(ii) $\dfrac{1}{2(1-3x)^2}$. [2]

Exercise 906. (8864 N2016/2.) (Answer on p. **1541**.) **Do not use a calculator in answering this question.**

Use the substitution $u = e^x$ to solve the inequality $2e^{2x} \geq 9 - 3e^x$, giving your answer in logarithmic form. [5]

Exercise 907. (8864 N2016/3.) (Answer on p. **1541**.)

The curve C has equation $y = e^{-x} - x^2$.

(i) Sketch the graph of C. [1]

(ii) Find the numerical value of the gradient of C at the point where $x = 0.5$. [1]

(iii) Find the equation of the normal to C at the point where $x = 0.5$. Give your answer in the form $y = mx + c$, with m and c correct to 3 significant figures. [3]

(iv) Find $\displaystyle\int_0^k \left(e^{-x} - x^2\right) dx$, where $k > 0$. Give your answer in terms of k. [3]

Exercise 908. (8864 N2016/4.) (Answer on p. **1541**.)

The curve C has equation $y = 1 + 6x - 3x^2 - 4x^3$.

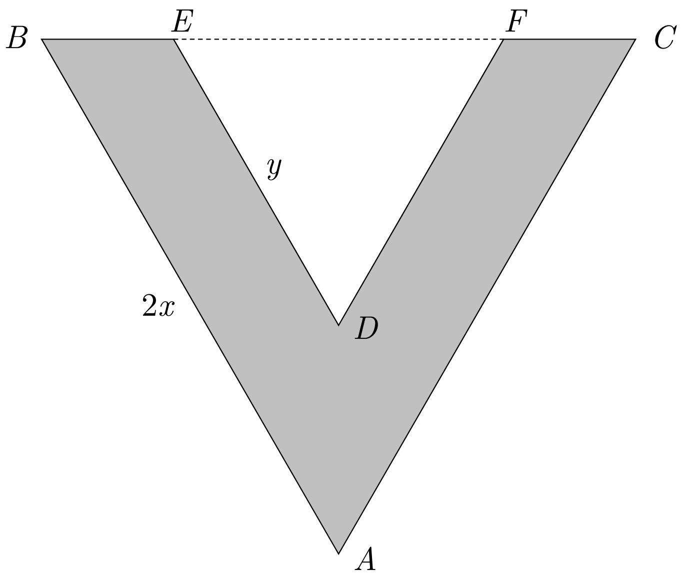
(i) Find $\dfrac{dy}{dx}$. Hence find the coordinates of the stationary points on the curve. [4]

(ii) Use a non-calculator method to determine the nature of each of the stationary points. [2]

(iii) Sketch the graph of C, stating the coordinates of any points where the curve crosses the x-axis. [2]

(iv) Find the numerical value of the area under the curve C between $x = 0.5$ and $x = 1$. [1]

Exercise 909. (8864 N2016/5.) (Answer on p. **1542**.)



The diagram shows a V-shape which is formed by removing the equilateral triangle DEF, in which $DE = y$ cm, from an equilateral triangle ABC, in which $AB = 2x$ cm. The points E and F are on BC such that $BE = FC$. The area of the V-shape $ABEDFCA$ is $2\sqrt{3}\,\text{cm}^2$.

(i) Show that $4x^2 - y^2 = 8$. [3]

(ii) Given that the perimeter of $ABEDFCA$ is 10 cm, find the values of x and y. [6]

Exercise 910. (8864 N2016/6.) (Answer on p. **1543**.)

A music store manager intend to carry out a survey to investigate how much money students at a local college spend on music each year. There are 1260 male students and 1140 female students at the college.

(i) Describe how to obtain a stratified sample of 80 students to take part in the survey. [2]

(ii) State, in this context, one advantage that stratified sampling has compared to simple random sampling. [1]

The amount of money, in hundreds of dollars, spent by a student in a year is X. For a simple random sample of 80 students, it is found that $\sum x = 312$ and $\sum x^2 = 1328$.

(iii) Calculate unbiased estimates for the population mean and variance of X. [3]

Exercise 911. (8864 N2016/7.) (Answer on p. **1543**.)

The events A and B are such that $\mathrm{P}(A) = 0.6$, $\mathrm{P}(B) = 0.25$ and $\mathrm{P}(A \cap B) = 0.05$.

(i) Draw a Venn diagram to represent this situation, showing the probability in each of the four regions. [3]

(ii) Find the probability that

(a) at least one of A and B occurs, [1]

(b) exactly one of A and B occurs. [1]

(iii) Find $\mathrm{P}(A|B')$. [2]

Exercise 912. (8864 N2016/8.) (Answer on p. **1543**.)

Two boxes, A and B, contain balls of different colours. Box A contains 5 blue balls, 3 red balls and 2 green balls. Box B contains 4 blue balls and 2 green balls. One of the boxes is selected at random. Two balls are then chosen at random, without replacement, from the selected box. Find the probability that

(i) both balls are red, [2]

(ii) the two balls are of different colours, [4]

(iii) both balls are red, given that they are the same colour. [3]

Exercise 913. (8864 N2016/9.) (Answer on p. **1543**.)

Watch batteries are supplied to a shop in packs of 8. The probability that any randomly chosen battery has a lifetime of less than two years is 0.6, independently of all other batteries.

(i) For a single pack of batteries, find the probability that

(a) all of the batteries have a lifetime of less than two years, [1]

(b) at least half of the batteries have a lifetime of less than two years. [2]

(ii) For any 4 packs of batteries, find the probability that, for no more than 2 of the packs, at least half of the batteries have a lifetime of less than two years. [2]

(iii) A customer buys 10 packs of these batteries. Use a suitable approximation to estimate the probability that at least 40 of these batteries have a lifetime of less than two years. State the mean and variance of the distribution that you use. [4]

Exercise 914. (8864 N2016/10.) (Answer on p. **1543**.)

A scientist claims that the mean top speed of cheetahs, in km/h, is 95. The top speed of each cheetah in a random sample of 40 cheetahs is recorded and the mean is found to be 96.3. It is known that the top speeds of cheetahs are normally distributed with standard deviation 4.1.

(i) Test the scientist's claim at the 5% significance level. [4]

The scientist now decides to test the claim that the mean top speed of cheetahs, in km/h, is greater than 95. He takes a second random sample of 40 cheetahs and records their top speeds. Using a 5% significance level, he finds that the mean top speed of cheetahs is *not* greater than 95.

(ii) Find the set of values within which the mean top speed of this second sample must lie. [5]

Exercise 915. (8864 N2016/11.) (Answer on p. **1543**.)

Members of an athletics club are training for a 'Swim-Run' charity event, in which each athlete has to complete a 1000-metre swim followed by a 1000-metre run. The times, x minutes, to swim 1000 metres and the times, y minutes, to run 1000 metres, for a random sample of 8 members of the club, are given in the following table.

Athlete	A	B	C	D	E	F	G	H
x	15.0	16.1	18.2	16.3	17.2	18.1	15.6	16.3
y	2.5	2.7	3.3	3.0	3.5	3.4	2.7	2.8

(i) Give a sketch of the scatter diagram for the data, as shown on your calculator. [2]

(ii) Find the product moment correlation coefficient. [1]

(iii) Find the equation of the regression line of y on x in the form $y = mx + c$, giving the values of m and c correct to 3 significant figures. [1]

(iv) Calculate an estimate of the time taken to run 1000 metres by an athletes who swims 1000 metres in 16.9 minutes. State two reasons why you would expect this to be a reliable estimate. [3]

The time taken by a new member of the club to swim 1000 metres and run 1000 metres are 18.4 minutes and 2.6 minutes respectively.

(v) Calculate the new product moment correlation coefficient when the times for the new member are included. [1]

(vi) State, with a reason, which of your answers to parts **(ii)** and **(v)** is more likely to represent the correlation between swimming and running times for all members of the club. [1]

Exercise 916. (8864 N2016/12.) (Answer on p. **1543**.)

Shortbread biscuits of a certain brand are sold in boxes containing 12 biscuits. The masses, in grams, of the individual biscuits and of the empty boxes have independent normal distributions with means and standard deviations as shown in the following table.

	Mean	Standard deviation
Individual biscuit	20	1.1
Empty box	5	0.8

(i) Find the probability that the mass of an individual biscuit is less than 19 grams. [2]

(ii) Find the probability that the total mass of a box containing 12 biscuits is more than 248 grams. State the mean and variance of the distribution that you use. [4]

The cost of producing biscuits if 0.6 cents per gram and the cost of producing empty boxes is 0.2 cents per gram.

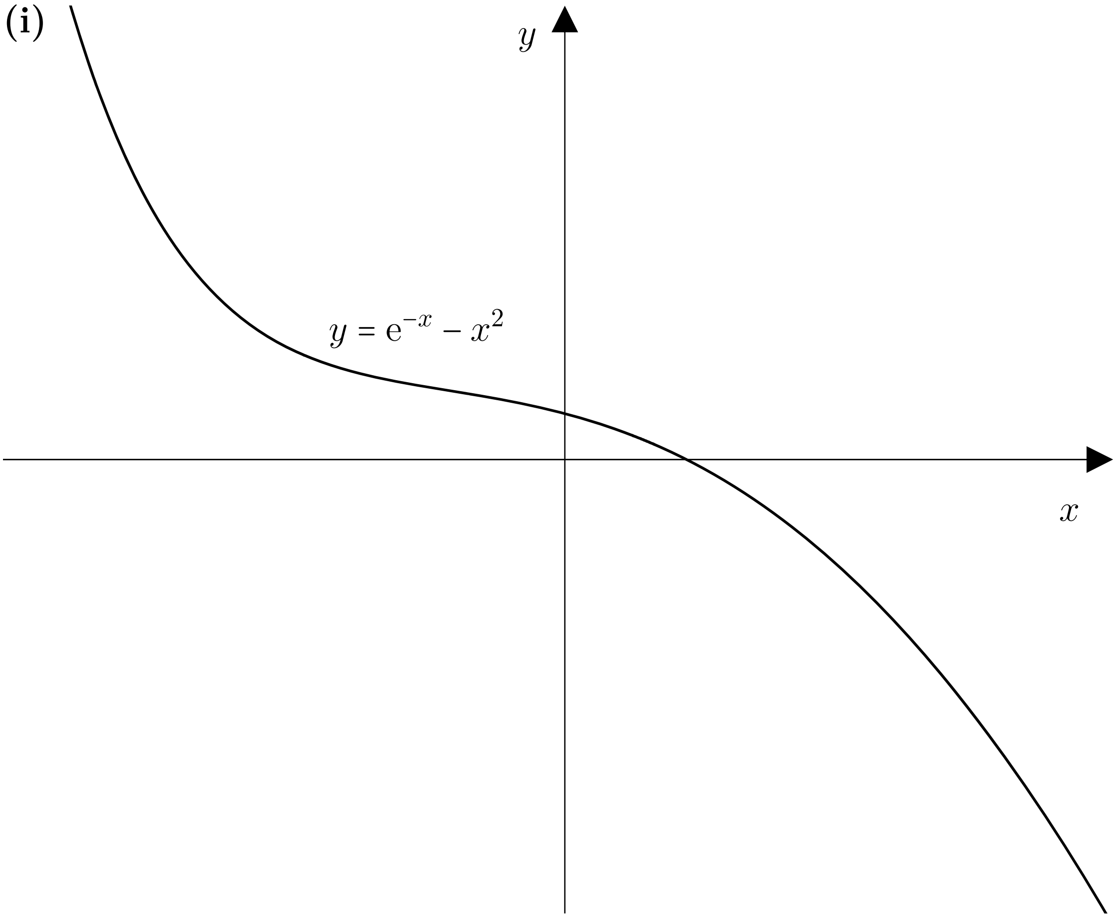
(iii) Find the probability that the total cost of producing a box containing 12 biscuits is between 142 cents and 149 cents. State the mean and variance of the distribution that you use. [5]

A905 (8864 N2016/1)(i) $\frac{d}{dx}2\ln\left(3x^2+4\right) = 2\frac{6x}{3x^2+4} = \frac{12x}{3x^2+4}$.

(ii) $\frac{d}{dx}\frac{1}{2(1-3x)^2} = \frac{-2\cdot(-3)}{2(1-3x)^3} = \frac{3}{(1-3x)^3}$.

A906 (8864 N2016/2). $2e^{2x} \geq 9-3e^x \iff 2u^2 \geq 9-3u \iff 0 \leq 2u^2+3u-9 = (2u-3)(u+3) \iff u \in (-\infty,-3] \cup [1.5,\infty) \iff e^x \in [1.5,\infty) \iff x \in [\ln 1.5, \infty)$.

(i)



A907 (8864 N2016/3)

(ii) Compute $\frac{dy}{dx} = -e^{-x}-2x$. So, $\left.\frac{dy}{dx}\right|_{x=0.5} = -e^{-0.5}-1 \approx -1.61$.

(iii) At $x = 0.5$, $y = e^{-0.5}-0.5^2$. So, the equation of the normal is approximately

$$y - \left(e^{-0.5}-0.5^2\right) = \frac{-1}{-e^{-0.5}-1}(x-0.5) \qquad \text{or} \qquad y = 0.622x + 0.453.$$

(iv) $\int_0^k \left(e^{-x}-x^2\right) dx = \left[-e^{-x}-\frac{x^3}{3}\right]_0^k = \left(-e^{-k}-\frac{k^3}{3}\right) - (-1-0) = 1-e^{-k}-\frac{k^3}{3}$.

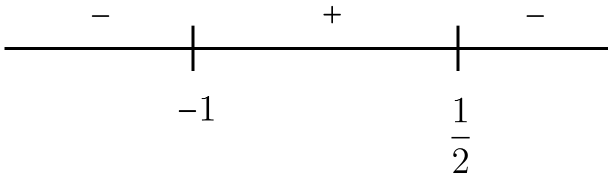
A908 (8864 N2016/4)(i) $\frac{dy}{dx} = 6-6x-12x^2$.

$\left.\frac{dy}{dx}\right|_{x=\bar{x}} = 0 \iff 0 = 6-6\bar{x}-12\bar{x}^2 = 2\bar{x}^2+\bar{x}-1 = (2\bar{x}-1)(\bar{x}+1) \iff \bar{x} = -1, \frac{1}{2}$. The corresponding y-coordinates are

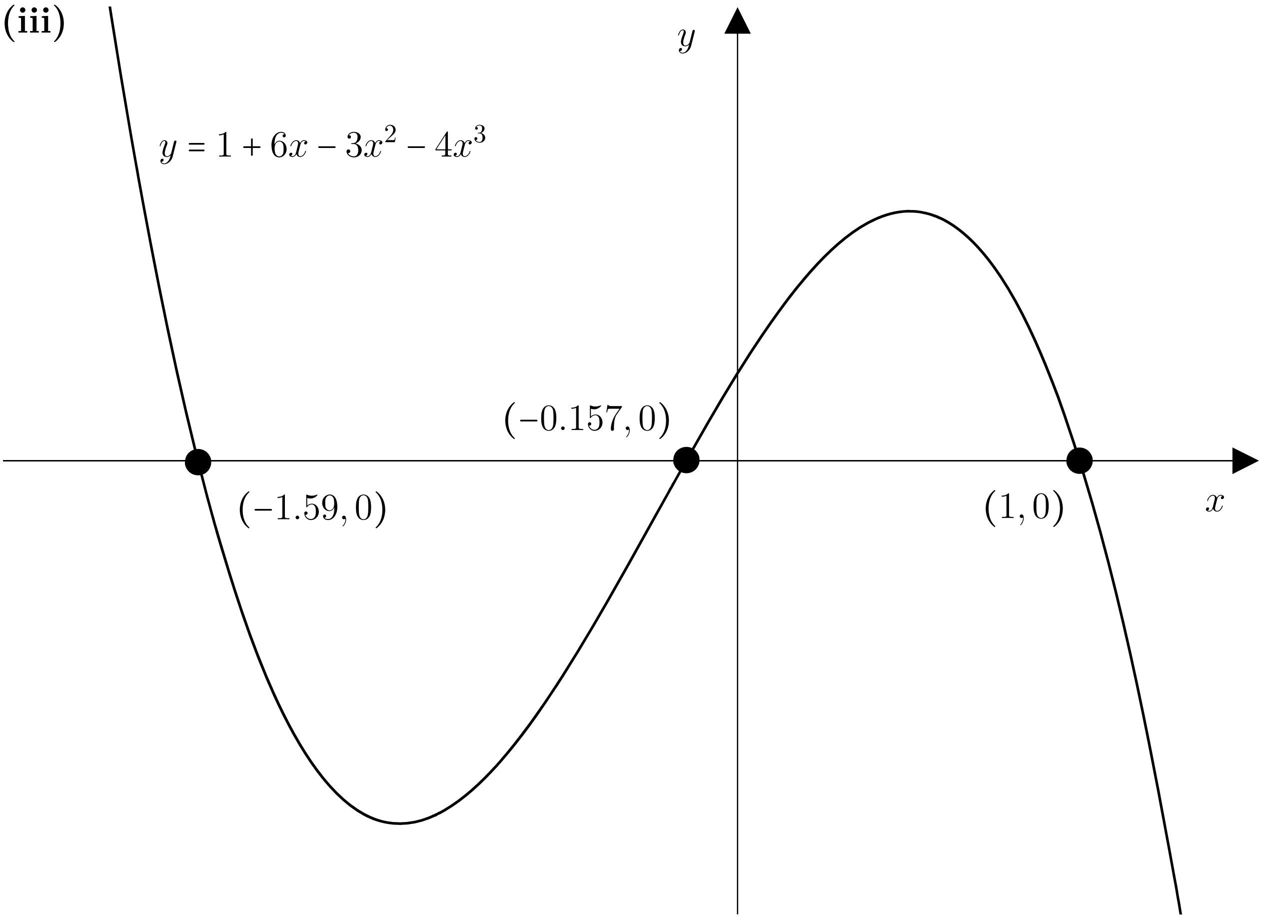
$$1+6(-1)-3(-1)^2-4(-1)^3 = -4 \qquad \text{and} \qquad 1+6\left(\frac{1}{2}\right)-3\left(\frac{1}{2}\right)^2-4\left(\frac{1}{2}\right)^3 = \frac{11}{4}.$$

Hence, the two stationary points are $(-1,-4)$ and $\left(\frac{1}{2},\frac{11}{4}\right)$.

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = 6-6x-12x^2$ is ∩-shaped. Hence, -1 is a strict minimum point, while $\frac{1}{2}$ is a strict maximum point.



(iii)



(iv) $\int_{0.5}^{1} 1+6x-3x^2-4x^3\,\mathrm{d}x = \left[x+3x^2-x^3-x^4\right]_{0.5}^{1} = 2-\left(\frac{1}{2}+\frac{3}{4}-\frac{1}{8}-\frac{1}{16}\right) = \frac{15}{16}$.

A909 **(8864 N2016/5)(i)** The heights of the triangles ABC and DEF are, respectively, $\sqrt{(2x)^2-x^2} = \sqrt{3}x$ and $\sqrt{y^2-(y/2)^2} = \frac{\sqrt{3}}{2}y$.

Hence, their areas are, respectively, $\frac{1}{2}\left(\sqrt{3}x\right)(2x) = \sqrt{3}x^2$ and $\frac{1}{2}\left(\frac{\sqrt{3}}{2}y\right)(y) = \frac{\sqrt{3}}{4}y^2$.

Thus, their difference is $\sqrt{3}x^2 - \frac{\sqrt{3}}{4}y^2 = 2\sqrt{3}$ or $4x^2-y^2 \stackrel{1}{=} 8$.

(ii) The perimeter is $10 = |AB|+|AC|+|BE|+|FC|+|DE|+|DF| = 2x+2x+2x-y+y+y = 6x+y$.

Rearrange to get $y \stackrel{2}{=} 10-6x$ and plug $\stackrel{2}{=}$ into $\stackrel{1}{=}$ to get

$$4x^2-(10-6x)^2 = 8 \qquad \text{or} \qquad 0 = -32x^2-108+120x = 8x^2-30x+27 = (4x-9)(2x-3)$$

$$\text{or} \qquad x = \frac{9}{4}, \frac{3}{2}.$$

Plug in $x = \frac{9}{4}, \frac{3}{2}$ into $\stackrel{2}{=}$ to get the corresponding values for y: $y = -\frac{7}{2}$ (reject) and $y = 1$.

Hence, $x = 3/2$ and $y = 1$.

A910 **(8864 N2016/6). XXX**

A911 **(8864 N2016/7). XXX**

A912 **(8864 N2016/8). XXX**

A913 **(8864 N2016/9). XXX**

A914 **(8864 N2016/10). XXX**

A915 **(8864 N2016/11). XXX**

A916 **(8864 N2016/12). XXX**

Part VIII. Appendices

The main text above has not always been complete, precise, or rigorous. In these appendices, I go some way towards filling in these gaps. In particular, I give formal definitions, statements of claims, and proofs of claims.

Where there is a trade-off between generality of a result and the simplicity of its proof, I usually favour the latter.

I've written these appendices mostly for the sake of completeness (and also to persuade myself that I probably haven't screwed up too much). Unlike the main text, I would not recommend that the student (or anyone) read these appendices from top to bottom. They are written rather tersely with few (and often zero) examples.

141. Appendices for Part 0. A Few Basics

141.1. Division

Definition 262. (Euclidean division) Let x and d be non-zero integers. If $x, d < 0$, then let q be the smallest integer that satisfies $dq \leq x$; otherwise, let q be the largest integer that satisfies $dq \leq x$. Then let $r = x - dq \geq 0$.

In the equation $x = dq + r$, we call x the *dividend*, d the *divisor*, q the *quotient*, and r the *remainder*.

And if $r = 0$, then we say that x divided by d *leaves no remainder* and that d is a *factor* of (or *divides*) x.

Theorem 50. (The Euclidean Division Theorem) *Let x and d be non-zero integers. Then there exists a unique pair of integers (q, r) such that*

$$x = dq + r \qquad \textit{and} \qquad 0 \leq r < d.$$

Proof. Let q and r be as constructed in Definition 262. This proves existence.

For uniqueness, we consider only the case where $x, d > 0$ (the other cases are similarly dealt with).

Let θ and ρ be integers that satisfy $x = d\theta + \rho$ and $0 \leq \rho < d$. Then

$$0 = x - x = (dq + r) - (d\theta + \rho) = d(q - \theta) + r - \rho.$$

Rearranging, $r - \rho = d(\theta - q)$.

If $\theta = q$, then $r - \rho = 0$ and $\rho = r$, so that (q, r) is indeed unique.

If $\theta < q$, then $\theta - q \leq -1$, so that $r - \rho \leq -d$ and $r \leq \rho - d < 0$, contradicting our assumption that $0 \leq r$.

If $\theta > q$, then $\theta - q \geq 1$, so that $r - \rho \geq d$ and $r \geq \rho + d \geq d$, contradicting our assumption that $r < d$. $\square$

Fact 1. *Suppose P and Q are statements. Then*

$$\text{NOT-}(P \text{ AND } Q) \iff \text{NOT-}P \text{ OR NOT-}Q.$$

Proof. We want to show that NOT−(P AND Q) and (NOT−P OR NOT−Q) always have the same truth value. That is, we want to show that

1. Whenever NOT−(P AND Q) is true, (NOT−P OR NOT−Q) is also true; and
2. Whenever NOT−(P AND Q) is false, (NOT−P OR NOT−Q) is also false.

We now do so:

1. Suppose NOT−(P AND Q) is true. Then P AND Q is false. So, either P or Q is false. So, either NOT−P or NOT−Q is true. So, NOT−P OR NOT−Q is true.
2. Next, suppose NOT−(P AND Q) is false. Then P AND Q is true. So, both P and Q are true. So, both NOT−P and NOT−Q are false. So, NOT−P OR NOT−Q is false.

Since NOT−(P AND Q) and as (NOT−P OR NOT−Q) always have the same truth value, they are equivalent. □

Letting "1" = true and "0" = false, we can also construct this **truth table** where there are four possible cases depending on whether P and Q are true or false:

P	Q	NOT−P	NOT−Q	P AND Q	NOT−(P AND Q)	NOT−P OR NOT−Q
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	0	1	1

This truth table is a compact way to prove Fact 1. From the table, we can quickly see that in each of the four possible cases, both

$$\text{NOT-}(P \text{ AND } Q) \qquad \text{and} \qquad (\text{NOT-}P \text{ OR NOT-}Q)$$

always have the same truth value. Thus, the two statements are equivalent.

By the way, we could also have defined equivalence using truth tables—two statements are equivalent if their truth "columns" are identical. We'd have to take care though to ensure that our truth table lists all possibilities.

Fact 2. *Suppose P and Q are statements. Then*

$$\text{NOT-}(P \text{ OR } Q) \iff \text{NOT-}P \text{ AND NOT-}Q.$$

Proof. Construct this truth table:

P	Q	NOT-P	NOT-Q	P OR Q	NOT-(P OR Q)	NOT-P AND NOT-Q
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	1

Since NOT-(P OR Q) and NOT-P AND NOT-Q always have the same truth value, they are equivalent. □

Fact 6. $(P \implies Q \text{ AND } Q \implies P) \iff (P \iff Q)$.

Proof. We show $P \iff Q$ always has the same truth value as $P \implies Q$ AND $Q \implies P$:

1. Suppose $P \iff Q$ is true. Then

   (a) Whenever P is true, Q is also true—so, $P \implies Q$ is always true.

   (b) Whenever P is false, Q is also false—so, $Q \implies P$ is always true.

   Altogether, $P \implies Q$ AND $Q \implies P$ is true.

2. Suppose $P \iff Q$ is false. Then there exists some instance where either

   (a) P is true while Q is false, so that $P \implies Q$ is false; or

   (b) P is false while Q is true, so that $Q \implies P$ is false.

   Altogether, $P \implies Q$ AND $Q \implies P$ is false. □

141.3. The Four Categorical Statements

Before Ch. 3.13, we were discussing only **propositional logic**.

Ch. 3.13 began discussing **predicate** (or **first-order**) **logic**. The terms **all** and **some** correspond to the **for every** ($\forall$) and **there is at least one** ($\exists$) **quantifiers**:

Definition 263. The *All Yes* statement, "All S are P," means, "For every x, if x is S, then x is P."

The *All No* statement, "No S is P," (or "All S is NOT−P") means, "For every x, if x is S, then x is NOT−P."

The *Some Yes* statement, "Some S is P," means, "There is at least one x such that x is S AND x is P."

The *Some No* statement, "Some S is NOT−P," means, "There is at least one x such that x is S AND x is NOT−P."

Note that thus defined, the All Yes and All No statements inherit from the implication the property that if the hypothesis (in the implication) is false, then the statement as a whole is automatically true:

Example 1565. The All Yes statement, "All unicorns are green," means,

"For every x, if x is a unicorn, then x is green."

Since unicorns don't exist, the statement "x is a unicorn" is always false. So, the implication "if x is a unicorn, then x is green" is always true. Hence, the above All Yes statement is also always true.

Example 1566. The All No statement, "No unicorns is green," means,

"For every x, if x is a unicorn, then x is not green."

Since unicorns don't exist, the statement "x is a unicorn" is always false. So, the implication "if x is a unicorn, then x is not green" is always true. Hence, the above All No statement is also always true.

Note that we are following modern logic. Which contrasts with traditional, pre-19th-century, Aristotelian logic, where the All Yes and All No statements in the last two examples would instead have been considered false. See e.g. Parsons (2017), "The Traditional Square of Opposition".

To negate the categorical statements, we'll use this axiom:[568]

Axiom 1. *The negation of "For every x, P" is "There is at least one x such that* NOT$-P$*."*

The negation of "For every x, NOT$-P$*" is "There is at least one x such that P."*

Fact 7. **(a)** *The negation of an All Yes statement is the corresponding Some No statement.*

(b) *The negation of an All No statement is the corresponding Some Yes statement.*

Proof. **(a)** By Definition 263, the All Yes statement, "All S are P," means, "For every x, x is $S \implies x$ is P". By Axiom 1 and Fact 3, this statement's negation is, "There is at least one x such that x is S AND x is NOT$-P$," which by Definition 263 is equivalent to the Some No statement, "Some S is NOT$-P$."

(b) Similar. □

[568]This could also be a result derived from more basic definitions and axioms. But to keep things short and simple, I'll simply assert this as an axiom.

141.4. Sets

The set is what mathematicians call a **primitive notion**. That is, sets are left **undefined** (though they do have to satisfy certain axioms). But having summoned out of the void this single undefined object called the set, we can then go on to define every other mathematical object based on the set. The set is thus the single Lego block out of which *all* of mathematics is built. *Every* mathematical object can be defined solely in terms of sets. The idea is to have just one undefined object, then define everything else based on this single undefined object.

And so for example, in conventional set theory, we first *define* the number 0 to be the empty set. We then *define* the number 1 as the set that contains 0; the number 2 as the set that contains 0 and 1; the number 3 as the set that contains 0, 1, and 2; etc.

$$\begin{aligned}
0 &= \{\} && = \varnothing. \\
1 &= \{0\} && = \{\{\}\} && = \{\varnothing\}. \\
2 &= \{0,1\} && = \{\{\},\{\{\}\}\} && = \{\varnothing,\{\varnothing\}\}. \\
3 &= \{0,1,2\} && = \{\{\},\{\{\}\},\{\{\},\{\{\}\}\}\} && = \{\varnothing,\{\varnothing\},\{\{\varnothing\}\}\}. \\
&\vdots
\end{aligned}$$

As another example, perhaps surprisingly the **function** is also defined to be a set. We'll see this shortly in Ch. 142.10 below.

In the main text (Ch. 4.9), we made this claim:

Proposition 25. *A real number is rational* $\iff$ *It eventually recurs.*

Proof. We'll formally prove this only below (see p. 1552). For now, we'll give only this proof sketch:

($\Longleftarrow$) Consider for example $x = 8.344\overline{57193} = 8.344\,571\,935\,719\,357\,193\ldots$ where the digits 57 193 eventually recur. Now consider $99\,999\,000x$. We have

$$\begin{aligned}
99\,999\,000x &= 100\,000\,000x - 1\,000x \\
&= 834\,457\,193.571\,935\,719\,3\cdots - 8\,344.571\,935\,719\,357\,193\ldots \\
&= 834\,457\,193 - 8\,344 = 834\,448\,849.
\end{aligned}$$

We've just shown that $x = 834\,448\,849/99\,999\,000$—$x$ is the ratio of two integers and is thus rational.

($\Longrightarrow$) Consider for example $9/7 = 1.\overline{285\,714} = 1.285\,714\,285\,714\ldots$.

Long division (the remainder at each step is highlighted in blue):

Line			Explanation
Line 1		1. 2 8 5 7 1 4	
2	7	9	Explanation
3		7	$1 \times 7 = 7$
4		2 0	$9 - 7 = 2$
5		1 4	$2 \times 7 = 14$
6		6 0	$20 - 14 = 6$
7		5 6	$8 \times 7 = 56$
8		4 0	$60 - 56 = 4$
9		3 5	$5 \times 7 = 35$
10		5 0	$40 - 35 = 5$
11		4 9	$7 \times 7 = 49$
12		1 0	$50 - 49 = 1$
13		7	$1 \times 7 = 7$
14		3 0	$10 - 7 = 3$
15		2 8	$4 \times 7 = 28$
16		2	$30 - 28 = 2$

At line 16, the division isn't complete. But observe that the remainder at line 16 is the same as in line 4—namely, 2. And so clearly, the process will simply repeat. Lines 16 through 28 of the long division will look exactly the same as lines 4 through 16. Lines 28 through 40 will again look the same. Etc. The digits 285 714 will thus recur.

The key insight here is that there are only finitely many possible remainder values (namely, $0, 1, \ldots, 9$) And thus, the remainder must eventually repeat (or hit zero). This completes the proof sketch. □

To formally prove Proposition 25, let's first work towards a formal definition of what the term *eventually recurs* means.

Definition 264. Let $x \in \mathbb{R}$. We say that x *has a decimal representation* if there exist unique $x_0 \in \mathbb{Z}$ and $x_i \in \{0, 1, 2, \ldots, 9\}$ for each $i \in \mathbb{Z}^+$ such that

$$x = x_0 + \sum_{i=1}^{\infty} x_i 10^{-i}.$$

Theorem 51. *Every real number has a decimal representation.*

Proof. Omitted. □

Definition 265. Let x be a real number with decimal representation $x = x_0.x_1x_2x_3\ldots$. We say that x *eventually recurs* if there exist $s, t \in \mathbb{Z}^+$ such that for every $n \in \mathbb{Z}^+$,

$$x_s = x_{s+nt}$$

We are now ready to formally prove Proposition 25 (reproduced for convenience):

Proposition 25. *A real number is rational $\iff$ It eventually recurs.*

Proof. Let x be a real number with decimal representation $x = x_0 + \sum_{i=1}^{\infty} x_i 10^{-i}$.

($\Longleftarrow$) Suppose x eventually recurs. Then there exist $s, t \in \mathbb{Z}^+$ such that for every $n \in \mathbb{Z}^+$,

$$x_s = x_{s+nt}.$$

And hence, $$x = x_0 + \sum_{i=1}^{s-1} x_i 10^{-i} + \left(\sum_{i=1}^{\infty} 10^{-it}\right)\left(\sum_{i=0}^{t-1} 10^{t-i} x_{s+i}\right).$$

Now, observe that $10^{s-1}x = \sum_{i=0}^{s-1} x_i 10^{s-1-i} + 10^{s-1}\left(\sum_{i=1}^{\infty} 10^{-it}\right)\left(\sum_{i=0}^{t-1} 10^{t-i} x_{s+i}\right)$

So, $$10^{s-1+t}x = 10^t \sum_{i=0}^{s-1} x_i 10^{s-1-i} + 10^t 10^{s+t-1}\left(\sum_{i=1}^{\infty} 10^{-it}\right)\left(\sum_{i=0}^{t-1} 10^{t-i} x_{s+i}\right)$$

$$= 10^t \sum_{i=0}^{s-1} x_i 10^{s-1-i} + 10^{s+t-1}\left(\sum_{i=1}^{\infty} 10^{-it}\right)\left(\sum_{i=1}^{t-1} 10^{t-i} x_{s+i}\right) + 10^{s+t-1}\left(\sum_{i=0}^{t-1} 10^{t-i} x_{s+i}\right)$$

So, $10^{s-1+t}x - 10^{s-1}x = \left(10^t - 1\right)\sum_{i=0}^{s-1} x_i 10^{s-1-i} + 10^{s+t-1}\left(\sum_{i=0}^{t-1} 10^{t-i} x_{s+i}\right) = A$, which is an integer.

Hence, $x = A/\left(10^{s-1+t} - 10^{s-1}\right)$ is the ratio of two integers and thus rational. ✓

($\Longrightarrow$) Let $x = a/b$, where a and b are positive integers. (The other cases are dealt with similarly.)

We'll make use of the Euclidean division algorithm (Definition 1):

Let q_0 be the largest integer such that $a = q_0 b + r_0$ and r_0 is a non-negative integer.

For each $i \in \mathbb{Z}^+$, let q_i be the largest integer such that $10r_{i-1} = q_i b + r_i$ and r_i is a non-negative integer. Hence,

$$x = \frac{a}{b} = q_0 + 10^{-1}q_1 + 10^{-2}q_2 + 10^{-3}q_3 + \cdots = q_0.q_1q_2q_3\ldots$$

Note that since $r_i \in \{0, 1, 2, \ldots b-1\}$, there must exist $s, t \in \mathbb{Z}^+$ $r_s = r_{s+t}$.

And since $r_s = r_{s+t}$, by definition of q_i,

$$q_{s+t+1} = q_{s+1}, \quad q_{s+t+2} = q_{s+2}, \quad \ldots, \quad q_{s+(n+1)t} = q_t.$$

We also have $$r_s = r_{s+t} = r_{s+2t} = r_{s+3ts} = \ldots$$

So for every $n \in \mathbb{Z}^+$, $q_{s+nt+1} = q_{s+1}, \quad q_{s+nt+2} = q_{s+2}, \quad \ldots, \quad q_{s+(n+1)t} = q_t$.

Hence, x eventually recurs. □

To prove **De Morgan's Laws** in set theory, we can actually reuse our earlier proofs (from logic). But here as an exercise, let's prove these two laws using the set theory notation we've just learnt.

Fact 9. *Suppose P and Q are sets. Then*

$$\textbf{(a)} \quad (P \cap Q)' = P' \cup Q' \qquad \textit{and} \qquad \textbf{(b)} \quad (P \cup Q)' = P' \cap Q'.$$

Proof.

$$\begin{array}{rlcrl}
\textbf{(a)} & x \in (P \cap Q)' & & \textbf{(b)} & x \in (P \cup Q)' \\
\iff & x \notin P \cap Q & & \iff & x \notin P \cup Q \\
\iff & x \notin P \text{ OR } x \notin Q & & \iff & x \notin P \text{ AND } x \notin Q \\
\iff & x \in P' \text{ OR } x \in Q' & & \iff & x \in P' \text{ AND } x \in Q' \\
\iff & x \in P' \cup Q'. & & \iff & x \in P' \cap Q'.
\end{array}$$

□

142. Appendices for Part I. Functions and Graphs

142.1. Ordered Pairs and Ordered n-tuples

As mentioned on p. 1550, *every* mathematical object can be defined in terms of sets. The **ordered pair** is directly defined as a set.

Definition 266. Let x and y be objects. The *ordered pair* (x, y) is defined by

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

There's actually more than one way we can define an ordered pair.[569] What's important is that our definition correctly captures the idea that $(a, b) = (x, y) \iff a = x$ and $b = y$. This is accomplished by the above definition:

Fact 24. *Suppose a, b, x, and y are objects; and (a, b) and (x, y) are ordered pairs. Then*

$$(a, b) = (x, y) \quad\iff\quad a = x \text{ AND } b = y.$$

Proof. ($\Longleftarrow$) If $a = x$ AND $b = y$, then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} = (x, y)$.

($\Longrightarrow$) We'll prove the contrapositive: If $a \neq x$ OR $b \neq y$, then $(a, b) \neq (x, y)$.

Suppose $a \neq x$. Then $\{a\} \neq \{x\}$.

Case 1. If $\{a\} \neq \{x, y\}$, then $\{a\}$ is an element of (a, b) but not of (x, y). So, $(a, b) \neq (x, y)$.

Case 2. If $\{a\} = \{x, y\}$, then $x \neq y$ (otherwise $\{a\} = \{x, y\} = \{x\}$, a contradiction). So $(a, b) = \{\{a\}, \{a, b\}\} = \{\{x, y\}, \{x, y, b\}\}$ does not contain $\{x\}$ and thus $(a, b) \neq (x, y)$.

Now suppose $a = x$ and $b \neq y$. Then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{x\}, \{x, b\}\}$ does not contain $\{x, y\}$ and therefore $(a, b) \neq (x, y)$. □

We then define the **ordered triple** (x, y, z) to be the ordered pair $((x, y), z)$. Similarly, we define the **ordered quadruple** (x_1, x_2, x_3, x_4) to be the ordered pair $((x_1, x_2, x_3), x_4)$. Etc. More generally,

Definition 267. Let $n \geq 3$ and x_1, x_2, ..., x_n be objects. The *ordered n-tuple* $(x_1, x_2, \ldots, x_n)$ is defined recursively to be the ordered pair whose first coordinate is the ordered $(n-1)$-tuple $(x_1, x_2, \ldots, x_{n-1})$ and second coordinate is x_n. That is,

$$(x_1, x_2, \ldots, x_n) = ((x_1, x_2, \ldots, x_{n-1}), x_n).$$

Given the ordered n-tuple $(x_1, x_2, \ldots, x_n)$, we call each x_i its *ith coordinate.*

Remark 206. Many writers simply call it a **tuple** instead of an **ordered n-tuple**.

We can easily prove the analogue of Fact 24 for ordered n-tuples:

[569] The above definition by Kuratowski (1921, p. 171) is today the one that's usually used.

Fact 249. *Let* x_1, x_2, ..., x_n *and* y_1, y_2, ..., y_n *be objects. Suppose* $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *are ordered* n*-tuples. Then*

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \qquad \Longleftrightarrow \qquad x_i = y_i \text{ for all } i = 1, 2, \ldots, n.$$

Proof. ($\Longleftarrow$) Trivial.

($\Longrightarrow$) Let $n \geq 3$. By Definition 267,

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \iff ((x_1, x_2, \ldots x_{n-1}), x_n) = ((y_1, y_2, \ldots, y_{n-1}), y_n).$$

By Fact 24, this last equation is true if and only if

$$(x_1, x_2, \ldots x_{n-1}) = (y_1, y_2, \ldots, y_{n-1}) \qquad \text{AND} \qquad x_n = y_n.$$

Applying the above recursively, we find that for all $i \geq 3$, $x_i = y_i$. We also find that $(x_1, x_2) = (y_1, y_2)$, whereupon Fact 24 tells us that $x_1 = y_1$ and $x_2 = y_2$. $\square$

142.2. The Cartesian Plane

Definition 268. The *cartesian product* of two sets A and B, denoted $A \times B$, is defined by

$$A \times B = \{(x, y) : x \in A, y \in B\}.$$

Definition 269. Given a set A, we will also write the cartesian product $A \times A$ more simply as A^2.

Definition 270. Given a set A and $n \geq 3$, we define A^n recursively to be the cartesian product A^{n-1} and A. That is,

$$A^n = A^{n-1} \times A.$$

Definition 271. An n-dimensional space is a subset of $\mathbb{R}^n$.

Definition 272. Given $x_1, x_2, \ldots, x_n \in \mathbb{R}$, the ordered n-tuple $(x_1, x_2, \ldots, x_n)$ is called a *point in n-dimensional space.*

We reproduce from Ch. 7.2 this definition of the cartesian plane:

Definition 36. The *cartesian plane* is the set of all points:

$$\{(x, y) : x, y \in \mathbb{R}\}.$$

In light of the preceding definitions, we could also have defined the **cartesian plane** more simply as this set:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\} = \left\{(x, y) \in \mathbb{R}^2\right\}.$$

142.3. Lines

This subchapter restricts attention to the **cartesian plane** $\mathbb{R}^2$.

Fact 25. *Suppose $ax + by + c = 0$ describes a line. Then*

(a) *The line is horizontal* $\iff a = 0$.

(b) *The line is vertical* $\iff b = 0$.

Proof. **(a)** Let (x_1, y_1) and (x_2, y_2) be points on the line with $x_1 \neq x_2$. Since both are points on the line, we have

$$ax_1 + by_1 + c = 0 \qquad \text{and} \qquad ax_2 + by_2 + c = 0.$$

($\Longleftarrow$) Suppose $a = 0$ (so that $b \neq 0$). Then $by_1 + c = 0$ and $by_2 + c = 0$, so that $y_1 = y_2 = -c/b$. Hence, the line is horizontal. ✓

($\Longrightarrow$) Suppose $a \neq 0$.

If $b = 0$, then the line also contains the point $(x_1, y_1 + 1)$, so that it contains two points with different y-coordinates and hence is not horizontal.

If $b \neq 0$, then

$$y_1 = -\frac{ax_1 + c}{b} \qquad \text{and} \qquad y_2 = -\frac{ax_2 + c}{b}.$$

But since $x_1 \neq x_2$ and $a \neq 0$, it follows that $y_1 \neq y_2$, so that again the line is not horizontal. ✓

The proof of **(b)** is similar and thus omitted. □

Fact 250. *Two distinct points can be contained by at most one line.*

Proof. Suppose the lines $ax + by + c = 0$ and $dx + ey + f = 0$ contain the distinct points (p, q) and (r, s). Then

$$ap + bq + c \overset{1}{=} 0, \qquad ar + bs + c \overset{2}{=} 0, \qquad dp + eq + f \overset{3}{=} 0, \quad \text{and} \quad dr + es + f \overset{4}{=} 0.$$

From $\overset{1}{=} - \overset{2}{=}$, we get $a(p - r) \overset{5}{=} b(s - q)$.

Similarly, from $\overset{3}{=} - \overset{4}{=}$, we get $d(p - r) \overset{6}{=} e(s - q)$.

Case 1. Suppose $p = r$. Then $s \neq q$ (because $(p, q) \neq (r, s)$). Since $p - r = 0$ and $s - q \neq 0$, $\overset{5}{=}$ and $\overset{6}{=}$ imply that $b = e = 0$. Our two lines are both vertical and are simply $x = p$ and $x = r$. But since $p = r$, the two lines are identical. ✓

Case 2. Suppose $p \neq r$. Then we can rewrite $\overset{5}{=}$ and $\overset{6}{=}$ as

$$a \overset{7}{=} b\frac{s - q}{p - r} \qquad \text{and} \qquad d \overset{8}{=} e\frac{s - q}{p - r}.$$

Since at least one of a or b must be non-zero, $\overset{7}{=}$ implies that *both* a and b must be non-zero. Similarly, since at least one of d or e must be non-zero, $\overset{8}{=}$ implies that *both* e and e must be non-zero. Now use $\overset{7}{=}$ and $\overset{8}{=}$ to rewrite $\overset{1}{=}$ and $\overset{3}{=}$ as:

$$\frac{s-q}{p-r}p+q+\frac{c}{b}\overset{9}{=}0 \qquad \text{and} \qquad \frac{s-q}{p-r}p+q+\frac{f}{e}\overset{10}{=}0.$$

And now, $\overset{10}{=} - \overset{9}{=}$ yields $c/b \overset{11}{=} f/e$.

We now show that a point (t,u) is in the line $ax+by+c=0$ if and only if it is also in the line $dx+ey+f=0$. Equivalently, (t,u) satisfies $ax+by+c=0$ if and only if it also satisfies $dx+ey+f=0$. We will thus have shown that the two lines are identical:

$$0=at+bu+c\overset{7}{=}b\frac{s-q}{p-r}t+bu+c=\frac{s-q}{p-r}t+u+\frac{c}{b}$$

$$\overset{11}{=}\frac{s-q}{p-r}t+u+\frac{f}{e}=e\left(\frac{s-q}{p-r}t+u+\frac{f}{e}\right)\overset{8}{=}dt+eu+f. \qquad \square$$

Fact 251. *Given two distinct points (p,q) and (r,s), the unique line that contains both points is $(q-s)x+(r-p)y+ps-qr=0$.*[570]

Proof. We need merely plug in and verify that the given line contains (p,q) and (r,s):

$$(q-s)p+(r-p)q+ps-qr=0. \checkmark \qquad\qquad (q-s)r+(r-p)s+ps-qr=0. \checkmark$$

By Fact 250 then, this is the unique line that contains both points. $\square$

Fact 26. *The unique line that contains both of the distinct points (x_1,y_1) and (x_2,y_2) is*

$$(x_2-x_1)(y-y_1)=(y_2-y_1)(x-x_1).$$

Proof. By Fact 251, the unique line that contains both points is $(y_1-y_2)x+(x_2-x_1)y+x_1y_2-y_1x_2=0$. Rearranging, $(x_2-x_1)(y-y_1)=(y_2-y_1)(x-x_1)$. $\square$

Definition 45. Two lines are *perpendicular* if

(a) Their gradients are negative reciprocals of each other; or

(b) One line is vertical while the other is horizontal.

If two lines l and m are perpendicular, then we will also write $l \perp m$.

[570]If $r \neq p$ (the line isn't vertical), then this line may be written as $y=\dfrac{s-q}{r-p}x+\dfrac{qr-ps}{r-p}$.

And if $r=p$ (the line is vertical), then it may be written as $x=\dfrac{qr-ps}{q-s}$.

Definition 273. Let $A = (p, q)$ and $B = (r, s)$ be distinct points. Then

1. The *line* AB is the graph of the equation $(q - s)x + (r - p)y + ps - qr = 0$.
2. The *line segment* AB is the graph of the equation $(q - s)x + (r - p)y + ps - qr = 0$ with the constraint $x \in [p, r]$.
3. The *ray* AB is the graph of the equation $(q - s)x + (r - p)y + ps - qr = 0$ with the constraint **(i)** $x \geq p$ if $p < r$; **(ii)** $x \leq p$ if $p > r$; **(iii)** $y \geq q$ if $p = r$ and $s > q$; and **(iv)** $y \leq q$ if $p = r$ and $s < q$.

Definition 274. Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be two points in $\mathbb{R}^n$ ($n \in \mathbb{Z}^+$). The *distance between A and B* is denoted $|AB|$ and is defined as this number:

$$|AB| = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}.$$

So, for $n = 1$ ($\mathbb{R}^1 = \mathbb{R}$, the real number line), the distance between two points A and B (A and B are simply real numbers) is simply

$$|AB| = \sqrt{(A - B)^2} = |A - B| = |B - A|.$$

(This corresponds to Definition 46 in the main text.)

For $n = 2$ ($\mathbb{R}^2$, the cartesian plane), the distance between two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ is

$$|AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

(This corresponds to Definition 48 in the main text.)

For $n = 3$ ($\mathbb{R}^3$), the distance between two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ is

$$|AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.$$

Etc.

Given a point a and $\varepsilon > 0$,[571] the **ε-neighbourhood** (or sometimes **ball**) of a point a is the set of points each of whose distance from a is less than ε. (So, this captures the idea of "nearby" points.) A bit more formally and precisely,

Definition 275. Let A be a point in $\mathbb{R}^n$ ($n \in \mathbb{Z}^+$) and $\varepsilon > 0$. The *ε-neighbourhood of A*, denoted $\mathrm{N}_\varepsilon(A)$, is the set of points in $\mathbb{R}^n$ defined by

$$\mathrm{N}_\varepsilon(A) = \{B \in \mathbb{R}^n : |AB| < \varepsilon\}.$$

So, for $n = 1$ ($\mathbb{R}^1 = \mathbb{R}$, the real number line), the ε-neighbourhood of the point A (A is simply a real number) is simply the open interval centred on A and with width 2ε:

$$\mathrm{N}_\varepsilon(A) = \{B \in \mathbb{R} : |AB| < \varepsilon\} = (A - \varepsilon, A + \varepsilon).$$

[571] The symbol ε is the Greek lower-case letter *epsilon*.

Figure to be
inserted here.

For $n = 2$ ($\mathbb{R}^2$, the cartesian plane), the ε-neighbourhood of the point $A = (a_1, a_2)$ is

$$\mathrm{N}_\varepsilon(A) = \left\{B \in \mathbb{R}^2 : |AB| < \varepsilon\right\} = \left\{B = (b_1, b_2) \in \mathbb{R}^2 : \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} < \varepsilon\right\}.$$

For $n = 3$ ($\mathbb{R}^3$), the ε-neighbourhood of the point $A = (a_1, a_2, a_3)$ is

$$\mathrm{N}_\varepsilon(A) = \left\{B \in \mathbb{R}^3 : |AB| < \varepsilon\right\} = \left\{B = (b_1, b_2, b_3) \in \mathbb{R}^2 : \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2} < \varepsilon\right\}.$$

Informally, an **isolated point** in a set S is one that isn't "close" to any other point in S. Formally,

Definition 276. Let $x \in S \subseteq \mathbb{R}^n$ $(n \in \mathbb{Z}^+)$. We call x an *isolated point of* S if there exists some $\varepsilon > 0$ such that

$$\mathrm{N}_\varepsilon(x) \cap S = \{x\}.$$

142.5. Maximum and Minimum Points

In Ch. 142.4, we learnt that given a point $P \in \mathbb{R}^2$ and $\varepsilon > 0$, the symbol $\mathrm{N}_\varepsilon(P)$ denotes the ε-neighbourhood of P (i.e. the set of points each of whose distance from P is less than ε).

Definition 277. Let $P = (a, b)$ be a point in the graph $G \subseteq \mathbb{R}^2$ (which may be of a function f). We say that P is

1. A *global maximum (point)* of G (and f) if for all $(x, y) \in G$, we have $b \geq y$.
2. The *strict global maximum (point)* of G (and f) if for all $(x, y) \in G$, we have $b > y$.
3. A *local maximum (point)* of G (and f) if there exists $\varepsilon > 0$ such that
$$\text{For all } (x, y) \in \mathrm{N}_\varepsilon(P) \cap G, \text{ we have } b \geq y.$$
4. A *strict local maximum (point)* of G (and f) if there exists $\varepsilon > 0$ such that
$$\text{For all } (x, y) \in \mathrm{N}_\varepsilon(P) \cap G, \text{ we have } b > y.$$
5. A *global minimum (point)* of G (and f) if for all $(x, y) \in G$, we have $b \leq y$.
6. The *strict global minimum (point)* of G (and f) if for all $(x, y) \in G$, we have $b < y$.
7. A *local minimum (point)* of G (and f) if there exists $\varepsilon > 0$ such that
$$\text{For all } (x, y) \in \mathrm{N}_\varepsilon(P) \cap G, \text{ we have } b \leq y.$$
8. A *strict local minimum (point)* of G (and f) if there exists $\varepsilon > 0$ such that
$$\text{For all } (x, y) \in \mathrm{N}_\varepsilon(P) \cap G, \text{ we have } b < y.$$

Note: A graph can have at most one strict global maximum and at most one strict global minimum (hence the use of the definite article *the*).

In contrast, a graph can have more than one of each of the other six types of extreme points (hence the use of the indefinite article *a*).

142.6. The Distance between a Line and a Point

This subchapter restricts attention to the **cartesian plane** $\mathbb{R}^2$.

Proposition 3. *The unique point on the line $ax+by+c=0$ that is closest to the point (p,q) is*

$$B=\left(p-a\frac{ap+bq+c}{a^2+b^2},q-b\frac{ap+bq+c}{a^2+b^2}\right).$$

Proof. Case 1. Suppose $b=0$. Then the line may be rewritten as $x=-c/a$.
The distance between (p,q) and an arbitrary point $(-c/a,y)$ on the line is

$$\sqrt{(p-c/a)^2+(q-y)^2}=\sqrt{p^2+c^2/a^2-2c/a+q^2+y^2-2qy},$$

which by Fact 34(d) is minimised at $y=-(-2q)/(2\times 1)=q$.
So, the unique closest point is simply $(-c/a,q)$, which, as claimed, is equal to

$$\left(p-a\frac{ap+bq+c}{a^2+b^2},q-b\frac{ap+bq+c}{a^2+b^2}\right)=\left(\cancel{p}-a\frac{\cancel{ap}+c}{a^2},q\right).$$

Case 2. Suppose $b\neq 0$. Then the line may be rewritten as $y\stackrel{1}{=}(-a/b)x-c/b$.
The distance between an arbitrary point (x,y) on the line and the point (p,q) is

$$\sqrt{(x-p)^2+(y-q)^2}=\sqrt{(x-p)^2+\left(-\frac{a}{b}x-\frac{c}{b}-q\right)^2}$$

$$=\sqrt{x^2+p^2-2px+\frac{a^2}{b^2}x^2+\frac{c^2}{b^2}+q^2+2\frac{ac}{b^2}x+2\frac{aq}{b}x+2\frac{cq}{b}}$$

$$=\sqrt{\left(1+\frac{a^2}{b^2}\right)x^2+2\left(\frac{ac}{b^2}+\frac{aq}{b}-p\right)x+p^2+\frac{c^2}{b^2}+q^2+2\frac{cq}{b}},$$

which by Fact 34(d) is minimised at

$$x=-\frac{2\left(\frac{ac}{b^2}+\frac{aq}{b}-p\right)}{2\left(1+\frac{a^2}{b^2}\right)}=\frac{b^2p-ac-abq}{a^2+b^2}=p-a\frac{ap+bq+c}{a^2+b^2}.$$

Plug this x-coordinate into $\stackrel{1}{=}$ to get

$$y\stackrel{1}{=}-\frac{a}{b}x-\frac{c}{b}=-\frac{a}{b}\left(p-a\frac{ap+bq+c}{a^2+b^2}\right)-\frac{c}{b}=\frac{-\frac{a^3}{b}p-abp+\frac{a^2}{b}(ap+bq+c)-\frac{a^3c}{b}-bc}{a^2+b^2}$$

$$=q+\frac{-q\left(\cancel{a^2}+b^2\right)\cancel{-\frac{a^3}{b}p}-abp+\frac{a^2}{b}\left(\cancel{ap}+\cancel{bq}+\cancel{c}\right)\cancel{-\frac{a^2c}{b}}-bc}{a^2+b^2}=q-b\frac{ap+bq+c}{a^2+b^2}.$$ ◻

Definition 61. The *distance between a point A and a graph G* is the distance between A and B, where B is the point on G that's closest to A.

Corollary 3. *The distance between a point (p, q) and a line $ax + by + c = 0$ is*

$$\frac{|ap + bq + c|}{\sqrt{a^2 + b^2}}.$$

Fact 35. *Let A be a point that is not on the line l. Suppose B is a point on l. Then*

$$B \text{ is the point on } l \text{ that is closest to } A \qquad \iff \qquad l \perp AB.$$

Proof. We already proved $\Longleftarrow$ in the main text. We now prove $\Longrightarrow$:

($\Longrightarrow$) Suppose B is the point on l that is closest to A. By Proposition 3,

$$B = \left(p - a\frac{ap + bq + c}{a^2 + b^2}, q - b\frac{ap + bq + c}{a^2 + b^2}\right).$$

By Fact 251, line AB is given by

$$\left(\cancel{q} - \cancel{q} + b\frac{ap + bq + c}{a^2 + b^2}\right)x + \left(\cancel{p} - a\frac{ap + bq + c}{a^2 + b^2} - \cancel{p}\right)y + p\left(q - b\frac{ap + bq + c}{a^2 + b^2}\right) - q\left(p - a\frac{ap + bq + c}{a^2 + b^2}\right) = 0.$$

If $a = 0$, then by Fact 25, l is horizontal and AB is vertical, so that by Definition 45, $l \perp AB$.

If instead $b = 0$, then again by Fact 25, l is vertical and AB is horizontal, so that again by Definition 45, $l \perp AB$.

So suppose $a \neq 0 \neq b$. Then by Fact 27, the gradient of AB is

$$-b\frac{ap + bq + c}{a^2 + b^2} \div \left(-a\frac{ap + bq + c}{a^2 + b^2}\right) = \frac{b}{a},$$

which is the negative reciprocal of the gradient $-a/b$ of the line l. So, by Definition 45, $l \perp AB$. □

In Ch. 144.13, we'll prove the above results again using the language of vectors.

142.7. Stationary and Turning Points of a Graph

So far, we've defined only **stationary points of functions**. We have *not* defined **stationary points of graphs**.

Example 1567. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = x^2$. Then f has the stationary point $(0, 0)$.

Now consider instead the graph of $y = x^2$—this is simply the set $\left\{(x, y) \in \mathbb{R}^2 : y = x^2\right\}$. We would quite sensibly like to say that this graph also has the stationary point $(0, 0)$. The problem though is that we haven't formally defined what a stationary point of a graph is. So, right now, we can't actually say that this graph has any stationary point.

It turns out that writing down general definitions of stationary points of graphs is tricky, because a graph is simply *any* set of points (in $\mathbb{R}^2$).

Example 1568. Below is a graph G. Intuitively and visually, we want to be able to say that P is a stationary point of G.

Figure to be inserted here.

But it's tricky to write down a general definition of stationary points where this is so.

Here's one possible definition of stationary points for graphs:

Definition 278. Let $G \subseteq \mathbb{R}^2$ be a graph. We call $P = (a, b) \in G$ a *turning point* of G if there exist $\varepsilon > 0$ and a function f whose graph is $G \cap \mathrm{N}_\varepsilon(P)$ and has turning point (a, b).

Having defined a graph's stationary points, we can also easily define its **turning points**:

Definition 279. A *turning point* of a graph is any point that is both a stationary point and a strict local maximum or minimum of that graph.

This subchapter will be paralleled by Ch. 146.5 ("Asymptotes of a Graph").

142.8. Reflections

This subchapter restricts attention to the **cartesian plane** $\mathbb{R}^2$.

Proposition 26. *Suppose P and Q be distinct points. Then there exists a unique point $R \neq P$ that satisfies these two properties:*

(a) $|PQ| = |QR|$ **(b)** *R is on the line PQ.*

Proof. In Exercise 93 (main text), we already proved existence—specifically, we proved that the point $R = (2c - a, 2d - b)$ satisfies properties **(a)** and **(b)**.

We now prove uniqueness. To do so, we'll prove that there are at most two points S that satisfy properties **(a)** and **(b)**—one is P and the other is R. We will thus have shown that R unique.

Suppose $S = (S_x, S_y)$ is a point that satisfies properties **(a)** and **(b)**.

Let $P = (a, b)$ and $Q = (c, d)$.

So, $|PQ| = \sqrt{(c-a)^2 + (d-b)^2} = |QS| = \sqrt{(S_x - c)^2 + (S_y - d)^2}$ or $(c-a)^2 + (d-b)^2 \overset{1}{=} (S_x - c)^2 + (S_y - d)^2$.

By Fact 26, the line PQ is $(c-a)(y-b) = (d-b)(x-a)$. So, $(c-a)(S_y - b) \overset{2}{=} (d-b)(S_x - a)$.

If $c = a$, then $S_x = a = c$, and $S_y = d \pm \sqrt{(c-a)^2 + (d-b)^2}$—these are the two points S that satisfy properties **(a)** and **(b)**.

If $c \neq a$, then $S_y \overset{3}{=} \dfrac{(d-b)(S_x - a)}{c-a} + b$. Plugging $\overset{3}{=}$ into $\overset{1}{=}$, we have a quadratic equation in S_x, so that by the Fundamental Theorem of Algebra, S_x has at most two real roots. □

Fact 252. *The graph of* $x^2 + y^2 = 1$ *is symmetric in* every *line through the origin.*

Proof. Let l be the line $y = kx$ (where $k \in \mathbb{R}$) (so that l is some line through the origin).

Let G be the graph of $x^2 + y^2 = 1$. Pick any point $A = \left(a, \pm\sqrt{1-a^2}\right) \in G$.

By Fact 41, the reflection of A in l is

$$\begin{aligned}
B &= \left(a - 2\frac{a - k\left(\pm\sqrt{1-a^2}\right)}{1+k^2}, \pm\sqrt{1-a^2} + 2k\frac{a - k\left(\pm\sqrt{1-a^2}\right)}{1+k^2}\right) \\
&= \left(\frac{ak^2 - a \pm 2k\sqrt{1-a^2}}{1+k^2}, \frac{\pm\sqrt{1-a^2} + 2ak \mp k^2\sqrt{1-a^2}}{1+k^2}\right),
\end{aligned}$$

which we now verify satisfies the equation $x^2 + y^2 = 1$ (and so $B \in G$):

$$\begin{aligned}
&\left(\frac{ak^2 - a \pm 2k\sqrt{1-a^2}}{1+k^2}\right)^2 + \left(\frac{\pm\sqrt{1-a^2} + 2ak \mp k^2\sqrt{1-a^2}}{1+k^2}\right)^2 \\
&= \frac{1}{(1+k^2)^2}\left[\cancel{a^2k^4} + \cancel{a^2} + \overset{2}{\cancel{4}}k^2\left(1\cancel{-a^2}\right)\cancel{-2a^2k^2}\cancel{\pm 4ak^3\sqrt{1-a^2}}\cancel{\mp 4ak\sqrt{1-a^2}}\right] \\
&+ \frac{1}{(1+k^2)^2}\left[1\cancel{-a^2} + \cancel{4a^2k^2} + k^4\left(1\cancel{-a^2}\right)\cancel{\pm 4ak\sqrt{1-a^2}} - 2k^2\left(\cancel{1}\cancel{-a^2}\right)\cancel{\mp 4ak^3\sqrt{1-a^2}}\right] \\
&= \frac{2k^2 + 1 + k^4}{(1+k^2)^2} = 1.
\end{aligned}$$

We've just shown that the reflection B of any $A \in G$ in l is also in G. So, l is a line of symmetry for G. $\square$

142.9. The Quadratic Equation

We reproduce Fact 34 (from Ch. 14):

Fact 34. *Consider the graph of the quadratic equation* $y = ax^2 + bx + c$.

(a) *The only* y*-intercept is* $(0, c)$.

(b) *There are two, one, or zero* x*-intercepts, depending on the sign of* $b^2 - 4ac$:

(i) *If* $b^2 - 4ac > 0$, *then there are two* x*-intercepts:*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

And $$ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right).$$

(ii) *If* $b^2 - 4ac = 0$, *then there is one* x*-intercept* $x = -b/2a$ *(where the graph just touches the* x*-axis).*

And $$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2.$$

(iii) *If* $b^2 - 4ac < 0$, *then there are no* x*-intercepts. There is also no way to factorise the quadratic polynomial* $ax^2 + bx + c$ *(unless we use complex numbers).*

(c) *It is symmetric in the vertical line* $x = -b/2a$.

(d) *The only turning point is* $\left(-b/2a, -b^2/4a + c\right)$, *which is a strict global* **(i)** *minimum if* $a > 0$; *or* **(ii)** *maximum if* $a < 0$.

Proof. **(a)** If $x = 0$, then $y = a \cdot 0^2 + b \cdot 0 + c = c$.

(b) On p. 168, we already showed that the two roots of the quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac > 0$, then both roots are real.

If $b^2 - 4ac = 0$, then there is only one real root, $-b/2a$.

If $b^2 - 4ac < 0$, then there are no real roots (but there are two complex roots).

(*Proof continues below ...*)

(... *Proof continued from above.*)

(c) Pick any point $\left(p, ap^2+bp+c\right)$ in the graph. Its reflection in the line $x = -\dfrac{b}{2a}$ is $\left(-\dfrac{b}{a}-p, ap^2+bp+c\right)$. This reflection point is also in the graph, as we now verify:

$$a\left(-\frac{b}{a}-p\right)^2+b\left(-\frac{b}{a}-p\right)+c = a\left(\frac{b^2}{a^2}+p^2+\frac{2b}{a}p\right)-\frac{b^2}{a}-bp+c = ap^2+bp+c. \qquad \checkmark$$

(d) Differentiate[572] the quadratic equation $y = ax^2+bx+c$ with respect to x to get

$$y'(x) = 2ax+b.$$

The sole stationary point (and hence also the only possible turning point) is given by $y'(\bar{x}) = 0$ or

$$\bar{x} = \frac{-b}{2a} \text{ and } y(\bar{x}) = a\left(\frac{-b}{2a}\right)^2+b\left(\frac{-b}{2a}\right)+c = \frac{b^2}{4a}-\frac{b^2}{2a}+c = -\frac{b^2}{4a}+c.$$

(i) Suppose $a > 0$. Then $y'(x) < 0$ everywhere to the left of $\bar{x}$ and $y'(x) > 0$ everywhere to the right of $\bar{x}$. Hence, by the First Derivative Test for Extrema, $\bar{x}$ is a strict local minimum and thus, by Definition 58, also a turning point.

(ii) Suppose $a < 0$. Then $y'(x) > 0$ everywhere to the left of $\bar{x}$ and $y'(x) < 0$ everywhere to the right of $\bar{x}$. Hence, by the First Derivative Test for Extrema, $\bar{x}$ is a strict local maximum and thus, by Definition 58, also a turning point. □

It was easy to prove that $y = ax^2+bx+c$ is symmetric in the vertical line $x = -b/2a$ (Fact 34(c)). What's harder to prove is that this is *the* unique line of symmetry:

Proposition 27. *There is only line of symmetry for $y = ax^2+bx+c$.*

Proof. Let G be the graph of $y = ax^2+bx+c$ and l be a line of symmetry for G.

Let $dx+ey+f = 0$ be the equation for l.

Since $a \neq 0$, we can rewrite G as $y = x^2+gx+h$, where $g = \dfrac{b}{a}$ and $h = \dfrac{c}{a}$.

Case 1. Suppose l is vertical (or $e = 0$). Then $d \neq 0$ and we can rewrite the equation of l as $x - \dfrac{f}{d} = 0$.

Let $P_1 = \left(x_1, x_1^2+gx_1+h\right) \in G$. Its reflection in l is the point

$$\tilde{P}_1 = \left(2\frac{f}{d}-x_1, x_1^2+gx_1+h\right).$$

The line l is a line of symmetry for G if and only if the above reflection point is also in G, i.e.

[572] Here we rely on results given only later on in Part V (Calculus).

$$x_1^2 + gx_1 + h = \left(2\frac{f}{d} - x_1\right)^2 + g\left(2\frac{f}{d} - x_1\right) + h$$

$$\vdots \text{ (omitted algebra)}$$

$$\iff \qquad 0 = (dx_1 - f)(dg + 2f)$$

The above equation holds for *all* $x_1 \in \mathbb{R}$. Hence, $dg + 2f = 0$ or $\dfrac{f}{d} = -\dfrac{g}{2} = -\dfrac{b}{2a}$.

We've just shown that there is exactly one vertical line of symmetry for G, namely $x = -\dfrac{b}{2a}$. This completes our examination of **Case 1**. ✓

Case 2. Suppose l is not vertical (i.e. $e \neq 0$). (We will show that this produces a contradiction. We will hence have shown that no non-vertical line can be a line of symmetry for G and thus that $x = -\dfrac{b}{2a}$ is the only line of symmetry for G.)

Then we can rewrite the equation of l as $ix + y + j = 0$, where $i = \dfrac{d}{e}$ and $j = \dfrac{f}{e}$.

Claim 1. *G and l intersect at most twice.*

Proof. Plug the equation for G into that for l to get

$$0 = ix + y + j = ix + x^2 + gx + h + j,$$

which is a quadratic equation in x and hence has at most two solutions. □

Claim 2. *There exist infinitely many points in G each of whose reflection in l is not itself.*

Proof. By Claim 1, G and l intersect at most twice. These are also the only two points each of whose reflection in l is itself. Since G has infinitely many points, the claim holds.□

Pick any point $P_2 = \left(x_2, x_2^2 + gx_2 + h\right) \in G$ whose reflection in l is not itself. By Fact 41,[573] its reflection in l is the point

$$\tilde{P}_2 = \left(x_2 - 2i\frac{ix_2 + x_2^2 + gx_2 + h + j}{i^2 + 1}, x_2^2 + gx_1 + h - 2\frac{ix_2 + x_2^2 + gx_2 + h + j}{i^2 + 1}\right) \in G$$

Observe that if $ix_2 + x_2^2 + gx_2 + h + j = 0$, then $P_2 = \tilde{P}_2$, contradicting our choice of P_2. So, $ix_2 + x_2^2 + gx_2 + h + j \overset{☺}{\neq} 0$.

Now, l is a line of reflection for G if and only if $\tilde{P}_2 \in G$, i.e.

[573] $\left(p - 2a\dfrac{ap + bq + c}{a^2 + b^2}, q - 2b\dfrac{ap + bq + c}{a^2 + b^2}\right)$

$$x_2^2 + gx_2 + h - 2\frac{ix_2 + x_2^2 + gx_2 + h + j}{i^2+1} = \left(x_2 - 2i\frac{ix_2 + x_2^2 + gx_2 + h + j}{i^2+1}\right)^2 + g\left(x_2 - 2i\frac{ix_2 + x_2^2}{i}\right.$$

$$\vdots \text{ (omitted algebra)}$$

$$\iff \quad 0 = \left(ix_2 + x_2^2 + gx_2 + h + j\right)\left[1 - 2ix_2 + \frac{2i^2}{i^2+1}\left(ix_2 + x_2^2 + \right.\right.$$

$$\overset{☺}{\iff} \quad 0 = 1 - 2ix_2 + \frac{2i^2}{i^2+1}\left(ix_2 + x_2^2 + gx_2 + h + j\right) + gi$$

This last equation in x_2 has at most two solutions for x_2, contradicting Claim 2. □

142.10. Functions

Definition 280. Let A and B be sets. A *function f with domain A and codomain B*—more simply denoted $f : A \to B$—is the ordered triple (A, B, G), where G is any subset of $A \times B$ that satisfies this property:

For every $x \in A$, there is exactly one $y \in B$ such that $(x, y) \in G$.

We call G the *graph* of f.

For each $(x, y) \in G$, we write $f(x) \stackrel{1}{=} y$ or $f : x \mapsto y$. We refer to the full specification of $\stackrel{1}{=}$ as the *mapping rule* of f.

Fact 45. (Vertical Line Test) *Suppose G is a graph (i.e. any set of points in the cartesian plane). Then G is the graph of a nice function $\iff$ No vertical line intersects G more than once.*

Proof. Let $D = \{x : x \text{ is the first coordinate of a point in } G\}$.

No vertical line intersects G more than once $\iff$ For any $a \in \mathbb{R}$, there exists at most one $b \in \mathbb{R}$ such that $(a, b) \in G$ $\iff$ G is the graph of the function $f : D \to \mathbb{R}$ defined by $f(x)$ equals the second coordinate of the point in G whose first coordinate is x. □

142.11. When a Function Is Increasing or Decreasing at a Point

Definition 73 defined what it means for a function to be increasing or decreasing **on a set**. But in the main text, we did **not** define what it means for a function to be increasing or decreasing **at a point**.

Nonetheless, Definition 73 *does* specify what it means for a function to be increasing or decreasing on a set that contains exactly one point. This is not very interesting:

Fact 253. *Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $S \subseteq D$. If $|S| = 1$, then f is increasing, decreasing, strictly increasing,* and *strictly decreasing on S.*

Proof. Suppose S contains only one point. Then there do not exist $a, b \in S$ with $a < b$. So, the conditions in Definition 73**(a)**, **(b)**, **(c)**, and **(d)** are vacuously true. □

Here's a possible definition of when a function is increasing or decreasing **at a point**:

Definition 281. Let $D, C \subseteq \mathbb{R}$, $f : D \to C$, and $a \in D$. We say that f is *increasing at a* if there exists $\varepsilon > 0$ such that f is increasing on $(a - \varepsilon, a + \varepsilon) \cap D$.

To obtain three more definitions, replace the two instances of the word *increasing* in the last sentence with *decreasing*, *strictly increasing*, or *strictly decreasing*.

Example 1569. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^2$.

At 0, f is not increasing, strictly increasing, decreasing, or strictly decreasing because for all $\varepsilon>0$, f is not increasing, strictly increasing, decreasing, or strictly decreasing on $(0-\varepsilon,0+\varepsilon)\cap\mathbb{R}=(-\varepsilon,\varepsilon)$.

The above definition implies that if a is an isolated point of D, then f is increasing, decreasing, strictly increasing, *and* strictly decreasing at a:

Example 1570. Define $f:\{1,2\}\to\mathbb{R}$ by $f(x)=1$.

At 1, f is increasing, strictly increasing, decreasing, *and* strictly decreasing because there exists $\varepsilon>0$ (e.g. $\varepsilon=0.5$) such that f is increasing, strictly increasing, decreasing, *and* strictly decreasing on $(1-\varepsilon,1+\varepsilon)\cap\{1,2\}=(0.5,1.5)\cap\{1,2\}=\{1\}$ (Fact 253).

Here's a second possible definition of when a function is increasing or decreasing at a point:

Definition 282. Let $D,C\subseteq\mathbb{R}$, $f:D\to C$, and $a\in D$. We say[574] that f is *increasing at a* if there exists $\varepsilon>0$ such that for every $s\in(a-\varepsilon,a)\cap D$ and every $t\in(a,a+\varepsilon)\cap D$, we have $f(s)\le f(a)\le f(t)$.

To obtain three more definitions, in the last sentence, replace **(a)** *increasing* with *decreasing*, *strictly increasing*, or *strictly decreasing*; and **(b)** the two instances of $\le$ with $\ge$, $<$, or $>$.

It turns out that this last definition is weaker than (or implied) by the previous definition:

Example 1571. Define[575] $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\begin{cases}x, & \text{for } x\in\mathbb{Q},\\ 2x, & \text{for } x\notin\mathbb{Q}.\end{cases}$.

At 0, f is increasing under Definition 282. but not under Definition 281.

Note that f is not differentiable and is, under Definition 282, increasing only at 0 and not increasing, decreasing, strictly increasing, or strictly decreasing at any other point.

[574]This definition (or similar) is given by Spivak (2006), Did (2013), and Sohrab (2014).
[575]This example was stolen from Did (2013).

142.12. One-to-One Functions

Fact 47. (Horizontal Line Test) *A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.*

Proof. Let $c \in \mathbb{R}$ and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = c$. Let F and G be the graphs of f and g. Observe that

$$\begin{aligned}
& F \text{ and } G \text{ intersect at least twice} \\
\iff & \text{There exist } a, b \text{ with } a \neq b \text{ such that } (a, c), (b, c) \in F \cap G \\
\iff & \text{There exist } a, b \text{ with } a \neq b \text{ such that } f(a) = c \text{ and } f(b) = c \\
\iff & \text{There exists } y \in \text{Codomain } f \text{ (namely, } y = c\text{) such that there are} \\
& \text{more than one } x \in \text{Domain } f \text{ (namely, } x = a, b\text{) with } f(x) = y. \\
\iff & f \text{ is not one-to-one.}
\end{aligned}$$

□

Lemma 2. *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function.*

(a) *If $f(x_1) < f(x_2) < f(x_3)$ for any $x_1 < x_2 < x_3$, then f is strictly increasing.*

(b) *If $f(x_1) > f(x_2) > f(x_3)$ for any $x_1 < x_2 < x_3$, then f is strictly decreasing.*

Proof. **(a)** Let $a, b \in D$ with $a < b$. Pick any $c \in (a, b)$. By assumption, $f(a) < f(c) < f(b)$. So, $f(a) < f(b)$ and by Definition 73, f is strictly increasing.

(b) Similar, omitted. □

Definition 283. Given a set $S \subseteq \mathbb{R}$, the largest and smallest numbers in S (if they exist) are denoted $\max S$ and $\min S$.

Example 1572. Let $S = \{1, 2, 3\}$. Then $\max S = 3$ and $\min S = 1$.

Example 1573. Let $T = [0, 1]$. Then $\max T = 1$ and $\min T = 0$.

Example 1574. Let $U = (0, 1)$. Then neither $\max U$ nor $\min U$ exists.

Lemma 3. *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. If f is neither strictly increasing nor strictly decreasing, then there exist $x_1 < x_2 < x_3$ such that either $f(x_2) \leq \min\{f(x_1), f(x_3)\}$ or $f(x_2) \geq \max\{f(x_1), f(x_3)\}$.*

Proof. Suppose f is neither strictly increasing nor strictly decreasing.

Then by Lemma 2, there exist $x_1, x_2, x_3 \in D$ with $x_1 < x_2 < x_3$ such that

$$
\begin{aligned}
& \text{NOT-“}f(x_1) < f(x_2) < f(x_3)\text{”} \quad \text{OR} \quad \text{NOT-“}f(x_1) > f(x_2) > f(x_3)\text{”} \\
\iff \quad & f(x_2) \le f(x_1) \text{ OR } f(x_2) \ge f(x_3) \quad \text{AND} \quad f(x_2) \ge f(x_1) \text{ OR } f(x_2) \le f(x_3) \\
\iff \quad & f(x_2) \le f(x_1) \text{ AND } f(x_2) \ge f(x_1) \quad \text{OR} \quad f(x_2) \le f(x_1) \text{ AND } f(x_2) \le f(x_3) \\
\iff \quad & f(x_2) = f(x_1) \quad \text{OR} \quad f(x_2) \le \min\{f(x_1), f(x_3)\} \quad \text{OR} \quad f(x_2) \ge \max\{f(\\
\iff \quad & f(x_2) \le \min\{f(x_1), f(x_3)\} \quad \text{OR} \quad f(x_2) \ge \max\{f(x_1), f(x_3)\}.
\end{aligned}
$$

□

Proposition 4. *Suppose a nice function is continuous and its domain is an interval. If this function is one-to-one, then it is also either strictly increasing or strictly decreasing.*

Proof. Let D be an interval and $f : D \to \mathbb{R}$ be a continuous function.

If D is empty or contains only one point, then the claim is vacuously true. So, assume D contains more than one point (and hence, infinitely many points).

Suppose for contradiction that f is neither strictly increasing nor strictly decreasing on D.

Then by Lemma 3, there exist $x_1 < x_2 < x_3$ such that either $f(x_2) \le \min\{f(x_1), f(x_3)\}$ or $f(x_2) \ge \max\{f(x_1), f(x_3)\}$. Since f is one-to-one, these last two weak inequalities may be replaced by strict ones—i.e. either $f(x_2) < \min\{f(x_1), f(x_3)\} = a$ or $f(x_2) > \max\{f(x_1), f(x_3)\} = b$.

If $f(x_2) < a$, then pick any $y \in (f(x_2), a)$. By the Intermediate Value Theorem (Theorem 12), there exists $x_4 \in (x_1, x_2)$ such that $f(x_4) = y$, so that f is not one-to-one (contradiction).

Similarly, if $f(x_2) > b$, then pick any $y \in (b, f(x_2))$. By the Intermediate Value Theorem (Theorem 12), there exists $x_5 \in (x_1, x_2)$ such that $f(x_5) = y$ and $f(x_5) = y$, so that f is not one-to-one (contradiction). □

Fact 50. *Suppose $f : A \to B$ is a function with range C. Then f is one-to-one if and only if there exists a function $g : C \to A$ where for every $y \in C$, we have*

$$y = f(x) \qquad \Longrightarrow \qquad g(y) = x.$$

(Moreover, if g exists, then by Definition 79, it is the inverse of f.)

Proof. ($\Longrightarrow$) Suppose f is one-to-one. Then for each $y \in C$, there exists a unique $x \in A$ such that $f(x) = y$. And so, we can construct the function $g : C \to A$, where for every $x \in C$, we have

$$y = f(x) \qquad \Longrightarrow \qquad g(y) = x.$$

($\Longleftarrow$) Suppose the function g exists. Let $a, b \in A$ with $f(a) \overset{1}{=} f(b)$.

Then $g(f(a)) = a$ and $g(f(b)) = b$. By $\overset{1}{=}$, $g(f(a)) = g(f(b))$ and $a = b$.

We've just shown that $f(a) = f(b)$ implies $a = b$. So, by Fact 48(b), f is one-to-one. □

Fact 254. *Every inverse is both* **(a)** *one-to-one; and* **(b)** *onto.*

Proof. Let $f : A \to B$ be a function with range C and inverse $f^{-1} : C \to A$.

(a) Pick any $y_1, y_2 \in C$ with $y_1 \neq y_2$.[576] Then there exist $x_1, x_2 \in A$ with $x_1 \neq x_2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Hence, $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$—$f^{-1}$ is one-to-one.

(b) For each $x \in A$, there exists $y \in C$ such that $f^{-1}(y) = x$. Hence, f^{-1} "hits" every element in $A = \text{Codomain}\, f^{-1}$ and is onto. □

Fact 51. *Suppose the function f has inverse f^{-1}. Then*

$$y = f(x) \qquad \Longleftrightarrow \qquad f^{-1}(y) = x.$$

Proof. ($\Longrightarrow$) By Definition 79 (of inverses).

($\Longleftarrow$) Suppose $f^{-1}(y) \overset{1}{=} x$. Then $x \in \text{Domain}\, f$.

Let $z \overset{2}{=} f(x)$. By Definition 79, $f^{-1}(z) \overset{3}{=} x$.

By Fact 254(b), f^{-1} is one-to-one. Hence, by $\overset{1}{=}$ and $\overset{3}{=}$, $z = y$. Thus, $y \overset{2}{=} f(x)$. □

Fact 57. *Suppose the function f has inverse f^{-1} and $\left(f^{-1}\right)^{-1}$ is the inverse of f^{-1}. Then*

$$f = \left(f^{-1}\right)^{-1} \qquad \Longleftrightarrow \qquad f \text{ is onto.}$$

[576] If no such y_1, y_2 exist, then C is either empty or contains exactly one element—in either case, f^{-1} is one-to-one.

Proof. Below, we'll prove that

(a) Domain f = Domain $\left(f^{-1}\right)^{-1}$.

(b) $f(x) = \left(f^{-1}\right)^{-1}(x)$ for all $x \in$ Domain f.

(c) Codomain f = Codomain $\left(f^{-1}\right)^{-1} \iff f$ is onto.

We'll thus have proven that $f = \left(f^{-1}\right)^{-1} \iff f$ is onto.

(a) By Definition 79 (of the inverse), Codomain $f^{-1} \overset{1}{=}$ Domain f and Domain $\left(f^{-1}\right)^{-1} \overset{2}{=}$ Range f^{-1}.

But by Fact 254, f^{-1} is onto, i.e. Range $f^{-1} \overset{3}{=}$ Codomain f^{-1}.

By $\overset{3}{=}$ and $\overset{1}{=}$, Range $f^{-1} \overset{4}{=}$ Domain f. By $\overset{4}{=}$ and $\overset{2}{=}$, get Domain $\left(f^{-1}\right)^{-1}$ = Domain f.

(b) Let $x \in$ Domain f = Domain $\left(f^{-1}\right)^{-1}$ and $y = f(x)$.

Then $f^{-1}(y) = x$. So, $\left(f^{-1}\right)^{-1}(x) = y$. Hence, $\left(f^{-1}\right)^{-1}(x) = f(x)$.

(c) Observe that Range f = Domain f = Codomain $\left(f^{-1}\right)^{-1}$.

So, Codomain f = Codomain $\left(f^{-1}\right)^{-1} \iff$ Codomain f = Range $f \iff f$ is onto. □

Fact 255. *Let f be a function and $A \subseteq$ Domain f. Suppose $f\big|_A$ has inverse $f\big|_A^{-1}$. Then*

(a) *The composite function $f \circ f\big|_A^{-1}$ exists and is defined by $\left(f \circ f\big|_A^{-1}\right)(x) = x$.*

(b) *The composite function $f\big|_A^{-1} \circ f$ may not exist.*

(c) *Nonetheless, for all $x \in A$, we have $f\big|_A^{-1}(f(x)) = x$.*

Proof. **(a)** Since Range $f\big|_A^{-1}$ = Domain $f\big|_A = A \subseteq$ Domain f, the composite function $f \circ f\big|_A^{-1}$ exists.

For all $x \in$ Domain $f\big|_A^{-1}$, if $y = f\big|_A^{-1}(x)$, then $x = f(y)$—and so, $\left(f \circ f\big|_A^{-1}\right)(x) = f\left(f\big|_A^{-1}(x)\right) = x$.

(b) Define $f : \{0,1\} \to \{2,3\}$ by $f(0) = 2$ and $f(1) = 3$. Let $A = \{0\}$.

Then Range $f = \{2,3\} \nsubseteq \{2\}$ = Range $f\big|_A$ = Domain $f\big|_A^{-1}$, so that $f\big|_A^{-1} \circ f$ does not exist.

(c) Suppose $x \in A$. Since Domain $f\big|_A^{-1}$ = Range $f\big|_A$, we have $f(x) \in$ Domain $f\big|_A^{-1}$.

For all $x \in$ Domain $f\big|_A^{-1}$, if $y = f(x)$, then $x = f\big|_A^{-1}(y)$—and so, $f\big|_A^{-1}(f(x)) = x$. □

142.14. Standard Definitions of the Identity and Inclusion Functions

In the main text, Definition 101 defined an identity function. But as mentioned in n. 246, that was not quite the standard definition of an identity function. The standard definition of an identity function is this:

Definition 284. If S is a set, then the *identity function on* S is the function $I_S : S \to S$ defined by $I_S(x) = x$.

The main text's Definition 101 (of an identity function) is actually the standard definition of the **inclusion function** from S to T:

Definition 285. Let S and T be sets. If $S \subseteq T$, then the *inclusion function from* S *to* T is the function $i_S : S \to T$ defined by $i_S(x) = x$.

The inclusion function (from S to T) differs from the identity function (on S) only in that its domain S might be a strict subset of the codomain T.

142.15. Left, Right, and "Full" Inverses

In the main text, Definition 79 gave a definition of the *inverse function* (or more simply *inverse*). But as noted in n. 178, that was not quite the standard definition. It turns out that the standard definition of an **inverse** (we might call this the "full" inverse) is that it's both a **left** and **right inverse**:

Definition 286. Let A and B be sets, $f : A \to B$ be a function, and I_A and I_B be the identity functions on A and B, respectively. The function g is

(a) A *left inverse of* f if $g \circ f = I_A$;

(b) A *right inverse of* f if $f \circ g = I_B$;

(c) An *inverse of* f if it is both a left inverse and right inverse of f.

Equivalently, the function g is an inverse of the function $f : A \to A$ if $f \circ g = g \circ f = I_A$.

Recall that a function is **(a)** a *surjection* (or *surjective* or *onto*) if every element in its codomain is "hit"; and **(b)** an *injection* (*injective* or *one-to-one*) if each element in its codomain is "hit" at most once. We call a function a *bijection* if every element in its codomain is "hit" exactly once—or equivalently,

Definition 287. We call a function a *bijection* (or *bijective*) if it's both a surjection and an injection.

Proposition 28. *A function has*

(a) *A left inverse* $\iff$ *It is injective.*

(b) *A right inverse* $\iff$ *It is surjective.*

(c) *An inverse* $\iff$ *It is bijective.*

Proof. Let $f : A \to B$ be a function with range C.

(a) ($\Longrightarrow$) Suppose f is not injective. Then there exist $x_1, x_2 \in A$ and $y \in B$ such that $x_1 \neq x_2$ but $y = f(x_1) = f(x_2)$.

So, for any function g for which the composite function $g \circ f$ is defined, we have $(g \circ f)(x_1) = (g)(f(x_1)) = g(y)$ and $(g \circ f)(x_2) = g(y)$. Hence, $(g \circ f)(x_1) = x_1$ and $(g \circ f)(x_2) = x_2$ cannot both be true. Thus, g cannot be a left inverse of f.

($\Longleftarrow$) Suppose f is injective. Then for each $y \in C$, there exists exactly one $x \in A$ such that $f(x) = y$.

Define $g : C \to A$ by $g(y) \stackrel{1}{=} x$. So, the composite function $g \circ f : A \to A$ exists and is defined by $(g \circ f)(x) = g(f(x)) = g(y) \stackrel{1}{=} x$. Hence, $g \circ f = I_A$ and g is a left inverse of f.

(b) ($\Longrightarrow$) Suppose f is not surjective. Then there exists $y \in B$ such that for all $x \in A$, $f(x) \neq y$. So, if h is a function for which the composite function $f \circ h$ is defined, then $(f \circ h)(y) = f(h(y)) \neq y$ and h cannot be a right inverse of f.

($\Longleftarrow$) Suppose f is surjective. Then for each $x \in B$, there exists some $y \in A$ such that $f(y) \stackrel{2}{=} x$.

Define $h : B \to A$ by $h(x) \overset{3}{=} y$. So, the composite function $f \circ h : B \to B$ exists and is defined by $(f \circ h)(x) = f(h(x)) \overset{3}{=} f(y) \overset{2}{=} x$. Hence, $f \circ h = I_B$ and h is a right inverse of f.

(c) follows from **(a)**, **(b)**, and Definition 286(c). □

Example 1575. Define $f : \{0\} \to \{1,2\}$ by $f(0) = 1$ and $g : \{1,2\} \to \{0\}$ by $g(1) = g(2) = 0$.

Observe that f is injective (but not surjective). So, f has a left inverse (but not a right inverse).

In contrast, g is surjective (but not injective). So, g has a right inverse (but not a left inverse).

Indeed, g is a left inverse of f and f is a right inverse of g, as we now show:

Since Range $f = \{1\} \subseteq$ Domain g, the composite function $g \circ f : \{0\} \to \{0\}$ exists and is defined by $(g \circ f)(0) = g(f(0)) = g(1) = 0$. Hence, $g \circ f = I_{\{0\}}$.

Proposition 29. *Let $f : A \to B$ be a function with range C and g be a function.*

(a) *Suppose f is injective. Then g is a left inverse of f $\iff$ g satisfies these three conditions hold:*

- **(i)** $C \subseteq$ Domain g*;*
- **(ii)** Codomain $g = A$*; and*
- **(iii)** *g maps each $x \in C$ to the unique $y \in A$ for which $f(y) = x$.*

(b) *Suppose f is surjective. Then g is a right inverse of f $\iff$ g satisfies these three conditions hold:*

- **(i)** Domain $g = B$*;*
- **(ii)** $A \subseteq$ Codomain g*; and*
- **(iii)** *g maps each $x \in B$ to some $y \in A$ for which $f(y) = x$.*

(c) *Suppose f is bijective. Then g is an inverse of f $\iff$ g satisfies these three conditions hold:*

- **(i)** Domain $g = B$*;*
- **(ii)** Codomain $g = A$*; and*
- **(iii)** *g maps each $x \in B$ to the unique $y \in A$ for which $f(y) = x$.*

Moreover, f has only one inverse. **(Uniqueness of Inverse)**

Proof. **(a)** ($\Longleftarrow$) Since Range $f = C \subseteq$ Domain g, the composite function $(g \circ f) : A \to A$ exists and is defined by $(g \circ f)(y) = g(f(y)) = g(x) = y$. So, $g \circ f = I_A$ and g is a left inverse of f. ✓

($\Longrightarrow$) We show that if any of the three conditions is violated, then $g \circ f \neq I_A$ and g is not be a left inverse of f:

- **(i)** If $C \nsubseteq$ Domain g, then $g \circ f$ is not defined.
- **(ii)** If Codomain $g \neq A$, then Codomain $(g \circ f) =$ Codomain $g \neq A$.

(iii) Suppose that for some $x \in C$, y is the unique element in A such that $f(y) = x$, but $g(x) \neq y$. Then $(g \circ f)(y) = g(f(y)) = g(x) \neq y$.

(b) ($\Longleftarrow$) Since Range $g \subseteq A$ = Domain f, the composite function $(f \circ g) : B \to B$ exists. Moreover, for each $x \in B$, we have $(f \circ g)(x) = f(g(x)) = f(y) = x$. So, $f \circ g = I_B$ and g is a right inverse of f. ✓

($\Longrightarrow$) We show that if any of the three conditions is violated, then $f \circ g \neq I_B$ and g is not a right inverse of f:

(i) If Domain $g \neq B$, then Domain $(f \circ g)$ = Domain $g \neq B$.

(ii) If Range $f = A \nsubseteq$ Codomain g, then $f \circ g$ is not defined.

(iii) Suppose that for some $x \in B$, $g(x) = y$ but $f(y) \neq x$. Then $(f \circ g)(x) = f(g(x)) = f(y) \neq x$.

(c) ($\Longleftarrow$) Suppose the three conditions hold. Then by **(a)** and **(b)**, g is a left and right inverse of f. And hence, by Definition 286(c), g is an inverse of f.

($\Longrightarrow$) Suppose any of the three conditions is violated. Then by **(a)** and **(b)**, g is either not a left inverse or not a right inverse of f. And hence, by Definition 286(c), g is not a inverse of f.

There is only one function that satisfies the three conditions, so that the inverse is indeed unique. □

It turns out that there is an asymmetry between left and right inverses. We can meaningfully single out a left inverse and call it *the* **Left Inverse**:

Definition 288. Let $f : A \to B$ be a function with range C. Suppose f is injective. Then *the Left Inverse* of f is the function $g : C \to A$ that maps each $x \in C$ to the unique $y \in A$ for which $f(y) = x$.

The above definition of *the* Left Inverse is what we've been using as our definition of "the" inverse throughout the main text of this textbook (and, in particular, corresponds to Definition 79).

In contrast, we cannot meaningfully single out a right inverse and call it *the* **Right Inverse**:

Example 1576. Define $f : [0,1] \to \{2\}$ by $f(x) = 2$. Then the following functions are right inverses of f:

- $g : \{2\} \to [0,1]$ defined by $g(x) = 0$;
- $h : \{2\} \to [0,1]$ defined by $h(x) = 0.5$; and more generally,
- $i : \{2\} \to [0,1]$ defined by $i(x) = k$, for any $k \in [0,1]$.

There's no meaningful criterion by which we can call one of these functions g, h, or i *the* Right Inverse.

142.16. When Do f and f^{-1} Intersect?

Fact 54. *Let D be an interval and $f: D \to \mathbb{R}$ be a continuous function with inverse f^{-1}. If f and f^{-1} intersect at least once, then* at least one *of their intersection points is on the line $y = x$.*

Proof. Pick any intersection point (a, b).

If (a, b) is on the line $y = x$, then we are done. So, suppose it is not, i.e. $a \neq b$, so that either $a > b$ or $a < b$.

Since (a, b) is in the graph of f^{-1}, by Fact 52, (b, a) is in the graph of f.

Now define $g(x) = f(x) - x$. By Theorem 26, g is also continuous on $[a, b]$.

- If $a < b$, then $g(a) = f(a) - a = b - a > 0$, while $g(b) = f(b) - b = a - b < 0$.
- If $a > b$, then $g(a) = f(a) - a = b - a < 0$, while $g(b) = f(b) - b = a - b > 0$.

Either way, by the Intermediate Value Theorem (Theorem 12), there exists $c \in (a, b)$ such that $g(c) = 0$ or equivalently, $f(c) = c$.

Hence, (c, c) is in the graph of f and, by Fact 52, is also in the graph of f^{-1}.Thus, f and f^{-1} also intersect at (c, c), which is on the line $y = x$. □

Fact 55. *Let D be an interval and $f: D \to \mathbb{R}$ be a continuous function with inverse f^{-1}. If f and f^{-1} intersect at an even number of points, then* all *of their intersection points are on the line $y = x$.*

Proof. Suppose (for contradiction) that there exists (a, b) with $a \neq b$ that is in the graphs of f and f^{-1}.

Then by Fact 52, (b, a) is in the graphs of f^{-1} and f.

Hence, any shared intersection points that aren't on the line $y = x$ must come in pairs.

Since the total number of intersection points is even, any shared intersection points that *are* on the line $y = x$ must also come in pairs.

By Fact 54, at least one of f and f^{-1}'s intersection points is on the line $y = x$. Thus, there exist at least two points on the line $y = x$ at which f and f^{-1} intersect. Call them (c, c) and (d, d) with $c < d$.

Now, since f is continuous on an interval, by Corollary 7, it is either strictly increasing or strictly decreasing.

Since $c < d$ and $f(c) = c < f(d) = d$, f is strictly increasing. ☺

Since $a \neq b$, we have either $a > b$ or $a < b$.

- If $a > b$, then $f(a) = b < f(b) = a$, contradicting ☺.
- If $a < b$, then $f(a) = b > f(b) = a$, contradicting ☺.

□

142.17. Transformations

We first formally define what a **translation**, a **stretch**, and a **compression** are.

Definition 289. Let $a > 0$. We say that the graph H is the graph G ...

(a) *Translated a units ...*

- **(i)** *Downwards* if $(p,q) \in G \iff (p, q-a) \in H$;
- **(ii)** *Upwards* if $(p,q) \in G \iff (p, q+a) \in H$;
- **(iii)** *Rightwards* if $(p,q) \in G \iff (p+a, q) \in H$;
- **(iv)** *Leftwards* if $(p,q) \in G \iff (p-a, q) \in H$;

(b) *Stretched by a factor of a, outwards from the ...*

- **(i)** *x-axis* if $(p,q) \in G \iff (p, aq) \in H$;
- **(ii)** *y-axis* if $(p,q) \in G \iff (ap, q) \in H$;

(c) *Compressed by a factor of a, inwards towards the ...*

- **(i)** *x-axis* if $(p,q) \in G \iff (p, q/a) \in H$;
- **(ii)** *y-axis* if $(p,q) \in G \iff (p/a, q) \in H$.

Fact 58. *Let $a, b > 0$, and $c, d \in \mathbb{R}$. Suppose f is a nice function. Then to get from the graph of f to the graph of $y = af(bx+c) + d$, follow these steps:*

1. *To get (the graph of) $y = f(x+c)$, translate* left*wards by c units.*
2. *To get $y = f(bx+c)$,* compress *inwards towards y-axis by a factor of b.*
3. *To get $y = af(bx+c)$, stretch outwards from x-axis by a factor of a.*
4. *To get $y = af(bx+c)+d$, translate upwards by d units.*

Proof. Let G_0 be the graph of f; G_1 of $y = f(x+c)$; G_2 of $y = f(bx+c)$; G_3 of $y = af(bx+c)$; and G_4 of $y = af(bx+c)+d$.

1. Observe that $q = f(p) \iff q = f(p-c+c)$. So, $(p,q) \in G_0 \iff (p-c, q) \in G_1$. Hence, by Definition 289(a)(iv), G_1 is G_0 translated leftwards by c units.
2. Observe that $q = f(p+c) \iff q = f(bp/b+c)$. So, $(p,q) \in G_1 \iff (p/b, q) \in G_2$. Hence, by Definition 289(c)(ii), G_2 is G_1 compressed by a factor of b, inwards towards the y-axis.
3. Observe that $q = f(bp+c) \iff aq = af(bp+c)$. So, $(p,q) \in G_2 \iff (p, aq) \in G_3$. Hence, by Definition 289(b)(i), G_3 is G_2 stretched by a factor of a, outwards from the x-axis.
4. Observe that $q = af(bp+c)+d \iff q+d = af(bp+c)+d$. So, $(p,q) \in G_3 \iff (p, q+d) \in G_4$. Hence, by Definition 289(a)(i), G_4 is G_3 translated upwards by d units.

□

Fact 256. *Let $d \in \mathbb{R}$ and f be a nice function. Suppose the graph of f has x-intercept $(p, 0)$, y-intercept $(0, q)$, line of symmetry $ax + by + c = 0$, turning point (r, s), and asymptote $\alpha x + \beta y + \gamma = 0$. Then*

(a) *The graph of $y = f(x) + d$ has y-intercept $(0, q + d)$, line of symmetry $ax + b(y - d) + c = 0$, turning point $(r, s + d)$, and asymptote $\alpha x + \beta(y + d) + \gamma = 0$.*

(b) *The graph of $y = f(x + d)$ has x-intercept $(p - d, 0)$, line of symmetry $a(x - d) + by + c = 0$, turning point $(r - d, s)$, and asymptote $\alpha(x - d) + \beta y + \gamma = 0$.*

(c) *The graph of $y = df(x)$ has y-intercept $(0, dq)$, line of symmetry $ax + by/d + c = 0$, turning point (r, ds), and asymptote $\alpha x + \beta y/d + \gamma = 0$.*

(d) *The graph of $y = f(dx)$ has x-intercept $(p/d, 0)$, line of symmetry $ax/d + by + c = 0$, turning point $(r/d, s)$, and asymptote $\alpha x/d + \beta y + \gamma = 0$.*

Proof. We prove only **(a)**—the proofs of **(b)**, **(c)**, and **(d)** are similar and thus omitted.

(a) Let F be the graph of f and G be the graph of $y = f(x) + d$.

Intercept. Since $(0, q) \in F$, we have $q = f(0)$ or $q + d = f(0) + d$. Hence, $(0, q + d)$ is in G (and is a y-intercept of G). ✓

Line of symmetry. Pick any point $P_1 = (u, v)$ in G. By Fact 41, the reflection of P_1 in the line $ax + b(y - d) + c = 0$ is the point

$$\bar{P}_1 = \left(u - 2a\frac{au + bv + c - bd}{a^2 + b^2}, v - 2b\frac{au + bv + c - bd}{a^2 + b^2}\right).$$

It suffices to show that $\bar{P}_1 \in G$.

Since $P_1 \in G$, we have $P_2 = (u, v - d) \in F$. And the reflection of P_2 in the line of symmetry $ax + by + c = 0$ is the point

$$\bar{P}_2 = \left(u - 2a\frac{au + b(v - d) + c}{a^2 + b^2}, (v - d) - 2b\frac{au + b(v - d) + c}{a^2 + b^2}\right) \in F.$$

Since $\bar{P}_2 \in F$, the translation of $\bar{P}_2$ upward by d units is in G. But this point is $\bar{P}_1$—so, we've shown that $\bar{P}_1 \in G$. ✓

Turning point. Define $g : \text{Domain} f \to \text{Codomain} f$ by $g(x) = f(x) + d$. Then the graph of g is G.

We are given that r is **(i)** a stationary point; and **(ii)** a strict local extremum of f. We want to show that r is, likewise, also a stationary point and a strict local extremum of g.

By **(i)**, $f'(r) = 0$ and hence $g'(r) = f'(r) + \frac{\mathrm{d}}{\mathrm{d}x}d = 0 + 0$,[577] so that r is also a stationary point of f. ✓

By **(ii)**, there exists $\varepsilon > 0$ such that for all $x \in \text{Domain} f \cap (r - \varepsilon, r + \varepsilon)$, either $f(r) > f(x)$ or $f(r) < f(x)$. Hence, it is also true that for all $x \in \text{Domain} g \cap (r - \varepsilon, r + \varepsilon)$, either $g(r) = f(r) + d > g(x) = f(x) + d$ or $g(r) = f(r) + d < g(x) = f(x) + d$. Thus, r is also a strict local extremum of g.

[577]Here we are using results from Part V (Calculus) or the accompanying Appendices.

Asymptote. We'll deal with two cases:

Case 1. $\alpha x + \beta y + \gamma = 0$ is vertical (so $\beta = 0$ and the line may be rewritten as $x = -\gamma/\alpha$).

By Definition 322,[578] the left- or right-hand limit of f at $-\gamma/\alpha$ is $\pm\infty$.

Suppose the left-hand limit of f at $-\gamma/\alpha$ is ∞ (the other three cases are dealt with similarly). Then by Definition 321, for every $N \in \mathbb{R}$, there exists $\delta > 0$ such that $x \in \text{Domain } f \cap \mathrm{N}_\delta^-(a)$ implies $f(x) > N - d$—and hence, also that $g(x) = f(x) + d > N$. Thus, the left-hand limit of g at $-\gamma/\alpha$ is ∞. And by Definition 322, the line $x = -\gamma/\alpha$ or $\alpha x + \beta y + \gamma = 0$ is an asymptote for g.

Case 2. $\alpha x + \beta y + \gamma = 0$ is not vertical (so $\beta \neq 0$ and the line may be rewritten as $y = -(\alpha/\beta)x - \gamma/\beta$).

By Definition 326, either $\lim_{x\to\infty} f(x) \stackrel{1}{=} -(\alpha/\beta)x - \gamma/\beta$ or $\lim_{x\to-\infty} f(x) \stackrel{2}{=} -(\alpha/\beta)x - \gamma/\beta$.

Suppose $\stackrel{1}{=}$ ($\stackrel{2}{=}$ is dealt with similarly). Then by Definition 325, for every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $\text{Domain } f \cap (N,\infty) \neq \varnothing$ and for every $x \in \text{Domain } f \cap (N,\infty)$, we have $f(x) \in \mathrm{N}_\varepsilon(-(\alpha/\beta)x - \gamma/\beta)$—and hence, also $g(x) = f(x) + d \in \mathrm{N}_\varepsilon(-(\alpha/\beta)x - \gamma/\beta + d)$. And so by Definition 326, $y = -(\alpha/\beta)x - \gamma/\beta + d$ or $\alpha x + \beta(y+d) + \gamma = 0$ is an asymptote for g. □

Fact 59. *Let $a \in \mathbb{R}$ and f be a nice function.*

(a) *Suppose $f(x) \stackrel{1}{=} a$ for all $x \in \text{Domain } f$. Then*

  (i) *f is symmetric in the horizontal line $y = a$; and*
  (ii) *f is not symmetric in any other horizontal line.*

(b) *Suppose f is not constant. Then f is not symmetric in any horizontal line.*

Proof. We already proved **(a)(i)** in the main text.

Let F be the graph of f.

(a)(ii) Let $b \in \mathbb{R}$ with $b \stackrel{2}{\neq} a$. Pick any $c \in \text{Domain } f$.

Consider the point $(c, f(c)) \in F$. Its reflection in $x = b$ is $R = (c, 2b - f(c))$.

Recall that as a function, f must map c to exactly one object. So, $R \in F \iff f(c) = 2b - f(b)$ or, rearranging, $b = f(c) \stackrel{1}{=} a$, which contradicts $\stackrel{2}{\neq}$. Hence, $R \notin F$. Thus, f is not symmetric in $x = b$.

(b) Let $b \in \mathbb{R}$. Since f is not constant, there exists some $c \in \text{Domain } f$ such that $f(c) \stackrel{3}{\neq} b$.[579]

Consider the point $(c, f(c)) \in F$. Its reflection in $x = b$ is $R = (c, 2b - f(c))$.

Recall that as a function, f must map c to exactly one object. So, $R \in F \iff f(c) = 2b - f(b)$ or, rearranging, $b = f(c)$, which contradicts $\stackrel{3}{\neq}$. Hence, $R \notin F$. Thus, f is not symmetric in $x = b$. □

[578] Here we are using results from Part V (Calculus) or the accompanying Appendices.

[579] If no such c exists, then f is constant with $f(x) = b$ for all $x \in \text{Domain } f$—contradicting our assumption that f is constant.

Fact 60. *Let $a \in \mathbb{R}$ and f be a nice, one-to-one function (whose domain is not empty).*

(a) *Suppose* Domain $f = \{a\}$. *Then*

(i) *f is symmetric in the vertical line $x = a$; and*

(ii) *f is not symmetric in any other vertical line.*

(b) *Suppose* Domain f *contains more than one point. Then f is not symmetric in any vertical line.*

Proof. We already proved **(a)** in the main text.

Let F be the graph of f.

(b) Let $c \in \mathbb{R}$. We'll show that f is not symmetric in $x = c$.

Since Domain f contains more than one point, pick any distinct $d, e \in$ Domain f with corresponding distinct points $(d, f(d)), (e, f(e)) \in F$.

By Fact 39, the reflections of $(d, f(d))$ and $(e, f(e))$ in $x = c$ are $(2c - d, f(d))$ and $(2c - e, f(e))$.

Suppose for contradiction f is symmetric in $x = c$. Then both $(2c - d, f(d))$ and $(2c - e, f(e))$ are also in F.

Then since f is one-to-one, we must have $2c - d = d$ and $2c - e = e$—or, $c = d$ and $c = e$, which implies $d = e$. But this contradicts our assumption that d and e are distinct. So, f is not symmetric in $x = c$. □

Fact 61. *Suppose f is a nice function. Then f is even $\iff$ f is symmetric in the y-axis.*

Proof. Let G be the graph of f. Then f is symmetric in the y-axis $\iff$ For each $(x, a) \in G$, we have $(-x, a) \in G$ $\iff$ For each $x \in$ Domain f, $f(x) = a = f(-x)$ $\iff$ f is even. □

Fact 62. *Suppose f is a nice function. Then f is odd $\iff$ f is symmetric about the origin.*

Proof. Let G be the graph of f. Then f is symmetric about the origin $\iff$ For each $(x, a) \in G$, we have $(-x, -a) \in G$ $\iff$ For each $x \in$ Domain f, $f(x) = a = -f(-x)$ $\iff$ f is odd. □

142.18. Trigonometry

We reproduce from Ch. 102.3 this textbook's official definitions of the sine and cosine functions:

Definition 219. The *sine function* $\sin : \mathbb{R} \to \mathbb{R}$ is defined by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Definition 220. The *cosine function* $\cos : \mathbb{R} \to \mathbb{R}$ is defined by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Definition 290. Let $a, b \in \mathbb{R}$, $r > 0$, and $p, q \in [0, 2\pi]$. Let $P = (r\cos p + a, r\sin p + b)$ and $Q = (r\cos q + a, r\sin q + b)$, so that P and Q are points on the circle of radius r centred on (a, b). We define the *arc* PQ to be this set of points:

$$PQ = \begin{cases} \{(r\cos t + a, r\sin t + b) : t \in [p, q)\}, & \text{for } p \le q, \\ \{(r\cos t + a, r\sin t + b) : t \in [p, q + 2\pi]\}, & \text{for } p > q, \end{cases}$$

Remark 207. The above definition formalizes the anticlockwise convention:

With the parametrization $C = (r\cos t + a, r\sin t + b)$, as t increases, we're "going anticlockwise":

t	0	$\pi/2$	π	$3\pi/2$	2π
C	$(r, 0)$	$(0, r)$	$(-r, 0)$	$(0, -r)$	$(r, 0)$

And the above definition of PQ is such that if $p \le q$, then we "go" from $t = p$ to $t = q$ (anticlockwise from P to Q).

And if $p > q$, then we "go" from $t = p$ to $t = q + 2\pi$ (also anticlockwise from P to Q).

We reproduce from Ch. 56 the definition of the angle between two vectors:

Definition 145. The *angle between two non-zero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ is this number:

$$\cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

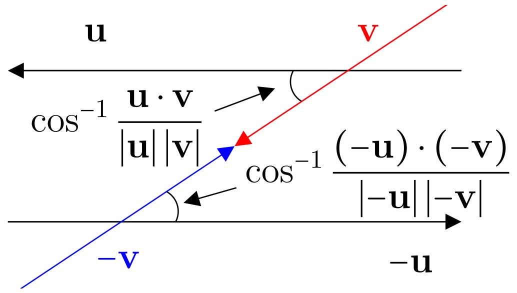
Much of the following assumes knowledge of what's covered in Part IV (Vectors).

Definition 291. Let AB be an arc of a circle with centre O. The *angle subtended by* AB is the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Fact 257. ("Z" Rule) *The angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *equals the angle between* $-\mathbf{u}$ *and* $-\mathbf{v}$.

Proof. Apply definition 145: $\cos^{-1}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \cos^{-1}\frac{(-\mathbf{u})\cdot(-\mathbf{v})}{|-\mathbf{u}|\,|-\mathbf{v}|}$. □

Picture to illustrate why the above result may be interpreted as the "Z" Rule:[580]



Definition 292. A *triangle* is any set $T = \{A, B, C\} \subseteq \mathbb{R}^2$ three distinct points that do not lie on the same line.[581] We call A, B, and C the triangle's *vertices* (singular: *vertex*). Its three *sides* are the line segments AB, BC, and AC. Each of the triangle's three *angles* α, β, and γ is defined as the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$; $\overrightarrow{BA}$ and $\overrightarrow{BC}$; and $\overrightarrow{CA}$ and $\overrightarrow{CB}$ (respectively). Moreover, we call each of α, β, and γ the angle *facing* or *opposite* to the side BC, AC, and AB (respectively).

So, by Definition 145,

$$\alpha = \cos^{-1}\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|},\ \beta = \cos^{-1}\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|},\ \text{and } \gamma = \cos^{-1}\frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|}.$$

Also, given that Range $\cos^{-1} = [0, \pi]$ and A, B, and C can't line on the same line (see footnote), we have

$$\alpha, \beta, \gamma \in (0, \pi).$$

Remark 208. We'll often refer to the triangle $T = \{A, B, C\}$ by such shorthand as $\triangle_T$ or $\triangle ABC$.

Fact 258. (Law of Cosines) *Let $\{A, B, C\}$ be a triangle. Suppose γ is the angle facing AB. Then*

(a) $$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 - 2\left|\overrightarrow{BC}\right|\left|\overrightarrow{AC}\right|\cos\gamma;$$

(b) $$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 - 2\overrightarrow{BC}\cdot\overrightarrow{AC};$$

(c) $$\cos\gamma = \frac{\left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AB}\right|^2}{2\left|\overrightarrow{BC}\right|\left|\overrightarrow{AC}\right|}.$$

[580] Which was informally stated as Fact 64 in the main text.
[581] A, B, and C lie on the same line $\iff$ The angle α, β, or γ is 0 or π.

Proof. **(a)** and **(b)**:

$$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AC} - \overrightarrow{BC}\right|^2 = \left(\overrightarrow{AC} - \overrightarrow{BC}\right) \cdot \left(\overrightarrow{AC} - \overrightarrow{BC}\right) = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 - 2\overrightarrow{CA} \cdot \overrightarrow{CB}$$

$$= \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 - 2\left|\overrightarrow{BC}\right|\left|\overrightarrow{AC}\right| \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|} = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 - 2\left|\overrightarrow{BC}\right|\left|\overrightarrow{AC}\right| \cos\gamma$$

$$= \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 - 2\left|\overrightarrow{BC}\right|\left|\overrightarrow{AC}\right| \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|} = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 - 2\overrightarrow{BC} \cdot \overrightarrow{AC}.$$

(c) Simply rearrange **(a)**. □

Plug $\gamma = \pi/2$ into Fact 258 to get Pythagoras' Theorem:

Corollary 51. (Pythagoras' Theorem) *Let $\{A, B, C\}$ be a triangle. Suppose $\gamma = \pi/2$ is the angle facing AB. Then*

$$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2.$$

Fact 65. *The sum of any triangle's three angles is π.*

Proof. Let $\{A, B, C\}$ be a triangle; $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{CA}$, and $\mathbf{c} = \overrightarrow{AB}$; and $a = |\mathbf{a}|$, $b = |\mathbf{b}|$, and $c = |\mathbf{c}|$.

So, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, $\mathbf{c} \stackrel{1}{=} -(\mathbf{a} + \mathbf{b})$, and $c^2 \stackrel{2}{=} a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b}$.

Let α, β, and γ be the angles facing the sides BC, AC, and AB. So,

$$\alpha = \cos^{-1} \frac{-\mathbf{c} \cdot \mathbf{b}}{cb}, \ \beta = \cos^{-1} \frac{-\mathbf{c} \cdot \mathbf{a}}{ca}, \text{ and } \gamma = \cos^{-1} \frac{-\mathbf{b} \cdot \mathbf{a}}{ba}.$$

Now,

$$\begin{aligned}\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \frac{-\mathbf{c} \cdot \mathbf{b}}{cb}\frac{-\mathbf{c} \cdot \mathbf{a}}{ca} - \sqrt{1 - \left(\frac{-\mathbf{c} \cdot \mathbf{b}}{cb}\right)^2}\sqrt{1 - \left(\frac{-\mathbf{c} \cdot \mathbf{a}}{ca}\right)^2} \\ &= \frac{(\mathbf{c} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{a})}{abc^2} - \frac{\sqrt{b^2c^2 - (\mathbf{c} \cdot \mathbf{b})^2}\sqrt{a^2c^2 - (\mathbf{c} \cdot \mathbf{a})^2}}{abc^2}.\end{aligned}$$

But,

$$(\mathbf{c} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{a}) \stackrel{1}{=} [(\mathbf{a} + \mathbf{b}) \cdot \mathbf{b}][(\mathbf{a} + \mathbf{b}) \cdot \mathbf{a}]$$

$$= \left(\mathbf{a} \cdot \mathbf{b} + b^2\right)\left(a^2 + \mathbf{a} \cdot \mathbf{b}\right) = (\mathbf{a} \cdot \mathbf{b})^2 + a^2b^2 + \left(a^2 + b^2\right)\mathbf{a} \cdot \mathbf{b}$$

And,

$$b^2c^2 - (\mathbf{c}\cdot\mathbf{b})^2 \overset{1,2}{=} a^2b^2 + b^4 + 2b^2\mathbf{a}\cdot\mathbf{b} - [(\mathbf{a}+\mathbf{b})\cdot\mathbf{b}]^2 = a^2b^2 + b^4 + 2b^2\mathbf{a}\cdot\mathbf{b} - \left(\mathbf{a}\cdot\mathbf{b}+b^2\right)^2$$
$$= a^2b^2 + b^4 + 2b^2\mathbf{a}\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})^2 - b^4 - 2b^2\mathbf{a}\cdot\mathbf{b} = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2$$

Similarly,

$$a^2c^2 - (\mathbf{c}\cdot\mathbf{a})^2 \overset{1,2}{=} a^4 + a^2b^2 + 2a^2\mathbf{a}\cdot\mathbf{b} - [(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}]^2 = a^4 + a^2b^2 + 2a^2\mathbf{a}\cdot\mathbf{b} - \left(a^2+\mathbf{a}\cdot\mathbf{b}\right)^2$$
$$= a^4 + a^2b^2 + 2a^2\mathbf{a}\cdot\mathbf{b} - a^4 - (\mathbf{a}\cdot\mathbf{b})^2 - 2a^2\mathbf{a}\cdot\mathbf{b} = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2.$$

Altogether,

$$\cos(\alpha+\beta) = \frac{(\mathbf{a}\cdot\mathbf{b})^2 + a^2b^2 + (a^2+b^2)\,\mathbf{a}\cdot\mathbf{b} - a^2b^2 + (\mathbf{a}\cdot\mathbf{b})^2}{ab\,(a^2+b^2+2\mathbf{a}\cdot\mathbf{b})}$$
$$= \frac{2(\mathbf{a}\cdot\mathbf{b})^2 + (a^2+b^2)\,\mathbf{a}\cdot\mathbf{b}}{ab\,(a^2+b^2+2\mathbf{a}\cdot\mathbf{b})}$$
$$= \frac{\mathbf{a}\cdot\mathbf{b}\left[2\mathbf{a}\cdot\mathbf{b} + (a^2+b^2)\right]}{ab\,(a^2+b^2+2\mathbf{a}\cdot\mathbf{b})} \overset{3}{=} \frac{\mathbf{a}\cdot\mathbf{b}}{ab}.$$

Below, we show that $\alpha+\beta \in [0,\pi]$. And so, we can apply Fact 99(b) to $\overset{3}{=}$: $\cos^{-1}(\cos(\alpha+\beta)) = \alpha+\beta = \cos^{-1}\dfrac{\mathbf{a}\cdot\mathbf{b}}{ab}$.

Thus, since $\gamma = \cos^{-1}\dfrac{-\mathbf{b}\cdot\mathbf{a}}{ba}$, by Fact 101(c), $\alpha+\beta+\gamma = \pi$.

We now show that $\alpha+\beta \in [0,\pi]$. In fact, we'll show that $\alpha+\beta \in (0,\pi]$.

First, note that since $\alpha,\beta \in (0,\pi)$, $\alpha+\beta \in (0,2\pi)$. Next, compute

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta - \sin\beta\sin\alpha = \sqrt{1-\left(\frac{-\mathbf{c}\cdot\mathbf{b}}{cb}\right)^2}\frac{-\mathbf{c}\cdot\mathbf{a}}{ca} - \sqrt{1-\left(\frac{-\mathbf{c}\cdot\mathbf{a}}{ca}\right)^2}\frac{-\mathbf{c}\cdot\mathbf{b}}{cb}$$

$$= \frac{\sqrt{a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2}}{b^2c^2}\frac{-\mathbf{c}\cdot\mathbf{a}}{ca} + \frac{\sqrt{a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2}}{a^2c^2}\frac{-\mathbf{c}\cdot\mathbf{b}}{cb} = \frac{\sqrt{a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2}}{abc^3}\left(-\frac{\mathbf{c}\cdot\mathbf{a}}{b} - \frac{\mathbf{c}\cdot\mathbf{b}}{a}\right)$$

Now, $\sqrt{a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2} > 0$ and $abc^3 > 0$. Moreover,

$$-\frac{\mathbf{c}\cdot\mathbf{a}}{b} - \frac{\mathbf{c}\cdot\mathbf{b}}{a} \overset{1}{=} \frac{(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}}{b} + \frac{(\mathbf{a}+\mathbf{b})\cdot\mathbf{b}}{a} = \frac{a^2+\mathbf{b}\cdot\mathbf{a}}{b} + \frac{\mathbf{a}\cdot\mathbf{b}+b^2}{a} = \frac{a^3+\mathbf{a}\cdot\mathbf{b}\,(a+b)+b^3}{ab} = \frac{(a+b)}{}$$
$$= = \frac{a+b}{ab}\left(a^2-ab+b^2+\mathbf{a}\cdot\mathbf{b}\right) = \frac{a+b}{ab}\left[(a-b)^2+ab+\mathbf{a}\cdot\mathbf{b}\right] \overset{4}{\geq} \frac{a+b}{ab}\left[(a-b)^2+ab-ab\right] = \frac{a+b}{ab}(a-$$

where at $\overset{4}{\geq}$, we use Fact 133 (Cauchy's Inequality).

So, $\sin(\alpha+\beta) \geq 0$, which implies $\alpha+\beta \notin (\pi,2\pi)$, which implies $\alpha+\beta \in (0,\pi]$. $\square$

Fact 67. *Two triangles are similar $\iff$ Suppose one triangle's sides have lengths a, b, and c. Then there exists $k > 0$ such that the other's sides' lengths' are ka, kb, and kc.*

Proof. Let the first triangle be $\{A, B, C\}$, with angles α, β, and γ facing BC, AC, and AB, respectively; also, $\left|\overrightarrow{BC}\right| = a$, $\left|\overrightarrow{AC}\right| = b$, and $\left|\overrightarrow{AB}\right| = c$.

Let the second triangle $\{D, E, F\}$ with angles δ, ε, and ζ facing EF, DF, and DE, respectively.

($\Longleftarrow$) Suppose there exists $k > 0$ such that $\left|\overrightarrow{EF}\right| = ka$, $\left|\overrightarrow{DF}\right| = kb$, and $\left|\overrightarrow{DE}\right| = kc$.

By the Law of Cosines,

$$\alpha = \cos^{-1}\frac{b^2+c^2-a^2}{2cb}. \qquad \text{and}$$

$$\delta = \frac{(kb)^2+(kc)^2-(ka)^2}{2(kc)(kb)} = \cos^{-1}\frac{k^2\left(b^2+c^2-a^2\right)}{k^2 2cb} = \cos^{-1}\frac{b^2+c^2-a^2}{2cb} = \alpha.$$

We can similarly show that $\varepsilon = \beta$ and $\zeta = \gamma$.

($\Longrightarrow$) Suppose $\delta = \alpha$, $\varepsilon = \beta$, and $\zeta = \gamma$. Then

$$\cos\alpha = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \cos\delta \overset{1}{=} \frac{\overrightarrow{DE}\cdot\overrightarrow{DF}}{\left|\overrightarrow{DE}\right|\left|\overrightarrow{DF}\right|}.$$

Similarly,

$$\cos\beta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|} = \cos\varepsilon \overset{2}{=} \frac{\overrightarrow{ED}\cdot\overrightarrow{EF}}{\left|\overrightarrow{ED}\right|\left|\overrightarrow{EF}\right|}.$$

Plug $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$ and $\overrightarrow{EF} = \overrightarrow{DF} - \overrightarrow{DE}$ into $\overset{2}{=}$:

$$\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|} = \frac{\overrightarrow{BA}\cdot\left(\overrightarrow{AC}-\overrightarrow{AB}\right)}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|} = \frac{\overrightarrow{BA}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|} + \frac{\left|\overrightarrow{AB}\right|}{\left|\overrightarrow{BC}\right|} = \frac{\overrightarrow{ED}\cdot\overrightarrow{EF}}{\left|\overrightarrow{ED}\right|\left|\overrightarrow{EF}\right|} = \frac{\overrightarrow{ED}\cdot\left(\overrightarrow{DF}-\overrightarrow{DE}\right)}{\left|\overrightarrow{ED}\right|\left|\overrightarrow{EF}\right|} \overset{3}{=} \frac{\overrightarrow{ED}\cdot\overrightarrow{DF}}{\left|\overrightarrow{ED}\right|\left|\overrightarrow{EF}\right|}$$

Now plug $\overset{1}{=}$ into $\overset{3}{=}$ to get $\dfrac{\left|\overrightarrow{AB}\right|}{\left|\overrightarrow{BC}\right|} = \dfrac{\left|\overrightarrow{ED}\right|}{\left|\overrightarrow{EF}\right|}$ or $\dfrac{c}{a} = \dfrac{\left|\overrightarrow{ED}\right|}{\left|\overrightarrow{EF}\right|}$ or $\left|\overrightarrow{ED}\right| \overset{4}{=} \dfrac{\left|\overrightarrow{EF}\right|}{a}c$.

By repeating steps similar to the above, we can also show that $\dfrac{\left|\overrightarrow{AC}\right|}{\left|\overrightarrow{BC}\right|} = \dfrac{\left|\overrightarrow{DF}\right|}{\left|\overrightarrow{EF}\right|}$ or $\dfrac{b}{a} = \dfrac{\left|\overrightarrow{DF}\right|}{\left|\overrightarrow{EF}\right|}$ or $\left|\overrightarrow{DF}\right| \overset{5}{=} \dfrac{\left|\overrightarrow{EF}\right|}{a}b$.

And of course, we also have $\left|\overrightarrow{EF}\right| \overset{6}{=} \dfrac{\left|\overrightarrow{EF}\right|}{a}a$.

From $\stackrel{4}{=}$, $\stackrel{5}{=}$, and $\stackrel{6}{=}$, we see that there exists $k > 0$ (namely $k = \left|\overrightarrow{EF}\right|/a$) such that the second triangle's sides have lengths ka, kb, and kc. □

Corollary 10. *Suppose two triangles share an angle. If both triangles also share the ratio of the lengths of the sides adjacent to that angle, then they are similar.*

Proof. Let $\triangle ABC$ have angle α facing BC.

Suppose $\triangle DEF$ also has angle α facing EF and that $\dfrac{|AB|}{|AC|} = \dfrac{|DE|}{|DF|}$.

Then $|DE| \stackrel{1}{=} \dfrac{|DF|}{|AC|}|AB|$ and $|DF| \stackrel{2}{=} \dfrac{|DF|}{|AC|}|AC|$.

Also, $\alpha = \cos^{-1}\dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \cos^{-1}\dfrac{\overrightarrow{DE}\cdot\overrightarrow{DF}}{\left|\overrightarrow{DE}\right|\left|\overrightarrow{DF}\right|}$ so that $\dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \dfrac{\overrightarrow{DE}\cdot\overrightarrow{DF}}{\left|\overrightarrow{DE}\right|\left|\overrightarrow{DF}\right|}$. Next,

$$\frac{|EF|^2}{|BC|^2} = \frac{|DE|^2 + |DF|^2 - 2\overrightarrow{DE}\cdot\overrightarrow{DF}}{|AB|^2 + |AC|^2 - 2\overrightarrow{AB}\cdot\overrightarrow{AC}} = \frac{\left(\frac{|DF|}{|AC|}|AB|\right)^2 + \left(\frac{|DF|}{|AC|}|AC|\right)^2 - 2\left(\frac{|DF|}{|AC|}|AB|\right)\cdot\left(\frac{|DF|}{|AC|}|AC|\right)}{|AB|^2 + |AC|^2 - 2\overrightarrow{AB}\cdot\overrightarrow{AC}}$$

$$= \frac{\frac{|DF|^2}{|AC|^2}|AB|^2 + \frac{|DF|^2}{|AC|^2}|AC|^2 - 2\frac{|DF|^2}{|AC|^2}|AB|\cdot|AC|}{|AB|^2 + |AC|^2 - 2\overrightarrow{AB}\cdot\overrightarrow{AC}} = \frac{|DF|^2}{|AC|^2}.$$

Hence, $|EF| \stackrel{3}{=} \dfrac{|DF|}{|AC|}|BC|$. Thus, by $\stackrel{1}{=}$, $\stackrel{2}{=}$, $\stackrel{3}{=}$, and Fact 67 ($\Longleftarrow$), $\triangle ABC$ and $\triangle DEF$ are similar. □

In primary school, we learnt that the area of a triangle is

$$\frac{1}{2} \times \text{Base} \times \text{Height}.$$

Let's now write this down as a formal definition:

Definition 293. Let $\{A, B, C\}$ be a triangle. Let D be the foot of the perpendicular[582] from C to AB. The *area* of the triangle is this number:

$$\frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|.$$

Fact 259. *Let $\{A, B, C\}$ be a triangle. If α is the angle facing BC, then the triangle's area is*

$$\frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\alpha.$$

[582] See Ch. 62.

Proof. Let D be the foot of the perpendicular from C to AB. Then the area of the triangle is $0.5\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|$.

So, it suffices to show that $\left|\overrightarrow{CD}\right| = \left|\overrightarrow{AC}\right|\sin\alpha$.

Now, $\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\frac{\overrightarrow{AD}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AD}\right|\left|\overrightarrow{AC}\right|}\right)^2 = 1 - \frac{\left(\overrightarrow{AD}\cdot\overrightarrow{AC}\right)^2}{\left|\overrightarrow{AD}\right|^2\left|\overrightarrow{AC}\right|^2}$. Next,

$$\frac{\left(\overrightarrow{AD}\cdot\overrightarrow{AC}\right)^2}{\left|\overrightarrow{AD}\right|^2} = \frac{\left[\overrightarrow{AD}\cdot\left(\overrightarrow{AD}+\overrightarrow{DC}\right)\right]^2}{\left|\overrightarrow{AD}\right|^2} = \frac{\left[\overrightarrow{AD}\cdot\overrightarrow{AD}+\overrightarrow{AD}\cdot\overrightarrow{DC}\right]^2}{\left|\overrightarrow{AD}\right|^2} = \frac{\left[\left|\overrightarrow{AD}\right|^2+0\right]^2}{\left|\overrightarrow{AD}\right|^2} = \left|\overrightarrow{AD}\right|^2.$$

So, $\left|\overrightarrow{AC}\right|^2\sin^2\alpha = \left|\overrightarrow{AC}\right|^2 - \frac{\left(\overrightarrow{AD}\cdot\overrightarrow{AC}\right)^2}{\left|\overrightarrow{AD}\right|^2} = \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AD}\right|^2$.

By Pythagoras' Theorem, $\left|\overrightarrow{CD}\right|^2 = \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AD}\right|^2$.

Hence, $\left|\overrightarrow{AC}\right|^2\sin^2\alpha = \left|\overrightarrow{CD}\right|^2$. Thus, $\left|\overrightarrow{AC}\right|\sin\alpha = \pm\left|\overrightarrow{CD}\right|$. (The negative value can be discarded since $\left|\overrightarrow{AC}\right|, \sin\alpha, \left|\overrightarrow{CD}\right| > 0$.) □

Fact 73. Suppose $A, B \in \mathbb{R}$. Then

Addition and Subtraction Formulae for Sine and Cosine

(a) $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

(b) $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

Double Angle Formulae for Sine and Cosine

(c) $\sin 2A = 2\sin A\cos A$

(d) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Proof. (We already proved **(c)** and **(d)** in Exercise 140.)

(a) and **(b)** We first prove the Addition Formulae for Sine and Cosine:

Fix $B \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by

$$f(A) \stackrel{1}{=} \sin(A+B) - [\sin A\cos B + \cos A\sin B] \qquad \text{and}$$
$$g(A) \stackrel{2}{=} \cos(A+B) - [\cos A\cos B - \sin A\sin B].$$

Then f and g are differentiable, with

$$f'(A) = \cos(A+B) - [\cos A\cos B - \sin A\sin B] = g(A) \qquad \text{and}$$
$$g'(A) = -\sin(A+B) - [-\sin A\cos B - \cos A\sin B] = -f(A).$$

Moreover, $(f)^2 + (g)^2$ is differentiable, with

$$\left((f)^2+(g)^2\right)'(A)=2f(A)f'(A)+2g(A)g'(A)=2f(A)g(A)-2g(A)f(A)=0.$$

By Proposition 8 (Derivative Is Zero Function Implies Constant Function), there exists some $c\in\mathbb{R}$ such that $\left((f)^2+(g)^2\right)(A)=c$ for all $A\in\mathbb{R}$. But

$$\left((f)^2+(g)^2\right)(0)=$$
$$\{\sin(0+B)-[\sin 0\cos B+\cos 0\sin B]\}^2+\{\cos(0+B)-[\cos 0\cos B-\sin 0\sin B]\}^2=0.$$

So, $\left((f)^2+(g)^2\right)(A)\stackrel{3}{=}0$ for all $A\in\mathbb{R}$.

Since for all $A\in\mathbb{R}$, $(f)^2(A)\geq 0$ and $(g)^2(A)\geq 0$, $\stackrel{3}{=}$ implies that $(f)^2(A)=0$ and $(g)^2(A)=0$—and hence, $f(A)=0$ and $g(A)=0$. And so, by $\stackrel{1}{=}$ and $\stackrel{2}{=}$, we have, for all $A\in\mathbb{R}$,

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \qquad \text{and} \qquad \cos(A+B)=\cos A\cos B-\sin A\sin B.$$

Since B was arbitrary, these last two equations also apply for any $B\in\mathbb{R}$. This completes the proof of the Addition Formulae for Sine and Cosine.

The proof of the Subtraction Formulae for Sine and Cosine is similar and omitted. □

Fact 79. *The lines of symmetry for*

(a) sin *are* $x=k\pi/2$*, for odd integers* k*;*

(b) cos *are* $x=k\pi$*, for integers* k*.*

Proof. **(a)** We will first show that no line of symmetry for sin can be **(i)** horizontal; or **(ii)** oblique. We will then show that **(iii)** sin is symmetric in **(I)** $x=k\pi/2$, for odd integers k; but **(II)** in no other vertical lines.

(a)(i) Since sin is not constant, by Fact 59, it is not symmetric in any horizontal line.

(a)(ii) Let $a\neq 0$, $b\neq 0$. Consider the oblique line $ax+by+c=0$ or $y=-\dfrac{a}{b}x-\dfrac{c}{b}$.

Suppose $\dfrac{a}{b}>0$. Then the line contains the point $P=\left(\dfrac{2b-c}{a},-2\right)$. This point is the point on the line that is closest to some point Q in the graph of sin.

Let R be the reflection of Q in the line, so that R is also the reflection of Q in P. But since the y-coordinate of Q is no lower than -1, that of R must be lower than -2. Hence, R is not in the graph of sin and this line cannot be a line of symmetry for sin.

The case where $\dfrac{a}{b}<0$ is similarly handled and thus omitted.

(a)(iii) By Fact 42, the vertical line $x=e$ is a line of symmetry for sin if and only if for all x, $\sin x\stackrel{\star}{=}\sin(2e-x)$.

(a)(iii)(I) Suppose e is an odd integer multiple of $\pi/2$. Then

$$\sin(2e-x)=\sin 2e\cos x-\cos 2e\sin x=0\cos x-(-1)\sin x=\sin x.$$

So, $x=e$ is a line of symmetry for sin.

(a)(iii)(II) Suppose e is not an odd integer multiple of $\pi/2$. Then $\cos e \neq 0$. Moreover, there exists $x \in \mathbb{R}$ such that $\sin(e-x) \neq 0$ and hence

$$\begin{aligned}\sin(2e-x) - \sin x &= \sin 2e \cos x - \cos 2e \sin x - \sin x \\ &= 2\sin e \cos e \cos x - \left(2\cos^2 e - 1\right)\sin x - \sin x \\ &= 2\cos e\left(\sin e \cos x - \cos e \sin x\right) \\ &= 2\cos e \sin(e-x) \neq 0.\end{aligned}$$

So, there exists $x \in \mathbb{R}$ such that $\stackrel{\star}{=}$ is false. Hence, $x = e$ is not a line of symmetry for sin.

(b) Similar, omitted. □

Fact 92. *The graph of* tan *has no lines of symmetry.*

Proof. We will show that tan is not symmetric in any **(a)** horizontal line; **(b)** vertical line; or **(c)** oblique line.

(a) Since tan is not constant, by Fact 59, it is not symmetric in any horizontal line.

(b) By Fact 42, the vertical line $x = d$ is a line of symmetry for tan if and only if for all x (in Domain tan),

$$\tan x \stackrel{\star}{=} \tan(2d - x).$$

Consider the points $0, \pi/4$ in the graph of tan. Suppose

$$\tan 0 = \tan(2d - 0) \qquad \text{and} \qquad \tan \pi/4 = \tan(2d - \pi/4),$$

or,

$$0 \stackrel{1}{=} \tan 2d \qquad \text{and} \qquad 1 = \frac{\tan 2d - \tan \pi/4}{1 + \tan 2d \tan \pi/4} \stackrel{2}{=} \frac{\tan 2d - 1}{1 + \tan 2d}.$$

Plug $\stackrel{1}{=}$ into $\stackrel{2}{=}$ to get $1 = -1$, a contradiction. Hence, no vertical line $x = d$ is a line of symmetry for tan.

(c) The reflection of the vertical asymptote $x = \pi/2$ in any oblique line is a non-vertical (asymptote). So, the reflection of tan in any oblique line cannot be tan. Hence, no oblique line can be a line of symmetry for tan. □

Fact 93. *The graph of* $\tan\big|_{(-\pi/2,\pi/2)}$ *has no lines of symmetry.*

Proof. Same as proof of Fact 92. □

Definition 294. Let $A \subseteq \mathbb{R}$ and $f : A \to B$ be a function. We say that f is *periodic with period* p if for any $a_1, a_2 \in A$,

$$a_1 - a_2 = kp \text{ for some } k \in \mathbb{Z} \qquad \Longrightarrow \qquad f(a_1) = f(a_2).$$

Fact 98. **(a)** $\sin\left(\sin^{-1}x\right) = x$ *for all* $x \in$ Domain $\sin^{-1} = [-1,1]$.
(b) $\cos\left(\cos^{-1}x\right) = x$ *for all* $x \in$ Domain $\cos^{-1} = [-1,1]$.
(c) $\tan\left(\tan^{-1}x\right) = x$ *for all* $x \in$ Domain $\tan^{-1} = \mathbb{R}$.

Proof. Apply Fact 255. □

Fact 99.
(a) $\sin^{-1}(\sin x) = x \iff x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.
(b) $\cos^{-1}(\cos x) = x \iff x \in [0,\pi]$.
(c) $\tan^{-1}(\tan x) = x \iff x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Proof. **(a)** ($\Longrightarrow$) If $x \notin [-\pi/2, \pi/2]$, then $\sin^{-1}(\sin x) \neq x$ because Range $\sin^{-1} = [-\pi/2, \pi/2]$.
($\Longleftarrow$) If $x \in [-\pi/2, \pi/2]$, then by Fact 99, $\sin^{-1}(\sin x) = x$.
The proofs of **(b)** and **(c)** are similar and thus omitted. □

Fact 101. **(Addition Formulae for Arcsine and Arccosine)** *Let* $x \in [-1,1]$.
(a) $\cos^{-1}x + \sin^{-1}x = \pi/2$.
(b) $\sin^{-1}x + \sin^{-1}(-x) = 0$.
(c) $\cos^{-1}x + \cos^{-1}(-x) = \pi$.

Proof. **(a)** Let $x \in [-1,1]$ and $A \stackrel{1}{=} \sin^{-1}x \in [-\pi/2, \pi/2]$, so that $x = \sin A = \cos(-A + \pi/2)$.
Since $-A + \pi/2 \in [0,\pi]$, by Fact 99(b), $\cos^{-1}x = \cos^{-1}(\cos(-A+\pi/2)) \stackrel{2}{=} -A + \pi/2$.
Taking $\stackrel{1}{=} + \stackrel{2}{=}$ yields the result.
(b) By the Addition Formula for Sine, Fact 98(a), and Fact 100(b),

$$\begin{aligned}\sin\left(\sin^{-1}x + \sin^{-1}(-x)\right) &= \sin\left(\sin^{-1}x\right)\cos\left(\sin^{-1}(-x)\right) + \cos\left(\sin^{-1}x\right)\sin\left(\sin^{-1}(-x)\right) \\ &= x\sqrt{1-(-x)^2} + \sqrt{1-x^2}(-x) = 0.\end{aligned}$$

Since Range $\sin^{-1} = [-\pi/2, \pi/2]$, we have $\sin^{-1}x + \sin^{-1}(-x) \in [-\pi, \pi]$. And so, $\sin^{-1}x + \sin^{-1}(-x)$ must equal $-\pi$, 0, or π.
But $\sin^{-1}x + \sin^{-1}(-x) = \pm\pi$ is not possible for any x. So, $\sin^{-1}x + \sin^{-1}(-x) = 0$.
(c) Let $x \in [-1,1]$ and $A \stackrel{1}{=} \cos^{-1}x \in [0,\pi]$, so that $x = \cos A = -\cos(\pi - A)$.
Since $\pi - A \in [0,\pi]$, by Fact 99(b), $\cos^{-1}(-x) = \cos^{-1}(\cos(\pi - A)) \stackrel{2}{=} \pi - A$.
Taking $\stackrel{1}{=} + \stackrel{2}{=}$ yields the result. □

Fact 260. *The composite functions* $\sin^{-1} \circ \sin$, $\cos^{-1} \circ \cos$, *and* $\tan^{-1} \circ \tan$ *exist and their domain, codomain, and mapping rules are as follows:*

		Domain	**Codomain**	**Mapp**
(a)	$\sin^{-1} \circ \sin$	$\mathbb{R}$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\sin^{-1}\ (\sin x) = \begin{cases} x - 2k\pi, \\ -x + (2k+1)\pi, \end{cases}$
(b)	$\cos^{-1} \circ \cos$	$\mathbb{R}$	$[0, \pi]$	$\cos^{-1}\ (\cos x) = \begin{cases} -x + 2k\pi, \\ x - 2k\pi \\ -x + (2k+2)\pi, \end{cases}$
(c)	$\tan^{-1} \circ \tan$	$\mathbb{R} \setminus \{(2a+1)\pi/2 : a \in \mathbb{Z}\}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\tan^{-1}\ (\tan x) = x - k\pi$, for for $x \in ($

Proof. **(a)** Since Range sin $= [-1, 1] \subseteq [-1, 1] =$ Domain $\sin^{-1}$, $\sin^{-1} \circ \sin$ exists and has domain Domain sin $= \mathbb{R}$ and codomain Codomain $\sin^{-1} = [-\pi/2, \pi/2]$.

For each $x \in \mathbb{R}$, define $t \in [-\pi/2, 3\pi/2]$ and $k \in \mathbb{Z}$ by $x = t + 2k\pi$.[583]

Then $\sin^{-1}(\sin x) = \sin^{-1}(\sin(t + 2k\pi)) = \sin^{-1}(\sin t)$.

If $t \in [-\pi/2, \pi/2]$, then by Fact 99,

$$\sin^{-1}(\sin t) = t = x - 2k\pi.$$

If instead $t \in [\pi/2, 3\pi/2]$, then $\pi - t \in [-\pi/2, \pi/2]$. Hence, by the Difference Formula for Sine and Fact 99,

$$\sin^{-1}(\sin t) = \sin^{-1}(\sin(\pi - t)) = \pi - t = -x + (2k+1)\pi.$$

(b) Since Range cos $= [-1, 1] \subseteq [-1, 1] =$ Domain $\cos^{-1}$, $\cos^{-1} \circ \cos$ exists and has domain Domain cos $= \mathbb{R}$ and codomain Codomain $\cos^{-1} = [0, \pi]$.

Again, for each $x \in \mathbb{R}$, define $t \in [-\pi/2, 3\pi/2]$ and $k \in \mathbb{Z}$ by $x = t + 2k\pi$.

Then $\cos^{-1}(\cos x) = \cos^{-1}(\cos(t + 2k\pi)) = \cos^{-1}(\cos t)$.

If $t \in [0, \pi]$, then by Fact 99,

$$\cos^{-1}(\cos t) = t = x - 2k\pi.$$

If instead $t \in [-\pi/2, 0]$, then $t + \pi \in [0, \pi]$. Hence, by the Difference Formula for Cosine, Fact 101(d), and Fact 99,

$$\cos^{-1}(\cos t) = \cos^{-1}(-\cos(t + \pi)) = \pi - \cos^{-1}(\cos(t + \pi)) = \pi - (t + \pi) = -t = -x + 2k\pi.$$

And if instead $t \in [\pi, 3\pi/2]$, then $t - \pi \in [0, \pi]$. Hence, by the Difference Formula for Cosine, Fact 101(d), and Fact 99,

[583]The existence of such t and k are given by the Euclidean Division Algorithm (Theorem 50).

$\cos^{-1}(\cos t) = \cos^{-1}(-\cos(t-\pi)) = \pi - \cos^{-1}(\cos(t-\pi)) = \pi - (t-\pi) = 2\pi - t = -x + (2k+2)\pi$.

(c) Since Range $\tan = \mathbb{R} \subseteq \mathbb{R} =$ Domain $\tan^{-1}$, $\tan^{-1} \circ \tan$ exists and has domain Domain $\tan = \mathbb{R} \setminus \{(2a+1)\pi/2 : a \in \mathbb{Z}\}$ and codomain Codomain $\tan^{-1} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For each $x \in \mathbb{R}$, define $s \in (-\pi/2, \pi/2)$ and $k \in \mathbb{Z}$ by $x = s + k\pi$.

Now, $\tan^{-1}(\tan x) = \tan^{-1}(\tan(s+k\pi)) = \tan^{-1}(\tan s) = s = x - k\pi$. □

Fact 100. **(a)** $\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}$ *for all* $x \in [-1,1]$.

(b) $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$ *for all* $x \in [-1,1]$.

(c) $\sin\left(\tan^{-1} x\right) = \dfrac{x}{\sqrt{1+x^2}}$ *for all* $x \in \mathbb{R}$.

(d) $\cos\left(\tan^{-1} x\right) = \dfrac{1}{\sqrt{1+x^2}}$ *for all* $x \in \mathbb{R}$.

(e) $\tan\left(\sin^{-1} x\right) = \dfrac{x}{\sqrt{1-x^2}}$ *for all* $x \in (-1,1)$.

(f) $\tan\left(\cos^{-1} x\right) = \dfrac{\sqrt{1-x^2}}{x}$ *for all* $x \in [-1,1] \setminus \{0\}$.

Proof. **(a)** Let $y = \cos^{-1} x$. Then $x = \cos y$ and $1 - x^2 = 1 - \cos^2 y = \sin^2 y$.

Hence, $\sin\left(\cos^{-1} x\right) = \sin y = \pm\sqrt{1-x^2}$.

For $x \in [-1,1]$, we have $\cos^{-1} x \in [0,\pi]$ and hence, $\sin\left(\cos^{-1} x\right) \geq 0$. So,

For all $x \in [-1,1]$, $$\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}.$$

(b) Let $y = \sin^{-1} x$. Then $x = \sin y$ and $1 - x^2 = 1 - \sin^2 y = \cos^2 y$.

Hence, $\cos\left(\sin^{-1} x\right) = \cos y = \pm\sqrt{1-x^2}$.

For $x \in [-1,1]$, we have $\sin^{-1} x \in [-\pi/2, \pi/2]$ and hence, $\cos\left(\sin^{-1} x\right) \geq 0$. So,

For all $x \in [-1,1]$, $$\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}.$$

(c) Let $y = \tan^{-1} x$. Then $x = \tan y$ and $1 + x^2 = 1 + \tan^2 y = \sec^2 y$. So,

$$\sec^2 y = \frac{1}{\cos^2 y} = \frac{1}{1-\sin^2 y} = \frac{1}{1-\sin^2 y} = \frac{1}{1-\left[\sin\left(\tan^{-1} x\right)\right]^2}.$$

And, $$1 + x^2 = \frac{1}{1-\left[\sin\left(\tan^{-1} x\right)\right]^2}.$$

Rearranging, $$\left[\sin\left(\tan^{-1} x\right)\right]^2 = 1 - \frac{1}{1+x^2} = \frac{1+x^2-1}{1+x^2} = \frac{x^2}{1+x^2}.$$

Hence, $$\sin\left(\tan^{-1}x\right)=\pm\sqrt{\frac{x^2}{1+x^2}}=\pm\frac{|x|}{\sqrt{1+x^2}}.$$

For $x\geq 0$, we have $\tan^{-1}x\in[0,\pi/2)$ and hence, $\sin\left(\tan^{-1}x\right)\geq 0$. So,

For all $x\geq 0$, $$\sin\left(\tan^{-1}x\right)=\frac{|x|}{\sqrt{1+x^2}}=\frac{x}{\sqrt{1+x^2}}.$$

For $x<0$, we have $\tan^{-1}x\in(-\pi/2,0)$ and hence, $\sin\left(\tan^{-1}x\right)<0$. So,

For all $x<0$, $$\sin\left(\tan^{-1}x\right)=-\frac{|x|}{\sqrt{1+x^2}}=-\frac{-x}{\sqrt{1+x^2}}=\frac{x}{\sqrt{1+x^2}}.$$

Altogether, for all $x\in\mathbb{R}$, $$\sin\left(\tan^{-1}x\right)=\frac{x}{\sqrt{1+x^2}}.$$

(d) Let $y=\tan^{-1}x$. Then $x=\tan y$ and $1+x^2=1+\tan^2 y=\sec^2 y$. So,

$$\sec^2 y=\frac{1}{\cos^2 y}=\frac{1}{\left[\cos\left(\tan^{-1}x\right)\right]^2}.$$

And, $$1+x^2=\frac{1}{\left[\cos\left(\tan^{-1}x\right)\right]^2}.$$

Rearranging, $$\cos\left(\tan^{-1}x\right)=\pm\frac{1}{\sqrt{1+x^2}}.$$

For $x\in\mathbb{R}$, we have $\tan^{-1}x\in(-\pi/2,\pi/2)$ and hence, $\cos\left(\tan^{-1}x\right)\geq 0$. So,

For all $x\in\mathbb{R}$, $$\cos\left(\tan^{-1}x\right)=\frac{1}{\sqrt{1+x^2}}.$$

(e) By **(b)** (and also Fact 98(a)), $\tan\left(\sin^{-1}x\right)=\dfrac{\sin\left(\sin^{-1}x\right)}{\cos\left(\sin^{-1}x\right)}=\dfrac{x}{\sqrt{1-x^2}}$ for all $x\in(-1,1)$.

(f) By **(a)** (and also Fact 98(a)), $\tan\left(\cos^{-1}x\right)=\dfrac{\sin\left(\cos^{-1}x\right)}{\cos\left(\cos^{-1}x\right)}=\dfrac{\sqrt{1-x^2}}{x}$ for all $x\in[-1,1]\setminus\{0\}$. □

Fact 261. *The composite functions* $\sin^{-1} \circ \cos$ *and* $\cos^{-1} \circ \sin$ *exist and their domain, codomain, and mapping rules are as follows:*

		Domain	**Codomain**	**Mapping rule**
(a)	$\sin^{-1} \circ \cos$	$\mathbb{R}$	$\mathbb{R}$	$\sin^{-1}(\cos x) = \begin{cases} x + (0.5 - 2k)\pi, & \text{for } x \in [2k\pi - \\ -x + (2k+0.5)\pi, & \text{for } x \in [2k\pi, 2 \end{cases}$
(b)	$\cos^{-1} \circ \sin$	$\mathbb{R}$	$\mathbb{R}$	$\cos^{-1}(\sin x) = \begin{cases} -x + (2k+0.5)\pi, & \text{for } x \in [2k\pi, 2 \\ x - (2k+0.5)\pi & \text{for } x \in [2k\pi + \\ -x + (2k+2.5)\pi, & \text{for } x \in [2k\pi + \end{cases}$

Proof. **(a)** Since Range cos $= [-1,1] \subseteq [-1,1] =$ Domain $\sin^{-1}$, $\sin^{-1} \circ \cos$ exists and has domain Domain cos $= \mathbb{R}$ and codomain Codomain $\sin^{-1} = [-\pi/2, \pi/2]$.

Let $x \in \mathbb{R}$. We have $\sin\left(x + \dfrac{\pi}{2}\right) = \cos x$.

Hence, $\sin^{-1}(\cos x) = \sin^{-1}\left(\sin\left(x + \dfrac{\pi}{2}\right)\right)$. And by Fact 260(a),

$$\sin^{-1}\left(\sin\left(x + \frac{\pi}{2}\right)\right) = \begin{cases} x + \pi/2 - 2k\pi, & \text{for } x + \pi/2 \in [2k\pi - \pi/2, 2k\pi + \pi/2], \\ -x - \pi/2 + (2k+1)\pi, & \text{for } x + \pi/2 \in [2k\pi + \pi/2, 2k\pi + 3\pi/2]. \end{cases}$$

The result follows.

(b) Since Range sin $= [-1,1] \subseteq [-1,1] =$ Domain $\cos^{-1}$, $\cos^{-1} \circ \sin$ exists and has domain Domain sin $= \mathbb{R}$ and codomain Codomain $\cos^{-1} = [0, \pi]$.

Let $x \in \mathbb{R}$. We have $\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$.

Hence, $\cos^{-1}(\sin x) = \cos^{-1}\left(\cos\left(x - \dfrac{\pi}{2}\right)\right)$. And by Fact 260(b),

$$\cos^{-1}\left(\cos\left(x - \frac{\pi}{2}\right)\right) = \begin{cases} -x + \pi/2 + 2k\pi, & \text{for } x - \dfrac{\pi}{2} \in [2k\pi - \pi/2, 2k\pi], \\ x - \pi/2 - 2k\pi & \text{for } x - \dfrac{\pi}{2} \in [2k\pi, 2k\pi + \pi], \\ -x + \pi/2 + (2k+2)\pi, & \text{for } x - \dfrac{\pi}{2} \in [2k\pi + \pi, 2k\pi + 3\pi/2]. \end{cases}$$

The result follows. ☐

Fact 102. (Addition Formulae for Arctangent) *Let $x \in \mathbb{R}$.*

(a) $\tan^{-1} x + \tan^{-1}(-x) = 0$.

(b) $\tan^{-1} x + \tan^{-1}\dfrac{1}{x} = \begin{cases} \pi/2, & \text{for } x > 0, \\ -\pi/2, & \text{for } x < 0. \end{cases}$

(c) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\dfrac{x+y}{1-xy}, & \text{for } xy < 1, \\ \tan^{-1}\dfrac{x+y}{1-xy} + \pi, & \text{for } xy > 1 \text{ AND } x > 0, \\ \tan^{-1}\dfrac{x+y}{1-xy} - \pi, & \text{for } xy > 1 \text{ AND } x < 0. \end{cases}$

Proof. **(a)** Since Range $\tan^{-1} = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, we have $\tan^{-1} x + \tan^{-1}(-x) \overset{1}{\in} (-\pi, \pi)$.

By the Addition Formula for Sine and Fact 100(c) and (d),

$$\begin{aligned} \sin\left(\tan^{-1} x + \tan^{-1}(-x)\right) &= \sin\left(\tan^{-1} x\right)\cos\left(\tan^{-1}(-x)\right) + \cos\left(\tan^{-1} x\right)\sin\left(\tan^{-1}(-x)\right) \\ &= \frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+(-x)^2}} + \frac{1}{\sqrt{1+x^2}}\frac{-x}{\sqrt{1+(-x)^2}} = 0. \end{aligned}$$

Given $\overset{1}{\in}$, we must have $\tan^{-1} x + \tan^{-1}(-x) = 0$.

(b) Since Range $\tan^{-1} = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, we have $\tan^{-1} x + \tan^{-1}\dfrac{1}{x} \overset{2}{\in} (-\pi, \pi)$.

By the Addition Formula for Sine and Fact 100(c) and (d),

$$\begin{aligned} \sin\left(\tan^{-1} x + \tan^{-1}\frac{1}{x}\right) &= \sin\left(\tan^{-1} x\right)\cos\left(\tan^{-1}\frac{1}{x}\right) + \cos\left(\tan^{-1} x\right)\sin\left(\tan^{-1}\frac{1}{x}\right) \\ &= \frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+(1/x)^2}} + \frac{1}{\sqrt{1+x^2}}\frac{1/x}{\sqrt{1+(1/x)^2}} \\ &= \frac{1}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+(1/x)^2}}\left(x + \frac{1}{x}\right) \\ &= \frac{1}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+(1/x)^2}}\frac{x^2+1}{x} \times \frac{|x|}{|x|} \\ &= \frac{1}{\sqrt{1+x^2}}\frac{|x|}{\sqrt{x^2+1}}\frac{x^2+1}{x} \\ &= \frac{|x|}{|x^2+1|}\frac{x^2+1}{x} = \frac{|x|}{x^2+1}\frac{x^2+1}{x} = \frac{|x|}{x}. \end{aligned}$$

So, if $x > 0$, then $\sin\left(\tan^{-1} x + \tan^{-1}\dfrac{1}{x}\right) = 1$ and hence, by $\overset{2}{\in}$, $\tan^{-1} x + \tan^{-1}\dfrac{1}{x} = \pi/2$.

While if $x>0$, then $\sin\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)=-1$ and hence, by $\overset{2}{\in}$, $\tan^{-1}x+\tan^{-1}\frac{1}{x}=-\pi/2$.

(c) We continue with the proof in the main text.

Case 2. $xy>1$ AND $x>0$.

Then by Lemma 1(c), $\tan^{-1}x+\tan^{-1}y\in(\pi/2,\pi)$.

So, by Fact 260(c), $\tan^{-1}\left(\tan\left(\tan^{-1}x+\tan^{-1}y\right)\right)=\tan^{-1}x+\tan^{-1}y-\pi$. Thus,

$$\tan^{-1}\frac{x+y}{1-xy}=\tan^{-1}x+\tan^{-1}y+\pi.$$

Case 3. $xy>1$ AND $x<0$.

Then by Lemma 1(d), $\tan^{-1}x+\tan^{-1}y\in(-\pi,-\pi/2)$.

So, by Fact 260(c), $\tan^{-1}\left(\tan\left(\tan^{-1}x+\tan^{-1}y\right)\right)=\tan^{-1}x+\tan^{-1}y+\pi$. Thus,

$$\tan^{-1}\frac{x+y}{1-xy}=\tan^{-1}x+\tan^{-1}y-\pi.$$

□

Lemma 1. *Suppose $x,y\in\mathbb{R}$. Then*

(a)	$\tan^{-1}x+\tan^{-1}y=\pm\pi/2$	$\Longleftrightarrow$	$xy=1$;
(b)	$\in(-\pi/2,\pi/2)$	$\Longleftrightarrow$	$xy<1$;
(c)	$\in(\pi/2,\pi)$	$\Longleftrightarrow$	$xy>1$ AND $x>0$;
(d)	$\in(-\pi,-\pi/2)$	$\Longleftrightarrow$	$xy>1$ AND $x<0$.

Proof. **(a)** ($\Longleftarrow$) If $xy=1$, then $y=1/x$ and by Fact 102(b),

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}x+\tan^{-1}\frac{1}{x}=\begin{cases}\pi/2, & \text{for } x>0,\\ -\pi/2, & \text{for } x<0.\end{cases}$$

(a) ($\Longrightarrow$) Suppose $\tan^{-1}x+\tan^{-1}y\overset{1}{=}\pm\pi/2$. It cannot be that $x=0$ or $y=0$ (because then $\tan^{-1}y=\pm\pi/2$ or $\tan^{-1}x=\pm\pi/2$).

Suppose $x>0$. If $y\neq 1/x$, then (because $\tan^{-1}$ is strictly increasing), $\tan^{-1}x+\tan^{-1}y\neq\tan^{-1}x+\tan^{-1}(1/x)=\pi/2$, a contradiction. So, $y=1/x$ or $xy=1$.

Similarly, if $x<0$, then again $xy=1$.

(b) If $x=0$ or $y=0$, then $\tan^{-1}x+\tan^{-1}y=\tan^{-1}y\in(-\pi/2,\pi/2)$ or $\tan^{-1}x+\tan^{-1}y=\tan^{-1}x\in(-\pi/2,\pi/2)$ and $xy=0<1$. So, suppose $x\neq 0$, $y\neq 0$.

($\Longleftarrow$) Suppose $xy<1$.

Case 1. Suppose $x>0$. Since $y<1/x$ and $\tan^{-1}$ is strictly increasing,

$$\tan^{-1}x+\tan^{-1}y<\tan^{-1}x+\tan^{-1}\frac{1}{x}=\pi/2.$$

Since $\tan^{-1}x>0$ and $\tan^{-1}y>-\pi/2$,

$$\tan^{-1} x + \tan^{-1} y > -\pi/2.$$

Case 2. Suppose $x < 0$. Since $y > 1/x$ and $\tan^{-1}$ is strictly increasing,

$$\tan^{-1} x + \tan^{-1} y > \tan^{-1} x + \tan^{-1} \frac{1}{x} = -\pi/2.$$

Since $\tan^{-1} x < 0$ and $\tan^{-1} y < \pi/2$,

$$\tan^{-1} x + \tan^{-1} y < \pi/2.$$

($\Longrightarrow$) Suppose $-\pi/2 < \tan^{-1} x + \tan^{-1} y < \pi/2$.

Case 1. Suppose $x > 0$.

Then $\tan^{-1} x + \tan^{-1} y < \pi/2 = \tan^{-1} x + \tan^{-1}(1/x)$.

Since $\tan^{-1}$ is strictly increasing, $y < 1/x$ or equivalently, $xy < 1$.

Case 2. Suppose $x < 0$.

Then $\tan^{-1} x + \tan^{-1} y > -\pi/2 = \tan^{-1} x + \tan^{-1}(1/x)$.

Since $\tan^{-1}$ is strictly increasing, $y > 1/x$ or equivalently, $xy < 1$.

(c) & **(d)** Since Range $\tan^{-1} = (-\pi/2, \pi/2)$, $\tan^{-1} x + \tan^{-1} y \in (-\pi, \pi)$. And so, by **(b)**,

$$\tan^{-1} x + \tan^{-1} y \in (-\pi, -\pi/2) \cup (\pi/2, \pi) \qquad \Longleftrightarrow \qquad xy > 1.$$

$$\begin{array}{ll} \text{Now,} & \tan^{-1} x + \tan^{-1} y \in (\pi/2, \pi) \\ \Longleftrightarrow & \tan^{-1} x > 0 \text{ OR } \tan^{-1} y > 0 \\ \Longleftrightarrow & x > 0 \text{ OR } y > 0. \end{array}$$

Given that $xy > 1 > 0$, this last condition is equivalent to $x > 0$ AND $y > 0$.

$$\begin{array}{ll} \text{Similarly,} & \tan^{-1} x + \tan^{-1} y \in (-\pi, -\pi/2) \\ \Longleftrightarrow & \tan^{-1} x < 0 \text{ OR } \tan^{-1} y < 0 \\ \Longleftrightarrow & x < 0 \text{ OR } y < 0. \end{array}$$

Given that $xy > 1 > 0$, this last condition is equivalent to $x < 0$ AND $y < 0$. □

Definition 295. *Quadrants I, II, III,* and *IV* are the subsets of the cartesian plane defined by

$$\begin{aligned} \text{Quadrant I} &= \{(x,y) : x \in \mathbb{R}^+, y \in \mathbb{R}^+\}, \\ \text{Quadrant II} &= \{(x,y) : x \in \mathbb{R}^-, y \in \mathbb{R}^+\}, \\ \text{Quadrant III} &= \{(x,y) : x \in \mathbb{R}^-, y \in \mathbb{R}^-\}, \\ \text{Quadrant IV} &= \{(x,y) : x \in \mathbb{R}^+, y \in \mathbb{R}^-\}. \end{aligned}$$

We say that an angle A is *in Quadrant I, II, III,* or *IV* if $A \in \left(2k\pi, \left(2k + \frac{1}{2}\right)\pi\right)$, $A \in \left(\left(2k + \frac{1}{2}\right)\pi, (2k+1)\pi\right)$, $A \in \left((2k+1)\pi, \left(2k + \frac{3}{2}\right)\pi\right)$, or $A \in \left(\left(2k + \frac{3}{2}\right)\pi, (2k+2)\pi\right)$ for some integer k.

Fact 262. *For all* $x \in \mathbb{R}$, $\sin\left(\cos^{-1} x\right) = \sqrt{1 - x^2}$

Proof. Let $\theta = \cos^{-1} x \in [0, \pi]$. Then $\cos\theta = x$ and $\sin\theta \geq 0$.

Now, from the identity $\sin^2\theta + \cos^2\theta = 1$, we have $\sin\theta = \pm\sqrt{1 - \cos^2\theta} = \pm\sqrt{1 - x}$. Since $\sin\theta \geq 0$, we can discard the negative value. Thus, $\sin\left(\cos^{-1} x\right) = \sin\theta = \sqrt{1 - x}$, as desired. □

Fact 263. *Suppose two polynomials have degrees a and b. Then their*

(a) *Sum is a polynomial of degree* $\max\{a,b\}$*; and*

(b) *Product is a polynomial of degree ab.*

Proof. **(a)** Omitted.[584]

(b) Omitted.[585] □

Theorem 8. (Euclidean Division Theorem for Polynomials.) *Let $p(x)$ and $d(x)$ be P- and D-degree polynomials in x with $D \leq P$. Then there exists a unique polynomial $q(x)$ of degree $P-D$ such that $r(x) = p(x) - d(x)q(x)$ is a polynomial with degree less than D.*

Proof. Let $p(x) = \sum_{i=0}^{P} p_i x^i$ and $d(x) = \sum_{i=0}^{D} d_i x^i$. Let $q(x) = \sum_{i=0}^{P-D} q_i x^i$ be a $(P-D)$-degree polynomial. We have

$$d(x)q(x) = q_{P-D}d_D x^P + (q_{P-D}d_{D-1} + q_{P-D-1}d_D)x^{P-1} + (q_{P-D}d_{D-2} + q_{P-D-1}d_{D-1} + q_{P-D-2}d_D)x^{P}$$
$$\cdots + (q_{P-D}d_0 + q_{P-D-1}d_1 + \cdots + q_1 d_{D-1} + q_0 d_D)x^D + r(x),$$

where $r(x)$ is a polynomial of degree less than D.

Now for $i \in \{0,1,2,\ldots,P-D\}$, pick q_i so that

$$\begin{aligned}
p_P &= q_{P-D}d_D,\\
p_{P-1} &= q_{P-D}d_{D-1} + q_{P-D-1}d_D,\\
p_{P-2} &= q_{P-D}d_{D-2} + q_{P-D-1}d_{D-1} + q_{P-D-2}d_D,\\
&\vdots\\
p_D &= q_{P-D}d_0 + q_{P-D-1}d_1 + \cdots + q_1 d_{D-1} + q_0 d_D.
\end{aligned}$$

Then $p(x) - d(x)q(x) = r(x)$. This completes the proof of existence.

To prove uniqueness, suppose $s(x)$ is a polynomial. Then by Fact 263, the polynomial

$$p(x) - d(x)s(x) = \overbrace{p(x) - d(x)q(x)}^{\text{Degree less than } D} + \overbrace{d(x)}^{\text{Degree } D}[q(x) - s(x)]$$

is of degree less than D if and only if $q(x) = s(x)$.Hence, the polynomial q is unique. □

[584] See e.g. ProofWiki.
[585] See e.g. ProofWiki.

We need Euclid's Lemma to prove the Rational Root Theorem:

Lemma 4. (Euclid's Lemma) *Let $a, b, d \in \mathbb{Z}$. Suppose a and d share no common factors greater than one. If d divides ab, then d also divides b.*

Proof. Omitted. (See e.g. XXX) □

Theorem 11. (Rational Root Theorem) *Let $p(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ with $p_0, p_1, \ldots, p_n \in \mathbb{Z}$ and also $N, D \in \mathbb{Z}$. Suppose N/D is fully reduced.*

If $p\left(\frac{N}{D}\right) = 0$, then $\frac{p_0}{N}, \frac{p_n}{D} \in \mathbb{Z}$.

Proof. Suppose $x - \frac{N}{D}$ is a factor of $p(x)$. (The proof of the case where instead $x + \frac{N}{D}$ factorises $p(x)$ is similar and omitted.)

Then by the Factor Theorem (Theorem 10), $p\left(\frac{N}{D}\right) = 0$ or

$$p_n\left(\frac{N}{D}\right)^n + p_{n-1}\left(\frac{N}{D}\right)^{n-1} + \cdots + p_1\left(\frac{N}{D}\right) + p_0 \overset{1}{=} 0.$$

Rearrange $\overset{1}{=}$ as

$$p_{n-1}N^{n-1} + \cdots + p_1 N D^{n-2} + p_0 D^{n-1} = -\frac{p_n N^n}{D}.$$

Since LHS is an integer, D divides $p_n N^n$. But since D doesn't divide N, by repeatedly applying Lemma 4, we find that D divides p_n.

Next, similarly rearrange $\overset{1}{=}$ as

$$p_n N^{n-1} + p_{n-1} D N^{n-2} + \cdots + p_1 D^{n-1} = -\frac{p_0 D^n}{N}.$$

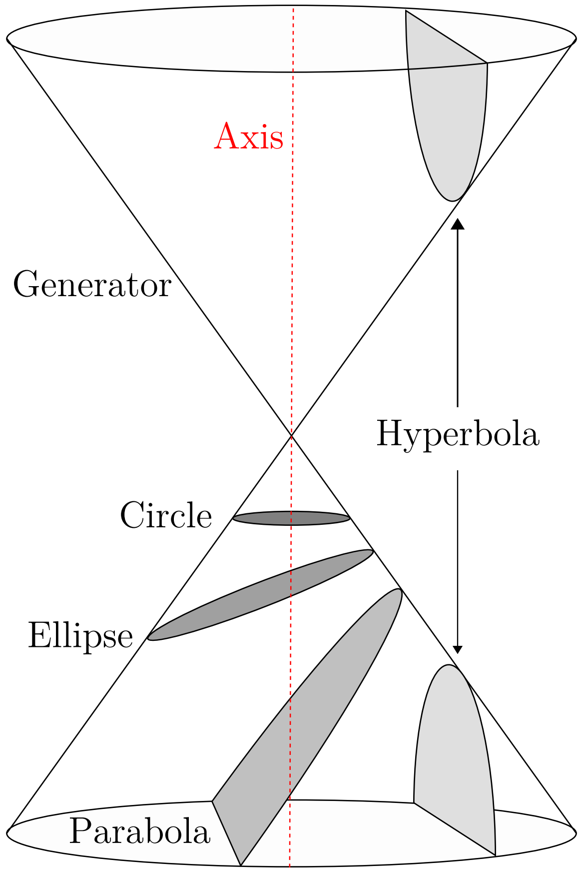
Since LHS is an integer, N divides $p_0 D^n$. But since N doesn't divide D, by repeatedly applying Lemma 4, we find that N divides p_0. □

142.20. Conic Sections

For a very brief introduction to conic sections, either watch this video[586] or read the following. (Or both.)

Take a vertical line and an oblique line. Rotate the oblique line about the vertical line to form an **infinite double cone**. We call the vertical line the **axis** and the oblique line the **generator**. The midpoint of the cone (or the point where the axis and generator meet) is called the **vertex**.

Now take a two-dimensional cartesian plane and slice the double cone from all conceivable positions and at all conceivable angles. The intersection of the plane and the outer surface of the double cone then form curves which we call **conic sections**.



We have three types of conic sections:

1. An **ellipse** if the plane is less steep than the generator. A special case is the **circle** which is obtained if the plane is perpendicular to the axis.
2. A **parabola** if the plane is exactly as steep as the generator.
3. A **hyperbola** if the plane is steeper than the generator.

[586]Not made by me and narrated by an female Indian robot, but awesome nonetheless.

The ellipse and the parabola are formed from only one half of the double cone. In contrast, the hyperbola is formed from both halves—it thus has two branches.[587]

We shall not do so, but it is possible to prove that in general, a conic section is the graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

We call $B^2 - 4AC$ the **discriminant**, because it is possible to show that

1. If $B^2 - 4AC < 0$, then we have an **ellipse**.
2. If $B^2 - 4AC = 0$, then we have a **parabola**. (The **quadratic equation** is an example of a parabola.)
3. If $B^2 - 4AC > 0$, then we have a **hyperbola**.

In general, for any $k \in \mathbb{R}$, $y = k/x$ is symmetric in the lines $y = x$ and $y = -x$:

Lemma 5. *For all $k \in \mathbb{R}$, $y = k/x$ is symmetric in the lines $y = x$ and $y = -x$.*

Proof. Recall (Fact 37) the reflection of (p, q) in $y = x$ is (q, p). But

$$p = \frac{k}{q} \qquad \Longleftrightarrow \qquad q = \frac{k}{p}.$$

So, $y = x$ is a line of symmetry for $y = k/x$.

Similarly, recall (Fact 38) the reflection of (p, q) in $y = -x$ is $(-q, -p)$. But

$$p = \frac{k}{q} \qquad \Longleftrightarrow \qquad -q = \frac{k}{-p}.$$

So, $y = -x$ is a line of symmetry for $y = k/x$. □

[587] There are also four types of **degenerate conic sections**. In each case, the plane cuts through (or contains) the vertex. We have

1. A **point** if the plane is less steep than the generator (this is the **degenerate ellipse**).
2. A **single straight line** if the plane is exactly as steep as the generator (this is the **degenerate parabola**).
3. A **pair of intersecting lines** if the plane is steeper than the generator (this is the **degenerate hyperbola**).

Now, suppose the generator, which is usually oblique, is now instead parallel to the axis. Then we get a **degenerate cone** that is the cylinder. Now, any plane that is perpendicular to the cylinder's base produces:

4. A **pair of vertical lines**. (This is considered a degenerate parabola, because the plane is exactly as steep as the generator).

Fact 108. *Let $b, c, d, e \in \mathbb{R}$ with $d \neq 0$ and $cd - be \neq 0$. Consider the graph of*

$$y = \frac{bx + c}{dx + e}.$$

(a) **Intercepts**. If $e \neq 0$, then there is one y-intercept $(0, c/e)$. (If $e = 0$, then there are no y-intercepts.) And if $b \neq 0$, then there is one x-intercept $(-c/b, 0)$. (If $b = 0$, then there are no x-intercepts.)

(b) There are no **turning points**.

(c) There is the **horizontal asymptote** $y = b/d$ and the **vertical asymptote** $x = -e/d$. (The asymptotes are perpendicular and so, this is a **rectangular hyperbola**.)

(d) The hyperbola's **centre** is $(-e/d, b/d)$.

(e) The **two lines of symmetry** are $y = \pm x + (b + e)/d$.

Proof. We already proved **(a)**, **(c)**, and **(d)** in the main text. We now prove **(b)** and **(e)**.

(b) $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{b}{d} + \frac{cd - be}{d^2}\frac{1}{x + e/d}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{b}{d} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{cd - be}{d^2}\frac{1}{x + e/d}\right) = \frac{cd - be}{d^2}\frac{-1}{(x + e/d)^2}.$$

By assumption, $cd - be = 0$. So, $\mathrm{d}y/\mathrm{d}x \neq 0$. Hence, by Definitions 278 and 279, this graph has no turning points.

(e) By Lemma 5, the following graph is symmetric in $y = x$ and $y = -x$:

$$y = \frac{cd - be}{d^2}\frac{1}{x}.$$

Now shift this graph leftwards by e/d units to get the graph of:

$$y = \frac{cd - be}{d^2}\frac{1}{x + e/d},$$

which, by Fact 58, has lines of symmetry $y = x + e/d$ and $y = -(x + e/d)$.

Now shift this last graph upwards by b/d units to get the graph of:

$$y = \frac{b}{d} + \frac{(cd - be)/d^2}{x + e/d} = \frac{bx + c}{dx + e},$$

which, by Fact 58, has the claimed lines of symmetry:

$$y = x + \frac{b + e}{d} \qquad \text{and} \qquad y = -x + \frac{b - e}{d}.$$

It remains to be shown that these are *the only* two lines of symmetry. If there were a third distinct line of symmetry, then there would be more than two asymptotes. But this is not the case. Thus, there can be at most two distinct lines of symmetry. ☐

Fact 264. *Let $a, d, ce \neq 0$. Consider the graph of*

$$y = \frac{ax^2 + bx + c}{dx + e}.$$

(a) ***Intercepts***. *If $e \neq 0$, then there is one y-intercept $(0, c/e)$. (If $e = 0$, then there are no y-intercepts.)*

If $b^2 - 4ac > 0$, then the two x-intercepts are

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0\right).$$

If $b^2 - 4ac = 0$, then there is only one x-intercept $(-b/2a, 0)$.

And if $b^2 - 4ac < 0$, then there are no x-intercepts.

(b) *The two* ***turning points*** *are*

$$\left(\frac{-e \pm \sqrt{\left(ae^2 + cd^2 - bde\right)/a}}{d}, \frac{bd - 2ae \pm 2\sqrt{a\left(ae^2 + cd^2 - bde\right)}}{d^2}\right).$$

If $ad > 0$, then the turning point on the left is a strict local maximum and the one on the right is a strict local minimum. And if $ad < 0$, then the one on the left is a strict local minimum and the one on the right is a strict local maximum.

(c) *There are* ***two asymptotes****, one oblique and one vertical:*

$$y = \frac{a}{d}x + \frac{bd - ae}{d^2} \quad \text{and} \quad x = -\frac{e}{d}.$$

(Note that since the asymptotes are ***not*** *perpendicular, this is* ***not*** *a rectangular hyperbola.)*

(d) *The hyperbola's* ***centre*** *is $\left(-\frac{e}{d}, \frac{bd - 2ae}{d^2}\right)$.*

(e) *The* ***two lines of symmetry*** *are*

$$y = \frac{a \pm \sqrt{a^2 + d^2}}{d}x + \frac{bd - ae \pm e\sqrt{a^2 + d^2}}{d^2}.$$

Proof on the next page:

Proof. We already proved **(a)**, **(c)**, and **(d)** above. Here we prove only **(b)** and **(e)**.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{ax^2+bx+c}{dx+e}\right) = \dfrac{(dx+e)(2ax+b)-(ax^2+bx+c)d}{(dx+e)^2} = \dfrac{adx^2+2aex+be-cd}{(dx+e)^2}$.

Hence, $\mathrm{d}y/\mathrm{d}x = 0 \iff adx^2+2aex+be-cd = 0 \iff$

$$x = \frac{-2ae \pm \sqrt{4a^2e^2-4ad(be-cd)}}{2ad} = \frac{-e \pm \sqrt{(ae^2+cd^2-bde)/a}}{d}. \qquad ☺$$

So, if $(ae^2+cd^2-bde)/a < 0$, then there are no stationary points.

If $(ae^2+cd^2-bde)/a = 0$, then $\mathrm{d}y/\mathrm{d}x = 0$ at $x = -e/d$. But there is no point in $y = (ax^2+bx+c)/(dx+e)$ at which $x = -e/d$. And so here, there is no stationary point.

If $(ae^2+cd^2-bde)/a > 0$, then there are two stationary points, given by ☺. Plugging these values of x into $y = (ax^2+bx+c)/(dx+e)$ and doing the algebra (omitted), we can find the y-values and thus conclude the two stationary points are

$$P, Q = \left(\frac{-e \pm \sqrt{(ae^2+cd^2-bde)/a}}{d}, \frac{bd-2ae \pm 2\sqrt{a(ae^2+cd^2-bde)}}{d^2}\right),$$

with Q being to the left of P.

Observe the numerator of $\mathrm{d}y/\mathrm{d}x$ is a quadratic expression with coefficient ad on the squared term. Hence, if $ad > 0$, then this quadratic is ∪-shaped, so that Q is a strict local maximum and P is a strict local minimum. Conversely, if $ad < 0$, then the quadratic is ∩-shaped, so that Q is a strict local minimum and P is a strict local maximum.

(e) Let (p, q) be a point in the hyperbola, i.e. it satisfies $y = (ax^2+bx+c)/(dx+e)$. Use Fact 41 to write down the reflections of the point (p, q) in the lines:

$$y = \frac{a \pm \sqrt{a^2+d^2}}{d}x + \frac{bd-ae \pm e\sqrt{a^2+d^2}}{d^2}.$$

Through an insane amount of algebra (omitted), it is possible to show that these reflection points also satisfy $y = (ax^2+bx+c)/(dx+e)$, thus proving that this hyperbola is indeed symmetric in the above lines.[588]

It remains to be shown that these are *the only* two lines of symmetry. If there were a third distinct line of symmetry, then there would be more than two asymptotes. But this is not the case. Thus, there can be at most two distinct lines of symmetry. □

[588] This painful, brute-force method is not exactly the "proper" or "usual" way to find the lines of symmetry (for which see my "forthcoming" *H2 Further Mathematics Textbook*), but does avoid having to use other facts about conic sections that we haven't discussed.

Fact 14.

Proof. **(a)** ($\Longrightarrow$) Suppose $a_1, a_2, \ldots, a_n \neq 0$.

Then applying Fact **??**(a)($\Longrightarrow$) to a_1 and a_2, we have $a_1 a_2 \neq 0$.

Next, applying Fact **??**(a)($\Longrightarrow$) to $a_1 a_2$ and a_3, we have $a_1 a_2 a_3 \neq 0$.

$\vdots$

Finally, applying Fact **??**(a)($\Longrightarrow$) to $a_1 a_2 a_3 \ldots a_{n-1}$ and a_n, we have $P = a_1 a_2 a_3 \ldots a_n \neq 0$.

($\Longleftarrow$) Suppose without loss of generality $a_1 = 0$.

Then applying Fact **??**(a)($\Longleftarrow$) to a_1 and a_2, we have $a_1 a_2 = 0$.

Next, applying Fact **??**(a)($\Longleftarrow$) to $a_1 a_2$ and a_3, we have $a_1 a_2 a_3 = 0$.

$\vdots$

Finally, applying Fact **??**(a)($\Longleftarrow$) to $a_1 a_2 a_3 \ldots a_{n-1}$ and a_n, we have $P = a_1 a_2 a_3 \ldots a_n = 0$.

(b) ($\Longrightarrow$) If any of a_1, a_2, ..., and a_n is zero, then $P = 0$ and in particular $P \ngtr 0$.

Suppose an odd number of a_1, a_2, ..., and a_n are negative (and the rest are positive). Let P_1 be the product of all the positive terms, $\bar{a}$ be one of the negative terms, and P_2 be the product of all the negative terms except $\bar{a}$.

By repeated application of Fact **??**(b), P_1 and P_2 are positive. So, by the same result, $P_1 P_2$ is positive.

Now, by Fact **??**(c), $P = P_1 P_2 \bar{a}$ is negative and in particular $P \ngtr 0$.

($\Longleftarrow$) Suppose an even number of a_1, a_2, ..., and a_n are negative (and the rest are positive). Let P_3 be the product of all the positive terms and P_4 be the product of all the negative terms.

By repeated application of Fact **??**(b), P_3 and P_4 are positive. So, by the same result, $P = P_3 P_4$ is positive.

(c) Similar to **(b)** and omitted. □

143. Appendices for Part II. Sequences and Series

Informally, a sequence **converges** if its terms "eventually" get "arbitrarily" close to some limit $L \in \mathbb{R}$. Formally,

Definition 296. Let (a_n) be a (real and infinite) sequence. Let $L \in \mathbb{R}$. Suppose that for all $\varepsilon > 0$, there exists N such that for all $n \geq N$, we have

$$|a_n - L| < \varepsilon.$$

Then we say that the sequence (a_n) is *convergent* and that *its limit exists*; moreover, it *converges to* L and its *limit* is L.

That the sequence (a_n) converges to L may be written as

$$a_n \to L \qquad \text{or} \qquad \lim_{n\to\infty} a_n = L.$$

If a sequence does not converge, then we say that it *diverges* or is *divergent*, and *its limit does not exist.*

Definition 297. Given the series $a_1 + a_2 + a_3 + \dots$, its *nth partial sum* is the finite series $a_1 + a_2 + \dots + a_n$.

A **convergent series** is then simply one whose partial sums converge:

Definition 298. Let $a_1 + a_2 + a_3 + \dots$ be a series and its nth partial sum be $s_n = a_1 + a_2 + \dots + a_n$. Consider the sequence $(s_n) = (s_1, s_2, s_3, \dots)$.

If the sequence (s_n) converges to some real number L, then we say that the series $a_1 + a_2 + a_3 + \dots$ is *convergent* and that *its limit exists*; moreover, it *converges to* L and its *limit* is L.

If a series does not converge, then we say that it *diverges* or is *divergent*, and *its limit does not exist.*

Example 1577. Recall Grandi's series: $\quad 1 - 1 + 1 - 1 + 1 - 1 + \dots$.

The corresponding sequence of partial sums is $(s_n) = (1, 0, 1, 0, 1, 0, \dots)$.

Pick $\varepsilon = 0.4$. Suppose that for some k, we have $|s_k - L| \overset{1}{<} \varepsilon$. Then

$$|s_{k+1} - L| = |s_k - s_{k+1} + s_k - L| \geq |s_k - s_{k+1}| - |s_k - L| = 1 - |s_k - L| \overset{1}{>} 1 - \varepsilon = 0.6 > \varepsilon.$$

We have just shown that (s_n) diverges. Hence, Grandi's series diverges.

Fact 266. (Reverse Triangle Inequality) *Suppose $x, y \in \mathbb{R}$. Then $|x - y| \geq ||x| - |y||$.*

Proof.

$$
\begin{aligned}
& & xy &\le |x|\,|y| \\
&\iff & -2xy &\ge -2\,|x|\,|y| \\
&\iff & x^2+y^2-2xy &\ge |x|^2+|y|^2-2\,|x|\,|y| \\
&\iff & (x-y)^2 &\ge (|x|-|y|)^2 \\
&\iff & |x-y| &\ge ||x|-|y||.
\end{aligned}
$$

□

Fact 111. *Other than the zero series $0+0+0+\dots$, every (infinite) arithmetic series diverges.*

Proof. Let (a_n) be a non-zero arithmetic series. Let $d = a_2 - a_1$ be the common difference and $s_n = \sum_{i=1}^{n} a_i$ be the nth partial sum. Below, $\overset{\text{r}}{\ge}$ denotes the use of the Reverse Triangle Inequality.

Case 1. Suppose $d = 0$ (so that $a_1 = a_2 = \dots$). Since (a_n) is non-zero, we must have $a_1 \neq 0$. Pick $\varepsilon = |a_1|/2$. Let $L \in \mathbb{R}$. Suppose that for some k, we have $|s_k - L| \overset{1}{<} \varepsilon = |a_1|/2$. Then

$$
\begin{aligned}
|s_{k+1}-L| = |s_k + a_{k+1} - L| = |s_k + a_1 - L| = |a_1 - (L - s_k)| &\overset{\text{r}}{\ge} ||a_1| - |L - s_k|| \\
= ||a_1| - |s_k - L|| \ge |a_1| - |s_k - L| \overset{1}{>} |a_1| - \frac{|a_1|}{2} &= \frac{|a_1|}{2} = \varepsilon.
\end{aligned}
$$

Case 2. Suppose $d \neq 0$. If $d > 0$ (or $d < 0$), then let j be the smallest integer such that a_j is positive (or negative). Pick $\varepsilon = |d|/2$. Suppose that for some $k > j$, we have $|s_k - L| < \varepsilon = |d|/2$. Then $|a_{k+1}| = |a_k + d| > |a_j + d| > |d|$ and

$$
\begin{aligned}
|s_{k+1}-L| = |s_k + a_{k+1} - L| = |a_{k+1} - (L - s_k)| &\overset{\text{r}}{\ge} ||a_{k+1}| - |L - s_k|| \\
= ||a_{k+1}| - |s_k - L|| \ge |a_{k+1}| - |s_k - L| > |d| - \frac{|d|}{2} &= \frac{|d|}{2} = \varepsilon.
\end{aligned}
$$

□

Fact 114. *Let $a \in \mathbb{R}$. If $|r| < 1$, then*

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}.$$

Proof. If $a = 0$ or $r = 0$, then the result clearly holds. So suppose $a \neq 0$ and $r \neq 0$.

Let $S_k = a + ar + ar^2 + ar^3 + \cdots + ar^{k-1}$.

By Fact 113, $S_k = a\dfrac{1-r^k}{1-r}$.

Let $\varepsilon > 0$. Pick K so that $\left|\dfrac{a}{1-r}\right| \left|r^K\right| = \varepsilon$. Then for all $k > K$,

$$
\begin{aligned}
\left|\frac{a}{1-r} - S_k\right| = \left|\frac{a}{1-r} - a\frac{1-r^k}{1-r}\right| = \left|\frac{a}{1-r}r^k\right| &= \left|\frac{a}{1-r}\right|\left|r^k\right| \\
&= \left|\frac{a}{1-r}\right|\left|r^K\right|\left|r^{k-K}\right| = \varepsilon\left|r^{k-K}\right| < \varepsilon.
\end{aligned}
$$

□

Fact 115. *Let $a \neq 0$. If $|r| \geq 1$, then $a + ar + ar^2 + ar^3 + \dots$ diverges.*

Proof. Let $L \in \mathbb{R}$ and s_n be the nth partial sum of the given series. Pick $\varepsilon = |a_1|/2$. Suppose that for some k, we have $|s_k - L| < \varepsilon$. Then

$$\begin{aligned} |s_{k+1} - L| &= \left|s_k + a_1 r^k - L\right| = \left|a_1 r^k - (L - s_k)\right| \stackrel{\text{r}}{\geq} \left|\left|a_1 r^k\right| - |L - s_k|\right| \\ &= \left|\left|a_1 r^k\right| - |s_k - L|\right| \geq \left|a_1 r^k\right| - |s_k - L| \geq |a_1| - |s_k - L| \geq |a_1| - \frac{|a_1|}{2} = \frac{|a_1|}{2} = \varepsilon, \end{aligned}$$

where $\stackrel{\text{r}}{\geq}$ uses the Reverse Triangle Inequality. □

Fact 267. *Suppose $(a_n) \to a$ and $(b_n) \to b$. If there exists N such that $a_n \geq b_n$ for every $n \geq N$, then $a \geq b$.*

Proof. Suppose for contradiction that $b > a$.

Then there exists $\bar{N}$ such that, letting $\varepsilon \stackrel{1}{=} b - a > 0$, for every $n \geq \bar{N}$,

$$\begin{array}{lccc} & |a_n - a| < \varepsilon/3 & \text{and} & |b_n - b| < \varepsilon/3, \\ \text{which implies} & a_n - a < \varepsilon/3 & \text{and} & b - b_n < \varepsilon/3, \\ \text{or equivalently,} & a + \varepsilon/3 \stackrel{2}{>} a_n & \text{and} & b_n \stackrel{3}{>} b - \varepsilon/3. \end{array}$$

Altogether, $b_n \stackrel{3}{>} b - \varepsilon/3 \stackrel{1}{=} a + \varepsilon - \varepsilon/3 = a + 2\varepsilon/3 \stackrel{2}{>} a_n$, contradicting $a_n \geq b_n$. □

144. Appendices for Part III. Vectors

Note that much of the discussion in these Appendices can be handled more easily with the machinery and terminology of linear algebra. But in these Appendices, I shall avoid the explicit use of linear algebra because it isn't in H2 Maths.

144.1. Some General Definitions

In the main text, we defined terms such as **vector**, **length**, and **unit vector** in $\mathbb{R}^2$ or $\mathbb{R}^3$. We now give their general definitions in $\mathbb{R}^n$:

Definition 299. A *point* in $\mathbb{R}^n$ is any ordered n-tuple of real numbers.

Definition 300. The point $(0, 0, \ldots, 0)$ in $\mathbb{R}^n$ is called the *origin* and is denoted O.

Definition 301. Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be points in $\mathbb{R}^n$. Then the *vector from* A *to* B, denoted $\overrightarrow{AB}$, is the ordered n-tuple defined by[589]

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n) \in \mathbb{R}^n.$$

Definition 302. Given the vector $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, its *length* (or *norm* or *magnitude*), denoted $|\mathbf{u}|$, is the number defined by

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

Definition 303. Given $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $c \in \mathbb{R}$, the vector $c\mathbf{u}$ is

$$c\mathbf{u} = (cu_1, cu_2, \ldots, cu_n).$$

Definition 304. Given the points $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$, the *difference* $B - A$ is defined as the vector $\overrightarrow{AB}$, i.e.

$$B - A = \overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n).$$

Definition 305. Given a point $A = (a_1, a_2, \ldots, a_n)$ and a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, their *sum* $A + \mathbf{v}$ is this point:

$$A + \mathbf{v} = (a_1 + v_1, a_2 + v_2, \ldots, a_n + v_n).$$

Definition 306. Given a point $B = (b_1, b_2, \ldots, b_n)$ and a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, the *difference* $B - \mathbf{v}$ is this point:

$$B - \mathbf{v} = (b_1 - v_1, b_2 - v_2, \ldots, b_n - v_n).$$

[589] Note that technically, the set $\mathbb{R}^n$ that contains the points A and B is different from the set $\mathbb{R}^n$ that contains the vector $\overrightarrow{AB}$. The former is a **Euclidean space**, while the latter is a **vector space**. But this is beyond the scope of A-Level Maths and so we shan't worry about it.

Definition 307. Given the vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$, their *sum*, denoted $\mathbf{u} + \mathbf{v}$, is this vector:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n).$$

Definition 308. Given the vector $\mathbf{u} = (u_1, u_2, \dots, u_n)$, its *additive inverse*, denoted $-\mathbf{u}$, is this vector:

$$-\mathbf{u} = (-u_1, -u_2, \dots, -u_n).$$

Definition 309. Given the vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$, the difference $\mathbf{u} - \mathbf{v}$ is defined as the sum of $\mathbf{u}$ and $\mathbf{v}$.

Fact 268. *If* $\mathbf{u} = (u_1, u_2, \dots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \dots, v_n)$ *are vectors, then*

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, \dots, u_n - v_n).$$

Proof. By Definition 308, $-\mathbf{v} = (-v_1, -v_2, \dots, -v_n)$. And now by Definition 309,

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, u_2 - v_2, \dots, u_n - v_n). \qquad \square$$

Fact 269. *Suppose A, B, and C be points. Then* $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$.

Proof. Let $A = (a_1, a_2, \dots, a_n)$, $B = (b_1, b_2, \dots, b_n)$, and $C = (c_1, c_2, \dots, c_n)$. Then by Definition 304, $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, \dots, b_n - a_n)$, $\overrightarrow{AC} = (c_1 - a_1, c_2 - a_2, \dots, c_n - a_n)$, and $\overrightarrow{CB} = (b_1 - c_1, b_2 - c_2, \dots, b_n - c_n)$. And now by Fact 268,

$$\begin{aligned}\overrightarrow{AB} - \overrightarrow{AC} &= (b_1 - a_1, b_2 - a_2, \dots, b_n - a_n) - (c_1 - a_1, c_2 - a_2, \dots, c_n - a_n) \\ &= (b_1 - c_1, b_2 - c_2, \dots, b_n - c_n) = \overrightarrow{CB}.\end{aligned} \qquad \square$$

Definition 310. Given the non-zero vector $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$, the *unit vector in its direction* (or *its unit vector*), denoted $\hat{\mathbf{u}}$, is defined by

$$\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|}\mathbf{u}.$$

So, given the vector $\mathbf{u}$, its unit vector $\hat{\mathbf{u}}$ is simply the vector that points in the same direction but has length 1.

144.2. Some Basic Results

Fact 270. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *be a vector and* $c \in \mathbb{R}$*. Then* $|c\mathbf{a}| = |c|\,|\mathbf{a}|$.

Proof. By Definitions 303 and 302,

$$|c\mathbf{a}| = |c(a_1, a_2, \ldots, a_n)| = |(ca_1, ca_2, \ldots, ca_n)| = \sqrt{\sum_{i=1}^{n} (ca_i)^2}$$

$$= \sqrt{\sum_{i=1}^{n} \left(c^2 a_i^2\right)} = \sqrt{c^2 \sum_{i=1}^{n} a_i^2} = |c| \sqrt{\sum_{i=1}^{n} a_i^2} = |c|\,|\mathbf{a}|.$$

□

Fact 124. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *be non-zero vectors. Then*

(a) $\hat{\mathbf{a}} = \hat{\mathbf{b}} \iff \mathbf{a}$ *and* $\mathbf{b}$ *point in the same direction;*

(b) $\hat{\mathbf{a}} = -\hat{\mathbf{b}} \iff \mathbf{a}$ *and* $\mathbf{b}$ *point in exact opposite directions;*

(c) $\hat{\mathbf{a}} = \pm\hat{\mathbf{b}} \iff \mathbf{a} \parallel \mathbf{b}$*;*

(d) $\hat{\mathbf{a}} \neq \pm\hat{\mathbf{b}} \iff \mathbf{a} \nparallel \mathbf{b}$.

Proof. **(a)** Suppose $\hat{\mathbf{a}} = \hat{\mathbf{b}}$. Then $\mathbf{a}/|\mathbf{a}| = \mathbf{b}/|\mathbf{b}|$ or $\mathbf{a} = (|\mathbf{a}|/|\mathbf{b}|)\,\mathbf{b}$. Since $\mathbf{a} = k\mathbf{b}$ for some $k > 0$, by Definition 137, they point in the same direction.

Now suppose instead that $\mathbf{a}$ and $\mathbf{b}$ point in the same direction. Then by Definition 137, there exists $k > 0$ such that $\mathbf{a} = k\mathbf{b}$. Thus,

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{1}{|k\mathbf{b}|}k\mathbf{b} = \frac{1}{|k|\,|\mathbf{b}|}k\mathbf{b} = \frac{1}{|\mathbf{b}|}\mathbf{b} = \hat{\mathbf{b}}.$$

(b) Similar, omitted.

(c) and **(d)** follow from **(a)** and **(b)**. □

Fact 117. *Suppose* $\mathbf{v}$ *is a vector. Then* $|\mathbf{v}| \geq 0$*. Moreover,* $|\mathbf{v}| = 0 \iff \mathbf{v} = \mathbf{0}$.

Proof. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Then $|\mathbf{v}| = \sqrt{\sum v_i^2} \geq 0$, with $|\mathbf{v}| \iff \mathbf{v} = \mathbf{0}$. □

The next result holds only in two-dimensional space:

Fact 125. *Let* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *be vectors. If* $\mathbf{a} \nparallel \mathbf{b}$, *then there exist* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}.$$

Proof. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. Let $\mathbf{c} = (c_1, c_2)$ be any vector.

Suppose $a_1 = 0$. Then $a_2 \neq 0$ (because $\mathbf{a} \neq \mathbf{0}$) and $b_1 \neq 0$ (because $\mathbf{a} \nparallel \mathbf{b}$).

Now pick $$\alpha = \frac{b_1c_2 - b_2c_1}{a_2b_1} \qquad \text{and} \qquad \beta = \frac{c_1}{b_1}.$$

Then $$\alpha\mathbf{a} + \beta\mathbf{b} = \frac{b_1c_2 - b_2c_1}{a_2b_1}\begin{pmatrix} 0 \\ a_2 \end{pmatrix} + \frac{c_1}{b_1}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 + c_1 \\ \dfrac{b_1c_2 - b_2c_1}{\cancel{a_2}b_1}\cancel{a_2} + \dfrac{c_1}{b_1}b_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{c}.$$

The cases where $a_2 = 0$, $b_1 = 0$, or $b_2 = 0$ are similarly handled.

Now suppose $a_1, a_2, b_1, b_2 \neq 0$. Since $\mathbf{a} \nparallel \mathbf{b}$, we have $a_1/a_2 \neq b_1/b_2$ and thus $a_1b_2 - a_2b_1 \neq 0$.

Now pick $$\alpha = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \qquad \text{and} \qquad \beta = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Then $$\alpha\mathbf{a} + \beta\mathbf{b} = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{a_1b_2c_1 \cancel{- a_1b_1c_2} \cancel{+ a_1b_1c_2} - a_2b_1c_1}{a_1b_2 - a_2b_1} \\[2ex] \dfrac{\cancel{a_2b_2c_1} - a_2b_1c_2 + a_1b_2c_2 \cancel{- a_2b_2c_1}}{a_1b_2 - a_2b_1} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{c}.$$ □

Fact 126. *Suppose* $\mathbf{v}$ *is a line's direction vector. Then*

$$\mathbf{u} \text{ is also that line's direction vector} \qquad \iff \qquad \mathbf{u} \parallel \mathbf{v}.$$

Proof. Suppose l is the line described by $R = P + \lambda\mathbf{v}$ ($\lambda \in \mathbb{R}$), where $P \in \mathbb{R}^n$ is some point.

Let A and B be distinct points on l. Since $A, B \in l$, there are distinct real numbers α and β such that $A = P + \alpha\mathbf{v}$ and $B = P + \beta\mathbf{v}$.

Thus, $\overrightarrow{AB} = (\beta - \alpha)\mathbf{v}$, where $\beta - \alpha \neq 0$. We have just proven that any direction vector of l must be a non-zero scalar multiple of the vector $\mathbf{v}$.

We next prove that any non-zero scalar multiple of the vector $\mathbf{v}$ must be a direction vector of l. Let $k \neq 0$. Let $C = A + k\mathbf{v} = P + (\alpha + k)\mathbf{v}$. Observe that $C \in l$. Thus, $k\mathbf{v} = \overrightarrow{AC}$ is a direction vector of l. □

144.3. Scalar Product

Definition 311. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be vectors. Then their *scalar product* $\mathbf{u} \cdot \mathbf{v}$ is the number defined by

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

Fact 129. *Suppose* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors. Then*

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. **(Commutative)**

(b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$. **(Distributive over Addition)**

Proof. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, and $\mathbf{c} = (c_1, c_2, \ldots c_n)$.

Then $$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} b_i a_i = \mathbf{b} \cdot \mathbf{a}.$$

And, $$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \sum_{i=1}^{n} a_i (b_i + c_i) = \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} a_i c_i = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$ □

Fact 130. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors and* $c \in \mathbb{R}$ *be a scalar. Then*

$$(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}).$$

Proof. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. By Fact 116(b), we have

$$(c\mathbf{a}) \cdot \mathbf{b} = \sum_{i=1}^{n} (ca_i) b_i = c \sum_{i=1}^{n} a_i b_i = c(\mathbf{a} \cdot \mathbf{b}).$$ □

A vector's length is the square root of its scalar product with itself:

Fact 131. *Suppose* $\mathbf{v}$ *be a vector. Then* $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ *and* $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$.

Proof. By Def. 302, $|\mathbf{v}| = \sqrt{\sum v_i^2}$. By Def. 311, $\mathbf{v} \cdot \mathbf{v} = \sum v_i v_i = \sum v_i^2$. Thus, $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. □

Fact 133. (Cauchy's Inequality.) *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are non-zero vectors. Then*

$$-1 \le \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \le 1.$$

Equivalently, $$-|\mathbf{u}||\mathbf{v}| \le \mathbf{u} \cdot \mathbf{v} \le |\mathbf{u}||\mathbf{v}| \quad \text{or} \quad (\mathbf{u} \cdot \mathbf{v})^2 \le |\mathbf{u}|^2 |\mathbf{v}|^2.$$

Proof. Let $x \in \mathbb{R}$ and $S = |\mathbf{u} + x\mathbf{v}|^2$. Write

$$S = |\mathbf{u} + x\mathbf{v}|^2 = (\mathbf{u} + x\mathbf{v}) \cdot (\mathbf{u} + x\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) x + (\mathbf{v} \cdot \mathbf{v}) x^2 = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) x + |\mathbf{v}|^2 x^2.$$

Observe that S is a quadratic expression in x. Moreover, $S \ge 0$. So, its discriminant must be non-positive. That is, $(2\mathbf{u} \cdot \mathbf{v})^2 - 4|\mathbf{v}|^2 |\mathbf{u}|^2 \le 0$.

Rearranging, $\dfrac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{u}|^2|\mathbf{v}|^2} \leq 1$. Now take square roots to get $-1 \leq \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \leq 1$. □

Fact 135. *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are non-zero vectors. Then*

(a) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = 1 \iff$ $\mathbf{u}$ *and* $\mathbf{v}$ *point in the same direction;*

(b) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = -1 \iff$ $\mathbf{u}$ *and* $\mathbf{v}$ *point in exact opposite directions;*

(c) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \pm 1 \iff \mathbf{u} \parallel \mathbf{v}$*;*

(d) $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \in (-1, 1) \iff$ $\mathbf{u}$ *and* $\mathbf{v}$ *point in different directions.*

Proof. We first prove $\Longleftarrow$ of **(a)**. If $\mathbf{u}$ and $\mathbf{v}$ point in the same direction, then by Definition 137, there exists $k > 0$ such that $\mathbf{u} = k\mathbf{v}$ and so,

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(k\mathbf{v})\cdot\mathbf{v}}{|k\mathbf{v}||\mathbf{v}|} = \frac{k(\mathbf{v}\cdot\mathbf{v})}{|k||\mathbf{v}||\mathbf{v}|} = \frac{k|\mathbf{v}||\mathbf{v}|}{k|\mathbf{v}||\mathbf{v}|} = 1 \qquad ✓$$

The proof of $\Longleftarrow$ of **(b)** is very similar. If $\mathbf{u}$ and $\mathbf{v}$ point in the exact opposite directions, then by Definition 137, there exists $k < 0$ such that $\mathbf{u} = k\mathbf{v}$ and so,

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(k\mathbf{v})\cdot\mathbf{v}}{|k\mathbf{v}||\mathbf{v}|} = \frac{k(\mathbf{v}\cdot\mathbf{v})}{|k||\mathbf{v}||\mathbf{v}|} = \frac{k|\mathbf{v}||\mathbf{v}|}{-k|\mathbf{v}||\mathbf{v}|} = -1 \qquad ✓$$

In the remainder of this proof, we prove $\Longrightarrow$ of **(a)** and **(b)**.

We first show that if $\mathbf{u} \nparallel \mathbf{v}$, then $\mathbf{u}\cdot\mathbf{v} \neq \pm|\mathbf{u}||\mathbf{v}|$. We'll use the same idea that was used in the proof of Cauchy's Inequality:

If $\mathbf{u} \nparallel \mathbf{v}$, then $\mathbf{u} \neq x\mathbf{v}$ for all $x \in \mathbb{R}$. So, $\mathbf{u} - x\mathbf{v} \neq \mathbf{0}$ and $|\mathbf{u} - x\mathbf{v}| > 0$. Thus,

$$S = |\mathbf{u} - x\mathbf{v}|^2 = (\mathbf{u} - x\mathbf{v})\cdot(\mathbf{u} - x\mathbf{v}) = |\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v})x + |\mathbf{v}|^2 x^2 > 0.$$

Observe that S is a quadratic expression in x, with positive coefficient on x^2. And since $S > 0$ for all x, its discriminant must be negative,

$$(-2\mathbf{u}\cdot\mathbf{v})^2 - 4|\mathbf{v}|^2|\mathbf{u}|^2 < 0.$$

Rearranging, we get $(\mathbf{u}\cdot\mathbf{v})^2 < |\mathbf{u}|^2|\mathbf{v}|^2$ and thus $\mathbf{u}\cdot\mathbf{v} \neq \pm|\mathbf{u}||\mathbf{v}|$.

We've just shown that if $\mathbf{u} \nparallel \mathbf{v}$, then $\mathbf{u}\cdot\mathbf{v} \neq \pm|\mathbf{u}||\mathbf{v}|$. And so, by the contrapositive, if $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$, then $\mathbf{u} \parallel \mathbf{v}$.

Now, if $\mathbf{u}\cdot\mathbf{v} \neq |\mathbf{u}||\mathbf{v}|$, then by $\Longleftarrow$ of **(a)**, $\mathbf{u}$ and $\mathbf{v}$ do not point in the same direction. Thus, if $\mathbf{u}\cdot\mathbf{v} = -|\mathbf{u}||\mathbf{v}|$, then $\mathbf{u}$ and $\mathbf{v}$ must point in the exact opposite directions. ✓

Similarly, if $\mathbf{u}\cdot\mathbf{v} \neq -|\mathbf{u}||\mathbf{v}|$, then by $\Longleftarrow$ of **(b)**, $\mathbf{u}$ and $\mathbf{v}$ do not point in the exact opposite directions. Thus, if $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|$, then $\mathbf{u}$ and $\mathbf{v}$ must point in the same direction. ✓ □

144.4. Angles

Definition 312. The *standard basis vector in the ith direction* (or *ith standard basis vector*), denoted $\mathbf{e}_i$, is the vector whose ith coordinate is 1 and other coordinates are 0.

Definition 313. The *ith-direction cosine* of the vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ is this number

$$\frac{v_i}{|\mathbf{v}|}.$$

Fact 271. *Suppose θ is the angle between a vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and $\mathbf{e}_i$. Then*

$$\cos\theta = \frac{v_i}{|\mathbf{v}|}.$$

Proof. By Definition 145, $\displaystyle \cos\theta = \frac{\mathbf{v}\cdot\mathbf{e}_i}{|\mathbf{v}|\,|\mathbf{e}_i|} = \frac{v_i \cdot 1 + \sum_{j\neq i} v_j \cdot 0}{|\mathbf{v}|\cdot 1} = \frac{v_i}{|\mathbf{v}|}.$ □

The next Fact says that given two lines, we can choose any direction vector for each and the calculated angle between the two chosen direction vectors will, as expected, be fixed:

Fact 272. *If one line has direction vectors $\mathbf{u}_1$ and $\mathbf{v}_1$, while another has $\mathbf{u}_2$ and $\mathbf{v}_2$, then*

$$\frac{|\mathbf{u}_1\cdot\mathbf{u}_2|}{|\mathbf{u}_1|\,|\mathbf{u}_2|} = \frac{|\mathbf{v}_1\cdot\mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|}.$$

Proof. There exist non-zero real numbers λ and μ such that $\mathbf{u}_1 = \lambda\mathbf{v}_1$ and $\mathbf{u}_2 = \mu\mathbf{v}_2$. So,

$$\frac{|\mathbf{u}_1\cdot\mathbf{u}_2|}{|\mathbf{u}_1|\,|\mathbf{u}_2|} = \frac{|(\lambda\mathbf{v}_1)\cdot(\mu\mathbf{v}_2)|}{|\lambda\mathbf{v}_1|\,|\mu\mathbf{v}_2|} = \frac{|(\lambda\mathbf{v}_1)\cdot(\mu\mathbf{v}_2)|}{|\lambda\mathbf{v}_1|\,|\mu\mathbf{v}_2|} = \frac{|\lambda|\,|\mu|\,|\mathbf{v}_1\cdot\mathbf{v}_2|}{|\lambda|\,|\mu|\,|\mathbf{v}_1|\,|\mathbf{v}_2|} = \frac{|\mathbf{v}_1\cdot\mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|}.$$ □

Corollary 23. *Suppose θ is the angle between two lines l_1 and l_2. Then **(a)** $\theta = 0 \iff l_1 \parallel l_2$; **(b)** $\theta = \pi/2 \iff l_1 \perp l_2$.*

Proof. Suppose the two lines have direction vectors $\mathbf{u}$ and $\mathbf{v}$.

(a) $\theta = 0 \iff \cos^{-1}\dfrac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} = 0 \iff \dfrac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} = \cos 0 = 1 \iff \mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}|\,|\mathbf{v}| \iff \mathbf{u}\parallel\mathbf{v} \iff l_1 \parallel l_2$.

(b) $\theta = \dfrac{\pi}{2} \iff \cos^{-1}\dfrac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} = \dfrac{\pi}{2} \iff \dfrac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} = \cos\dfrac{\pi}{2} = 0 \iff \mathbf{u}\cdot\mathbf{v} = 0 \iff \mathbf{u}\perp\mathbf{v} \iff l_1 \perp l_2$. □

144.5. The Relationship Between Two Lines

Fact 139. *Suppose two lines are ...*

(a) *Identical. Then they are also parallel.*

(b) *Distinct and parallel. Then they do not intersect.*

(c) *Distinct. Then they share at most one intersection point.*

Proof. **(a)** If two lines are identical, then they also share a direction vector, so that by Definition 150, they are parallel.

(b) Suppose two lines are parallel. Then by Definition 150 and Fact 126, they share some direction vector $\mathbf{u}$.

Suppose also that they intersect at some point S. Then both lines can be described by $\mathbf{r} = \overrightarrow{OS} + \lambda\mathbf{u}$ and are identical.

Thus, if two parallel lines are distinct, then they cannot intersect.

(c) We already showed that two distinct and parallel lines do not intersect. We now show that two distinct and non-parallel lines share at most one intersection point.

Suppose for contradiction that two distinct lines share two distinct intersection points P and Q. Then $\overrightarrow{PQ}$ is a direction vector of both lines. Thus, both lines can be described by $\mathbf{r} = \overrightarrow{OP} + \lambda\overrightarrow{PQ}$ and must thus be identical. □

Fact 140. *If two lines (in the cartesian plane) are distinct and non-parallel, then they must share exactly one intersection point.*

Proof. Suppose two lines are described by

$$\mathbf{r} = (p_1, p_2) + \lambda (u_1, u_2) \qquad \text{and} \qquad \mathbf{r} = (q_1, q_2) + \mu (v_1, v_2) \qquad (\lambda, \mu \in \mathbb{R}).$$

By definition of a line, at least one of u_1 or u_2 must be non-zero. So, suppose without loss of generality that $u_1 \neq 0$. Now,

- If $v_1 = 0$, then $v_2 \neq 0$ and so $u_1 v_2 - u_2 v_1 = u_1 v_2 \neq 0$.
- If $v_1 \neq 0$, then $u_2/u_1 \neq v_2/v_1$ (because the two lines are not parallel) and so $u_1 v_2 - u_2 v_1 \neq 0$.

Thus, $u_1 v_2 - u_2 v_1 \neq 0$. The reader can verify (through rather tedious algebra) that the two lines intersect at these parameter values:

$$\hat{\mu} = \frac{u_1 (p_2 - q_2) + u_2 (q_1 - p_1)}{u_1 v_2 - u_2 v_1} \qquad \text{and} \qquad \hat{\lambda} = \frac{q_1 + \hat{\mu} v_1 - p_1}{u_1}.$$

This intersection point is also unique because by Fact 139, two lines can share at most one intersection point. □

Fact 155. *Suppose the line l is described by* $\mathbf{r} = (p_1, p_2, p_3) + \lambda(v_1, v_2, v_3)$ $(\lambda \in \mathbb{R})$.

(1) *If $v_1, v_2, v_3 \neq 0$, then l can be described by*

$$\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}.$$

(2) *If $v_1 = 0$ and $v_2, v_3 \neq 0$, then l is perpendicular to the x-axis and can be described by*

$$x = p_1 \qquad \text{and} \qquad \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}.$$

(3) *If $v_2 = 0$ and $v_1, v_3 \neq 0$, then l is perpendicular to the y-axis and can be described by*

$$y = p_2 \qquad \text{and} \qquad \frac{x-p_1}{v_1} = \frac{z-p_3}{v_3}.$$

(4) *If $v_3 = 0$ and $v_1, v_2 \neq 0$, then l is perpendicular to the z-axis and can be described by*

$$z = p_3 \qquad \text{and} \qquad \frac{x-p_1}{v_1} = \frac{y-p_2}{v_2}.$$

(5) *If $v_1, v_2 = 0$, then l is perpendicular to the x- and y-axes and can be described by*

$$x = p_1 \qquad \text{and} \qquad y = p_2.$$

(6) *If $v_1, v_3 = 0$, then l is perpendicular to the x- and z-axes and can be described by*

$$x = p_1 \qquad \text{and} \qquad z = p_3.$$

(7) *If $v_2, v_3 = 0$, then l is perpendicular to the y- and z-axes and can be described by*

$$y = p_2 \qquad \text{and} \qquad y = p_2.$$

Proof. Write $x \overset{1}{=} p_1 + \lambda v_1$, $y \overset{2}{=} p_2 + \lambda v_2$, and $z \overset{3}{=} p_3 + \lambda v_3$.

Now, $v_1 \times \overset{2}{=}$ minus $v_2 \times \overset{1}{=}$ yields

$$v_1 y - v_2 x = v_1 (p_2 + \lambda v_2) - v_2 (p_1 + \lambda v_1) = v_1 p_2 - v_2 p_1 \qquad \text{or} \qquad v_2 (x - p_1) \overset{4}{=} v_1 (y - p_2).$$

Similarly, $v_2 \times \overset{3}{=}$ minus $v_3 \times \overset{2}{=}$ and $v_1 \times \overset{3}{=}$ minus $v_3 \times \overset{1}{=}$ yield

$$v_2 (z - p_3) \overset{5}{=} v_3 (y - p_2) \qquad \text{and} \qquad v_1 (z - p_3) \overset{6}{=} v_3 (x - p_1).$$

(*Proof continues below ...*)

(... *Proof continued from above.*)

(1) If $v_1, v_2, v_3 \neq 0$, then divide $\stackrel{4}{=}$ by $v_1 v_2$ and $\stackrel{5}{=}$ by $v_2 v_3$ to get

$$\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2} \qquad \text{and} \qquad \frac{z-p_3}{v_3} = \frac{y-p_2}{v_2}.$$

(2) If $v_1 = 0$ and $v_2, v_3 \neq 0$, then $\stackrel{4}{=}$ becomes $x = p_1$ and divide $\stackrel{5}{=}$ by $v_2 v_3$ to get

$$\frac{z-p_3}{v_3} = \frac{y-p_2}{v_2}.$$

Since $(0, v_2, v_3) \cdot \mathbf{i} = 0$, l is perpendicular to the x-axis.

(3) If $v_2 = 0$ and $v_1, v_3 \neq 0$, then $\stackrel{5}{=}$ becomes $y = p_2$ and divide $\stackrel{6}{=}$ by $v_1 v_3$ to get

$$\frac{z-p_3}{v_3} = \frac{x-p_1}{v_1}.$$

Since $(v_1, 0, v_3) \cdot \mathbf{j} = 0$, l is perpendicular to the y-axis.

(4) If $v_3 = 0$ and $v_1, v_2 \neq 0$, then $\stackrel{6}{=}$ becomes $z = p_3$ and divide $\stackrel{4}{=}$ by $v_1 v_2$ to get

$$\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2}.$$

Since $(v_1, v_2, 0) \cdot \mathbf{k} = 0$, l is perpendicular to the z-axis.

(5) If $v_1, v_2 = 0$, then $\stackrel{4}{=}$ and $\stackrel{5}{=}$ become

$$x = p_1 \qquad \text{and} \qquad y = p_2.$$

Since $(0, 0, v_3) \cdot \mathbf{i} = 0$ and $(0, 0, v_3) \cdot \mathbf{j} = 0$, l is perpendicular to both the x- and y-axes.

(6) If $v_1, v_3 = 0$, then $\stackrel{4}{=}$ and $\stackrel{6}{=}$ become

$$x = p_1 \qquad \text{and} \qquad z = p_3.$$

Since $(0, v_2, 0) \cdot \mathbf{i} = 0$ and $(0, v_2, 0) \cdot \mathbf{k} = 0$, l is perpendicular to both the x- and z-axes.

(7) If $v_2, v_3 = 0$, then $\stackrel{5}{=}$ and $\stackrel{6}{=}$ become

$$y = p_2 \qquad \text{and} \qquad z = p_3.$$

Since $(v_1, 0, 0) \cdot \mathbf{j} = 0$ and $(v_1, 0, 0) \cdot \mathbf{k} = 0$, l is perpendicular to both the y- and z-axes. □

Fact 273. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are non-zero vectors, then* $(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}$ *is the unique vector such that*

$$\textbf{(a)}\ (\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} \parallel \mathbf{v}; \qquad \textit{and} \qquad \textbf{(b)}\ \mathbf{u}-(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} \perp \mathbf{v}.$$

Proof. We first verify that $(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}$ satisfies **(a)** and **(b)**:

(a) Let θ be the angle between $(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{[(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}]\cdot\mathbf{v}}{|(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}|\,|\mathbf{v}|} = \frac{(\mathbf{u}\cdot\hat{\mathbf{v}})\,(\hat{\mathbf{v}}\cdot\mathbf{v})}{|\mathbf{u}\cdot\hat{\mathbf{v}}|\,|\hat{\mathbf{v}}|\,|\mathbf{v}|} = \frac{\mathbf{u}\cdot\hat{\mathbf{v}}}{|\mathbf{u}\cdot\hat{\mathbf{v}}|} = \pm 1.$$

Thus, $\theta = 0$ and $(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} \parallel \mathbf{v}$.

(b) By Definition 146, $\mathbf{u}-(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} \perp \mathbf{v} \iff [\mathbf{u}-(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}]\cdot\mathbf{v} = 0$. But this last equation is true, as we now verify:

$$\begin{aligned}[\mathbf{u}-(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}]\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v}-(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}\cdot\mathbf{v} \qquad &= \mathbf{u}\cdot\mathbf{v}-\left(\mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}\right)\left(\frac{\mathbf{v}}{|\mathbf{v}|}\cdot\mathbf{v}\right)\\ = \mathbf{u}\cdot\mathbf{v}-\frac{1}{|\mathbf{v}|^2}(\mathbf{u}\cdot\mathbf{v})\,(\mathbf{v}\cdot\mathbf{v}) &= \mathbf{u}\cdot\mathbf{v}-\frac{1}{|\mathbf{v}|^2}(\mathbf{u}\cdot\mathbf{v})\,|\mathbf{v}|^2 = 0 \qquad \checkmark\end{aligned}$$

We now show that no other vector satisfies **(a)** and **(b)**.

Suppose $\mathbf{w}$ is also a vector that satisfies **(a)** and **(b)**. That is,

$$\textbf{(a)}\ \mathbf{w}\parallel\mathbf{v}; \qquad \text{and} \qquad \textbf{(b)}\ \mathbf{u}-\mathbf{w}\perp\mathbf{v}.$$

Since $\mathbf{w}\parallel\mathbf{v}$, there exists $\lambda\neq 0$ such that $\mathbf{w}=\lambda\mathbf{v}$. And since $\mathbf{u}-\mathbf{w}\perp\mathbf{v}$, we have

$$(\mathbf{u}-\lambda\mathbf{v})\cdot\mathbf{v}=0 \qquad \text{or} \qquad \mathbf{u}\cdot\mathbf{v}-\lambda\mathbf{v}\cdot\mathbf{v}=0 \qquad \text{or} \qquad \mathbf{u}\cdot\mathbf{v}=\lambda\mathbf{v}\cdot\mathbf{v} \qquad \text{or} \qquad \lambda=\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}.$$

Thus,

$$\mathbf{w}=\lambda\mathbf{v}=\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}=(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}.$$

□

Fact 274. (Lagrange's Identity) Suppose $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$. Then

(a) $\left(a_1^2+a_2^2\right)\left(b_1^2+b_2^2\right)-\left(a_1b_1+a_2b_2\right)^2=\left(a_1b_2-a_2b_1\right)^2$;

(b) $\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)-\left(a_1b_1+a_2b_2+a_3b_3\right)^2$
$=\left(a_1b_2-a_2b_1\right)^2+\left(a_3b_1-a_1b_3\right)^2+\left(a_2b_3-a_3b_2\right)^2$.

Proof. We proved **(a)** in Exercise 242(e) and **(b)** in Exercise 268(e). □

Fact 149. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are vectors. Then*

$$|\text{rej}_{\mathbf{b}}\mathbf{a}| = \left|\mathbf{a} \times \hat{\mathbf{b}}\right|.$$

Proof. By Pythagoras' Theorem (Theorem 17),

$$|\text{rej}_{\mathbf{b}}\mathbf{a}| = \sqrt{|\mathbf{a}|^2 - |\text{proj}_{\mathbf{b}}\mathbf{a}|^2}.$$

We will prove this claim twice, once in the 2D case and again in the 3D case. In each case, we will use Lagrange's Identity (LI).

2D case. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. Then

$$\begin{aligned}
|\text{rej}_{\mathbf{b}}\mathbf{a}| &= \sqrt{|\mathbf{a}|^2 - |\text{proj}_{\mathbf{b}}\mathbf{a}|^2} = \sqrt{a_1^2 + a_2^2 - \frac{(a_1b_1 + a_2b_2)^2}{b_1^2 + b_2^2}}. \\
&= \sqrt{\frac{(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2}{b_1^2 + b_2^2}} \\
&\overset{\text{LI}}{=} \sqrt{\frac{(a_1b_2 - a_2b_1)^2}{b_1^2 + b_2^2}} = \frac{|a_1b_2 - a_2b_1|}{\sqrt{b_1^2 + b_2^2}} = \left|\mathbf{a} \times \hat{\mathbf{b}}\right|.
\end{aligned}$$

3D case. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Then

$$\begin{aligned}
|\text{rej}_{\mathbf{b}}\mathbf{a}| &= \sqrt{|\mathbf{a}|^2 - |\text{proj}_{\mathbf{b}}\mathbf{a}|^2} = \sqrt{a_1^2 + a_2^2 + a_3^2 - \frac{(a_1b_1 + a_2b_2 + a_3b_3)^2}{b_1^2 + b_2^2 + b_3^2}} \\
&= \sqrt{\frac{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2}{b_1^2 + b_2^2 + b_3^2}} \\
&\overset{\text{LI}}{=} \frac{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{b_1^2 + b_2^2 + b_3^2}} = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|} = \left|\mathbf{a} \times \hat{\mathbf{b}}\right|.
\end{aligned}$$

□

Fact 146. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be non-zero vectors. If* $\mathbf{a}\times\mathbf{b}=\mathbf{0}$*, then* $\mathbf{a}\parallel\mathbf{b}$*.*

Proof. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, then by Fact 148, $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta=0$. Since $|\mathbf{a}|\neq 0$ and $|\mathbf{b}|\neq 0$, we have $\sin\theta=0$ and thus $\mathbf{a}\parallel\mathbf{b}$. □

Fact 157. *Suppose* $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{c}$ *are vectors, with* $\mathbf{a}\nparallel\mathbf{b}$*. Then*

$$\mathbf{c}\parallel\mathbf{a}\times\mathbf{b}\qquad\Longleftrightarrow\qquad\mathbf{c}\perp\mathbf{a},\mathbf{b}.$$

Proof. $\Longleftarrow$ follows from Fact 275 (below).

For $\Longrightarrow$, suppose $\mathbf{c}\parallel\mathbf{a}\times\mathbf{b}$. Then there exists $k\neq 0$ such that $\mathbf{c}=k(\mathbf{a}\times\mathbf{b})$. So, $\mathbf{c}\cdot\mathbf{a}=[k(\mathbf{a}\times\mathbf{b})]\cdot\mathbf{a}=k(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=0$ and thus $\mathbf{c}\perp\mathbf{a}$. We can similarly show that $\mathbf{c}\cdot\mathbf{b}=0$ and thus $\mathbf{c}\perp\mathbf{b}$. □

Fact 275. *Suppose* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$*, and* $\mathbf{d}$ *are non-zero vectors with* $\mathbf{a}\nparallel\mathbf{b}$*. If* $\mathbf{c}\perp\mathbf{a},\mathbf{b}$ *and* $\mathbf{d}\perp\mathbf{a},\mathbf{b}$*, then* $\mathbf{c}\parallel\mathbf{d}$*.*

Proof. Let $\mathbf{a}=(a_1,a_2,a_3)$, $\mathbf{b}=(b_1,b_2,b_3)$, $\mathbf{c}=(c_1,c_2,c_3)$, and $\mathbf{d}=(d_1,d_2,d_3)$.

Since $\mathbf{c}\perp\mathbf{a},\mathbf{b}$, we have $\mathbf{c}\cdot\mathbf{a}\overset{1}{=}0$ and $\mathbf{c}\cdot\mathbf{b}\overset{2}{=}0$. Or,

$$c_1a_1+c_2a_2+c_3a_3\overset{1}{=}0\qquad\text{and}\qquad c_1b_1+c_2b_2+c_3b_3\overset{2}{=}0.$$

Let $x=a_2b_3-a_3b_2$, $y=a_3b_1-a_1b_3$, and $z=a_1b_2-a_2b_1$, so that $\mathbf{a}\times\mathbf{b}=(x,y,z)$.

Now, $b_1\times\overset{1}{=}$ minus $a_1\times\overset{2}{=}$ yields

$$0=b_1(c_2a_2+c_3a_3)-a_1(c_2b_2+c_3b_3)=-c_2z+c_3y\qquad\text{or}\qquad c_2z\overset{3}{=}c_3y.$$

Similarly, $b_2\times\overset{1}{=}$ minus $a_2\times\overset{2}{=}$ and $b_3\times\overset{1}{=}$ minus $a_3\times\overset{2}{=}$ yield

$$c_1z\overset{4}{=}c_3x\qquad\text{and}\qquad c_1y\overset{5}{=}c_2x.$$

Since $\mathbf{d}\perp\mathbf{a},\mathbf{b}$, we have, similarly,

$$d_2z\overset{6}{=}d_3y,\qquad d_1z\overset{7}{=}d_3x,\quad\text{and}\quad d_1y\overset{8}{=}d_2x.$$

We will break down the remainder of the proof into four cases, depending on whether any of x, y, and z are zero. We will show that wherever no contradiction arises, $\mathbf{c}$ can be written as a non-zero scalar multiple of $\mathbf{d}$, so that $\mathbf{c}\parallel\mathbf{d}$.

(*Proof continues below ...*)

(... *Proof continued from above.*)

Case 1. All of x, y, and z are zero.

Then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, so that by Fact 146, $\mathbf{a} \parallel \mathbf{b}$, contradicting our assumption that $\mathbf{a} \nparallel \mathbf{b}$.

Case 2. Exactly two of x, y, and z are zero.

Suppose $x = y = 0$ and $z \neq 0$. Then from $\stackrel{3}{=}$, $\stackrel{4}{=}$, $\stackrel{6}{=}$, and $\stackrel{7}{=}$, we have $c_1 = c_2 = 0$ and $d_1 = d_2 = 0$. And now, $\mathbf{c}$ can be written as a non-zero scalar multiple of $\mathbf{d}$:

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} = \frac{c_3}{d_3} \begin{pmatrix} 0 \\ 0 \\ d_3 \end{pmatrix} = \frac{c_3}{d_3} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \frac{c_3}{d_3} \mathbf{d}.$$

Case 3. Exactly one of x, y, and z is zero.

Suppose $x = 0$ and $y, z \neq 0$. Then from $\stackrel{4}{=}$ and $\stackrel{7}{=}$, we have $c_1 = 0$ and $d_1 = 0$.

Note that $c_2 \neq 0$, because otherwise, from $\stackrel{3}{=}$, we have $c_3 = 0$ and now $\mathbf{c} = \mathbf{0}$, contradicting our assumption that $\mathbf{c}$ is non-zero. Similarly, $c_3, d_2, d_3 \neq 0$.

And now $\stackrel{3}{=}$ divided by $\stackrel{6}{=}$ yields: $$\frac{c_2}{d_2} = \frac{c_3}{d_3}.$$

And so again, $\mathbf{c}$ can be written as a non-zero scalar multiple of $\mathbf{d}$:

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 \\ c_3 \end{pmatrix} = \frac{c_3}{d_3} \begin{pmatrix} 0 \\ d_2 \\ d_3 \end{pmatrix} = \frac{c_3}{d_3} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \frac{c_3}{d_3} \mathbf{d}.$$

Case 4. None of x, y, and z is zero.

Then $\stackrel{3}{=}$ divided by $\stackrel{6}{=}$ yields: $$\frac{c_2}{d_2} = \frac{c_3}{d_3}.$$

Similarly, $\stackrel{4}{=}$ divided by $\stackrel{7}{=}$ yields: $$\frac{c_1}{d_1} = \frac{c_3}{d_3}.$$

Thus, $$\frac{c_1}{d_1} = \frac{c_2}{d_2} = \frac{c_3}{d_3}.$$

And now, $\mathbf{c}$ can be written as a non-zero scalar multiple of $\mathbf{d}$:

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{c_3}{d_3} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \frac{c_3}{d_3} \mathbf{d}. \qquad \square$$

144.9. Planes in General

The definition of a **plane**, reproduced:

Definition 175. A *plane* is any set of points that can be written as

$$\left\{R : \overrightarrow{OR} \cdot \mathbf{n} = d\right\} \qquad \text{or} \qquad \{R : \mathbf{r} \cdot \mathbf{n} = d\},$$

where $\mathbf{n}$ is some non-zero vector and $d \in \mathbb{R}$.

Fact 158. *Let A and B be distinct points and q be a plane. If q contains A and B, then it also contains all the points in the line AB.*

Proof. Suppose the points are A and B, so that the line AB may be described by $\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$ $(\lambda \in \mathbb{R})$.

Suppose the plane can be described by $\mathbf{r} \cdot \mathbf{n} = d$.

We will prove that any point on the line AB is also on the plane q. (We will thus have shown that q contains the line AB.) To do so, we need merely verify that the generic point $\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$ on the line AB satisfies q's vector equation:

$$\mathbf{r} = \left(\overrightarrow{OA} + \lambda\overrightarrow{AB}\right) \cdot \mathbf{n} \qquad = \left[\overrightarrow{OA} + \lambda\left(\overrightarrow{OB} - \overrightarrow{OA}\right)\right] \cdot \mathbf{n}$$
$$= \overrightarrow{OA} \cdot \mathbf{n} + \lambda\overrightarrow{OB} \cdot \mathbf{n} - \lambda\overrightarrow{OA} \cdot \mathbf{n} = d + \lambda d - \lambda d = d.$$

□

Fact 159. *If $q = \left\{R : \overrightarrow{OR} \cdot \mathbf{n} = d\right\}$ is a plane, then $\mathbf{n} \perp q$.*

Proof. Let $\mathbf{v}$ be any vector on q. Then there are points $A, B \in q$ such that $\mathbf{v} = \overrightarrow{AB}$. Since $A, B \in q$, we have $\overrightarrow{OA} \cdot \mathbf{n} = d$ and $\overrightarrow{OB} \cdot \mathbf{n} = d$.

So, $$\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \overrightarrow{AB} = \mathbf{n} \cdot \left(\overrightarrow{OB} - \overrightarrow{OA}\right) = \mathbf{n} \cdot \overrightarrow{OB} - \mathbf{n} \cdot \overrightarrow{OA} = d - d = 0.$$

We've just proven that given any vector $\mathbf{v}$ on q, we have $\mathbf{n} \cdot \mathbf{v} = 0$ or equivalently $\mathbf{n} \perp \mathbf{v}$. Hence, by Definition 177, $\mathbf{n}$ is a normal vector of q (i.e. $\mathbf{n} \perp q$). □

Fact 160. *Let q be a plane and $\mathbf{n}$ and $\mathbf{m}$ be vectors. Suppose $\mathbf{n} \perp q$. Then*

$$\mathbf{m} \parallel \mathbf{n} \implies \mathbf{m} \perp q.$$

Proof. By Definition 177, $\mathbf{n} \perp q \iff \mathbf{n} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ on q.

By Definition 138, $\mathbf{m} \parallel \mathbf{n} \iff \mathbf{m} = k\mathbf{n}$ for some $k \neq 0$.

And so, for any vector $\mathbf{v}$ on q, we have

$$\mathbf{m} \cdot \mathbf{v} = (k\mathbf{n}) \cdot \mathbf{v} = k(\mathbf{n} \cdot \mathbf{v}) = k \cdot 0 = 0.$$

Thus, by Definition 177, $\mathbf{m} \perp q$. □

Fact 163. *Let q be a plane and P and R be points. Suppose $P \in q$. Then*

$$R \in q \qquad \iff \qquad \text{The vector } \overrightarrow{PR} \text{ is on } q.$$

Proof. Let $\mathbf{n}$ be a normal vector of q. Since $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$, we have

$$\overrightarrow{OR} \cdot \mathbf{n} = \left(\overrightarrow{OP} + \overrightarrow{PR}\right) \cdot \mathbf{n} = \overrightarrow{OP} \cdot \mathbf{n} + \overrightarrow{PR} \cdot \mathbf{n}. \qquad ☺$$

And now,

$$\begin{array}{lll} & R \in q & \\ \iff & \overrightarrow{OR} \cdot \mathbf{n} = \overrightarrow{OP} \cdot \mathbf{n} & \text{(Definition 175)} \\ \iff & \overrightarrow{PR} \cdot \mathbf{n} = 0 & (☺) \\ \iff & \overrightarrow{PR} \perp \mathbf{n} & \text{(Definition 146)} \\ \iff & \overrightarrow{PR} \text{ is on } q. & \text{(Fact 162)} \end{array}$$ □

How to go back and forth between a plane's vector and cartesian forms:

Fact 276. *If $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ is a non-zero vector, then*

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix} = d \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} : \sum_{i=1}^{k} n_i x_i = d \right\}.$$

Proof. By Definition 311, $(x_1, x_2, \ldots, x_k) \cdot (n_1, n_2, \ldots, n_k) = \sum_{i=1}^{k} n_i x_i$. □

Remark 209. Note that Definition 175 actually serves as the general definition of the $k-1$-dimensional **hyperplane** in $\mathbb{R}^k$. In general, the $k-1$-dimensional hyperplane in $\mathbb{R}^k$ has cartesian equation $\sum_{i=1}^{k} n_i x_i = d$, where $\mathbf{n} = (n_1, n_2, \ldots, n_k)$.

And so the hyperplane in $\mathbb{R}^3$ is the "flat" two-dimensional plane with cartesian equation $ax + by + cz = d$. While the hyperplane in $\mathbb{R}^2$ is the one-dimensional line with cartesian equation $ax + by = c$.

Here is the general Definition of a two-dimensional plane in $\mathbb{R}^n$:

Definition 314. A *two-dimensional plane* in $\mathbb{R}^n$ is any set that can be written as

$$\left\{ R : \overrightarrow{OR} = \overrightarrow{OA} + \lambda \mathbf{u} + \mu \mathbf{v} \quad (\lambda, \mu \in \mathbb{R}) \right\},$$

for some point A and some non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$.

Fact 277. *In 3D space, Definitions 175 and 314 are equivalent.*

Proof. Corollary 33 shows that Definition 175 implies Definition 314.

We now show the converse. Let $q = \left\{R : \overrightarrow{OR} = \overrightarrow{OA} + \lambda\mathbf{u} + \mu\mathbf{v} \quad (\lambda, \mu \in \mathbb{R})\right\}$, for some point A and some non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$.

Let $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ (which is non-zero because $\mathbf{u} \nparallel \mathbf{v}$), $d = \overrightarrow{OA} \cdot \mathbf{n}$, and $s = \left\{R : \overrightarrow{OR} \cdot \mathbf{n} = d\right\}$. We will show that **(a)** $P \in q \implies P \in s$; and then **(b)** $P \in s \implies P \in q$. We will thus have shown that $s = q$.

(a) Suppose $P \in q$, i.e. $\overrightarrow{OP} = \overrightarrow{OA} + \alpha\mathbf{u} + \beta\mathbf{v}$ for some $\alpha, \beta \in \mathbb{R}$. Then

$$\overrightarrow{OP} \cdot \mathbf{n} = \overrightarrow{OA} \cdot \mathbf{n} + \alpha\mathbf{u} \cdot \mathbf{n} + \beta\mathbf{v} \cdot \mathbf{n} = d + 0 + 0 = d.$$

So, $P \in s$.

(b) Now suppose $P \in s$, i.e. $\overrightarrow{OP} \cdot \mathbf{n} = d$. Rearranging, $\overrightarrow{OP} \cdot \mathbf{n} = \overrightarrow{OA} \cdot \mathbf{n}$ or $\overrightarrow{AP} \cdot \mathbf{n} = 0$. And so by Corollary 32, $\overrightarrow{AP}$ is on q. By Theorem 19 then, $\overrightarrow{AP}$ can be written as the linear combination of $\mathbf{u}$ and $\mathbf{v}$. That is, there exist real numbers α and β such that $\overrightarrow{AP} = \alpha\mathbf{u} + \beta\mathbf{v}$. Rearranging, $\overrightarrow{OP} = \overrightarrow{OA} + \alpha\mathbf{u} + \beta\mathbf{v}$. So, $P \in q$. □

Fact 162. *Suppose q is a plane with normal vector $\mathbf{n}$. Then*

$$\mathbf{v} \perp \mathbf{n} \qquad \implies \qquad \mathbf{v} \textit{ is on } q.$$

Proof. Let $\mathbf{n} = (n_1, n_2, \ldots, n_k) \perp q$ and suppose $\mathbf{v} \perp \mathbf{n}$. Our goal is to show that $\mathbf{v}$ is on q.

Suppose q is described by $\mathbf{r} \cdot \mathbf{n} \stackrel{1}{=} d$. Since $\mathbf{n} \neq \mathbf{0}$, pick any $n_i \neq 0$. Let P be the point whose ith coordinate is d/n_i and other coordinates are 0. Then $P \in q$ because

$$\overrightarrow{OP} \cdot \mathbf{n} = \frac{d}{n_i} n_i + \sum_{j \neq i} 0 \cdot n_j \stackrel{1}{=} d.$$

Next, let $Q = P + \mathbf{v}$. Then we also have $Q \in q$ because

$$\overrightarrow{OQ} \cdot \mathbf{n} = \left(\overrightarrow{OP} + \mathbf{v}\right) \cdot \mathbf{n} = \overrightarrow{OP} \cdot \mathbf{n} + \mathbf{v} \cdot \mathbf{n} = d + 0 \stackrel{1}{=} d.$$

Since $P, Q \in q$ and $\mathbf{v} = \overrightarrow{PQ}$, $\mathbf{v}$ is a vector on the plane. □

Fact 278. *Let $\mathbf{m}$ be a vector and q be a plane. Suppose $\mathbf{m} \perp q$. Then there exists $e \in \mathbb{R}$ such that $\overrightarrow{OR} \cdot \mathbf{m} = e$ for all $R \in q$.*

Proof. Let $A, R \in q$ and $e = \overrightarrow{OA} \cdot \mathbf{m}$. On the one hand, $\overrightarrow{AR} \cdot \mathbf{m} = 0$ (because $\mathbf{m} \perp q$). On the other, $\overrightarrow{AR} \cdot \mathbf{m} = \left(\overrightarrow{OR} - \overrightarrow{OA}\right) \cdot \mathbf{m} = \overrightarrow{OR} \cdot \mathbf{m} - \overrightarrow{OA} \cdot \mathbf{m}$. Thus, $\overrightarrow{OR} \cdot \mathbf{m} = \overrightarrow{OA} \cdot \mathbf{m} = e$. □

Suppose $\mathbf{n} \perp q$. By Fact 160, $\mathbf{m} \parallel \mathbf{n} \implies \mathbf{m} \perp q$. The converse is also true:[590]

Theorem 18. *Let q be a plane and $\mathbf{n}$ and $\mathbf{m}$ be vectors. Suppose $\mathbf{n} \perp q$. Then*

$$\mathbf{m} \perp q \quad \implies \quad \mathbf{m} \parallel \mathbf{n}.$$

Proof. Let $q = \left\{R : \overrightarrow{OR} \cdot \mathbf{n} = d\right\}$, $\mathbf{n} = (n_1, n_2, \ldots, n_k)$, and $\mathbf{m} = (m_1, m_2, \ldots, m_k)$. Suppose $\mathbf{m} \perp q$. By Fact 278, there exists some $e \in \mathbb{R}$ such that for all $R \in q$, $\overrightarrow{OR} \cdot \mathbf{m} \stackrel{1}{=} e$.

We will use Lemmata 6 and 7 to prove Theorem 18:

Lemma 6. $n_i = 0 \iff m_i = 0$.

Proof of Lemma 6. Suppose $n_i = 0$. Since $\mathbf{n} \neq \mathbf{0}$, there exists some j for which $n_j \neq 0$.

Let $w \in \mathbb{R}$. Let S_w be the point whose ith coordinate is w, jth coordinate is d/n_j, and other coordinates are 0. Then $S_w \in q$ because

$$\overrightarrow{OS_w} \cdot \mathbf{n} = wn_i + \frac{d}{n_j}n_j + \sum_{l \notin \{i,j\}} 0 \cdot n_l = 0 + d + 0 = d.$$

Since $S_w \in q$, we have $\overrightarrow{OS_w} \cdot \mathbf{m} \stackrel{1}{=} e$ or

$$\overrightarrow{OS_w} \cdot \mathbf{m} = wm_i + \frac{d}{n_j}m_j + \sum_{l \notin \{i,j\}} 0 \cdot m_l = wm_i + \frac{d}{n_j}m_j + 0 = wm_i + \frac{d}{n_j}m_j \stackrel{2}{=} e.$$

Since $\stackrel{2}{=}$ holds for all $w \in \mathbb{R}$, it must be that $m_i = 0$.

The proof that $m_i = 0 \implies n_i = 0$ is similar and thus omitted. □

Lemma 7. $n_i \neq 0 \implies m_i d/n_i = e$.

Proof of Lemma 7. Suppose $n_i \neq 0$. Let Q be the point whose ith coordinate is d/n_i and other coordinates are 0. Then $Q \in q$ because

$$\overrightarrow{OQ} \cdot \mathbf{n} = \frac{d}{n_i}n_i + \sum_{l \neq i} 0 \cdot n_l = d + 0 = d.$$

Since $Q \in q$, $\overrightarrow{OQ} \cdot \mathbf{m} \stackrel{1}{=} e$ or $\overrightarrow{OQ} \cdot \mathbf{m} = \frac{d}{n_i}m_i + \sum_{l \neq i} 0 \cdot m_l = \frac{d}{n_i}m_i + 0 = \frac{d}{n_i}m_i = e.$ □

We've completed our proofs of Lemmata 6 and 7. On the next page, we continue with our proof of Theorem 18.

(*Proof continues below …*)

[590]Note that this is really just a restatement of a fundamental result from linear algebra. The proof is rather long, but uses only material and language we've already covered in this textbook.

(... *Proof continued from above.*)

We will show that whether **(a)** $d \neq 0$; or **(b)** $d = 0$, we have $\mathbf{m} \parallel \mathbf{n}$.

(a) Suppose $d \neq 0$. By Lemma 6, if $n_i = 0$, then $m_i = 0$, so that $m_i = n_i (e/d)$.

And by Lemma 7, if $n_i \neq 0$, then $m_i = n_i (e/d)$.

Hence, for all $i \in \{1, 2, \ldots, k\}$, $m_i = n_i (e/d)$. Thus, we may write $\mathbf{m} \stackrel{4}{=} \dfrac{e}{d}\mathbf{n}$.

Since $\mathbf{m} \neq 0$, it must be that $e \neq 0$. So, $\stackrel{4}{=}$ shows that $\mathbf{m} \parallel \mathbf{n}$.

(b) Suppose $d = 0$. Since $\mathbf{n} \neq \mathbf{0}$, there is some j for which $n_j \neq 0$. Write $m_j \stackrel{5}{=} \dfrac{m_j}{n_j} n_j$.

By Lemma 6, $n_j \neq 0 \implies m_j \neq 0$.

By Lemma 7, $m_j d/n_j = e$. Since $d = 0$, we have $e \stackrel{6}{=} 0$.

By Lemma 6, for i such that $n_i = 0$, we have $m_i = 0$. And so, for such i, $m_i \stackrel{7}{=} \dfrac{m_j}{n_j} n_i$.

Now consider any $s \neq j$ such that $n_s \neq 0$. Let T be the point whose sth coordinate is n_j, jth coordinate is $-n_s$, and other coordinates are 0. Then $T \in q$ because

$$\overrightarrow{OT} \cdot \mathbf{n} = n_j n_s + (-n_s) n_j + \sum_{l \notin \{s,j\}} 0 \cdot n_l = n_j n_s - n_s n_j + 0 = 0.$$

Since $T \in q$, we have $\overrightarrow{OT} \cdot \mathbf{m} \stackrel{1}{=} e \stackrel{6}{=} 0$ or

$$\overrightarrow{OT} \cdot \mathbf{m} = n_j m_s + (-n_s) m_j + \sum_{l \notin \{i,j\}} 0 \cdot m_l = n_j m_s - n_s m_j + 0 \stackrel{8}{=} 0.$$

Thus, for any $s \neq j$ such that $n_s \neq 0$, we can rearrange $\stackrel{8}{=}$ to write $m_s \stackrel{9}{=} \dfrac{m_j}{n_j} n_s$.

Altogether then, $\stackrel{5}{=}$, $\stackrel{7}{=}$, and $\stackrel{9}{=}$ show that for all $i \in \{1, 2, \ldots, k\}$, we have $m_i = (m_j/n_j) n_i$.

Hence,
$$\mathbf{m} \stackrel{10}{=} \frac{m_j}{n_j}\mathbf{n}.$$

Since $m_j \neq 0$, $\stackrel{10}{=}$ proves that $\mathbf{m} \parallel \mathbf{n}$. □

Fact 279. *The unique plane that contains the point A and has normal vector $\mathbf{n}$ is*

$$\left\{R : \overrightarrow{OR} \cdot \mathbf{n} = \overrightarrow{OA} \cdot \mathbf{n}\right\}.$$

Proof. The given plane contains A, because $\overrightarrow{OA} \cdot \mathbf{n} = \overrightarrow{OA} \cdot \mathbf{n}$.

Let $\mathbf{v}$ be a vector on the plane. Then there exist points P and Q on the plane such that $\mathbf{v} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$. Now, $\mathbf{v} \cdot \mathbf{n} = \left(\overrightarrow{OQ} - \overrightarrow{OP}\right) \cdot \mathbf{n} = \overrightarrow{OQ} \cdot \mathbf{n} - \overrightarrow{OP} \cdot \mathbf{n} = \overrightarrow{OA} \cdot \mathbf{n} - \overrightarrow{OA} \cdot \mathbf{n} = 0$. Thus, $\mathbf{v} \perp \mathbf{n}$. We've just shown that the given plane has normal vector $\mathbf{n}$.

We now prove uniqueness. Suppose the plane $\left\{R : \overrightarrow{OR} \cdot \mathbf{m} = d\right\}$ contains A and has normal vector $\mathbf{n}$. Then by Theorem 18, $\mathbf{m} = k\mathbf{n}$ for some $k \neq 0$. So, $d = \overrightarrow{OA} \cdot \mathbf{m} = \overrightarrow{OA} \cdot (k\mathbf{n}) = k\overrightarrow{OA} \cdot \mathbf{n}$.

And now, $\left\{R : \overrightarrow{OR} \cdot \mathbf{m} = d\right\} = \left\{R : \overrightarrow{OR} \cdot k\mathbf{n} = k\overrightarrow{OA} \cdot \mathbf{n}\right\} = \left\{R : \overrightarrow{OR} \cdot \mathbf{n} = \overrightarrow{OA} \cdot \mathbf{n}\right\}$. □

How to go back and forth between a (hyper)plane's cartesian and parametric forms:

Fact 280. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ be a vector. Let $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ be a non-zero vector. Without loss of generality, suppose $n_1 \neq 0$. Let $\mathbf{v}_1 = (d/n_1, 0, 0, \ldots, 0)$. And for each $i \in \{2, \ldots, k\}$, let $\mathbf{v}_i$ be the vector whose 1st coordinate is $-n_i/n_1$, ith coordinate is 1, and remaining coordinates are 0. Suppose*

$$S = \left\{\mathbf{x} : \sum_{i=1}^{k} n_i x_i = d\right\} \qquad \text{and} \qquad T = \left\{\mathbf{x} : \mathbf{x} = \mathbf{v}_1 + \sum_{i=2}^{k} \lambda_i \mathbf{v}_i \quad (\lambda_2, \ldots, \lambda_k \in \mathbb{R})\right\}.$$

Then $S = T$.

Proof. We will show that **(a)** $\mathbf{a} \in S$ implies $\mathbf{a} \in T$; and **(b)** $\mathbf{a} \in T$ implies $\mathbf{a} \in S$.

(a) Suppose $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in S$. Then $\sum_{i=1}^{k} n_i a_i = d$ or $a_1 = \left(d - \sum_{i=2}^{k} a_i n_i\right)/n_1$.

For each $i = 2, 3, \ldots, k$, let $\lambda_i = a_i$. Then we have

$$\mathbf{v}_1 + \sum_{i=2}^{k} \lambda_i \mathbf{v}_i = \left(\frac{d}{n_1} - \sum_{i=2}^{k} a_i \frac{n_i}{n_1}, a_2, a_3, \ldots, a_k\right) = \mathbf{a}.$$

We've just shown that $\mathbf{a} \in T$.

(b) Now suppose $\mathbf{a} \in T$. Then $\mathbf{a} = \left(\frac{d}{n_1} - \sum_{i=2}^{k} \lambda_i \frac{n_i}{n_1}, -\lambda_2, -\lambda_3, \ldots, -\lambda_k\right)$ for some $\lambda_2, \ldots, \lambda_k \in \mathbb{R}$.

And now, $$\sum_{i=1}^{k} n_i a_i = n_1 \left(\frac{d}{n_1} - \sum_{i=2}^{k} \lambda_i \frac{n_i}{n_1}\right) + \sum_{i=2}^{k} n_i \lambda_i = d.$$

We've just shown that $\mathbf{a} \in S$. □

The results given in this subchapter were general. In contrast, the results in the next subchapter will apply only to planes in $\mathbb{R}^3$.

144.10. Planes in Three-Dimensional Space

Fact 164. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are non-parallel vectors on a plane* q*, then* $\mathbf{a} \times \mathbf{b} \perp q$.

Proof. Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{n} = (n_1, n_2, n_3) \perp q$, so that $\mathbf{a}, \mathbf{b} \perp \mathbf{n}$, or

$$a_1 n_1 + a_2 n_2 + a_3 n_3 \overset{1}{=} 0 \qquad \text{and} \qquad b_1 n_1 + b_2 n_2 + b_3 n_3 \overset{2}{=} 0.$$

Now, $b_2 \times \overset{1}{=}$ minus $a_2 \times \overset{2}{=}$ yields

$$0 = b_2 (a_1 n_1 + a_3 n_3) - a_2 (b_1 n_1 + b_3 n_3) = (a_1 b_2 - a_2 b_1) n_1 - (a_2 b_3 - a_3 b_2) n_3.$$

Similarly, $b_1 \times \overset{1}{=}$ minus $a_1 \times \overset{2}{=}$ and $b_3 \times \overset{1}{=}$ minus $a_3 \times \overset{2}{=}$ yield

$$(a_3 b_1 - a_1 b_3) n_3 - (a_1 b_2 - a_2 b_1) n_2 = 0 \qquad \text{and} \qquad (a_2 b_3 - a_3 b_2) n_2 - (a_3 b_1 - a_1 b_3) n_1 = 0.$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{n} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \times \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} (a_3 b_1 - a_1 b_3) n_3 - (a_1 b_2 - a_2 b_1) n_2 \\ (a_1 b_2 - a_2 b_1) n_1 - (a_2 b_3 - a_3 b_2) n_3 \\ (a_2 b_3 - a_3 b_2) n_2 - (a_3 b_1 - a_1 b_3) n_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

Since $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$, $\mathbf{n} \neq \mathbf{0}$, and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{n} = \mathbf{0}$, by Fact 146, $\mathbf{a} \times \mathbf{b} \parallel \mathbf{n}$. □

Theorem 19. *Let* q *be a plane and* $\mathbf{a}$ *and* $\mathbf{b}$ *be non-parallel vectors on* q*. Suppose* $\mathbf{c}$ *is a non-zero vector. Then*

$$\mathbf{c} \text{ is a vector on } q \quad \Longleftrightarrow \quad \text{There exist } \lambda, \mu \in \mathbb{R} \text{ such that } \mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b}.$$

Proof. In the main text, we already proved $\Longleftarrow$. Here[591] we prove $\Longrightarrow$.

Let $\mathbf{n} = (n_1, n_2, n_3)$ be the plane's normal vector. Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$, so that $\mathbf{a} \cdot \mathbf{n} = 0$, $\mathbf{b} \cdot \mathbf{n} = 0$, $\mathbf{c} \cdot \mathbf{n} = 0$, and also $\mathbf{a} \neq k\mathbf{b}$ for all $k \neq 0$.

Write $\quad a_1 n_1 + a_2 n_2 + a_3 n_3 \overset{1}{=} 0, \quad b_1 n_1 + b_2 n_2 + b_3 n_3 \overset{2}{=} 0, \quad$ and $\quad c_1 n_1 + c_2 n_2 + c_3 n_3 \overset{3}{=} 0.$

Since $\mathbf{n} \neq 0$, suppose WLOG that $n_3 \neq 0$. Now rewrite $\overset{1}{=}$, $\overset{2}{=}$, and $\overset{3}{=}$ as

$$a_3 \overset{1}{=} -\frac{a_1 n_1 + a_2 n_2}{n_3}, \qquad b_3 \overset{2}{=} -\frac{b_1 n_1 + b_2 n_2}{n_3}, \qquad \text{and} \qquad c_3 \overset{3}{=} -\frac{c_1 n_1 + c_2 n_2}{n_3}.$$

We will use Lemmata 8 and 9 to prove Theorem 19:

Lemma 8. **(a)** *a_1 and a_2 are not both zero.* **(b)** *b_1 and b_2 are not both zero.*

Proof of Lemma 8. **(a)** If $a_1, a_2 = 0$, then $a_3 \neq 0$ (because $\mathbf{a} \neq \mathbf{0}$) and $a_1 n_1 + a_2 n_2 + a_3 n_3 = 0 + 0 + a_3 n_3 \neq 0$, contradicting $\overset{1}{=}$. The proof of **(b)** is similar. □

(*Proof continues below ...*)

[591]Note that again, this long proof is just a fundamental result from linear algebra (applied to the 3D case), but written using only material and language we've introduced in this textbook.

(... *Proof continued from above.*)

Lemma 9. $a_1b_2 - a_2b_1 \neq 0$.

Proof of Lemma 9. Suppose for contradiction that $a_1b_2 - a_2b_1 \stackrel{4}{=} 0$. We will show that whether **(a)** $a_1 \neq 0$ or **(b)** $a_1 = 0$, a contradiction arises and hence $a_1b_2 - a_2b_1 \neq 0$.

(a) If $a_1 \neq 0$, then rearranging $\stackrel{4}{=}$, we have $b_2 \stackrel{5}{=} a_2b_1/a_1$. If $b_1 = 0$, then $b_2 = 0$, but this contradicts Lemma 8. So, $b_1 \neq 0$.

We now show that $\mathbf{a} = k\mathbf{b}$ for $k \neq 0$, contradicting our assumption that $\mathbf{a} \nparallel \mathbf{b}$:

$$\mathbf{a} = (a_1, a_2, a_3) \stackrel{1}{=} \left(a_1, a_2, -\frac{a_1n_1 + a_2n_2}{n_3}\right) = \frac{a_1}{b_1}\left(b_1, \frac{a_2b_1}{a_1}, -\frac{b_1n_1 + (a_2b_1/a_1)\,n_2}{n_3}\right)$$

$$\stackrel{5}{=} \frac{a_1}{b_1}\left(b_1, b_2, -\frac{b_1n_1 + b_2}{n_3}\right) \stackrel{2}{=} \frac{a_1}{b_1}(b_1, b_2, b_3) = \frac{a_1}{b_1}\mathbf{b}.$$

(b) If $a_1 = 0$, then by Lemma 8, $a_2 \neq 0$ and the same contradictions as in **(a)** arise. □

The proofs of Lemmata 8 and 9 are complete. We now resume our proof of Theorem 19.

Pick: $\mu \stackrel{6}{=} \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$ and $\lambda \stackrel{7}{=} \begin{cases} (c_1 - \mu b_1)/a_1 & \text{if } a_1 \neq 0 \\ (c_2 - \mu b_2)/a_2 & \text{if } a_1 = 0. \end{cases}$

We now verify that $\lambda\mathbf{a} + \mu\mathbf{b} = \mathbf{c}$, or equivalently, that

$$\lambda a_1 + \mu b_1 \stackrel{8}{=} c_1, \qquad \lambda a_2 + \mu b_2 \stackrel{9}{=} c_2, \quad \text{and} \quad \lambda a_3 + \mu b_3 \stackrel{10}{=} c_3.$$

We now show that if $\stackrel{8}{=}$ and $\stackrel{9}{=}$ hold, then so too does $\stackrel{10}{=}$:

$$\lambda a_3 + \mu b_3 \stackrel{1,2}{=} -\lambda\frac{a_1n_1 + a_2n_2}{n_3} - \mu\frac{b_1n_1 + b_2n_2}{n_3}$$

$$= -\frac{(\lambda a_1 + \mu b_1)\,n_1 + (\lambda a_2 + \mu b_2)\,n_2}{n_3} \stackrel{8,9}{=} -\frac{c_1n_1 + c_2n_2}{n_3} \stackrel{10}{=} c_3.$$

It thus suffices to show that $\stackrel{8}{=}$ and $\stackrel{9}{=}$ hold. And we now do so, in each of two cases:

(i) Suppose $a_1 \neq 0$. Then $\stackrel{8}{=}$ holds: $\lambda a_1 + \mu b_1 \stackrel{7}{=} c_1 - \mu b_1 + \mu b_1 \stackrel{8}{=} c_1$. And so too does $\stackrel{9}{=}$:

$$\lambda a_2 + \mu b_2 \stackrel{7}{=} (c_1 - \mu b_1)\frac{a_2}{a_1} + \mu b_2 = \frac{a_2c_1}{a_1} + \mu\frac{a_1b_2 - a_2b_1}{a_1} \stackrel{6}{=} \frac{a_2c_1}{a_1} + \frac{a_1c_2 - a_2c_1}{a_1} \stackrel{9}{=} c_2.$$

(ii) Suppose $a_1 = 0$. Then $\stackrel{9}{=}$ holds: $\lambda a_2 + \mu b_2 \stackrel{7}{=} c_2 - \mu b_2 + \mu b_2 \stackrel{9}{=} c_2$. And so too does $\stackrel{8}{=}$:

$$\lambda a_1 + \mu b_1 = (c_2 - \mu b_2)\frac{a_1}{a_2} + \mu b_1 = \frac{a_1c_2}{a_2} + \mu\frac{a_2b_1 - a_1b_2}{a_2} = \frac{a_1c_2}{a_2} + \frac{a_2c_1 - a_1c_2}{a_2} \stackrel{8}{=} c_1.$$ □

144.11. The Relationship Between a Line and a Plane

Fact 171. *Given a line and a plane, exactly one of the three following possibilities holds: The line and plane are*

(a) *Parallel and do not intersect at all; or*

(b) *Parallel and the line lies entirely on the plane; or*

(c) *Non-parallel and intersect at exactly one point.*

Proof. Describe the line l and plane q by

$$\mathbf{r} \stackrel{1}{=} \mathbf{p} + \lambda \mathbf{v} \qquad \text{and} \qquad \mathbf{r} \cdot \mathbf{n} \stackrel{2}{=} d.$$

To find any points at which l and q intersect, plug $\stackrel{1}{=}$ into $\stackrel{2}{=}$ to get

$$(\mathbf{p} + \lambda \mathbf{v}) \cdot \mathbf{n} = d \qquad \text{or} \qquad \mathbf{p} \cdot \mathbf{n} + \lambda \mathbf{v} \cdot \mathbf{n} = d \qquad \text{or} \quad \lambda \mathbf{v} \cdot \mathbf{n} \stackrel{3}{=} d - \mathbf{p} \cdot \mathbf{n}.$$

Thus, the intersection points of l and q correspond to the values of λ for which $\stackrel{3}{=}$ holds.

Suppose $l \parallel q$. Then by Fact 170, $\mathbf{v} \cdot \mathbf{n} = 0$ and $\stackrel{3}{=}$ becomes $\mathbf{p} \cdot \mathbf{n} = d$.

(a) If $\mathbf{p} \cdot \mathbf{n} \neq d$, then l and q do not intersect at any value of λ. So, l and q do not intersect.

(b) If $\mathbf{p} \cdot \mathbf{n} = d$, then l and q intersect at *all* values of λ. So, l lies completely on q.

(c) Now suppose instead that $l \nparallel q$. Then by Fact 170, $\mathbf{v} \cdot \mathbf{n} \neq 0$.

And so, we can rearrange $\stackrel{3}{=}$ to get $\quad \lambda = \dfrac{d - \mathbf{p} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}.$

This shows that there is only one value of λ at which the line and plane intersect. And this unique intersection point is given by

$$\mathbf{p} + \lambda \mathbf{v} = \mathbf{p} + \frac{d - \mathbf{p} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v}. \qquad \square$$

144.12. The Relationship Between Two Planes

Fact 173. *Suppose q and r are planes with normal vectors $\mathbf{u}$ and $\mathbf{v}$. Then*

$$\text{(a) } q \parallel r \iff \mathbf{u} \parallel \mathbf{v}; \qquad \textit{and} \qquad \text{(b) } q \perp r \iff \mathbf{u} \perp \mathbf{v}.$$

Proof. Let θ be the angle between q and r. By Definitions 181 and Facts 172 and 135,

(a) $$q \parallel r \iff \theta = \cos^{-1} \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} = 0 \iff \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} = \cos 0 = 1 \iff \mathbf{u} \parallel \mathbf{v}.$$

(b) $$q \perp r \iff \theta = \cos^{-1} \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{\pi}{2} \iff \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|} = \cos \frac{\pi}{2} = 0 \iff \mathbf{u} \cdot \mathbf{v} = 0 \iff \mathbf{u} \perp \mathbf{v}.$$ □

Fact 174. *If two planes are parallel, then they are either identical or do not intersect.*

Proof. Suppose two planes are parallel. Then they share some normal vector $\mathbf{n}$.

Suppose they are described by $\overrightarrow{OR} \cdot \mathbf{n} = d_1$ and $\overrightarrow{OR} \cdot \mathbf{n} = d_2$. If $d_1 = d_2$, then they are identical.

So suppose $d_1 \neq d_2$. If the point P is on the first plane, then $\overrightarrow{OP} \cdot \mathbf{n} = d_1 \neq d_2$, so that P is not on the second plane. Thus, the two planes do not intersect. □

Lemma 10. *Let $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ be vectors. If $\mathbf{n} \nparallel \mathbf{m}$, then there are i and j such that $n_i m_j - n_j m_i \neq 0$.*

Proof. Suppose for contradiction that $n_i m_j - n_j m_i \stackrel{1}{=} 0$ for all i, j.

Pick any s such that $n_s \neq 0$. Then by $\stackrel{1}{=}$, we have $n_i m_s - n_s m_i \stackrel{1}{=} 0$ for all i. Rearranging, $(m_s / n_s)\, n_i \stackrel{1}{=} m_i$ for all i. Thus, $\mathbf{m} = (m_s / n_s)\, \mathbf{n}$, contradicting $\mathbf{n} \nparallel \mathbf{m}$. □

Fact 175. *If two planes are not parallel, then they must intersect.*

Proof. Let the two planes be described by $\overrightarrow{OR} \cdot \mathbf{n} = d$ and $\overrightarrow{OR} \cdot \mathbf{m} = e$, where $\mathbf{n} = (n_1, n_2, \dots, n_k)$, $\mathbf{m} = (m_1, m_2, \dots, m_k)$, and $\mathbf{n} \nparallel \mathbf{m}$. By Lemma 10, there exist i and j such that $n_i m_j - n_j m_i \neq 0$. And since $n_i m_j - n_j m_i \neq 0$, at least one of n_i or n_j must be non-zero.

Suppose without loss of generality that $n_i \neq 0$. Let $P = (p_1, p_2, \dots, p_k)$ be the point with

$$p_j = \frac{en_i - dm_i}{n_i m_j - n_j m_i}, \qquad p_i = \frac{d - p_j n_j}{n_i}, \quad \text{and} \quad p_l = 0 \text{ for all } l \notin \{i, j\}.$$

We now verify that both planes contain the point P:

$$\begin{aligned}
\overrightarrow{OP} \cdot \mathbf{n} &= \sum p_l n_l = p_i n_i + p_j n_j + \sum_{l \notin \{i,j\}} p_l n_l \\
&= \frac{d - p_j n_j}{n_i} n_i + p_j n_j + 0 = d - p_j n_j + p_j n_j = d, \qquad \checkmark \\
\overrightarrow{OP} \cdot \mathbf{m} &= \sum p_l m_l = p_i m_i + p_j m_j + \sum_{l \notin \{i,j\}} p_l m_l = \frac{d - p_j n_j}{n_i} m_i + p_j m_j + 0 \\
&= \frac{dm_i + p_j (n_i m_j - n_j m_i)}{n_i} = \frac{dm_i + en_i - dm_i}{n_i} = e. \qquad \checkmark
\end{aligned}$$

□

Fact 176. *Suppose two non-parallel planes have normal vectors* $\mathbf{n}$ *and* $\mathbf{m}$*. Then their intersection is a line with direction vector* $\mathbf{n} \times \mathbf{m}$*.*

Proof. Here (Appendices) we'll actually go a little further by fully specifying the line along which the two planes intersect.

Let the two planes q_1 and q_2 be described by $\overrightarrow{OR} \cdot \mathbf{n} = d$ and $\overrightarrow{OR} \cdot \mathbf{m} = e$. Let P be the point constructed in the proof of Fact 175.

Then, we claim, q_1 and q_2 intersect at the line described by

$$\mathbf{r} = \overrightarrow{OP} + \lambda \mathbf{n} \times \mathbf{m} \qquad (\lambda \in \mathbb{R}).$$

To prove this claim, we first verify that q_1 and q_2 contain the above line. To do so, plug the generic point of the above line into each plane's vector equation:

$$\begin{aligned}
\left(\overrightarrow{OP} + \lambda \mathbf{n} \times \mathbf{m}\right) \cdot \mathbf{n} &= \overrightarrow{OP} \cdot \mathbf{n} + (\lambda \mathbf{n} \times \mathbf{m}) \cdot \mathbf{n} \ = d + 0 = d, \qquad \checkmark \\
\left(\overrightarrow{OP} + \lambda \mathbf{n} \times \mathbf{m}\right) \cdot \mathbf{m} &= \overrightarrow{OP} \cdot \mathbf{m} + (\lambda \mathbf{n} \times \mathbf{m}) \cdot \mathbf{m} = d + 0 = d. \qquad \checkmark
\end{aligned}$$

Next, let $S \in q_1 \cap q_2$ with $S \neq P$. We will prove that S is on the given line.

Since $P, S \in q_1 \cap q_2$, we have $\overrightarrow{PS} \perp \mathbf{n}, \mathbf{m}$. And so by Fact 157, $\overrightarrow{PS} \parallel \mathbf{n} \times \mathbf{m}$. That is,

$$\overrightarrow{PS} = \lambda \mathbf{n} \times \mathbf{m} \text{ for some } \lambda \in \mathbb{R}.$$

Rearranging, we have $\overrightarrow{OS} = \overrightarrow{OP} + \lambda \mathbf{n} \times \mathbf{m}$. Hence, S is on the given line. □

144.13. Distances

As mentioned in Remark 209, in 2D space, the hyperplane $\mathbf{r} \cdot \mathbf{n} = d$ describes a line. So, Fact 178, which applies generally to n-dimensional space, can actually also be applied to 2D space to prove the following two results, which are now reproduced from our Appendices for Part I (Functions and Graphs):

Proposition 3. *The unique point on the line $ax + by + c = 0$ that is closest to the point (p, q) is*

$$B = \left(p - a\frac{ap + bq + c}{a^2 + b^2}, q - b\frac{ap + bq + c}{a^2 + b^2} \right).$$

Proof. Replace $\mathbf{n}$, d, and A in Fact 178 with (a, b), $-c$, and (p, q) to get

$$k = \frac{d - \overrightarrow{OA} \cdot \mathbf{n}}{|\mathbf{n}|^2} = \frac{-c - (p, q) \cdot (a, b)}{a^2 + b^2} = \frac{-c - ap - bq}{a^2 + b^2} = -\frac{ap + bq + c}{a^2 + b^2}.$$

So, by Facts 178 and **??**, the point on the given line that's closest to the given point is

$$B = A + k\mathbf{n} = (p, q) - \frac{ap + bq + c}{a^2 + b^2}(a, b) = \left(p - a\frac{ap + bq + c}{a^2 + b^2}, q - b\frac{ap + bq + c}{a^2 + b^2} \right). \qquad \square$$

Corollary 3. *The distance between a point (p, q) and a line $ax + by + c = 0$ is*

$$\frac{|ap + bq + c|}{\sqrt{a^2 + b^2}}.$$

Proof. Continue with the above proof and apply Fact 178:

$$\left| \overrightarrow{AB} \right| = |k| \, |\mathbf{n}| = \left| -\frac{ap + bq + c}{a^2 + b^2} \right| \sqrt{a^2 + b^2} = \frac{|ap + bq + c|}{\sqrt{a^2 + b^2}}. \qquad \square$$

Example 1578. Let $A = (1,2,3)$ be a point, q be the plane $\mathbf{r} \cdot \mathbf{n} = \mathbf{r} \cdot (1,1,1) = 3$, and B be the foot of the perpendicular from A to q. Find B and $\left|\overrightarrow{AB}\right|$ (the distance between A and q).

In Ch. 77, we already showed that $B = (0,1,2)$ and $\left|\overrightarrow{AB}\right| = \sqrt{3}$. We'll now show this again using calculus.

First, write q into parametric form: $R = (0,0,3) + \lambda(1,-1,0) + \mu(0,1,-1) \qquad (\lambda, \mu \in \mathbb{R})$.

The distance between A and any arbitrary point $R \in q$ is

$$\left|\overrightarrow{AR}\right| = \sqrt{(\lambda-1)^2 + (-\lambda+\mu-2)^2 + (-\mu)^2} \stackrel{1}{=} \sqrt{2\lambda^2 + 2\mu^2 + 2\lambda - 4\mu - 2\mu\lambda + 5}.$$

The values of λ and μ that minimise $\left|\overrightarrow{AR}\right|$ correspond to the point B. Our goal then is to find these values. We'll do so using calculus—this will be very similar to what we did in Chs. 62 and 69, one difference being that we'll take **two** derivatives w.r.t. λ and μ.

Another difference is that these derivatives are **partial derivatives**. Loosely, when taking a partial derivative with respect to a variable, we treat any other variable as a constant. So,

$$\frac{\partial}{\partial\lambda}\left(2\lambda^2 + 2\mu^2 + 2\lambda - 4\mu - 2\mu\lambda + 5\right) = 4\lambda + 2 - 2\mu,$$

$$\frac{\partial}{\partial\mu}\left(2\lambda^2 + 2\mu^2 + 2\lambda - 4\mu - 2\mu\lambda + 5\right) = 4\mu - 4 - 2\lambda.$$

Setting these last two expressions equal to zero, we get these First Order Conditions:

$$4\lambda + 2 - 2\mu\Big|_{\lambda=\tilde{\lambda},\mu=\tilde{\mu}} \stackrel{1}{=} 0 \qquad \text{and} \qquad 4\mu - 4 - 2\lambda\Big|_{\lambda=\tilde{\lambda},\mu=\tilde{\mu}} \stackrel{2}{=} 0.$$

Take $\stackrel{2}{=}$ plus $2\times \stackrel{1}{=}$ to get $\tilde{\lambda} = 0$ and $\tilde{\mu} = 1$. Hence,

$$B = (0,0,3) + \tilde{\lambda}(1,-1,0) + \tilde{\mu}(0,1,-1) = (0,0,3) + 0(1,-1,0) + 1(0,1,-1) = (0,1,2).$$

And, $$\left|\overrightarrow{AB}\right| = \sqrt{2\tilde{\lambda}^2 + 2\tilde{\mu}^2 + 2\tilde{\lambda} - 4\tilde{\mu} - 2\tilde{\mu}\tilde{\lambda} + 5} = \sqrt{0+2+0-4-0+5} = \sqrt{3}.$$

Happily, these results are the same as before.

Note that this textbook does not explain why the above method works. We will merely note that the intuition for why it works is similar to that given for calculus (of a single variable) in in Part V (Calculus).

144.15. The Relationship Between Two Lines in 3D Space

Fact 181. *Suppose l_1 and l_2 are distinct lines described by $R = P + \alpha\mathbf{u}$ and $R = Q + \beta\mathbf{v}$ $(\alpha, \beta \in \mathbb{R})$. Then exactly one of the following three possibilities holds: The two lines are*

(a) *Parallel and do not intersect; moreover, the unique plane that contains both lines is $q_a = \left\{R : R = P + \lambda\mathbf{u} + \mu\overrightarrow{PQ} \quad (\lambda, \mu \in \mathbb{R})\right\}$; or*

(b) *Not parallel and share exactly one intersection point; moreover, the unique plane that contains both lines is described by $q_b = \{R : R = P + \lambda\mathbf{u} + \mu\mathbf{v} \quad (\lambda, \mu \in \mathbb{R})\}$; or*

(c) *Skew (i.e. neither parallel nor intersect) and are not coplanar.*

Proof. **(a)** Suppose $l_1 \parallel l_2$. Then by Fact 139, they do not intersect.[592] So, $\mathbf{u} \nparallel \overrightarrow{PQ}$.[593]

By Fact 169, q_a is the unique plane that contains P, $\mathbf{u}$, and $\overrightarrow{PQ}$.

Set $\mu = 0$ to see that q_a contains l_1.

Next, since $\mathbf{u} \parallel \mathbf{v}$, l_2 can also be described by $R = Q + \beta\mathbf{u}$ $(\beta \in \mathbb{R})$. Now set $\mu = 1$ to see that q_a also contains l_2.

In the remainder of this proof, we'll suppose instead that $l_1 \nparallel l_2$.

Then $\mathbf{u} \nparallel \mathbf{v}$ and by Corollary 33, q_b is the unique plane that contains P, $\mathbf{u}$, and $\mathbf{v}$. So, q_b is the only possible plane that contains both l_1 and l_2.

Set $\mu = 0$ to see that q_b contains l_1.

By Fact 139, l_1 and l_2 intersect at most once.[594]

(b) Suppose l_1 and l_2 share an intersection point S. Then $\overrightarrow{PS}$ is a direction vector of l_1. So, $\overrightarrow{PS} \parallel \mathbf{v}$. Hence,

$$q_b = \{R : R = P + \lambda\mathbf{u} + \mu\mathbf{v} \quad (\lambda, \mu \in \mathbb{R})\} = \left\{R : R = P + \lambda\mathbf{u} + \mu_1\mathbf{v} + \mu_2\overrightarrow{PS} \quad (\lambda, \mu_1, \mu_2 \in \mathbb{R})\right\}.$$

Set $\mu_2 = 1$ and $\lambda = 0$ to get $R = P + \mu_1\mathbf{v} + \overrightarrow{PS} = S + \mu_1\mathbf{v}$ and see that q_b contains l_2.

Thus, q_b is the unique plane that contains both l_1 and l_2.

(c) Suppose q_b contains l_2. Then there exist $\hat{\lambda}$ and $\hat{\mu}$ such that

$$Q = P + \hat{\lambda}\mathbf{u} + \hat{\mu}\mathbf{v}.$$

Now, consider the point $T = Q - \hat{\mu}\mathbf{v} \in l_2$. It is also in l_1 because

$$T = Q - \hat{\mu}\mathbf{v} = P + \hat{\lambda}\mathbf{u} + \hat{\mu}\mathbf{v} - \hat{\mu}\mathbf{v} = P + \hat{\lambda}\mathbf{u}.$$

We've just shown that if q_b contains l_2, then l_1 and l_2 intersect.

So, if l_1 and l_2 do not intersect (and are thus skew), then q_b does not contain l_2. □

[592] This eliminates the possibility that the two lines are parallel and intersect.

[593] If $\mathbf{u} \parallel \overrightarrow{PQ}$, then $k\mathbf{u} = \overrightarrow{PQ}$ for some $k \in \mathbb{R}$ and $Q = P + \overrightarrow{PQ} = P + k\mathbf{u}$ is on l_1, so that the two lines intersect.

[594] This eliminates the possibility that the two lines are not parallel and intersect more than once.

144.16. A Necessary and Sufficient Condition for Skew Lines

Fact 281. *Suppose the lines l_1 and l_2 are described by $\mathbf{r} = \overrightarrow{OP} + \lambda\mathbf{u}$ and $\mathbf{r} = \overrightarrow{OQ} + \lambda\mathbf{v}$ $(\lambda \in \mathbb{R})$. Then*

$$l_1 \text{ and } l_2 \text{ are skew} \qquad \iff \qquad \overrightarrow{PQ} \cdot (\mathbf{u} \times \mathbf{v}) \neq 0.$$

Proof. (\impliedby) If $\overrightarrow{PQ} \cdot (\mathbf{u} \times \mathbf{v}) \neq 0$, then $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, so that by Corollary 25, $\mathbf{u} \nparallel \mathbf{v}$ and $l_1 \nparallel l_2$.

Suppose for contradiction that l_1 and l_2 intersect at some point S. Then there are numbers α and β such that

$$S = P + \alpha\mathbf{u} = Q + \beta\mathbf{v} \qquad \text{or} \qquad \overrightarrow{PQ} = \alpha\mathbf{u} - \beta\mathbf{v}.$$

And so, $$\overrightarrow{PQ} \cdot (\mathbf{u} \times \mathbf{v}) = \alpha\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) - \beta\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0 - 0 = 0.$$

But this contradicts $\overrightarrow{PQ} \cdot (\mathbf{u} \times \mathbf{v}) \neq 0$. So, l_1 and l_2 do not intersect.

Since l_1 and l_2 are non-parallel and do not intersect, they are skew.

($\implies$) Now suppose $\overrightarrow{PQ} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

If $\overrightarrow{PQ} = \mathbf{0}$, then $P = Q$, so that l_1 and l_2 intersect and are not skew. And if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then by Corollary 25, l_1 and l_2 are parallel and are again not skew.

So, suppose $\overrightarrow{PQ}, \mathbf{u} \times \mathbf{v} \neq \mathbf{0}$. Then by Corollary 25, $\mathbf{u} \nparallel \mathbf{v}$.

Also, $\overrightarrow{PQ} \perp (\mathbf{u} \times \mathbf{v})$. By Corollary 32 then, $\overrightarrow{PQ}$ lies on the same plane as $\mathbf{u}$ and $\mathbf{v}$.

And so by Theorem 19, there exist α and β such that

$$\overrightarrow{PQ} = \alpha\mathbf{u} + \beta\mathbf{v}.$$

Now, let q be the plane described by

$$\left\{R : \overrightarrow{OR} = \overrightarrow{OP} + \lambda\mathbf{u} + \mu\mathbf{v}\right\} \qquad (\lambda, \mu \in \mathbb{R}).$$

Clearly, q contains l_1 (to see this, set $\mu = 0$). It also contains the point Q, because

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OP} + \alpha\mathbf{u} + \beta\mathbf{v}.$$

Hence, q also contains l_2 (to see this, set $\lambda = \alpha$).

We've just shown that l_1 and l_2 are coplanar. And so by Corollary 36, they aren't skew. □

Fact 282. *Suppose $a \in \mathbb{R}$ and $b > 0$. Then*

(a) *The two square roots of $a + \mathrm{i}b$ (i.e. solutions to $x^2 = a + \mathrm{i}b$) are*

$$\pm\frac{\sqrt{2}}{2}\left(\sqrt{\sqrt{a^2+b^2}+a}+\mathrm{i}\sqrt{\sqrt{a^2+b^2}-a}\right).$$

(b) *The two square roots of $a - \mathrm{i}b$ (i.e. solutions to $x^2 = a - \mathrm{i}b$) are*

$$\pm\frac{\sqrt{2}}{2}\left(\sqrt{a+\sqrt{a^2-b^2}}-\mathrm{i}\sqrt{a-\sqrt{a^2-b^2}}\right).$$

Proof. **(a)**

$$\begin{aligned}&\left[\pm\frac{\sqrt{2}}{2}\left(\sqrt{\sqrt{a^2+b^2}+a}+\mathrm{i}\sqrt{\sqrt{a^2+b^2}-a}\right)\right]^2\\ &=\frac{1}{2}\left[\sqrt{a^2+b^2}+a-\left(\sqrt{a^2+b^2}-a\right)+2\mathrm{i}\sqrt{\left(\sqrt{a^2+b^2}+a\right)\left(\sqrt{a^2+b^2}-a\right)}\right]\\ &=\frac{1}{2}\left(2a+2\mathrm{i}\sqrt{a^2+b^2-a^2}\right)=a+\mathrm{i}\sqrt{b^2}=a+\mathrm{i}b.\end{aligned}$$

(b)

$$\begin{aligned}&\left[\pm\frac{\sqrt{2}}{2}\left(\sqrt{a+\sqrt{a^2-b^2}}-\mathrm{i}\sqrt{a-\sqrt{a^2-b^2}}\right)\right]^2\\ &=\frac{1}{2}\left[a+\sqrt{a^2-b^2}+a-\sqrt{a^2+b^2}-2\mathrm{i}\sqrt{\left(a+\sqrt{a^2-b^2}\right)\left(a-\sqrt{a^2-b^2}\right)}\right]\\ &=\frac{1}{2}\left[2a-2\mathrm{i}\sqrt{a^2-\left(a^2-b^2\right)}\right]=a-\mathrm{i}\sqrt{b^2}=a-\mathrm{i}b.\end{aligned}$$

□

Fact 283. *Suppose $a, b \in \mathbb{R}$ with $b \neq 0$. Then the two square roots of $a + b\mathrm{i}$ are*

$$\pm\frac{\sqrt{2}}{2}\left(\sqrt{\sqrt{a^2+b^2}+a}+\mathrm{i}\frac{b}{|b|}\sqrt{\sqrt{a^2+b^2}-a}\right).$$

Proof.

$$\begin{aligned}&\left[\pm\frac{\sqrt{2}}{2}\left(\sqrt{\sqrt{a^2+b^2}+a}+\mathrm{i}\frac{b}{|b|}\sqrt{\sqrt{a^2+b^2}-a}\right)\right]^2\\ &=\frac{1}{2}\left[\sqrt{a^2+b^2}+a-\left(\sqrt{a^2+b^2}-a\right)+2\mathrm{i}\frac{b}{|b|}\sqrt{\left(\sqrt{a^2+b^2}+a\right)\left(\sqrt{a^2+b^2}-a\right)}\right]\\ &=\frac{1}{2}\left(2a+2\mathrm{i}\frac{b}{|b|}\sqrt{a^2+b^2-a^2}\right)=a+\mathrm{i}\frac{b}{|b|}|b|=a+\mathrm{i}b.\end{aligned}$$

□

Lemma 11. *If $p, q \in \mathbb{C}$, then $(p+q)^* = p^* + q^*$ and $(pq)^* = p^*q^*$.*

Proof. Let $p = (a, b)$ and $q = (c, d)$. Then

$$(p+q)^* = (a+c, b+d)^* = (a+c, -b-d) = (a, -b) + (c, -d) = p^* + q^*.$$

$$(pq)^* = [(a,b)(c,d)]^* = (ac-bd, ad+bc)^* = (ac-bd, -ad-bc) = (a,-b)(c,-d) = p^*q^*.$$

□

Theorem 21. (Complex Conjugate Root Theorem.) *Suppose* $c_0, c_1, \ldots, c_n \in \mathbb{R}$. *If* $z = a + ib$ *solves* $c_n x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0 = 0$, *then so does* $z^* = a - ib$.

Proof. We are given that $a + ib$ solves $\sum_{k=0}^{n} c_k x^k = 0$. Or equivalently: $\sum_{k=0}^{n} c_k (a+ib)^k \overset{1}{=} 0$.

Taking conjugates of both sides of $\overset{1}{=}$, we have $\left[\sum_{k=0}^{n} c_k (a+ib)^k\right]^* = 0^* \overset{2}{=} 0$.

We now apply Lemma 11(a) and (b) to show that $\left[\sum_{k=0}^{n} c_k (a+ib)^k\right]^* \overset{3}{=} \sum_{k=0}^{n} c_k (a-ib)^k$.

$$\left[\sum_{k=0}^{n} c_k (a+ib)^k\right]^* \overset{(a)}{=} \sum_{k=0}^{n} \left[c_k (a+ib)^k\right]^* \overset{(b)}{=} \sum_{k=0}^{n} c_k^* \left[(a+ib)^k\right]^*$$

$$= \sum_{k=0}^{n} c_k \left[(a+ib)^k\right]^* \overset{(b)}{=} \sum_{k=0}^{n} a_k \left[(a+ib)^*\right]^k = \sum_{k=0}^{n} c_k (a-ib)^k.$$

Together, $\overset{2}{=}$ and $\overset{3}{=}$ show that $\sum_{k=0}^{n} c_k (a-ib)^k = 0$. Hence, $a - ib$ also solves $\sum_{k=0}^{n} c_k x^k = 0$. □

Fact 190. *Suppose* z *is a non-zero complex number with* $|z| = r$ *and* $\arg z = \theta$. *Then*

$$z = r(\cos\theta + i\sin\theta).$$

Proof. Let $z = a + ib$. Then by Definitions 193 and 194, we have

$$r \overset{1}{=} |z| = \sqrt{a^2+b^2} \quad \text{and} \quad \theta \overset{2}{=} \arg z = \begin{cases} \cos^{-1}\dfrac{a}{|z|} = \cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}, & \text{if } b \ge 0, \\ -\cos^{-1}\dfrac{a}{|z|} = -\cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}, & \text{if } b < 0. \end{cases}$$

Rearranging $\overset{2}{=}$,

$$a \overset{3}{=} \begin{cases} r\cos\theta, & \text{if } b \ge 0, \\ r\cos(-\theta) = r\cos\theta, & \text{if } b < 0. \end{cases}$$

So, $a \overset{3}{=} \cos\theta$. Plugging $\overset{3}{=}$ into $\overset{1}{=}$, we have

$$r = \sqrt{r^2\cos^2\theta + b^2} \iff r^2 = r^2\cos^2\theta + b^2 \iff b^2 = r^2\sin^2\theta \iff b \overset{4}{=} \pm r\sin\theta.$$

Observe that $b \ge 0 \iff \sin\theta \ge 0$ and $b < 0 \iff \sin\theta < 0$. That is, $\sin\theta$ has the same sign as b. Hence, we can discard the negative value in $\overset{4}{=}$ to get $b \overset{5}{=} r\sin\theta$.

And now by $\overset{3}{=}$ and $\overset{5}{=}$, the result follows: $z = a + ib = r(\cos\theta + i\sin\theta)$. □

Theorem 22. (Euler's Formula) *Suppose* $\theta \in \mathbb{R}$. *Then* $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$.

Proof. Define the function $f : \mathbb{R} \to \mathbb{C}$ by $\theta \mapsto \mathrm{e}^{-\mathrm{i}\theta}(\cos\theta + \mathrm{i}\sin\theta)$. Then[595]

$$\begin{aligned} f'(\theta) &= (-\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\theta}(\cos\theta + \mathrm{i}\sin\theta) + \mathrm{e}^{-\mathrm{i}\theta}(-\sin\theta + \mathrm{i}\cos\theta) \\ &= \mathrm{e}^{-\mathrm{i}\theta}(-\mathrm{i}\cos\theta + \sin\theta) + \mathrm{e}^{-\mathrm{i}\theta}(-\sin\theta + \mathrm{i}\cos\theta) = 0. \end{aligned}$$

But by Proposition 8, the only functions whose derivatives are zero are constant functions. Thus, $\mathrm{e}^{-\mathrm{i}\theta}(\cos\theta + \mathrm{i}\sin\theta) = C$ for some constant C.

To find what C is, plug in $\theta = 0$ to get $C = \mathrm{e}^{-0}(\cos 0 + \mathrm{i}\sin 0) = 1 \cdot (1+0) = 1$.

Thus, $\mathrm{e}^{-\mathrm{i}\theta}(\cos\theta + \mathrm{i}\sin\theta) = 1$. Rearranging, $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$. □

Lemma 12. *Suppose* $x \in [k\pi, (k+1)\pi]$, *where* $k \in \mathbb{Z}$. *Then*

$$\cos^{-1}(\cos x) = \begin{cases} x - k\pi, & \text{for } k \text{ even}, \\ (k+1)\pi - x, & \text{for } k \text{ odd}. \end{cases}$$

Proof. First, note that $x - k\pi$ and $(k+1)\pi - x$ are both in $[0,\pi]$.

Moreover, if $y \in [0,\pi]$, then $\cos^{-1}(\cos y) \overset{1}{=} y$.

(a) If k is even, then $\cos(x - k\pi) = \cos x \underbrace{\cos(k\pi)}_{1} + \sin x \underbrace{\sin(k\pi)}_{0} = \cos x$.

And so, $$\cos^{-1}(\cos x) = \cos^{-1}[\cos(x - k\pi)] \overset{1}{=} x - k\pi. \quad ✓$$

(b) If k is odd, then $\cos[(k+1)\pi - x] = \underbrace{\cos[(k+1)\pi]}_{1}\cos x + \underbrace{\sin[(k+1)\pi]}_{0} = \cos x$.

And so, $$\cos^{-1}(\cos x) = \cos^{-1}[\cos((k+1)\pi - x)] \overset{1}{=} (k+1)\pi - x. \quad ✓$$ □

[595] We're actually cheating a little with this proof here, because we haven't explained how the derivatives of complex-valued functions work. We simply assume that they work "fairly similarly".

Fact 193. *Let z and w be non-zero complex numbers. Then*

$$\textbf{(a)}\ |zw| = |z|\,|w|; \qquad \textit{and} \qquad \textbf{(b)}\ \arg(zw) = \arg z + \arg w + 2k\pi,$$

where in **(b)**,
$$k = \begin{cases} -1, & \textit{if } \arg z + \arg w > \pi, \\ 0, & \textit{if } \arg z + \arg w \in (-\pi, \pi], \\ 1, & \textit{if } \arg z + \arg w \leq -\pi. \end{cases}$$

Proof. We already proved **(a)** in Exercise 322. We now prove **(b)**.

As in **(a)**, let $r = |z|$, $s = |w|$, $\theta = \arg z$, and $\phi = \arg w$.

Note that $\theta, \phi \in (-\pi, \pi]$ and so $\theta + \phi \in (-2\pi, 2\pi]$.

Case 1. Suppose $\sin(\theta + \phi) \geq 0$. Then by Definition 194,

$$\arg(zw) \overset{1}{=} \cos^{-1}\frac{rs\cos(\theta+\phi)}{rs} = \cos^{-1}[\cos(\theta+\phi)].$$

Also, since $\sin(\theta + \phi) \geq 0$, we have $\theta + \phi \in (-2\pi, -\pi] \cup [0, \pi]$.

Case 1a. If $\theta + \phi \in (-2\pi, -\pi]$, then by Lemma 12,

$$\arg(zw) \overset{1}{=} \cos^{-1}[\cos(\theta+\phi)] = \theta + \phi + 2\pi.$$

Case 1b. If $\theta + \phi \in [0, \pi]$, then by Lemma 12,

$$\arg(zw) \overset{1}{=} \cos^{-1}[\cos(\theta+\phi)] = \theta + \phi.$$

Case 2. Suppose $\sin(\theta + \phi) < 0$. Then by Definition 194,

$$\arg(zw) \overset{2}{=} -\cos^{-1}\frac{rs\cos(\theta+\phi)}{rs} = -\cos^{-1}[\cos(\theta+\phi)].$$

Also, since $\sin(\theta + \phi) < 0$, we have $\theta + \phi \in (-\pi, 0) \cup (\pi, 2\pi)$.

Case 2a. If $\theta + \phi \in (-\pi, 0)$, then by Lemma 12,

$$\arg(zw) \overset{2}{=} -\cos^{-1}[\cos(\theta+\phi)] = -[-(\theta+\phi)] = \theta + \phi.$$

Case 2b. If $\theta + \phi \in (\pi, 2\pi)$, then by Lemma 12,

$$\arg(zw) \overset{2}{=} -\cos^{-1}[\cos(\theta+\phi)] = -[2\pi - (\theta+\phi)] = \theta + \phi - 2\pi.$$

We've just shown that $\arg(zw) = \arg z + \arg w + 2k\pi$, with

$$k = \begin{cases} -1, & \text{for } \arg z + \arg w > \pi, \\ 0, & \text{for } \arg z + \arg w \in (-\pi, \pi], \\ 1, & \text{for } \arg z + \arg w \leq -\pi. \end{cases}$$ □

Fact 194. *Suppose w is a non-zero complex number. Then*

(a)
$$\left|\frac{1}{w}\right| = \frac{1}{|w|}.$$

(b) *If w is not a negative real number, then* $\arg\frac{1}{w} = -\arg w$.

(c) *If w is a negative real number, then* $\arg\frac{1}{w} = \arg w = \pi$.

Proof. Let $w = a + ib \neq 0$. By Fact 186(b),

$$\frac{1}{w} = \left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}\right).$$

(a) By Definition 193, $|w| = \sqrt{a^2+b^2}$ and

$$\left|\frac{1}{w}\right| = \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(-\frac{b}{a^2+b^2}\right)^2} = \frac{\sqrt{a^2+b^2}}{a^2+b^2} = \frac{1}{\sqrt{a^2+b^2}} = \frac{1}{|w|}. \quad \checkmark$$

(b) By Definition 194, $\arg w = \begin{cases} \cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}, & \text{for } b \geq 0, \\ -\cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}, & \text{for } b < 0. \end{cases}$

And,
$$\arg\frac{1}{w} = \begin{cases} \cos^{-1}\dfrac{a/(a^2+b^2)}{1/\sqrt{a^2+b^2}} = \cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}, & \text{for } \dfrac{-b}{a^2+b^2} \geq 0 \quad\text{or}\quad b \leq 0, \\ -\cos^{-1}\dfrac{a/(a^2+b^2)}{1/\sqrt{a^2+b^2}} = -\cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}, & \text{for } \dfrac{-b}{a^2+b^2} < 0 \quad\text{or}\quad b > 0. \end{cases}$$

Thus, if $b < 0$, then $\quad \arg\dfrac{1}{w} = \cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}} = -\arg w. \quad \checkmark$

And if $b > 0$, then $\quad \arg\dfrac{1}{w} = -\cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}} = -\arg w. \quad \checkmark$

If $b = 0$, $a > 0$, then $\arg w = \arg a = 0$ and $\arg\frac{1}{w} = \arg\frac{1}{a} = 0$ so that indeed,

$$\arg\frac{1}{w} = -\arg w. \quad \checkmark$$

And in the exceptional case where $b = 0$, $a < 0$, we have $\arg\frac{1}{w} = \pi = \arg w$. □

Fact 195. *Suppose z and w are non-zero complex numbers. Then*

$$\textbf{(a)}\ \left|\frac{z}{w}\right| = \frac{|z|}{|w|}; \qquad \textit{and} \qquad \textbf{(b)}\ \arg\frac{z}{w} = \arg z - \arg w + 2k\pi,$$

where in **(b)**,
$$k = \begin{cases} -1, & \textit{if } \arg z - \arg w > \pi, \\ 0, & \textit{if } \arg z - \arg w \in (-\pi, \pi], \\ 1, & \textit{if } \arg z - \arg w \le -\pi. \end{cases}$$

Proof. We already proved **(a)** in Exercise 325. We now prove **(b)**:

Suppose w is not a negative real number. Then by Facts 193 and 194,

$$\arg\frac{z}{w} = \arg\left(z\frac{1}{w}\right) = \arg z + \arg\frac{1}{w} + 2k\pi = \arg z - \arg w + 2k\pi,$$

where
$$k = \begin{cases} -1, & \text{for } \arg z + \arg\dfrac{1}{w} = \arg z - \arg w > \pi, \\ 0, & \text{for } \arg z + \arg\dfrac{1}{w} = \arg z - \arg w \in (-\pi, \pi], \\ 1, & \text{for } \arg z + \arg\dfrac{1}{w} = \arg z - \arg w \le -\pi. \quad ✓ \end{cases}$$

Now suppose instead that w is a negative real number. Then $\arg w = -\pi$.

And now by Corollary 40,

$$\arg\frac{z}{w} = \arg(-z) = \begin{cases} \arg z - \pi = \overbrace{\arg z - \arg w}^{\in(-\pi,\pi]} + 2\overbrace{k}^{0}\pi, & \text{for } \arg z > 0, \\ \arg z + \pi = \underbrace{\arg z - \arg w}_{\le -\pi} + 2\underbrace{k}_{1}\pi, & \text{for } \arg z \le 0. \quad ✓ \end{cases}$$

□

Revision in progress (November 2021).

And hence messy at the moment.
Appy polly loggies for any inconvenience caused.

146.1. Some Useful Terms

Figure to be
inserted here.

Let c be a point and $\varepsilon > 0$. Previously, Definition 275 already defined $\mathrm{N}_\varepsilon(c)$, the *$\varepsilon$-neighbourhood of c*. We now define several similar and related terms:[596]

Definition 315. Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Define the

- *left ε-neighbourhood of c*, denoted $\mathrm{N}_\varepsilon^-(c)$, by $\mathrm{N}_\varepsilon^-(c) = (c-\varepsilon, c)$;
- *right ε-neighbourhood of c*, denoted $\mathrm{N}_\varepsilon^+(c)$, by $\mathrm{N}_\varepsilon^+(c) = (c, c+\varepsilon)$; and
- *deleted* (or *punctured*) *ε-neighbourhood of c*, denoted $\not{\mathrm{N}}_\varepsilon(c)$, $\not{\mathrm{N}}_\varepsilon(c) = (c-\varepsilon, c) \cup (c, c+\varepsilon) = (c-\varepsilon, c+\varepsilon) \setminus \{c\}$.

Remark 210. The $\mathrm{N}_\varepsilon(x)$ notation is fairly standard, though instead of the letter "N", other writers may instead use "B" or "V".

Unfortunately, there is no standard notation for deleted neighbourhoods. Here in these Appendices I'll use $\not{\mathrm{N}}_\varepsilon(x)$, which is completely non-standard and possibly used by no one else.[597]

Definition 316. Let $c \in \mathbb{R}$. We shall say

- *Some neighbourhood of c* to mean $(c-\varepsilon, c+\varepsilon)$ for some $\varepsilon > 0$;
- *Some left neighbourhood of c* to mean $(c-\varepsilon, c)$ for some $\varepsilon > 0$;
- *Some right neighbourhood of c* to mean $(c, c+\varepsilon)$ for some $\varepsilon > 0$; and
- *Some deleted neighbourhood of c* to mean $(c-\varepsilon, c) \cup (c, c+\varepsilon)$ for some $\varepsilon > 0$.

Example 1579. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is

- Positive on some neighbourhood of 1;
- Negative on some left neighbourhood of 0;
- Positive on some right neighbourhood of 0;
- Non-zero on some deleted neighbourhood of 0.

Informally, given a point x and a set S, if we can always find another point in S that's "arbitrarily" close to x, then we call x a **limit point** of the set S. Formally,

[596]Though for simplicity, we restrict these definitions to the case of the real number line $\mathbb{R}$.
[597]Velleman (2016) uses $x \to a^{\neq}$, which is also non-standard.

Definition 317. Let $S \subseteq \mathbb{R}$. We call x a *limit point of S* if for every $\varepsilon > 0$,

$$\cancel{N}_\varepsilon(x) \cap S \neq \varnothing.$$

In words, x is a limit point of S if (and only if) *every* ε-neighbourhood of x intersects S at some point other than x.

Important note: For x to be a limit point of S, there is no requirement that $x \in S$. That is, it is possible that $x \notin S$ is a limit point of S.

Earlier, Definition 276 defined **isolated points**. Together with Definition 317, we have the result that an isolated point is never a limit point:

Fact 284. *Let x be a point and S be a set. If x is an isolated point of S, then x is not a limit point of S.*

An isolated point of a set must be in that set. In contrast, a limit point of a set need not be in that set.

So, the converse of Fact 284 result is false. That is, the following claim is false:

"If x is not a limit point of S, then x is an isolated point of S."

Or, "If x is not an isolated point of S, then x is a limit point of S."

Instead, we have this partial converse:

Fact 285. *If x is not a limit point of S. then either x is an isolated point of S or $x \notin S$.*

Or, If x is not an isolated point of S and $x \in S$, then x is a limit point of S.

Remark 211. Some writers treat the terms **limit point**, **cluster point**, and **accumulation point** as synonyms. But unfortunately and confusingly, other writers assign different meanings to these three terms. Fortunately, in these appendices, we will only mention limit points. We will never mention cluster or accumulation points. So, we can ignore the issue of whether these three terms "should" be synonyms or have different meanings.

We already discussed *degenerate* and *non-degenerate intervals* in Exercise 42. Formally,

Definition 318. A *degenerate interval* is an interval that is empty or contains one number. A *non-degenerate interval* is an interval that contains more than one number.

Or equivalently (and as already discussed in Exercise 42), a *degenerate interval* is an interval that has **(i)** equal endpoints; or **(iii)** finitely many elements. A *non-degenerate interval* is an interval that has **(i)** distinct endpoints; or **(iii)** infinitely many elements.

146.2. Limits

In the main text, we described the statement $\lim_{x \to a} f(x) = L \in \mathbb{R}$ informally as

For **all** values of x that are **"close"** but **not equal** to a,
$f(x)$ is "close" (or possibly even equal) to L.

We now formalise the idea contained in the above informal statement:

Definition 319. Let f be a nice function with domain D and a be a limit point of D. We say that *the limit of f at a is $L \in \mathbb{R}$* and write $\lim_{x \to a} f(x) = L$ if

For every $\varepsilon > 0$, there exists $\delta > 0$ such that
$x \in D \cap \cancel{\mathrm{N}}_{\delta}(a)$ implies $f(x) \in \mathrm{N}_{\varepsilon}(L)$.

The above definition is called the ε-δ definition and is usually credited to Bernard Bolzano (1781–1848) and Augustin-Louis Cauchy (1789–1857).

Subtle but important point: The above definition includes the requirement that a be a limit point of D. If a is not a limit point of D (e.g. when a is an isolated point of D), then $\lim_{x \to a} f(x)$ is simply **undefined** (or **does not exist**).

The following result says that if the limit is defined (or exists), then it must be unique:

Fact 286. *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} f(x) = L_2$, then $L_1 = L_2$.*

Proof. Suppose for contradiction that $L_1 \neq L_2$. Let $\varepsilon = |L_1 - L_2| / 2$. Observe that $\mathrm{N}_{\varepsilon}(L_1) \cap \mathrm{N}_{\varepsilon}(L_2) \stackrel{1}{=} \varnothing$.

By Definition 319, there exist $\delta_1, \delta_2 > 0$ such that $x \in D \cap \cancel{\mathrm{N}}_{\min\{\delta_1, \delta_2\}}(a)$ implies $f(x) \in \mathrm{N}_{\varepsilon}(L_1)$ AND $f(x) \in \mathrm{N}_{\varepsilon}(L_2)$. But this contradicts $\stackrel{1}{=}$. □

We tweak Definition 319 to get **left-** and **right-hand limits**:

Definition 320. Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and a be a limit point of D.

We say that *the left-hand limit of f at a is $L \in \mathbb{R}$* and write $\lim_{x \to a^-} f(x) = L$ if

For every $\varepsilon > 0$, there exists $\delta > 0$ such that
$x \in D \cap \mathrm{N}_{\delta}^{-}(a)$ implies $f(x) \in \mathrm{N}_{\varepsilon}(L)$.

We say that *the right-hand limit of f at a is $L \in \mathbb{R}$* and write $\lim_{x \to a^+} f(x) = L$ if

For every $\varepsilon > 0$, there exists $\delta > 0$ such that
$x \in D \cap \mathrm{N}_{\delta}^{+}(a)$ implies $f(x) \in \mathrm{N}_{\varepsilon}(L)$.

Remark 212. When we write $\lim_{x\to a} f(x) \overset{1}{=} L$, the symbol "$x$" is, as usual, a dummy variable that can be replaced by any other symbol, such as y, z, or ☺. So for example, these four statements are equivalent:

$$\lim_{x\to a} f(x) = L, \quad \lim_{y\to a} f(y) = L, \quad \lim_{z\to a} f(z) = L, \quad \lim_{☺\to a} f(☺) = L.$$

And so, really, the symbol "x" is superfluous. We could very well rewrite $\overset{1}{=}$ more simply as

$$\lim_{a} f = L.$$

Similarly, we could very well rewrite $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$ as

$$\lim_{a^-} f = L \qquad \text{and} \qquad \lim_{a^+} f = L.$$

Fact 196. *Suppose f is a nice function. Then*

$$\lim_{x\to a} f(x) = L \qquad \iff \qquad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

Proof. Immediate from Definitions 319 and 320. □

So far, we've only defined what the statement $\lim_{x\to a} f(x) = L$ means. Strictly and pedantically speaking, we have not yet defined what a statement such as the following means:

$$\lim_{x\to a} kf(x) \overset{1}{=} M.$$

We shall assign to $\overset{1}{=}$ this meaning:

$$\text{Let } g = kf. \text{ Then } \lim_{x\to a} g(x) = M.$$

Similarly for other statements. For example, the statement $\lim_{x\to a} c \overset{2}{=} c$ shall mean this:

$$\text{Define } h : \mathbb{R} \to \mathbb{R} \text{ by } h(x) = c. \text{ Then } \lim_{x\to a} h(x) = c.$$

146.3. Infinite Limits and Vertical Asymptotes

Definition 321. Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and a be a limit point of D.

(a) We say that *the left-hand limit of f at a is ∞* and write $\lim_{x \to a^-} f(x) = \infty$ if

$$\text{For every } N \in \mathbb{R}, \text{ there exists } \delta > 0 \text{ such that}$$
$$x \in D \cap \mathrm{N}_\delta^-(a) \text{ implies } f(x) > N.$$

(b) We say that *the right-hand limit of f at a is ∞* and write $\lim_{x \to a^+} f(x) = \infty$ if

$$\text{For every } N \in \mathbb{R}, \text{ there exists } \delta > 0 \text{ such that}$$
$$x \in D \cap \mathrm{N}_\delta^+(a) \text{ implies } f(x) > N.$$

(c) We say that *the limit of f at a is ∞* and write $\lim_{x \to a} f(x) = \infty$ if

$$\text{For every } N \in \mathbb{R}, \text{ there exists } \delta > 0 \text{ such that}$$
$$x \in D \cap \not{\mathrm{N}}_\delta(a) \text{ implies } f(x) > N.$$

(d) We say that *the left-hand limit of f at a is $-\infty$* and write $\lim_{x \to a^-} f(x) = -\infty$ if

$$\text{For every } N \in \mathbb{R}, \text{ there exists } \delta > 0 \text{ such that}$$
$$x \in D \cap \mathrm{N}_\delta^-(a) \text{ implies } f(x) < N.$$

(e) We say that *the right-hand limit of f at a is $-\infty$* and write $\lim_{x \to a^+} f(x) = -\infty$ if

$$\text{For every } N \in \mathbb{R}, \text{ there exists } \delta > 0 \text{ such that}$$
$$x \in D \cap \mathrm{N}_\delta^+(a) \text{ implies } f(x) < N.$$

(f) We say that *the limit of f at a is $-\infty$* and write $\lim_{x \to a} f(x) = -\infty$ if

$$\text{For every } N \in \mathbb{R}, \text{ there exists } \delta > 0 \text{ such that}$$
$$x \in D \cap \not{\mathrm{N}}_\delta(a) \text{ implies } f(x) < N.$$

Definition 322. If the left- or right-hand limit of f at a is $\pm\infty$, then the line $x = a$ is called a *vertical asymptote* of f.

146.4. Limits at Infinity and Horizontal and Oblique Asymptotes

Definition 323. Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $L \in \mathbb{R}$. We say that

(a) *The limit of f as x approaches* ∞ is L and write $\lim_{x\to\infty} f(x) = L$ if

For every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that
$D \cap (N, \infty) \neq \varnothing$ and $f(x) \in \mathrm{N}_\varepsilon(L)$ for every $x \in D \cap (N, \infty)$;

(b) *The limit of f as x approaches* $-\infty$ is L and write $\lim_{x\to-\infty} f(x) = L$ if

For every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that
$D \cap (N, \infty) \neq \varnothing$ and $f(x) \in \mathrm{N}_\varepsilon(L)$ for every $x \in D \cap (-\infty, N)$.

Again, we can, at long last, write down a formal definition of **horizontal asymptotes**:

Definition 324. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is called a *horizontal asymptote* of f.

We can similarly define **oblique** asymptotes.[598]

Definition 325. Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say that

(a) *The limit of f as x approaches* ∞ is $ax + b$ and write $\lim_{x\to\infty} f(x) = ax + b$ if

For every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that
$D \cap (N, \infty) \neq \varnothing$ and $f(x) \in \mathrm{N}_\varepsilon(ax + b)$ for every $x \in D \cap (N, \infty)$.

(b) *The limit of f as x approaches* $-\infty$ is $ax + b$ and write $\lim_{x\to-\infty} f(x) = ax + b$ if

For every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that
$D \cap (N, \infty) \neq \varnothing$ and $f(x) \in \mathrm{N}_\varepsilon(ax + b)$ for every $x \in D \cap (-\infty, N)$.

Definition 326. If $\lim_{x\to\infty} f(x) = ax + b$ or $\lim_{x\to-\infty} f(x) = ax + b$, then the line $y = ax + b$ is called an *oblique asymptote* of f.

[598]Actually, with Definitions 325 and 326, we have no need for Definitions 323 and 324.

146.5. Asymptotes of a Graph

This chapter parallels Ch. 142.7.

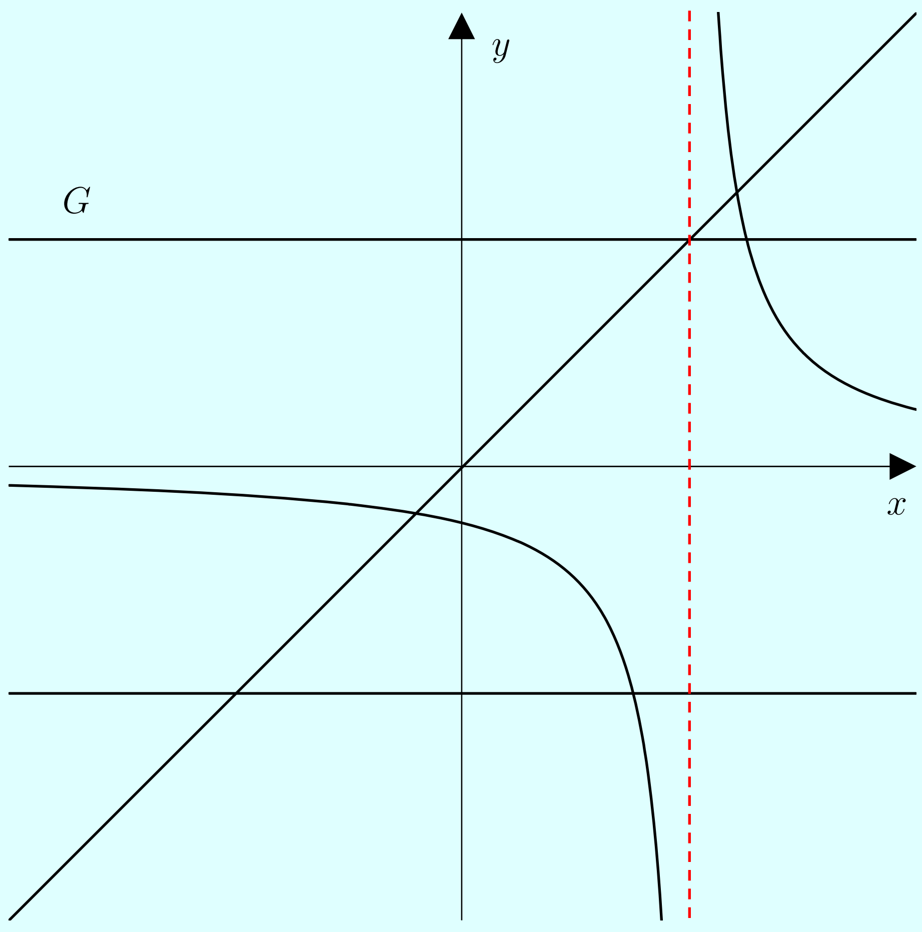
So far, we've defined only **asymptotes of functions**. We have *not* defined **asymptotes of graphs**.

Example 1580. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = 1/x$. Then f has asymptotes $y = 0$ and $x = 0$.

Now consider instead the graph of $y = 1/x$—this is simply the set $\left\{(x, y) \in \mathbb{R}^2 : y = 1/x\right\}$. We would quite sensibly like to say that this graph also has asymptotes $y = 0$ and $x = 0$. The problem though is that we haven't formally defined what any asymptote of a graph is. So, right now, we can't actually say that this graph has any asymptotes.

It turns out that writing down general definitions of asymptotes of graphs is pretty tricky, because a graph is simply *any* set of points (in $\mathbb{R}^2$).

Example 1581. Below is a graph G. Intuitively and visually, we want to be able to say that the red line is a vertical asymptote for G.



But it's tricky to write down a general definition of (vertical) asymptotes where this is so.

Here's one possible set of definitions of vertical, horizontal, and oblique asymptotes:

Definition 327. Let $G \subseteq \mathbb{R}^2$ be a graph. We call

(a) $x = a$ a *vertical asymptote* of G if there exist $\varepsilon > 0$ and a nice function f with graph F such that

  (i) $x = a$ is a vertical asymptote of f; and

  (ii) for every $P \in F$, $\mathrm{N}_\varepsilon(P) \cap G = \{P\}$;

(b) $y = a$ a *horizontal asymptote* of G if there exist $\varepsilon > 0$ and a nice function f with graph F such that

  (a) $y = a$ is a horizontal asymptote of f; and

  (b) for every $P \in F$, $\mathrm{N}_\varepsilon(P) \cap G = \{P\}$;

(c) $y = ax + b$ an *oblique asymptote* of G if there exist $\varepsilon > 0$ and a nice function f with graph F such that

  (a) $y = ax + b$ is an oblique asymptote of f; and

  (b) for every $P \in F$, $\mathrm{N}_\varepsilon(P) \cap G = \{P\}$.

The condition in each **(ii)** is to rule out "thick" lines or even "shaded areas" in G:

Example 1582. Let $G = [1,2] \times \mathbb{R}$. Absent **(a)(ii)** in the above definition, every vertical line $x = a$ for any $a \in [1,2]$ would be a vertical asymptote of G.

With **(a)(ii)**, we'd say instead that G has no vertical asymptotes (which is probably what we want).

Theorem 23. (Rules for Limits) *Let f and g be nice functions; and $k, L, M \in \mathbb{R}$. Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then*

(a) $\lim_{x\to a} [kf(x)] \overset{\text{F}}{=} kL$ **(Constant Factor Rule for Limits)**

(b) $\lim_{x\to a} [f(x) \pm g(x)] \overset{\pm}{=} L + M$ **(Sum and Difference Rules for Limits)**

(c) $\lim_{x\to a} [f(x) g(x)] \overset{\times}{=} LM$ **(Product Rule for Limits)**

(d) $\lim_{x\to a} \dfrac{1}{g(x)} \overset{\text{R}}{=} \dfrac{1}{M}$ *(for $M \neq 0$)* **(Reciprocal Rule for Limits)**

(e) $\lim_{x\to a} \dfrac{f(x)}{g(x)} \overset{\div}{=} \dfrac{L}{M}$ *(for $M \neq 0$)* **(Quotient Rule for Limits)**

(f) $\lim_{x\to a} k \overset{\text{C}}{=} k$ **(Constant Rule for Limits)**

(g) $\lim_{x\to a} x^k \overset{\text{P}}{=} a^k$ **(Power Rule for Limits)**

Proof. First, note that the statements $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ say the following:

☺ For every $\varepsilon_f > 0$, there exists $\delta_f > 0$ such that $x \in D \cap \mathcal{N}_{\delta_f}(a)$ implies $|f(x) - L| < \varepsilon_f$.

★ For every $\varepsilon_g > 0$, there exists $\delta_g > 0$ such that $x \in D \cap \mathcal{N}_{\delta_g}(a)$ implies $|g(x) - M| < \varepsilon_g$.

Fix $\varepsilon > 0$. For each Rule, we shall find some $\delta > 0$ such that if $x \in D \cap \mathcal{N}_\delta(a)$, then the value of the given function at x is less than ε away from the claimed limit.

(a) Pick $\varepsilon_f = \varepsilon / |k|$. Let δ_f be as given by ☺. Pick $\delta = \delta_f$.

Suppose $x \in D \cap \mathcal{N}_\delta(a)$. Then by ☺, $|f(x) - L| < \varepsilon_f$ and hence

$$|kf(x) - kL| = |k| |f(x) - L| < |k| \varepsilon_f = |k| \varepsilon / |k| = \varepsilon.$$

(*Proof continues below ...*)

(... *Proof continued from above.*)

(b) Pick $\varepsilon_f = \varepsilon/2$, $\varepsilon_g = \varepsilon/2$. Let δ_f and δ_g be as given by ☺ and ★. Pick $\delta = \min\{\delta_f, \delta_g\}$. Suppose $x \in D \cap \not{\mathrm{N}}_\delta(a)$. Then by ☺ and ★, $|f(x) - L| < \varepsilon_f$, $|g(x) - M| < \varepsilon_g$, and hence

$$|f(x) \pm g(x) - (L \pm M)| \underset{\substack{\downarrow \\ \text{Triangle Inequality}}}{\leq} |f(x) - L| + |g(x) - M| < \varepsilon_f + \varepsilon_g = \varepsilon.$$

(c) Pick $\varepsilon_f \overset{1}{=} \varepsilon/|2M|$ and $\varepsilon_g \overset{2}{=} \varepsilon/(2\varepsilon_f + 2|L|)$.
Let δ_f and δ_g be as given by ☺ and ★. Pick $\delta = \min\{\delta_f, \delta_g\}$.
Suppose $x \in D \cap \not{\mathrm{N}}_\delta(a)$. Then by ☺ and ★, $|f(x) - L| < \varepsilon_f$ and $|g(x) - M| < \varepsilon_g$. And so,

$$\begin{aligned}
& |f(x)g(x) - LM| \\
&= |f(x)g(x) - f(x)M + f(x)M - LM| && \text{(Plus Zero Trick)} \\
&\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| && \text{(Triangle Inequality)} \\
&= |f(x)||g(x) - M| + |f(x) - L||M| && (|ab| = |a||b|) \\
&= |f(x) - L + L||g(x) - M| + |f(x) - L||M| && \text{(Plus Zero Trick)} \\
&\leq [|f(x) - L| + |L|]|g(x) - M| + |f(x) - L||M| && \text{(Triangle Inequality)} \\
&< (\varepsilon_f + |L|)\varepsilon_g + \varepsilon_f|M| \overset{1,2}{=} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

(d) Pick $\varepsilon_g \overset{1}{=} \dfrac{\varepsilon|M|^2}{1 + \varepsilon|M|}$. Let δ_g be as given by ★.

Assume $\varepsilon < 1/|M|$ (without loss of generality). Then $\varepsilon_g \overset{1}{=} \dfrac{\varepsilon|M|^2}{1+\varepsilon|M|} < \dfrac{|M|}{1+\varepsilon|M|} < \dfrac{|M|}{1} = |M|$.

Let $\hat{\varepsilon} = |M|/2$. Since $\lim_{x \to a} g(x) = M$, there exists $\hat{\delta} > 0$ such that if $x \in D \cap \not{\mathrm{N}}_{\hat{\delta}}(a)$, then $g(x) \in \mathrm{N}_{\hat{\varepsilon}}(M)$ and, in particular, $g(x) \neq 0$.

Pick $\delta = \{\delta_g, \hat{\delta}\}$. Suppose $x \in D \cap \not{\mathrm{N}}_\delta(a)$. Then $|g(x) - M| \overset{2}{<} \varepsilon_g$ and $g(x) \neq 0$. Also, $|g(x)| \overset{3}{>} |M| - \varepsilon_g > 0$. And so,

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{g(x)M}\right| = \frac{|g(x) - M|}{|g(x)||M|} \overset{2}{<} \frac{\varepsilon_g}{|g(x)||M|} \overset{3}{<} \frac{\varepsilon_g}{(|M| - \varepsilon_g)|M|}$$

$$\overset{1}{=} \frac{\frac{\varepsilon|M|^2}{1+\varepsilon|M|}}{\left(|M| - \frac{\varepsilon|M|^2}{1+\varepsilon|M|}\right)|M|} = \frac{\varepsilon|M|^2}{\left[(1+\varepsilon|M|)|M| - \varepsilon|M|^2\right]|M|} = \varepsilon.$$

(*Proof continues below ...*)

(... *Proof continued from above.*)

(e) follows from **(c)** and **(d)**.

(f) For any x, $|k - k| = 0 < \varepsilon$.

(g) We shall prove the Power Rule for Limits only in the case where $a > 0$:

Let $a > 0$. Pick $\delta = \min\left\{\frac{\varepsilon}{2\exp k}, \frac{a}{2}\right\}$.

Suppose $x \in D \cap \mathcal{N}_\delta(a)$. Note that $x > 0$ (because $\delta \le a/2$). And so,

$$\begin{aligned}
\left|x^k - a^k\right| &= \left|\exp(k \ln x) - \exp(k \ln a)\right| && \text{(Definition 335)} \\
&= \left|(\exp k)\left[\exp(\ln x) - \exp(\ln a)\right]\right| && \text{(Fact 224}\textbf{(c)}\text{)} \\
&= \left|(\exp k)(x - a)\right| && \text{(Definition 85)} \\
&= |\exp k|\,|x - a| \le |\exp k| \frac{\varepsilon}{2\exp k} = \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

It remains to be proven that the Power Rule for Limits also holds in the case where $a \le 0$. Unfortunately, such a proof must be omitted altogether from this textbook, for reasons briefly discussed in Remark **??**. □

146.7. Why the Geometric Series "Trick" Doesn't Always Work

We now reproduce Examples 737 and 738 from Part II (Sequences and Series). We explain why the "trick" works in the first but not the second example:

Example 737. Suppose $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. Then

1. $2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$
2. So, $S - 2S = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\right) - \left(2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$ or $-S = -2$.
3. Hence, $S = 2$.

The above argument is mostly correct, but is informal and lacks some details. Here is a fuller and more formal presentation of the same argument:

For each $k \in \mathbb{Z}_0^+$, define $S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k-1}}$. Suppose we are told that $\lim_{k\to\infty} S_k$ is convergent, i.e. is equal to some real number S. Then

1. $\lim_{k\to\infty} (2S_k) = \left(\lim_{k\to\infty} 2\right)\left(\lim_{k\to\infty} S_k\right) = 2S$.

   Here we use the Product Rule for Limits.[599] This is legitimate if and only if $\lim_{k\to\infty} S_k$ is actually convergent.
2. $S_k - 2S_k = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k-1}}\right) - \left(2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k}\right) = -2 - \frac{1}{2^k}$.

   $\lim_{k\to\infty} (S_k - 2S_k) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} 2S_k = S - 2S = -S$. Here we use the Product and Difference Rules for Limits. This is legitimate if and only if $\lim_{k\to\infty} S_k$ and $\lim_{k\to\infty} 2S_k$ are actually convergent.

   $\lim_{k\to\infty}\left(-2 - \frac{1}{2^k}\right) = \lim_{k\to\infty} -2 - \lim_{k\to\infty} \frac{1}{2^k} = -2$. Here we use the Difference Rule for Limits. This is legitimate if and only if $\lim_{k\to\infty} \frac{1}{2^k}$ is actually convergent.
3. Hence, $-S = -2$ or $S = 2$.

[599] We also use the Constant Rule for Limits. These two Rules are stated as part of Theorem 23 (Part V, Calculus).

Example 738. Suppose $S = 1 + 2 + 4 + 8 + 16 + \dots$. Then

1. $2S = 2 + 4 + 8 + 16 + 32 + \dots$
2. So, $S - 2S = (1 + 2 + 4 + 8 + 16 + \dots) - (2 + 4 + 8 + 16 + 32 + \dots)$ or $-S = -1$.
3. Hence, $S = 1$.

Let's see why the above argument does not work.

For each $k \in \mathbb{Z}_0^+$, define $S_k = 1 + 2 + 4 + 8 + \dots + 2^{k-1}$. We do not know whether $\lim_{k\to\infty} S_k$ is convergent (indeed, it is not). Now,

1. $\lim_{k\to\infty} (2S_k) = \left(\lim_{k\to\infty} 2\right)\left(\lim_{k\to\infty} S_k\right) = 2S$. ✗

   Here we use the Product Rule for Limits.[600] This is legitimate if and only if $\lim_{k\to\infty} S_k$ is actually convergent (it isn't).
2. $S_k - 2S_k = \left(1 + 2 + 4 + 8 + \dots + 2^{k-1}\right) - \left(2 + 4 + 8 + 16 + \dots + 2^k\right) = 1 - 2^k$.

   $\lim_{k\to\infty} (S_k - 2S_k) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} 2S_k = S - 2S = -S$. ✗

   Here we use the Product and Difference Rules for Limits. This is legitimate if and only if $\lim_{k\to\infty} S_k$ and $\lim_{k\to\infty} 2S_k$ are actually convergent (they aren't).

   $\lim_{k\to\infty} \left(1 - 2^k\right) = \lim_{k\to\infty} 1 - \lim_{k\to\infty} 2^k = 1$. ✗

   Here we use the Difference Rule for Limits. This is legitimate if and only if $\lim_{k\to\infty} 2^k$ is actually convergent (it isn't).
3. Hence, $-S = 1$ or $S = -1$. ✗

146.8. Continuity

Definition 198 is the definition of continuity used in this textbook and also most introductory calculus texts. However, past the introductory level, the preferred definition is usually this ε-δ characterisation of continuity:

Proposition 30. *Suppose $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $c \in D$. Then statements* **(a)** *and* **(b)** *are equivalent:*

(a) *f is continuous at c.*

(b) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in D \cap \cancel{\mathrm{N}}_\delta(c)$ implies $f(x) \in \mathrm{N}_\varepsilon(f(c))$.*

Proof. If c is an isolated point of D, then **(a)** is true (by Definition 198) and **(b)** is vacuously true (because there exists $\delta > 0$ such that $D \cap \cancel{\mathrm{N}}_\delta(c) = \varnothing$).

So, suppose c is not an isolated point of D. By Fact 285, c is a limit point of D. So, by Definitions 319 and 198,

$$\textbf{(b)} \qquad \Longleftrightarrow \qquad \lim_{x \to c} f(x) = f(c) \qquad \Longleftrightarrow \qquad \textbf{(a)}. \qquad \square$$

Fact 198. *Every constant function is continuous.*

Proof. Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$.

Let $\varepsilon > 0$. Pick any $\delta > 0$. If $x \in D \cap \cancel{\mathrm{N}}_\delta(a)$, then

$$|f(x) - f(a)| = |c - c| = 0 < \varepsilon. \qquad \square$$

Fact 199. *Every identity function is continuous.*

Proof. Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$.

Let $\varepsilon > 0$. Pick any $\delta = \varepsilon$. If $x \in D \cap \cancel{\mathrm{N}}_\delta(a)$, then

$$|f(x) - f(a)| = |x - a| < \delta = \varepsilon. \qquad \square$$

Theorem 28. *Let f and g be nice functions with* $\operatorname{Range} g \subseteq \operatorname{Domain} f$. *If g is continuous at a and f is continuous at $g(a)$, then fg is also continuous at a.*

Proof. Let D_f and D_g be the domains of f and g. Let $\varepsilon > 0$.

Since f is continuous at $g(a)$, there exists $\hat{\delta} > 0$ such that for every $y \in D_f \cap \cancel{\mathrm{N}}_{\hat{\delta}}(g(a))$, we have $|f(y) - f(g(a))| < \varepsilon$.

Since g is continuous at a, there exists $\delta > 0$ such that for every $x \in D_g \cap \cancel{\mathrm{N}}_\delta(a)$, we have $|g(x) - g(a)| < \hat{\delta}$.

Hence, for every $x \in D_g \cap \cancel{\mathrm{N}}_\delta(a)$, we have $|(fg)(x) - (fg)(a)| < \varepsilon$. $\square$

Fact 287. *Let $D \subseteq \mathbb{R} \setminus \{0\}$. Suppose $f : D \to \mathbb{R}$ is defined by $f(x) = 1/x$. Then f is continuous.*

Proof. Let $a \in D$ and $\varepsilon > 0$. Pick $\delta \stackrel{1}{=} \min\left\{\frac{a^2\varepsilon}{2}, \frac{|a|}{2}\right\}$. Let $x \stackrel{2}{\in} D \cap \mathcal{N}_\delta(a)$.

Since $\delta < \frac{|a|}{2}$ and $x \in (a-\delta, a+\delta)$, we have $|x| > \frac{|a|}{2}$ and hence $\left|\frac{1}{x}\right| \stackrel{3}{<} \frac{2}{|a|}$. Now,

$$|f(x) - f(a)| = \left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{ax}\right| = \left|\frac{1}{a}\right|\left|\frac{1}{x}\right||x-a| \stackrel{2}{<} \left|\frac{1}{a}\right|\left|\frac{1}{x}\right|\delta$$

$$\stackrel{1}{<} \left|\frac{1}{a}\right|\left|\frac{1}{x}\right|\frac{a^2\varepsilon}{2} \stackrel{3}{<} \left|\frac{1}{a}\right|\frac{2}{|a|}\frac{a^2\varepsilon}{2} = \varepsilon.$$

□

Theorem 26. *Suppose f and g are continuous at $a \in \mathbb{R}$. Then so too is each of these functions:*

(a) $f \pm g$ **(b)** $f \cdot g$ **(c)** $\frac{f}{g}$ *(provided $g(a) \neq 0$)* **(d)** cf

Proof. Let $\varepsilon > 0$; D_f and D_g be the domains of f and g; and $\stackrel{\triangle}{\leq}$ and $\stackrel{+0}{=}$ denote the uses of the Triangle Inequality and Plus Zero Trick.

(a) Since f and g are continuous at a, there exists $\delta_1 > 0$ such that for every $x \in \mathcal{N}_{\delta_1}(a) \cap D_f \cap D_g$, we have $f(x) \in \mathrm{N}_{\varepsilon/2}(f(a))$ and $g(x) \in \mathrm{N}_{\varepsilon/2}(g(a))$, so that

$$|f(x) \pm g(x) - [f(a) \pm g(a)]| \stackrel{\triangle}{\leq} |f(x) - f(a)| + |g(x) - g(a)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(b) Let $\varepsilon_g \stackrel{1}{=} \frac{\varepsilon/2}{1+|f(a)|} > 0$ and $\varepsilon_f \stackrel{2}{=} \frac{\varepsilon/2}{\varepsilon_g + |g(a)|} > 0$.

Since f and g are continuous at a, there exists $\delta_2 > 0$ such that for every $x \in \mathcal{N}_{\delta_2}(a) \cap D_f \cap D_g$, we have $f(x) \stackrel{3}{\in} \mathrm{N}_{\varepsilon_f}(f(a))$ and $g(x) \stackrel{4}{\in} \mathrm{N}_{\varepsilon_g}(g(a))$, so that

$$\begin{aligned}
|f(x)g(x) - f(a)g(a)| &\stackrel{+0}{=} |f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)| \\
&\stackrel{\triangle}{\leq} |f(x)g(x) - f(a)g(x)| + |f(a)g(x) - f(a)g(a)| \\
&= |f(x) - f(a)||g(x)| + |f(a)||g(x) - g(a)| \\
&\stackrel{+0}{=} |f(x) - f(a)||g(x) - g(a) + g(a)| + |f(a)||g(x) - g(a)| \\
&\stackrel{\triangle}{\leq} |f(x) - f(a)|[|g(x) - g(a)| + |g(a)|] + |f(a)||g(x) - g(a)| \\
&\stackrel{3,4}{<} \varepsilon_f[\varepsilon_g + |g(a)|] + |f(a)|\varepsilon_g \stackrel{2}{=} \frac{\varepsilon}{2} + |f(a)|\varepsilon_g \stackrel{1}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$ [601]

(c) From Fact 287 and Theorem 28, the function $1/g$ is continuous at a. The result then follows by also using **(b)**.

(d) Since f is continuous, there exists $\delta_3 > 0$ such that for every $x \in \mathcal{N}_{\delta_3}(a) \cap D_f$, we have $f(x) \in \mathrm{N}_{\varepsilon/|c|}(f(a))$, so that

$$|cf(x) - cf(a)| \le |c|\,|f(x) - f(a)| < |c| \frac{\varepsilon}{|c|} = \varepsilon. \qquad \square$$

Theorem 25. *Let f and g be nice functions such that the composite function fg is well-defined. Let $b \in \mathbb{R}$. Suppose $\lim_{x\to a} g(x) = b$ and f is continuous at b. Then*

$$\lim_{x\to a} f(g(x)) = f(b).$$

Remark 213. It turns out that Theorem 25 is true not only in the case where $a \in \mathbb{R}$, but also in those cases where $a = \pm\infty$,[602] as is proven below.

We deliberately failed to mention this in the main text (and in particular did not specify $a \in \mathbb{R}$), so as not to confuse the average student.

Proof. Let D_f, D_g, and D_{fg} be the domains of f, g, and fg. Since fg is well-defined, we have $D_{fg} = D_g$.

Let $\varepsilon > 0$.

Since f is continuous at b, by Proposition 30, there exists $\alpha > 0$ such that

$$y \in D_f \cap \not\mathrm{N}_\alpha(b) \qquad \stackrel{1}{\Longrightarrow} \qquad f(y) \in \mathrm{N}_\varepsilon(f(b)).$$

We will consider three separate cases:

Case 1. $a \in \mathbb{R}$.

Since $\lim_{x\to a} g(x) = b$, by Definition 319, there exists $\delta > 0$ such that

$$\begin{aligned} x \in D_g \cap \not\mathrm{N}_\delta(a) \quad &\Longrightarrow \quad g(x) \in \mathrm{N}_\alpha(b) \stackrel{??}{=} D_f \cap \mathrm{N}_\alpha(b) = D_f \cap \not\mathrm{N}_\alpha(b) \cup \{b\} \\ &\stackrel{1}{\Longrightarrow} \quad f(g(x)) \in \mathrm{N}_\varepsilon(f(b)) \cup \{f(b)\} = \mathrm{N}_\varepsilon(f(b)). \end{aligned}$$

We've just shown that

$$x \in D_{fg} \cap \not\mathrm{N}_\delta(a) \qquad \Longrightarrow \qquad (fg)(x) \in \mathrm{N}_\varepsilon(f(b)).$$

And so, by Definition 319,

$$\lim_{x\to a} f(g(x)) = f(b).$$

Case 2. $a = \infty$.

Since $\lim_{x\to a} g(x) = \lim_{x\to\infty} g(x) = b$, by Definition 323, there exists $N \in \mathbb{R}$ such that $D_g \cap (N, \infty) \ne \varnothing$ and

$$\begin{aligned} x > N \quad &\Longrightarrow \quad g(x) \in D_f \cap \mathrm{N}_\alpha(b) = D_f \cap \not\mathrm{N}_\alpha(b) \cup \{b\} \\ &\stackrel{1}{\Longrightarrow} \quad f(g(x)) \in \mathrm{N}_\varepsilon(f(b)) \cap \{f(b)\} = \mathrm{N}_\varepsilon(f(b)). \end{aligned}$$

We've just shown that

$$x > N \qquad \Longrightarrow \qquad (fg)(x) \in \mathrm{N}_\varepsilon(f(b)).$$

Also, $D_{fg} \cap (N, \infty) \neq \varnothing$.

And so, by Definition 323,

$$\lim_{x \to a} f(g(x)) = f(b).$$

Case 3. $a = -\infty$.

Similar to Case 2. □

Take any continuous function f. Restrict its domain to create a new function g. Then g is also continuous. A bit more formally and precisely,

Theorem 52. *Let $E \subseteq D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a continuous function. Suppose $g : E \to \mathbb{R}$ is defined by $g(x) = f(x)$. Then g is also continuous.*

Proof. Let $a \in E$ and $\varepsilon > 0$.

By the continuity of f and Proposition 30, there exists $\delta > 0$ such that

$$x \in D \cap \cancel{\mathrm{N}}_{\delta}(a) \qquad \Longrightarrow \qquad f(x) \in \mathrm{N}_{\varepsilon}(f(a)).$$

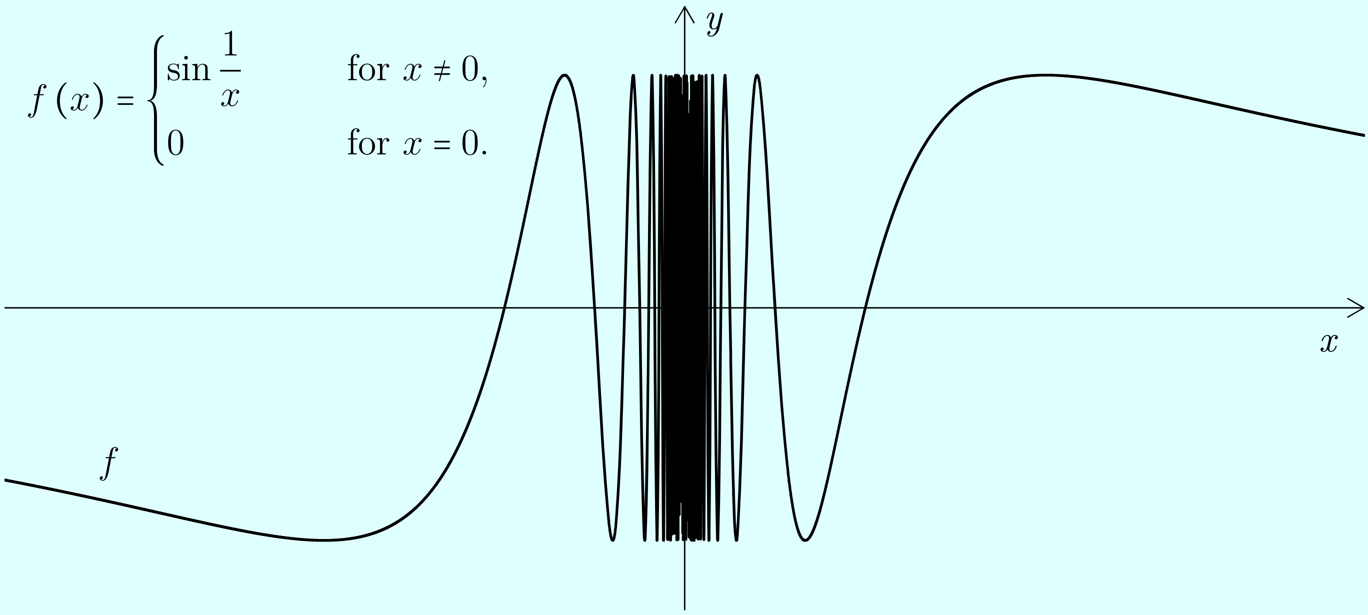
Since $E \subseteq D$ and $g(x) = f(x)$ for all $x \in E$, we have, in particular

$$x \in E \cap \cancel{\mathrm{N}}_{\delta}(a) \subseteq D \cap \cancel{\mathrm{N}}_{\delta}(a) \qquad \Longrightarrow \qquad g(x) = f(x) \in \mathrm{N}_{\varepsilon}(f(a)) = \mathrm{N}_{\varepsilon}(g(a)).$$

Thus, by Proposition 30, g is continuous at a. □

Example 1583. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$



Suppose (for contradiction) that $\lim_{x \to 0^+} f(x) = L \in \mathbb{R}$. Let $\varepsilon = 0.1$. Then there exists $\delta > 0$ such that

$$x \in D \cap \mathrm{N}_\delta^+(0) = (0, 0 + \delta) \qquad \Longrightarrow \qquad f(x) \in \mathrm{N}_\varepsilon(L) = (L - 0.1, L + 0.1).$$

But this is false, because given any $\delta > 0$, the function f takes on *all* values between -1 and 1 on $(0, 0 + \delta)$.[603]

With this contradiction, we've shown that $\lim_{x \to 0^+} f(x)$ does not exist. By Fact 197 then, 0 is an essential discontinuity.

Proposition 31. *Suppose that a function is increasing or decreasing. If the function's range is an interval, then it is continuous.*

Proof. Suppose $f : D \to \mathbb{R}$ is an increasing function. (The proof of the case where f is decreasing is similar and thus omitted.)

Let $a \in D$[604] and $\varepsilon > 0$. We will show that f is continuous at $a \in D$.

We first make two observations:

Observation I. Suppose f is not minimised at a.

Then since the range of f is an interval and f is increasing, there must exist $b < a$ such that $f(b) \in (f(a) - \varepsilon, f(a))$. And moreover, since f is increasing, for every $x \in (b, a)$, we must have $f(x) \in [f(b), f(a)]$ and in particular $f(x) \in \mathrm{N}_\varepsilon(a)$.

Similarly,

[603] Let $b \in [-1, 1]$. Pick any integer k such that $\sin^{-1} b + 2k\pi > 1/\delta$ (the existence of such an integer k is guaranteed by the Archimedean Property). Now, let $c = 1/\left(\sin^{-1} b + 2k\pi\right)$. Observe that $c < \delta$ and $f(c) = \sin\left(\sin^{-1} b + 2k\pi\right) = \sin\left(\sin^{-1} b\right) = b$. We've just shown that for any $b \in [-1, 1]$, there exists $c \in (0, 0 + \delta)$ such that $f(c) = b$.

[604] If no such a exists, then D is empty and f is continuous.

Observation II. Suppose f is not maximised at a.

Then since the range of f is an interval and f is increasing, there must exist $c > a$ such that $f(c) \in (f(a), f(a) + \varepsilon)$. And moreover, since f is increasing, for every $x \in (a, c)$, we must have $f(x) \in [f(a), f(c)]$ and in particular $f(x) \in \mathrm{N}_{\varepsilon}(a)$.

Depending on whether f is minimised and/or maximised at a, we have four possible cases to examine:

Case 1. Suppose f is both minimised *and* maximised at a.

Then f is a constant function and hence continuous (Fact 198).

Case 2. Suppose f is not minimised but is maximised at a.

Then using Observation I, pick $\delta = a - b$ so that for every $x \in D \cap \not{\mathrm{N}}_{\delta}^{-}(a)$, we have $f(x) \in \mathrm{N}_{\varepsilon}(a)$.

Since f is maximised at a and is an increasing function, for every $x > a$ (with $x \in D$), we must have $f(x) = f(a) \in \mathrm{N}_{\varepsilon}(a)$.

Altogether then, for every $x \in D \cap \not{\mathrm{N}}_{\delta}(a)$, we have $f(x) \in \mathrm{N}_{\varepsilon}(a)$.

Case 3. Suppose f is minimised but not maximised at a.

Then using Observation II, pick $\delta = c - a$ so that for every $x \in D \cap \not{\mathrm{N}}_{\delta}^{+}(a)$, we have $f(x) \in \mathrm{N}_{\varepsilon}(a)$.

Since f is minimised at a and is an increasing function, for every $x < a$ (with $x \in D$), we must have $f(x) = f(a) \in \mathrm{N}_{\varepsilon}(a)$.

Altogether then, for every $x \in D \cap \not{\mathrm{N}}_{\delta}(a)$, we have $f(x) \in \mathrm{N}_{\varepsilon}(a)$.

Case 4. Suppose f is neither minimised nor maximised at a.

Then using Observations I and II, pick $\delta = \min\{a - b, c - a\}$ so that for every $x \in D \cap \not{\mathrm{N}}_{\delta}(a)$, we have $f(x) \in \mathrm{N}_{\varepsilon}(a)$. □

Theorem 27. *Let D be an interval and $f : D \to \mathbb{R}$ be a one-to-one function. If f is continuous, then so too is its inverse f^{-1}.*

Proof. Since f is one-to-one and continuous on an interval, by Proposition 4, it is either strictly increasing or strictly decreasing. And so, by Proposition 6, the inverse f^{-1} is also either strictly increasing or strictly decreasing.

Observe that the range of f^{-1} is D, which is an interval. So, by Proposition 31, f^{-1} is continuous. □

The **Extreme Value Theorem** is the intuitively plausible result that if a function is continuous on a closed interval, then it attains both its minimum and maximum on that interval:

Theorem 53. (Extreme Value Theorem) *Suppose the function f is continuous on the closed interval $[a, b]$. Then there exist $m, M \in [a, b]$ such that for every $x \in [a, b]$, we have*

$$f(m) \le f(x) \le f(M).$$

Figure to be
inserted here.

Proof. Omitted.[605] □

[605]See e.g. Wikipedia.

146.9. The Derivative

In the main text, we made this informal claim:

$$\text{A function } f \text{ is differentiable at a point } a \qquad \Longleftrightarrow \qquad f \text{ is approximately linear at } a.$$

Here's a formal version of the above claim:

Proposition 32. *Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. Suppose $L \in \mathbb{R}$. Then statements* **(a)** *and* **(b)** *are equivalent:*

(a) $f'(a) = L$.

(b) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$x \in D \cap \cancel{N}_{\delta}(a) \qquad \Longrightarrow \qquad |f(x) - [f(a) + L(x-a)]| < \varepsilon |x-a|.$$

Proof. Statement **(a)** is equivalent to this statement:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in D \cap \cancel{N}_{\delta}(a) \qquad \Longrightarrow \qquad \left| \frac{f(x) - f(a)}{x-a} - L \right| < \varepsilon.$$

The above statement is in turn equivalent to **(b)**, because

$$\left| \frac{f(x) - f(a)}{x-a} - L \right| = \left| \frac{1}{x-a} [f(x) - f(a) - L(x-a)] \right|$$

$$= \left| \frac{1}{x-a} \right| |f(x) - f(a) - L(x-a)| = \left| \frac{1}{x-a} \right| |f(x) - [f(a) + L(x-a)]|. \qquad \square$$

Now, it's not completely obvious why Proposition 32 corresponds to our above informal claim. So, let's parse it a little.

Let t be the tangent line at a. The line t is described by this equation:

$$y = f(a) + L(x-a).$$

Let x_1 be any point that's "near" a. Then the value taken by t at x_1 is

$$f(x_1) + L(x_1 - a).$$

Figure to be inserted here.

The vertical distance between f and t at x_1 is

$$f(x) - [f(x_1) + L(x_1 - a)].$$

We can think of line t as our attempt to approximate f. We can thus also think of the above quantity as the **error** made by line t when attempting to approximate f.

Proposition 32 then says that this error is smaller than $\varepsilon |x_1 - a|$. Since ε is usually assumed to be a small quantity and x_1 is "near" a, $\varepsilon |x_1 - a|$ is itself a tiny quantity. Thus, Proposition 32 says that the error (made by line t when attempting to approximate f) is tiny and hence, t is a good approximation for f (at least for x "near" a).

Lemma 13. $\lim_{x\to a}(x-a) = 0$.

Proof. By the Difference, Power, and Constant Rules for Limits,

$$\lim_{x\to a}(x-a) = \lim_{x\to a} x - \lim_{x\to a} a = a - a = 0. \qquad \square$$

Lemma 14. *Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$. If $\lim_{x\to a}[f(x) - c] = 0$, then $\lim_{x\to a} f(x) = c$.*

Proof. Let $\varepsilon > 0$. If $\lim_{x\to a}[f(x)-c] = 0$, then by Definition 319, there exists $\delta > 0$ such that

$$x \in D \cap \cancel{\mathrm{N}}_\delta(a) \qquad \Longrightarrow \qquad f(x) - c \in \mathrm{N}_\varepsilon(0) \qquad \Longleftrightarrow \qquad f(x) \in \mathrm{N}_\varepsilon(c).$$

So, $\lim_{x\to a} f(x) = c$ (Definition 319 again). $\square$

We now use the last two lemmata to prove that differentiability implies continuity:

Theorem 29. *If a function is differentiable at a point, then it is also continuous at that point.*

Proof. Let $D \subseteq \mathbb{R}$, $f: D \to \mathbb{R}$, and $a \in D$. Suppose f is differentiable at a.

For $x \neq a$, we have

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x-a).$$

Since f is differentiable at a, $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists. Now,

$$0 \overset{1}{=} \lim_{x\to a}(x-a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a) \overset{\times}{=} \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)\right] = \lim_{x\to a}[f(x)-f(a)],$$

where $\overset{1}{=}$ uses Lemma 13 and $\overset{\times}{=}$ uses the Product Rule for Limits.

We've just shown that $\lim_{x\to a}[f(x)-f(a)] = 0$. So, by Lemma 14, $\lim_{x\to a} f(x) = f(a)$. Hence, by Definition 198, f is continuous at a. $\square$

In the main text (p. 884), we proved the Power Rule of Differentiation only in the special case where the exponent c is a non-negative integer. We now also prove it in the case where the base x is positive and c is *any* real exponent.

Fact 206. (Power Rule for Differentiation) *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a differentiable function. Suppose f is defined by $f(x) = x^c$ for some $c \in \mathbb{R}$. Then the derivative of f is the function $f' : D \to \mathbb{R}$ defined by*

$$f'(x) \stackrel{\text{P}}{=} cx^{c-1}.$$

In Ch. 146.27 (below), Definition 335 will give the **general definition of exponentiation**:

$$x^c \stackrel{\star}{=} \exp(c \ln x), \qquad \text{for } x > 0.$$

We will use $\stackrel{\star}{=}$ to prove Fact 206:

Proof. Suppose $x > 0$. Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} x^c &\stackrel{\star}{=} \frac{\mathrm{d}}{\mathrm{d}x}\left[\exp(c \ln x)\right] \\
&= \left[\exp(c \ln x)\right] \frac{\mathrm{d}}{\mathrm{d}x}(c \ln x) && \text{(Chain Rule)} \\
&= c \exp(c \ln x) \frac{\mathrm{d}}{\mathrm{d}x} \ln x && \text{(Constant Factor Rule for Differentiation)} \\
&= c \exp(c \ln x) \frac{1}{x} && \text{(Fact 221)} \\
&\stackrel{\star}{=} cx^c \frac{1}{x} \\
&= cx^{c-1}. && \text{(Laws of Exponents)}
\end{aligned}$$

We've just proven that the Power Rule of Differentiation holds in the case where $x > 0$.

It remains to be proven that the Power Rule also holds in the case where $x \le 0$. Unfortunately, this proof shall be omitted altogether from this textbook, for reasons that were already discussed in Remark **??**. □

146.10. "Most" Elementary Functions are Differentiable

In the main text, we made these imprecise claims:

"Most" elementary functions are differentiable.
Moreover, their derivatives are themselves elementary.

The following results are my attempt to justify the above claims.

Fact 288. *Let D be an interval. Suppose $f : D \to \mathbb{R}$ is a polynomial function. Then f is differentiable. Moreover, f' is elementary.*

Proof. Since f is a polynomial function, let $a_0, a_1, \ldots, a_k \in \mathbb{R}$ and $k \in \mathbb{Z}_0^+$, and suppose $f(x) = a_0 + a_1 x + \cdots + a_k x^k$ for every $x \in D$.

For each $i = 0, 1, \ldots, k$, define $h_i : D \to \mathbb{R}$ by $h_i(x) = a_i x^i$, so that $f(x) = \sum_{i=0}^{k} h_i(x)$ Observe that by the Constant, Constant Factor, and Power Rules for Differentiation, each h_i is differentiable with derivative $h_i' : \mathbb{R} \to \mathbb{R}$ defined by $h_i'(x_i) = i a_i x^{i-1}$.

Hence, by the Sum Rule for Differentiation, the derivative of f is the function $f' : D \to \mathbb{R}$ defined by

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=0}^{k} h_i(x) = \sum_{i=0}^{k} h_i'(x) = \sum_{i=0}^{k} i a_i x^{i-1}.$$

We've just shown that f is differentiable. Moreover, the derivative of f is itself a polynomial function and hence also an elementary function. □

Fact 289. *Let D be an interval. Suppose $f : D \to \mathbb{R}$ is a trigonometric function. Then f is differentiable. Moreover, f' is elementary.*

Proof. In H2 Maths, the only trigonometric functions we consider are sin, cos, tan, cosec, sec, and cot.

$\sin' = \cos$ (Fact 211) and cos is an elementary function.

$\cos' = -\sin$ (Fact 212) and $-\sin$ is an elementary function.

$\tan' = \sec^2$ and $\sec^2$ is an elementary function.

$\operatorname{cosec}' = \cos$ and $-\operatorname{cosec}\cot$ is an elementary function.

$\sec' = \cos$ and $\sec\tan$ is an elementary function.

$\cot' = -\operatorname{cosec}^2$ and $-\operatorname{cosec}^2$ is an elementary function. □

Fact 290. *Let D be an interval. Suppose $f : D \to \mathbb{R}$ is defined by $f(x) = \tan^{-1} x$. Then f is differentiable. Moreover, f' is elementary.*

Proof. The derivative of f is the function $f' : D \to \mathbb{R}$ defined by $f'(x) = 1/\left(1 + x^2\right)$, which is an elementary function. □

Fact 291. *Let $D \subseteq (-1,1)$ be an interval. Suppose $f, g : D \to \mathbb{R}$ are defined by $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$. Then f and g are differentiable. Moreover, f' and g' are elementary.*

Proof. The derivative of f is the function $f' : D \to \mathbb{R}$ defined by $f'(x) = 1/\sqrt{1-x^2}$, which is an elementary function.

The derivative of g is the function $g' : D \to \mathbb{R}$ defined by $g'(x) = -1/\sqrt{1-x^2}$, which is an elementary function. □

Fact 292. *Let $D \subseteq \mathbb{R}^+$ be an interval. Suppose $f : D \to \mathbb{R}$ is defined by $f(x) = \ln x$. Then f is differentiable. Moreover, the derivative of f is itself an elementary function.*

Proof. The derivative of f is the function $f' : D \to \mathbb{R}$ defined by $f'(x) = 1/x$, which is an elementary function. □

Fact 293. *Let D be an interval. Suppose $f : D \to \mathbb{R}$ is defined by $f(x) = \exp x$. Then f is differentiable. Moreover, the derivative of f is itself an elementary function.*

Proof. The derivative of f is the function $f' : D \to \mathbb{R}$ defined by $f'(x) = \exp x$, which is an elementary function. □

Fact 294. *Let D be an interval that does not contain zero and $c \in \mathbb{R}$. Suppose $f : D \to \mathbb{R}$ is defined by $f(x) = x^c$. Then f is differentiable. Moreover, the derivative of f is itself an elementary function.*

Proof. The derivative of f is the function $f' : D \to \mathbb{R}$ defined by $f'(x) = cx^{c-1}$, which is an elementary function. □

The next Corollary is immediate from the Sum, Difference, Product, and Quotient Rules for Differentiation and our definition of elementary functions:

Corollary 52. *Let D be an interval. Suppose $f, g : D \to \mathbb{R}$ are differentiable functions. Then $f+g$, $f-g$, $f \cdot g$ are differentiable and elementary.*

If moreover $g(x) \neq 0$ for all $x \in D$, then f/g is also differentiable and elementary.

The next Corollary is immediate from the Chain Rule:

Corollary 53. *Let D be an interval. Suppose $f, g : D \to \mathbb{R}$ are differentiable functions. Then fg is differentiable.*

One way to see why not all elementary functions are differentiable is to examine the following gaps in the above results:

- The arcsine and arccosine functions have domain $[-1,1]$. However, in Fact 291, we place the restriction that the domain D must be a subset of $(-1,1)$.

- In Fact 294, we place the restriction that the domain D does not contain zero.
- In Corollary 52, for the quotient f/g, we place the restriction that $g(x) \neq 0$ for all x.

Fact 295. *Let D be an interval. Suppose $f : D \to \mathbb{R}$ is a polynomial function of order k. Then f is smooth. Moreover, for every integer $n \geq k+1$, the nth derivative of f is the function $f^{(n)} : D \to \mathbb{R}$ defined by $f^{(n)}(x) = 0$.*

Proof. The first k derivatives of f domain D, are real-valued, and are defined by

$$\begin{aligned}
f'(x) &= ka_k x^{k-1} + (k-1)a_{k-1}x^{k-2} + \cdots + 2a_2 x + a_1 \\
f''(x) &= k(k-1)a_k x^{k-2} + (k-1)(k-2)a_{k-1}x^{k-3} + \cdots + 2a_2 \\
&\vdots \\
f^{(k-1)}(x) &= a_k k(k-1)\cdots\cdot 2x + a_{k-1}(k-1)! \\
f^{(k)}(x) &= a_k k!
\end{aligned}$$

For any $n \geq k+1$, the nth derivative of f is the function $f^{(n)} : D \to \mathbb{R}$ defined by

$$f^{(n)}(x) = 0.$$

We've just shown that for *every* positive integer n, the nth derivative of f is defined at every point in D, the domain of f. Hence, by Definition 209, f is smooth. $\square$

146.11. Leibniz Got the Product Rule Wrong

In Ch. 90.2, we mentioned that Leibniz initially got the Product Rule wrong. Here are more details on this story:

In an answer to a MathOverflow question, Francois Ziegler writes,

> In the manuscript "Determinationum progressio in infinitum" (...), Leibniz writes ... (with "$\sqcap$" in place of "="):
>
> $$\odot = \overline{dt} \int \frac{a^2}{a^2+t^2}. \quad \text{Hence} \quad \overline{d\odot} = \frac{a^2}{a^2+t^2}\overline{d\overline{dt}}$$
>
> This amounts to asserting that $d[uv] = dv\,du$ where $u = dt$ and $v = \int \frac{a^2}{a^2+t^2}$; and thus differentiating the product wrong, as the editors comment in footnote 14.

"Footnote 14" is from a 1923 edition of Leibniz's collected writings and letters. It reads (translation by Google Translate):

> Leibniz differenziert das Produkt zunächst falsch; in der weiteren Überlegung nähert er sich trotz zweier Fehler der richtigen Formel.
>
> *Leibniz initially differentiates the product wrongly; in the further consideration he approaches the right formula despite two mistakes.*

The 1923 editors date the above manuscript to "*Anfang* [Early] November 1675".

Note though that by 1675-11-11, Leibniz had clearly recognised that the naïve Product Rule was wrong. From Edwards (1979, p. 254):

> In the manuscript of November 11, he poses the questions as to whether $d(uv) = (du)(dv)$ and $d(v/u) = (dv)(du)$, and answers in the negative by noting that
>
> $$d\left(x^2\right) = (x+dx)^2 - x^2 = 2x\,dx + (dx)^2 = 2x\,dx,$$
>
> ignoring the higher-order infinitesimal, while
>
> $$(dx)(dx) = (x+dx-x)(x+dx-x) = (dx)^2.$$
>
> At this time Leibniz is still searching for the correct product and quotient rules.

By 1675-11-27, Leibniz had arrived at the correct Product Rule (translation from Child, 1920, p. 107):

> Ergo $d\overline{xy} \sqcap d\overline{xy} - xd\overline{y}$. Quod Theorema sane memorabile omnibus curvis commune est.
>
> *Therefore $d\overline{xy} \sqcap d\overline{xy} - xd\overline{y}$. Now this is really a noteworthy theorem and a general one for all curves.*

146.12. Proving the Chain Rule

In Ch. 90.4, we gave an incorrect "proof" of the Chain Rule that contained two flaws. The first flaw concerns the possibility that $g(x) - g(a) = 0$ and hence that we might be committing the Cardinal Sin of Dividing by Zero (see Ch. 2.2).

The second flaw concerns the use of this equation:

$$\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \stackrel{2}{=} f'(g(a)).$$

While plausible, $\stackrel{2}{=}$ requires some additional justification, as we now explain: By our definition of the derivative,

$$f'(g(a)) = \lim_{y\to g(a)} \frac{f(y) - f(g(a))}{y - g(a)}.$$

Hence, $\stackrel{2}{=}$ holds if and only if the following equation holds:

$$\lim_{y\to g(a)} \frac{f(y) - f(g(a))}{y - g(a)} = \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)},$$

which is again plausible, but not obviously true.

The following proof fixes both of these flaws.

Theorem 32. (Chain Rule) *Let f and g be nice functions with* $\text{Range}\, g \subseteq \text{Domain}\, f$. *Suppose f and g are differentiable. Then the function $f \circ g$ is differentiable and its derivative is*

$$(f' \circ g) \cdot g'.$$

Proof. Let $D = \text{Domain}\, f$, $E = \text{Domain}\, g$, and $a \in E$.

Define the function $h : E \to \mathbb{R}$ by

$$h(x) \stackrel{1}{=} \begin{cases} \dfrac{f(g(x)) - f(g(a))}{g(x) - g(a)}, & \text{for } g(x) \neq g(a), \\[2ex] f'(g(a)), & \text{for } g(x) = g(a). \end{cases}$$

We now state and prove two Lemmata:

Lemma 15. *If $x \in E$ with $x \neq a$, then*

$$h(x) \frac{g(x) - g(a)}{x - a} \stackrel{2}{=} \frac{f(g(x)) - f(g(a))}{x - a}.$$

Proof. If $g(x) \neq g(a)$, then

$$h(x)\frac{g(x)-g(a)}{x-a} \overset{1}{=} \frac{f(g(x))-f(g(a))}{g(x)-g(a)}\frac{g(x)-g(a)}{x-a} = \frac{f(g(x))-f(g(a))}{x-a}.$$

While if $g(x)=g(a)$, then

$$h(x)\frac{g(x)-g(a)}{x-a} \overset{1}{=} f'(g(a))\frac{0}{x-a} = 0 = \frac{f(g(x))-f(g(a))}{x-a}.$$

This completes the proof of Lemma 15. □

Lemma 16. $\lim_{x\to a} h(x) \overset{3}{=} f'(g(a))$.

Proof. Let $\varepsilon > 0$. Since $f'(g(a))$ exists,[606] there exists $\bar{\delta} > 0$ such that for every $y \in D \cap \cancel{N}_{\bar{\delta}}(g(a))$, we have

$$\left|\frac{f(y)-f(g(a))}{y-g(a)} - f'(g(a))\right| \overset{4}{<} \varepsilon.$$

Next, by the continuity of g, there exists $\delta > 0$ such that for every $x \in E \cap \cancel{N}_{\delta}(a)$, we have $g(x) \in D \cap N_{\bar{\delta}}(g(a))$ and hence also

$$|h(x)-f'(g(a))| \overset{1}{=} \begin{cases} \left|\dfrac{f(g(x))-f(g(a))}{g(x)-g(a)} - f'(g(a))\right| \overset{4}{<} \varepsilon & \text{for } g(x) \neq g(a), \\ |f'(g(a))-f'(g(a))| = 0 & \text{for } g(x) = g(a). \end{cases}$$

We've just shown that for any $\varepsilon > 0$, we can find $\delta > 0$ such that for every $x \in E \cap \cancel{N}_{\delta}(a)$, we have $|h(x)-f'(g(a))| < \varepsilon$. Hence,

$$\lim_{x\to a} h(x) \overset{3}{=} f'(g(a)).$$

This completes the proof of Lemma 16. □

We now complete the proof of the Chain Rule:[607]

[606] $f'(g(a))$ exists because Range $g \subseteq$ Domain f and f is differentiable.

[607] At $\overset{\times}{=}$, we can use the Product Rule because these two limits exist:

1. $\lim_{x\to a} h(x) \overset{3}{=} f'(g(a))$ (by Lemma 16)
2. $\lim_{x\to a} \dfrac{g(x)-g(a)}{x-a} = g'(a)$ (by the definition of the derivative)

$$
\begin{aligned}
(f\circ g)'(a) &= \lim_{x\to a}\frac{(f\circ g)(x)-(f\circ g)(a)}{x-a} = \lim_{x\to a}\frac{f(g(x))-f(g(a))}{x-a} \\
&\overset{2}{=} \lim_{x\to a}\left[h(x)\frac{g(x)-g(a)}{x-a}\right] \\
&\overset{\times}{=} \underbrace{\lim_{x\to a}h(x)}_{\overset{3}{=}f'(g(a))}\underbrace{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}_{g'(a)} = f'(g(a))\,g'(a).
\end{aligned}
$$

We've just shown that for any $a\in E$, the derivative of $f\circ g$ at a exists and is equal to $f'(g(a))\,g'(a)$.

Hence, the derivative of $f\circ g$ is the function $(f\circ g)':E\to\mathbb{R}$ defined by

$$(f\circ g)'(x) = f'(g(x))\,g'(x).$$

Or equivalently, the derivative of $f\circ g$ is

$$(f'\circ g)\cdot g'. \qquad \square$$

Below is a weak version of the **Inverse Function Theorem (IFT)**. It is weak because of the two strong assumptions in the second sentence.[608] These strong assumptions allow for an easy proof (using the Chain Rule.)

Theorem 54. (Inverse Function Theorem) *Let D be an open interval and $f:D\to\mathbb{R}$ be a differentiable function. Suppose f is one-to-one and its inverse f^{-1} is differentiable. Then for all $a\in D$ such that $f'(a)\neq 0$,*

$$\left(f^{-1}\right)'(f(a)) = \frac{1}{f'(a)}.$$

Proof. By Proposition XXX, $f^{-1}\circ f:D\to\mathbb{R}$ is defined by $\left(f^{-1}\circ f\right)(x) = x$. And so, by the Power Rule of Differentiation, $\left(f^{-1}\circ f\right)':D\to\mathbb{R}$ is differentiable and defined by $\left(f^{-1}\circ f\right)'(x)\overset{1}{=}1$.

Let $a\in D$ be such that $f'(a)\neq 0$. By the Chain Rule, $\left(f^{-1}\circ f\right)'(a)\overset{2}{=}\left(f^{-1}\right)'(f(a))\,(f)'(a)$.

Plug $\overset{1}{=}$ into $\overset{2}{=}$ to get $1=\left(f^{-1}\right)'(f(a))\,(f)'(a)$. Rearrange to get

$$\left(f^{-1}\right)'(f(a)) = \frac{1}{f'(a)}. \qquad \square$$

Corollary 54. (Parametric Differentiation Rule) *Let f and g be nice functions with* Range $g\subseteq$ Domain f. *Suppose f and g are differentiable. If $a\in$* Domain g *and $g'(a)\neq 0$, then*

$$(f'\circ g)(a) = \frac{(f\circ g)'(a)}{g'(a)}.$$

[608]These strong assumptions are usually absent from versions of the IFT usually presented elsewhere.

Proof. By the Chain Rule, $(f \circ g)'(a) = (f' \circ g)(a)\, g'(a)$. Rearrangement yields the result. $\square$

It's not at all obvious why the above result corresponds to our Parametric Differentiation Rule. To see why, replace f and g with the more familiar y and x. Then

$$\underbrace{f'(g)}_{\frac{\mathrm{d}y}{\mathrm{d}x}} = \underbrace{\frac{(fg)'}{g'}}_{\frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}}.$$

(As usual, t is a dummy variable that can be replaced by any other symbol.)

146.13. Fermat's Theorem on Extrema

Theorem 35. (Fermat's Theorem on Extrema) *Let f be a function that is differentiable on (a,b). Suppose $c \in (a,b)$.*

If c is an extremum of f, then $f'(c) = 0$.

Proof. We'll prove the contrapositive—i.e. if $f'(c) \neq 0$, then c is not an extremum of f.

Suppose $f'(c) > 0$. That is, suppose

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0.$$

Then there exists $\delta > 0$ such that for every $x \in (a,b) \cap \cancel{\mathrm{N}}_{\delta}(c)$,

$$\frac{f(x) - f(c)}{x - c} > 0,$$

and hence, $\quad x > c \iff f(x) > f(c) \quad$ and $\quad x < c \iff f(x) < f(c)$.

So, c is not an extremum of f.

If instead $f'(c) < 0$, then we'll similarly find that c is not an extremum of f. □

Definition 328. Let $S \subseteq \mathbb{R}$. We call $x \in S$ an *interior point of S* if there exists $\varepsilon > 0$ such that $\mathrm{N}_{\varepsilon}(x) \subseteq S$.

(Also, a point is called an *interior point of a function* if it is an interior point of that function's domain.)

146.14. Rolle's Theorem and the Mean Value Theorem

Theorem 55. (Rolle's Theorem) *Let $a < b$. Suppose f is a function that is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Figure to be
inserted here.

Proof. Since f is continuous on $[a, b]$, by the Extreme Value Theorem, f attains a maximum and a minimum on $[a, b]$.

Suppose f attains its maximum and minimum at a and b. Then since $f(a) = f(b)$, it must be that f is a constant function. And so, by the Constant Rule for Differentiation, $f'(x) = 0$ for all $x \in (a, b)$.

If either the maximum or minimum occurs at some $c \in (a, b)$, then by Fermat's Theorem on Extrema, $f'(c) = 0$. □

Using Rolle's Theorem, we can prove that **a function whose derivative is a zero function is itself is a constant function**:

Proposition 8. *Let $D, E \subseteq \mathbb{R}$ and $f : D \to E$ be a function. Suppose D is an interval. If the derivative of f is the function $f' : D \to \mathbb{R}$ defined by $f'(x) = 0$, then f is a constant function—i.e. f defined by $f(x) = c$ (for some $c \in E$).*

Proof. Since f is differentiable, D must contain more than one element.

So, pick any $a, b \in D$ with $a < b$. Note that since D is an interval, $[a, b] \subseteq D$.

Define $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = f(a + x(b - a)) - f(a) - x[f(b) - f(a)].$$

Observe that g is a differentiable function, with

$$g'(x) = (b - a) f'(a + x(b - a)) - [f(b) - f(a)] = 0 - [f(b) - f(a)] \overset{1}{=} f(a) - f(b).$$

Also, $g(0) = f(a) - f(a) - 0 = 0$ and

$$g(1) = f(a + 1(b - a)) - f(a) - 1[f(b) - f(a)] = 0.$$

Observe that g satisfies the assumptions of Rolle's Theorem. And so, there exists $d \in (0, 1)$ such that $g'(d) \overset{2}{=} 0$.

Together, $\overset{1}{=}$ and $\overset{2}{=}$ imply that $f(a) = f(b)$.

We've just shown that $f(a) = f(b)$ for any $a, b \in D$. Hence, f is constant on D.[609] $\square$

The **Mean Value Theorem** is a more general version of Rolle's Theorem:

Theorem 56. (Mean Value Theorem) *Let $a < b$. Suppose the function f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Figure to be
inserted here.

Proof. Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

The function g is continuous on $[a, b]$ and differentiable on (a, b), with

$$g'(x) \overset{1}{=} f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Also, $$g(a) = f(a) - \left[\frac{f(b) - f(a)}{b - a}(a - a) + f(a)\right] = 0.$$

And, $$g(b) = f(b) - \left[\frac{f(b) - f(a)}{b - a}(b - a) + f(a)\right] = 0.$$

Altogether, g satisfies the assumptions of Rolle's Theorem.

And so, there exists $c \in (a, b)$ such that $g'(c) \overset{2}{=} 0$. Together with $\overset{1}{=}$, we have

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \qquad \text{or} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}.$$ $\square$

609 This proof was stolen from copper.hat.

Fact 208. **(Increasing/Decreasing Test, IDT)** *Let f be a function and D be an interval. Suppose f is differentiable on D.*

(a) $f'(x) \geq 0$ for all $x \in D$	$\iff$	f is	increasing on D.
(b) $f'(x) > 0$ for all $x \in D$	$\implies$	f is strictly	increasing on D.
(c) $f'(x) \leq 0$ for all $x \in D$	$\iff$	f is	decreasing on D.
(d) $f'(x) < 0$ for all $x \in D$	$\implies$	f is strictly	decreasing on D.

Proof. **(a)** ($\implies$) Pick any $a, b \in D$ with $a < b$. By the Mean Value Theorem, there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Since $f'(c) \geq 0$ and $b - a > 0$, we have $f(b) \geq f(a)$.

(a) ($\impliedby$) We'll prove the contrapositive:

Pick any $e \in D$. Suppose $f'(e) < 0$. Then there exists some $\delta > 0$ such that for every $x \in D \cap \cancel{N}_\delta(e)$, we have

$$\frac{f(x) - f(e)}{x - e} < 0.$$

Hence, f is not increasing on D.

The proof of **(b)** is identical to **(a)** ($\implies$) .

The proofs of **(c)** and **(d)** are similar to **(a)** and **(b)** (respectively) and thus omitted. $\square$

Lemma 17. *Let $l < a < r$,[610] D be the interval (l,a) or $[l,a)$, and E be the interval (a,r) or $(a,r]$.*

Suppose $f : D \cup \{a\} \to \mathbb{R}$ is continuous.

(a) *If f is increasing on D, then $f(a) \geq f(x)$ for all $x \in D$.*

(b) *If f is strictly increasing on D, then $f(a) > f(x)$ for all $x \in D$.*

(c) *If f is decreasing on D, then $f(a) \leq f(x)$ for all $x \in D$.*

(d) *If f is strictly decreasing on D, then $f(a) < f(x)$ for all $x \in D$.*

Suppose $g : E \cup \{a\} \to \mathbb{R}$ is continuous.

(e) *If g is decreasing on E, then $g(a) \geq g(x)$ for all $x \in E$.*

(f) *If g is strictly decreasing on E, then $g(a) > g(x)$ for all $x \in E$.*

(g) *If g is increasing on E, then $g(a) \leq g(x)$ for all $x \in E$.*

(h) *If g is strictly increasing on E, then $g(a) < g(x)$ for all $x \in E$.*

Proof. **(a)** We'll prove the contrapositive: Suppose there exists $b \in D$ such that $f(a) < f(b)$. Let $\xi = [f(b) - f(a)]/2$. By the continuity of f, there exists $\delta \in (0, a-b)$ such that for all $x \in (a-\delta, a)$, we have $|f(x) - f(a)| < \xi$ and hence also $f(b) > f(x)$.

So, $b \notin (a-\delta, a)$. In particular, $b \leq a - \delta$.

Hence, for any $x \in (a-\delta, a)$, we have $b < x$ and $f(b) > f(x)$—so, f is not increasing on D.

(b) If f is strictly increasing on D, then it is also increasing on D, so that by **(a)**, $f(a) \overset{1}{\geq} f(x)$ for all $x \in D$.

Suppose (for contradiction) there exists $c \in D$ such that $f(c) \geq f(a)$. Since f is strictly increasing on D, for any $x \in (c,a)$, we have $f(x) > f(c) \geq f(a)$, contradicting $\overset{1}{\geq}$.

The proofs of **(c)** and **(d)** are similar to those of **(a)** and **(b)** and thus omitted.

The proofs of **(e)**–**(h)** are similar to those of **(a)**–**(d)** and thus omitted. □

Lemma 18. *Let D an interval, $f : D \to \mathbb{R}$ be a continuous function, and $a \in D$. We shall call $D \cap (-\infty, a)$ and $D \cap (a, \infty)$ "a's left" and as "a's right", respectively.*

(a) *If f is increasing on a's left and decreasing on a's right, then a is a global maximum of f.*

(b) *If f is strictly increasing on a's left and strictly decreasing on a's right, then a is a strict global maximum of f.*

(c) *If f is decreasing on a's left and increasing on a's right, then a is a global minimum of f.*

(d) *If f is strictly decreasing on a's left and strictly increasing on a's right, then a is a strict global minimum of f.*

[610] We allow for $l = -\infty$ and $r = \infty$. In such cases, D and E must be open.

Proof. **(a)** If f is increasing on a's left, then by Lemma 17**(a)**, $f(a) \geq f(x)$ for all $x \in D$. And if f is decreasing on a's right, then by Lemma 17**(e)**, $f(a) \geq f(x)$ for all $x \in D$. Hence, a is a local maximum of f.

The proofs of **(b)**–**(d)** are similar and thus omitted. □

Proposition 33. (First Derivative Test for Extrema, FDTE) *Let $a \in \mathbb{R}$, $\varepsilon > 0$, and f be a nice function that's continuous on $(a - \varepsilon, a + \varepsilon)$.*

(a) *If $f' \geq 0$ on $(a - \varepsilon, a)$ and $f' \leq 0$ on $(a, a + \varepsilon)$, then a is a local maximum of f.*

(b) *If $f'(x) > 0$ on $(a - \varepsilon, a)$ and $f'(x) < 0$ on $(a, a + \varepsilon)$, then a is a strict local maximum of f.*

(c) *If $f'(x) \leq 0$ on $(a - \varepsilon, a)$ and $f'(x) \geq 0$ on $(a, a + \varepsilon)$, then a is a local minimum of f.*

(d) *If $f'(x) < 0$ on $(a - \varepsilon, a)$ and $f'(x) > 0$ on $(a, a + \varepsilon)$, then a is a strict local minimum of f.*

Proof. **(a)** By the Increasing/Decreasing Test (IDT), f is increasing on a's left and decreasing on a's right. Hence, by Lemma 18, a is a global maximum of f.

The proofs of **(b)**–**(d)** are similar and thus omitted. □

Example 1584. Define $g: \mathbb{R} \to \mathbb{R}$ by[611]

$$g(x) = \begin{cases} x^4 \left(2 + \sin \dfrac{1}{x}\right), & \text{for } x \neq 0, \\ 0, & \text{for } x = 0, \end{cases}$$

Observe that for any $x \neq 0$, $g(x) = x^4 \left(2 + \sin \dfrac{1}{x}\right) \geq x^4 (2 - 1) = x^4 > 0$. Hence, 0 is a strict local (and also global) minimum of g.

Figure to be
inserted here.

Also, g is differentiable (and thus continuous) everywhere, as we now show. For $x \neq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^4 \left(2 + \sin \frac{1}{x}\right)\right] = 4x^3 \left(2 + \sin \frac{1}{x}\right) + x^4 \left(\cos \frac{1}{x}\right) \frac{-1}{x^2} = x^2 \left[4x \left(2 + \sin \frac{1}{x}\right) - \cos \frac{1}{x}\right].$$

(This last expression is defined for all $x \neq 0$.)

For $x = 0$, we have

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} \frac{x^4 \left(2 + \sin \frac{1}{x}\right)}{x} = \lim_{x \to 0} \left[x^3 \left(2 + \sin \frac{1}{x}\right)\right] = 0.$$

Altogether then,

$$g'(x) = \begin{cases} x^2 \left[4x \left(2 + \sin \dfrac{1}{x}\right) - \cos \dfrac{1}{x}\right], & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

Given any $\varepsilon > 0$, we can always find some $(a, b) \subseteq (0, \varepsilon)$ such that $g'(x) < 0$ for all $x \in (a, b)$,[612] so that by the Increasing/Decreasing Test (IDT), g is (strictly) decreasing on $(a, b) \subseteq (0, \varepsilon)$. Thus, the converses of **(c)** and **(d)** of Lemma 18 and Proposition 33 are false.

To similarly show that the converses of **(a)** and **(b)** of of Lemma 18 and Proposition 33 are false, consider $-g$.

[611] This example was stolen from Gelbaum and Olmsted, *Counterexamples in Analysis* (1964, p. 36).

[612] This should be easy to see by examining $g'(x)$. But lest you are not convinced, here is one possible construction:

Let $\varepsilon > 0$. Let $b = 1/(2k\pi)$, where k is any positive integer such that $b < \min\{\varepsilon, 1/8\}$. (The existence of such an integer k is guaranteed by the Archimedean Property.)

Proposition 10. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function and $c\in(a,b)$. Suppose c is the only stationary point of f in (a,b).*

(a) *If $f(c)$ is greater than $f(a)$ and $f(b)$, then c is the strict global maximum of f.*

(b) *If $f(c)$ is smaller than $f(a)$ and $f(b)$, then c is the strict global minimum of f.*

Proof. **(a)** Suppose for contradiction that there exists $x_1\in(a,c)$ such that $f(x_1)=f(c)$. Then by the Mean Value Theorem, there exists $x_2\in(x_1,c)$ such that $f'(x_2)=[f(c)-f(x_1)]/(c-$ 0, contradicting our assumption that c is the unique stationary point.

Suppose for contradiction that there exists $x_3\in(a,c)$ such that $f(x_3)>f(c)>f(a)$. Then by the Intermediate Value Theorem, there exists $x_4\in(a,x_3)\subseteq(a,c)$ such that $f(x_4)=f(c)$. We will then arrive at the contradiction given in the first paragraph of this proof.

The above two paragraphs show that for all $x\in(a,c)$, $f(x)<f(c)$. We can similarly show that for all $x\in(c,b)$, $f(x)<f(c)$. Since $f(c)$ is also greater than $f(a)$ and $f(b)$, we conclude that $f(c)>f(x)$ for all $x\in[a,b]$. Thus, c is the strict global maximum of f.

(b) Similar, omitted. □

Proposition 11. (Second Derivative Test for Extrema, SDTE) *Let f be a function that is twice differentiable at c. Suppose $f'(c)=0$ (i.e. c is a stationary point of f).*

(a) *If $f''(c)<0$, then c is a strict local maximum of f.*

(b) *If $f''(c)>0$, then c is a strict local minimum of f.*

(c) *If $f''(c)=0$, then the SDTE is inconclusive—more specifically, c could be any of the following:*

- *a strict local maximum;*
- *a strict local minimum;*
- *an inflexion point; or*
- *none of the above.*

Proof. **(a)** Suppose $f''(c)<0$. Then

$$0\overset{1}{>}f''(c)=\lim_{x\to c}\frac{f'(x)-f'(c)}{x-c}=\lim_{x\to c}\frac{f'(x)}{x-c}.$$

Since f is twice-differentiable at c, f' is continuous at c. And so, by $\overset{1}{>}$, there exists $\varepsilon>0$ such that $f'(x)>0$ for all $x\in(c-\varepsilon,c)$ and $f'(x)<0$ for all $x\in(c,c+\varepsilon)$. By the FDTE**(b)** then, c is a strict local maximum of f.

Observe that

$$g'(b)=b^2[4b(2+0)-1]=\underbrace{b^2}_{+}\underbrace{(8b-1)}_{-}<0.$$

By the continuity of g', there exists $\hat{\varepsilon}>0$ such that if we let $a=1/(2k\pi+\hat{\varepsilon})$, then $g'(x)<0$ for all $x\in(a,b)$.

The proof of **(b)** is similar and thus omitted.

(c) If $f''(c) = 0$, then c could be a strict local maximum (Example 1186), strict local minimum (Example 1187), inflexion point (Example 1188), or none of the above (Example 1189).

To prove that "c is *exactly* one of the following", simply note that **(i)** a strict local maximum cannot also be a strict local minimum; and **(ii)** by Fact 297, an inflexion point cannot be an extremum. $\square$

Definition 329. Let f be a function and D be an interval that contains more than one number.

If for every $a, b \in D$ and every $\lambda \in (0,1)$,

(a) $f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda) f(b)$, then f is *concave on D*.

(b) $f(\lambda a + (1-\lambda)b) > \lambda f(a) + (1-\lambda) f(b)$, then f is *strictly concave on D*.

(c) $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b)$, then f is *convex on D*.

(d) $f(\lambda a + (1-\lambda)b) < \lambda f(a) + (1-\lambda) f(b)$, then f is be *strictly convex on D*.

(d) $f(\lambda a + (1-\lambda)b) = \lambda f(a) + (1-\lambda) f(b)$, then f is *linear on D*.

A *concave function* is one that's concave on its domain.

A *strictly concave function* is one that's strictly concave on its domain.

A *convex function* is one that's convex on its domain.

A *strictly convex function* is one that's strictly convex on its domain.

A *linear function* is one that's linear on its domain.

It is immediate from the above definition that "linearity $\iff$ concavity and convexity"

Fact 296. *Suppose f is a function and D is an interval that contains more than one number. Then*

$$f \text{ is linear on } D \qquad \iff \qquad f \text{ is both concave and convex on } D.$$

The following lemma characterises concavity and can serve as an alternative definition thereof:

Lemma 19. *Suppose f is a function and D is an interval that contains more than one number. Then*

(a) *f is concave on D $\iff$ For any $x_1, x_2, x_3 \in D$ with $x_1 < x_2 < x_3$, we have*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

(b) *Same as* **(a)**, *but replace* concave *and* $\geq$ *with* strictly concave *and* $>$.

(c) *Same as* **(a)**, *but replace* concave *and* $\geq$ *with* convex *and* $\leq$.

(d) *Same as* **(a)**, *but replace* concave *and* $\geq$ *with* strictly convex *and* $<$.

Proof. **(a)** ($\Longrightarrow$) Let $x_1, x_2, x_3 \in D$ with $x_1 < x_2 < x_3$ and $\lambda = \dfrac{x_3 - x_2}{x_3 - x_1}$.

Observe that $x_3 - x_1 > x_3 - x_2 > 0$, so that $\lambda \in (0,1)$.

Also, $\lambda x_1 + (1-\lambda) x_3 = \dfrac{x_3 - x_2}{x_3 - x_1} x_1 + \dfrac{x_2 - x_1}{x_3 - x_1} x_3 = x_2$.

If f is concave on D, then

$$\begin{aligned}
& & f(x_2) &\geq \lambda f(x_1) + (1-\lambda) f(x_3) \\
& & &= \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3) \\
&\iff & (x_3 - x_1) f(x_2) &\geq (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3) \\
&\iff & (x_3 - x_2 + x_2 - x_1) f(x_2) &\geq (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3) \\
&\iff & (x_3 - x_2)[f(x_2) - f(x_1)] &\geq (x_2 - x_1)[f(x_3) - f(x_2)] \\
&\iff & \frac{f(x_2) - f(x_1)}{x_2 - x_1} &\geq \frac{f(x_3) - f(x_2)}{x_3 - x_2}. \qquad ✓
\end{aligned}$$

(a) ($\Longleftarrow$) Let $x_1, x_3 \in D$ with $x_1 < x_3$ and $\lambda \in (0,1)$. Let $x_2 = \lambda x_1 + (1-\lambda) x_3$.

If $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \dfrac{f(x_3) - f(x_2)}{x_3 - x_2}$, then as shown above,

$$f(\lambda x_1 + (1-\lambda) x_3) = f(x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_3).$$

So, f is concave. ✓

To prove **(b)**, simply replace the appropriate weak inequalities in our proof of **(a)** with strict ones.

The proofs of **(c)** and **(d)** are similar to **(a)** and **(d)** (respectively) and thus omitted. □

Proposition 12. (First Derivative Test for Concavity, FDTC) *Let D be an interval with endpoints l and r. Suppose $f : D \to \mathbb{R}$ is differentiable. Then*

(a)	*f' is decreasing on (l,r)*	$\iff$	*f is concave on D.*
(b)	*f' is strictly decreasing on (l,r)*	$\iff$	*f is strictly concave on D.*
(c)	*f' is increasing on (l,r)*	$\iff$	*f is convex on D.*
(d)	*f' is strictly increasing on (l,r)*	$\iff$	*f is strictly convex on D.*

Proof. Let $a, b \in D$ with $a < b$.

(a) ($\Longrightarrow$) Suppose f is concave on D.

By Lemma 19(a), for every $c \in (a,b)$,

$$\frac{f(c) - f(a)}{c - a} \geq \frac{f(b) - f(c)}{b - c}.$$

$$\begin{aligned}
\text{Hence,} & & \lim_{c \to a^+} \frac{f(c) - f(a)}{c - a} &\geq \lim_{c \to b^-} \frac{f(b) - f(c)}{b - c} \\
\iff & & f'(a) &\geq f'(b). \qquad ✓
\end{aligned}$$

(a) ($\Longleftarrow$) Suppose f' is decreasing on (l,r).

Let $\lambda \in (0,1)$.

By the MVT, there exist d and e with $a < d < (\lambda a + (1-\lambda) b) < e < b$ such that

$$f'(d) = \frac{f(\lambda a + (1-\lambda) b) - f(a)}{\lambda a + (1-\lambda) b - a} = \frac{f(\lambda a + (1-\lambda) b) - f(a)}{(1-\lambda)(b-a)}$$

and $$f'(e) = \frac{f(b) - f(\lambda a + (1-\lambda) b)}{b - [\lambda a + (1-\lambda) b]} = \frac{f(b) - f(\lambda a + (1-\lambda) b)}{\lambda (b-a)}$$

Since f' is decreasing, $f'(d) \geq f'(e)$ or

$$\frac{f(\lambda a + (1-\lambda) b) - f(a)}{(1-\lambda)(b-a)} \geq \frac{f(b) - f(\lambda a + (1-\lambda) b)}{\lambda (b-a)}$$

$$\iff \qquad \lambda [f(\lambda a + (1-\lambda) b) - f(a)] \geq (1-\lambda)[f(b) - f(\lambda a + (1-\lambda) b)]$$

$$\iff \qquad f(\lambda a + (1-\lambda) b) \geq \lambda f(a) + (1-\lambda) f(b). \qquad \checkmark$$

(b) ($\Longrightarrow$) Suppose f is strictly concave on D.

By Lemma 19(a), for every c, d, e such that $a < c < d < e < b$,

$$\frac{f(c) - f(a)}{c-a} \overset{1}{>} \frac{f(d) - f(c)}{d-c} \overset{2}{>} \frac{f(e) - f(d)}{e-d} \overset{3}{>} \frac{f(b) - f(e)}{b-e}.$$

By $\overset{1}{>}$,

$$\overbrace{\lim_{c \to a^+} \frac{f(c) - f(a)}{c-a}}^{f'(a)} \overset{4}{\geq} \frac{f(d) - f(c)}{d-c}.$$

Similarly, by $\overset{3}{>}$,

$$\frac{f(e) - f(d)}{e-d} \overset{5}{\geq} \overbrace{\lim_{e \to b^-} \frac{f(b) - f(e)}{b-e}}^{f'(b)}.$$

Together, $\overset{2}{>}$, $\overset{4}{\geq}$, and $\overset{4}{\geq}$ imply that $f'(a) > f'(b)$. $\checkmark$

(b) ($\Longleftarrow$) Similar to **(a)** (just replace the appropriate weak inequalities with strict ones). $\checkmark$

The proofs of **(c)** and **(d)** are similar to **(a)** and **(b)** (respectively) and thus omitted. $\square$

Fact **209**. **(Graphical Test for Concavity, GTC)** *Let D be an interval and f be a function. Then*

(a)	*f is concave*	*on D*	$\iff$	*f is on or above the line segment connecting any two points in D.*
(b)	*f is strictly concave*	*on D*	$\iff$	*f is above the line segment connecting any two points in D (excluding the two endpoints).*
(c)	*f is convex*	*on D*	$\iff$	*f is on or below the line segment connecting any two points in D.*
(d)	*f is strictly convex*	*on D*	$\iff$	*f is below the line segment connecting any two points in D (excluding the two endpoints).*

(a) f is concave on D

Proof. $\iff$ For every $a, b \in D$ and $\lambda \in (0,1)$, $f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda) f(b)$[6]

$\iff$ f lies on or above the line segment connecting any two points in D.

(b) Same proof as **(a)**, but replace "concave", "$\geq$", and "on or above" with "strictly concave", "$>$", and "above", respectively.

The proofs of **(c)** and **(d)** are similar to those of **(a)** and **(b)**. $\square$

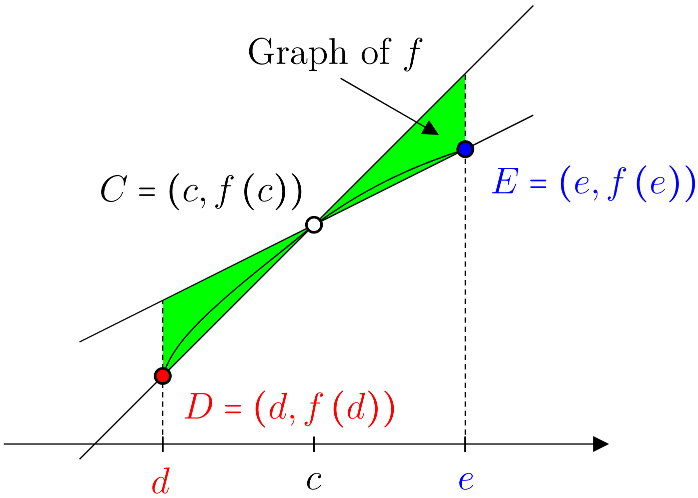
Concavity (or convexity) implies continuity:

Proposition 34. *Suppose $a < b$. If $f : (a,b) \to \mathbb{R}$ is concave, then f is continuous.*

The previous sentence is also true if concave *is replaced by* strictly concave *or* convex *or* strictly convex.

Proof. Suppose f is concave and $c \in (a,b)$. We will show that f is continuous at c.

Pick any $d \in (a,c)$ and any $e \in (c,b)$.



Consider the points $C = (c, f(c))$, $D = (d, f(d))$, and $E = (e, f(e))$; and the lines DC and CE. Observe that the lines DC and CE are described, respectively, by

[613] At $\lambda a + (1-\lambda) b$, the height of the graph of f is $f(\lambda a + (1-\lambda) b)$ and that of the line segment from a to b is $\lambda f(a) + (1-\lambda) f(b)$.

$$y = f(c) + \frac{f(c) - f(d)}{c - d}(x - c) \qquad \text{and} \qquad y = f(c) + \frac{f(e) - f(c)}{e - c}(x - c).$$

By the GTC (Fact 209), for any $x \in (d, c)$, the point $(x, f(x))$ must lie **(i)** on or above the line DC; *and* **(ii)** on or below the line CE.[614]

Similarly, for any $x \in (c, e)$, the point $(x, f(x))$ must lie **(i)** on or above the line CE; and **(ii)** on or below the line DC.

Altogether, for any $x \in (d, e)$, the point $(x, f(x))$ must lie in the green region. So, it is intuitively "obvious" that as x gets "nearer" to c, $f(x)$ must also get "nearer" to $f(c)$. In other words, f is continuous at c.[615]

Let's prove this last bit of "obvious" intuition a little more rigorously. For any $x \in (d, e)$, we have

$$\min\left\{f(c) + \frac{f(c) - f(d)}{c - d}(x - c), f(c) + \frac{f(e) - f(c)}{e - c}(x - c)\right\} \leq f(x) \leq \max\left\{f(c) + \frac{f(c) - f(d)}{c - d}\right.$$

Hence,

$$\begin{aligned} |f(x) - f(c)| &\leq \max\left\{\frac{|f(c) - f(d)|}{c - d}|x - c|, \frac{|f(e) - f(c)|}{e - c}|x - c|\right\} \\ &= |x - c| \max\left\{\frac{|f(c) - f(d)|}{c - d}, \frac{|f(e) - f(c)|}{e - c}\right\}. \end{aligned}$$

If $f(c) - f(d) = f(e) - f(c) = 0$, then by Lemma 20 (below), f is constant on (d, e) and hence also continuous on (d, e).

So, suppose that at least one of $f(c) - f(d)$ or $f(e) - f(c)$ is non-zero.

Let $\varepsilon > 0$. Pick

$$\delta = \frac{\varepsilon}{2 \max\left\{\frac{|f(c) - f(d)|}{c - d}, \frac{|f(e) - f(c)|}{e - c}\right\}}.$$

Then for any $x \in \cancel{\mathrm{N}}_\delta(c) \subseteq (d, c) \cup (c, e)$, we have $|x - c| < \delta$ and

$$\begin{aligned} |f(x) - f(c)| &\leq |x - c| \max\left\{\frac{|f(c) - f(d)|}{c - d}, \frac{|f(e) - f(c)|}{e - c}\right\} \\ &< \delta \max\left\{\frac{|f(c) - f(d)|}{c - d}, \frac{|f(e) - f(c)|}{e - c}\right\} = \frac{\varepsilon}{2}. \end{aligned}$$

We've just shown that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in (a, b) \cap \cancel{\mathrm{N}}_\delta(c)$ implies $f(x) \in \mathrm{N}_\varepsilon(f(c))$. That is, we've just shown that f is continuous at c. □

Lemma 20. *Let $d < c < e$. Suppose $f : [d, e] \to \mathbb{R}$ is concave. If $f(d) = f(c) = f(e)$, then f is constant.*

[614] **(i)** is direct from the GTC. To see **(ii)**, suppose for contradiction that the point is above the line CE—then C would be below the line segment connecting this point and E, contradicting the GTC.

[615] This "proof by picture" was stolen from .

Proof. Suppose for contradiction there exists $a \in (d, c) \cup (c, e)$ such that $f(a) \neq f(d)$.

- If $f(a) > f(d)$ and $a \in (d, c)$, then the line segment connecting $(a, f(a))$ and $(e, f(e))$ lies above $(c, f(c))$, contradicting the Graphical Test for Concavity (GTC, Fact 209).
- If $f(a) > f(d)$ and $a \in (c, e)$, then the line segment connecting $(d, f(d))$ and $(a, f(a))$ lies above $(c, f(c))$, contradicting the GTC.
- If $f(a) < f(d)$ and $a \in (d, c)$, then the line segment connecting $(d, f(d))$ and $(c, f(c))$ lies above $(a, f(a))$, contradicting the GTC.
- If $f(a) < f(d)$ and $a \in (c, e)$, then the line segment connecting $(c, f(c))$ and $(e, f(e))$ lies above $(a, f(a))$, contradicting the GTC.

Hence, no such a exists and f is constant. □

Definition 330. We call c an *inflexion point* of a function if the function is continuous at c and there exists $\varepsilon > 0$ such that the function satisfies either statement **(a)** or **(b)**:

(a) Strictly concave on $(c-\varepsilon, c)$ & strictly convex on $(c, c+\varepsilon)$.

(b) Strictly convex on $(c-\varepsilon, c)$ & strictly concave on $(c, c+\varepsilon)$.

Proposition 35. (First Derivative Test for Inflexion Points, FDTI) *Let f be a function and c be a point. If c is an inflexion point of f, then c is a strict local extremum of f'.*

Proof. Since c is an inflexion point, by Definition 330, there exists $\bar{\varepsilon} > 0$ with $\bar{\varepsilon} < \varepsilon$ such that either **(a)** f is strictly concave on $(c-\bar{\varepsilon}, c)$ and strictly convex on $(c, c+\bar{\varepsilon})$; OR **(b)** f is strictly convex on $(c-\bar{\varepsilon}, c)$ and strictly concave on $(c, c+\bar{\varepsilon})$.

Suppose **(a)**. Then by the FDTC (Proposition 12), f' is strictly decreasing on $(c-\bar{\varepsilon}, c)$ and strictly increasing on $(c, c+\bar{\varepsilon})$. And so, by the FDTE (Proposition 33), c is a strict local minimum of f'.

Suppose **(b)**. Then we can similarly show that c is a strict local maximum of f'. □

As noted in Remark 141, the converse of the FDTI is false:

Example 1585. Define $f : \mathbb{R} \to \mathbb{R}$ by[616]

$$f(x) = \begin{cases} x^3 + x^4 \sin \dfrac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

With some work, we can verify that

- f is differentiable
- f' is continuous
- 0 is a strict local minimum of f'

However, 0 is **not** an inflexion point of f.

Proposition 36. (Second Derivative Test for Inflexion Points, SDTI) *Let f be a function and c be a point. Suppose f is twice-differentiable on some neighbourhood of c. If c is an inflexion point of f, then $f''(c) = 0$.*

Proof. By the First Derivative Test for Inflexion Points (FDTI), c is a strict local extremum of f'. So, by Fermat's Theorem on Extrema, $f''(c) = 0$. □

We now discuss the **Tangent Line Test (TLT)**.

Let f be a function and c be a point. Consider the tangent line of f at c. It is described by

[616] This example was stolen from Kouba (1995), "Can We Use the First Derivative to Determine Inflection Points?"

$$y = f(c) + f'(c)(x - c).$$

If at $t \neq c$, the tangent line is strictly

- Above f, then $f(c) + f'(c)(t - c) > f(t)$.
- Below f, then $f(c) + f'(c)(t - c) < f(t)$.

Hence to say that the tangent line is strictly above f on c's "immediate left" and strictly below f on c's "immediate right" is equivalent to this statement:

There exists $\varepsilon > 0$ such that

$$f(c) + f'(c)(x - c) \overset{1}{>} f(x) \qquad \text{for all } x \in (c - \varepsilon, c).$$

and
$$f(c) + f'(c)(x - c) \overset{2}{<} f(x) \qquad \text{for all } x \in (c, c + \varepsilon).$$

Or equivalently, there exists $\varepsilon > 0$ such that[617]

$$f'(c) \overset{3}{<} \frac{f(x) - f(c)}{x - c} \qquad \text{for all } x \in (c - \varepsilon, c) \cup (c, c + \varepsilon).$$

Conversely and similarly, to say that the tangent line is strictly above f on c's "immediate right" and strictly below f on c's "immediate left" is equivalent to this statement:

There exists $\varepsilon > 0$ such that

$$f'(c) > \frac{f(x) - f(c)}{x - c} \qquad \text{for all } x \in (c - \varepsilon, c) \cup (c, c + \varepsilon).$$

Hence, the TLT which was given informally in the main text may be rewritten formally as follows:

Proposition 37. (Tangent Line Test, TLT) *Let f be a function and c be a point. If c is an inflexion point of f, then there exists some $\varepsilon > 0$ such that for all $x \in (c - \varepsilon, c) \cup (c, c + \varepsilon)$, either* **(a)** *or* **(b)** *is true:*

$$\textbf{(a)} \quad f'(c) < \frac{f(x) - f(c)}{x - c} \qquad\qquad \textbf{(b)} \quad f'(c) > \frac{f(x) - f(c)}{x - c}.$$

Proof. By the First Derivative Test for Inflexion Points (FDTI), c is a strict local extremum of f'. That is, there exists $\varepsilon > 0$ such that for all $x \in (c - \varepsilon, c) \cup (c, c + \varepsilon)$, either statement **(i)** or **(ii)** is true:

$$\textbf{(i)} \quad f'(c) < f'(x) \qquad\qquad \textbf{(ii)} \quad f'(c) > f'(x).$$

Let $x \in (c - \varepsilon, c) \cup (c, c + \varepsilon)$. By the Mean Value Theorem (MVT), there exists a between x and c such that

[617] To get $\overset{3}{<}$, rearrange $\overset{1}{>}$ and $\overset{2}{<}$, in each case dividing by $x - c$. Note that in the case of $\overset{1}{>}$, $x - c < 0$ and so when dividing by $x - c$, we must reverse the inequality.

$$f'(a) \overset{1}{=} \frac{f(x) - f(c)}{x - c}.$$

So, if **(i)** is true, then **(a)** $f'(c) < \dfrac{f(x) - f(c)}{x - c}$ is true.

Similarly, if **(ii)** is true, then **(b)** is true. □

There is the possibility that **an inflexion point is an extremum**:

Example 1586. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{for } x \le 0, \\ \sqrt{x} & \text{for } x > 0. \end{cases}$$

Figure to be
inserted here.

Clearly, 0 is a strict local minimum (and indeed a strict global minimum) of f.[618]

Observe that f is continuous. Moreover, f is strictly convex on $\mathbb{R}^-$ and strictly concave on $\mathbb{R}^+$. And so by Definition 330, 0 is an inflexion point of f.

And so, awkwardly enough, we have a point that is both an inflexion point and a strict local minimum.

(By the way, f is smooth everywhere except at 0, where it is not differentiable.)

The above awkward possibility is eliminated if we assume the function is twice-differentiable on some neighbourhood of the point:

Fact 297. *Let f be a function and c be a point. Suppose f is twice-differentiable on some neighbourhood of c. If c is an inflexion point of f, then c is not an extremum of f.*

Proof. Since c is an inflexion point of f, by Proposition 35, c is a strict local extremum of f'.

Suppose for contradiction that c is an extremum of f. Then by Fermat's Theorem on Extrema, $f'(c) \varnothing 10$.

Since c is a strict local extremum of f', we have on some deleted neighborhood of c either **(i)** $f' < 0$; or **(ii)** $f' > 0$.

If **(i)**, then $f > f(c)$ on some left neighborhood of c and $f < f(c)$ on some right neighborhood of c, so that c is not an extremum of f (contradiction).

[618] For all $x \neq 0$, we have $f(x) > 0 = f(0)$.

Similarly, if **(ii)**, then $f < f(c)$ on some left neighborhood of c and $f > f(c)$ on some right neighborhood of c, so that again c is not an extremum of f (contradiction). $\square$

146.19. Sine and Cosine

In this appendix, we'll show that our formal definitions of the sine and cosine functions (Definitions 219 and 220 on p. 1012) correspond to our earlier unit-circle definitions.

XXX To be written

Definition 331. Given the two series $A = \sum_{n=0}^{\infty} a_n$ and $B = \sum_{n=0}^{\infty} b_n$, define

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}, \qquad \text{for each } n \in \mathbb{Z}_0^+.$$

Then the *Cauchy product* of A and B is the series $\sum_{n=0}^{\infty} c_n$,

Definition 332. We say that the series $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* and *converges absolutely to* L if:

$$\sum_{n=0}^{\infty} |a_n| = L.$$

Theorem 57. *Suppose the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge to A and B. For each $n \in \mathbb{Z}_0^+$, let $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. If $\sum_{n=0}^{\infty} a_n$ is also absolutely convergent, then $\sum_{n=0}^{\infty} c_n$ converges to AB.*

Proof. Omitted.[619] □

Remark 214. If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent but neither is *absolutely* convergent, then $\sum_{n=0}^{\infty} c_n$ may not converge.

Theorem 58. *Let f and g be functions and $R_F, R_G \in \mathbb{R}^+ \cup \{\infty\}$. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all $x \in (-R_F, R_F)$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ for all $x \in (-R_G, R_G)$, If $c \in (-R_G, R_G)$ and $\sum_{n=0}^{\infty} b_n c^n$ converges absolutely to some number in $(-R_F, R_F)$, then*

$$(fg)(c) = \sum_{n=0}^{\infty} a_n \left(\sum_{m=0}^{\infty} b_m c^m \right)^n.$$

Proof. Omitted.[620] □

[619] See e.g. Apostol (1974, p. 204, Theorem 8.46) or Rudin (1976, p. 74, Theorem 3.50).
[620] See e.g. Apostol (1974, p. 238, Theorem 9.25: "Substitution Theorem") or Lang (1999, p. 66, Theorem 3.4: "Composition of Series").

146.21. Antidifferentiation

Example of a function that is antidifferentiable but not continuous:[621]

Example 1587. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

For $x \neq 0$, we have

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

Also,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0.$$

Hence, f is differentiable and its derivative is the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

Thus, g is an antidifferentiable function. However, g is not continuous.

Fact 215. *Let $f(x)$ and $g(x)$ be expressions containing the variable x. If $\dfrac{\mathrm{d}}{\mathrm{d}x} g(x) = f(x)$, then $\int f(x)\,\mathrm{d}x = g(x) + C$.*

Proof. Suppose h is a differentiable function defined by $h(x) = g(x)$. Since $\dfrac{\mathrm{d}}{\mathrm{d}x} g(x) = f(x)$, the derivative of h is defined by $h'(x) = f(x)$.

Now, suppose the function i is defined on an interval and by $i(x) = f(x)$. Then by Corollary 47, its antiderivatives are exactly those functions defined by $i'(x) = g(x) + C$ (for $C \in \mathbb{R}$). That is,

$$\int f(x)\,\mathrm{d}x = g(x) + C. \qquad \square$$

The converse of Fact 215 is not true for the simple and pedantic reason that in Definition 223 but not in Definition 206, there is the requirement that the function's domain be an interval.

[621] See Wikipedia for more examples.

146.22. The Definite Integral

The following definition of the definite integral heavily favours simplicity over generality. Its chief advantage is that it simply formalises the intuition discussed in Ch. 104.4. Another is that it avoids any mention of suprema and infima (concepts we haven't discussed in this textbook).

Definition 333. Let $a < b$, f be a real-valued function defined on $[a, b]$, $n \in \mathbb{Z}_0^+$, and

$$S_n = \frac{b-a}{2^n} \sum_{i=1}^{2^n} f\left(a + \left(i - \frac{1}{2}\right) \frac{b-a}{2^n}\right).$$

Suppose $\lim_{n\to\infty} S_n$ exists. Then we say that f is *integrable on* $[a, b]$. Moreover, we call $\lim_{n\to\infty} S_n$ the *definite integral of f from a to b* is S and will often denote it by

$$\int_a^b f, \qquad \int_a^b f \,\mathrm{d}x, \quad \text{or} \quad \int_a^b f(x) \,\mathrm{d}x.$$

If $\lim_{n\to\infty} S_n$ does not exist, then f is *not integrable on* $[a, b]$ and the definite integral of f from a to b does not exist.

Explanation. Partition $[a, b]$ into 2^n closed intervals of equal width. Then each such interval has width $\frac{b-a}{2^n}$ and midpoint $a + \left(i - \frac{1}{2}\right) \frac{b-a}{2^n}$.

For each such interval, consider the corresponding rectangle that has width $\frac{b-a}{2^n}$ and height $f\left(a + \left(i - \frac{1}{2}\right) \frac{b-a}{2^n}\right)$. This rectangle's area is $\frac{b-a}{2^n} f\left(a + \left(i - \frac{1}{2}\right) \frac{b-a}{2^n}\right)$.
We have in total 2^n such rectangles. And their total area is S_n.

Figure to be
inserted here.

To prove some of the following results, we'll need **uniform continuity**:

Definition 334. Let f be a function and S be a set. We say that f is *uniformly continuous on S* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $a, b \in S$ with $|a - b| < \delta$, we have $|f(a) - f(b)| < \varepsilon$.

Theorem 59. *If a function is continuous on a closed interval, then it is also uniformly continuous on that interval.*

Proof. Omitted. (See e.g. Abbott, 2015, pp. 132–133.) □

Theorem 42. *Let $a < b$. If the function f is continuous on $[a,b]$, then $\int_a^b f$ exists.*

Proof. Let $\varepsilon > 0$. Since f is continuous on $[a,b]$, by Theorem 59, f is also uniformly continuous on $[a,b]$. That is, there exists $\delta > 0$ such that if $c, d \in [a,b]$ and $|c-d| < \delta$, then $|f(c) - f(d)| \overset{1}{<} \dfrac{\varepsilon}{b-a}$.

For each $n \in \mathbb{Z}_0^+$ and $i \in \{1, 2, \ldots, 2^n\}$, define the interval $I_{n,i} = \left[a + (i-1)\dfrac{b-a}{2^n}, a + i\dfrac{b-a}{2^n}\right]$.

By the Extreme Value Theorem, for each $i \in \{1, 2, \ldots, 2^n\}$, there exist $l_{n,i}, u_{n,i} \in I_{n,i}$ such that $f(x) \overset{3}{\in} [f(l_{n,i}), f(u_{n,i})]$ for every $x \in I_{n,i}$. Let

$$L_n \overset{4}{=} \frac{b-a}{2^n} \sum_{i=1}^{2^n} f(l_{n,i}) \qquad \text{and} \qquad U_n \overset{5}{=} \frac{b-a}{2^n} \sum_{i=1}^{2^n} f(u_{n,i}).$$

Pick $N \in \mathbb{Z}_0^+$ such that $\delta \overset{2}{>} \dfrac{b-a}{2^N}$.[622] Consider $I_{N,i}$. Since this interval is of length $\dfrac{b-a}{2^N}$, it must be that $|u_{N,i} - l_{N,i}| \le \dfrac{b-a}{2^N} \overset{2}{<} \delta$. Hence, by $\overset{1}{<}$, $f(u_{N,i}) - f(l_{N,i}) < \dfrac{\varepsilon}{b-a}$. Thus,

$$U_N - L_N = \frac{b-a}{2^N} \sum_{i=1}^{2^N} [f(u_{N,i}) - f(l_{N,i})] < \frac{b-a}{2^N} \sum_{i=1}^{2^N} \frac{\varepsilon}{b-a} = \frac{b-a}{2^N} \frac{2^N \varepsilon}{b-a} = \varepsilon.$$

We've just shown that $U_N - L_N \overset{6}{<} \varepsilon$—we'll use this later.

For each $n \ge N$ and each $i \in \{1, 2, \ldots, 2^N\}$, let $T_{n,i} = \{2^{n-N}(i-1) + 1, 2^{n-N}(i-1) + 2, \ldots, 2^{n-N} i\}$ ($T_{n,i}$ is a set of 2^{n-N} integers).

Consider the intervals $I_{n,j}$. They are disjoint and each has width $\dfrac{b-a}{2^n}$. Moreover, $\cup_{j \in T_{n,i}} I_{n,j} = I_{N,i}$.

For any $j \in T_{n,i}$, we have $a + \left(j - \dfrac{1}{2}\right)\dfrac{b-a}{2^n} \in \cup_{j \in T_{n,i}} I_{n,j} = I_{N,i}$ and hence, by $\overset{3}{\in}$,

$$f\left(a + \left(j - \frac{1}{2}\right)\frac{b-a}{2^n}\right) \in [f(l_{N,i}), f(u_{N,i})].$$

Thus, $$\frac{1}{2^{n-N}} \sum_{j \in T_{n,i}} f\left(a + \left(j - \frac{1}{2}\right)\frac{b-a}{2^n}\right) \overset{7}{\in} [f(l_{N,i}), f(u_{N,i})].$$

Now,

[622] The existence of N is given by the Archimedean Property.

$$\begin{aligned}
S_n &= \frac{b-a}{2^n}\sum_{i=1}^{2^n} f\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{2^n}\right) \\
&= \frac{b-a}{2^n}\sum_{i=1}^{2^N}\sum_{j\in T_{n,i}} f\left(a+\left(j-\frac{1}{2}\right)\frac{b-a}{2^n}\right) \\
&= \frac{b-a}{2^N}\sum_{i=1}^{2^N}\frac{1}{2^{n-N}}\sum_{j\in T_{n,i}} f\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{2^n}\right) \\
&\overset{7}{\in} \left[\frac{b-a}{2^N}\sum_{i=1}^{2^N} f\left(l_{N,i}\right), \frac{b-a}{2^N}\sum_{i=1}^{2^N} f\left(u_{N,i}\right)\right] \\
&\overset{4,5}{=} \left[L_N, U_N\right].
\end{aligned}$$

We've just shown for every $n \geq N$, $S_n \in [L_N, U_N]$ and hence, by $\overset{6}{<}$, $|S_n - L| < \varepsilon$.[623] So, $\lim_{n\to\infty} S_n = \int_a^b f$ exists. □

Theorem 43. (Rules of Integration) *Let $a < b$, $c \in (a,b)$, and $d, e \in \mathbb{R}$. Suppose the functions f and g are continuous on $[a,b]$, so that by Theorem 42, $\int_a^b f$, $\int_a^b g$, $\int_a^c f$, and $\int_c^b f$ exist. Then*

(a) $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g.$ **(Sum and Difference Rules)**

(b) $\int_a^b f = \int_a^c f + \int_c^b f.$ **(Adjacent Intervals Rule)**

(c) $\int_a^b (df) = d\int_a^b f.$ **(Constant Factor Rule)**

(d) $\int_a^b d = (b-a)\,d.$ **(Constant Rule)**

(e) *If $f \geq g$ on $[a,b]$, then* $\int_a^b f \geq \int_a^b g.$ **(Comparison Rule I)**

(f) *If $d \leq f(x) \leq e$ for every $x \in [a,b]$, then* **(Comparison Rule II)**

$$(b-a)\,d \leq \int_a^b f \leq (b-a)\,e.$$

Proof. For each $n \in \mathbb{Z}_0^+$, let

$$F_n = \frac{b-a}{2^n}\sum_{i=1}^{2^n} f\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{2^n}\right) \qquad \text{and} \qquad G_n = \frac{b-a}{2^n}\sum_{i=1}^{2^n} g\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{2^n}\right).$$

Since f and g are continuous on $[a,b]$, by Theorem 42,

[623] It is also true that $|S_n - U| < \varepsilon$.

$$\lim_{n\to\infty} F_n \stackrel{1}{=} \int_a^b f \in \mathbb{R} \qquad \text{and} \qquad \lim_{n\to\infty} G_n = \int_a^b g \in \mathbb{R}.$$

(a) Define $h = f \pm g$. For each $n \in \mathbb{Z}_0^+$, let $H_n = \frac{b-a}{2^n}\sum_{i=1}^{2^n} h\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{2^n}\right) = F_n \pm G_n$.

By the Sum and Difference Rules for Limits,

$$\int f \pm g = \int h = \lim_{n\to\infty} H_n = \lim_{n\to\infty}(F_n \pm G_n) \stackrel{\pm}{=} \lim_{n\to\infty} F_n \pm \lim_{n\to\infty} G_n = \int_a^b f \pm \int_a^b g. \qquad \checkmark$$

(b) The proof of the Adjacent Intervals Rule is long so we'll put it at the bottom.

(c) Define $i = df$. For each $n \in \mathbb{Z}_0^+$, let $I_n = \frac{b-a}{2^n}\sum_{i=1}^{2^n} i\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{2^n}\right) = dF_n$.

By the Constant Factor Rule for Limits,

$$\int (df) = \int i = \lim_{n\to\infty} I_n = \lim_{n\to\infty}(dF_n) \stackrel{\text{C}}{=} d\lim_{n\to\infty} F_n = d\int_a^b f. \qquad \checkmark$$

(d) Define $j : [a,b] \to \mathbb{R}$ by $j(x) = d$. For each $n \in \mathbb{Z}_0^+$, let

$$J_n = \frac{b-a}{2^n}\sum_{i=1}^{2^n} j\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{2^n}\right) = \frac{b-a}{2^n}\sum_{i=1}^{2^n} d = \frac{b-a}{2^n}2^n d = (b-a)\,d.$$

Hence,
$$\int_a^b d = \int_a^b j = \lim_{n\to\infty} J_n = (b-a)\,d. \qquad \checkmark$$

(e) Suppose $f \geq g$ on $[a,b]$. Then for every $n \in \mathbb{Z}_0^+$ and $i \in \{1, 2, \ldots, 2^n\}$,

$$f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{2^n}\right) \geq g\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{2^n}\right).$$

So, for every $n \in \mathbb{Z}_0^+$,

$$F_n = \frac{b-a}{2^n}\sum_{i=1}^{2^n} f\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{2^n}\right) \geq \frac{b-a}{2^n}\sum_{i=1}^{2^n} g\left(a + \left(i - \frac{1}{2}\right)\frac{b-a}{2^n}\right) = G_n. \qquad \checkmark$$

Hence, by Fact 267, $\int_a^b f = \lim_{n\to\infty} F_n \geq \lim_{n\to\infty} G_n = \int_a^b g$.

(f) follows from **(d)** and **(e)**. ✓

(b) We now prove the **Adjacent Intervals Rule**.

For each $n \in \mathbb{Z}_0^+$ and $p, q \in [a,b]$ with $p < q$, partition $[p,q]$ into 2^n intervals of equal width. Specifically, define the leftmost interval to be $I_{[p,q],n,1} = \left[p, q + \frac{q-p}{2^n}\right]$ and the remaining $2^n - 1$ intervals to be $I_{[p,q],n,i} = \left(p + (i-1)\frac{q-p}{2^n}, p + i\frac{q-p}{2^n}\right]$, for $i \in \{2, \ldots, 2^n\}$.

The midpoint of each of these 2^n intervals is $p + \left(i - \frac{1}{2}\right)\frac{q-p}{2^n}$.

Corresponding to each of these 2^n intervals is a rectangle with width $\frac{q-p}{2^n}$, height $f\left(p+\left(i-\frac{1}{2}\right)\frac{q}{2}\right.$ and hence area $\frac{q-p}{2^n}f\left(p+\left(i-\frac{1}{2}\right)\frac{q-p}{2^n}\right)$.

These 2^n rectangles have total area

$$S_{n,[p,q]}=\frac{q-p}{2^n}\sum_{i=1}^{2^n}f\left(p+\left(i-\frac{1}{2}\right)\frac{q-p}{2^n}\right).$$

Since f is continuous on $[p,q]$, by Theorem 42,

$$\lim_{n\to\infty}S_{n,[p,q]}\overset{1}{=}\int_p^q f\in\mathbb{R}.$$

Next, for any $n\in\mathbb{Z}_0^+$, suppose c is in the $j(n)$th interval. Then $j(n)=\left\lceil\frac{c-a}{b-a}2^n\right\rceil$.

Define $l_n=\frac{b-a}{2^n}\sum_{i=1}^{j(n)}f\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{2^n}\right)$ and $r_n=\frac{b-a}{2^n}\sum_{i=j(n)}^{2^n}f\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{2^n}\right)$.

Observe that l_n and r_n are approximately the portions of $S_{n,[a,b]}$ that are to the left and right of c. Moreover,

$$l_n+r_n\overset{2}{=}S_{n,[a,b]}+\frac{b-a}{2^n}f\left(a+\left(j(n)-\frac{1}{2}\right)\frac{b-a}{2^n}\right).$$

Below we state three Lemmata. If they are true, then we can also easily prove the Adjacent Intervals Rule:

$$\int_a^b f=\lim_{n\to\infty}(l_n+r_n)=\lim_{n\to\infty}l_n+\lim_{n\to\infty}r_n=\int_a^c f+\int_c^b f.$$

Lemma 21. $\int_a^b f=\lim_{n\to\infty}(l_n+r_n)$.

Proof. Since $\lim_{n\to\infty}S_{n,[a,b]}\overset{1}{=}\int_a^b f$ and $\lim_{n\to\infty}\frac{b-a}{2^n}f\left(a+\left(j(n)-\frac{1}{2}\right)\frac{b-a}{2^n}\right)=0$, by the Sum Rule for Limits,

$$\lim_{n\to\infty}(l_n+r_n)\overset{2}{=}\lim_{n\to\infty}\left[S_{n,[a,b]}+\frac{b-a}{2^n}f\left(a+\left(j(n)-\frac{1}{2}\right)\frac{b-a}{2^n}\right)\right]\overset{\pm}{=}\int_a^b f+0=\int_a^b f.$$ □

Lemma 22. $\lim_{n\to\infty}l_n=\int_a^c f$.

Lemma 23. $\lim_{n\to\infty}r_n=\int_c^b f$.

Proof. The proofs of Lemmata 22 and 23 are similar. So, we'll only prove Lemma 22:

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} S_{n,[a,c]} = \int_a^c f$, there exists N_1 such that for every $m \geq N_1$,

$$\left| S_{m,[a,c]} - \int_a^c f \right| \overset{1}{<} \frac{\varepsilon}{2}.$$

By uniform continuity, there exists $\delta > 0$ such that if $|x_1 - x_2| < \delta$ (and $x_1, x_2 \in [a,b]$), then

$$|f(x_1) - f(x_2)| \overset{2}{<} \frac{\varepsilon}{16(c-a)}.$$

Let $M = \max_{x\in[a,b]} f(x)$.[624]

Let m be any positive integer greater than $\max\left\{\ln \frac{2(c-a)}{\delta} / \ln 2, N_1\right\}$, so that $\overset{1}{<}$ holds.

Let t be any positive integer greater than $\max\left\{\ln \frac{64(b-a)M}{\varepsilon} / \ln 2, \ln \frac{16(b-a)}{\delta} / \ln 2, \ln \frac{8(b-a)}{c-a} /\right.$

Then these inequalities hold (as verified in the footnote):[625]

$$\frac{c-a}{2^m} + \frac{b-a}{2^{m+t-3}} \overset{3}{<} \delta, \quad \frac{b-a}{2^{m+t-3}} M \overset{4}{<} \frac{\varepsilon}{2^{m+3}}, \text{ and } \frac{b-a}{c-a}\frac{\varepsilon}{2^{m+t+1}} \overset{5}{<} \frac{\varepsilon}{2^{m+4}}.$$

Fix $k \in \{1, 2, \ldots, 2^m\}$.

Let k_l be the largest integer such that the interval $I_{[a,b],m+t,k_l}$ is strictly to the left of the interval $I_{[a,c],m,k}$.

Similarly, let k_r be the smallest integer such that the interval $I_{[a,b],m+t,k_r}$ is strictly to the right of the interval $I_{[a,c],m,k}$.

The left endpoint of the interval $I_{[a,b],m+t,k_l}$ is $a + (k_l - 1)\frac{b-a}{2^{m+t}}$. The right endpoint of the interval $I_{[a,b],m+t,k_r}$ is $a + k_r \frac{b-a}{2^{m+t}}$. The distance between these two endpoints is[626]

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The existence of M is given by the Extreme Value Theorem.

625 Below, we'll use this identity a few times:

$$\alpha^{\ln\beta/\ln\alpha} = \exp\left(\ln \alpha^{\ln\beta/\ln\alpha}\right) = \exp\left(\frac{\ln\beta}{\ln\alpha}\ln\alpha\right) = \exp(\ln\beta) = \beta.$$

For $\overset{3}{<}$, $\frac{c-a}{2^m} < \frac{c-a}{2^{\ln\frac{2(c-a)}{\delta}/\ln 2}} = \frac{c-a}{\frac{2(c-a)}{\delta}} = \frac{\delta}{2}$ and $\frac{b-a}{2^{m+t-3}} < \frac{b-a}{2^{t-3}} = \frac{8(b-a)}{2^t} < \frac{8(b-a)}{2^{\ln\frac{16(b-a)}{\delta}/\ln 2}} = \frac{8(b-a)}{\frac{16(b-a)}{\delta}} = \frac{\delta}{2}$. (So, each of the two terms on the left of the inequality $\overset{3}{<}$ is less than $\frac{\delta}{2}$).

For $\overset{4}{<}$, $\frac{b-a}{2^{m+t-3}}M < \frac{M(b-a)}{2^{m-3+\ln\frac{64(b-a)M}{\varepsilon}/\ln 2t}} = \frac{M(b-a)}{2^{m-3}\times\frac{64(b-a)M}{\varepsilon}} = \frac{\varepsilon}{2^{m+3}}$.

For $\overset{5}{<}$, $\frac{b-a}{c-a}\frac{\varepsilon}{2^{m+t+1}} < \frac{b-a}{c-a}\frac{\varepsilon}{2^{m+1+\ln\frac{8(b-a)}{c-a}/\ln 2}} = \frac{b-a}{c-a}\frac{\varepsilon}{2^{m+1}\times\frac{8(b-a)}{c-a}} = \frac{\varepsilon}{2^{m+4}}$

626 For $k_r - k_l + 1 \overset{6}{<} \frac{c-a}{b-a}2^t + 8$, observe that the ratio of the width of the bigger interval $I_{[a,c],m,i}$ to that of

$$a+k_r\frac{b-a}{2^{m+t}}-\left[a+(k_l-1)\frac{b-a}{2^{m+t}}\right]=(k_r-k_l+1)\frac{b-a}{2^{m+t}}\overset{6}{<}\left(\frac{c-a}{b-a}2^t+8\right)\frac{b-a}{2^{m+t}}=\frac{c-a}{2^m}+\frac{b-a}{2^{m+t-3}}\overset{3}{<}\delta.$$

So by $\overset{2}{<}$, for every $s\in\{k_l,k_l+1,k_l+2,\dots,k_r\}$,

$$\left|f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)-f\left(a+\left(k-\frac{1}{2}\right)\frac{c-a}{2^m}\right)\right|\overset{7}{<}\frac{\varepsilon}{16(c-a)}.$$

Now,

$$\left|\frac{c-a}{2^m}f\left(a+\left(k-\frac{1}{2}\right)\frac{c-a}{2^m}\right)-\frac{b-a}{2^{m+t}}\sum_{s=k_l}^{k_r}f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)\right|$$

$$=\left|\frac{c-a}{2^m}f\left(a+\left(k-\frac{1}{2}\right)\frac{c-a}{2^m}\right)-\frac{b-a}{2^{m+t}}\sum_{s=k_l}^{k_r}\left[f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)-f\left(a+\left(k-\frac{1}{2}\right)\frac{c-a}{2^m}\right)+f\left(a+\left(k-\right.\right.\right.\right.$$

$$=\left|\left[\frac{c-a}{2^m}-\frac{b-a}{2^{m+t}}(k_r-k_l+1)\right]f\left(a+\left(k-\frac{1}{2}\right)\frac{c-a}{2^m}\right)-\frac{b-a}{2^{m+t}}\sum_{s=k_l}^{k_r}\left[f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)-f\left(a+\left(k-\right.\right.\right.\right.$$

$$\overset{7}{<}\left|\left[\frac{c-a}{2^m}-\frac{b-a}{2^{m+t}}(k_r-k_l+1)\right]f\left(a+\left(k-\frac{1}{2}\right)\frac{c-a}{2^m}\right)\right|+\frac{b-a}{2^{m+t}}(k_r-k_l+1)\frac{\varepsilon}{16(c-a)}$$

$$<\left|\left[\frac{c-a}{2^m}-\frac{b-a}{2^{m+t}}(k_r-k_l+1)\right]M\right|+\frac{b-a}{2^{m+t}}(k_r-k_l+1)\frac{\varepsilon}{16(c-a)}$$

$$\overset{6}{<}\left|\left[\frac{c-a}{2^m}-\frac{b-a}{2^{m+t}}\left(\frac{c-a}{b-a}2^t+8\right)\right]M\right|+\frac{b-a}{2^{m+t}}\left(\frac{c-a}{b-a}2^t+8\right)\frac{\varepsilon}{16(c-a)}$$

$$=\frac{b-a}{2^{m+t-3}}M+\frac{\varepsilon}{2^{m+4}}+\frac{b-a}{c-a}\frac{\varepsilon}{2^{m+t+1}}$$

$$\overset{4,5}{<}\frac{\varepsilon}{2^{m+3}}+\frac{\varepsilon}{2^{m+4}}+\frac{\varepsilon}{2^{m+4}}=\frac{\varepsilon}{2^{m+2}}$$

Thus,

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the smaller interval $I_{[a,c],m,i}$ is $\frac{c-a}{2^m}\div\frac{b-a}{2^{m+t}}=\frac{c-a}{b-a}2^t$. This ratio should in turn be close to the number k_r-k_l+1, which is the number of the smaller intervals that "just" more than cover one of the bigger interval. Hence, $k_r-k_l+1\overset{6}{<}\frac{c-a}{b-a}2^t+8$—where 8 is just a somewhat arbitrary but sufficiently big number that guarantees $\overset{6}{<}$ is true.

$$\left|S_{m,[a,c]} - \frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}\sum_{s=k_l}^{k_r} f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)\right|$$

$$= \left|\sum_{k=1}^{2^m}\frac{c-a}{2^m}f\left(a+\left(k-\frac{1}{2}\right)\frac{c-a}{2^m}\right) - \frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}\sum_{s=k_l}^{k_r} f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)\right|$$

$$= \sum_{k=1}^{2^m}\left|\frac{c-a}{2^m}f\left(a+\left(k-\frac{1}{2}\right)\frac{c-a}{2^m}\right) - \frac{b-a}{2^{m+t}}\sum_{s=k_l}^{k_r} f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)\right| \overset{8}{<} 2^m\frac{\varepsilon}{2^{m+2}} = \frac{\varepsilon}{4}.$$

Also,

$$\frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}\sum_{s=k_l+2}^{k_r-2} f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right) \le l_{m+t} = \frac{b-a}{2^{m+t}}\sum_{i=1}^{j(m+t)} f\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right) \le \frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}\sum_{s=k_l}^{k_r} f\left(a+\left(s\right.\right.$$

Hence,

$$\left|l_{m+t} - \frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}\sum_{s=k_l}^{k_r} f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)\right|$$

$$\le \left|\frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}\sum_{s=k_l}^{k_r} f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right) - \frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}\sum_{s=k_l+2}^{k_r-2} f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)\right|$$

$$= \frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}\left|\sum_{s=k_l}^{k_r} f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right) - \sum_{s=k_l+2}^{k_r-2} f\left(a+\left(s-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right)\right|$$

$$= \frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}\left|f\left(a+\left(k_l-\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right) + f\left(a+\left(k_l+\frac{1}{2}\right)\frac{b-a}{2^{m+t}}\right) + f\left(a+\left(k_r-\frac{3}{2}\right)\frac{b-a}{2^{m+t}}\right) + f\left(a+\left(k_r-\frac{1}{2}\right)\right.\right.$$

$$\le \frac{b-a}{2^{m+t}}\sum_{k=1}^{2^m}|4M| = \frac{b-a}{2^{m+t}}2^m|4M| = \frac{b-a}{2^{t-2}}|M| \overset{9}{<} \frac{\varepsilon}{4}.$$

Thus, by $\overset{8}{<}$ and $\overset{9}{<}$,

$$\left|l_{m+t} - S_{m,[a,c]}\right| \overset{42}{<} \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

And

$$\left|l_{m+t} - \int_a^c f\right| \le \left|l_{m+t} - S_{m,[a,c]}\right| + \left|S_{m,[a,c]} - \int_a^c f\right| \overset{42,1}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We've just shown that given any $\varepsilon > 0$, we can find $n = m + t$ such that $\left|l_n - \int_a^c f\right| < \varepsilon$.

Hence,

$$\lim_{n\to\infty} l_n = \int_a^c f$$

This completes the proof of Lemma 22. $\square$

Hence, this also completes the proof of the Adjacent Intervals Rule. $\square$

Theorem 44. **(First Fundamental Theorem of Calculus, FTC1)** *Let $a < b$, $f : [a,b] \to \mathbb{R}$ be a continuous function, and $c \in [a,b]$. Suppose the function $g : [a,b] \to \mathbb{R}$ is defined by*

$$g(x) = \int_c^x f.$$

Then $g' = f$ (i.e. the derivative of g is f or equivalently, g is an antiderivative of f).

Proof. Define $h : [a,b] \setminus \{c\} \to \mathbb{R}$ by

$$h(x) = \frac{g(x) - g(c)}{x - c} - f(c).$$

For any $x \in [a,b] \setminus \{c\}$, we have

$$\begin{aligned}
h(x) &= \frac{\int_a^x f - \int_a^c f}{x-c} - f(c) && \text{(Definition of } g\text{)}\\
&= \frac{\int_c^x f}{x-c} - f(c) && \text{(Adjacent Intervals Rule)}\\
&= \frac{\int_c^x f}{x-c} - \frac{x-c}{x-c} f(c) && \text{(Times One Trick)}\\
&= \frac{1}{x-c}\left[\int_c^x f - \int_c^x f(c)\,\mathrm{d}t\right] && \text{(Factorise, Constant Rule)}\\
&= \frac{1}{x-c}\int_c^x f(t) - f(c)\,\mathrm{d}t. && \text{(Difference Rule)}
\end{aligned}$$

Let $\varepsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that for every $t \in [a,b] \cap (c-\delta, c+\delta)$, we have

$$-\frac{\varepsilon}{2} < f(t) - f(c) < \frac{\varepsilon}{2}\,.$$

So, by Constant Rule and Comparison Rule II, for every $x \in [a,b] \cap \cancel{N}_\delta(c)$,

$$-\frac{\varepsilon}{2} \le \frac{1}{x-c}\int_c^x f(t) - f(c)\,\mathrm{d}t = h(x) \le \frac{\varepsilon}{2}\,.$$

We've just shown that given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in [a,b] \cap \cancel{N}_\delta(c)$, we have $h(x) \in \mathrm{N}_\varepsilon(0)$. So, we've just shown that $\lim_{x\to c} h(x) = 0$.

Or equivalently, $$\lim_{x\to c}\left[\frac{g(x)-g(c)}{x-c} - f(c)\right] \stackrel{1}{=} 0.$$

By the Constant Rule for Limits, $\lim_{x\to c} f(c) \stackrel{2}{=} f(c)$. Hence, by the Difference Rule for Limits,

$$\lim_{x\to c}\frac{g(x)-g(c)}{x-c} = f(c)\,.$$

But of course, LHS is simply $g'(c)$. Altogether then, we've shown that $g'(c) = f(c)$. And since c was chosen arbitrarily from $[a, b]$, we've also shown that $g'(x) = f(x)$ for every $x \in [a, b]$. Thus, $g' = f$. $\square$

146.24. $\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x$

Fact 216. *Let $a, b, c \in \mathbb{R}$ with $a \neq 0$ and $ax^2+bx+c \neq 0$. If $d = \sqrt{|b^2-4ac|}$, then*

$$\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x = \begin{cases} \dfrac{1}{d}\ln\left|\dfrac{x+\frac{b-d}{2a}}{x+\frac{b+d}{2a}}\right| + C, & \text{for } b^2-4ac>0, \\[2ex] \dfrac{-2}{2ax+b} + C, & \text{for } b^2-4ac=0, \\[2ex] \dfrac{2}{d}\tan^{-1}\dfrac{2ax+b}{d} + C, & \text{for } b^2-4ac<0. \end{cases}$$

Proof. Observe that $ax^2+bx+c = a\left(x+\dfrac{b}{2a}\right)^2 + c - \dfrac{b^2}{4a} = a\left[\left(x+\dfrac{b}{2a}\right)^2 - \dfrac{b^2-4ac}{4a^2}\right]$.

So,

$$ax^2+bx+c \stackrel{1}{=} \begin{cases} a\left[\left(x+\dfrac{b}{2a}\right)^2 - \left(\dfrac{d}{2a}\right)^2\right], & \text{for } b^2-4ac>0, \\[2ex] a\left(x+\dfrac{b}{2a}\right)^2, & \text{for } b^2-4ac=0, \\[2ex] a\left[\left(x+\dfrac{b}{2a}\right)^2 + \left(\dfrac{d}{2a}\right)^2\right], & \text{for } b^2-4ac<0. \end{cases}$$

There are three possible cases. We examine each on the next page.

(*Proof continues below …*)

(... *Proof continued from above.*)

(a) If $b^2 - 4ac > 0$, then

$$\frac{1}{ax^2+bx+c} \overset{1}{=} \frac{1}{a\left(x+\frac{b+d}{2a}\right)\left(x+\frac{b-d}{2a}\right)} = \frac{1}{a}\left(\frac{A}{x+\frac{b+d}{2a}} + \frac{B}{x+\frac{b-d}{2a}}\right) = \frac{(A+B)\,x + A\frac{b-d}{2a} + B\frac{b+d}{2a}}{a\left(x+\frac{b+d}{2a}\right)\left(x+\frac{b-d}{2a}\right)}.$$

Comparing coefficients, we have $A + B \overset{2}{=} 0$ and

$$A\frac{b-d}{2a} + B\frac{b+d}{2a} = \frac{1}{2a}\left[(A+B)\,b + (B-A)\,d\right] \overset{2}{=} (B-A)\,\frac{d}{2a} = 1.$$

So, $B - A = 2a/d$. Adding $\overset{2}{=}$, we get

$$2B = \frac{2a}{d} \text{ or } B = \frac{a}{d} \qquad \text{and} \qquad A = \frac{-a}{d}.$$

Thus,

$$\begin{aligned}
\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x &= \frac{1}{a}\left(\int \frac{A}{x+\frac{b+d}{2a}}\,\mathrm{d}x + \int \frac{B}{x+\frac{b-d}{2a}}\,\mathrm{d}x\right) \\
&= \int \frac{\frac{-1}{d}}{x+\frac{b+d}{2a}}\,\mathrm{d}x + \int \frac{\frac{1}{d}}{x+\frac{b-d}{2a}}\,\mathrm{d}x \\
&= \frac{-1}{d}\ln\left|x+\frac{b+d}{2a}\right| + \frac{1}{d}\ln\left|x+\frac{b-d}{2a}\right| + C \\
&= \frac{1}{d}\ln\left|\frac{x+\frac{b-d}{2a}}{x+\frac{b+d}{2a}}\right| + C
\end{aligned}$$

(b) If $b^2 - 4ac = 0$, then

$$\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x \overset{1}{=} \int \frac{1}{a}\frac{1}{\left(x+\frac{b}{2a}\right)^2}\,\mathrm{d}x = -\frac{1}{a}\frac{1}{x+\frac{b}{2a}} + C = \frac{-2}{2ax+b} + C.$$

(c) If $b^2 - 4ac < 0$, then

$$\int \frac{1}{ax^2+bx+c}\,\mathrm{d}x \overset{1}{=} \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2 + \left(\frac{d}{2a}\right)^2}\,\mathrm{d}x \overset{4}{=} \frac{1}{a}\left(\frac{2a}{d}\tan^{-1}\frac{x+b/2a}{d/2a}\right) + C = \frac{2}{d}\tan^{-1}\frac{2ax+b}{d} + C,$$

where $\overset{4}{=}$ uses Proposition 16(a). □

146.25. $\displaystyle\int \frac{1}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x$

Fact 217. *Suppose $a, b, c \in \mathbb{R}$ with $a<0$ and $ax^2+bx+c>0$. Then*

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x = \frac{1}{\sqrt{-a}}\sin^{-1}\frac{-(2ax+b)}{\sqrt{b^2-4ac}}+C.$$

Proof.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\sqrt{-a}}\sin^{-1}\frac{-(2ax+b)}{\sqrt{b^2-4ac}}+C\right]\\
&=\frac{1}{\sqrt{-a}}\frac{1}{\sqrt{1-\left[\frac{-(2ax+b)}{\sqrt{b^2-4ac}}\right]^2}}\frac{-2a}{\sqrt{b^2-4ac}}\frac{2\sqrt{-a}}{\sqrt{b^2-4ac-(2ax+b)^2}}\\
&=\frac{2\sqrt{-a}}{\sqrt{-4a^2x^2-4abx-4ac}}=\frac{1}{\sqrt{ax^2+bx+c}}.
\end{aligned}$$

□

The above claim—the second sentence especially—may be considered a little imprecise. For a more precise statement of the above claim, see footnote.[627]

Remark 215. In Fact 217, the expression $\dfrac{1}{\sqrt{|a|}}\sin^{-1}\dfrac{-2ax-b}{\sqrt{b^2-4ac}}$ raises two possible concerns:

(a) If $b^2-4ac\le 0$, then this expression is undefined.

(b) If $\dfrac{-2ax-b}{\sqrt{b^2-4ac}}\notin[-1,1]=\text{Domain}\sin^{-1}$, then again this expression is undefined.

Fortunately, we can dismiss these concerns:

(a) Since $a<0$ ($\cap$-shaped quadratic curve) and there exists $x\in\mathbb{R}$ such that $ax^2+bx+c>0$ (curve intersects y-axis twice), it must be that $b^2-4ac>0$ (otherwise the curve would intersect the y-axis at most once).

(b) $ax^2+bx+c>0 \iff 4a^2x^2+4abx+4ac<0 \iff \overbrace{4a^2x^2+4abx+b^2}^{(2ax+b)^2}<b^2-4ac$

$$\overset{(a)}{\iff}\frac{(2ax+b)^2}{b^2-4ac}<1\iff\left|\frac{2ax+b}{\sqrt{b^2-4ac}}\right|<1\iff\left|\frac{-(2ax+b)}{\sqrt{b^2-4ac}}\right|<1$$

$$\iff\frac{-(2ax+b)}{\sqrt{b^2-4ac}}\in(-1,1)\subseteq[-1,1]=\text{Domain}\sin^{-1}.$$

[627] Let $a, b, c\in\mathbb{R}$ with $a<0$. Let D be an interval with $ax^2+bx+c>0$ for all $x\in D$. Suppose the functions $F:D\to\mathbb{R}$ and $f:D\to\mathbb{R}$ are defined by

$$F(x)=\frac{1}{\sqrt{ax^2+bx+c}}\qquad\text{and}\qquad f(x)=\frac{1}{\sqrt{|a|}}\sin^{-1}\frac{-2ax-b}{\sqrt{b^2-4ac}}.$$

Then an antiderivative of F is f.

The next result says that if $a \neq 0$ and $x^2 + a > 0$, then

$$\int \frac{1}{\sqrt{x^2+a}}\,\mathrm{d}x = \ln\left|\sqrt{x^2+a}+x\right| + C.$$

Fact 298. *Let $a \neq 0$ and D be an interval with $x^2 + a > 0$ for all $x \in D$. Suppose the functions $F : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$ are defined by*

$$F(x) = \frac{1}{\sqrt{x^2+a}} \qquad \text{and} \qquad f(x) = \ln\left|\sqrt{x^2+a}+x\right|.$$

Then an antiderivative of F is f.

Proof. If $\sqrt{x^2+a} + x > 0$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\ln\left|\sqrt{x^2+a}+x\right| &= \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\sqrt{x^2+a}+x\right) = \frac{\frac{2x}{2\sqrt{x^2+a}}+1}{\sqrt{x^2+a}+x} \\ &= \frac{x+\sqrt{x^2+a}}{\sqrt{x^2+a}\left(\sqrt{x^2+a}+x\right)} = \frac{1}{\sqrt{x^2+a}}.\end{aligned}$$

If $\sqrt{x^2+a} + x < 0$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\ln\left|\sqrt{x^2+a}+x\right| &= \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(-\sqrt{x^2+a}-x\right) = \frac{\frac{-2x}{2\sqrt{x^2+a}}-1}{-\sqrt{x^2+a}-x} \\ &= \frac{x+\sqrt{x^2+a}}{\sqrt{x^2+a}\left(\sqrt{x^2+a}+x\right)} = \frac{1}{\sqrt{x^2+a}}.\end{aligned}$$

□

The next result says that if $a > 0$, $b^2 - 4ac \neq 0$, and $ax^2 + bx + c \neq 0$, then

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x = \frac{1}{\sqrt{a}} \ln \left| \sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} + x + \frac{b}{2a} \right| + C.$$

Fact 299. *Let $a, b, c \in \mathbb{R}$ with $a > 0$ and $b^2 - 4ac \neq 0$. Let D be an interval with $ax^2 + bx + c \neq 0$ for all $x \in D$. Suppose the functions $F : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$ are defined by*

$$F(x) = \frac{1}{\sqrt{ax^2+bx+c}} \qquad \text{and} \qquad f(x) = \frac{1}{\sqrt{a}} \ln \left| \sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} + x + \frac{b}{2a} \right|.$$

Then an antiderivative of F is f.

Proof. If $\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} + x + \frac{b}{2a} > 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{a}} \ln \left| \sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} + x + \frac{b}{2a} \right| \right) = \frac{1}{\sqrt{a}} \frac{\frac{2x+\frac{b}{a}}{2\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}} + 1}{\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} + x + \frac{b}{2a}}$$

$$= \frac{1}{\sqrt{a}} \frac{\frac{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}+x+\frac{b}{2a}}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}}{\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} + x + \frac{b}{2a}} = \frac{1}{\sqrt{a}} \frac{1}{\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}}} = \frac{1}{\sqrt{ax^2+bx+c}}.$$

If $\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} + x + \frac{b}{2a} < 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{a}} \ln \left| \sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} + x + \frac{b}{2a} \right| \right) = \frac{1}{\sqrt{a}} \frac{\frac{2x+\frac{b}{a}}{-2\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}} - 1}{-\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} - x - \frac{b}{2a}}$$

$$= \frac{1}{\sqrt{a}} \frac{\frac{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}+x+\frac{b}{2a}}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}}{\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} + x + \frac{b}{2a}} = \frac{1}{\sqrt{a}} \frac{1}{\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}}} = \frac{1}{\sqrt{ax^2+bx+c}}.$$

□

146.26. The Substitution Rule

Proposition 38. (Substitution Rule) *Let G and u be differentiable functions and $g = G'$. Suppose the composite function $G \circ u$ exists. Then*

$$\int [(g \circ u) \cdot u'] \, \mathrm{d}x \overset{\star}{=} G \circ u + C.$$

Proof. By the Chain Rule, $\dfrac{\mathrm{d}}{\mathrm{d}x}(G \circ u) = (G' \circ u) \cdot u' = (g \circ u) \cdot u'$. □

Here's why $\overset{\star}{=}$ corresponds to Steps 3, 4, and 5 of our Five-Step Substitution Rule Recipe:

$$\begin{aligned} \int [(g \circ u) \cdot u'] \, \mathrm{d}x &= \int g(u) \cdot \frac{\mathrm{d}u}{\cancel{\mathrm{d}x}} \cancel{\mathrm{d}x} = \int g(u) \, \mathrm{d}u && \text{(Step 3)} \\ &= G + C && \text{(Step 4)} \\ &= G \circ u + C && \text{(Step 5)} \end{aligned}$$

Formally and strictly speaking,

- Step 3 is wrong: We can't just magically cancel out the dx's.
- Step 5 is also wrong: We can't just suddenly plug u back in.

Only $\overset{\star}{=}$ is actually correct. Nonetheless, the Five-Step Substitution Rule Recipe serves as a (very) convenient mnemonic for $\overset{\star}{=}$.[628] The Recipe does give us the correct answer because here it so happens that two wrongs do make a right.

[628] Here once again Leibniz's notation shines, just as it did with the Chain Rule and the Inverse Function Theorem.

Remark 216. Many calculus textbooks[629] (and probably also many high-school/JC maths teachers) incorrectly present the Substitution Rule as

$$\int f(g(x))\, g'(x)\, \mathrm{d}x \stackrel{1}{=} \int f(u)\, \mathrm{d}u.$$

The above equation says that the functions $(f \circ g) \cdot g'$ and f have the same antiderivatives, which is clearly false!

This observation was already made by David Gale in his 1994 article "Teaching Integration by Substitution". I can do no better than reproduce his remarks:

> *Of course the equation is false. The expression* $\int f(x)\, dx$ *stands for antiderivative, as in a table of integrals, and the variable, be it* x, t, u *or anything else is a dummy. Clearly the antiderivatives on the left and right above are not equal. What the books mean, no doubt, is that if you substitute* $g(x)$ *for* u *after taking the antiderivative on the right you get the antiderivative on the left. I expect some readers will say I am being pedantic or that there is no need to be so rigorous at the freshman level, but I think this kind of lapse is symptomatic of a rather strange set of standards and perhaps it sheds light on why none of the books proves the inverse substitution theorem. It is because none of them formulates it.*

Here are two corrected versions of $\stackrel{1}{=}$ (without changing it too much):

$$\int f(g(x))\, g'(x)\, \mathrm{d}x = \int f(u)\, \mathrm{d}u\Big|_{u=g(x)} \qquad \text{or} \qquad \int (f \circ g) \cdot g' = \left(\int f\right) \circ g + C.$$

Letting $F' = f$, we can also produce two more corrected versions of $\stackrel{1}{=}$:

$$\int f(g(x))\, g'(x)\, \mathrm{d}x = F(g(x)) + C \qquad \text{or} \qquad \int (f \circ g) \cdot g' = F \circ g + C.$$

[629] Some textbooks that make this error: Stewart (*Single Variable Calculus*, 2011, p. 331); Thomas and Finney (*Calculus and Analytic Geometry*, 1998, p. 294)—see also Hass, Heil, and Weir (*Thomas' Calculus*, 2018, p. 291). Also, ProofWiki (retrieved 2018-10-06-1058).

Some textbooks that do **not** make this error: Apostol (*Calculus: Volume I*, 1967, p. 212); Larson and Edwards (*Calculus of a Single Variable*, 2016, p. 296).

146.27. Revisiting Logarithms and Exponentiation

In Ch. 5.4, we only defined what b^x means in these two cases:

1. $x \in \mathbb{Z}$—so e.g., we know that $5^2 = 25$, $(-5)^3 = 125$, and $4.5^{-2} = 4/81$; or
2. $b \geq 0$ and $x \in \mathbb{Q}$—so e.g., we know that $5^{2.6} = \left(\sqrt[10]{5}\right)^{26}$.

We did **not** define what b^x in these two cases:

3. $x \notin \mathbb{Q}$—so e.g., we **don't** know what $5^{\sqrt{3}}$ is; or
4. $b < 0$ and $x \notin \mathbb{Z}$—so e.g., we **don't** know what $(-5)^{2.6}$ is.

And so, we're actually cheating when we take for granted that the Laws of Exponents (Proposition 1) hold for *all* positive bases and real exponents. For example, we haven't defined what $5^{\sqrt{3}}$ and $5^{-\sqrt{3}}$ are, yet we cavalierly take for granted that

$$5^{\sqrt{3}} \cdot 5^{-\sqrt{3}} = 5^{\sqrt{3}+\left(-\sqrt{3}\right)} = 5^0 = 1.$$

Let us think about what b^x might mean if $x \notin \mathbb{Q}$.

Consider $5^{\sqrt{3}}$. Say we know that $\sqrt{3} \approx 1.732\,050\,8\ldots$ and also that

$$\begin{aligned}
5^1 &= 5, \\
5^{1.7} &= 5^{\frac{17}{10}} &&= \left(\sqrt[10]{5}\right)^{17} &&\approx (1.174\,619\ldots)^{17} &&\approx 15.425\ldots \\
5^{1.73} &= 5^{\frac{173}{100}} &&= \left(\sqrt[100]{5}\right)^{173} &&\approx (1.016\,224\,6\ldots)^{173} &&\approx 16.188\ldots \\
5^{1.732} &= 5^{\frac{1\,732}{1\,000}} &&= \left(\sqrt[1\,000]{5}\right)^{1\,732} &&\approx (1.001\,610\,73\ldots)^{173} &&\approx 16.241\ldots \\
5^{1.732\,0} &= 5^{\frac{17\,320}{10\,000}} &&= \left(\sqrt[10\,000]{5}\right)^{17\,320} &&\approx (1.000\,160\,957\ldots)^{1\,730} &&\approx 16.241\ldots \\
5^{1.732\,05} &= 5^{\frac{173\,205}{100\,000}} &&= \left(\sqrt[100\,000]{5}\right)^{173\,205} &&\approx (1.000\,016\,094\,5\ldots)^{17\,305} &&\approx 16.242\ldots \\
5^{1.732\,050} &= 5^{\frac{1\,732\,050}{1\,000\,000}} &&= \left(\sqrt[1\,000\,000]{5}\right)^{1\,732\,050} &&\approx (1.000\,016\,094\,5\ldots)^{173\,050} &&\approx 16.242\ldots
\end{aligned}$$

And so informally, we might say that $5^{\sqrt{3}} \approx 16.242\,4\ldots$

A little more formally, we might say that $5^{\sqrt{3}}$ is the limit of the following sequence:

$$5^1, 5^{1.7}, 5^{1.73}, 5^{1.732}, 5^{1.732\,0}, 5^{1.732\,05}, 5^{1.732\,050}, 5^{1.732\,050\,8}, \ldots$$

And so, following the above discussion, one possible approach for defining b^x in the case where $x \notin \mathbb{Q}$ might go like this:

- Assume (prove) there is a sequence of rational numbers $(x_0, x_1, \ldots)$ that converges to x.
- Use that sequence to form the sequence $(b^{x_0}, b^{x_1}, \ldots)$.
- Assume (prove) this latter sequence converges to some number y. That is,

$$(b^{x_0}, b^{x_1}, \ldots) \to y.$$

- Now define b^x to be equal to y.

This seems like a perfectly sensible approach. But strangely, it is **not** the approach we will take. Instead, somewhat surprisingly, we'll define exponents based on the **natural logarithm** and **exponential** functions:[630]

Definition 335. Let $b > 0$ and $x \in \mathbb{R}$. Then b *raised to the power of* x is denoted b^x and is the number defined by

$$b^x = \exp(x \ln b).$$

Given b^x, we call b the *base* and x the *exponent*.

Separately, we define

$$0^x = \begin{cases} 0, & \text{for } x > 0, \\ 1, & \text{for } x = 0, \\ \text{Undefined}, & \text{for } x < 0. \end{cases}$$

Fact 223. *Suppose $x > 0$ and $n \in \mathbb{R}$. Then $\ln x^n = n \ln x$.*

Proof. Use Definitions 335 (above) and 85 (exp is inverse of ln):

$$\ln x^n = \ln[\exp(n \ln x)] = n \ln x. \qquad \square$$

[630] Which were in turn formally defined as Definitions 229 and 85 in the main text.

The following Proposition shows that for $b > 0$ and *any* $x \in \mathbb{R}$, Definitions 335 and 230 imply the four definitions of exponents and logarithms given in Ch. 5.4. So, we'll let Definitions 335 and 230 supersede those earlier definitions.

Proposition 39. *In those cases where the base is positive, Definition 335 implies Definitions* **(a)** 27*;* **(b)** 29*;* **(c)** 30*; and* **(d)** 32*.*

Proof. Below, $\overset{\star}{=}$ and $\overset{\circ}{=}$ denote the use of Definitions 335 and 85 (exp is the inverse of ln), respectively.

(a) Let $b \in \mathbb{R} \setminus \{0\}$.

Case 1. If $x \in \mathbb{Z}^+$, then

$$b^x \overset{\star}{=} \exp(x \ln b) = \exp\Big(\underbrace{\ln b + \ln b + \cdots + \ln b}_{x \text{ times}}\Big)$$
$$\overset{2}{=} \underbrace{\exp(\ln b)\exp(\ln b)\ldots\exp(\ln b)}_{x \text{ times}} \overset{\circ}{=} \underbrace{b \cdot b \cdot \cdots \cdot b}_{x \text{ times}}.$$

Case 2. If $x \in \mathbb{Z}^-$, then

$$b^x \overset{\star}{=} \exp(x \ln b) = \exp\Big(\underbrace{-\ln b - \ln b - \cdots - \ln b}_{|x| \text{ times}}\Big) \overset{3}{=} \exp\Bigg(\underbrace{\ln\frac{1}{b} + \ln\frac{1}{b} + \cdots + \ln\frac{1}{b}}_{|x| \text{ times}}\Bigg)$$
$$\overset{4}{=} \underbrace{\exp\left(\ln\frac{1}{b}\right)\exp\left(\ln\frac{1}{b}\right)\ldots\exp\left(\ln\frac{1}{b}\right)}_{|x| \text{ times}} \overset{\circ}{=} \underbrace{\frac{1}{b}\cdot\frac{1}{b}\cdot\cdots\cdot\frac{1}{b}}_{|x| \text{ times}} = \frac{1}{b^{|x|}}.$$

Case 3. If $x = 0$, then $b^x = b^0 \overset{\star}{=} \exp(0 \ln b) = \exp 0 \overset{5}{=} 1$.

Above, $\overset{2}{=}$ and $\overset{4}{=}$ used Fact 224(c), while $\overset{3}{=}$ and $\overset{5}{=}$ used Facts 222(c) and 224(a).

(b) Let $b > 0$ and $x \in \mathbb{Z} \setminus \{0\}$. We'll show that $\left(b^{\frac{1}{x}}\right)^x = b$:

$$\left(b^{\frac{1}{x}}\right)^x \overset{\star}{=} \left[\exp\left(\frac{1}{x}\ln b\right)\right]^x \overset{\star}{=} \exp\left\{x \ln\left[\exp\left(\frac{1}{x}\ln b\right)\right]\right\} \overset{\circ}{=} \exp\left[x\left(\frac{1}{x}\ln b\right)\right] = \exp(\ln b) \overset{\circ}{=} b.$$

(c) If $b > 0$ and $x = m/n$ for some $m, n \in \mathbb{Z}$, then

$$b^x \overset{\star}{=} \exp(x \ln b) = \exp\left(\frac{m}{n}\ln b\right) \overset{\circ}{=} \exp\left\{m \ln\left[\exp\left(\frac{1}{n}\ln b\right)\right]\right\} = \left[\exp\left(\frac{1}{n}\ln b\right)\right]^m \overset{\star}{=} \left(b^{\frac{1}{n}}\right)^m.$$

(d) $b^x = n \overset{\star}{\iff} \exp(x \ln b) = n \iff \ln[\exp(x \ln b)] = \ln n \overset{\circ}{\iff} x \ln b = \ln n \iff x = \dfrac{\ln n}{\ln b}$ $\overset{6}{\iff} x = \log_b n$. (The last step $\overset{6}{\iff}$ uses Definition 230.) □

In the main text, we gave a partial proof of the following Laws of Exponents, covering only the case where the exponents x and y were positive integers. Here now is a full proof covering also the case where x and y are *any* real numbers:

Proposition 1. (Laws of Exponents) *Suppose $a, b > 0$ and $x, y \in \mathbb{R}$. Then*

(a) $b^x b^y = b^{x+y}$. **(b)** $b^{-x} = \dfrac{1}{b^x}$. **(c)** $\dfrac{b^x}{b^y} = b^{x-y}$. **(d)** $(b^x)^y = b^{xy}$. **(e)** $(ab)^x = a^x b^x$.

Proof. Below, $\overset{\star}{=}$ and $\overset{\circ}{=}$ denote the use of Definitions 335 and 85 (exp is the inverse of ln), respectively.

(a) $b^x b^y \overset{\star}{=} \exp(x \ln b) \exp(y \ln b) \overset{1}{=} \exp(x \ln b + y \ln b) = \exp[(x+y) \ln b] \overset{\star}{=} b^{x+y}$.[631]

(b) $b^{-x} \overset{\star}{=} \exp(-x \ln b) \overset{2}{=} \exp(-\ln b^x) \overset{3}{=} \exp\left(\ln \dfrac{1}{b^x}\right) \overset{\circ}{=} \dfrac{1}{b^x}$.[632]

(c) $b^{x-y} \overset{\star}{=} \exp[(x-y) \ln b] = \exp(x \ln b - y \ln b) \overset{4}{=} \dfrac{\exp(x \ln b)}{\exp(y \ln b)} \overset{\star}{=} \dfrac{b^x}{b^y}$.[633]

(d) $(b^x)^y \overset{\star}{=} \exp[y(\ln b^x)] \overset{5}{=} \exp[xy(\ln b)] \overset{\star}{=} b^{xy}$.[634]

(e) $(ab)^x \overset{\star}{=} \exp[x \ln(ab)] \overset{6}{=} \exp[x(\ln a + \ln b)] = \exp(x \ln a + x \ln b)$

$\overset{7}{=} [\exp(x \ln a)][\exp(x \ln b)] \overset{\star}{=} a^x b^x$.[635] □

Fact 63. *For every $x \in \mathbb{R}$, $\mathrm{e}^x = \exp x$.*

Proof. By Definitions 335 and 86,

$$\mathrm{e}^x = \exp(x \ln \mathrm{e}) = \exp(x \cdot 1) = \exp x.$$ □

Fact 23. *Let $x > 0$.*

(a) *If $b > 0$, then $b^x > 0$.*

(b) *If $b > 1$, then $b^x > 1$.*

Proof. **(a)** $b^x = \exp(x \ln b) > 0$ because Range exp $= \mathbb{R}^+$.

(b) $\ln b > 0$ and $x \ln b > 0$, so that $b^x = \exp(x \ln b) > 1$. □

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[631] In $\overset{1}{=}$, use Fact 224(c).

[632] In $\overset{2}{=}$ and $\overset{3}{=}$, use Facts 223 and 222(c), respectively.

[633] In $\overset{4}{=}$, use Fact 224(e).

[634] In $\overset{5}{=}$, use Fact 223.

[635] In $\overset{6}{=}$ and $\overset{7}{=}$, use Facts 222(b) and 224(c), respectively.

147. Appendices for Part VI. Probability and Statistics

147.1. How to Count

Theorem 60. (AP.) *If A and B are disjoint, finite sets, then* $|A \cup B| = |A| + |B|$.

Proof. Let $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_q\}$. Then

$$A \cup B = \{a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q\}.$$

We have $|A| = p$, $|B| = q$, and $|A \cup B| = p + q$. The result follows. □

Corollary 55. *If A_1, A_2, $\ldots$, A_n are disjoint, finite sets, then* $|\cup_{i=1}^n A_i| = \sum_{i=1}^n |A_i|$.

Proof. By induction (details omitted). □

Theorem 61. (MP.) *If A and B are finite sets, then* $|A \times B| = |A| \times |B|$.

Proof. Let $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_q\}$. Then

$$A \times B = \left\{ (a_1, b_1), (a_1, b_2), \ldots, (a_1, b_q), (a_2, b_1), (a_2, b_2), \ldots, (a_2, b_q), \ldots, \right.$$
$$\left. \ldots, (a_p, b_1), (a_p, b_2), \ldots, (a_p, b_q) \right\}.$$

We have $|A| = p$, $|B| = q$, and $|A \times B| = pq$. The result follows. □

Corollary 56. *If A_1, A_2, $\ldots$, A_n are finite sets, then* $|\times_{i=1}^n A_i| = \pi_{i=1}^n |A_i|$.

Proof. By induction (details omitted). □

Theorem 62. (IEP.) *If A and B are finite sets, then* $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof. $A \cup B = (A \setminus (A \cap B)) \cup B$. So by the AP, $|A \cup B| \overset{1}{=} |A \setminus (A \cap B)| + |B|$.
Now, $(A \setminus (A \cap B)) \cup (A \cap B) = A$. So also by the AP, $|A \setminus (A \cap B)| + |A \cap B| = |A|$ or $|A \setminus (A \cap B)| \overset{2}{=} |A| - |A \cap B|$. Plug $\overset{2}{=}$ into $\overset{1}{=}$ to get the desired result. □

Corollary 57. *If A_1, A_2, A_3, are finite sets, then*

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Proof. Similar to the previous proof, just more tedious. □

And here's the generalisation of the IEP:

Corollary 58. *If A_1, A_2, ..., A_n, are finite sets, then*

$$|\cup_{i=1}^n A_i| = \sum_{i=1}^n |A_i| - \sum_{i,j \text{ distinct}} |A_i \cap A_j| + \sum_{i,j,k \text{ distinct}} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |\cap_{i=1}^n A_i|.$$

Proof. By induction (details omitted). □

Theorem 63. (CP.) *If A and B are finite sets and $A \subseteq B$, then* $|A \setminus B| = |A| - |B|$.

Proof. B and $A \setminus B$ are disjoint, finite sets. Moreover, $B \cup (A \setminus B) = A$. So by the AP, $|B| + |A \setminus B| = |A|$. Rearranging yields the desired result. □

Corollary 59. *If A is a finite set and $B_1, B_2, \ldots B_n \subseteq A$ are disjoint, then*

$$|A \setminus \cup_{i=1}^n B_i| = |A| - \sum_{i=1}^n |B_i|.$$

Proof. By the corollary to the AP, $|\cup_{i=1}^n B_i| = \sum_{i=1}^n |B_i|$. The result then follows by the CP. □

147.2. Circular Permutations

Consider n objects, only k of which are distinct. Let r_1, r_2, ..., and r_k be the numbers of times the 1st, 2nd, ..., and kth distinct objects appear. We already know from Fact 227 that the number of (linear) permutations of these n objects is

$$\frac{n!}{r_1!r_2!\dots r_k!}.$$

We also know that m distinct objects have $m!$ (linear) permutations and $(m-1)!$ **circular** permutations.

A reasonable conjecture might thus be that the number of **circular** permutations of the above n objects is

$$\frac{(n-1)!}{r_1!r_2!\dots r_k!}.$$

The above conjecture **sometimes** "works"—e.g. SEE has $3!/2! = 3$ (linear) permutations and SEE indeed also has $(3-1)!/2! = 1$ circular permutation. However and unfortunately, this conjecture is, in general, incorrect. Here are two counter-examples.

Example 1588. There are $3!/3! = 1$ (linear) permutations of the three letters AAA.

If the above conjecture were true, then there ought to be $(3-1)!/3! = 2!/3! = 1/3$ circular permutations of AAA. But this is not even an integer, so obviously it cannot be the number of circular permutations of AAA. In fact, there is also exactly 1 circular permutation of AAA.

Example 1589. There are $6!/(3!3!) = 20$ (linear) permutations of the six letters AAABBB.

If the above conjecture were true, then there ought to be $(6-1)!/(3!3!) = 10/3$ circular permutations of AAABBB. But this is not even an integer, so obviously it cannot be the number of circular permutations of AAABBB. In fact, there are exactly 4 circular permutations of AAABBB.

A general solution (i.e. formula) is possible but is a bit too advanced for A-Levels.[636]

[636] See e.g. this Handbook on Combinatorics.

Proposition 21 (p. 1197 above). *Let S be the sample space, Σ be the corresponding event space, and A, B be events. If the probability function $P : \Sigma \to \mathbb{R}$ satisfies the Kolmogorov Axioms, then P also satisfies the following properties:*

1. **Complements.** $\mathrm{P}(A) = 1 - \mathrm{P}(A^c)$.
2. **Probability of Empty Event is Zero.** $\mathrm{P}(\varnothing) = 0$.
3. **Monotonicity.** If $B \subseteq A$, then $\mathrm{P}(B) \leq \mathrm{P}(A)$.
4. **Probabilities Are At Most One.** $\mathrm{P}(A) \leq 1$.
5. **Inclusion-Exclusion.** $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.

Proof. (Continued from p. 1197.)

2. Probability of Empty Event is Zero. $\varnothing \cap A = \varnothing$ and $\varnothing \cup A = A$. And so again by the Additivity Axiom, $\mathrm{P}(\varnothing \cup A) = \mathrm{P}(A) = \mathrm{P}(\varnothing) + \mathrm{P}(A)$. Thus, $\mathrm{P}(\varnothing) = 0$.

But also by definition, $A \cup A^c = S$. Hence, $\mathrm{P}(A \cup A^c) = \mathrm{P}(s)$.

By the Normalisation Axiom, $\mathrm{P}(s) = 1$.

3. Monotonicity. $A \cap \{B \setminus A\} = \varnothing$ and $A \cup \{B \setminus A\} = B$. Thus, by the Additivity Axiom, $\mathrm{P}(B) = \mathrm{P}(A) + \mathrm{P}(B \setminus A)$. By the Non-Negativity Axiom, $\mathrm{P}(B \setminus A) \geq 0$. Hence, $\mathrm{P}(B) \geq \mathrm{P}(A)$.

4. Probabilities Are At Most One. Any event A is a subset of S. And so by Monotonicity, $\mathrm{P}(A) \leq \mathrm{P}(s)$. But by the Normalisation Axiom, $\mathrm{P}(s) = 1$. Thus, $\mathrm{P}(A) \leq 1$.

5. Inclusion-Exclusion Principle. By the Additivity Axiom, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B \setminus A)$.

Also by the Additivity Axiom, $\mathrm{P}(A \cap B) + \mathrm{P}(B \setminus A) = \mathrm{P}(B)$.

Altogether then, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$. $\square$

Proposition 22 (p. 1233). *The expectation operator E is linear. That is, if X and Y are random variables and c is a constant, then*

(a) Additivity: $E[X+Y] = E[X] + E[Y]$,

(b) Homogeneity of degree 1: $E[cX] = cE[X]$.

Proof. This proposition applies even for non-discrete random variables. But we'll prove this proposition only for the case where the random variable is discrete.

We'll use the linearity of the expectation operator. We prove (b) first.

$$(b) \quad \mathbf{E}[cX] = \sum_{k \in \text{Range}(x)} \mathrm{P}(X = k) \cdot (ck) = c \sum_{k \in \text{Range}(x)} \mathrm{P}(X = k) \cdot k = c\mathbf{E}[X].$$

$$(a) \quad \mathbf{E}[X+Y]$$

$$= \sum_{k \in \text{Range}(x)} \sum_{l \in \text{Range}(y)} \mathrm{P}(X = k, Y = l) \cdot (k + l)$$

$$= \sum_{k \in \text{Range}(x)} k \sum_{l \in \text{Range}(y)} \mathrm{P}(X = k, Y = l) + \sum_{l \in \text{Range}(y)} l \sum_{k \in \text{Range}(x)} \mathrm{P}(X = k, Y = l)$$

$$= \sum_{k \in \text{Range}(x)} k\mathrm{P}(X = k) + \sum_{l \in \text{Range}(y)} l\mathrm{P}(Y = l)$$

$$= \mathbf{E}[X] + \mathbf{E}[Y].$$

□

Proposition 23 (p. 1241). Let X and Y be ***independent*** random variables. Let c be a constant. Then

(a) Additivity: $Var[X+Y] = Var[X] + Var[Y]$,

(b) Homogeneity of degree 2: $Var[cX] = c^2 Var[X]$.

Proof. We use Fact 236 and the linearity of the expectation operator.

$$(b) \quad \mathbf{V}[cX] = \mathbf{E}\left[(cX)^2\right] - (c\mu_X)^2 = c^2\mathbf{E}\left[X^2\right] - c^2\mu_X^2 = c^2\left(\mathbf{E}\left[X^2\right] - \mu_X^2\right) = c^2\mathbf{V}[X].$$

To prove **(a)**, we'll also use Lemma 24:

$$\begin{aligned}
\mathbf{V}[X+Y] &= \mathbf{E}\left[(X+Y)^2\right] - (\mathbf{E}[X+Y])^2 \\
&= \mathbf{E}\left[X^2 + Y^2 + 2XY\right] - (\mathbf{E}[X] + \mathbf{E}[Y])^2 \\
&= \mathbf{E}\left[X^2\right] + \mathbf{E}\left[Y^2\right] + 2\mathbf{E}[XY] - \left(\mu_X^2 + \mu_Y^2 + 2\mu_X\mu_Y\right) \\
&= \underbrace{\mathbf{E}\left[X^2\right] - \mu_X^2}_{\mathbf{V}[X]} + \underbrace{\mathbf{E}\left[Y^2\right] - \mu_Y^2}_{\mathbf{V}[Y]} + 2\underbrace{(\mathbf{E}[XY] - \mu_X\mu_Y)}_{0}.
\end{aligned}$$

□

Lemma 24. *If X and Y are independent random variables, then $E[XY] = E[X]E[Y]$.*

Proof. We prove this Lemma only for the case where X and Y are both discrete.

$$\begin{aligned}
\mathbf{E}[XY] &= \sum_k \sum_l \mathrm{P}(X=k, Y=l) \cdot kl \\
&= \sum_k \sum_l \mathrm{P}(X=k)\,\mathrm{P}(Y=l) \cdot kl \qquad \text{(independence)} \\
&= \sum_k \left(\mathrm{P}(X=k)\, k \sum_l \mathrm{P}(Y=l) \cdot l\right) = \sum_k \left(\mathrm{P}(X=k)\, k\mathbf{E}[Y]\right) \\
&= \mathbf{E}[Y] \sum_k \mathrm{P}(X=k)\, k = \mathbf{E}[Y]\,\mathbf{E}[X].
\end{aligned}$$

□

Fact 300. *(a) Let X be the number of fair coin-flips until we get two consecutive heads. Let Y be the number of fair coin-flips until we get HT consecutively. Then $E[X] = \mu_X = 6$ and $E[Y] = \mu_Y = 4$.*

(b) Flip a fair coin $n+1$ times. This gives us n pairs of consecutive coin-flips. Let A be the proportion of these n pairs of consecutive coin-flips that are HH. Let B be the proportion that are HT. Then $E[A] = \mu_A = 1/4$ and $E[B] = \mu_B = 1/4$.

Proof. **(a)** To find μ_X actually requires a clever, new trick. Let

$$\begin{aligned} p &= \mathbf{E}\left[\text{Additional number of flips to get } HH | \text{Last flip was } T\right], \\ q &= \mathbf{E}\left[\text{Additional number of flips to get } HH | \text{Last two flips were } TH\right]. \end{aligned}$$

Observe that p is the number of flips, if we're "restarting" . Thus, $p = \mu_X$. Now,

$$\begin{aligned} q &= \mathrm{P}\left(\text{Next flip is } H\right) \times 1 + \mathrm{P}\left(\text{Next flip is } T\right) \times (1+p) \\ &= 0.5 \times 1 + 0.5 \times (1+p) = 1 + 0.5p. \end{aligned}$$

(Explanation: If the next flip is H, then we've completed HH and this took us only 1 more flip. If instead the next flip is T, then we start all over again; we've already taken 1 flip and are expected to take another p flips.) Similarly, observe that

$$\begin{aligned} p &= \mathrm{P}\left(\text{Next flip is } H\right) \times (1+q) + \mathrm{P}\left(\text{Next flip is } T\right) \times (1+p) \\ &= 0.5 \times (2 + 0.5p) + 0.5 \times (1+p) = 1.5 + 0.75p. \end{aligned}$$

(Explanation: If the next flip is H, then we expect to take, in addition, another q flips. If instead the next flip is T, then we start all over again; we've already taken 1 flip and are expected to take another p flips.)

Hence, $p = 6 = \mu_X$. The reasoning used above is illustrated by the probability tree below.

Let's now find μ_Y. Again, let

$$\begin{aligned} r &= \mathbf{E}\left[\text{Additional number of flips to get } HT | \text{Last two flips were } TT\right], \\ s &= \mathbf{E}\left[\text{Additional number of flips to get } HT | \text{Last flip was } H\right]. \end{aligned}$$

Observe that r is the number of flips, if we're "restarting". Thus, $r = \mu_X$.

(... Proof continued on the next page ...)

(... Proof continued from the previous page ...)

Now, also observe that

$$\begin{aligned} s &= \mathrm{P}\left(\text{Next flip is } T\right) \times 1 + \mathrm{P}\left(\text{Next flip is } H\right) \times (1+s) \\ &= 0.5 \times 1 + 0.5 \times (1+s) = 1 + 0.5s. \end{aligned}$$

(Explanation: If the next flip is T, then we've completed HT and this took us only 1 more flip. If instead the next flip is H, then we've already taken 1 flip and are expected to take another s flips.)

So $s = 2$. Similarly, observe that

$$\begin{aligned} r &= \mathrm{P}\left(\text{Next flip is } H\right) \times (1+s) + \mathrm{P}\left(\text{Next flip is } T\right) \times (1+r) \\ &= 0.5 \times (1+2) + 0.5 \times (1+r) = 2 + 0.5r. \end{aligned}$$

(Explanation: If the next flip is H, then we've already taken 1 flip and are expected to take another s flips. If the next flip is T, then we've already taken 1 flip and are expected to take another r flips.)

So $r = 4 = \mu_Y$.

(b) Let S_i be the random variable that indicates whether the ith pair of consecutive coin-flips is HH. That is, $S_i = 1$ if so and $S_i = 0$ if not. Then

$$A = \frac{S_1 + S_2 + \cdots + S_n}{n}.$$

$$\text{And so,} \qquad \mathbf{E}[A] = \mathbf{E}\left[\frac{S_1 + S_2 + \cdots + S_n}{n}\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[S_i].$$

But $\mathrm{E}[S_i] = 1/4$. Thus, $\mathrm{E}[A] = 1/4$.

The proof that $\mathrm{E}[B] = 1/4$ is similar. □

Fact 301. $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.

Proof. Omitted. See ✿ or WIKIPEDIA. □

Fact 241 (p. 1260). *Let $Z \sim N(0,1)$ and ϕ and Φ be its PDF and CDF.*

1. $\Phi(\infty) = 1$. *(As with any random variable, the area under the entire PDF is 1.)*
2. $\phi(a) > 0$, *for all* $a \in \mathbb{R}$. *(The PDF is positive* <u>everywhere</u>. *This has a surprising implication: however large a is, there is always some non-zero probability that $Z \geq a$.)*
3. $E[Z] = 0$. *(The mean of Z is 0.)*
4. *The PDF ϕ reaches a global maximum at the mean 0. (In fact, we can go ahead and compute $\phi(0) = 1/\sqrt{2\pi} \approx 0.399$.)*
5. $Var[Z] = 1$. *(The variance of Z is 1.)*
6. $P(Z \leq a) = P(Z < a)$. *(We've already discussed this earlier. It makes no difference whether the inequality is strict. This is because $P(Z = a) = 0$.)*
7. *The PDF ϕ is symmetric about the mean. This has several implications:*
   (a) $P(Z \geq a) = P(Z \leq -a) = \Phi(-a)$.
   (b) *Since* $P(Z \geq a) = 1 - P(Z \leq a) = 1 - \Phi(a)$, *it follows that* $\Phi(-a) = 1 - \Phi(a)$ *or, equivalently,* $\Phi(a) = 1 - \Phi(-a)$.
   (c) $\Phi(0) = 1 - \Phi(0) = 0.5$.
8. $P(-1 \leq Z \leq 1) = \Phi(1) - \Phi(-1) \approx 0.6827$. *(There is probability 0.6827 that Z takes on values within 1 standard deviation of the mean.)*
9. $P(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2) \approx 0.9545$. *(There is probability 0.9545 that Z takes on values within 2 standard deviations of the mean.)*
10. $P(-3 \leq Z \leq 3) = \Phi(3) - \Phi(-3) \approx 0.9973$. *(There is probability 0.9973 that Z takes on values within 3 standard deviations of the mean.)*
11. *The PDF ϕ has two points of inflexion, namely at ± 1. (The points of inflexion are one standard deviation away from the mean.)*

Proof. **1.** Let $u = x/\sqrt{2}$. We have $u^2 = 0.5x^2$ and $du/dx = 1/\sqrt{2}$. And using Fact 301:

$$\Phi(\infty) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} e^{-0.5x^2} \sqrt{2} \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \frac{1}{\sqrt{\pi}} \int_{u=-\infty}^{u=\infty} e^{-u^2}\,\mathrm{d}u \stackrel{1}{=} \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1.$$

(... Proof continued on the next page ...)

(... *Proof continued from the next page* ...) **2.** Obvious.

3. $$\mathbf{E}[Z] = \int_{-\infty}^{\infty} x\phi(x)\,\mathrm{d}x = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(-xe^{-0.5x^2}\right)\mathrm{d}x = \frac{-1}{\sqrt{2\pi}}\left[e^{-0.5x^2}\right]_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}}[0-0] = 0.$$

4. $$\frac{d}{da}\phi(a) = \frac{d}{da}\frac{1}{\sqrt{2\pi}}e^{-0.5a^2} = \frac{-a}{\sqrt{2\pi}}e^{-0.5a^2}\begin{cases} >0, & \text{if } a<0, \\ =0, & \text{if } a=0, \\ <0, & \text{if } a>0. \end{cases}$$

5. $$\mathbf{V}[Z] = \int_{-\infty}^{\infty}(x-0)^2\phi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}e^{-0.5x^2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overbrace{x}^{u}\,\overbrace{xe^{-0.5x^2}}^{v'}\,\mathrm{d}x$$
$$= \frac{1}{\sqrt{2\pi}}\left[e^{-0.5x^2} - \int e^{-0.5x^2}\,\mathrm{d}x\right]_{-\infty}^{\infty} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-0.5x^2}\,\mathrm{d}x = 1.$$

ϕ is continuous, increasing for $a<0$ and decreasing for $a>0$. Thus, ϕ reaches a global maximum at 0. By plugging in $a=0$, we can compute this global maximum value to be $\phi(0) = 1/\sqrt{2\pi} \approx 0.399$.

6. By the Additivity Axiom, $\mathrm{P}(Z\le a) = \mathrm{P}(Z<a, Z=a) = \mathrm{P}(Z<a)+\mathrm{P}(Z=a) = \mathrm{P}(Z<a)+0 = \mathrm{P}(Z<a)$, as desired.

7. Clearly, $\phi(a) = \mathrm{e}^{-0.5a^2}/\sqrt{2\pi} = \mathrm{e}^{-0.5(-a)^2}/\sqrt{2\pi} = \phi(-a)$ for all $a\in\mathbb{R}$. Thus, ϕ is symmetric about the vertical axis $x=0$, which is also the mean.

7(a). Using the substitution $u=-x$, we have $du/dx=-1$ and

$$\mathrm{P}(Z\ge a) = \int_{x=a}^{x=\infty}\frac{e^{-0.5x^2}}{\sqrt{2\pi}}\,\mathrm{d}x = \int_{u=-a}^{u=-\infty}\frac{-e^{-0.5u^2}}{\sqrt{2\pi}}\,\mathrm{d}u = \int_{u=-\infty}^{u=-a}\frac{e^{-0.5u^2}}{\sqrt{2\pi}}\,\mathrm{d}u = \mathrm{P}(Z\le -a) = \Phi(-a).$$

7(b) and 7(c). Obvious.

8, 9, and 10. These can be computed numerically, using a computer.

11. $$\frac{d^2}{da^2}\phi(a) = \frac{d}{da}\frac{-a}{\sqrt{2\pi}}e^{-0.5a^2} = \frac{1}{\sqrt{2\pi}}e^{-0.5a^2}\left(a^2-1\right)\begin{cases} >0, & \text{if } a<-1, \\ =0, & \text{if } a=-1, \\ <0, & \text{if } -1<a<1, \\ =0, & \text{if } a=1, \\ >0, & \text{if } a>1. \end{cases}$$

Hence, ± 1 are the only two points of inflexion since ϕ changes concavity only here. □

Theorem 64. *Let $a, b \in \mathbb{R}$ be constants with $a \neq 0$ and X be a continuous random variable with PDF f_X and CDF F_X. Let $Y = aX + b$. Then*

$$f_Y(c) = \frac{1}{|a|} f_X\left(\frac{c-b}{a}\right).$$

Proof. $F_Y(c) = \mathrm{P}(Y \leq c) = \mathrm{P}(aX + b \leq c) = \mathrm{P}(aX \leq c - b)$.

Case #1. If $a > 0$, then $F_Y(c) = \cdots = \mathrm{P}(aX \leq c-b) = \mathrm{P}\left(X \leq \frac{c-b}{a}\right) = F_X\left(\frac{c-b}{a}\right)$.

Now differentiate:

$$\frac{d}{da} F_Y(c) = \frac{d}{dc} F_X\left(\frac{c-b}{a}\right) = f_Y(c) = \frac{1}{a} f_X\left(\frac{c-b}{a}\right) = \frac{1}{|a|} f_X\left(\frac{c-b}{a}\right).$$

Case #2. If $a < 0$, then $F_Y(c) = \cdots = \mathrm{P}(aX \leq c-b) = \mathrm{P}\left(X \geq \frac{c-b}{a}\right) = 1 - F_X\left(\frac{c-b}{a}\right)$.

Now differentiate:

$$\frac{d}{da} F_Y(c) = \frac{d}{dc}\left[1 - F_X\left(\frac{c-b}{a}\right)\right] = f_Y(c) = -\frac{1}{a} f_X\left(\frac{c-b}{a}\right) = \frac{1}{|a|} f_X\left(\frac{c-b}{a}\right).$$

□

Fact 242 (p. 1270). *Let $X \sim N(\mu, \sigma^2)$ and $a, b \in \mathbb{R}$ be constants. Then $aX + b \sim N(a\mu + b, a^2\sigma^2)$.*

Proof. By Theorem 64, the PDF of $aX + b$ is given by

$$f_{aX+b}(c) = \frac{1}{|a|} f_X\left(\frac{c-b}{a}\right) = \frac{1}{|a|} \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5\left(\frac{\frac{c-b}{a}-\mu}{\sigma}\right)^2} = \frac{1}{|a|\,\sigma\sqrt{2\pi}} e^{-0.5\left[\frac{c-(a\mu+b)}{a\sigma}\right]^2}.$$

But this lattermost expression is indeed the PDF of the random variable with distribution $\mathrm{N}(a\mu + b, a^2\sigma^2)$. □

Fact 244 (p. 1306). *Let $S = (X_1, X_2, \ldots, X_n)$ be a random sample of size n. Let $\bar{X}$ be the sample mean and S^2 be the sample variance. Let $a \in \mathbb{R}$ be a constant. Then*

$$(a)\ S^2 = \frac{\sum_{i=1}^n X_i^2 - \frac{\left[\sum_{i=1}^n X_i\right]^2}{n}}{n-1} \quad \text{and} \quad (b)\ S^2 = \frac{\sum_{i=1}^n (X_i - a)^2 - \frac{\left[\sum_{i=1}^n (X_i - a)\right]^2}{n}}{n-1}.$$

Proof. This proof may look intimidating but it's really just a bunch of tedious algebra. (I've also tried to go slow with the algebra, so more steps are explicitly listed than is typical in a proof.)

(a) Start from the definition of the sample variance and do the algebra:

$$\begin{aligned}
S^2 &= \frac{\sum_{i=1}^n \left(X_i - \bar{X}\right)^2}{n-1} = \frac{\sum_{i=1}^n \left(X_i^2 + \bar{X}^2 - 2\bar{X}X_i\right)}{n-1} = \frac{\sum_{i=1}^n X_i^2 - \sum_{i=1}^n \bar{X}^2 - \sum_{i=1}^n \left(2\bar{X}X_i\right)}{n-1} \\
&= \frac{\sum_{i=1}^n X_i^2 - n\bar{X}^2 - 2\bar{X}\sum_{i=1}^n X_i}{n-1} = \frac{\sum_{i=1}^n X_i^2 - n\bar{X}^2 - 2\bar{X}\left(n\bar{X}\right)}{n-1} = \frac{\sum_{i=1}^n X_i^2 - n\bar{X}^2}{n-1} \\
&= \frac{\sum_{i=1}^n X_i^2 - n\left[\frac{\sum_{i=1}^n X_i}{n}\right]^2}{n-1} = \frac{\sum_{i=1}^n X_i^2 - \frac{\left[\sum_{i=1}^n X_i\right]^2}{n}}{n-1}.
\end{aligned}$$

(b) Start from the formula found in (a) and do the algebra:

$$\begin{aligned}
S^2 &= \frac{\sum_{i=1}^n X_i^2 - \frac{\left[\sum_{i=1}^n X_i\right]^2}{n}}{n-1} = \frac{\sum_{i=1}^n (X_i - a + a)^2 - \frac{\left[\sum_{i=1}^n (X_i - a + a)\right]^2}{n}}{n-1} \\
&= \frac{\sum_{i=1}^n \left[(X_i - a)^2 + a^2 + 2(X_i - a)a\right] - \frac{\left[\sum_{i=1}^n (X_i - a) + \sum_{i=1}^n a\right]^2}{n}}{n-1} \\
&= \frac{\sum_{i=1}^n (X_i - a)^2 + \sum_{i=1}^n a^2 + 2a\sum_{i=1}^n (X_i - a) - \frac{\left[\sum_{i=1}^n (X_i - a)\right]^2 + \left(\sum_{i=1}^n a\right)^2 + 2\sum_{i=1}^n (X_i - a)\sum_{i=1}^n a}{n}}{n-1} \\
&= \frac{\sum_{i=1}^n (X_i - a)^2 + na^2 + 2a\sum_{i=1}^n (X_i - a) - \frac{\left[\sum_{i=1}^n (X_i - a)\right]^2 + (na)^2 + 2na\sum_{i=1}^n (X_i - a)}{n}}{n-1} \\
&= \frac{\sum_{i=1}^n (X_i - a)^2 - \frac{\left[\sum_{i=1}^n (X_i - a)\right]^2}{n}}{n-1}.
\end{aligned}$$

□

Proposition 24 (p. 1312). *Let $(X_1, X_2, \ldots, X_n)$ be a random sample drawn from a distribution with population mean μ and population variance σ^2. Let $\bar{X}$ be the sample mean and S^2 be the sample variance. Then*

$$\text{(a) } E\left[\bar{X}\right] = \mu. \qquad \text{And} \qquad \text{(b) } E\left[S^2\right] = \sigma^2.$$

Proof. (a) was proven in Exercise 508. We prove only (b).

Equation $\stackrel{1}{=}$ is the key piece of intuition (and is formally proven below):

$$\overbrace{\mathbf{E}\left[\left(X_i - \bar{X}\right)^2\right]}^{\substack{\text{The degree to which} \\ X_i \text{ varies from } \bar{X}}} + \overbrace{\mathbf{E}\left[\left(\bar{X} - \mu\right)^2\right]}^{\substack{\text{The degree to which} \\ \bar{X} \text{ varies from } \mu}} \stackrel{1}{=} \overbrace{\mathbf{E}\left[(X_i - \mu)^2\right]}^{\substack{\text{The degree to which} \\ X_i \text{ varies from } \mu}}.$$

□

Rearranging,

Proof.

$$\mathbf{E}\left[\left(X_i - \bar{X}\right)^2\right] = \overbrace{\mathbf{E}\left[(X_i - \mu)^2\right]}^{\text{Population variance}} - \overbrace{\mathbf{E}\left[\left(\bar{X} - \mu\right)^2\right]}^{\text{Variance of sample mean}} = \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2.$$

We've just shown that $\left(X_i - \bar{X}\right)^2$ is a biased estimator for σ^2. And in turn, S^2 is not:

$$\mathbf{E}\left[S^2\right] = \mathbf{E}\left[\frac{\sum_{i=1}^n \left(X_i - \bar{X}\right)^2}{n-1}\right] = \mathbf{E}\left[\frac{\sum_{i=1}^n \frac{n}{n-1}\left(X_i - \bar{X}\right)^2}{n}\right] = \frac{\sum_{i=1}^n \mathbf{E}\left[\frac{n}{n-1}\left(X_i - \bar{X}\right)^2\right]}{n} = \frac{n\sigma^2}{n} = \sigma^2.$$

As promised, here is the proof of equation $\stackrel{1}{=}$:

$$\begin{aligned}
\mathbf{E}\left[\left(X_i - \bar{X}\right)^2\right] + \mathbf{E}\left[\left(\bar{X} - \mu\right)^2\right] &= \mathbf{E}\left[\left(X_i - \bar{X}\right)^2 + \left(\bar{X} - \mu\right)^2\right] \\
&= \mathbf{E}\left[\left(\left(X_i - \bar{X}\right) + \left(\bar{X} - \mu\right)\right)^2 - 2\left(X_i - \bar{X}\right)\left(\bar{X} - \mu\right)\right] \\
&= \mathbf{E}\left[(X_i - \mu)^2 - 2\left(X_i - \bar{X}\right)\left(\bar{X} - \mu\right)\right] \\
&= \mathbf{E}\left[(X_i - \mu)^2 - 2\left(X_i\bar{X} - \mu X_i - \bar{X}^2 + \mu\bar{X}\right)\right] \\
&= \mathbf{E}\left[(X_i - \mu)^2 - 2\left(X_i\bar{X} - \bar{X}^2\right)\right] \\
&= \mathbf{E}\left[(X_i - \mu)^2\right] + 2\left\{\mathbf{E}\left[\bar{X}^2\right] - \mathbf{E}\left[\bar{X}X_i\right]\right\} \\
&= \mathbf{E}\left[(X_i - \mu)^2\right].
\end{aligned}$$

The last equality follows because $\mathrm{E}\left[\bar{X}X_i\right] = \dfrac{\sum_{i=1}^n \mathrm{E}\left[\bar{X}X_i\right]}{n} = \mathrm{E}\left[\bar{X}\dfrac{\sum_{i=1}^n X_i}{n}\right] = \mathrm{E}\left[\bar{X}^2\right]$. □

Definition 336. The *random variable T_ν with Student's t-distribution with ν degrees of freedom* has PDF $f:\mathbb{R}\to\mathbb{R}$ given by mapping rule

$$f(t) = \frac{\int_0^\infty x^{\frac{\nu+1}{2}-1}e^{-x}dx}{\sqrt{\nu\pi}\int_0^\infty x^{\frac{\nu}{2}-1}e^{-x}dx}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

147.8. Calculating the Margin of Error

Let μ be the true population proportion (of votes for Dr. Chee). Say we take a random sample of size 900.[637] Let X be the sample number of votes for Dr. Chee. We know that $X \sim \mathrm{B}(900, \mu)$.

Our **confidence level** is 95%. So we want to find the smallest k such that

$$\mathrm{P}(900\mu - k \leq X \leq 900\mu + k) \geq 0.95.$$

And $\pm k/900$ will be our **margin of error**.

Case #1: Perfect hindsight: $\mu = 9142/23570$.

With perfect hindsight, we now know that $\mu = 9142/23570$. So $X \sim \mathrm{B}(900, 9142/23570)$.

We want to find the smallest k such that $\mathrm{P}(349 - k \leq X \leq 349 + k) \geq 0.95$.

where $900 \times 9142/23570 \approx 349$. Using the "Binomial" sheet at the usual link, we have

$$\mathrm{P}(349 - 28 \leq X \leq 349 + 28) \approx 0.9488,$$
$$\mathrm{P}(349 - 29 \leq X \leq 349 + 29) \approx 0.9565.$$

Thus, $k = 29$. Now, $29/900 \approx 3.2\%$. Thus, at a 95% confidence level, the margin of error is $\pm 3.2\%$. This is the "true" margin of error, assuming we know μ. But this assumption defeats the point of sampling—we don't know μ, which is why we're doing sampling in the first place!

What we want instead is the margin of error in the case where μ is unknown.

Case #2: Without perfect hindsight: μ **unknown.**

With μ unknown, a conservative interpretation would be to find the smallest k such that **for all** μ, $\mathrm{P}(900\mu - k \leq X \leq 900\mu + k) \geq 0.95$.

(... Analysis continued on the next page ...)

[637] This is slightly different from what actually happened: (1) The actual random sampling was most likely without replacement (which would change the maths slightly). (2) 100 votes were taken from each of 9 different polling stations (which would also change the maths slightly).

(... Analysis continued from the previous page ...)

Observe that $\text{Var}[X] = 900\mu(1-\mu)$ is maximised at $\mu = 0.5$. Thus, it is plausible[638] that if k satisfies

$$X \sim \text{B}(900, 0.5) \implies \text{P}(900 \times 0.5 - k \le X \le 900 \times 0.5 + k) \ge 0.95,$$

then k also satisfies

$$X \sim \text{B}(900, \mu) \implies \text{P}(900 \times 0.5 - k \le X \le 900 \times 0.5 + k) \ge 0.95.$$

Our problem thus boils down to finding the smallest k such that for $X \sim \text{B}(900, 0.5)$ implies

$$\text{P}(450 - k \le X \le 450 + k) \ge 0.95.$$

$$\text{We have} \qquad \begin{aligned} &\text{P}(450 - 29 \le X \le 450 + 29) \approx 0.9508, \\ &\text{P}(450 - 28 \le X \le 450 + 28) \approx 0.9426. \end{aligned}$$

We conclude that the smallest such k is 29. Now, $29/900 \approx 3.2\%$. So the margin of error may be given as $\pm 3.2\%$. This is the same as what was calculated above, which is not surprising, since $9142/23570 \approx 0.388$ is close to 0.5.

The reader will, of course, wonder why the Elections Department stated that the margin of error was $\pm 4\%$, rather than $\pm 3.2\%$ as I calculated here. I am not sure myself. My guess is that they probably don't bother going through all the above calculations afresh each time. Instead, each time they report a sample count, they simply read off the margin of error from a table that looks something like this:

Sample Size	**Approximate Margin of Error**
400 – 599	±5%
600 – 999	±4%
1000 – 2000	±3%

(By the way, note that it is common to use the CLT approximation when calculating the margin of error. I have not done so here. Instead, I've stuck with using the original, exact binomial distribution.)

[638] Proving this would need a little work though.

Fact 302. *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be numbers. Let $\bar{x} = \sum x_i/n$ and $\bar{y} = \sum y_i/n$. Then*

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} \in [-1, 1].$$

Proof. Let $\mathbf{u} = (x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x})$ and $\mathbf{v} = (y_1 - \bar{y}, y_2 - \bar{y}, \ldots, y_n - \bar{y})$ be n-dimensional vectors. Then

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

But from what we learnt about vectors,[639] if θ is the angle between two vectors, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Since $\cos\theta \in [-1, 1]$, the result follows. □

[639] Of course, in this textbook, we've only shown that this is true for two- and three-dimensional vectors. But let's just wave our hands and say that this is also true for higher-dimensional vectors.

Fact 248. *Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two ordered sets of data. The OLS regression line of y on x is $y - \bar{y} = \hat{b}(x - \bar{x})$, where*

$$\textbf{(i)} \quad \hat{b} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

$$\textbf{(ii)} \quad \hat{b} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2}.$$

Moreover, the regression line can also be written in the form $y = \hat{a} + \hat{b}x$, where $\hat{b}$ is a given above and $\hat{a} = \bar{y} - \hat{b}\bar{x}$.

Proof. (Continued from the proof begun on p. 1350.) Remember that the data $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are given. Thus, we can treat all the x_is and y_is as constants. We have

$$\frac{\partial}{\partial \hat{a}} \sum \hat{u}_i^2 = \sum \frac{\partial}{\partial \hat{a}} \hat{u}_i^2 = \sum \left(2\hat{u}_i \frac{\partial \hat{u}_i}{\partial \hat{a}}\right) = \sum -2\left[y_i - \left(\hat{a} + \hat{b}x_i\right)\right].$$

Thus, $\dfrac{\partial}{\partial \hat{a}} \sum \hat{u}_i^2 = 0 \iff y_i - \left(\hat{a} + \hat{b}x_i\right) = 0 \iff \hat{a} \overset{1}{=} \bar{y} - \hat{b}\bar{x}$.

We also have

$$\frac{\partial}{\partial \hat{b}} \sum \hat{u}_i^2 = \sum \frac{\partial}{\partial \hat{b}} \hat{u}_i^2 = \sum \left(2\hat{u}_i \frac{\partial \hat{u}_i}{\partial \hat{b}}\right) = \sum -2x_i\left[y_i - \left(\hat{a} + \hat{b}x_i\right)\right].$$

Thus, $\dfrac{\partial}{\partial \hat{b}} \sum \hat{u}_i^2 = 0 \iff \sum \left[y_i - \left(\hat{a} + \hat{b}x_i\right)\right] x_i = 0$. Plugging $\overset{1}{=}$ into this last equation, we have $\sum \left[y_i - \left(\bar{y} - \hat{b}\bar{x} + \hat{b}x_i\right)\right] x_i = 0$. Tedious algebra yields Formula (ii):

$$\hat{b} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2}.$$

More algebra yields Formula (i). □

147.10. Deriving a Linear Model from the Barometric Formula

According to NASA (1976), "U.S. Standard Atmosphere", p. 12, eq. (33a) (PDF), the barometric formula (relating pressure P to height H above sea level), in the case where $L_{M,b} \neq 0$ is given by

$$P = P_b \left[\frac{T_{M,b}}{T_{M,b} + L_{M,b}(h - h_b)} \right]^{\frac{g_0' M}{R^* L_{M,b}}},$$

where P_b, $T_{M,b}$, $L_{M,b}$, h_b, g_0', R^* are simply constants. Now, do the algebra:

$$\begin{aligned} P &= P_b \left[\frac{T_{M,b}}{T_{M,b} + L_{M,b}(h - h_b)} \right]^{\frac{g_0' M}{R^* L_{M,b}}} \\ &= P_b \left[\frac{T_{M,b} + L_{M,b}(h - h_b)}{T_{M,b}} \right]^{-\frac{g_0' M}{R^* L_{M,b}}} \\ &= P_{M,b} \left[1 + \frac{L_{M,b}}{T_{M,b}}(h - h_b) \right]^{-\frac{g_0' M}{R^* L_{M,b}}} \end{aligned}$$

$$\ln P = \ln P_{M,b} - \frac{g_0' M}{R^* L_{M,b}} \ln \left[1 + \frac{L_{M,b}}{T_{M,b}}(h - h_b) \right].$$

Now, for heights up to 11 000 m above sea level, h_b is simply the height *at* sea level. That is, $h_b = 0$ m. If we also let $a = \ln P_{M,b}$ and $b = -\dfrac{g_0' M}{R^* L_{M,b}}$ and get rid of the subscripts in $L_{M,b}$ and $T_{M,b}$ (just to make it neater), then we have

$$\ln P = a + b \ln \left(1 + \frac{L}{T} h \right).$$

For heights up to 11 000 m above sea level, $L = -0.000\,65$ kelvin per metre is the temperature lapse rate (the rate at which the temperature falls, as we go up in altitude; see p.3, Table 4) and $T = 288.15$ kelvin is the standard sea-level temperature (also precisely equal to 15 °C; see p. 4).

Part IX.
Answers to Exercises

148. Part 0 Answers (A Few Basics)

148.1. Ch. 1 Answers (Just To Be Clear)

(*This chapter had no exercises.*)

148.2. Ch. 2 Answers (PSLE Review: Division)

Tip: Click on the exercise number to return to that exercise.

A1. Long division for $8\,057 \div 39$. The dividend is $8\,057$ and the **divisor** is 39.

$$\begin{array}{r|r} & 206 \\ \hline 39 & 8057 \\ & 7800 \\ \cline{2-2} & 257 \\ & 0 \\ \cline{2-2} & 257 \\ & 234 \\ \cline{2-2} & 23 \end{array}$$

Thus,
$$\frac{8\,057}{39} = 206\frac{23}{39}.$$

The **quotient** is 206 and the **remainder** is 23.

A2. The error is in Step 5. Since $x = y$, we have $x - y = 0$. And so we cannot simply divide both sides by $x - y$. (Step 7 is not wrong because we already declared that $x > 0$, so we can go ahead and divide by x.)

A3. The error is in the first step, when he "divide[s] both numerators by x". The given equation's solutions are $x = 8$ *and* $x = 0$.

A4. Every conjunction (AND statement) is false because (at least) one of the two statements forming the conjunction is false.

In contrast, every disjunction (OR statement) is true because (at least) one of the two statements forming the disjunction is true.

(a)	B AND C:	"Germany is in Asia AND $1+1=2$."	✗
(b)	A AND D:	"Germany is in Europe AND $1+1=3$."	✗
(c)	C AND D:	"$1+1=2$ AND $1+1=3$."	✗
(d)	B OR C:	"Germany is in Asia OR $1+1=2$."	✓
(e)	A OR D:	"Germany is in Europe OR $1+1=3$."	✓
(f)	C OR D:	"$1+1=2$ OR $1+1=3$."	✓

A5. NOT–E: "It's not raining." NOT–F: "The grass is not wet." NOT–G: "I'm not sleeping." NOT–H: "My eyes are not shut."

A6(a) If $x=0.5$, then O is true while N is false. Since O and N do not always have the same truth value, we say that are **not** equivalent and write $O \not\Longleftrightarrow N$.

(b) If $x=-3$, then γ is true while α is false. Since γ and α do not always have the same truth value, we say that γ and α are **not** equivalent and write $\gamma \not\Longleftrightarrow \alpha$.

(c) Both statements are always true. So, we say they're equivalent and write ☺ $\Longleftrightarrow$ ☹.

(d) Statement ☼ is always true, while ● isn't always true (e.g. $x=1$). So, ☼ $\not\Longleftrightarrow$ ●.

(e) Statement ✡ is always false, while ● isn't always false (in particular, ● is true if $x=5$). So, ✡ $\not\Longleftrightarrow$ ●.

A7(a) NOT–(B AND C) is true. There are two ways to see this:

- Since B AND C is false, its negation NOT–(B AND C) must be true.
- By Fact 1, NOT–(B AND C) is equivalent to NOT–B OR NOT–C: "Germany is not in Asia OR $1+1\neq 2$". Which is true because NOT–B: "Germany is not in Asia" is true.

(b) NOT–(A AND D) is true. Two ways to see this:

- Since A AND D is false, its negation NOT–(A AND D) must be true.
- By Fact 1, NOT–(A AND D) is equivalent to NOT–A OR NOT–D: "Germany is not in Europe OR $1+1\neq 3$". Which is true because NOT–D: "$1+1\neq 3$" is true.

(c) NOT–(B AND D) is true. Two ways to see this:

- Since B AND D is false, its negation NOT–(B AND D) must be true.
- By Fact 1, NOT–(B AND D) is equivalent to NOT–B OR NOT–D: "Germany is not in Asia OR $1+1\neq 3$". Which is true because NOT–B: "Germany is not in Asia" is true. (Indeed, NOT–D: "$1+1\neq 3$" is also true.)

A8(a) NOT−$(B$ OR $C)$ is false. There are two ways to see this:

- Since B OR C is true, its negation NOT−$(B$ OR $C)$ must be false.
- By Fact 2, NOT−$(B$ OR $C)$ is equivalent to NOT−B AND NOT−C: "Germany is not in Asia AND $1+1\neq 2$". Which is false because NOT−C: "$1+1\neq 2$" is false.

(b) NOT−$(A$ OR $D)$ is false. Two ways to see this:

- Since A OR D is true, its negation NOT−$(A$ OR $D)$ must be false.
- By Fact 2, NOT−$(A$ OR $D)$ is equivalent to NOT−A AND NOT−D: "Germany is not in Europe AND $1+1\neq 3$". Which is false because NOT−A: "Germany is not in Europe" is false.

(c) NOT−$(B$ OR $D)$ is true. Two ways to see this:

- Since B OR D is false, its negation NOT−$(B$ OR $D)$ must be true.
- By Fact 2, NOT−$(B$ OR $D)$ is equivalent to NOT−B AND NOT−D: "Germany is not in Asia AND $1+1\neq 3$". Which is true because both NOT−B: "Germany is not in Asia" and NOT−D: "$1+1\neq 3$" are true.

A9. Remember: An implication $P \implies Q$ is true if either its hypothesis P is false or its conclusion Q is true.

Here, the hypothesis of each statement **(a)**–**(d)** is false (TPL is not a genius, π is not rational). Hence, each statement is true.

A10. Maths/Logic	**Everyday English**
$G \implies H$	That I'm sleeping implies that my eyes are shut.
$G \implies H$	I'm sleeping only if my eyes are shut.
If G, then H.	If I'm sleeping, then my eyes are shut.
If G, H.	If I'm sleeping, my eyes are shut.
H if G.	My eyes are shut if I'm sleeping.
H when G.	My eyes are shut when (or whenever) I'm sleeping.
H follows from G.	That my eyes are shut follows from the fact that I'm sleeping.
G is sufficient for H.	That I'm sleeping is sufficient for my eyes to be shut.
H is necessary for G.	It is necessary that my eyes are shut, for me to be sleeping.

A11. By Definition 5, $P \implies Q$ is equivalent to NOT−P OR Q. So, by Fact 2, the negation of NOT−P OR Q is

$$P \text{ AND NOT−} Q.$$

A12. By Fact 3, NOT−$(K \implies L)$ is equivalent to

$$K \text{ AND NOT-} L.$$

That's "x is donzer and not kiki." So the answer is **(d)**.

A13(a) $G \implies H$ is true (at least for most humans). The converse $H \implies G$ is false:

"If my eyes are shut, then I'm sleeping."
Or, "That my eyes are shut implies that I'm sleeping."

Two counterexamples to $H \implies G$: (1) I may be resting my eyes. (2) I may be blinking. In either counterexample, my eyes are shut, but I'm not sleeping.

(b) $M \implies N$ is false (counterexample: $x = 0.5$). The converse $N \implies M$ is true:

"If $x > 1$, then $x > 0$." Or, "That $x > 1$ implies that $x > 0$."

(c) $\gamma \implies \alpha$ is false (counterexample: $x = -3$). The converse $\alpha \implies \gamma$ is true:

"If $x = 3$, then $x^2 = 9$." Or, "That $x = 3$ implies that $x^2 = 9$."

A14(a) The converse is "If the Nazis won World War II (WW2), then Tin Pei Ling (TPL) is a genius." True because the hypothesis is false.

(b) The converse is "If the Allies won WW2, then TPL is a genius." False because the hypothesis is true AND the conclusion is false.

(c) The converse is "If I am the king of the world, then π is rational." True because the hypothesis is false.

(d) The converse is "If Lee Hsien Loong is Lee Kuan Yew's son, then π is rational." False because the hypothesis is true AND the conclusion is false.

A15(a) Since A is true while B is false, the implication , $A \implies B$ is false.

Since B is false, the converse $B \implies A$ is true. (Alternate answer: Since A is true, the converse $B \implies A$ is true.)

(b) Since C is true, the implication , $A \implies C$ is true.

Since A is true, the converse $C \implies A$ is true.

(c) Since A is true while D is false, the implication , $A \implies D$ is false.

Since D is false, the converse $D \implies A$ is true. (Alternate answer: Since A is true, the converse , $D \implies A$ is true.)

(d) Since C is true while D is false, the implication $C \implies D$ is false.

Since D is false, the converse $D \implies C$ is true. (Alternate answer: Since C is true, the converse , $D \implies C$ is true.)

A16. Remember: An implication $P \implies Q$ is true if either its hypothesis P is false or its conclusion Q is true.

(a)	If P is true, then	$P \implies Q$	(iii) could be true or false and
		$Q \implies P$	(i) must be true.
(b)	If P is false, then	$P \implies Q$	(i) must be true and
		$Q \implies P$	(iii) could be true or false.
(c)	If $P \implies Q$ is true, then	$Q \implies P$	(iii) could be true or false.
(d)	If $P \implies Q$ is false, then	$Q \implies P$	(i) must be true.

Parts **(c)** and **(d)** here correspond to Fact 4 (a) and (b).

A17. No. This is another example of **affirming the consequent** or the **fallacy of the converse**.

A18. Given the statement "If x is German, then x is European", its contrapositive is

(d) "If x is not European, then x is not German" (must be true).

(a) "If x is European, then x is German" could be true (e.g. x is Xi Jinping) or false (e.g. x is Boris Johnson).

(b) "If x is not German, then x is not European" could be true (e.g. x is Angela Merkel) or false (e.g. x is Boris Johnson).

(c) "If x is not German, then x is European" could be true (e.g. x is Boris Johnson) or false (e.g. x is Xi Jinping)

(d) "If x is not European, then x is German" could be true (e.g. x is Boris Johnson) or false (e.g. x is Xi Jinping)

By the way, **(a)** is the original statement's converse, while **(b)** is the **inverse** = "Negate both".

A19. $O \implies N$ is false (counterexample: $x = 0.5$) and so by Fact 6, $N \not\Longleftrightarrow O$.

A20. No two of the three statements are equivalent:

- $Y \not\Longrightarrow X$ because John could have a NRIC but be a permanent resident (and hence non-citizen).
- $X \not\Longrightarrow Z$ because John may be a newborn Singapore citizen who hasn't yet obtained his pink NRIC.
- $Y \not\Longrightarrow Z$ because John may have a NRIC but one that's *blue* rather than pink.

Likewise for the non-Singaporean version—no two of the three statements are equivalent:

- $X \not\Longrightarrow Y$ because John could be a snake (and hence an animal) but not human.
- $Z \not\Longrightarrow X$ because John could be a tree (and hence a living thing) but not an animal.
- $Z \not\Longrightarrow Y$ because John could be a tree (and hence a living thing) but not human.

A21	All Yes	All No	Some Yes	SomeNo
(a)	"All donzers are kiki."	"No donzer is kiki."	"Some donzer is kiki."	"Some donzer is not kiki."
(b)	"All donzers cause cancer."	"No donzer causes cancer."	"Some donzer causes cancer."	"Some donzer does not cause cancer."
(c)	"All bachelors are married."	"No bachelor is married."	"Some bachelor is married."	"Some bachelor is not married."
(d)	"All bachelors smoke."	"No bachelor smokes."	"Some bachelor smokes."	"Some bachelor does not smoke."

A22. By Definition 3, a true statement's negation is false and a false statement's negation is true.

(a) Consider the statements, "All animals are dogs," (All Yes) and, "No animal is a dog," (All No). Both are false. So, All Yes and the All No statements are not always negations of each other.

(b) Consider the statements, "Some animals are dogs," (Some Yes) and, "Some animal is a dog," (Some No). Both are true. So, Some Yes and the Some No statements are not always negations of each other.

A23.	Statement	Negation
(a)	All Yes: "All donzers are kiki."	Some No: "Some donzer is not kiki."
(b)	All No: "No donzer is kiki."	Some Yes: "Some donzer is kiki."
(c)	Some Yes: "Some donzer is kiki."	All No: "All donzers are not kiki."
(d)	Some No: "Some donzer is not kiki."	All Yes: "All donzers are kiki."
(e)	All Yes: "All bachelors are married."	Some No: "Some bachelor is not married."
(f)	All No: "No bachelor is married."	Some Yes: "Some bachelor is married."
(g)	Some Yes: "Some bachelor is married."	All No: "All bachelors are not married."
(h)	Some No: "Some bachelor is not married."	All Yes: "All bachelors are married."
(i)	All Yes: "All donzers cause cancer."	Some No: "Some donzer does not cause cancer."
(j)	All No: "No donzer causes cancer."	Some Yes: "Some donzer causes cancer."
(k)	Some Yes: "Some donzer causes cancer."	All No: "All donzers do not cause cancer."
(l)	Some No: "Some donzer does not cause cancer."	All Yes: "All donzers cause cancer."
(m)	All Yes: "All bachelors smoke."	Some No: "Some bachelor does not smoke."
(n)	All No: "No bachelor smokes."	Some Yes: "Some bachelor smokes."
(o)	Some Yes: "Some bachelor smokes."	All No: "All bachelors do not smoke."
(p)	Some No: "Some bachelor does not smoke."	All Yes: "All bachelors smoke."

A24(a) "No person is LeBron James" is an All No statement with subject "person" and predicate "LeBron James".

(b) The negation of All No is Some Yes: "Some person is LeBron James". Which is true, since there *is* a person who is LeBron James—namely LeBron James himself.

(c) Since the negation is true, the commentator's statement is false.

Where we place the word **not** is crucial. The commentator wanted to negate the statement "Everybody is LeBron James", but placed the word **not** in the wrong position. He incorrectly said, "Everybody is **not** LeBron James," but should instead have said, "**Not** everybody is LeBron James". This latter statement is true and is equivalent to the Some No statement, "Some person is not LeBron James".

Moral of the story: "All fruit are not apples" is different from "Not all fruit are apples". Don't get them confused.

A25. Each of **(a)**–**(n)** could be true or false. (None must be true or must be false.)

A26(a) "Some (i.e. at least one) prime number is even."

(b) "Some prime number greater than 2 is even."

(c) "No rational number is greater than π" or "All rational numbers are not greater than π."

(d) "Some number is either negative or imaginary."

(e) "Some number is either non-positive or rational".

148.4. Ch. 4 Answers (Sets)

A27. The set of the first seven integers is $C = \{1, 2, 3, 4, 5, 6, 7\}$.

A28. There is only one even prime number, namely 2. Hence, $D = \{2\}$.

A29. $X =$ {Lee Kuan Yew, Goh Chok Tong, Lee Hsien Loong}.

A30(a) H contains two elements, namely F and G. (You can think of H as a box that itself contains two boxes, namely F and G.)

(b) $H = \{\{\{1, 3, 5\}, \{100, 200\}\}, \{1, 3, 5, 100, 200\}\}$. (Note that the braces go three-deep.)

A31(a) I contains three elements, namely A, B, and G. (You can think of I as a box that itself contains three boxes, namely A, B, and G.)

(b) $I = \{\{1, 3, 5\}, \{100, 200\}, \{1, 3, 5, 100, 200\}\}$.

(c) Nope. H is a box that contains the two boxes F and G, while I is a box that contains the three boxes A, B, and G. So the sets H and I are not the same.

A32(a) Los Angeles $\in$ The set of the four largest cities in the US.

(b) Tharman $\notin$ The set of Singapore Prime Ministers (past and present).

A33(a) Yes, because $\{1, 2, 3\}$ and $\{3, 2, 1\}$ both contain the exact same elements, namely the numbers 1, 2, and 3. The order in which we write out the elements of a set doesn't matter.

(b) No. The set $\{\{1\}, 2, 3\}$ contains a set containing the number 1 and the two numbers 2 and 3.

In contrast, the set $\{\{3\}, 2, 1\}$ contains a set containing the number 3 and the two numbers 2 and 1.

Since the two sets contain different elements, they are not the same set.

A34. $\text{n}(x) = \text{n}(\{\text{LKY, GCT, LHL}\}) = 3$.

A35. L is the set containing the first 50 odd positive integers; hence, $\text{n}(L) = 50$. And M is the set containing the first 99 negative integers; hence, $\text{n}(M) = 99$.

A36. $N = \{102, 104, 106, 108, \ldots, 996, 998\}$.

A37. The set $W =$ {Apple, Apple, Apple, Banana, Banana, Apple} has only two distinct elements. Hence, $\text{n}(W) = 2$. We can rewrite the set more simply as $W =$ {Apple, Banana}.

A38. There is only one even prime number, namely 2. Hence, $C = \{2\}$ and $\text{n}(C) = 1$.

A39. The set of all primes is $H = \{2, 3, 5, 7, 11, 13, 17, 23, 29, \ldots\}$.

A40. None. All of them contain infinitely many elements and so all are infinite.

A41. The set $\{\{\{\}\}, \varnothing, \{\varnothing\}, \{\}\}$ contains only two elements—$\text{n}(S) = 2$.

Observe that $\{\varnothing\} = \{\{\}\}$ and $\{\} = \varnothing$. Hence, $\{\varnothing\}$ and $\{\}$ are repeated elements that we may ignore. Hence, the set $\{\{\{\}\}, \varnothing, \{\varnothing\}, \{\}\} = \{\{\{\}\}, \varnothing\}$ contains only two elements—namely, (i) the set that contains the empty set; and (ii) the empty set.

A42. The set $A = [1, 1]$ contains the real numbers that are ≥ 1 and ≤ 1. There is only one such number, namely the number 1. And so, $\mathrm{n}(D) = 1$. We can also write $X = \{1\}$.

The set $B = (1, 1)$ contains the real numbers that are > 1 and < 1. There are no such numbers. And so, $\mathrm{n}(B) = 0$. We can also write $B = \{\} = \varnothing$.

The set $C = (1, 1]$ contains the real numbers that are > 1 and ≤ 1. There are no such numbers. And so, $\mathrm{n}(C) = 0$. We can also write $C = \{\} = \varnothing$.

The set $D = [1, 1)$ contains the real numbers that are ≥ 1 and < 1. There are no such numbers. And so, $\mathrm{n}(D) = 0$. We can also write $D = \{\} = \varnothing$.

The set $E = (1, 1.01)$ contains the real numbers that are > 1 and < 1.01. There are infinitely many such numbers (e.g. 1.001, 1.0001, 1.0002). And so, $\mathrm{n}(Z) = \infty$.

For each term, we give three possible definitions:

A **degenerate interval** is an interval that has **(i)** equal endpoints; **(ii)** 0 or 1 elements; **(iii)** finitely many elements.

A **non-degenerate interval** is an interval that has **(i)** distinct endpoints; **(ii)** more than 1 element; **(iii)** infinitely many elements.

For this textbook's official definitions of degenerate and non-degenerate intervals, see Definition 318.

A43. $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}_0^+ = [0, \infty)$, $\mathbb{R}^- = (-\infty, 0)$, and $\mathbb{R}_0^- = (-\infty, 0]$.

A44(a) Every integer is also a rational number and a real number; hence, $\mathbb{Z} \subseteq \mathbb{Q}, \mathbb{R}$.

(b) A rational number is also a real number; hence, $\mathbb{Q} \subseteq \mathbb{R}$.

Some rational numbers are not integers (e.g. 1.5); hence, $\mathbb{Q} \nsubseteq \mathbb{Z}$.

(c) Some real numbers are neither rational nor integers (e.g. π); hence, $\mathbb{R} \nsubseteq \mathbb{Z}, \mathbb{Q}$.

A45. True. The set of current Singapore Prime Minister(s) is $S = \{\text{Lee Hsien Loong}\}$.The set of current Singapore Ministers is

$$T = \{\text{Lee Hsien Loong, Tharman, Teo Chee Hean, Khaw Boon Wan}, \dots\}.$$

Every element in S is in T; hence, $S \subseteq T$.

A46(a) False. Counterexample: Let $A = \{1\}$ and $B = \{1, 2\}$. Then $A \subseteq B$, but $A \neq B$.

(b) False. Counterexample: Let $A = \{1, 2\}$ and $B = \{1\}$. Then $B \subseteq A$, but $A \neq B$.

(c) True. If $A = B$, then by Def. 8, every element in A is also in B; so, by Def. 17, $A \subseteq B$.

(d) True. If $A = B$, then by Def. 8, every element in B is also in A; so, by Def. 17, $B \subseteq A$.

(e) False (in particular, $\Longleftarrow$ is false). Counterexample: same as in **(a)**.

(f) False (in particular, $\Longleftarrow$ is false). Counterexample: same as in **(b)**.

A47. Every square is a rectangle; so, $S \subseteq R$.

Moreover, some rectangles are not squares; so, $S \neq R$.

Hence, by Definition 18, $S \subset R$.

A48. No. Counterexample: if $A = \{1, 2\}$ and $B = \{1, 2\}$, then $A \subseteq B$, but $A \not\subset B$.

A49. Yes. By Definition 18, $A \subset B$ requires that $A \subseteq B$.

A50. True. Suppose $A \subseteq B$. Then either $A = B$ or if not, then by Definition 18, $A \subset B$.

A51(a) $[1, 2] \cup [2, 3] = [1, 3]$.**(b)** $(-\infty, -3) \cup [-16, 7) = (-\infty, 7)$. **(c)** $\{0\} \cup \mathbb{Z}^+ = \mathbb{Z}_0^+$.

A52(a) Every square is a rectangle. Hence, "the set of all squares and all rectangles" is itself simply "the set of all rectangles", i.e. $S \cup R = R$.

(b) Every rational and every irrational is a real number. Moreover, every real number is either rational or irrational. Hence, the set of all rationals and irrationals is $\mathbb{R}$ (the set of all real numbers).

A53(a) $(4, 7] \cap (6, 9) = (6, 7]$. **(b)** $[1, 2] \cap [5, 6] = \varnothing$. **(c)** $(-\infty, -3) \cap [-16, 7) = [-16, -3)$.

A54(a) The only objects that are both squares AND rectangles are squares. Hence, the intersection of the set of squares and the set of rectangles is simply the set of all squares, i.e. $S \cap R = S$.

(b) It is the empty set $\varnothing$. This is because no object is both rational AND irrational.

A55. $V \setminus T = \{3\}$ and $V \setminus U = \{1, 2\}$.

A56(a) $\mathscr{E} = \{0001, 0002, 0003, \ldots, 9999\}$, $|\mathscr{E}| = 10\,000$.

(b) $\mathscr{E} = \{\text{Male, Female}\}$, $|\mathscr{E}| = 2$.

(c) $\mathscr{E} = \{\text{Malay, Mandarin, Tamil, English}\}$, $|\mathscr{E}| = 4$ (source).

(d) $\mathscr{E} = \{\text{Malay}\}$, $|\mathscr{E}| = 1$.

A57(a) $S' = \{0000, 0001, 0002, \ldots, 2999, 4000, 4001, 4002, \ldots, 9999\}$.

(b) $S' = (-273.15, 30]$. The lowest possible temperature (absolute zero) is $-273.15\,°\text{C}$.

A58. $S = \{\text{Lee Hsien Loong, Lee Wei Ling, Lee Hsien Yang}\}$.

$T = \{\text{Lee Wei Ling, Lee Hsien Yang}\}$.

A59. $\{\text{Lee Kuan Yew, Lee Hsien Loong}\}$.

A60. In words,

- $\mathbb{Q}^+$ is the set of all rationals x such that x is greater than 0.
- $\mathbb{Q}_0^+$ is the set of all rationals x such that x is greater than or equal to 0.
- $\mathbb{Z}^+$ is the set of all integers x such that x is greater than 0.
- $\mathbb{Z}_0^+$ is the set of all integers x such that x is greater than or equal to 0.

A61(a) $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$ (b) $\mathbb{Q}^- = \{x \in \mathbb{Q} : x < 0\}$

(c) $\mathbb{Z}^- = \{x \in \mathbb{Z} : x < 0\}$ (d) $\mathbb{R}_0^- = \{x \in \mathbb{R} : x \le 0\}$

(e) $\mathbb{Q}_0^- = \{x \in \mathbb{Q} : x \le 0\}$ (f) $\mathbb{Z}_0^- = \{x \in \mathbb{Z} : x \le 0\}$

(g) $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ (h) $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$

(i) $(a, b] = \{x \in \mathbb{R} : a < x \le b\}$ (j) $[a, b) = \{x \in \mathbb{R} : a \le x < b\}$

(k) $(-\infty, -3) \cup (5, \infty) = \{x \in \mathbb{R} : x < -3 \text{ OR } x > 5\}$

(l) $(-\infty, 1] \cup (2, 3) \cup (3, \infty) = \{x : x \le 1 \text{ OR } 2 < x < 3 \text{ OR } x > 3\} = \{x \in \mathbb{R} : x \notin (1, 2], x \ne 3\}$

(m) $(-\infty, 3) \cap (0, 7) = \{x \in \mathbb{R} : 0 < x < 3\}$

(n) $\{\text{Negative even numbers}\} = \{x : x = 2k, k \in \mathbb{Z}^-\} = \{2k : k \in \mathbb{Z}^-\}$

(o) $\{\text{Positive odd numbers}\} = \{x : x = 2k - 1, k \in \mathbb{Z}^+\} = \{x : x = 2k + 1, k \in \mathbb{Z}_0^+\}$
$= \{2k - 1 : k \in \mathbb{Z}^+\} = \{2k + 1 : k \in \mathbb{Z}_0^+\}$

(p) $\{\text{Negative odd numbers}\} = \{x : x = 2k - 1, k \in \mathbb{Z}_0^-\} = \{x : x = 2k + 1, k \in \mathbb{Z}^-\}$
$= \{2k - 1 : k \in \mathbb{Z}_0^-\} = \{2k + 1 : k \in \mathbb{Z}^-\}$

(q) $\{\pi, 4\pi, 7\pi, 10\pi, \dots\} = \{(1 + 3k)\pi : k \in \mathbb{Z}_0^+\}$

(r) $\{-2\pi, \pi, 4\pi, 7\pi, 10\pi, \dots\} = \{(1 + 3k)\pi : k \in \mathbb{Z}, k \ge -1\}$

A62(a) $\mathbb{R} \setminus \mathbb{R}^+ = \mathbb{R}_0^-$.

(b) $\mathbb{R} \setminus (\mathbb{Q} \cup \mathbb{Z}) = \mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}'$.

(c) $[1, 6] \setminus ((3, 5) \cap (1, 4)) = [1, 6] \setminus (3, 4) = [1, 3) \cup (4, 6]$.

(d) $\{1, 5, 9, 13, \dots\} \cap \{2, 4, 6, 8, \dots\} = \varnothing$.

(e) $\{2, 5, 8, 11, \dots\} \cap \{2, 4, 6, 8, \dots\} = \{2, 8, 14, 20 \dots\} = \{2 + 6k : k \in \mathbb{Z}_0^+\}$.

(f) $(0, 5] \cap ([1, 8] \cap [5, 9))' = (0, 5] \cap [5, 8]' = (0, 5)$.

A65. False. Counterexample: If $x = -1$, then $\sqrt{x^2} = \sqrt{(-1)^2} = \sqrt{1} = 1 \neq x$. More generally, any $x < 0$ would work as a counterexample.

A66. False. Counterexample: If $x = -1$, then

$$\frac{\sqrt{x^2}}{x} = \frac{\sqrt{(-1)^2}}{-1} = \frac{\sqrt{1}}{-1} = \frac{1}{-1} = -1 \neq 1.$$

(Again, any $x < 0$ would serve as a counterexample.)

The error is at $\stackrel{1}{=}$: it is not generally true that $x = \sqrt{x^2}$.

A67. False. Counterexample: If $x = -1$, then

$$\frac{x}{\sqrt{x^2}} = \frac{-1}{\sqrt{(-1)^2}} = \frac{-1}{\sqrt{1}} = \frac{-1}{1} = -1 \neq 1.$$

(Again, any $x < 0$ would serve as a counterexample.)

The error is at $\stackrel{1}{=}$: it is not generally true that $x = \sqrt{x^2}$.

A63(a)

$$\begin{aligned}
\frac{5^{4x} \cdot 25^{1-x}}{5^{2x+1} + 3 \cdot 25^x + 17 \cdot 5^{2x}} &= \frac{5^{4x} \cdot 5^{2(1-x)}}{5^{2x+1} + 3 \cdot 5^{2x} + 17 \cdot 5^{2x}} \\
&= \frac{5^{2+2x}}{5^{2x+1} + 3 \cdot 5^{2x} + 17 \cdot 5^{2x}} \\
&= \frac{5^{2+2x}}{5^{2x}\left(5^1 + 3 + 17\right)} = \frac{5^{2+x}}{5^{2x} \cdot 25} = \frac{5^{2+2x}}{5^{2x+2}} = 1.
\end{aligned}$$

(b)

$$\begin{aligned}
\sqrt{2}\frac{8^{x+2} - 34 \cdot 2^{3x}}{\sqrt{8}^{2x+1}} &= \sqrt{2}\frac{8^{x+2} - 34 \cdot 2^{3x}}{\sqrt{8}^{2x}\sqrt{8}^{1}} = \sqrt{2}\frac{8^{x+2} - 34 \cdot 8^x}{8^x\sqrt{8}} \\
&= \sqrt{2}\frac{8^x\left(8^2 - 34\right)}{8^x\sqrt{8}} = \sqrt{2}\frac{8^2 - 34}{\sqrt{8}} = \sqrt{2}\frac{64 - 34}{\sqrt{8}} \\
&= \sqrt{2}\frac{30}{\sqrt{8}} = \sqrt{2}\frac{30}{2\sqrt{2}} = \sqrt{2}\frac{15}{\sqrt{2}} = 15.
\end{aligned}$$

A64(a) False. Counterexample: Let $b = 2$, $x = 1$, $y = 2$. Then $b^{\left(x^y\right)} = 2^{\left(1^2\right)} = 2^1 = 2$, but $b^{xy} = 2^{1 \cdot 2} = 2^2 = 4$. Hence, $b^{\left(x^y\right)} \neq b^{xy}$.

(b) True—see Proposition 1(d).

$$\frac{1}{\frac{x}{y}\pm\sqrt{\frac{x^2}{y^2}+1}} \overset{\text{TOT}}{=} \frac{1}{\frac{x}{y}\pm\sqrt{\frac{x^2}{y^2}+1}}\frac{\frac{x}{y}\mp\sqrt{\frac{x^2}{y^2}+1}}{\frac{x}{y}\mp\sqrt{\frac{x^2}{y^2}+1}}$$

$$= \frac{\frac{x}{y}\mp\sqrt{\frac{x^2}{y^2}+1}}{\left(\frac{x}{y}\right)^2-\left(\sqrt{\frac{x^2}{y^2}+1}\right)^2} = \frac{\frac{x}{y}\mp\sqrt{\frac{x^2}{y^2}+1}}{\frac{x^2}{y^2}-\left(\frac{x^2}{y^2}+1\right)}$$

$$= \frac{\frac{x}{y}\mp\sqrt{\frac{x^2}{y^2}+1}}{-1} = -\frac{x}{y}\pm\sqrt{\frac{x^2}{y^2}+1}.$$

At the last step, the -1 in the denominator flips the $\mp$ into a $\pm$.

A69.

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} \overset{\text{TOT}}{=} \frac{-b \pm \sqrt{b^2-4ac}}{2a}\frac{-b \mp \sqrt{b^2-4ac}}{-b \mp \sqrt{b^2-4ac}}$$

$$= \frac{b^2-(b^2-4ac)}{2a\left(-b \mp \sqrt{b^2-4ac}\right)}$$

$$= \frac{4ac}{2a\left(-b \mp \sqrt{b^2-4ac}\right)}$$

$$= \frac{2c}{-b \mp \sqrt{b^2-4ac}}.$$

A70(a) $\log_2 32 + \log_3 \frac{1}{27} = 5 - \log_3 27 = 5 - 3 = 2.$

(b) First, $\log_3 45 = \log_3 5 + \log_3 9 = \log_3 5 + 2.$

Next, $\log_9 25 = \dfrac{\log_3 25}{\log_3 9} = \dfrac{2\log_3 5}{2} = \log_3 5.$

So, $\log_3 45 - \log_9 25 = 2.$

(c) First, $\log_{16} 768 = \log_{16}(256 \times 3) = \log_{16} 256 + \log_{16} 3 = 2 + \dfrac{\log_2 3}{\log_2 16} = 2 + \dfrac{\log_2 3}{4}.$

Next, $\log_2 \sqrt[4]{3} = \log_2 3^{1/4} = \dfrac{1}{4}\log_2 3.$

So, $\log_{16} 768 - \log_2 \sqrt[4]{3} = 2.$

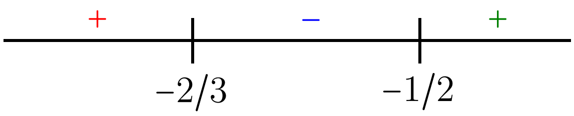
A71. If $x > 0$, then $7x > 0$ and $7x < 0$. If $x = 0$, then $7x = 0$ and $7x = 0$. If $x < 0$, then $7x < 0$ and $7x > 0$.

A73(a) $(x-1)/-4 > 0 \iff x - 1 < 0 \iff x < 1.$

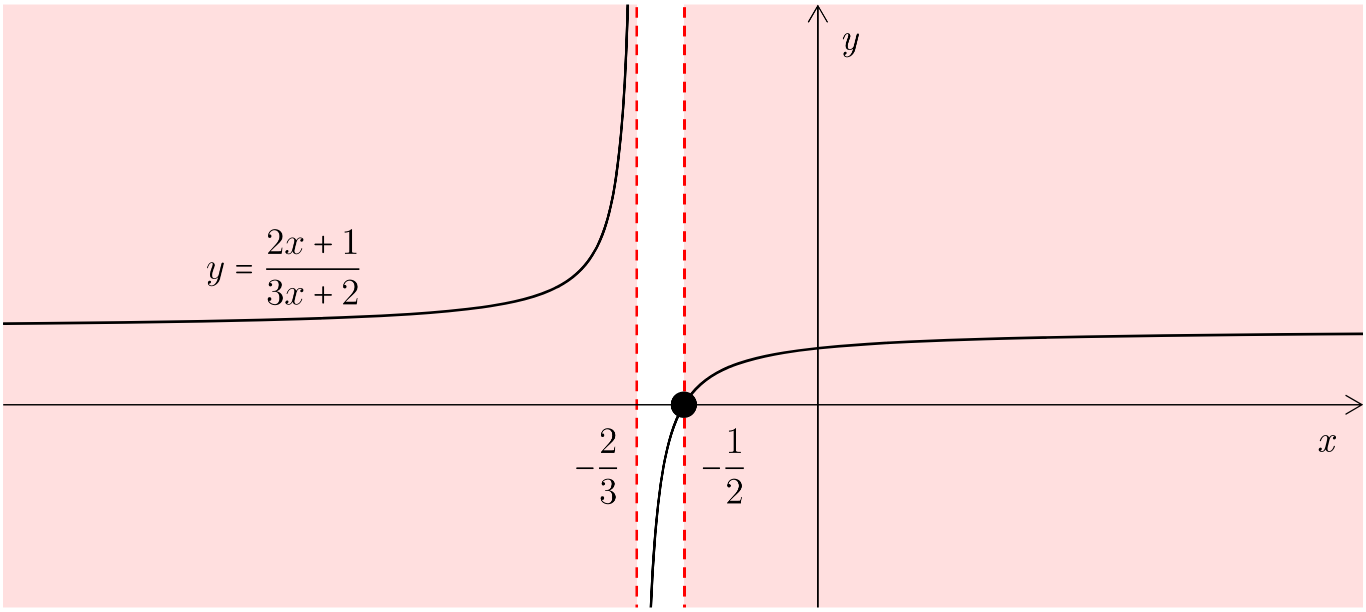
(b) $-1/-4 > 0$ is always true.

(c) $1/-4 > 0$ is always false.

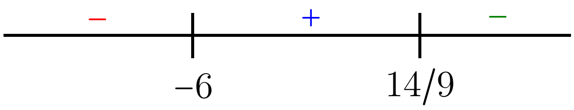
(d) The numerator and denominator equal zero at $-1/2$ and $-2/3$. Draw the **sign diagram**:



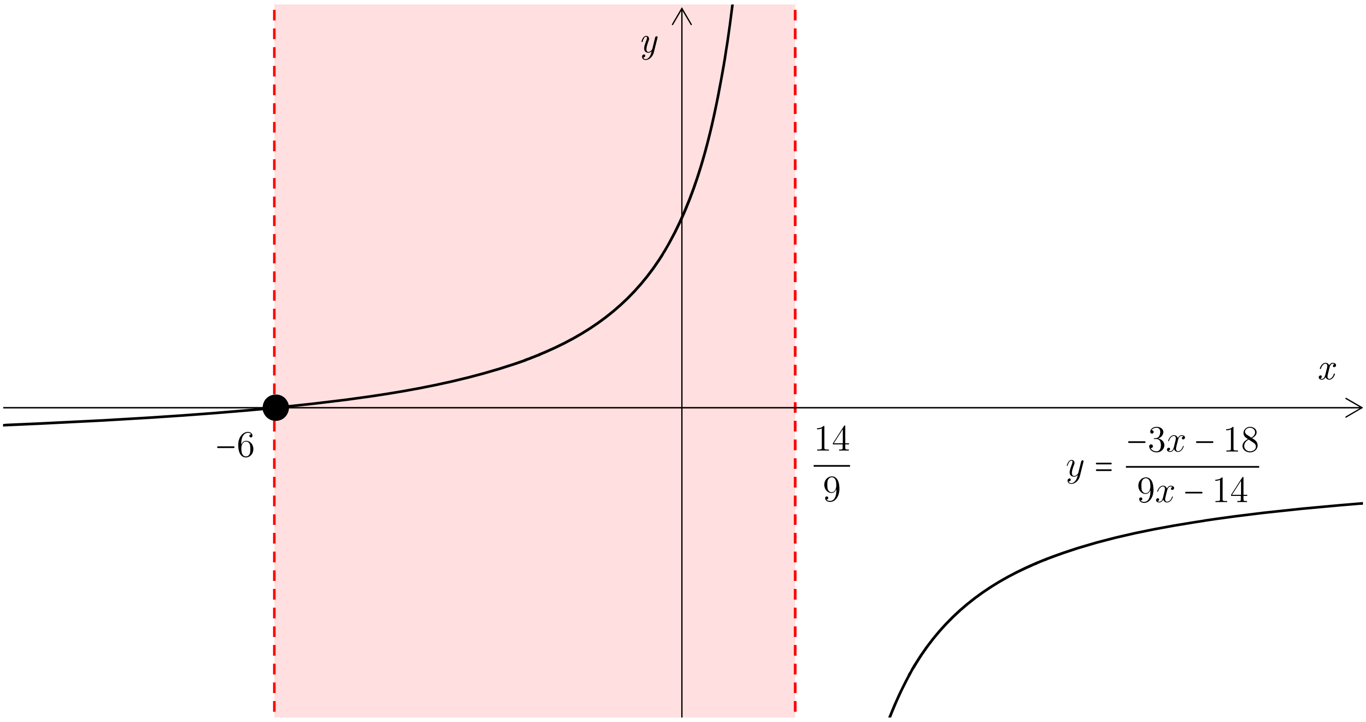
Hence, $x < -2/3$ or $x > -1/2$.



(e) The numerator and denominator equal zero at -6 and $14/9$. Draw the **sign diagram**:



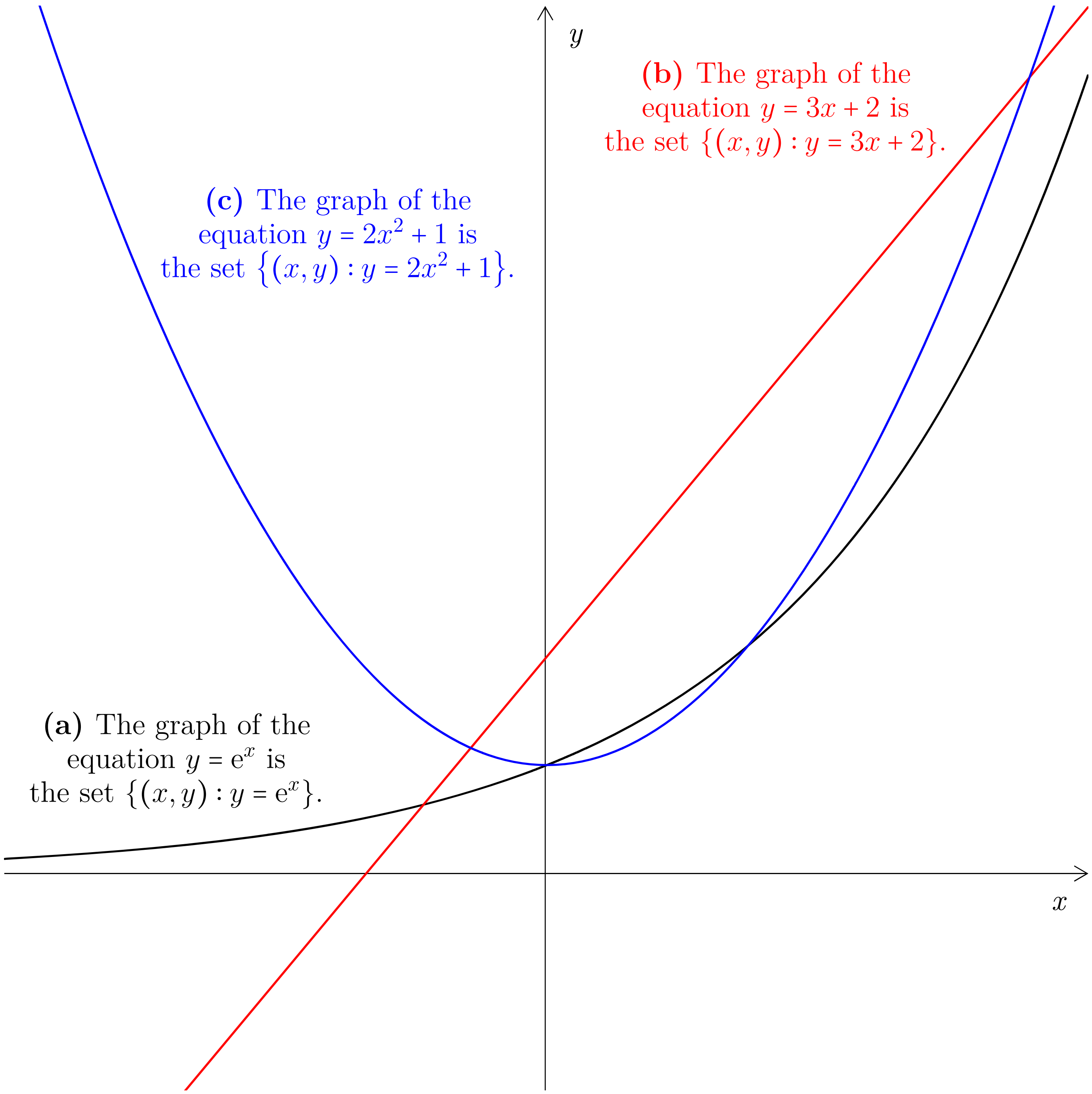
Hence, $-6 < x < 14/9$.



149. Part I Answers (Functions and Graphs)

149.1. Ch. 7 Answers (Graphs)

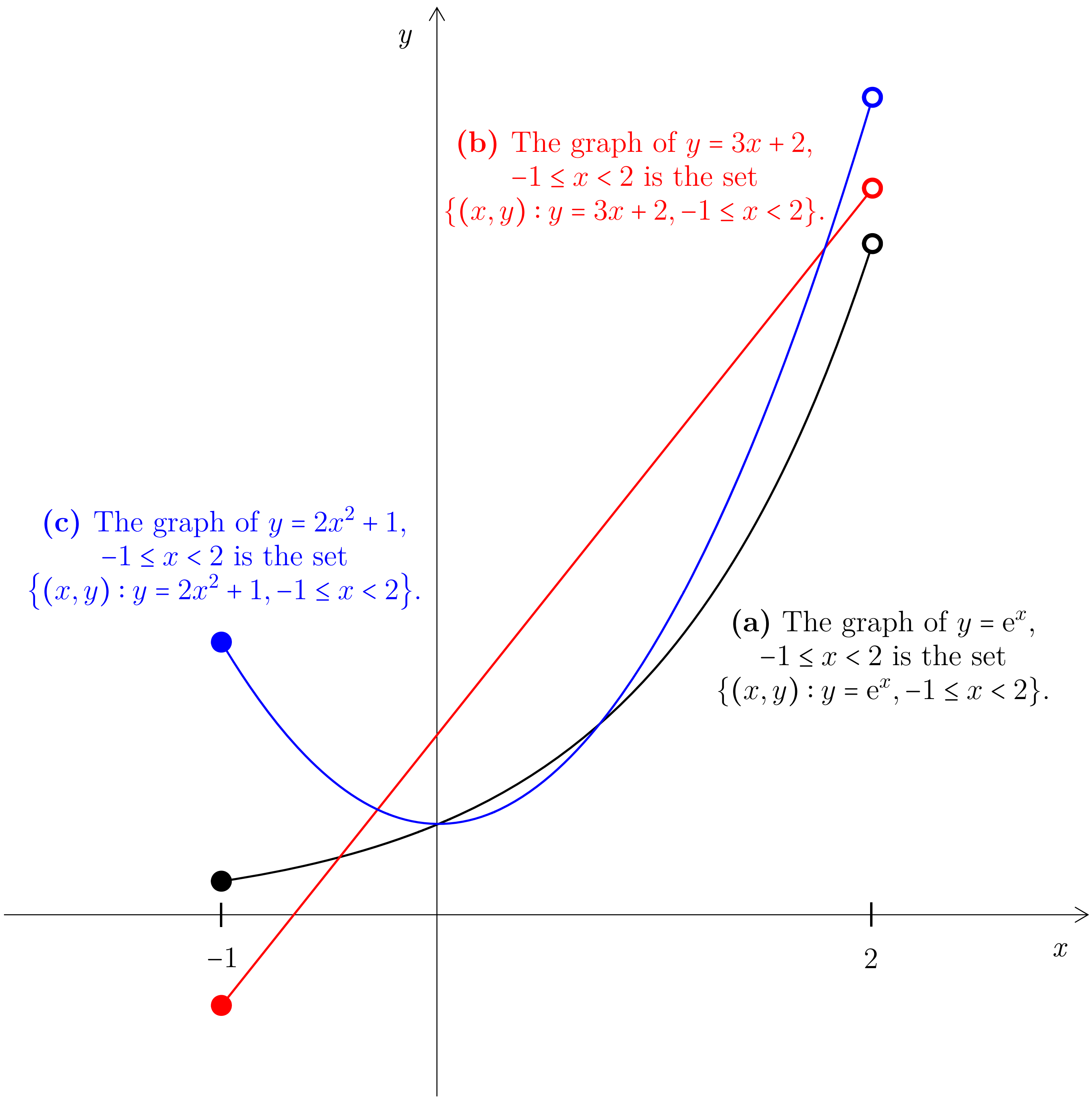
A75(a) The graph of the equation $y = e^x$ is the set $\{(x, y) : y = e^x\}$.
(b) The graph of the equation $y = 3x + 2$ is the set $\{(x, y) : y = 3x + 2\}$.
(c) The graph of the equation $y = 2x^2 + 1$ is the set $\{(x, y) : y = 2x^2 + 1\}$.



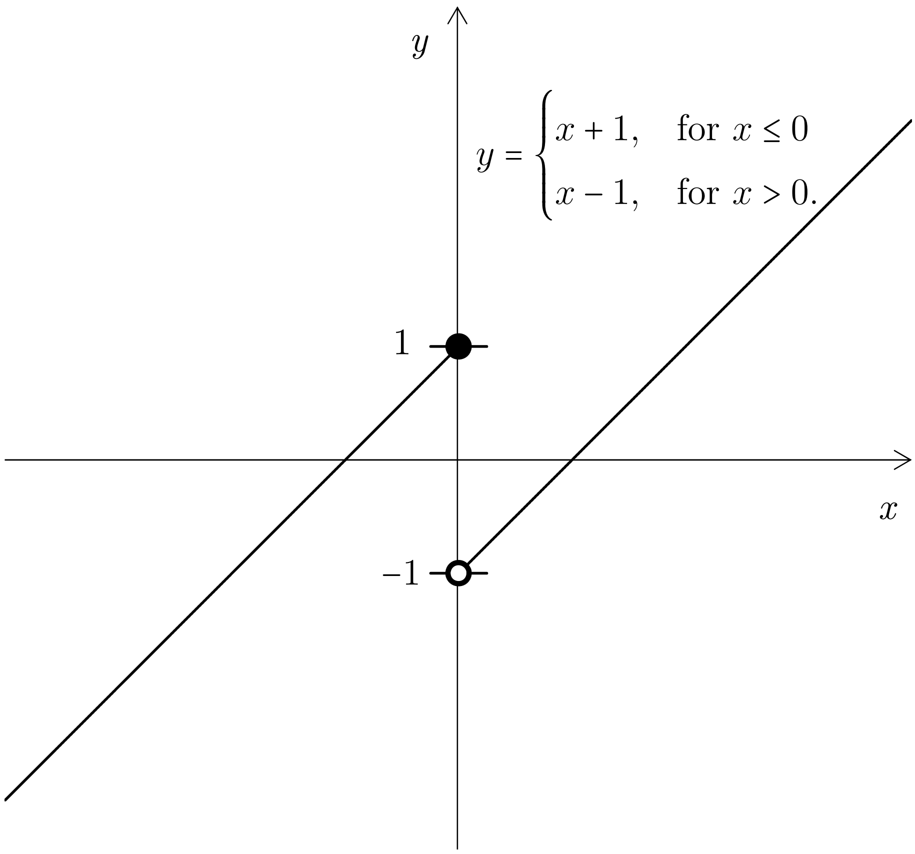
A76(a) The graph of $y = e^x$, $-1 \le x < 2$ is the set $\{(x, y) : y = e^x,\ -1 \le x < 2\}$.

(b) The graph of $y = 3x + 2$, $-1 \le x < 2$ is the set $\{(x, y) : y = 3x + 2,\ -1 \le x < 2\}$.

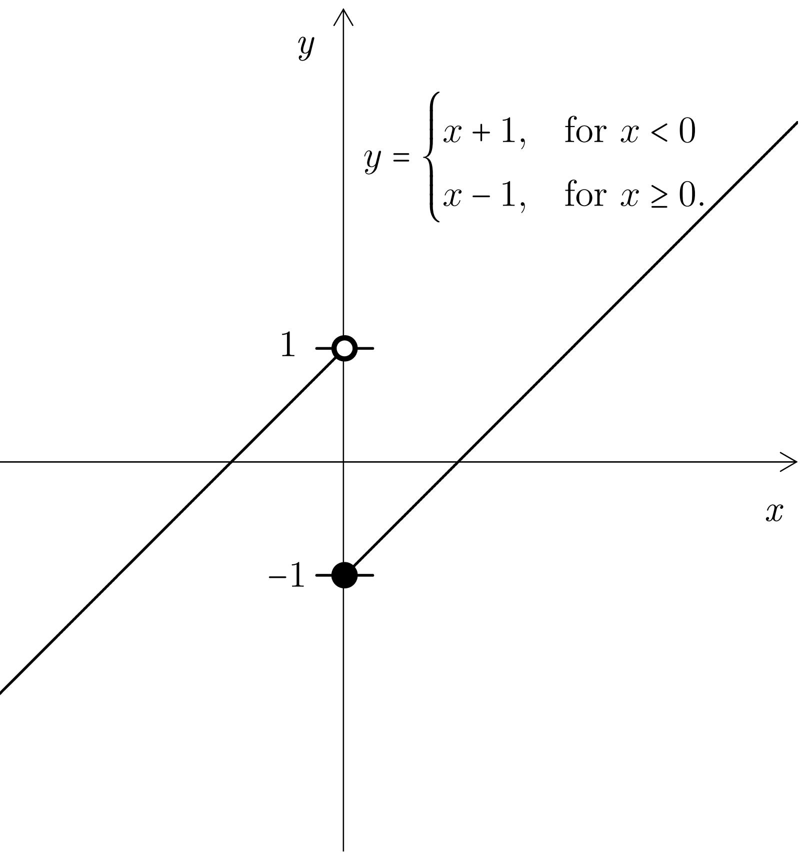
(c) The graph of $y = 2x^2 + 1$, $-1 \le x < 2$ is the set $\{(x, y) : y = 2x^2 + 1,\ -1 \le x < 2\}$.



A77(a)



(b)



A78(a) $y = 2$ has no x-intercepts, one y-intercept $(0, 2)$, and no solutions (or roots).

(b) $y = x^2 - 4$ has two x-intercepts $(-2, 0)$ and $(2, 0)$, one y-intercept $(0, -4)$, and two solutions (or roots) -2 and 2.

(c) $y = x^2 + 2x + 1$ has one x-intercept $(-1, 0)$, one y-intercept $(0, 1)$, and one solution (or root) -1.

(d) $y = x^2 + 2x + 2$ has no x-intercepts, one y-intercept $(0, 2)$, and no solutions (or roots).

A79(a) $(7-4)(y-5)=(9-5)(x-4)$ or $3y-15=4x-16$ or $y=\frac{4}{3}x-\frac{1}{3}$.

(b) $(-1-1)(y-2)=(-3-2)(x-1)$ or $-2y+4=-5x+5$ or $y=\frac{5}{2}x-\frac{1}{2}$.

A81(a) $(y-5)=3(x-4)$ or $y=3x-7$.

(b) $(y-2)=-2(x-1)$ or $y=-2x+4$.

149.3. Ch. 9 Answers (Distance)

A83. $|AB| = 12$, $|AC| = 4$, and $|BC| = 8$.

A84. $|AB| = \sqrt{12^2 + 1^2} = \sqrt{145}$, $|AC| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$, and $|BC| = \sqrt{8^2 + 2^2} = \sqrt{68}$.

A85. In Exercise 84, we already found

$$|AB| = \sqrt{145}, \qquad |AC| = 5, \qquad |BC| = \sqrt{68}.$$

Hence, $|AB| = |BA| = \sqrt{145} > |BC| = \sqrt{68}$ and $|AC| = 5 < |AB| = \sqrt{145}$. Thus,

(a) A is **further from** B than C.

(b) C is **closer to** A than B.

A86. $(3, 6)$.

149.4. Ch. 10 Answers (Circles)

A87(a)(i) The circle of radius 4 centred on $(5, 1)$.

(a)(ii) The circle of radius $\sqrt{2}$ centred on $(-3, -2)$.

(a)(iii) The circle of radius 9 centred on $(-1, 0)$.

(b)(i) $(x+1)^2 + (y-2)^2 = 25$.

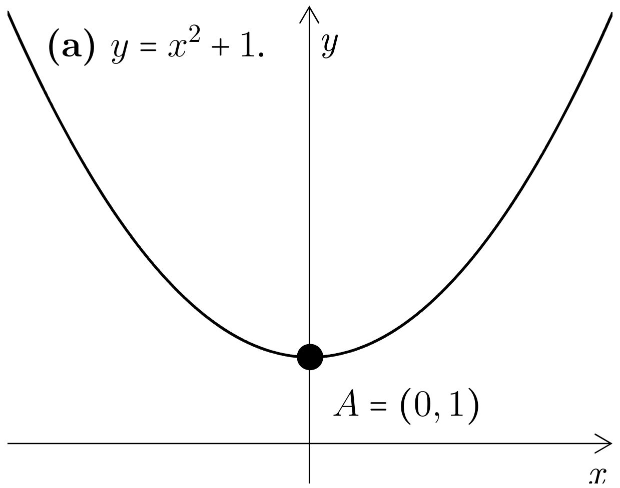
(b)(ii) $(x-3)^2 + (y+2)^2 = 1$.

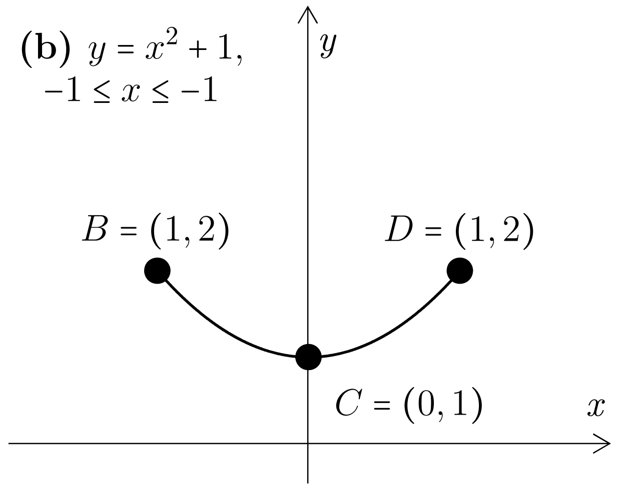
(b)(iii) $x^2 + y^2 = 3$.

	(c) Contains $(8, 2)$?	**(d)** Centre	Radius	Diameter
(a)(i)	$(8-5)^2 + (2-1)^2 \neq 16$, so no.	$(5, 1)$	4	8
(a)(ii)	$(8+3)^2 + (2+2)^2 \neq 2$, so no.	$(-3, -2)$	$\sqrt{2}$	$2\sqrt{2}$
(a)(iii)	$(8+1)^2 + 2^2 \neq 81$, so no.	$(-1, 0)$	9	18
(b)(i)	$(8+1)^2 + (2-2)^2 \neq 25$, so no.	$(-1, 2)$	5	10
(b)(ii)	$(8-3)^2 + (2+2)^2 \neq 1$, so no.	$(3, -2)$	1	2
(b)(iii)	$8^2 + 2^2 \neq 3$, so no.	$(0, 0)$	$\sqrt{3}$	$2\sqrt{3}$

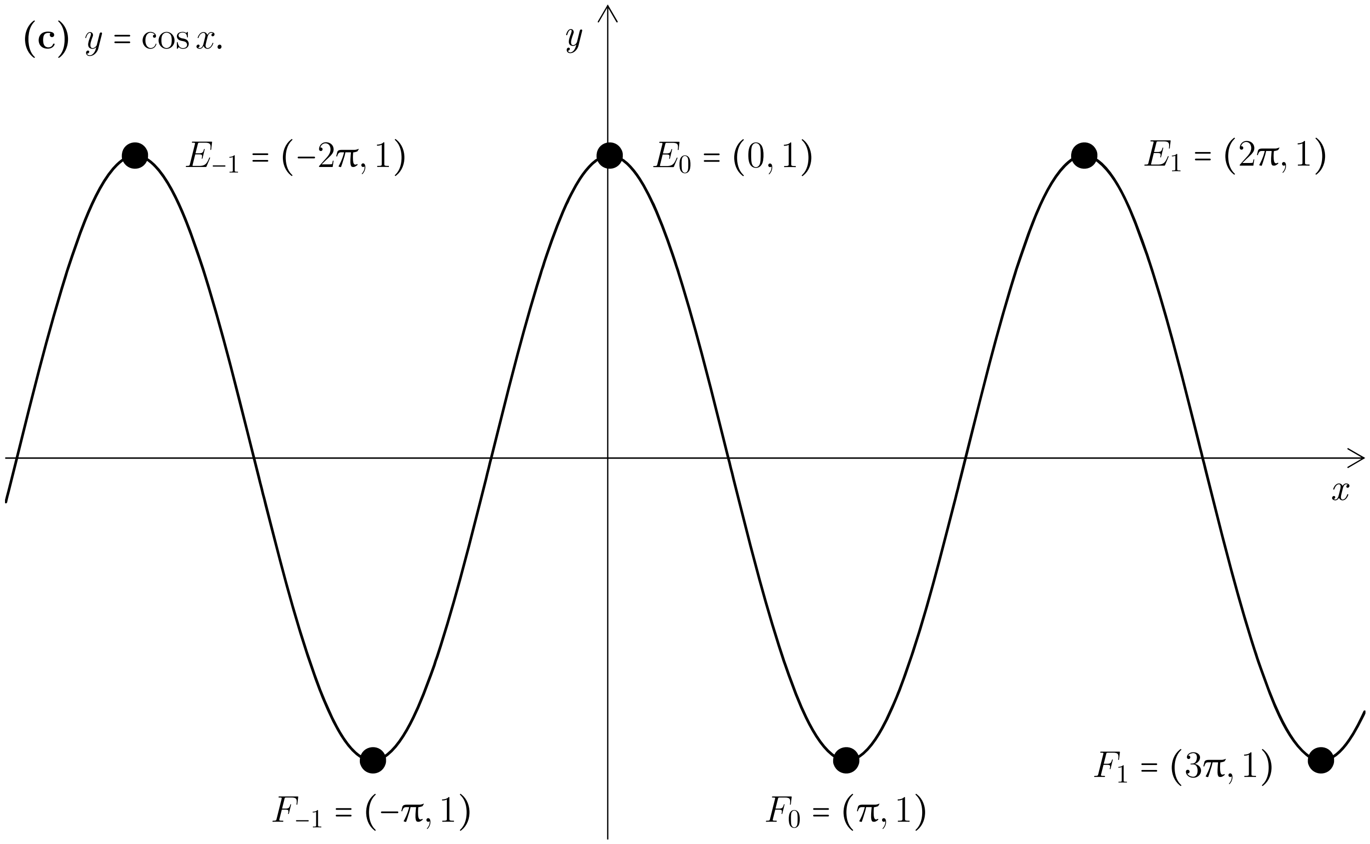
149.5. Ch. 12 Answers (Maximum and Minimum Points)

A90. Refer to graphs and table below.



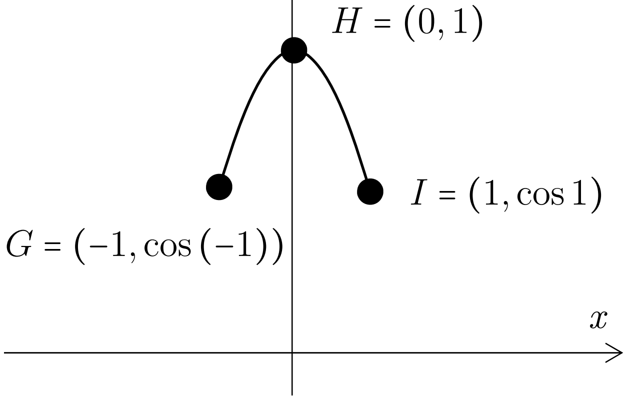


For **(c)**, for each $k \in \mathbb{Z}$, let $E_k = (2k\pi, 1)$ and $F_k = ((2k+1)\pi, 1)$—note that there are infinitely many points E_k and F_k.



	A	B	C	D	E_k	F_k	G	H	I
GMax		✓		✓	✓			✓	
SGMax								✓	
LMax		✓		✓	✓			✓	
SLMax		✓		✓	✓			✓	
GMin	✓		✓			✓	✓		✓
SGMin	✓		✓						
LMin	✓		✓			✓	✓		✓
SLMin	✓		✓			✓	✓		✓
Turning	✓		✓		✓	✓		✓	

(d) $y = \cos x$, $-1 \le x \le 1$



A89(a) False. In Example 235, $G = (1, 3)$ is a global maximum but not a strict local maximum.

(b) True. A point that's at least as high as every other point must also be at least as high as any "nearby" point.

(c) True. A point that's higher than *every* other point must also be **(i)** at least as high as any other point; **(ii)** higher than any "nearby" point; and **(iii)** at least as high as any "nearby" point. Hence, a strict global maximum must also be a **(i)** global maximum; **(ii)** strict local maximum; and **(iii)** local maximum.

(d) False. In Example 235, $G = (1, 3)$ is a global maximum and also a local minimum.

149.6. Ch. 16 Answers (Reflection and Symmetry)

A92. $(-12, 3)$.

A93(a) $|PQ| = \sqrt{(a-c)^2 + (b-d)^2} = \sqrt{[c-(2c-a)]^2 + [d-(2d-b)]^2} = |QR|$. ✓

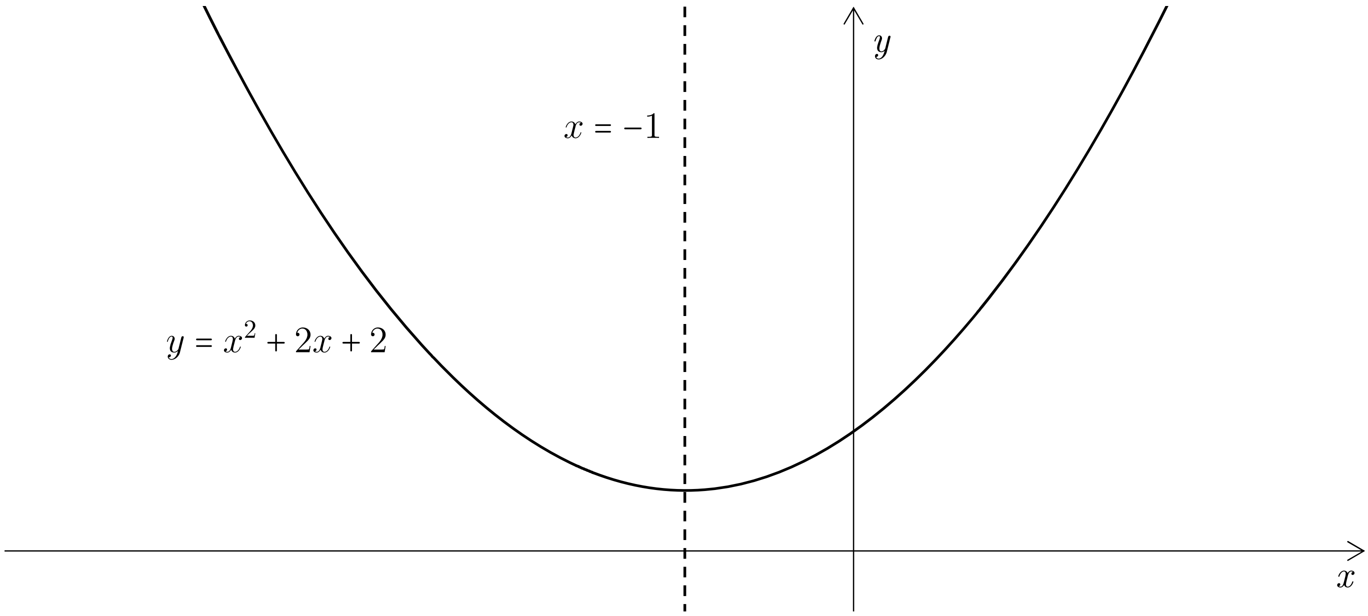
(b) By Fact 26, the line PQ is $(c-a)(y-b) = (d-b)(x-a)$.

To verify that $R = (2c-a, 2d-b)$ is on line PQ is to verify that it satisfies the above equation, as we do now:

$$(c-a)(2d-b-b) = 2(c-a)(d-b) = (d-b)(2c-a-a) = 2(d-b)(c-a). \quad ✓$$

A94. $(2, 3)$ and $(-2, -3)$.

A95. By informal observation,[640] $y = x^2 + 2x + 2$ is symmetric in the line $x = -1$.



149.7. Ch. 13 Answers (Solutions and Solution Sets)

(*This chapter had no exercises.*)

[640]Formal proof: Pick any point $(a, a^2 + 2a + 2)$ in the graph of $y = x^2 + 2x + 2$. It is possible to show that the closest point on the line $x = -1$ is $(-1, a^2 + 2a + 2)$. The reflection of the point $(a, a^2 + 2a + 2)$ in $(-1, a^2 + 2a + 2)$ is the point $(-2 - a, a^2 + 2a + 2)$, which we can verify is also a point in the graph of $y = x^2 + 2x + 2$:

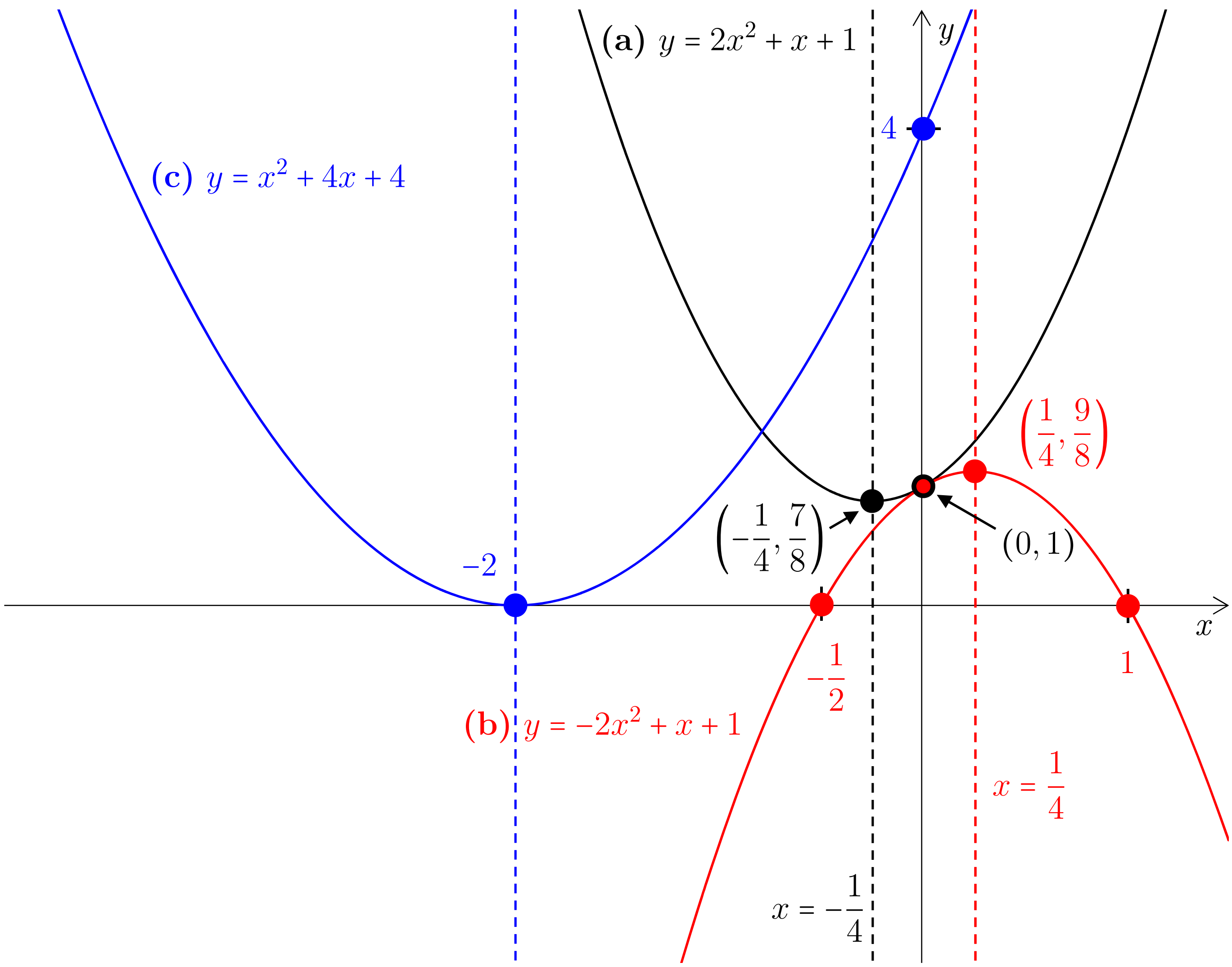
$$(-2-a)^2 + 2(-2-a) + 2 = 4 + a^2 + 4a - 4 - 2a + 2 = a^2 + 2a + 2.$$

We've just shown that the reflection of any point in the graph of $y = x^2 + 2x + 2$ in the line $x = -1$ is itself also in the same graph. Hence, the graph of $y = x^2 + 2x + 2$ is its own reflection in the line $x = -1$.

A91. In each case, the y-intercept is given by $(0, c)$, the discriminant by $b^2 - 4ac$, the x-intercepts by $\left(\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a, 0\right)$, the line of symmetry by $x = -b/2a$, and the turning point by $\left(-b/2a, c - b^2/4a\right)$.

	(a) $y = 2x^2 + x + 1$	**(b)** $y = -2x^2 + x + 1$	**(c)** $y = x^2 + 4x + 4$
y-intercept	$(0, 1)$	$(0, 1)$	$(0, 4)$
Discriminant	$-7 < 0$	$9 > 0$	0
x-intercepts	None	$\left(-\frac{1}{2}, 0\right)$, $(1, 0)$	$(-2, 0)$
Line of symmetry	$x = -\frac{1}{4}$	$x = \frac{1}{4}$	$x = -2$
Turning point	$\left(-\frac{1}{4}, \frac{7}{8}\right)$	$\left(\frac{1}{4}, \frac{9}{8}\right)$	$(-2, 0)$

The turning point in each of **(a)** and **(c)** is also the strict global minimum; and in **(b)**, it is also the strict global maximum.



A96. A function consists of <u>three</u> objects: namely, the <u>domain</u>, the <u>codomain</u>, and the <u>mapping rule</u>.

A97. In general, the domain <u>can be any set</u>; and the codomain <u>can be any set</u>.

A98. A function maps <u>every element</u> in its domain to <u>exactly one element</u> in its codomain.

A99(a) Yes, a is well-defined because it maps **every** element in the domain to **(exactly) one** element in the codomain—we have $a(\text{Cow}) = \text{Produces milk}$, $a(\text{Chicken}) = \text{Produces eggs}$, and $a(\text{Dog}) = \text{Guards the home}$.

(b) No, b isn't well-defined, because it isn't clear what $b(\text{Dog})$ is.

(c) No, c isn't well-defined, because it isn't clear what each state's "most splendid" city is.

(d) No, d isn't well-defined. China has more than one city with over 10M people, while Iceland has none. So, China would be mapped to more than one element in the codomain, while Iceland would be mapped to none—in either case, we'd violate the requirement that a function map **every** element in the domain to **(exactly) one** element in the codomain.

A100. $f(1) = 1 + 1 = 2$; $g(1) = 17(1) = 17$; $h(1) = 3^1 = 3$; $i(1)$ is undefined because $1 \notin \mathbb{Z}^- = \{-1, -2, -3, \ldots\}$; $j(1) = 17$.

A101. Below, each function is written out explicitly.

(a) Each function maps each element in its domain to exactly one element in the codomain and is thus well-defined.

(b) From below, it is clear that only $b = c$.

Function	Domain	Codomain	Mapping rule
a	$\{1,2\}$	$\{1,2,3,4\}$.	$a(1) = 2$, $a(2) = 4$
b	$\{1,2,3\}$	$\{1,2,3,4,5,6\}$	$b(1) = 2$, $b(2) = 4$, $b(3) = 6$
c	$\{1,2,3\}$	$\{1,2,3,4,5,6\}$	$c(1) = 2$, $c(2) = 4$, $c(3) = 6$
d	$\{0,1,2,3\}$	$\{1,2,3,4,5,6\}$.	$d(0) = 0$, $d(1) = 2$, $d(2) = 4$, $d(3) = 6$

A102(a) $f(3) = 3$, $f(\pi) = 3$, $f(3.5) = 4$, $f(3.88) = 4$, and $f(0)$ is undefined (because $0 \notin \mathbb{R}^+$).

(b) Yes. **(c)** Yes.

A103(a) Define $f : \mathbb{Z}^+ \to \mathbb{Z}$ by $f(x) = 3x$.

(b) Define $g : \mathbb{R}^- \to \mathbb{R} \setminus \{0\}$ by $g(x) = x^2$.

(c) Define $h : \{2k : k \in \mathbb{Z}\} \to \{2k : k \in \mathbb{Z}^+\}$ by $h(x) = x^2$.

A104(a)(i) Domain i contains objects that aren't real numbers. So, i is **not** a function of a real variable.

(a)(ii) Codomain i contains only real numbers. So, i **is** a real-valued function.

(a)(iii) Either Domain i or Codomain i contains objects that aren't real numbers. So, i is **not** a nice function.

(b)(i) Domain j contains only real numbers. So, j **is** a function of a real variable.

(b)(ii) Codomain i contains objects that aren't real numbers. So, j is **not** a real-valued function.

(b)(iii) Either Domain j or Codomain j contains objects that aren't real numbers. So, j is **not** a nice function.

(c)(i) Domain k contains only real numbers. So, k **is** a function of a real variable.

(c)(ii) Codomain k contains only real numbers. So, k **is** a real-valued function.

(c)(iii) Both Domain i and Codomain i contain only real numbers. So, j **is** a nice function.

(d)(i) Domain l contains only real numbers. So, l **is** a function of a real variable.

(d)(ii) Codomain l contains objects that aren't real numbers. So, l is **not** a real-valued function.

(d)(iii) Either Domain i or Codomain i contains objects that aren't real numbers. So, l is **not** a nice function.

	i	j	k	l
Function of a real variable		✓	✓	✓
Real-valued function	✓		✓	
Nice function			✓	

A105. Typically, we denote a function by a single letter, such as f or g.

And so, typically, "$f(x)$" denotes not a function, but instead the value of f at x. If f is a real-valued function, then $f(x)$ is simply a real number.

A106. This is a trick question. The answer is that yes, of course it is possible—as we stressed earlier, the mapping rule need not "make any sense".

Indeed, there exist exactly $2 \times 2 = 4$ possible functions with domain $A = \{\text{Lion, Eagle}\}$ and codomain $B = \{\text{Fat, Tall}\}$:

$$f(\text{Lion}) = \text{Fat},\ f(\text{Eagle}) = \text{Fat} \qquad g(\text{Lion}) = \text{Fat},\ g(\text{Eagle}) = \text{Tall}.$$
$$h(\text{Lion}) = \text{Tall},\ h(\text{Eagle}) = \text{Fat}. \qquad i(\text{Lion}) = \text{Tall},\ i(\text{Eagle}) = \text{Tall}.$$

In general, given any two finite sets S and T, we can construct $\text{n}(T)^{\text{n}(S)}$ possible functions using S as the domain and T as the codomain. (Can you see why?)[641]

A107(a) A function of a real variable. **(b)** A real-valued function. **(c)** A nice function.

A108(a)(i) Yes, a is well-defined. **Every** element in the domain is mapped to **(exactly) one** element in the codomain—5 to $10 \in \mathbb{Z}$, 6 to $12 \in \mathbb{Z}$, and 7 to $14 \in \mathbb{Z}$. **(ii)** Define $a : \{5, 6, 7\} \to \mathbb{Z}$ by $a(x) = 2x$.

(b)(i) Yes, b is well-defined. **Every** element in the domain is mapped to **(exactly) one** element in the codomain—5 to $10 \in \mathbb{Z}^+$, 6 to $12 \in \mathbb{Z}^+$, and 7 to $14 \in \mathbb{Z}^+$. **(ii)** Define $b : \{5, 6, 7\} \to \mathbb{Z}^+$ by $b(x) = 2x$.

(c)(i) No, c is not well-defined, because for example, the element 5 in the domain is allegedly mapped to 10, but 10 isn't an element in the codomain $\mathbb{Z}^-$.

[641] For each element x in the domain S, we have $\text{n}(T)$ possible elements in the codomain to which x can be mapped. And so, with $\text{n}(S)$ elements, the total number of possible functions is $\underbrace{\text{n}(T) \times \text{n}(T) \times \cdots \times \text{n}(T)}_{\text{n}(S) \text{ times}}$ or $\text{n}(T)^{\text{n}(S)}$.

(d)(i) No, d is not well-defined, because the element 5.4 in the domain is allegedly mapped to 10.8, but 10.8 isn't an element in the codomain $\mathbb{Z}$.

(e)(i) Yes, e is well-defined. **Every** element in the domain is mapped to **(exactly) one** element in the codomain—5.5 to $11 \in \mathbb{Z}$, 6 to $12 \in \mathbb{Z}$, and 7 to $14 \in \mathbb{Z}$. **(ii)** Define $e : \{5.5, 6, 7\} \to \mathbb{Z}$ by $e(x) = 2x$.

(f)(i) Yes, f is well-defined. From the mapping rule and the codomain, it is unambiguous that 3 is to be mapped to $4 \in \{3, 4\}$. **(ii)** Define $f : \{3\} \to \{3, 4\}$ by $f(3) = 4$.

(g)(i) Yes, g is well-defined. From the mapping rule and the codomain, it is unambiguous that 3 and 3.1 are to be mapped to $4 \in \{3, 4\}$. **(ii)** Define $g : \{3, 3.1\} \to \{3, 4\}$ by $g(3) = 4$. and $g(3.1) = 4$.

(h)(i) No, h is not well-defined. It is unclear whether 0 should be mapped to 3 or 4.

(i)(i) No, i is not well-defined. It is unclear what we should map 4 to, since there is no number larger than 4 in the codomain.

(j)(i) No, j is not well-defined. It is unclear what we should map 2 to, since there is no number smaller than 2 in the codomain.

(k)(i) Yes, k is well-defined. From the mapping rule and the codomain, it is unambiguous that 1 is to be mapped to $1 \in \{1\}$. **(ii)** Define $k : \{1\} \to \{1\}$ by $k(x) = x$.

(l)(i) Yes, l is well-defined. From the mapping rule and the codomain, it is unambiguous that 1 is to be mapped to $1 \in \{1, 2\}$. **(ii)** Define $l : \{1\} \to \{1, 2\}$ by $l(x) = x$.

(m)(i) No, m is not well-defined. It is unclear what 2 should be mapped to, since there is no number that is "the same" as 2 in the codomain.

(n)(i) No, n is not well-defined, because for example, the element -1 in the domain is allegedly mapped to $\sqrt{-1}$, but $\sqrt{-1}$ isn't an element in the codomain $\mathbb{R}$.[642]

(o)(i) No, o is not well-defined, because the element 0 in the domain is allegedly mapped to $1 \div 0$, but $1 \div 0$ isn't an element in the codomain $\mathbb{R}$.[643]

(p)(i) No, p is not well-defined, because for example, the element 3 in the domain is allegedly mapped to 4, but 4 isn't an element of the codomain $[0, 1]$.

(q)(i) Yes, q is well-defined. **Every** element x in the domain $[0, 1]$ is mapped to **(exactly) one** element in the codomain $\mathbb{R}$, namely $x + 1 \in \mathbb{R}$. **(ii)** Define $q : [0, 1] \to \mathbb{R}$ by $q(x) = x + 1$.

A109. Change the domain of n to $\mathbb{R}_0^+$, the set of non-negative reals. We then have the function $n : \mathbb{R}_0^+ \to \mathbb{R}$ that is (well-)defined by $n(x) = \sqrt{x}$.[644]

Change the domain of o to $\mathbb{R} \setminus \{0\}$, the set of all reals *except* zero. We then have the function $o : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ that is (well-)defined by $o(x) = 1/x$.[645]

A110(a) Range $a = \mathbb{R}_0^+$. **(b)** Range $b = \{0, 1, 4, 9, 16, 25, 49, \dots\}$.
(c) Range $c = \{0, 1, 4, 9, 16, 25, 49, \dots\}$. **(d)** Range $d = \mathbb{Z}$.
(e) Range $e = \mathbb{Z}$. **(f)** Range $f = \{100, 200\}$. **(g)** Range $g = \{100\}$.

[642] As noted earlier and as we'll learn later, $\sqrt{-1}$ is not a real but an imaginary number.

[643] As discussed in Ch. 8, $1 \div 0$ is not a real number. Indeed, it is not even a number; it is undefined.

[644] Note that the new domain $\mathbb{R}_0^+$ is indeed the "largest" subset of $\mathbb{R}$ such that the function n is well-defined. The addition of any negative number to this new domain would render the function ill-defined.

[645] Note that the new domain $\mathbb{R} \setminus \{0\}$ is indeed the "largest" subset of $\mathbb{R}$ such that the function o is well-defined. The addition of zero to this new domain would render the function ill-defined.

Only d and f are onto.

A111. Only **(b)** "Range $f \subseteq$ Codomain f" must be true.

149.10. Ch. 18 Answers (An Introduction to Continuity)

(*This chapter had no exercises.*)

149.11. Ch. 19Answers (When A Function Is Increasing or Decreasing)

A112(a) Pick any $a, b \in (-\infty, 0]$ with $a < b$. Observe that

$$a^4 > b^4 \qquad \text{or} \qquad j(a) > j(b).$$

And so, by Definition 73, i is strictly decreasing on $(-\infty, 0]$.

(b) Pick any $a, b \in [0, \infty)$ with $a < b$. Observe that

$$a^4 < b^4 \qquad \text{or} \qquad j(a) < j(b).$$

And so, by Definition 73, i is strictly increasing on $[0, \infty)$.

(c) By Fact 46, j is also decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

149.12. Ch. 20 Answers (Arithmetic Combinations of Functions)

A113.

(a) $(f+g)(2) = 7\cdot 2+5+2^3 = 27$

(b) $(g-f)(1) = 1^3-(7\cdot 1+5) = -11$

(c) $(g\cdot f)(2) = \left(2^3\right)(7\cdot 2+5) = 152$

(d) $(kg)(1) = 2\left(1^3\right) = 2$

(e) $\left(\dfrac{g}{f}\right)(1) = \dfrac{1^3}{7\cdot 1+5} = \dfrac{1}{12}$

(f) $(f+k)(1) = 7\cdot 1+5+2 = 14$

(g) $(h+i)(2) = 2+1+\sqrt{2+1} = 3+\sqrt{3}$

(h) $(i-h)(1) = \sqrt{1+1}-(1+1) = \sqrt{2}-2$

(i) $(i\cdot h)(2) = \sqrt{2+1}(2+1) = 3\sqrt{3}$

(j) $(li)(1) = 5\sqrt{1+1} = 5\sqrt{2}$

(k) $\left(\dfrac{i}{h}\right)(1) = \dfrac{\sqrt{1+1}}{1+1} = \dfrac{\sqrt{2}}{2}$

(l) $(h+l)(1) = 1+1+5 = 7$

(k) For each of $f+h$, $f-h$, and $f\cdot h$, the domain is Domain $f\cap$ Domain $h = \mathbb{R}\cap[-1,\infty) = [-1,\infty)$. For f/h, the domain is Domain $f\cap$ Domain $h\setminus\{x : h(x)=0\} = [-1,\infty)\setminus\{-1\} = (-1,\infty)$. So, define

$$(f+h):[-1,\infty)\to\mathbb{R} \text{ by } (f+h)(x) = f(x)+h(x) = 8x+6;$$

$$(f-h):[-1,\infty)\to\mathbb{R} \text{ by } (f-h)(x) = f(x)-h(x) = 6x+4;$$

$$(f\cdot h):[-1,\infty)\to\mathbb{R} \text{ by } (f\cdot h)(x) = f(x)\cdot h(x) = (7x+5)(x+1);$$

$$\left(\frac{f}{h}\right):(-1,\infty)\to\mathbb{R} \text{ by } \left(\frac{f}{h}\right)(x) = \frac{f(x)}{h(x)} = \frac{7x+5}{x+1}.$$

149.13. Ch. 22 Answers (Composite Functions)

A115(a)(i) The composite function $g^4 = gg^3$ exists because Range $g^3 = \mathbb{R} \subseteq$ Domain $g = \mathbb{R}$.
(ii) Define $g^4 : \mathbb{R} \to \mathbb{R}$ by

$$g^4(x) = gg^3(x) = g\left(g^3(x)\right) = g\left(\frac{3}{4} - \frac{x}{8}\right) = 1 - \frac{3/4 - x/8}{2} = \frac{5}{8} + \frac{x}{16}.$$

(iii) $g^4(1) = 11/16$ and $g^4(3) = 13/16$.

(b)(i) The composite function $g^5 = gg^4$ exists because Range $g^4 = \mathbb{R} \subseteq$ Domain $g = \mathbb{R}$.
(ii) Define $g^5 : \mathbb{R} \to \mathbb{R}$ by

$$g^5(x) = gg^4(x) = g\left(g^4(x)\right) = g\left(\frac{5}{8} + \frac{x}{16}\right) = 1 - \frac{5/8 + x/16}{2} = \frac{11}{16} - \frac{x}{32}.$$

(iii) $g^5(1) = 21/32$ and $g^5(3) = 19/32$.

(c)(i) The composite function $g^6 = gg^5$ exists because Range $g^5 = \mathbb{R} \subseteq$ Domain $g = \mathbb{R}$.
(ii) Define $g^6 : \mathbb{R} \to \mathbb{R}$ by

$$g^6(x) = gg^5(x) = g\left(g^5(x)\right) = g\left(\frac{11}{16} - \frac{x}{32}\right) = 1 - \frac{\frac{11}{16} - \frac{x}{32}}{2} = \frac{21}{32} + \frac{x}{64}.$$

(iii) $g^6(1) = 43/64$ and $g^6(3) = 45/64$.

(d) $b_n = 2^{n-1}$, $c_n = \left(-\frac{1}{2}\right)^n$, and hence $g^n(x) = \dfrac{\frac{2^n - (-1)^n}{3}}{2^{n-1}} + \left(-\frac{1}{2}\right)^n x = \dfrac{2}{3} + \dfrac{1}{3}\left(-\dfrac{1}{2}\right)^{n-1} + \left(-\dfrac{1}{2}\right)^n x$.

(e) As n approaches ∞, each of $\left(-\frac{1}{2}\right)^{n-1}$ and $\left(-\frac{1}{2}\right)^n$ approaches zero. Hence, $g^n(x)$ approaches $\dfrac{2}{3}$.

A116(a) Range $g = [1, \infty) \subseteq$ Domain $f = \mathbb{R}$. Hence, the composite function $fg : \mathbb{R} \to \mathbb{R}$ exists and is defined by

$$(fg)(x) = f(g(x)) = f\left(x^2 + 1\right) = \mathrm{e}^{x^2+1}.$$

We have $fg(1) = \mathrm{e}^{1^2+1} = \mathrm{e}^2$ and $fg(2) = \mathrm{e}^{2^2+1} = \mathrm{e}^5$.

Also, Range $f = \mathbb{R}^+ \subseteq$ Domain $g = \mathbb{R}$. Hence, the composite function $gf : \mathbb{R} \to \mathbb{R}$ exists and is defined by

$$(gf)(x) = g(f(x)) = g(\mathrm{e}^x) = (\mathrm{e}^x)^2 + 1 = \mathrm{e}^{2x} + 1.$$

We have $gf(1) = \mathrm{e}^{2\cdot 1} + 1 = \mathrm{e}^2 + 1$ and $gf(2) = \mathrm{e}^{2\cdot 2} + 1 = \mathrm{e}^4 + 1$.

(b) Range $g = \mathbb{R} \setminus \{0\} \subseteq$ Domain $f = \mathbb{R} \setminus \{0\}$. Hence, the composite function $fg : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ exists and is defined by

$$(fg)(x) = f(g(x)) = f\left(\frac{1}{2x}\right) = \frac{1}{1/2x} = 2x.$$

We have $fg(1) = 2 \cdot 1 = 2$ and $fg(2) = 2 \cdot 2 = 4$.

Also, Range $f = \mathbb{R} \setminus \{0\} \subseteq$ Domain $g = \mathbb{R} \setminus \{0\}$. Hence, the composite function $gf : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ exists and is defined by

$$(gf)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{2 \cdot (1/x)} = \frac{x}{2}.$$

We have $gf(1) = 1/2$ and $gf(2) = 2/2 = 1$.

(c) Range $g = [1, \infty) \subseteq$ Domain $f = \mathbb{R} \setminus \{0\}$. Hence, the composite function $fg : \mathbb{R} \to \mathbb{R}$ exists and is defined by

$$(fg)(x) = f(g(x)) = f\left(x^2 + 1\right) = \frac{1}{x^2 + 1}.$$

We have $fg(1) = 1/\left(1^2 + 1\right) = 1/2$ and $fg(2) = 1/\left(2^2 + 1\right) = 1/5$.

Also, Range $f = \mathbb{R} \setminus \{0\} \subseteq$ Domain $g = \mathbb{R}$. Hence, the composite function $gf : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ exists and is defined by

$$(gf)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x^2} + 1.$$

We have $gf(1) = 1/1^2 + 1 = 2$ and $gf(2) = 1/2^2 + 1 = 5/4$.

(d) Range $g = [-1, \infty) \nsubseteq$ Domain $f = \mathbb{R} \setminus \{0\}$. Hence, the composite function $fg : \mathbb{R} \to \mathbb{R}$ does not exist.

Range $f = \mathbb{R} \setminus \{0\} \subseteq$ Domain $g = \mathbb{R}$. Hence, the composite function $gf : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ exists and is defined by

$$(gf)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x^2} - 1.$$

We have $gf(1) = 1/1^2 - 1 = 0$ and $gf(2) = 1/2^2 - 1 = -3/4$.

A117(a) Range $f = \mathbb{R}^+ \subseteq$ Domain $f = \mathbb{R}$. Hence, the composite function $f^2 : \mathbb{R} \to \mathbb{R}$ exists and is defined by

$$f^2(x) = f(f(x)) = f(e^x) = e^{e^x}.$$

We have $f^2(1) = e^{e^1} = e^e$ and $f^2(2) = e^{e^2}$.

(b) Range $f = \mathbb{R} \subseteq$ Domain $f = \mathbb{R}$. Hence, the composite function $f^2 : \mathbb{R} \to \mathbb{R}$ exists and is defined by

$$f^2(x) = f(f(x)) = f(3x+2) = 3(3x+2)+2 = 9x+8.$$

We have $f^2(1) = 9 \cdot 1 + 8 = 17$ and $f^2(2) = 9 \cdot 2 + 8 = 26$.

(c) Range $f = [1, \infty) \subseteq$ Domain $f = \mathbb{R}$. Hence, the composite function $f^2 : \mathbb{R} \to \mathbb{R}$ exists and is defined by

$$f^2(x) = f(f(x)) = f\left(2x^2+1\right) = 2\left(2x^2+1\right)^2 + 1 = 2\left(4x^4+4x^2+1\right)+1 = 8x^4+8x^2+3.$$

We have $f^2(1) = 8 \cdot 1^4 + 8 \cdot 1^2 + 3 = 19$ and $f^2(2) = 8 \cdot 2^4 + 8 \cdot 2^2 + 3 = 128 + 32 + 3 = 163$.

(d) Range $f = \mathbb{R} \nsubseteq$ Domain $f = \mathbb{R}^+$. Hence, the composite function $f^2 : \mathbb{R} \to \mathbb{R}$ does not exist.

A??. No: $4 \div (3 \div 2) = 8/3$ but $(4 \div 3) \div 2 = 4/6 = 2/3$.

149.14. Ch. ?? Answers (one-to-one Functions)

A123(a) Write $y = a(x) = 5x$. Do the algebra: $x = y/5$.

So, a is one-to-one and its inverse is the function $a^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $a^{-1}(y) = y/5$.

(b) Write $y = b(x) = x^3$. Do the algebra: $x = \sqrt[3]{y}$.

So, b is one-to-one and its inverse is the function $b^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $b^{-1}(y) = \sqrt[3]{y}$.

(c) Write $y = c(x) = x^2 + 1$. Do the algebra: $x = \pm\sqrt{y-1}$.

Since Domain $c = \mathbb{R}_0^+$, we have $x \geq 0$. So, we discard the negative value and are left with $x = \sqrt{y-1}$.

Hence, c is one-to-one and its inverse is the function $c^{-1} : [1, \infty) \to \mathbb{R}_0^+$ defined by $c^{-1}(y) = \sqrt{y-1}$.

(d) Write $y = d(x) = x^2 + 1$. Do the algebra: $x = \pm\sqrt{y-1}$.

Since Domain $d = \mathbb{R}_0^-$, we have $x \leq 0$. So, we discard the positive value and are left with $x = -\sqrt{y-1}$.

Hence, d is one-to-one and its inverse is the function $d^{-1} : [1, \infty) \to \mathbb{R}_0^-$ defined by $d^{-1}(y) = -\sqrt{y-1}$.

(e) Write $y = e(x) = 1/\left(x^2 + 1\right)$. Do the algebra: $x = \pm 1/\sqrt{y-1}$.

Since Domain $e = \mathbb{R}_0^+$, we have $x \geq 0$. So, we discard the negative value and are left with $x = 1/\sqrt{y-1}$.

Hence, e is one-to-one and its inverse is the function $e^{-1} : (0, 1] \to \mathbb{R}_0^+$ defined by $e^{-1}(y) = 1/\sqrt{y-1}$.

(f) Write $y = f(x) = 1/\left(x^2 + 1\right)$. Do the algebra: $x = \pm 1/\sqrt{y-1}$.

Since Domain $f = \mathbb{R}_0^-$, we have $x \leq 0$. So, we discard the positive value and are left with $x = -1/\sqrt{y-1}$.

Hence, f is one-to-one and its inverse is the function $f^{-1} : (0, 1] \to \mathbb{R}_0^+$ defined by $f^{-1}(y) = -1/\sqrt{y-1}$.

A124(a) No, a is not one-to-one because $0 = a(-1) = a(1)$.

(b) No, b is not one-to-one because $0 = b(-1) = b(1)$.

(c) Write $y = c(x) = x^2 - 1$. Do the algebra: $x = \pm\sqrt{y+1}$.

Since Domain $c = \mathbb{R}_0^+$, we have $x \geq 0$. So, we discard the negative value and are left with $x = \sqrt{y+1}$.

Hence, c is one-to-one and its inverse is the function $c^{-1} : [-1, \infty) \to \mathbb{R}_0^+$ defined by $c^{-1}(y) = \sqrt{y+1}$.

(d) Write $y = d(x) = x^2 - 1$. Do the algebra: $x = \pm\sqrt{y+1}$.

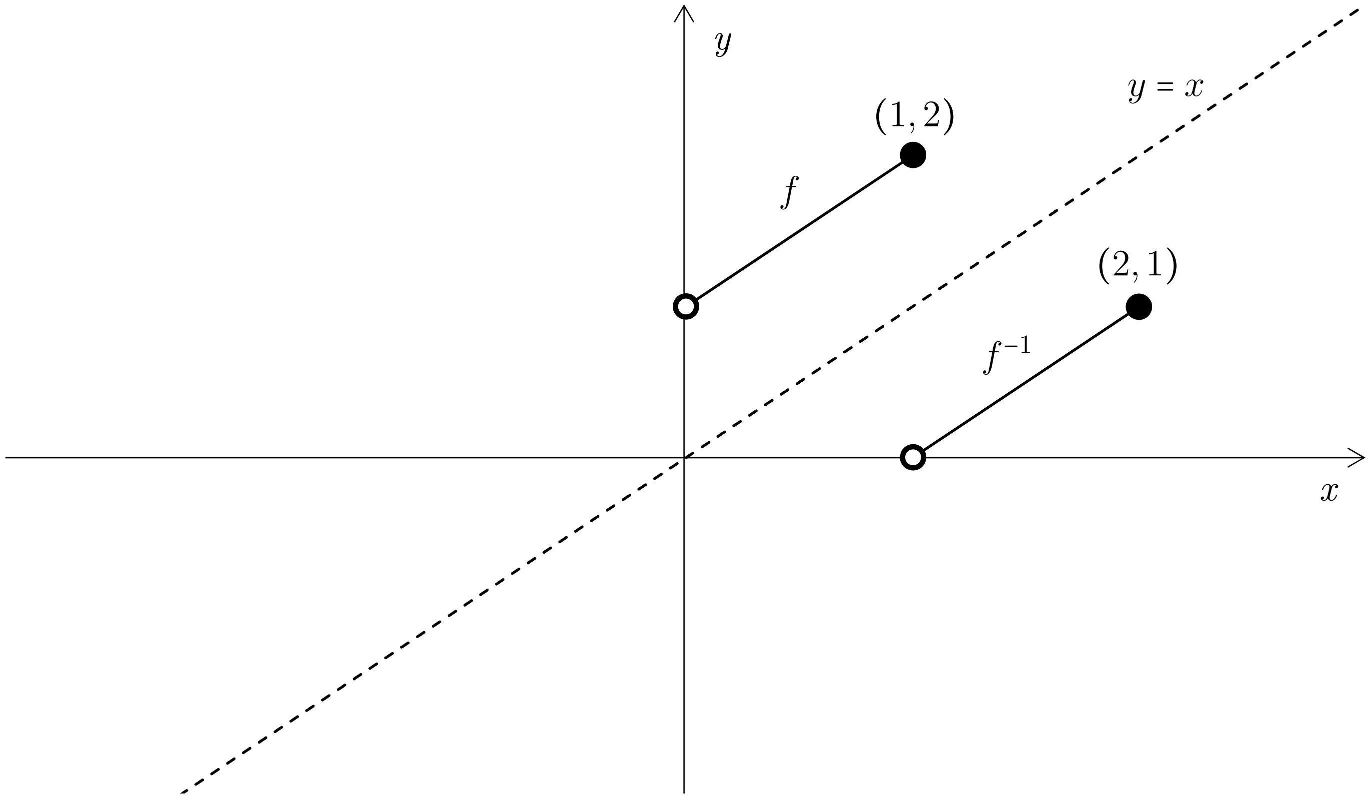
Since Domain $d = \mathbb{R}_0^-$, we have $x \leq 0$. So, we discard the positive value and are left with $x = -\sqrt{y+1}$.

Hence, d is one-to-one and its inverse is the function $d^{-1} : [-1, \infty) \to \mathbb{R}_0^-$ defined by $d^{-1}(y) =$

$-\sqrt{y+1}$.

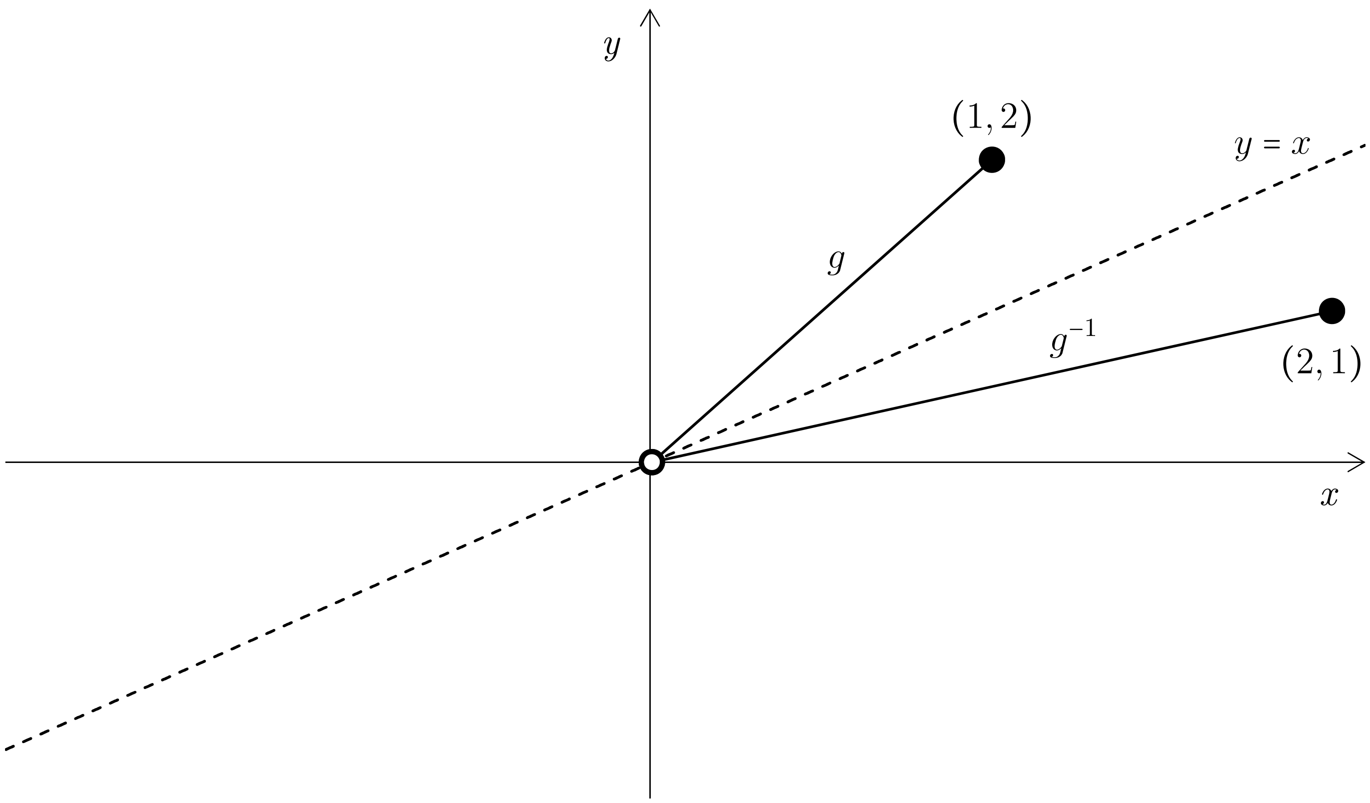
A**126****(a)** Write $y = f(x) = x + 1$. Do the algebra: $x = y - 1$.

So, f is one-to-one and its inverse is $f^{-1} : (1, 2] \to (0, 1]$ defined by $f^{-1}(y) = y - 1$.



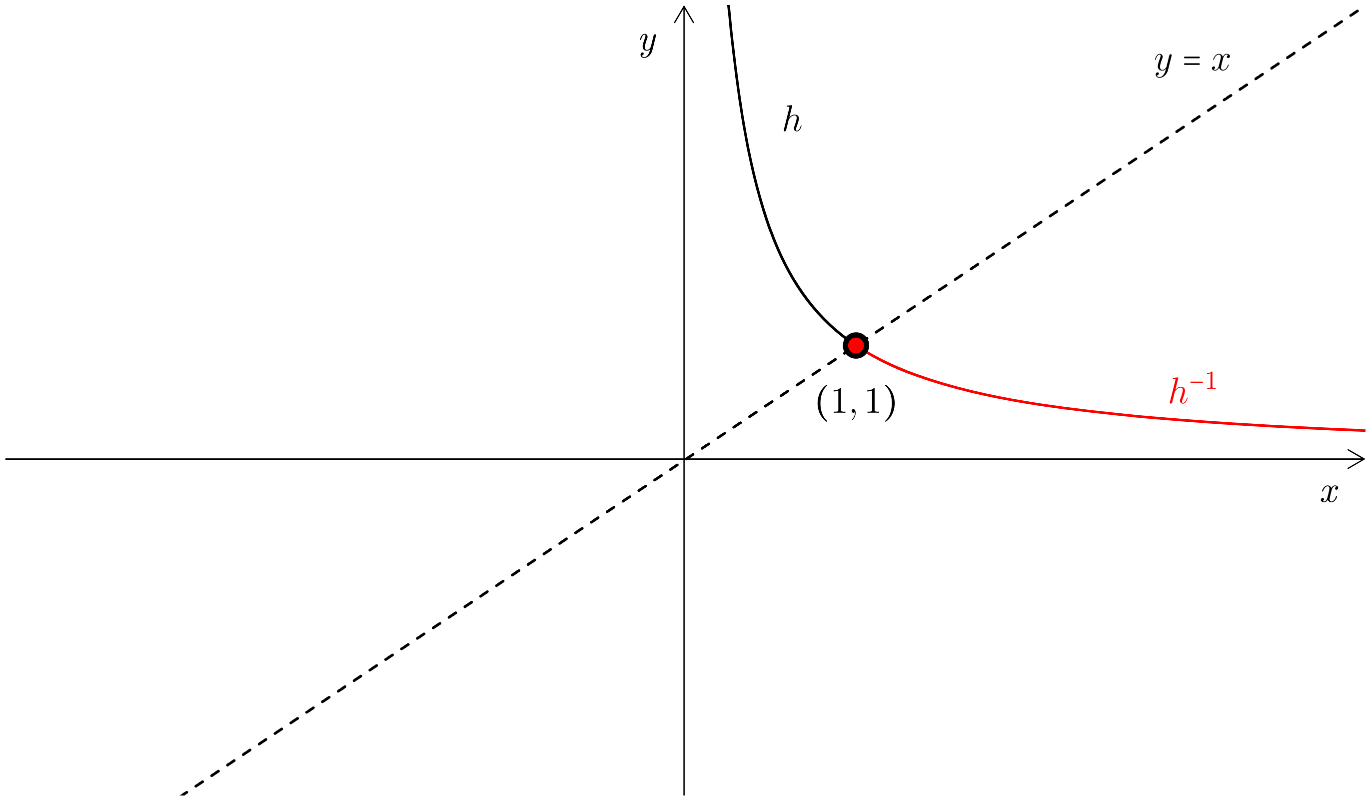
(b) Write $y = g(x) = 2x$. Do the algebra: $x = y/2$.

So, g is one-to-one and its inverse is $g^{-1} : (0, 2] \to (0, 1]$ defined by $g^{-1}(y) = y/2$.



(c) Write $y = h(x) = 1/x$. Do the algebra: $x = 1/y$.

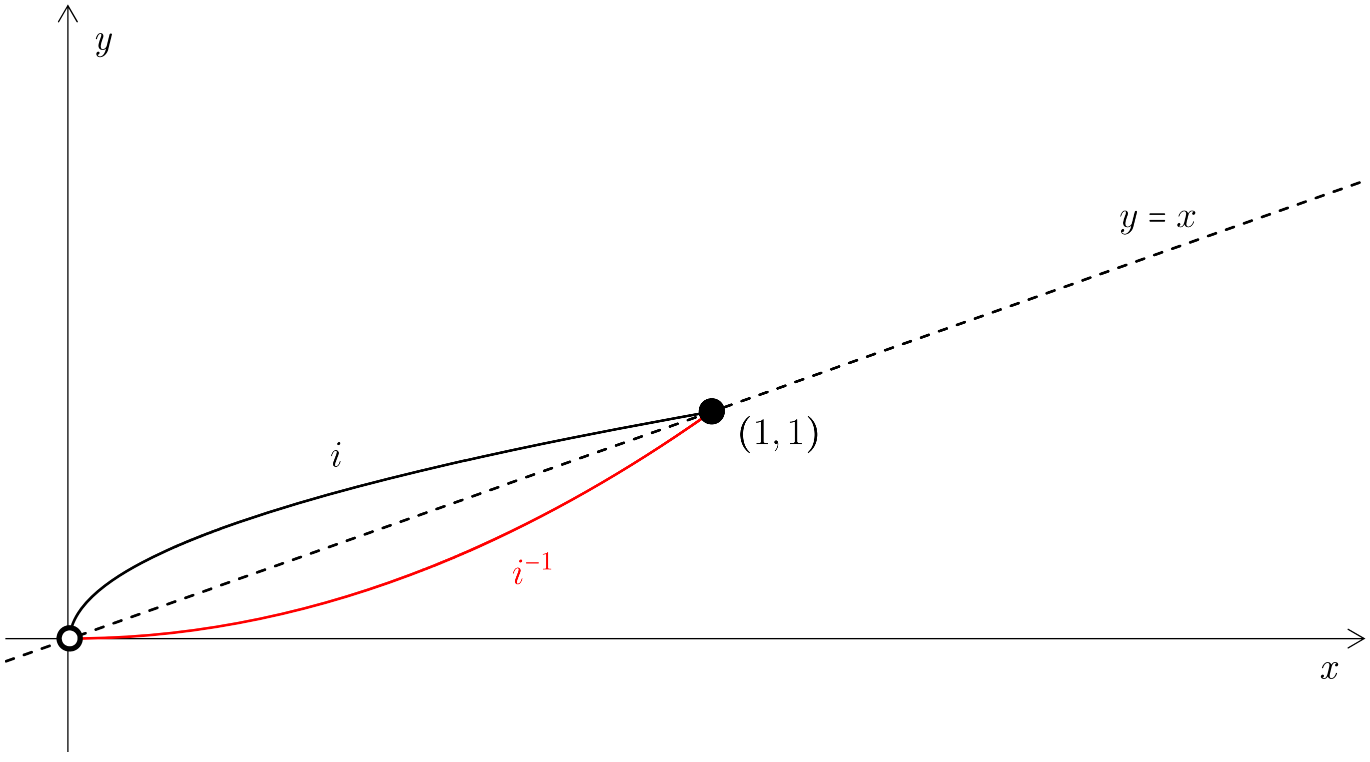
So, h is one-to-one and its inverse is $h^{-1} : [1, \infty) \to (0, 1]$ defined by $h^{-1}(y) = 1/y$.



(d) Write $y = i(x) = x^2$. Do the algebra: $x = \pm\sqrt{y}$.

Since $x \in \text{Domain}\, i = (0,1]$, we may discard the negative value and be left with $x = \sqrt{y}$.

So, i is one-to-one and its inverse is $i^{-1} : (0,1] \to (0,1]$ defined by $i^{-1}(x) = \sqrt{x}$.



A129(a)(i) Write $y = h(x) = 1/x$. Do the algebra: $x = 1/y$.

So, h is one-to-one and its inverse is $h^{-1} : [1,\infty) \to (0,1]$ defined by $h^{-1}(y) = 1/y$.

(a)(ii) Write $x = h^{-1}(y) = 1/y$. Do the algebra: $y = 1/x$.

So, h^{-1} is one-to-one and its inverse is $\left(h^{-1}\right)^{-1} : (0,1] \to [1,\infty)$ defined by $\left(h^{-1}\right)^{-1}(x) = 1/x$.

(a)(iii) Yes, Range $h = [1, \infty)$ = Codomain h.

(a)(iv) Yes, $h = \left(h^{-1}\right)^{-1}$.

(b)(i) Write $y = i(x) = x^2$. Do the algebra: $x = \pm\sqrt{y}$.

Since $x \in$ Domain $i = (0, 1]$, we may discard the negative value and be left with $x = \sqrt{y}$.

So, i is one-to-one and its inverse is $i^{-1} : (0, 1] \to (0, 1]$ defined by $i^{-1}(y) = \sqrt{y}$.

(b)(ii) Write $x = i^{-1}(y) = \sqrt{y}$. Do the algebra: $y = x^2$.

So, i^{-1} is one-to-one and its inverse is $\left(i^{-1}\right)^{-1} : (0, 1] \to (0, 1]$ defined by $\left(i^{-1}\right)^{-1}(x) = x^2$.

(b)(iii) Yes, Range $i = (0, 1]$ = Codomain i.

(b)(iv) Yes, $i = \left(i^{-1}\right)^{-1}$.

A130(a) The restriction of the function f to the set of non-positive reals.

(b) Pick any distinct $x_1, x_2 \in$ Domain $f|_{\mathbb{R}_0^-} = \mathbb{R}_0^-$. Suppose $x_1 < x_2$. Then $f|_{\mathbb{R}_0^-}(x_1) = x_1^2 > x_2^2 = f|_{\mathbb{R}_0^-}(x_2)$.

We've just shown that $f|_{\mathbb{R}_0^-}$ is strictly decreasing. Hence, $f|_{\mathbb{R}_0^-}$ is also one-to-one.

(c) Write $y = f|_{\mathbb{R}_0^-}(x) = x^2$. Do the algebra: $x = \pm\sqrt{y}$.

Since $x \in$ Domain $f|_{\mathbb{R}_0^-} = \mathbb{R}_0^-$, we may discard the positive value and be left with $x = -\sqrt{y}$.

So, $f|_{\mathbb{R}_0^-}$ is one-to-one and its inverse is the function $f|_{\mathbb{R}_0^-}^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^-$ defined by $f|_{\mathbb{R}_0^-}^{-1}(y) = -\sqrt{y}$.

(d) Yes. In fact, there are infinitely many such sets S. For example, S could be *any* subset of $\mathbb{R}_0^+$ or *any* subset of $\mathbb{R}_0^-$. It could also be $[-2, -1] \cup [3, 4]$ (why?).[646] And as mentioned in n. 194, S could also be $\varnothing$.

A131(a) The restriction of the function g to the set of non-positive reals.

(b) Pick any distinct $x_1, x_2 \in$ Domain $g|_{\mathbb{R}_0^-} = \mathbb{R}_0^-$. Suppose $x_1 < x_2$. Then $g|_{\mathbb{R}_0^-}(x_1) = |x_1| = x_1 > x_2 = |x_2| = g|_{\mathbb{R}_0^-}(x_2)$.

We've just shown that $g|_{\mathbb{R}_0^-}$ is strictly decreasing. Hence, $g|_{\mathbb{R}_0^-}$ is also one-to-one.

(c) Write $y = g|_{\mathbb{R}_0^-}(x) = |x|$. Do the algebra: $x = \pm y$.

Since $x \in$ Domain $g|_{\mathbb{R}_0^-} = \mathbb{R}_0^-$, we may discard the positive value and be left with $x = -y$.

So, $g|_{\mathbb{R}_0^-}$ is one-to-one and its inverse is the function $g|_{\mathbb{R}_0^-}^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^-$ defined by $g|_{\mathbb{R}_0^-}^{-1}(y) = -y$.

(d) Yes. In fact, there are infinitely many such sets S. For example, S could be *any* subset of $\mathbb{R}_0^+$ or *any* subset of $\mathbb{R}_0^-$. It could also be $[-2, -1] \cup [3, 4]$ (why?).[647] And as mentioned in n. 194, S could also be $\varnothing$.

A132(a) No, h is not one-to-one because for example, the element 1 in the codomain is "hit" twice (by 0 and 2):

[646] The range of $f|_{[-2,-1]\cup[3,4]}$ is $[1, 4] \cup [9, 16]$ and each element in this range is "hit" exactly once.

[647] The range of $g|_{[-2,-1]\cup[3,4]}$ is $[1, 2] \cup [3, 4]$ and each element in this range is "hit" exactly once.

$$1 = h(0) = \frac{1}{(0-1)^2} \qquad \text{and} \qquad 1 = h(2) = \frac{1}{(2-1)^2}.$$

(b) The restriction of the function g to the set of real numbers greater than one.

(c) Pick any distinct $x_1, x_2 \in \text{Domain}\, h|_{(1,\infty)} = (1,\infty)$. Suppose $x_1 < x_2$. Then $h|_{(1,\infty)}(x_1) = \frac{1}{(x_1-1)^2} > \frac{1}{(x_2-1)^2} = h|_{(1,\infty)}(x_2)$.

We've just shown that $h|_{(1,\infty)}$ is strictly decreasing. Hence, $h|_{(1,\infty)}$ is also one-to-one.

(d) Write $y = h|_{(1,\infty)}(x) = 1/(1-x)^2$. Do the algebra: $x = 1 \pm 1/\sqrt{y}$.

Since $x \in \text{Domain}\, h|_{(1,\infty)} = (1,\infty)$, we may discard $x = 1 - 1/\sqrt{y}\,(<1)$ and be left with $x = 1 + 1/\sqrt{y}$.

So, $h|_{(1,\infty)}$ is one-to-one and its inverse is the function $h|_{(1,\infty)}^{-1} : \mathbb{R}_0^+ \to (1,\infty)$ defined by $h|_{(1,\infty)}^{-1}(y) = 1 + 1/\sqrt{y}$.

(e) The restriction of the function g to the set of real numbers less than one.

(f) Pick any distinct $x_1, x_2 \in \text{Domain}\, h|_{(-\infty,1)} = (-\infty,1)$. Suppose $x_1 < x_2$. Then $h|_{(-\infty,1)}(x_1) = \frac{1}{(x_1-1)^2} < \frac{1}{(x_2-1)^2} = h|_{(-\infty,1)}(x_2)$.

We've just shown that $h|_{(-\infty,1)}$ is strictly increasing. Hence, $h|_{(-\infty,1)}$ is also one-to-one.

(g) Write $y = h|_{(-\infty,1)}(x) = 1/(1-x)^2$. Do the algebra: $x = 1 \pm 1/\sqrt{y}$.

Since $x \in \text{Domain}\, h|_{(-\infty,1)} = (-\infty,1)$, we may discard $x = 1 + 1/\sqrt{y}\,(>1)$ and be left with $x = 1 - 1/\sqrt{y}$.

So, $h|_{(-\infty,1)}$ is one-to-one and its inverse is the function $h|_{(-\infty,1)}^{-1} : \mathbb{R}_0^+ \to (-\infty,1)$ defined by $h|_{(-\infty,1)}^{-1}(y) = 1 - 1/\sqrt{y}$.

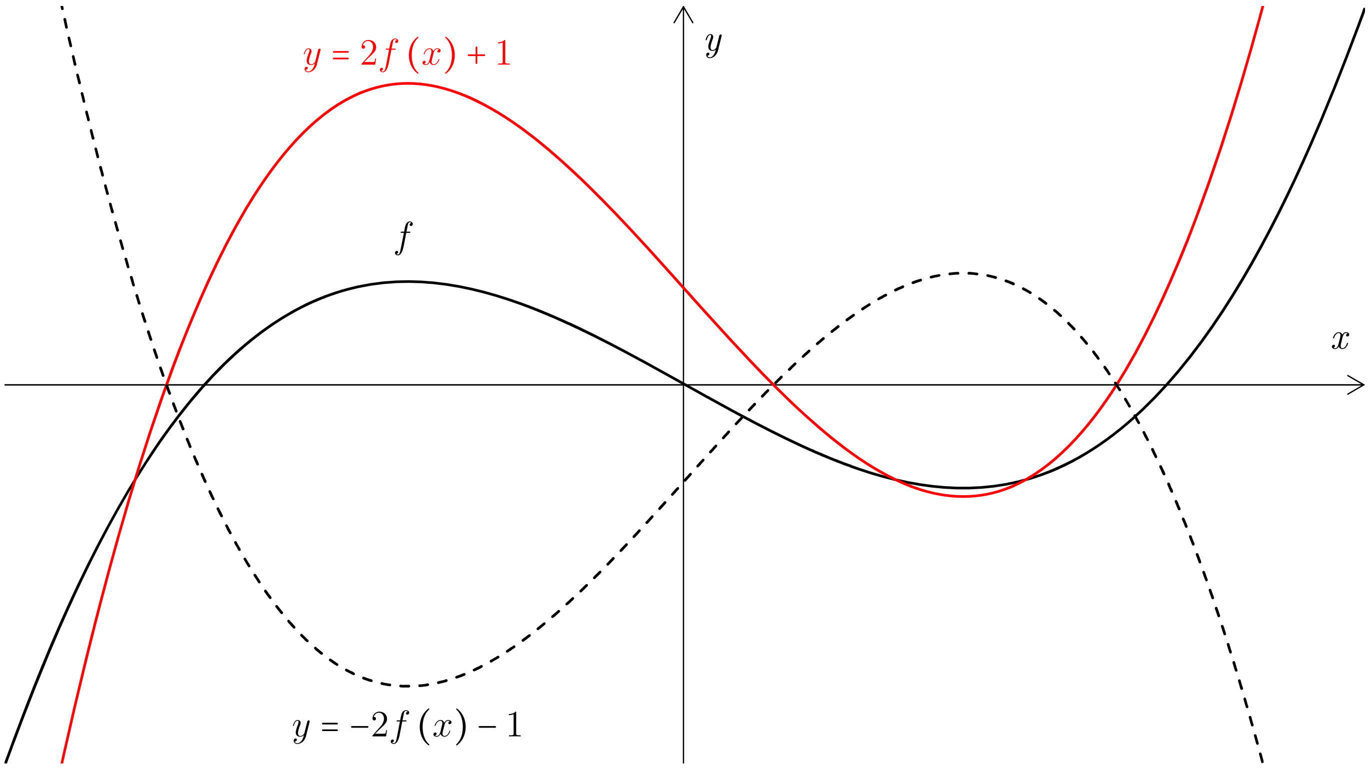
(i) Yes. In fact, there are infinitely many such sets S. For example, S could be *any* subset of $(1,\infty)$ or *any* subset of $(-\infty,1)$. It could also be $(0,1)\cup(2,3)$ (why?).[648] And as mentioned in n. 194, S could also be $\varnothing$.

[648]The range of $h|_{(0,1)\cup(2,3)}$ is $(1,\infty)\cup(1/4,1)$ and each element in this range is "hit" exactly once.

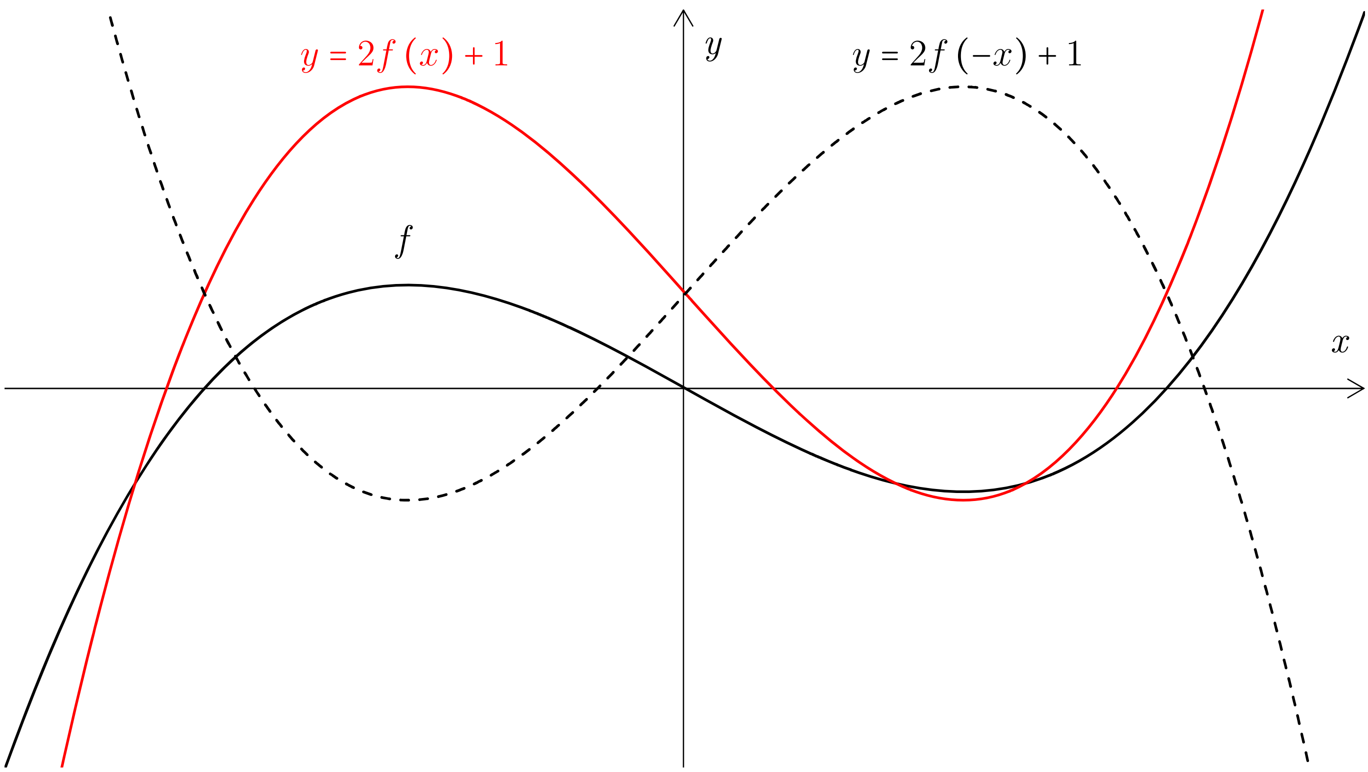
149.16. Ch. 25 Answers (Asymptotes and Limit Notation)

XXX

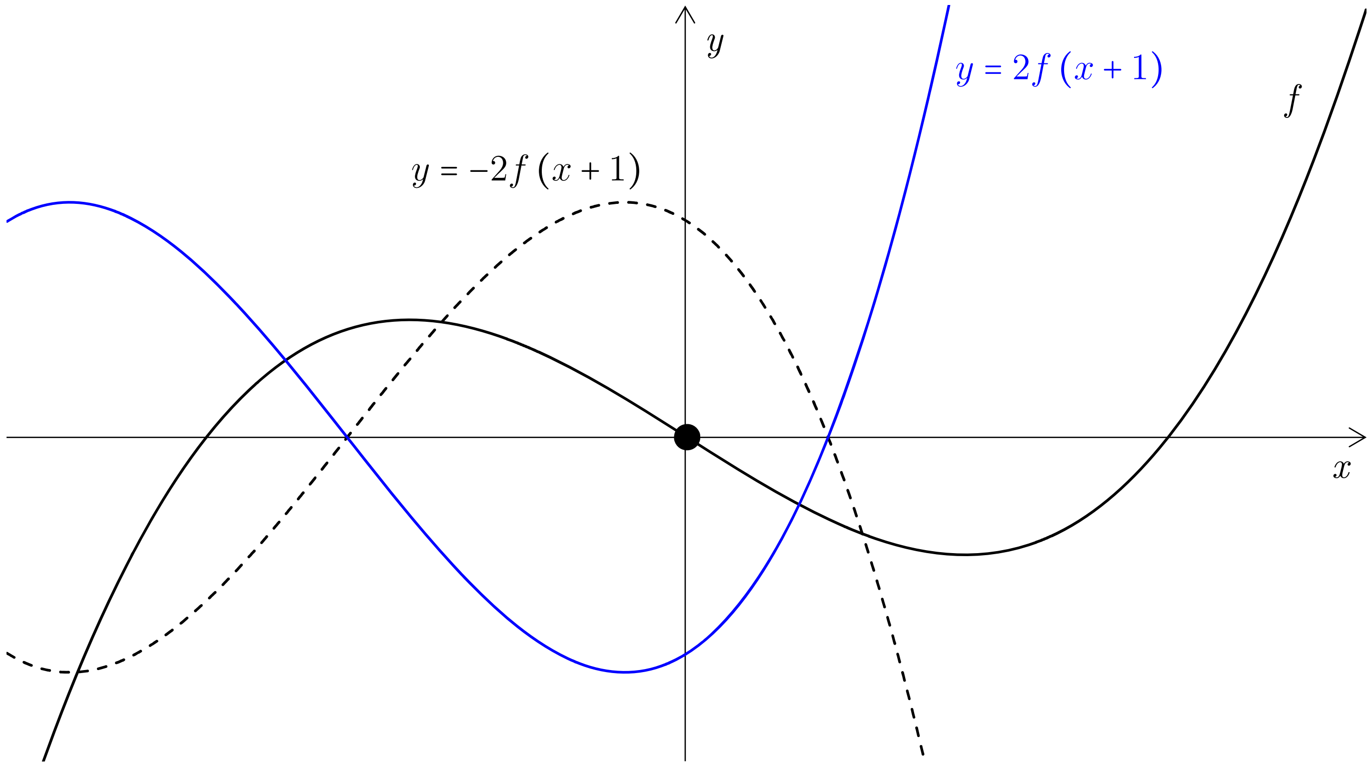
A133(a) We already graphed $y = 2f(x) + 1$ in the above example. Simply reflect that in the x-axis to get the graph of $y = -2f(x - 1)$.



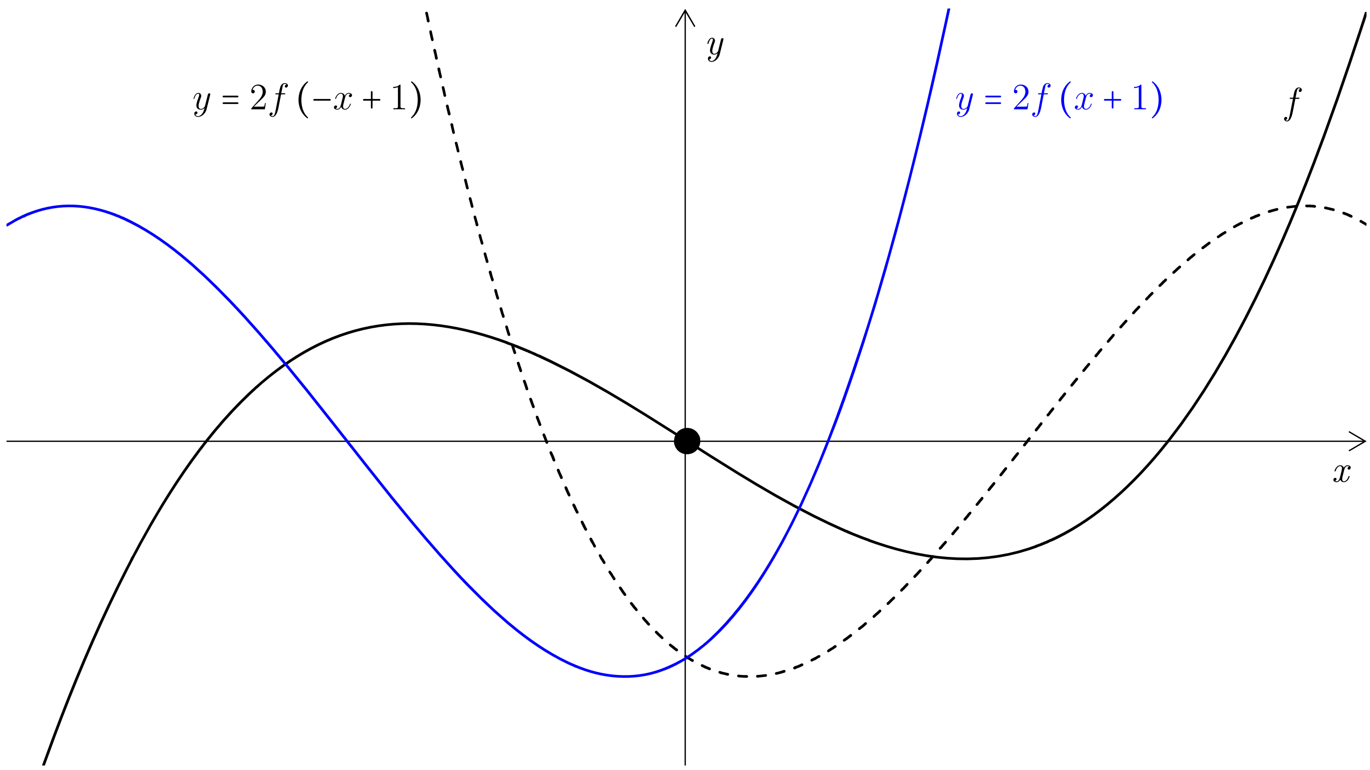
(b) We already graphed $y = 2f(x) + 1$ in the above example. Simply reflect that in the y-axis to get the graph of $y = 2f(-x) + 1$.



A133(c) We already graphed $y = 2f(x+1)$ in the above example. Simply reflect that in the x-axis to get the graph of $y = -2f(x+1)$.

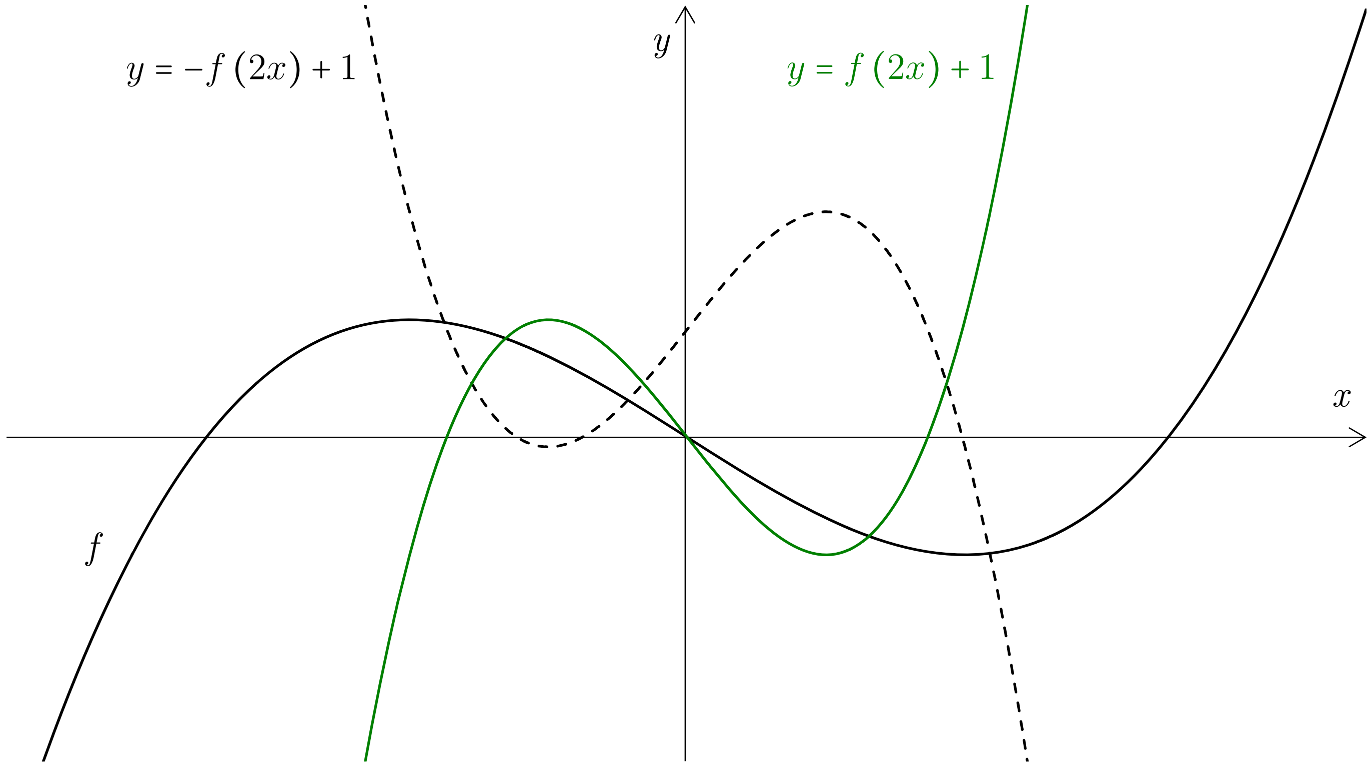


(d) We already graphed $y = 2f(x+1)$ in the above example. Simply reflect that in the x-axis to get the graph of $y = 2f(-x+1)$.

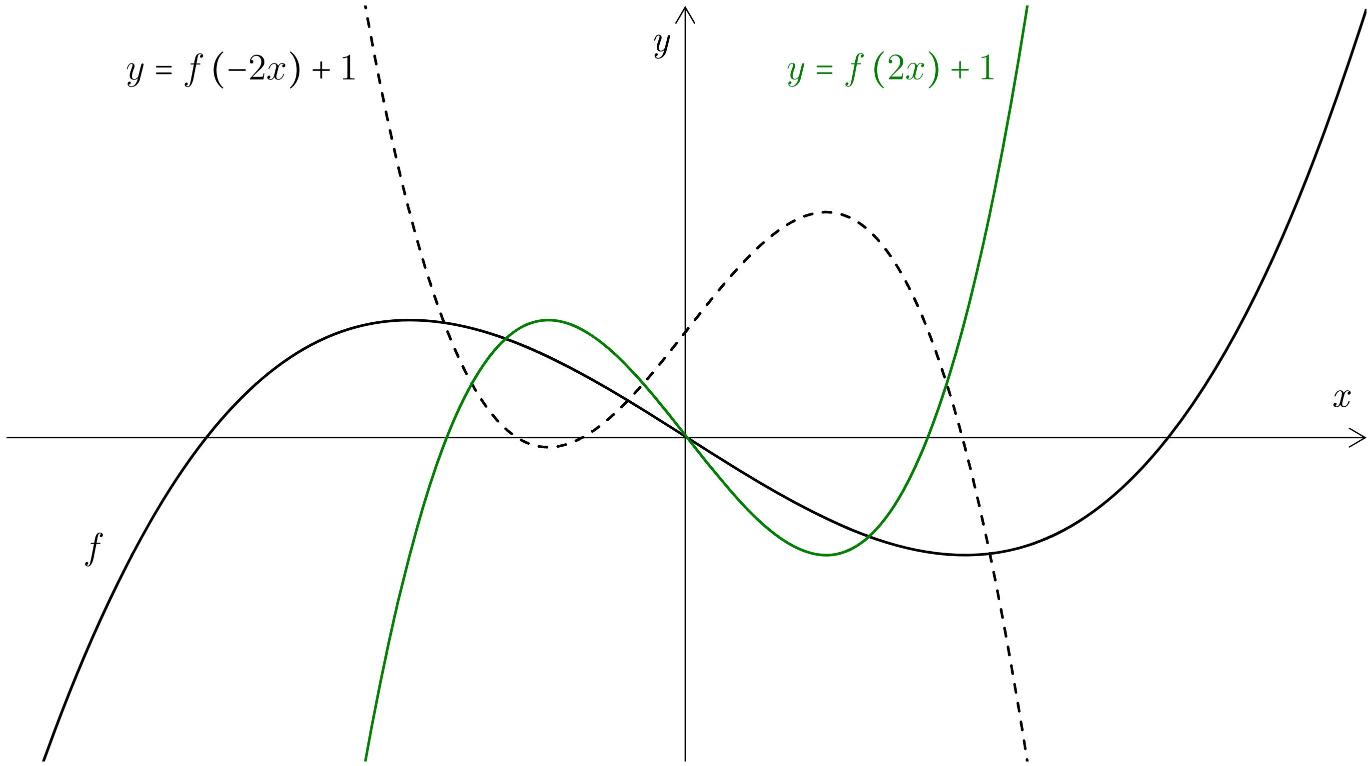


A133(e) We already graphed $y = f(2x) + 1$ in the above example. Reflect that in the x-axis to get $y = -f(2x) - 1$.

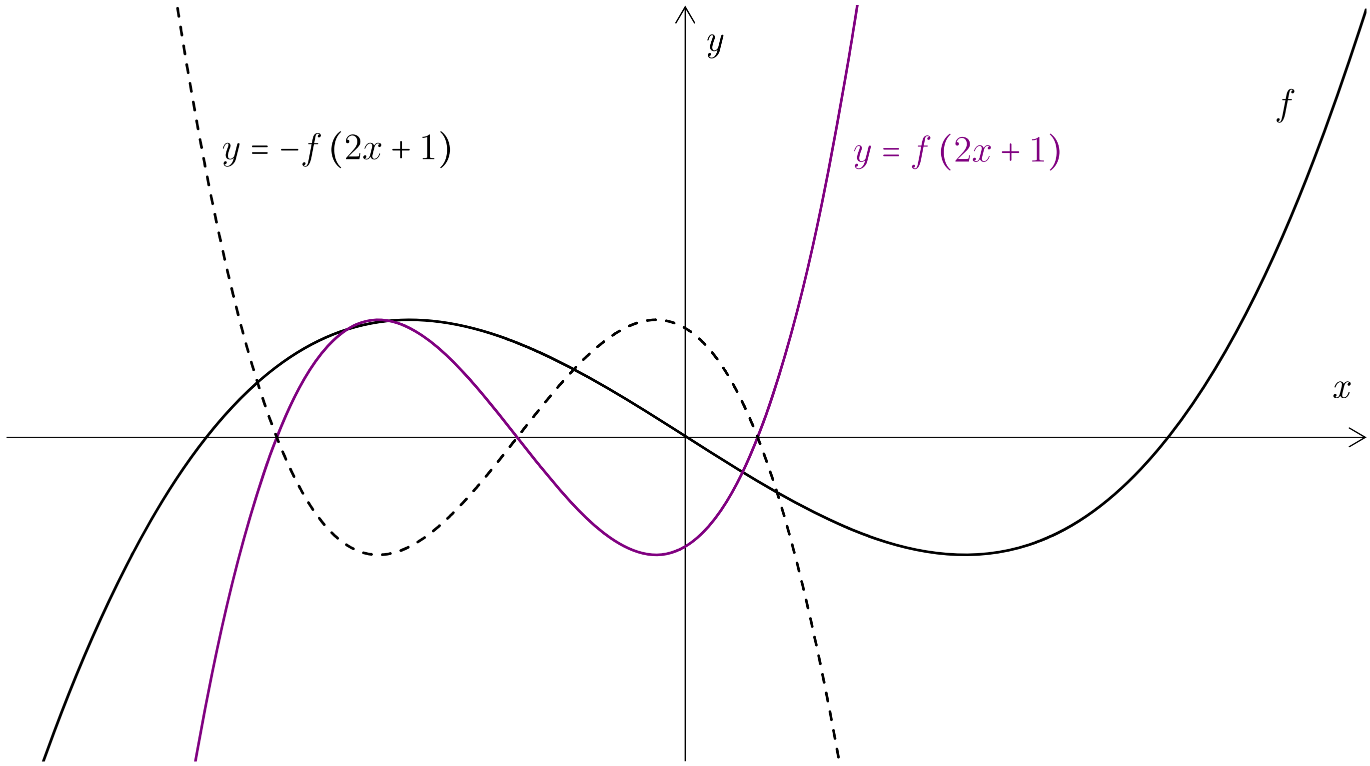
Then translate upwards by 2 units to get the graph of $y = -f(2x) + 1$.



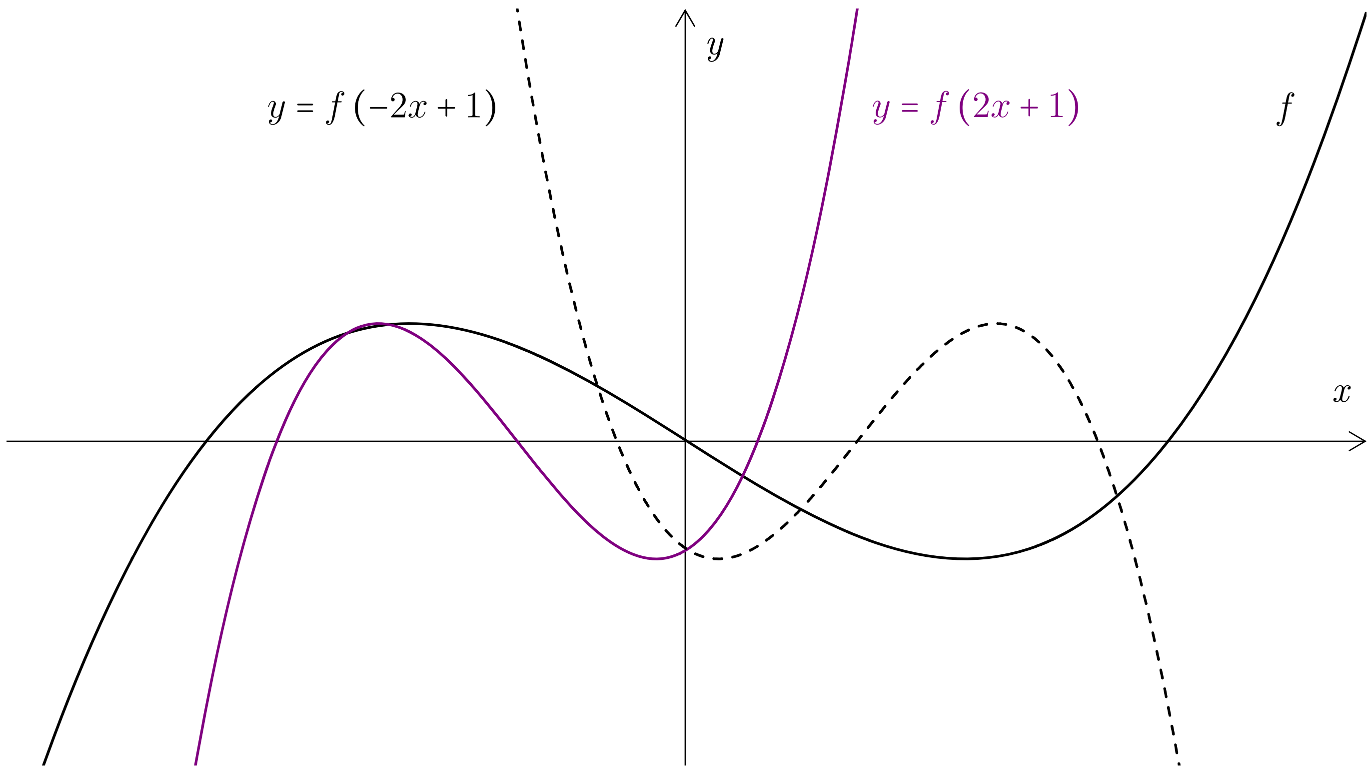
(f) We already graphed $y = f(2x) + 1$ in the above example. Reflect that in the y-axis to get $y = f(-2x) + 1$.



A133(g) We already graphed $y = f(2x+1)$ in the above example. Reflect that in the x-axis to get $y = -f(2x+1)$.



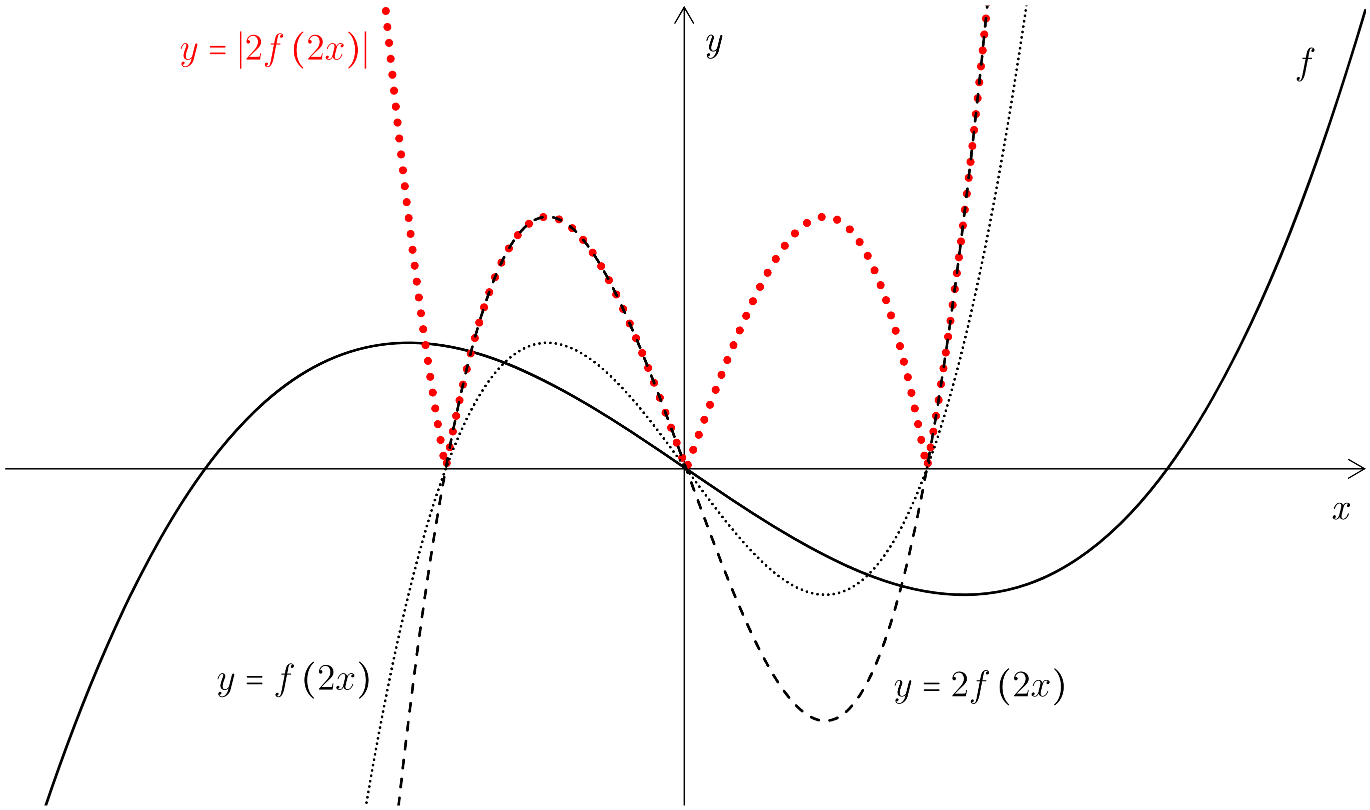
(h) We already graphed $y = f(2x+1)$ in the above example. Reflect that in the y-axis to get $y = f(-2x+1)$.



A134(a) First compress horizontally by a factor of 2 to get $y = f(2x)$.

Then stretch vertically by a factor of 2 to get $y = 2f(2x)$.

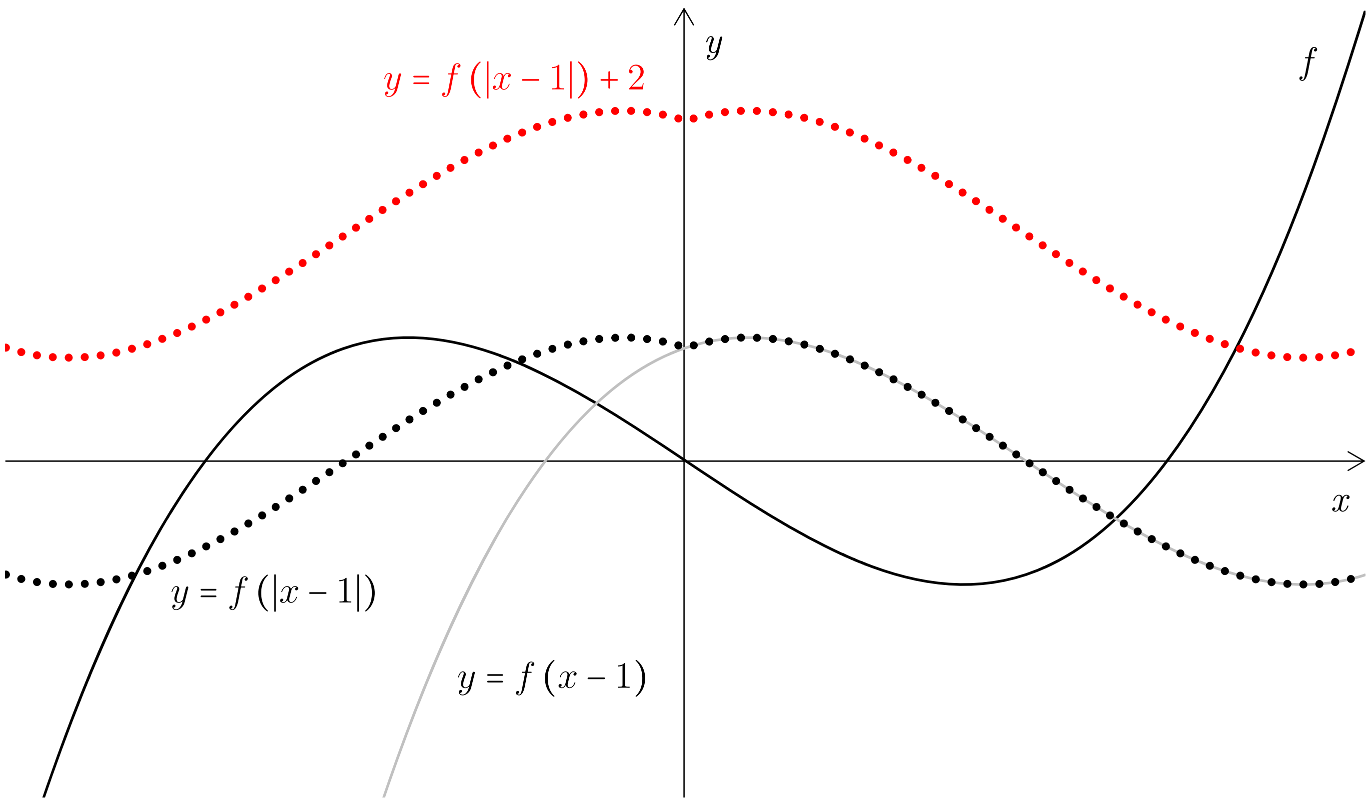
Now reflect the portion below the x-axis in the x-axis to get $y = |2f(2x)|$.



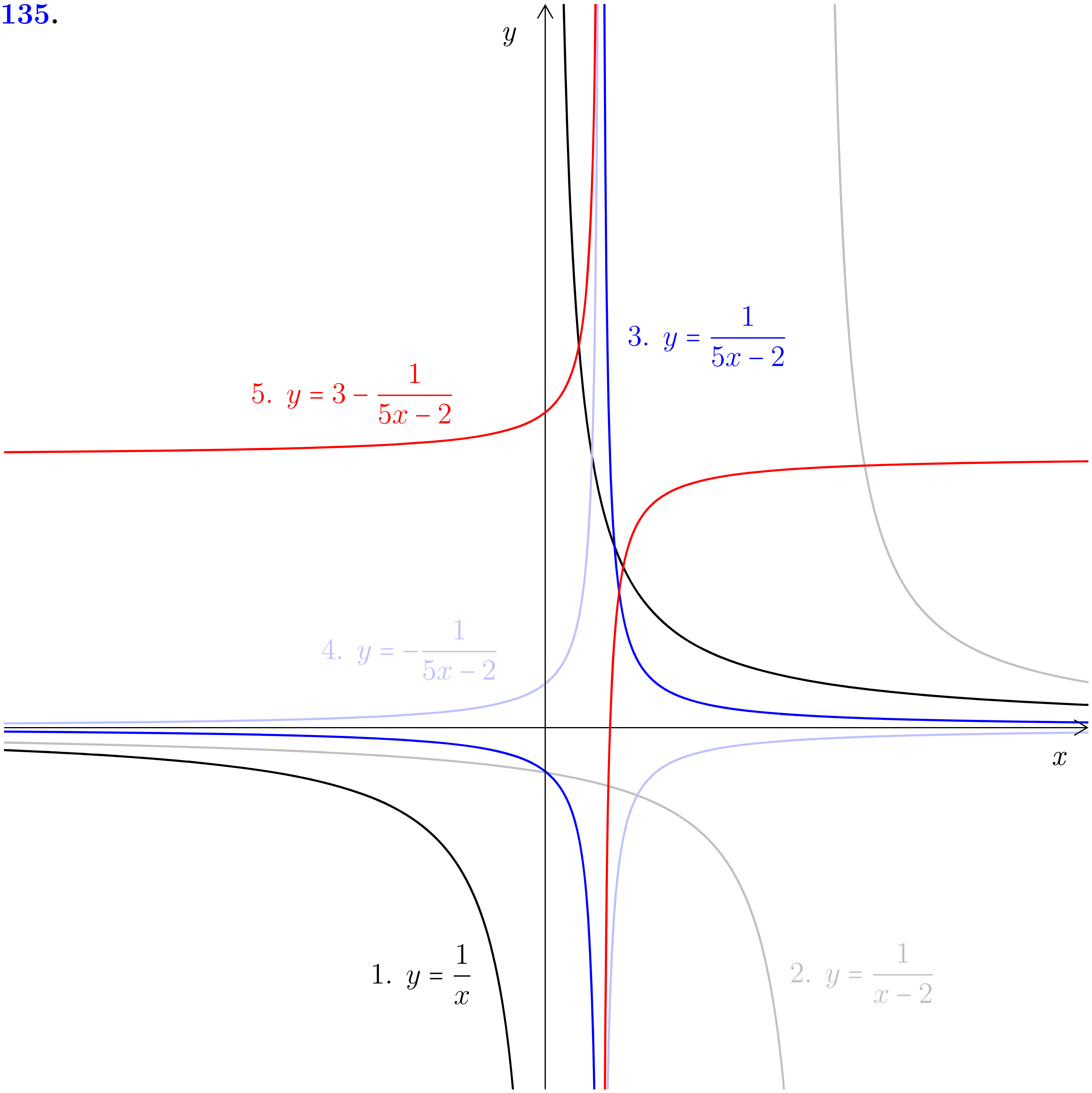
(b) First translate 1 unit rightwards to get $y = f(x-1)$.

Next reflect the right portion in the y-axis to get the left portion of $y = f(|x-1|)$.

Now translate 2 units upwards to get $y = f(|x-1|) + 2$.



A135.



1. Start with the graph of $y = 1/x$.
2. Translate rightwards by 2 units to get $y = 1/(x-2)$.
3. Compress horizontally by a factor of 5 to get $y = 1/(5x-2)$.
4. Reflect in the x-axis to get $y = -1/(5x-2)$.
5. Translate upwards by 3 units to get $y = 3 - 1/(5x-2)$.

149.18. Ch. 28 Answers (ln, exp, and e)

(*This chapter had no exercises.*)

149.19. Ch. 43 Answers (O-Level Review: The Derivative)

A184. f is both continuous everywhere and differentiable everywhere. ■ g is continuous everywhere but not differentiable everywhere. But it is differentiable everywhere *except* at $x = 0$. ■ h is neither continuous everywhere nor differentiable everywhere. But it is continuous *and* differentiable everywhere *except* at $x = 0$.

A185(a)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x, \text{ so } \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 2(0) = 0.$$

(b)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 15x^4 - 8x + 7, \text{ so } \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 3\cdot 0 - 8\cdot 0 + 7 = 7.$$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 3x + 4\right) = 2x + 3$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^5 - 4x^2 + 7x - 2\right) = 15x^4 - 8x + 7$. Thus,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &\stackrel{\times}{=} (2x+3)\left(3x^5 - 4x^2 + 7x - 2\right) + \left(x^2 + 3x + 4\right)\left(15x^4 - 8x + 7\right)\\ &= 21x^6 + 54x^5 + 60x^4 - 16x^3 - 15x^2 + 6x + 22.\end{aligned}$$

And so,
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 22.$$

A186. Use the product rule for **(a)** and **(b)**; and the quotient rule for **(c)**–**(e)**.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$ and $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$. Thus, $\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{\times}{=} \mathrm{e}^x\frac{1}{x} + \mathrm{e}^x \ln x = \mathrm{e}^x\left(\frac{1}{x} + \ln x\right)$.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x$ and $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x \ln x = \mathrm{e}^x\left(\frac{1}{x} + \ln x\right)$. Thus,
$$\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{\times}{=} x^2\mathrm{e}^x\left(\frac{1}{x} + \ln x\right) + 2x\mathrm{e}^x \ln x = x\mathrm{e}^x\left(1 + 2\ln x + x\ln x\right).$$

(c)
$$\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{\div}{=} \frac{x\frac{d\sin x}{dx} - \sin x\frac{dx}{dx}}{x^2} = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x} - \frac{\sin x}{x^2}.$$

(d)
$$\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{\div}{=} \frac{\cos x\frac{d\sin x}{dx} - \sin x\frac{d\cos x}{dx}}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

(e) $\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{\div}{=} \frac{z\frac{\mathrm{d}}{\mathrm{d}x}1 - 1\frac{\mathrm{d}z}{\mathrm{d}x}}{z^2} = -\frac{\mathrm{d}z/\mathrm{d}x}{z^2}$. By the way, this is called the **Reciprocal Rule**.

(f)
$$\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{\text{(e)}}{=} -\frac{\mathrm{d}\sin x/\mathrm{d}x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}.$$

(g)
$$\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{\text{(e)}}{=} -\frac{\mathrm{d}\cos x/\mathrm{d}x}{\cos^2 x} = -\frac{-\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}.$$

(h):
$$\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{\text{(e)}}{=} -\frac{\mathrm{d}\tan x/\mathrm{d}x}{\tan^2 x} = -\frac{1/\cos^2 x}{\sin^2 x/\cos^2 x} = -\frac{1}{\sin^2 x}.$$

A187(a)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}1 + \frac{d\left[x-\ln(x+1)\right]^2}{dx}\\
&\overset{\text{Ch}}{=} 0 + \frac{d\left[x-\ln(x+1)\right]^2}{d\left[x-\ln(x+1)\right]}\frac{d\left[x-\ln(x+1)\right]}{dx}\\
&= 2\left[x-\ln(x+1)\right]\left[\frac{dx}{dx}-\frac{d\ln(x+1)}{dx}\right]\\
&\overset{\text{Ch}}{=} 2\left[x-\ln(x+1)\right]\left[1-\frac{d\ln(x+1)}{d(x+1)}\frac{d(x+1)}{x}\right]\\
&= 2\left[x-\ln(x+1)\right]\left(1-\frac{1}{x+1}\times 1\right)\\
&= 2\left[x-\ln(x+1)\right]\frac{x}{x+1} = \left[x-\ln(x+1)\right]\frac{2x}{x+1}.\end{aligned}$$

So $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 2\left[0-\ln(0+1)\right]\frac{0}{0+1} = 2\left[0-0\right]\times 0 = 0.$

(b) Let $z = 1+\left[x-\ln(x+1)\right]^2$. Then,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{d\sin\frac{x}{z}}{dx} \overset{\text{Ch}}{=} \frac{d\sin\frac{x}{z}}{d\frac{x}{z}}\frac{d\frac{x}{z}}{dx} = \cos\frac{x}{z}\frac{d\frac{x}{z}}{dx} \overset{\div}{=} \cos\frac{x}{z}\frac{z\frac{dx}{dx}-x\frac{\mathrm{d}z}{\mathrm{d}x}}{z^2}\\
&\overset{(a)}{=} \cos\frac{x}{z}\frac{z-x\left\{\left[x-\ln(x+1)\right]\frac{2x}{x+1}\right\}}{z^2}\\
&= \left(\cos\frac{x}{1+\left[x-\ln(x+1)\right]^2}\right)\frac{1+\left[x-\ln(x+1)\right]^2-x\left\{\left[x-\ln(x+1)\right]\frac{2x}{x+1}\right\}}{\left\{1+\left[x-\ln(x+1)\right]^2\right\}^2},\end{aligned}$$

where in the last step we simply plugged in $z = 1+\left[x-\ln(x+1)\right]^2$. (We could do a little more algebra to simplify a little further, but we wouldn't get far.)

Observe that $$z\big|_{x=0} = 1+\left[0-\ln(0+1)\right]^2 = 1.$$

Thus, $$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = \cos\frac{0}{1}\frac{1-0\cdot 0}{1^2} = 1.$$

A188(a) $\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})$.

(b) By definition, $$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}.$$

If mass is constant (i.e. mass is not changing over time), then $\frac{\mathrm{d}m}{\mathrm{d}t} = 0$.

Altogether, $$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) \overset{\times}{=} \mathbf{v}\frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0 + m\mathbf{a} = m\mathbf{a}.$$

A189(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(\exp x) \stackrel{1}{=} \dfrac{1}{\exp x}\cdot\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\exp x\right)$.

(b) Since exp is defined to be the inverse of ln, we have $\ln(\exp x) = x$. Thus, $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(\exp x) = \dfrac{\mathrm{d}}{\mathrm{d}x}x = 1$.

(c) Putting our answers in (a) and (b) together, we have

$$\frac{1}{\exp x}\cdot\left(\frac{\mathrm{d}}{\mathrm{d}x}\exp x\right) = 1 \qquad \text{or} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\exp x = \exp x.$$

A190(a) The only turning point is $(0,1)$.

(b) The only turning point is $(0,1)$.

(c) For every $k \in \mathbb{Z}$, the points $(2k\pi, 1)$ and $((2k+1)\pi, -1)$ are turning points. (Hence, there are infinitely many turning points.)

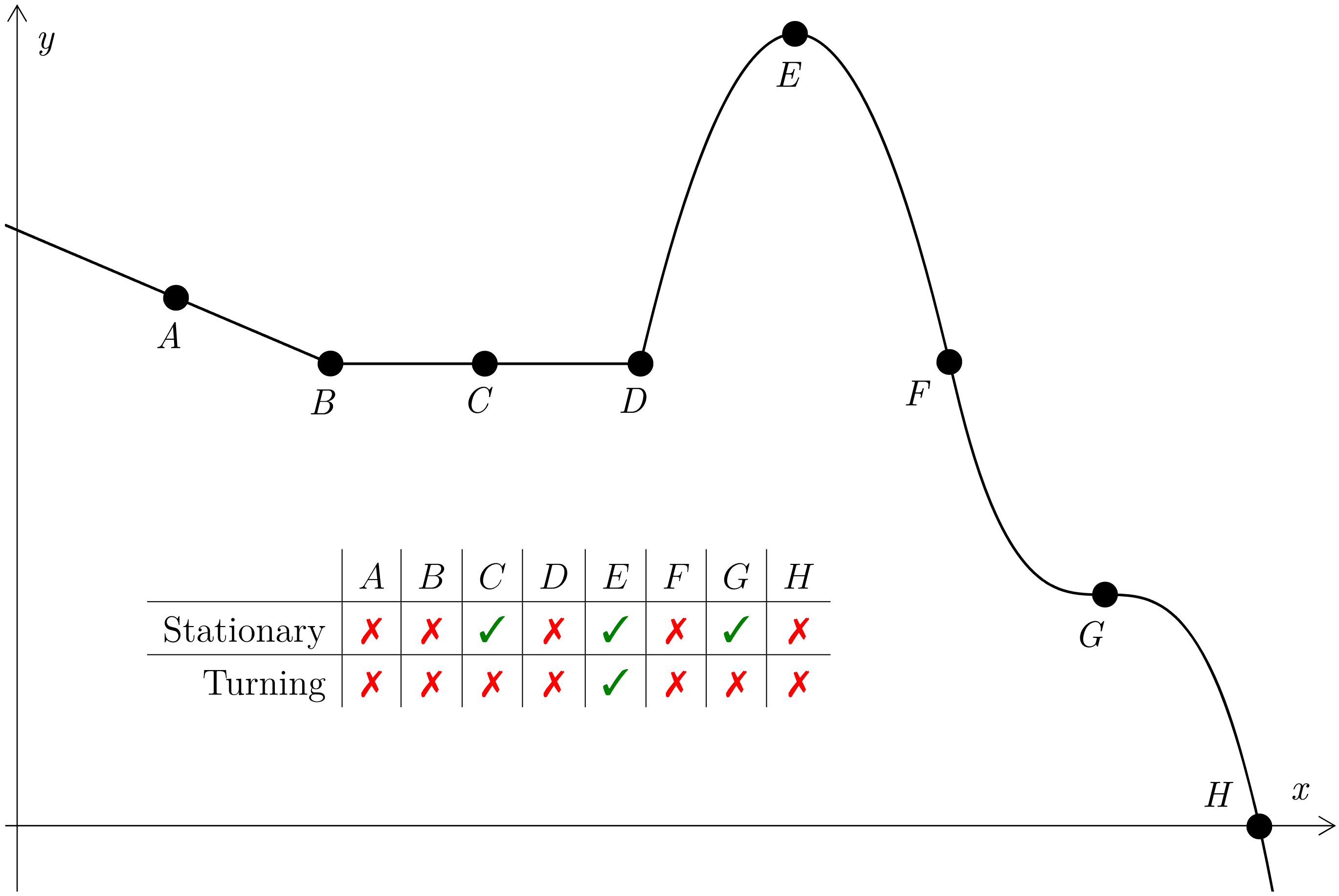
(d) The only turning point is $(0,1)$.

A191. At A, F, and H, the graph is decreasing. So A, F, and H are not stationary points. Thus, they cannot be turning points either.

B and D are "kinks" at which the derivative doesn't exist. So B and D cannot be stationary points. They are thus not turning points either.

At C and G, the derivative is zero. Thus, C and G are stationary points. However, they are not turning points (because the derivative isn't changing from negative to positive or positive to negative).

At E, the derivative is zero and changing from positive to negative. Thus, E is both a stationary and a turning point.



	A	B	C	D	E	F	G	H
Stationary	✗	✗	✓	✗	✓	✗	✓	✗
Turning	✗	✗	✗	✗	✓	✗	✗	✗

A192. For convenience, we reproduce the graph of f from Example 659:

Figure to be
inserted here.

(a) False. Points A and E (in the above graph) are a maximum and a minimum point, but not stationary points.

(b) False. Again, points A and E are a maximum and a minimum point, but not turning points.

(c) False. Point D is a stationary point, but is neither a maximum point nor minimum point.

(d) By Definition 58, it is true that every turning point is an extremum and therefore either a maximum or a minimum point.

(e) By Definition 58, it is true that every turning point is a stationary point.

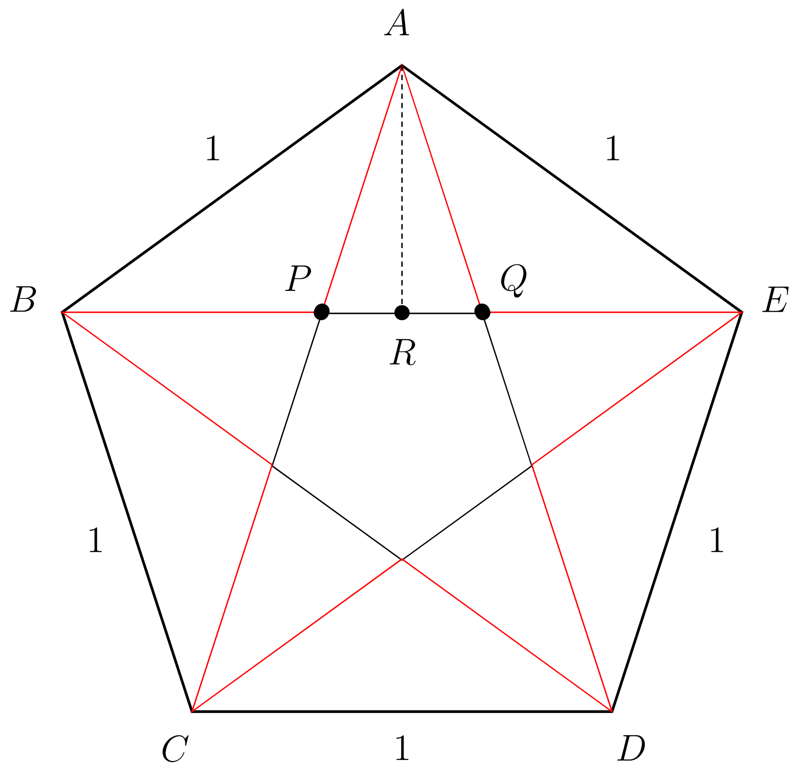
(f) False. Point D is a stationary point, but is not a turning point.

A137(a)(i) π. **(a)(ii)** $(n-2)\pi$. **(b)(i)** $(5-2)\pi = 3\pi$.

(b)(ii) By symmetry, the pentagon's five interior angles are equal. Hence, $\angle BAE = 3\pi/5$.

(b)(iii) Since $\triangle ABE$ is isosceles, $\angle ABE = (\pi - \angle BAE)/2 = \pi/5$.

(b)(iv) $\cos\dfrac{\pi}{5} = \cos\angle ABE = \dfrac{|BR|}{|AB|} = \dfrac{|BE|/2}{1} = \dfrac{|BE|}{2}$.



(c)(i) $\angle PAQ = \angle BAE - (\angle BAP + \angle QAE) = \dfrac{3\pi}{5} - \left(\dfrac{\pi}{5} + \dfrac{\pi}{5}\right) = \dfrac{\pi}{5}$.

(c)(ii) Both triangles already share the angle $\angle AQP$. In addition, $\angle ABQ = \dfrac{\pi}{5} = \angle PAQ$. Since the two triangles share two equal angles, they are similar.

(c)(iii) Since $\triangle APQ$ is isosceles, so too is $\triangle ABQ$. Hence, $|BQ| = |AB| = 1$.

(c)(iv) Since $\triangle ABQ$ and $\triangle APQ$ are similar, $\dfrac{|AB|}{|AQ|} \overset{\star}{=} \dfrac{|AQ|}{|PQ|}$.

Plug in $|AB| = 1$, $|AQ| = x$, and $|PQ| = |BQ| - |BP| = 1 - x$ to get $\dfrac{1}{x} \overset{\circ}{=} \dfrac{x}{1-x}$.

(c)(v) Rearrange $\overset{\circ}{=}$ to get $1 - x = x^2$ or $x^2 + x - 1 = 0$. Solving,

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $x > 0$, we can discard the negative value and conclude: $x = \left(-1 + \sqrt{5}\right)/2$.

(c)(vi) $|BE| = |BQ| + |QE| = 1 + x = \dfrac{1+\sqrt{5}}{2}$.

(c)(vii) From **(b)(iv)**, $\cos\frac{\pi}{5} = \frac{|BE|}{2} = \frac{1+\sqrt{5}}{4}$.

(d) $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4} = \frac{|BR|}{|AB|} = |BR|$.

Hence, $\sin\frac{\pi}{5} = \frac{|AR|}{|AB|} = \frac{\sqrt{|AB|^2 - |BR|^2}}{1} = \sqrt{1 - \left(\frac{1+\sqrt{5}}{4}\right)^2} = \sqrt{1 - \frac{1+5+2\sqrt{5}}{16}} = \sqrt{\frac{10-2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$.

A138(a) By the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$. Rearranging, $\cos C = \frac{a^2+b^2-c^2}{2ab}$.

(b) Recall that $(a+b)^2 \overset{1}{=} a^2 + b^2 + 2ab$ and $(a-b)^2 \overset{2}{=} a^2 + b^2 - 2ab$.

Now, $1 + \cos C = 1 + \frac{a^2+b^2-c^2}{2ab} = \frac{2ab + a^2 + b^2 - c^2}{2ab} \overset{1}{=} \frac{(a+b)^2 - c^2}{2ab}$.

And, $1 - \cos C = 1 - \frac{a^2+b^2-c^2}{2ab} = \frac{2ab - a^2 - b^2 + c^2}{2ab} \overset{2}{=} \frac{c^2 - (a-b)^2}{2ab}$.

(c) Recall that $x^2 - y^2 = (x+y)(x-y)$.

So, $(a+b)^2 - c^2 = (a+b+c)(a+b-c) = 2s(2s+2c) = 4s(s+c)$.

And, $c^2 - (a-b)^2 = (c+a-b)(c-a+b) = (2s-2b)(2s-2a) = 4(s-b)(s-a)$.

(d) By the First Pythagorean Identity, $\sin^2 C + \cos^2 C = 1$ or $\sin^2 C = 1 - \cos^2 C$ or $\sin C = \pm\sqrt{1-\cos^2 C}$.

Since $C \in [0, \pi]$, we have $\sin C \geq 0$. So, we can discard the negative value and conclude $\sin C = \sqrt{1-\cos^2 C}$.

(e)
$$\begin{aligned}
\text{Area} &= \frac{1}{2}ab\sin C \overset{\textbf{(d)}}{=} \frac{1}{2}ab\sqrt{1-\cos^2 C} = \frac{1}{2}ab\sqrt{(1+\cos C)(1-\cos C)} \\
&\overset{\textbf{(b)}}{=} \frac{1}{2}ab\sqrt{\frac{(a+b)^2-c^2}{2ab}\frac{c^2-(a-b)^2}{2ab}} = \frac{1}{2}ab\frac{1}{2ab}\sqrt{\left[(a+b)^2-c^2\right]\left[c^2-(a-b)^2\right]} \\
&= \frac{1}{4}\sqrt{4s(s+c)4(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(s-c)}.
\end{aligned}$$

A139. Since $\angle QPR = A - B$ and $|PR| = 1$, we have

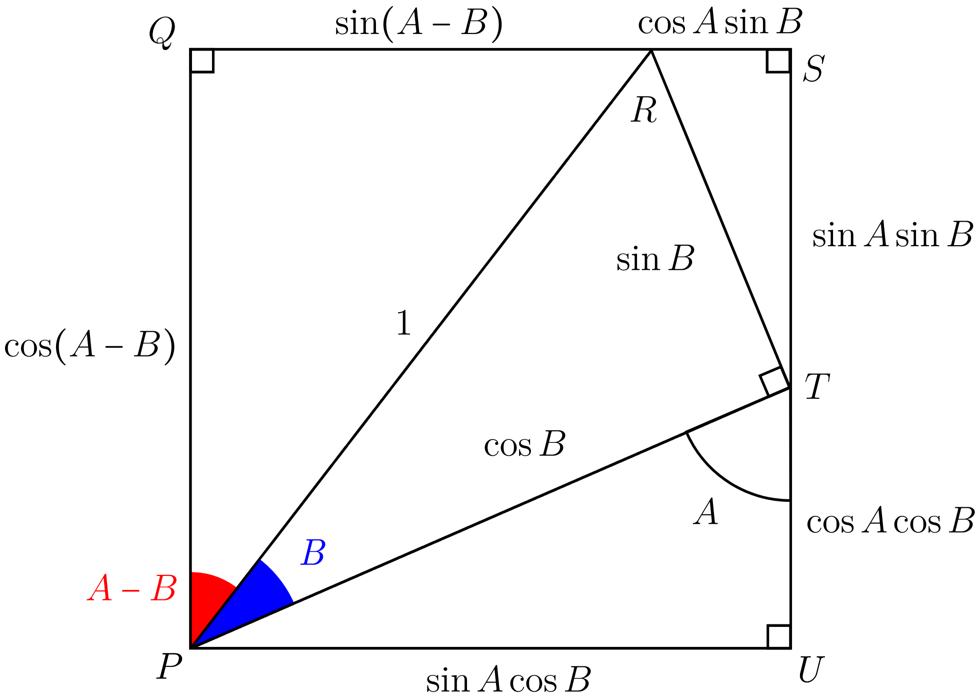
$$|QR| = \sin(A-B) \qquad \text{and} \qquad |PQ| = \cos(A-B).$$

1. $\angle PTU$ and $\angle QPT$ are alternate; hence, $\angle PTU = \angle QPT = A$. Moreover, $|PT| = \cos B$.

   Thus, $$|PU| = \sin A\cos B \text{ and } |TU| = \cos A\cos B.$$

2. $\angle SRT$ is complementary to $\angle STR$, which is in turn complementary to $\angle PTU$. Hence, $\angle SRT = \angle PTU = A$. Moreover, $|RT| = \sin B$.

   Thus, $$|ST| = \sin A\sin B \text{ and } |RS| = \cos A\sin B.$$



We now have $\quad PQ = UT + TS$ or $\cos(A-B) = \sin A \sin B + \cos A \cos B$.

And, $\quad QR = PU - RS$ or $\sin(A-B) = \sin A \cos B - \cos A \sin B$.

A140(a) $\sin 2A = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A$.

(b) $\cos 2A = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A = \cos^2 A - \left(1 - \cos^2 A\right) = 2\cos^2 A - 1$.

Also, $\cos 2A = \cos^2 A - \sin^2 A = \left(1 - \sin^2 A\right) - \sin^2 A = 1 - 2\sin^2 A$.

A141(a)

$$\begin{aligned}
\sin 3A &= \sin (A + 2A) \\
&= \sin A \cos 2A + \cos A \sin 2A && \text{(Addition Formula)} \\
&= \sin A \left(1 - 2\sin^2 A\right) + \cos A (2 \sin A \cos A) && \text{(Double-Angle Formulae)} \\
&= \sin A - 2\sin^3 A + 2 \sin A \cos^2 A \\
&= \sin A - 2\sin^3 A + 2 \sin A \left(1 - \sin^2 A\right) && \left(\sin^2 A + \cos^2 A = 1\right) \\
&= 3 \sin A - 4 \sin^3 A.
\end{aligned}$$

(b)

$$\begin{aligned}
\cos 3A &= \cos (A + 2A) \\
&= \cos A \cos 2A - \sin A \sin 2A && \text{(Addition Formula)} \\
&= \cos A \left(2\cos^2 A - 1\right) + \sin A (2 \sin A \cos A) && \text{(Double-Angle Formulae)} \\
&= 2\cos^3 A - \cos A + 2 \sin^2 A \cos A \\
&= 2\cos^3 A - \cos A + 2 \left(1 - \cos^2 A\right) \cos A && \left(\sin^2 A + \cos^2 A = 1\right) \\
&= 4 \cos^3 A - 3 \cos A.
\end{aligned}$$

A142. Take the hint and apply the Addition and Subtraction Formulae:

$$\sin P = \sin \left(\frac{P+Q}{2} + \frac{P-Q}{2}\right) = \sin \frac{P+Q}{2} \cos \frac{P-Q}{2} + \cos \frac{P+Q}{2} \sin \frac{P-Q}{2},$$

$$\sin Q = \sin \left(\frac{P+Q}{2} - \frac{P-Q}{2}\right) = \sin \frac{P+Q}{2} \cos \frac{P-Q}{2} - \cos \frac{P+Q}{2} \sin \frac{P-Q}{2},$$

$$\cos P = \cos \left(\frac{P+Q}{2} + \frac{P-Q}{2}\right) = \cos \frac{P+Q}{2} \cos \frac{P-Q}{2} - \sin \frac{P+Q}{2} \sin \frac{P-Q}{2},$$

$$\cos Q = \cos \left(\frac{P+Q}{2} - \frac{P-Q}{2}\right) = \cos \frac{P+Q}{2} \cos \frac{P-Q}{2} + \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}.$$

You can easily verify that the four S2P or P2S Formulae now follow.

A143. Let $2x = \dfrac{P+Q}{2}$ and $5x = \dfrac{P-Q}{2}$.

Then $$P = \frac{P+Q}{2} + \frac{P-Q}{2} = 2x + 5x = 7x;$$

And, $$Q = \frac{P+Q}{2} - \frac{P-Q}{2} = 2x - 5x = -3x.$$

And so, by the P2S Formulae, we have

(a) $$\sin 2x\cos 5x = \frac{\sin 7x + \sin(-3x)}{2} = \frac{\sin 7x - \sin 3x}{2}.$$

(b) $$\cos 2x\sin 5x = \frac{\sin 7x - \sin(-3x)}{2} = \frac{\sin 7x + \sin 3x}{2}.$$

(c) $$\cos 2x\cos 5x = \frac{\cos 7x + \cos(-3x)}{2} = \frac{\cos 7x + \cos 3x}{2}.$$

(d) $$\sin 2x\sin 5x = -\frac{\cos 7x - \cos(-3x)}{2} = \frac{\cos 3x - \cos 7x}{2}.$$

A144. Domain tan $= \mathbb{R} \setminus \{x : \cos x = 0\} = \mathbb{R} \setminus \{k\pi/2 : k$ is an odd integer$\}$.
Codomain tan $= \mathbb{R}$.

A145.
$$\begin{aligned}
\tan(A \pm B) &= \frac{\sin(A \pm B)}{\cos(A \pm B)} && \text{(By definition of tangent)} \\
&= \frac{\sin A\cos B \pm \cos A\sin B}{\cos A\cos B \mp \sin A\sin B} && \left(\begin{array}{c}\text{By Add. and Sub.} \\ \text{Form for sin and cos}\end{array}\right) \\
&= \frac{\sin A/\cos A \pm \sin B/\cos B}{1 \mp (\sin A\sin B)/(\cos A\cos B)} && \text{(Divide by } \cos A\cos B \neq 0\text{)} \\
&= \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} && \text{(By definition of tangent).}
\end{aligned}$$

A146. $\tan 2A = \dfrac{\tan A + \tan A}{1 - \tan A\tan A} = \dfrac{2\tan A}{1 - \tan^2 A}$.

A147.

$$\begin{aligned}
\tan 3A &= \tan(A + 2A) \\
&= \frac{\tan A + \tan 2A}{1 - \tan A\tan 2A} && \text{(Addition Formula)} \\
&= \frac{\tan A + 2\tan A/\left(1 - \tan^2 A\right)}{1 - \tan A\left[2\tan A/\left(1 - \tan^2 A\right)\right]} && \text{(Double-Angle Formula)} \\
&= \frac{\tan A\left(1 - \tan^2 A\right) + 2\tan A}{1\left(1 - \tan^2 A\right) - \tan A(2\tan A)} \\
&= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}. && \text{(Multiply by } 1 - \tan^2 A\text{)}
\end{aligned}$$

A148(a) $\dfrac{a+b}{a-b} = \dfrac{a+b}{a-b} \times \dfrac{\sin A\sin B}{\sin A\sin B} = \dfrac{a\sin A\sin B + b\sin A\sin B}{a\sin A\sin B - b\sin A\sin B} \overset{1}{=} \dfrac{a\sin A\sin B + a\sin B\sin B}{a\sin A\sin B - a\sin B\sin B} = \dfrac{\sin A + \sin B}{\sin A - \sin B}$.

(b) By Fact 75 (S2P or P2S Formulae),

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}} = \frac{\sin\frac{A+B}{2}/\cos\frac{A+B}{2}}{\sin\frac{A-B}{2}/\cos\frac{A-B}{2}} = \frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}$$

(As stated on p. 344, "Whenever you see a question with trigonometric functions, put MF26 (p. 3) next to you!")

A149(c)(i) $\dfrac{\sin x}{1+\cos x} = \dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1} = \dfrac{\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}} = \dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\dfrac{x}{2}.$

(c)(ii) $\dfrac{1-\cos x}{\sin x} = \dfrac{1-\left(1-2\sin^2\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \dfrac{\sin^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}} = \dfrac{\sin^2\frac{x}{2}}{\cos\frac{x}{2}} = \tan\dfrac{x}{2}.$

(c)(iii) $\tan^2\dfrac{x}{2} = \dfrac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \dfrac{1-\cos x}{1+\cos x}$. So, $\tan\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{1+\cos x}}$. By Corollary 13,

$$\tan\frac{x}{2} \geq 0 \text{ if } \frac{x}{2} \in \left[0,\frac{\pi}{2}\right) \qquad \text{and} \qquad \tan\frac{x}{2} \leq 0 \text{ if } \frac{x}{2} \in \left(-\frac{\pi}{2},0\right].$$

Thus,
$$\tan\frac{x}{2} = \begin{cases} \sqrt{\dfrac{1-\cos x}{1+\cos x}}, & \text{for } \dfrac{x}{2} \in \left[0,\dfrac{\pi}{2}\right), \\ -\sqrt{\dfrac{1-\cos x}{1+\cos x}}, & \text{for } \dfrac{x}{2} \in \left(-\dfrac{\pi}{2},0\right]. \end{cases}$$

(Note: Since $x/2$ is not an odd integer multiple of $\pi/2$, $1+\cos x \neq 0$, so that there is no danger that the denominator in the surd equals zero.)

A154. The sole error is in Step 6.

Since $x \in [-1,1)$, we have $A = \cos^{-1} x \in (0,\pi]$ and hence $-A \in [-\pi,0)$. So, by Fact 99(b), $\cos^{-1} x = \cos^{-1}(\cos(-A)) \neq -A$ and Step 6 is incorrect.

149.21. Ch. 38 Answers (Factorising Polynomials)

A156(a) In the expression $\frac{16x+3}{5x-2}$, the **dividend** is $16x+3$ and the **divisor** is $5x-2$.

$$
\begin{array}{r|rr}
\textbf{Terms:} & x^1 & x^0 \\
 & & 3.2 \\ \cline{2-3}
5x-2 & 16x & +3 \\
 & 16x & -6.4 \\ \cline{2-3}
 & & 9.4
\end{array}
$$

The **quotient** is 3.2 and 9.4 is the **remainder**. We have

$$\frac{16x+3}{5x-2} = 3.2 + \frac{9.4}{5x-2}.$$

(b) In the expression $\frac{4x^2-3x+1}{x+5}$, the **dividend** is $4x^2-3x+1$ and the **divisor** is $x+5$. Long division:

$$
\begin{array}{r|rrr}
\textbf{Terms:} & x^2 & x^1 & x^0 \\
 & & 4x & -23 \\ \cline{2-4}
x+5 & 4x^2 & -3x & +1 \\
 & 4x^2 & +20x & \\ \cline{3-4}
 & & -23x & +1 \\
 & & -23x & -115 \\ \cline{3-4}
 & & & 116
\end{array}
$$

The **quotient** is $4x-23$ and 116 is the **remainder**. We have

$$\frac{4x^2-3x+1}{x+5} = 4x-23 + \frac{116}{x+5}.$$

(c) In the expression $\frac{x^2+x+3}{-x^2-2x+1}$, the **dividend** is x^2+x+3 and the **divisor** is $-x^2-2x+1$. Long division:

$$
\begin{array}{r|rrr}
\textbf{Terms:} & x^2 & x^1 & x^0 \\
 & & & -1 \\ \cline{2-4}
-x^2-2x+1 & x^2 & +x & +3 \\
 & x^2 & +2x & -1 \\ \cline{3-4}
 & & -x & +4
\end{array}
$$

The **quotient** is -1 and $-x+4$ is the **remainder**. We have

$$\frac{x^2+x+3}{-x^2-2x+1} = -1 + \frac{-x+4}{-x^2-2x+1}.$$

A159(a) $\left(2x^3+7x^2-3x+5\right)\div(x-3)$ leaves $2\cdot3^3+7\cdot3^2-3\cdot3+5=113$.

(b) $\left(-2x^4+3x^2-7x-1\right)\div(x+2)$ leaves $-2\cdot(-2)^4+3\cdot(-2)^2-7\cdot(-2)-1=-7$.

A160(a) I'll use the SSGACM:

$$(2x-3)(x+1)=2x^2-x-3. \text{ ✗}$$

Aiyah, sian. Now try again, but switch −3 and 1:

$$(2x+1)(x-3)=2x^2-5x-3. \text{ ✓}$$

Yay! Done!

(b) I'll use the quadratic formula. We have $b^2-4ac=(-19)^2-4(7)(-6)=361+168=529>0$ and $\sqrt{529}=23$. Thus,

$$7x^2-19x-6=7\left(x-\frac{19-23}{14}\right)\left(x-\frac{19+23}{14}\right)=7\left(x+\frac{2}{7}\right)(x-3)=(7x+2)(x-3).$$

(c) I'll use the SSGACM: $(2x+1)(3x-1)=6x^2+x-1$. ✓ Yay! Done!

(d) I'll start by using the FTGACM. Since the constant term is $-14=-2\times7$, let's try plugging in 2:

$$p(2)=2\cdot2^3-2^2-17\cdot2-14<0. \text{ ✗}$$

Aiyah, sian. Doesn't work—by the FT, $x-2$ is not a factor for $2x^3-x^2-17x-14$.

Let's instead try −2:

$$p(-2)=2\cdot(-2)^3-(-2)^2-17\cdot(-2)-14=-16-4+34-14=0. \text{ ✓}$$

Yay, works! By the FT, $x+2$ is a factor for $2x^3-x^2-17x-14$.

Now, as usual, write $2x^3-x^2-17x-14=(x+2)\left(ax^2+bx+c\right)$.

The coefficients on the cubed and constant terms are $a=2$ and $2c=-14$. And so, $c=-7$. To find b, look at the coefficients on the squared term, which are $2a+b=-1$ and so $b=-5$. Thus, $ax^2+bx+c=2x^2-5x-7$.

To factorise $2x^2-5x-7$, I'll use the SSGACM:

$$(2x-7)(x+1)=2x^2-5x-7. \text{ ✓}$$

(*Wah*! So "lucky"! Success on the very first try!)

Altogether, we have $2x^3-x^2-17x-14=(x+2)(2x-7)(x+1)$.

A163**(a)** By **(i)** and the RT, $p(1) = a + b - 31 + 3 + 3 = a + b - 25 = 5$. And so, $b \stackrel{1}{=} 30 - a$.

By **(ii)**:
$$0 = p\left(\frac{1}{2}\right) = \frac{a}{16} + \frac{b}{8} - \frac{31}{4} + \frac{3}{2} + 3$$
$$= \frac{a}{16} + \frac{b}{8} - \frac{13}{4} \stackrel{1}{=} \frac{a}{16} + \frac{30-a}{8} - \frac{13}{4}$$
$$= \frac{60-a}{16} - \frac{13}{4} \stackrel{\times 16}{=} 60 - a - 52 = 8 - a.$$

Thus, $a = 8$ and $b = 22$. And we have
$$p(x) = 8x^4 + 22x^3 - 31x^2 + 3x + 3.$$

(b) Observe that $p(0) > 0$. Given also **(iii)** $p(-1/3) < 0$, the IVT says there *must* be some $-1/3 < c < 0$ such that $p(c) = 0$.

So, let's try the FTGACM, by plugging in $-1/4$:
$$8\left(-\frac{1}{4}\right)^4 + 22\left(-\frac{1}{4}\right)^3 - 31\left(-\frac{1}{4}\right)^2 + 3\left(-\frac{1}{4}\right) + 3 = 0. \checkmark$$

Yay, works! By the FT, $x + 1/4$ or $4(x + 1/4) = 4x + 1$ is a factor of $p(x)$.

From **(ii)**, we also already knew that $x - 1/2$ or $2(x - 1/2) = 2x - 1$ is a factor of $p(x)$.

So write
$$\begin{aligned} p(x) &= 8x^4 + 22x^3 - 31x^2 + 3x + 3 \\ &= (2x-1)(4x+1)\left(dx^2 + ex + f\right) \\ &= \left(8x^2 - 2x - 1\right)\left(dx^2 + ex + f\right). \end{aligned}$$

The coefficients on the 4th-degree and constant terms are $8d = 8$ and $-f = 3$. And so, $d = 1$ and $f = -3$. To find e, look at the coefficients on the linear term, which are $-2f - e = 3$. And so, $e = -2f - 3 = 3$. Thus, $dx^2 + ex + f = x^2 + 3x - 3$.

To factorise this last quadratic polynomial, we observe that $b^2 - 4ac = 3^2 - 4(1)(-3) = 21 > 0$. And so, by the quadratic formula,
$$x^2 + 3x - 3 = \left(x - \frac{-3-\sqrt{21}}{2}\right)\left(x - \frac{-3+\sqrt{21}}{2}\right) = \left(x + \frac{3+\sqrt{21}}{2}\right)\left(x + \frac{3-\sqrt{21}}{2}\right).$$

Thus $p(x) = 8x^4 + 22x^3 - 31x^2 + 3x + 3 = (2x-1)(4x+1)\left(x + \frac{3+\sqrt{21}}{2}\right)\left(x + \frac{3-\sqrt{21}}{2}\right)$.

A165. Let A, B, and C be Apu, Beng, and Caleb's ages today. Let k be the number of years between today and the day Apu turned 40.

The second sentence says that $A - k \stackrel{1}{=} 40$ and $B - k \stackrel{2}{=} 2(C - k)$.

The third sentence says that $A \stackrel{3}{=} 2B$ and $C \stackrel{4}{=} 28$.

Plug $\stackrel{3}{=}$ into $\stackrel{1}{=}$ to get $2B - k \stackrel{5}{=} 40$.

Plug $\stackrel{4}{=}$ into $\stackrel{2}{=}$ to get $B - k \stackrel{6}{=} 2(28 - k) = 56 - 2k$ or $B + k \stackrel{6}{=} 56$.

Take $\stackrel{5}{=} + \stackrel{6}{=}$ to get $3B = 96$ or $B = 32$.

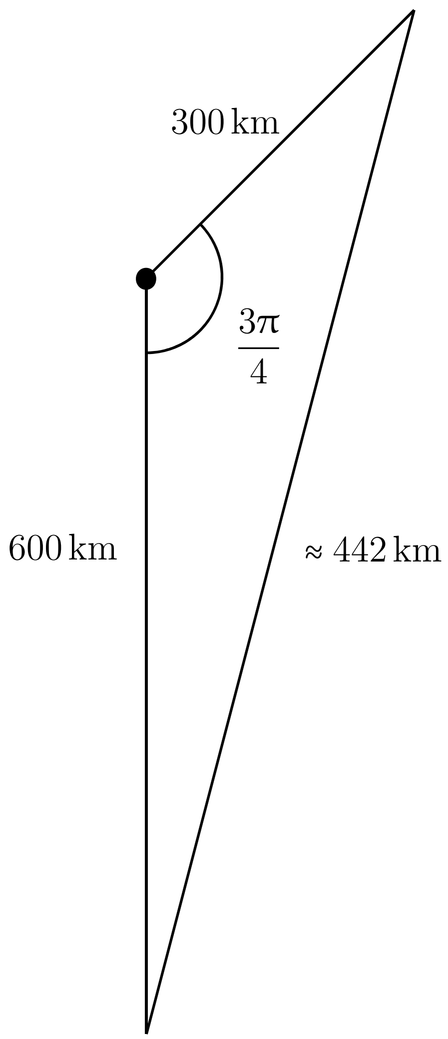
So, Beng turns 32 today. And from $\stackrel{3}{=}$, Apu turns 64 today.

A166. At the instant they turn, Plane A is 300 km northeast of the starting point, while Plane B is 600 km south of it. The angle formed by their flight paths is $3\pi/4$. So, by the Law of Cosines (Proposition 7), at this instant, the distance between the two planes is

$$\sqrt{300^2 + 600^2 - 2(300)(600)\cos(3\pi/4)} \approx 442\,\text{km}.$$

Thereafter, this distance shrinks at a rate of $100 + 200 = 300\,\text{km}\,\text{h}^{-1}$. So, they'll collide in another

$$\frac{442\,\text{km}}{300\,\text{km}\,\text{h}^{-1}} \approx 1.47\,\text{h}.$$



A168(a) From the given information, we have this system of equations:

$$\begin{aligned}2 &\stackrel{1}{=} a\cdot 1^2 + b\cdot 1 + c = a + b + c,\\ 5 &\stackrel{2}{=} a\cdot 3^2 + b\cdot 3 + c = 9a + 3b + c,\\ 9 &\stackrel{3}{=} a\cdot 6^2 + b\cdot 6 + c = 36a + 6b + c.\end{aligned}$$

Take $\stackrel{2}{=} - \stackrel{1}{=}$ to get $8a + 2b = 3$ or $b \stackrel{4}{=} 1.5 - 4a$.

Plug $\stackrel{4}{=}$ into $\stackrel{1}{=}$ to get $a + 1.5 - 4a + c = 2$ or $c \stackrel{5}{=} 3a + 0.5$.

Now plug $\stackrel{4}{=}$ and $\stackrel{5}{=}$ into $\stackrel{3}{=}$ to get $36a + 6(1.5 - 4a) + 3a + 0.5 = 9$ or $15a = -0.5$.

Hence, $a = -1/30$, $b = 49/30$, and $c = 0.4$.

(b) First, $2 \stackrel{1}{=} a + b + c$.

Next, recall that the strict global minimum point of a quadratic equation occurs at $x = -b/2a$. Hence, $b = 0$ and

$$y\Big|_{x=-b/2a} = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = \frac{b^2}{4a} - \frac{b^2}{2a} + c = c - \frac{b^2}{4a} = 0.$$

So, $c = 0$. Now from $\stackrel{1}{=}$, we also have $a = 2$.

A169(a) We'll use Method 2 from the last example—we'll rewrite the two equations as:

$$y = x^5 - x^3 + 2 - \frac{1}{1+\sqrt{x}},$$

then graph this equation on our GC. It looks like there are no x-intercepts. We thus conclude that this system of equations has no solutions and its solution set is $\varnothing$.

(b) Again, use Method 2 and rewrite the two equations as:

$$y = \frac{1}{1-x^2} - x^3 - \sin x,$$

then graph this equation on our GC. It looks like there is only one x-intercept. Using the "zero" function, we find that it's at $x \approx -1.179$.

Plug this value of x back into either of the original equations to get $y \approx -2.563$. We conclude that the (unique) solution is $\sim (-1.179, -2.563)$.

(*Answer continues below ...*)

(... *Answer continued from above.*)

A169(c) Recall that $x^2 + y^2 = 1$ describes the unit circle centred on the origin. Recall also that to get the TI84 to graph it, we must break it up into two equations:

$$y = \sqrt{1 - x^2} \text{ and } y = -\sqrt{1 - x^2}.$$

1. Enter the three equations $y = \sqrt{1 - x^2}$, $y = -\sqrt{1 - x^2}$, and $y = \sin x$.
2. Press GRAPH to graph the three equations.
3. Zoom in by pressing ZOOM, 2, then ENTER.

   It looks like $y = \sin x$ intersects the circle $x^2 + y^2 = 1$ at two points. To find what these two points are, we will use the "intersect" function.
4. Press 2ND, then CALC to bring up the CALCULATE menu. Then press 5 to select the "intersect" function.

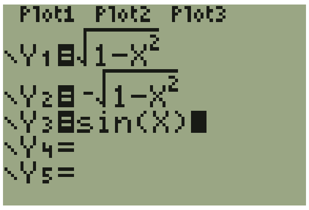
   As usual, the TI84 asks, "First curve?" I'll first look for the intersection that's to the right of the y-axis. So,
5. Press ENTER to confirm that we're selecting $y_1 = \sqrt{1 - x^2}$ as our first curve.

   The TI84 now asks, "Second curve?" For our second curve, we want to select $y_3 = \sin(x)$. To do so, simply:
6. Press the down arrow key once.
7. Now as usual, use the left and right arrow keys to move the cursor to approximately where we think the intersection point is. In my case, I've moved it to $(x, y) \approx (0.745, 0.678)$.
8. Now press ENTER. The TI84 now annoyingly now asks, "Guess?" So press ENTER once more to learn that the intersection point is $(x, y) \approx (0.739, 0.674)$.

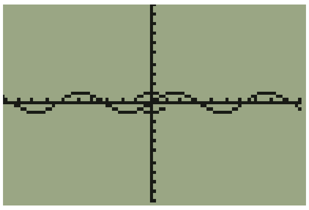
By repeating Steps 4–8 (and making the necessary changes), you should be able to find that the other intersection point is $(x, y) \approx (-0.739, -0.674)$. (Alternatively, you can save yourself some time by observing the symmetries here and immediately infer that this is the other intersection point.)

We conclude that this system of equations has two solutions: $(\pm 0.739\ldots, \pm 0.673\ldots)$.

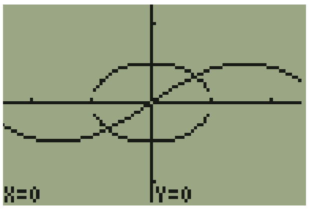
After Step 1.



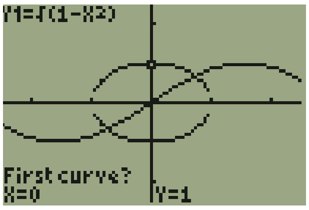
After Step 2.



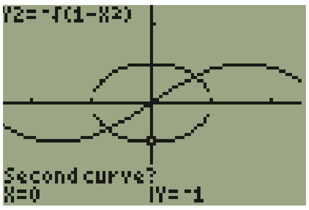
After Step 3.



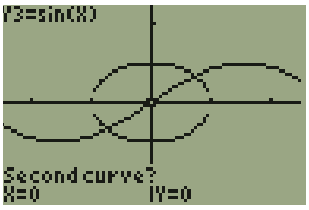
After Step 4.



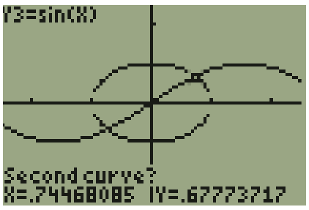
After Step 5.



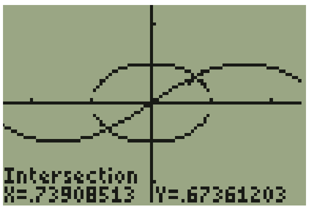
After Step 6.



After Step 7.



After Step 8.



149.23. Ch. 40 Answers (Partial Fractions Decomposition)

A170(a) Since $x^2+x-6=(x-2)(x+3)$, write

$$\frac{8}{x^2+x-6}=\frac{A}{x-2}+\frac{B}{x+3}=\frac{A(x+3)+B(x-2)}{(x-2)(x+3)}=\frac{(A+B)x+3A-2B}{(x-2)(x+3)}.$$

Comparing coefficients, $A+B=0$ and $3A-2B=8$. So, $A=8/5$ and $B=-8/5$.

Thus, $$\frac{8}{x^2+x-6}=\frac{8}{5(x-2)}-\frac{8}{5(x+3)}.$$

(b) Since $3x^2-8x-3=(3x+1)(x-3)$, write

$$\frac{17x-5}{3x^2-8x-3}=\frac{A}{3x+1}+\frac{B}{x-3}=\frac{A(x-3)+B(3x+1)}{(3x+1)(x-3)}=\frac{(A+3B)x-3A+B}{(3x+1)(x-3)}$$

Comparing coefficients, $A+3B\overset{1}{=}17$ and $-3A+B\overset{2}{=}-5$. From $3\times\overset{1}{=}+\overset{2}{=}$, we get $10B=46$ or $B=4.6$ and then also $A=3.2$.

Thus, $$\frac{17x-5}{3x^2-8x-3}=\frac{3.2}{3x+1}+\frac{4.6}{x-3}.$$

A171. To factorise the denominator x^3-x^2-x+1, try plugging in 1:

$$p(1)=1-1-1+1=0. \qquad \checkmark$$

Yay, works! By the Factor Theorem, $x-1$ is a factor for $p(x)$. So, write

$$x^3-x^2-x+1=(x-1)\left(ax^2+bx+c\right).$$

Comparing coefficients, $a=1$, $-c=1$, and $c-b=-1$. So, $c=-1$ and $b=0$. Hence, $ax^2+bx+c=x^2-1=(x+1)(x-1)$. Thus,

$$x^3-x^2-x+1=(x-1)(x+1)(x-1)=(x+1)(x-1)^2.$$

Now, write

$$\frac{2x^2-x+7}{x^3-x^2-x+1}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}=\frac{A(x-1)^2+B(x+1)(x-1)+C(x+1)}{(x+1)(x-1)^2}$$

$$=\frac{Ax^2-2Ax+A+Bx^2-B+Cx+C}{(x+1)(x-1)^2}=\frac{(A+B)x^2+(-2A+C)x+A-B+C}{(x+1)(x-1)^2}.$$

Comparing coefficients, $A+B=2$, $-2A+C=-1$, and $A-B+C=7$. Summing up these three equations, we get $2C=8$ or $C=4$ and then also $A=5/2$ and $B=-1/2$.

Thus, $$\frac{2x^2-x+7}{x^3-x^2-x+1}=\frac{5}{2(x+1)}-\frac{1}{2(x-1)}+\frac{4}{(x-1)^2}.$$

A172. To factorise the denominator x^3-x^2-x+1, try plugging in 1:

$$p(1) = 1 - 2 + 4 - 8 < 0. \qquad ✗$$

Aiyah, sian, doesn't work: By the Factor Theorem, $x - 1$ is **not** a factor for $p(x)$.

Second try—plug in 2:

$$p(2) = 2^3 - 2 \cdot 2^2 + 4 \cdot 2 - 8 = 0. \qquad ✓$$

Yay, works! By the Factor Thm, $x - 2$ is a factor for $p(x)$. Next, write

$$x^3 - 2x^2 + 4x - 8 = (x - 2)\left(ax^2 + bx + c\right).$$

Comparing coefficients, $a = 1$, $-2c = -8$, and $c - 2b = 4$. So $c = 4$ and $b = 0$.

The quadratic polynomial $ax^2 + bx + c = x^2 + 4$ has negative discriminant and cannot be further factorised.

Thus, $x^3 - 2x^2 + 4x - 8 = (x - 2)\left(x^2 + 4\right)$. Now, write

$$\frac{-3x^2 + 5}{x^3 - 2x^2 + 4x - 8} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 4} = \frac{A\left(x^2 + 4\right) + (Bx + C)(x - 2)}{(x - 2)\left(x^2 + 4\right)}$$

$$= \frac{Ax^2 + 4A + Bx^2 + (C - 2B)x - 2C}{(x - 2)\left(x^2 + 4\right)} = \frac{(A + B)x^2 + (C - 2B)x + 4A - 2C}{(x - 2)\left(x^2 + 4\right)}.$$

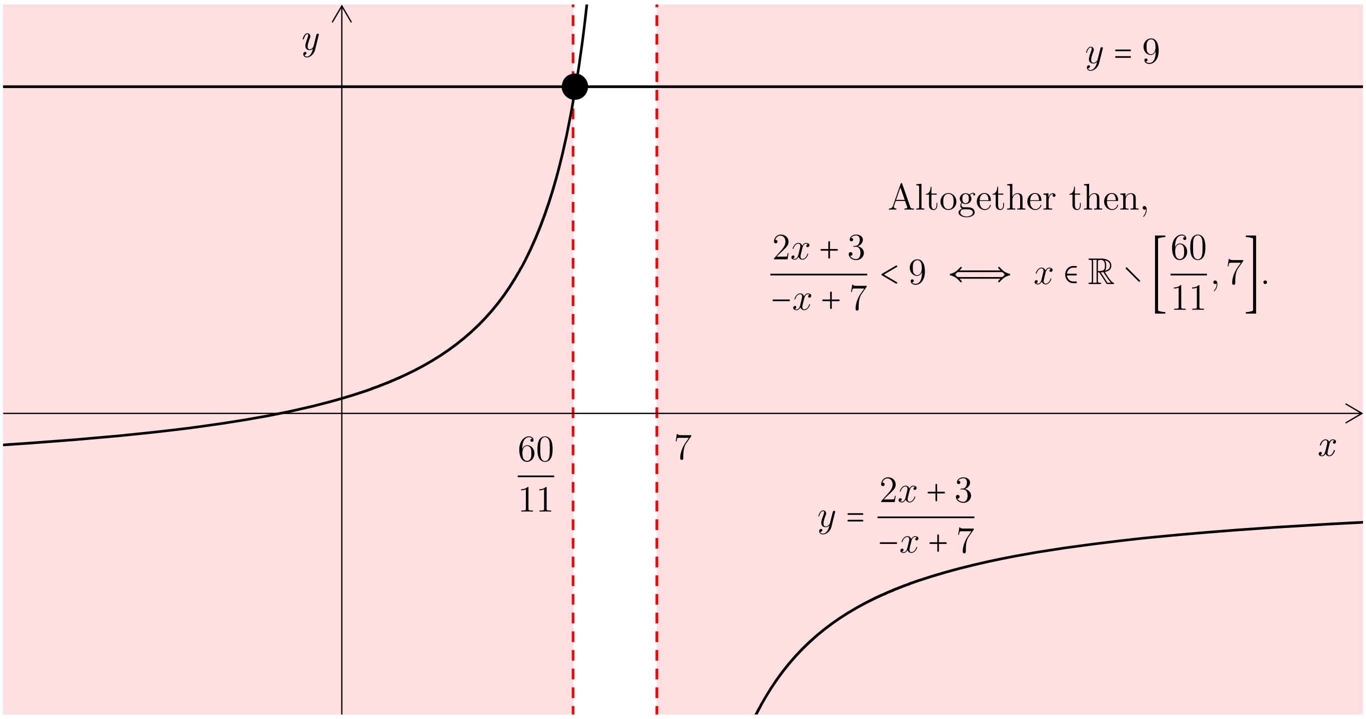
Comparing coefficients, $A + B \stackrel{1}{=} -3$, $C - 2B \stackrel{2}{=} 0$, and $4A - 2C \stackrel{3}{=} 5$. From $2 \times \stackrel{1}{=} + \stackrel{2}{=}$, we get $2A + C \stackrel{4}{=} -6$. Next, $2 \times \stackrel{4}{=} + \stackrel{3}{=}$ yields $8A = -7$ or $A = -7/8$. Now we also have $B = -17/8$ and $C = -17/4$.

Thus, $$\frac{-3x^2 + 5}{x^3 - 2x^2 + 4x - 8} = -\frac{7}{8(x - 2)} - \frac{17(x + 2)}{8\left(x^2 + 4\right)}.$$

A173(a) $(2x+3)/(-x+7) < 9 \iff 9+(2x+3)/(x-7) > 0 \iff (9x-63+2x+3)/(x-7) > 0 \iff (11x-60)/(x-7) > 0 \iff (11x-60, x-7 \overset{1}{>} 0$ OR $11x-60, x-7 \overset{2}{<} 0)$.

$11x-60, x-7 \overset{1}{>} 0 \iff (x > 60/11$ AND $x > 7) \iff x > 7$.

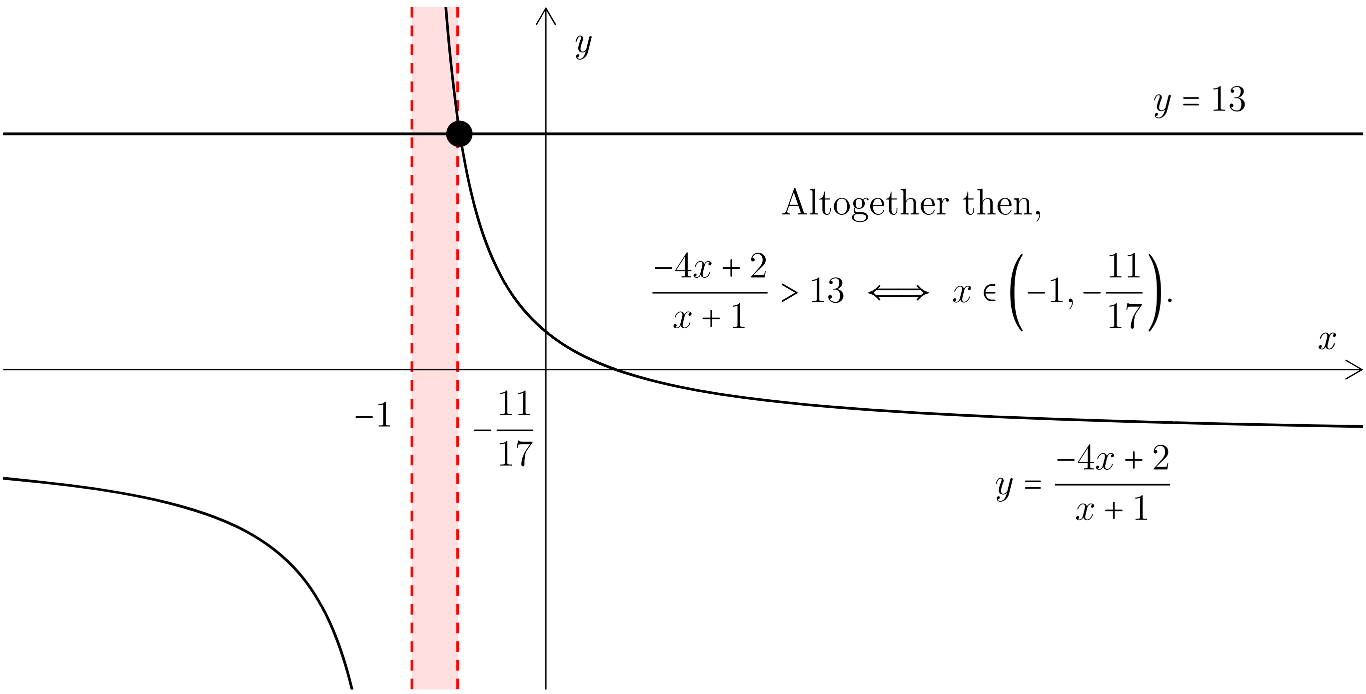
$11x-60, x-7 \overset{2}{<} 0 \iff (x < 60/11$ AND $x < 7) \iff x < 60/11$.



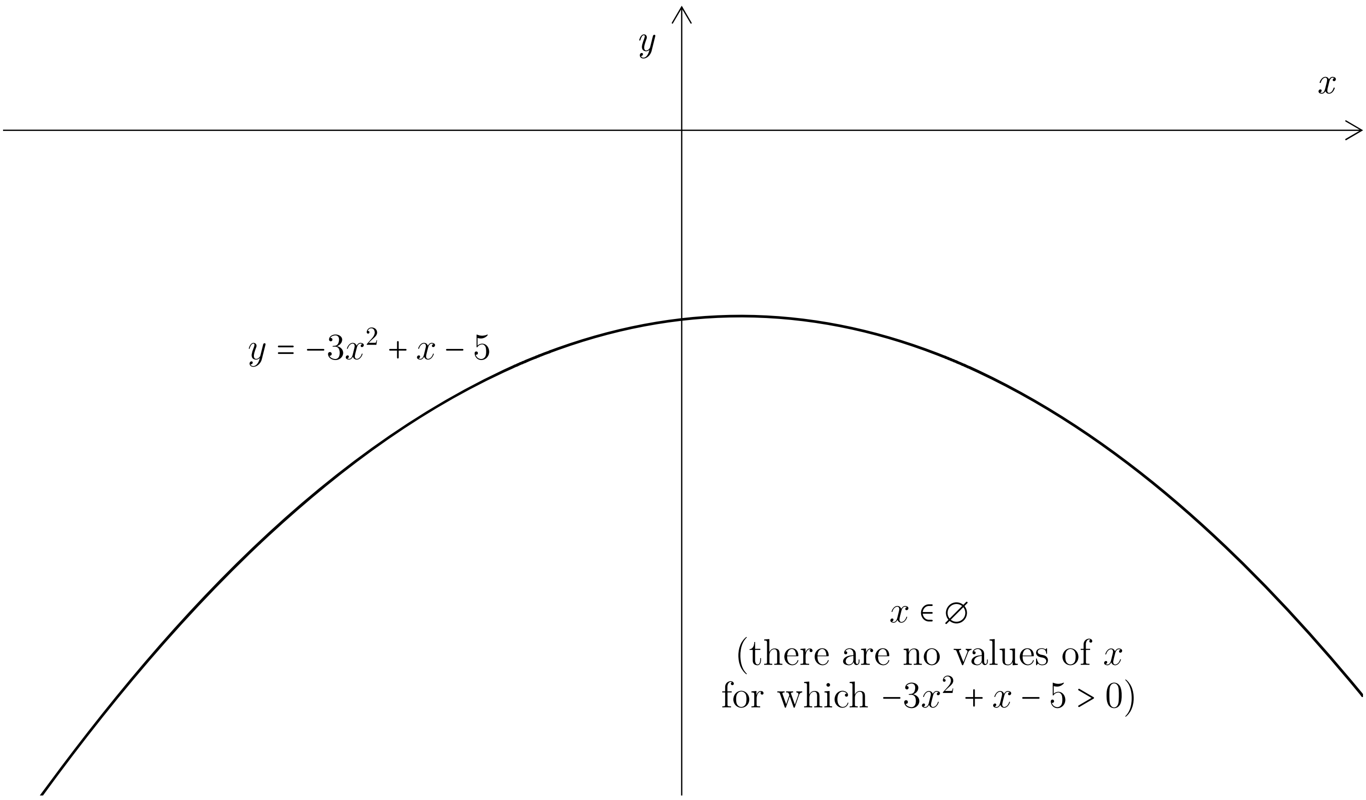
(b) $(-4x+2)/(x+1) > 13 \iff (-4x+2-13x-13)/(x+1) > 0 \iff (-17x-11)/(x+1) > 0 \iff -17x-11, x+1 \overset{1}{>} 0$ OR $-17x-11, x+1 \overset{2}{<} 0$.

$-17x-11, x+1 \overset{1}{>} 0 \iff (x < -11/17$ AND $x > -1) \iff x \in (-1, -11/17)$.

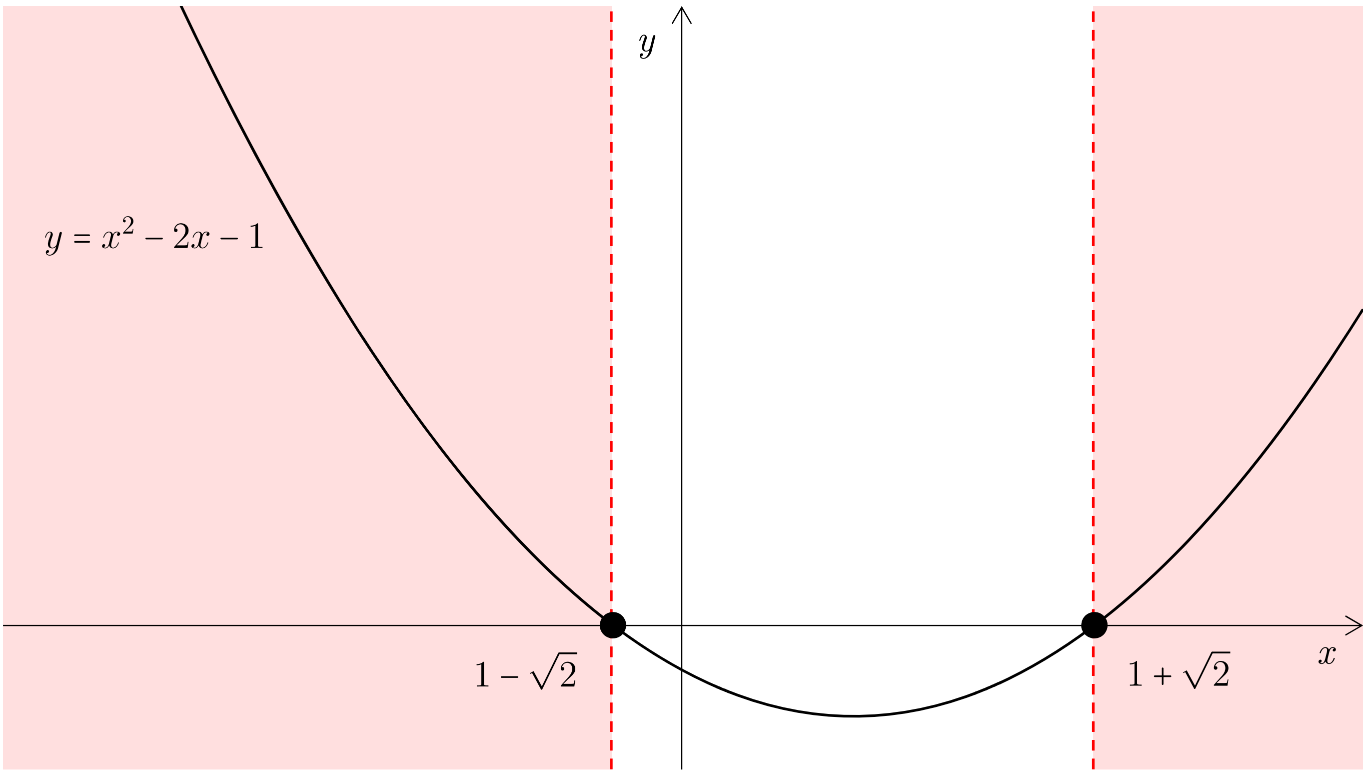
$-17x-11, x+1 \overset{2}{<} 0 \iff (x > -11/17$ AND $x < -1) \iff$ Contradiction.



A175(a) Since $a < 0$, this is a $\cap$-shaped quadratic. Since $b^2 - 4ac = 1^2 - 4(-3)(-5) = -59 < 0$, the graph doesn't touch the x-axis at all and is completely below the x-axis. Hence, there are no values of x for which this inequality is true—the solution set is $\varnothing$.



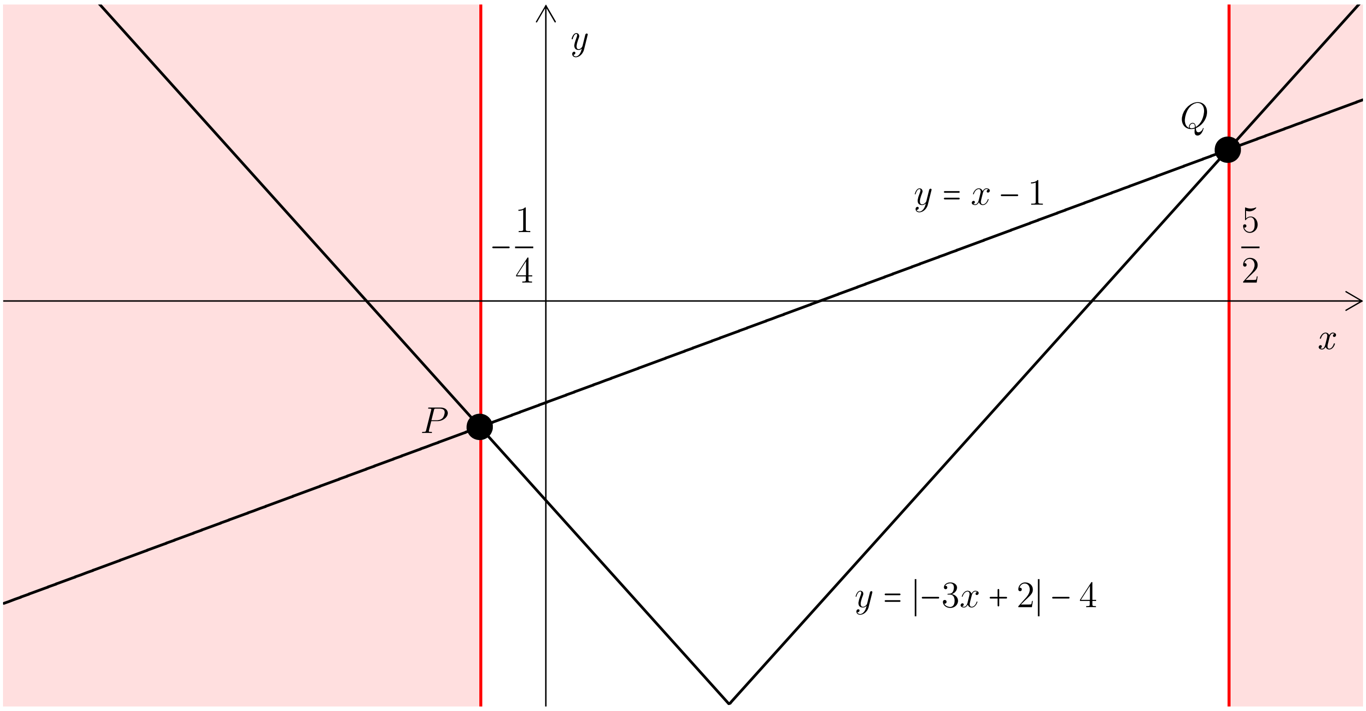
(b) Since $a > 0$, this is a $\cup$-shaped quadratic. Since $b^2 - 4ac = (-2)^2 - 4(1)(-1) = 8 > 0$, the graph intersects the x-axis at two points—these are simply given by the two roots of the quadratic, which are $x = \left(2 \pm \sqrt{8}\right)/2 = 1 \pm \sqrt{2}$. Hence, $x^2 - 2x - 1 > 0$ is true **"outside"** those two roots—the solution set is $\mathbb{R} \setminus \left[1 - \sqrt{2}, 1 + \sqrt{2}\right]$.



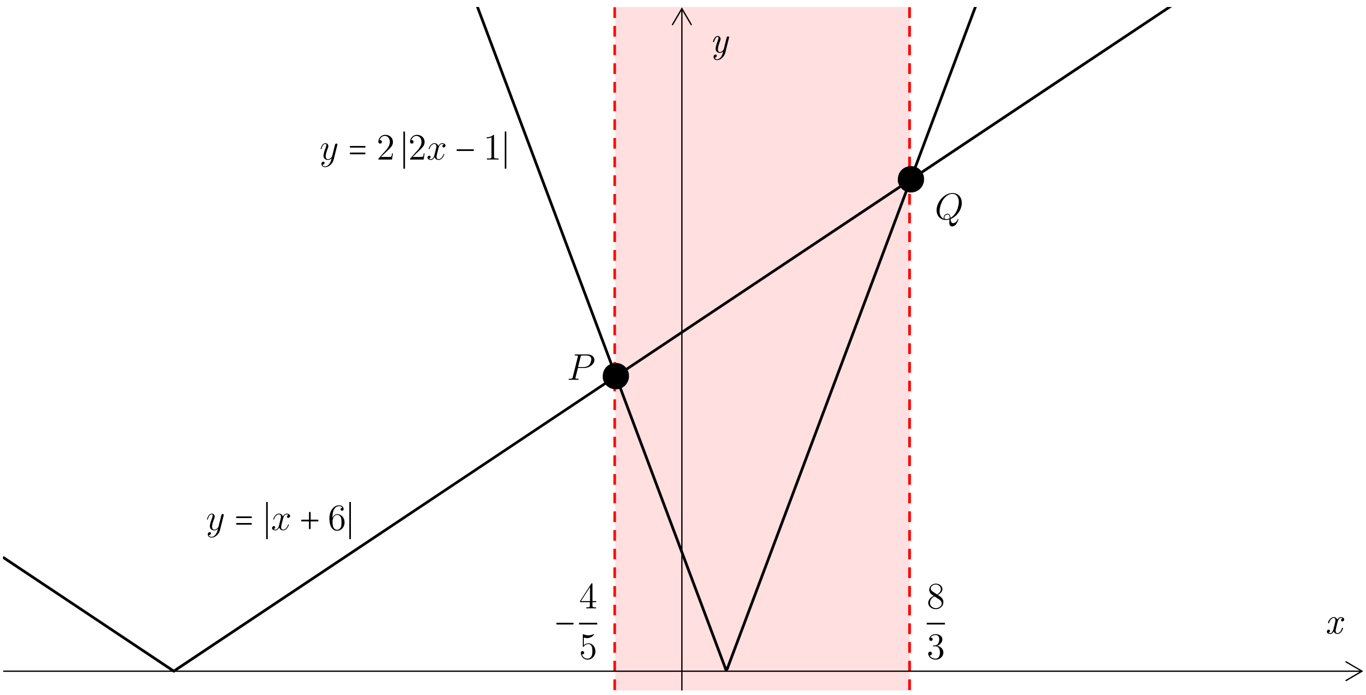
A176(a) $|x - 4| \leq 71 \iff -71 \leq x - 4 \leq 71 \iff -67 \leq x \leq 75$. The solution set is $[-67, 75]$.

(b) $|5-x|>13 \iff (5-x>13 \text{ OR } 5-x<-13) \iff (-8>x \text{ OR } 18<x)$. The solution set is $(-\infty,-8)\cup(18,\infty)$ or $\mathbb{R}\setminus[-8,18]$.

(c) Sketch $y=|-3x+2|-4$ and $y=x-1$. Observe $|-3x+2|-4\geq x-1 \iff x$ is to the left or right of the two intersection points P and Q. P is given by $-3x+2-4=x-1$ or $-1=4x$ or $x=-1/4$. Q is given by $3x-2-4=x-1$ or $2x=5$ or $x=5/2$. Thus, the solution set is $(-\infty,-1/4]\cup[5/2,\infty)$ or $\mathbb{R}\setminus(-1/4,5/2)$.



(d) Sketch $y=|x+6|$ and $y=2|2x-1|$. Observe that $|x+6|>2|2x-1| \iff x$ is **between** the two intersection points$P$ and Q. P is given by $-x-6=2(2x-1)$ or $-4=5x$ or $x=-4/5$. Q is given by $x+6=2(2x-1)$ or $8=3x$ or $x=8/3$. Thus, the solution set is $(-4/5,8/3)$.



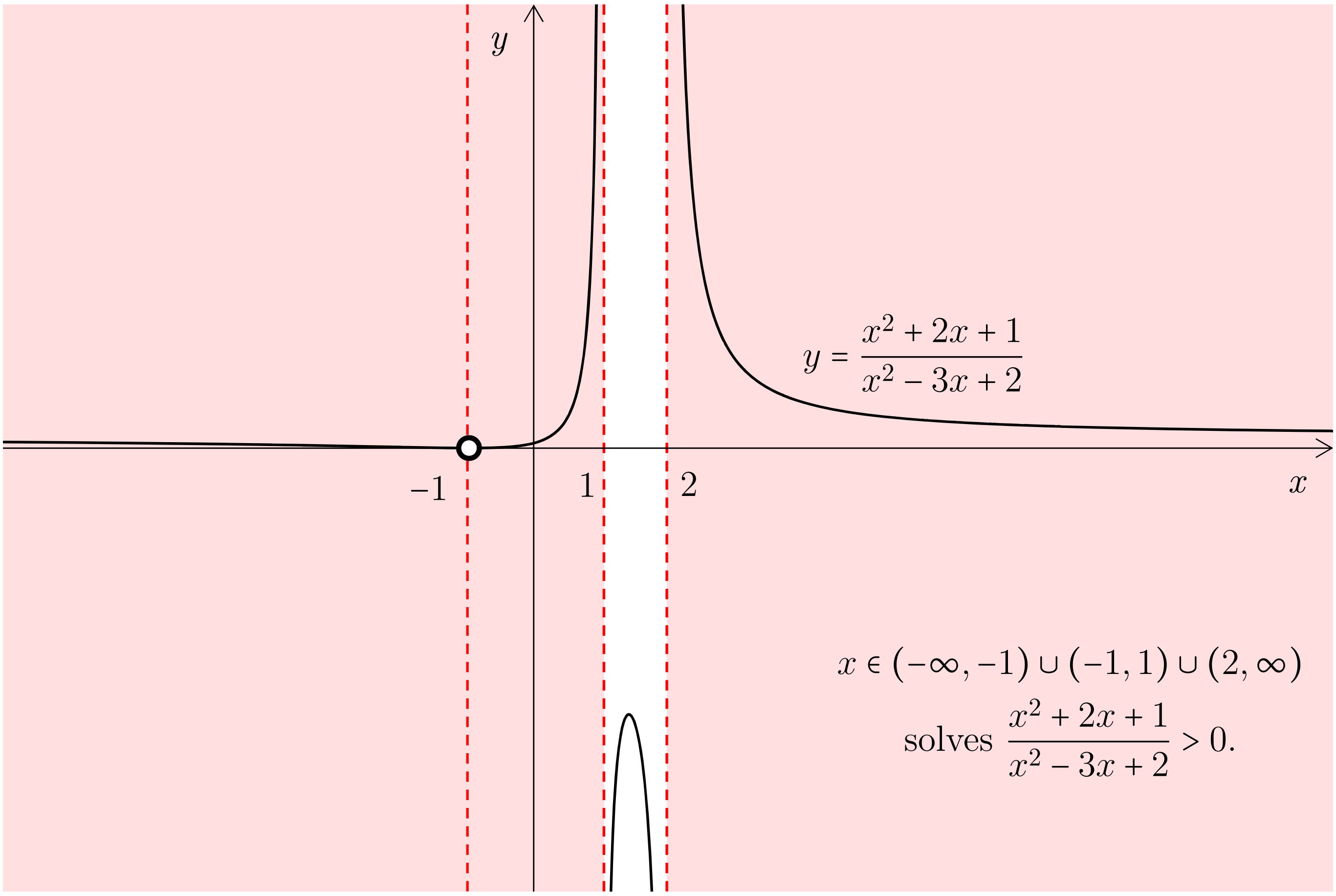
A177. $\dfrac{|x+y|}{1+|x+y|}=1-\dfrac{1}{1+|x+y|}\overset{1}{\leq}1-\dfrac{1}{1+|x|+|y|}=\dfrac{|x|}{1+|x|+|y|}+\dfrac{|y|}{1+|x|+|y|}\overset{2}{\leq}\dfrac{|x|}{1+|x|}+\dfrac{|y|}{1+|y|}$,

where $\overset{1}{\leq}$uses the Triangle Inequality and $\overset{2}{\leq}$ uses $1+|x|+|y|\geq 1+|x|$ and $1+|x|+|y|\geq 1+|y|$.

A178(a) $N = x^2 + 2x + 1 = (x+1)^2$ is positive everywhere except at $x = -1$, where it equals zero. So, the inequality is equivalent to $x^2 - 3x + 2 > 0$ AND $x \neq -1$.

Observe that $x^2 - 3x + 2 = (x-1)(x-2)$ is a ∪-shaped quadratic which intersects the x-axis at 1 and 2. Thus, $x^2 - 3x + 2 > 0 \iff x \in \mathbb{R} \backslash [1,2]$.

Thus, the inequality's solution set is $\mathbb{R} \backslash [1,2] \backslash \{-1\}$ or $(-\infty, 1) \cup (-1, 1) \cup (2, \infty)$.



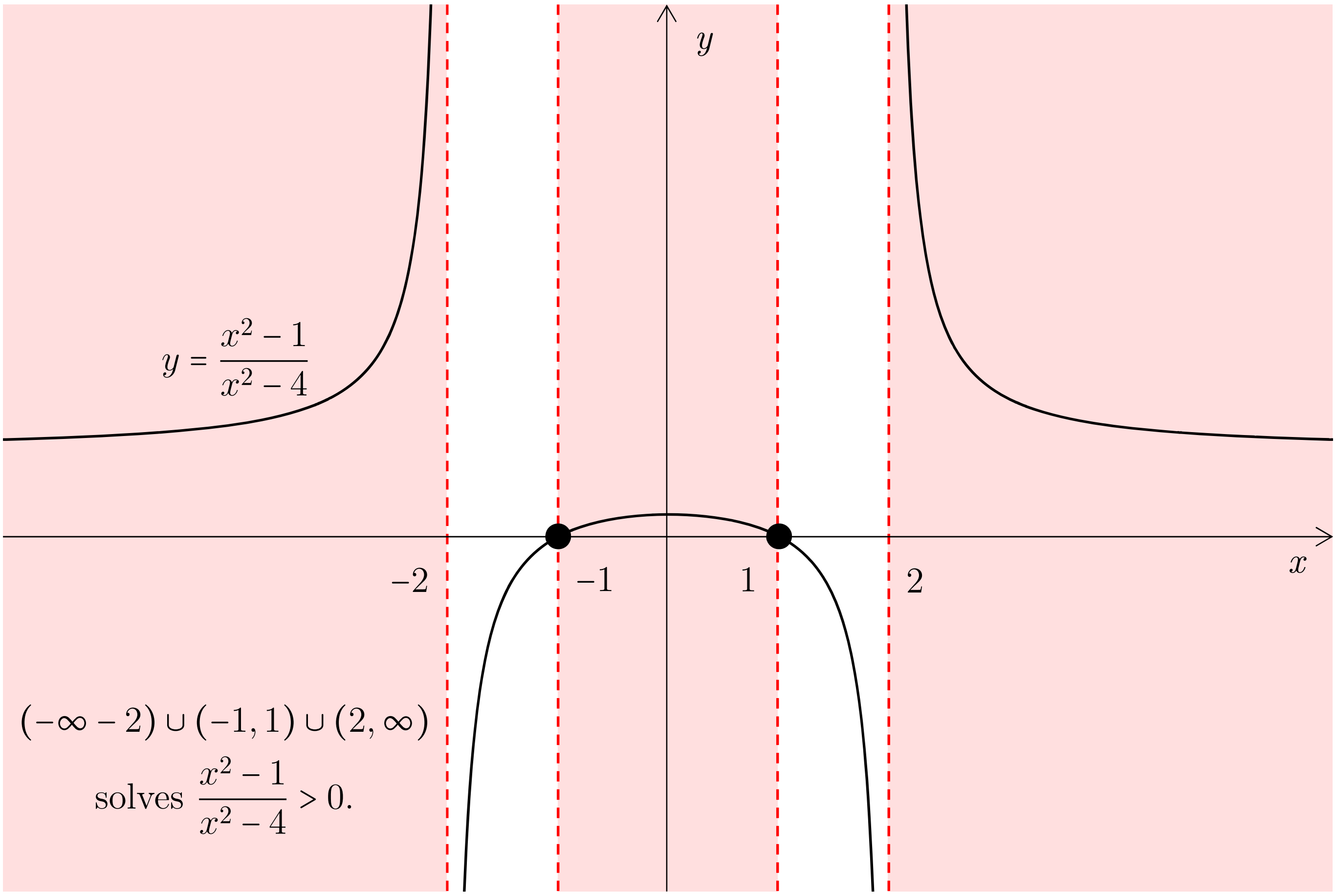
A178(b) $N = x^2 - 1 = (x+1)(x-1)$ is positive if $x \in \mathbb{R} \backslash [-1, 1]$ and negative if $x \in (-1, 1)$. $D = x^2 - 4 = (x+2)(x-2)$ is positive if $x \in \mathbb{R} \backslash [-2, 2]$ and negative if $x \in (-2, 2)$.

The given inequality is true if $N, D > 0$ or $N, D < 0$. We have

$$N, D > 0 \iff x \in \mathbb{R} \backslash [-1, 1] \text{ AND } x \in \mathbb{R} \backslash [-2, 2] \iff x \in \mathbb{R} \backslash [-2, 2].$$

$$N, D < 0 \iff x \in (-1, 1) \text{ AND } x \in (-2, 2) \iff x \in (-1, 1).$$

Thus, the inequality's solution set is $\mathbb{R} \backslash [-2, 2] \cup (-1, 1)$ or $(-\infty, -2) \cup (-1, 1) \cup (2, \infty)$.



A178(c) $N = x^2 - 3x - 18 = (x-6)(x+3)$ is positive if $x \in \mathbb{R}\setminus[-3,6]$ and negative if $x \in (-3,6)$.

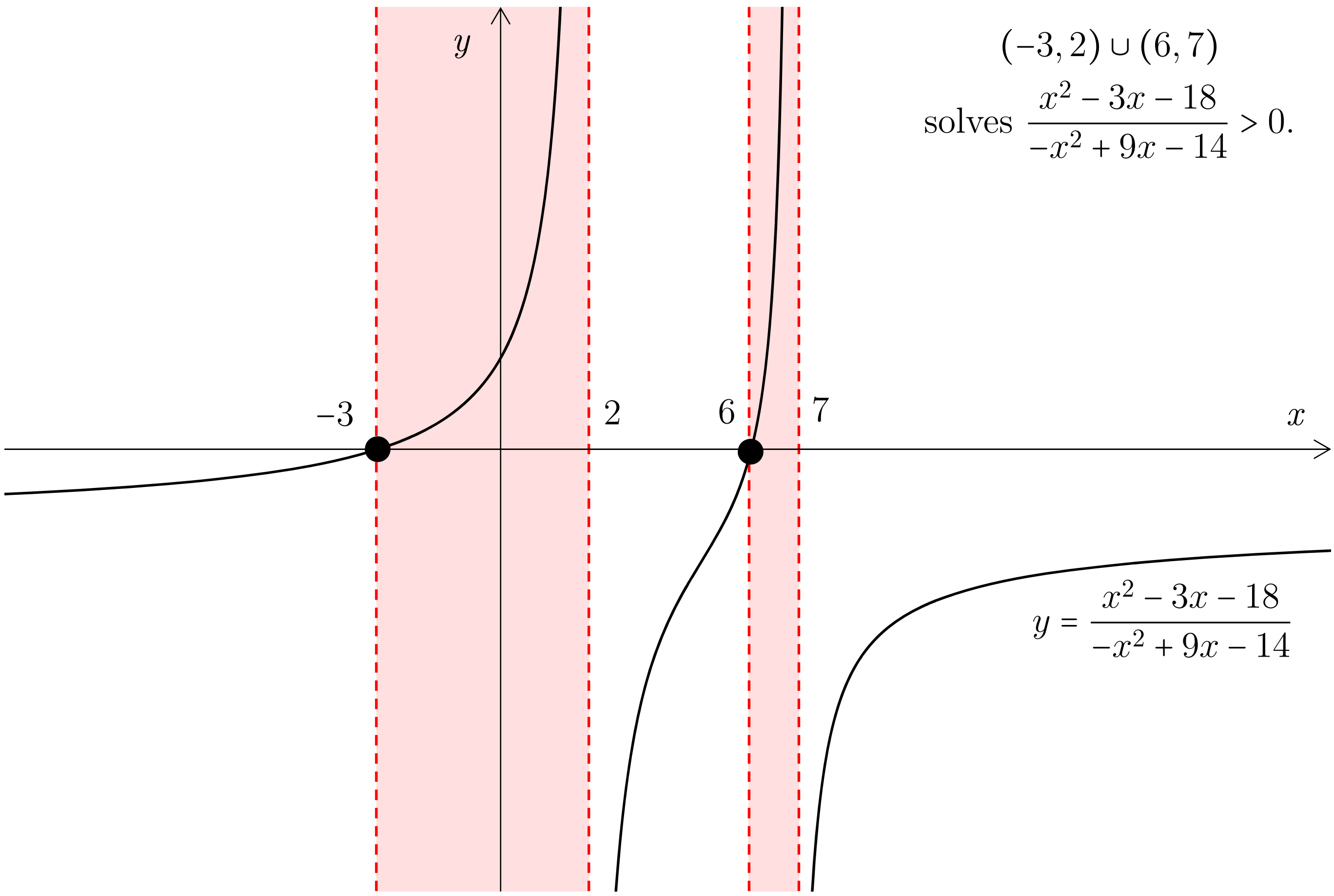
$D = -x^2 + 9x - 14 = -(x-2)(x-7)$ is positive if $x \in (2,7)$ and negative if $x \in \mathbb{R}\setminus[2,7]$.

The given inequality is true if $N, D > 0$ or $N, D < 0$. We have

$$N, D > 0 \iff x \in \mathbb{R}\setminus[-3,6] \text{ AND } x \in (2,7) \iff x \in (2,7)\setminus[-3,6] = (6,7).$$

$$N, D < 0 \iff x \in (-3,6) \text{ AND } x \in \mathbb{R}\setminus[2,7] \iff x \in (-3,6)\setminus[2,7] = (-3,2).$$

Thus, the inequality's solution set is $(-3,2) \cup (6,7)$.



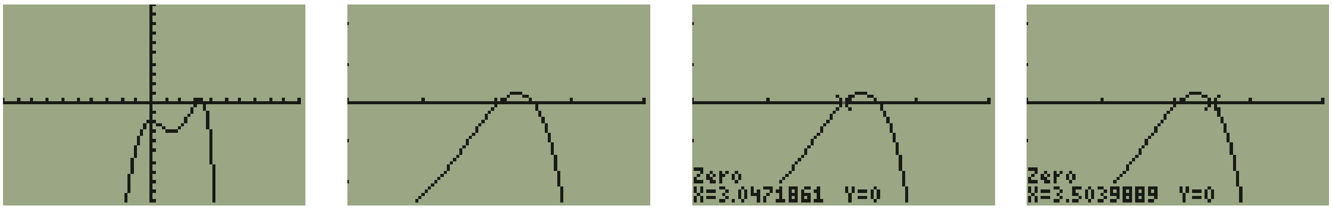
A179(a) Rewrite the inequality as $x^3 - x^2 + x - 1 - e^x > 0$, then graph $y = x^3 - x^2 + x - 1 - e^x$ on your TI84. It looks like the graph may be a little above the x-axis near 3, so let's zoom in. But we'll zoom in not through the ZOOM function, but by adjusting Xmin and Xmax in the WINDOW menu (I've adjusted them to 0 and 4).

And now as usual, we can "simply" find the two roots using the ZERO function. They are $x \approx 3.047, 3.504$.

Altogether then, based on these two roots and what the graph looks like, we conclude:

$$x^3 - x^2 + x - 1 > e^x \iff x^3 - x^2 + x - 1 - e^x > 0 \iff x \in (3.047\ldots, 3.503\ldots).$$

After graphing. **After zooming in.** **The two roots.**



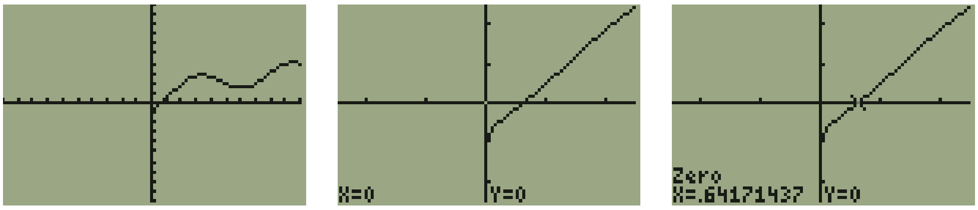
(b) Rewrite the inequality as $\sqrt{x} - \cos x > 0$, then graph $y = \sqrt{x} - \cos x$ on your TI84. It looks like there's at least one x-intercept near the origin, maybe more. So let's zoom in.

Alright, it looks like there's only one x-intercept. We can as usual find it using the ZERO function—$x \approx 0.642$.

Altogether then, based on this one root and what the graph looks like, we conclude:

$$\sqrt{x} > \cos x \iff \sqrt{x} - \cos x > 0 \iff x \gtrsim 0.642.$$

After graphing. **Zoom in.** **The only root.**



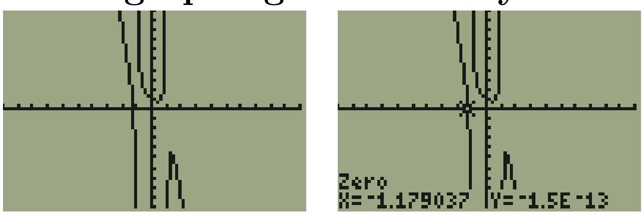
(c) Rewrite the inequality as $1/\left(1 - x^2\right) - x^3 - \sin x > 0$, then graph $y = 1/\left(1 - x^2\right) - x^3 - \sin x$ on your TI84. It looks like there's only one x-intercept near $x = -1$.

So let's simply find it using the ZERO function—$x \approx -1.179$.

Altogether then, based on this one root and what the graph looks like, we conclude:

$$\frac{1}{1 - x^2} > x^3 + \sin x \iff \frac{1}{1 - x^2} - x^3 - \sin x > 0 \iff x \in (-\infty, -1.179\ldots) \cup (-1, 1).$$

After graphing. **The only root.**



149.25. Ch. 42 Answers (Extraneous Solutions)

A180. We verify that $x \overset{2}{=} 0$ and $x \overset{4}{=} \pi/2$ satisfy $\overset{1}{=}$, while $x \overset{3}{=} \pi$ and $x \overset{4}{=} \pi/2$ do not and are extraneous solutions.

As usual, the squaring operation in Step 1 is an irreversible step. That is, this implication is true:

$$\sin x + \cos x = 1 \qquad \Longrightarrow \qquad (\sin x + \cos x)^2 = 1^2.$$

But its converse is not: $$(\sin x + \cos x)^2 = 1^2 \qquad \not\Rightarrow \qquad \sin x + \cos x = 1.$$

For example, $(\sin\pi + \cos\pi)^2 = (-1)^2 = 1$, but $\sin\pi + \cos\pi = -1$.

And so, we may (and indeed do) introduce extraneous solutions in Step 1.

A181. We verify that $x \overset{2}{=} 1$ satisfies $\overset{1}{=}$ but $x \overset{3}{=} 64$ does not and is an extraneous solution.

Plugging $x \overset{2}{=} 1$ back into the original equation $\overset{1}{=}$, we have

$$1^2\sqrt{1} + 1^2 + \sqrt{1} + 1 = 1 + 1 + 1 + 1 = 4 \neq 0,$$

so that $x \overset{2}{=} 1$ does **not** solve $\overset{1}{=}$.

The error is in Step 3, where raising to the power of 6 is an irreversible step (similar to squaring). That is, this implication is true:

$$a = b \qquad \Longrightarrow \qquad a^6 = b^6.$$

But its converse is not: $$a^6 = b^6 \qquad \not\Rightarrow \qquad a = b.$$

For example, $(-1)^6 = 1^6$ but $-1 \neq 1$.

And so, we may (and indeed do) introduce extraneous solutions in Step 3.

A182. We verify that $x \overset{4}{=} 1$ does not satisfy $\overset{1}{=}$ and so is an extraneous solution.

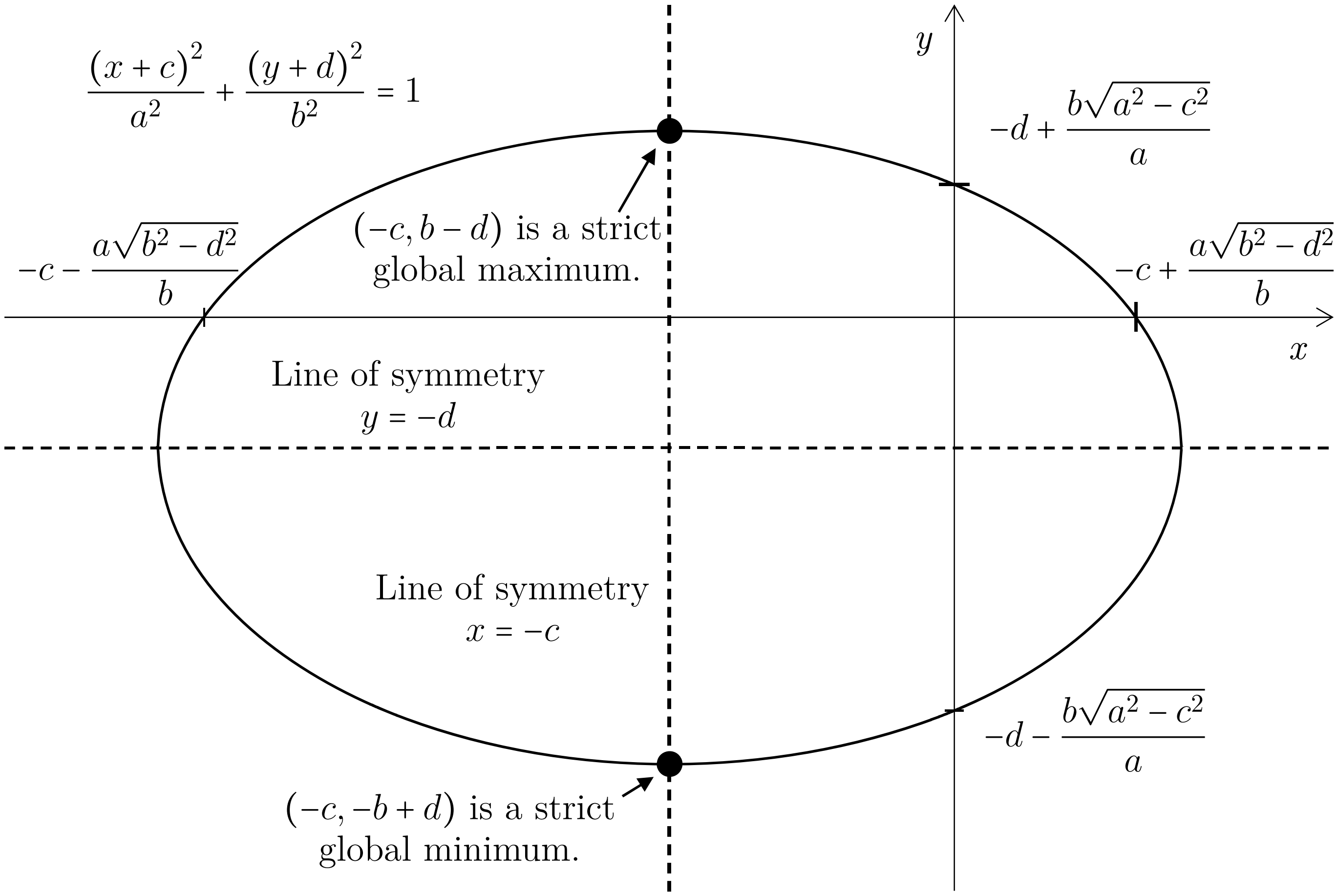
The error is in Step 3: Plugging $\overset{3}{=}$ into $\overset{1}{=}$ is an irreversible step.

It is true that $\overset{3}{=}$ implies $\overset{4}{=}$. (More explicitly, if there exists $x \in \mathbb{R}$ that satisfies $\overset{3}{=}$, then any such x also satisfies $\overset{4}{=}$.)

However, the converse may not be true and indeed isn't. There may exist $x \in \mathbb{R}$ that satisfies $\overset{4}{=}$ (and there does, namely $x = 1$), but such x may not (and indeed does not) satisfy $\overset{3}{=}$.

And so, we may (and indeed do) introduce extraneous solutions in Step 3.

A193. Translate $x^2/a^2 + y^2/b^2 = 1$ leftwards by c units to get $(x+c)^2/a^2 + y^2/b^2 = 1$. Then further translate downwards by d units to get $(x+c)^2/a^2 + (y+d)^2/b^2 = 1$.



So, $(x+c)^2/a^2 + (y+d)^2/b^2 = 1$ is the exact same ellipse as $x^2/a^2 + y^2/b^2 = 1$, but now centred on the point $(-c, -d)$ (instead of the origin).

$x^2/a^2 + y^2/b^2 = 1$ had two turning points—the strict global maximum $(0, b)$ and the strict global minimum $(0, -b)$. Since $(x+c)^2/a^2 + (y+d)^2/b^2 = 1$ is simply $x^2/a^2 + y^2/b^2 = 1$ translated c units leftwards and d units downwards, $(x+c)^2/a^2 + (y+d)^2/b^2 = 1$ again has two turning points—the strict global maximum $(-c, b-d)$ and the strict global minimum $(-c, -b-d)$.

By observation, there are no asymptotes.

By observation, there are two lines of symmetry $y = -d$ and $x = -c$.

(*Answer continues below ...*)

(... *Answer continued from above.*)

Following the hint, to find the y-intercepts, plug in $x = 0$:

$$\frac{(0+c)^2}{a^2} + \frac{(y+d)^2}{b^2} = 1 \iff \frac{(y+d)^2}{b^2} = 1 - \frac{c^2}{a^2} = \frac{a^2-c^2}{a^2}$$

$$\iff \frac{y+d}{b} = \frac{\pm\sqrt{a^2-c^2}}{a} \iff y = -d \pm \frac{b\sqrt{a^2-c^2}}{a}.$$

So, if $|a| > |c|$, then the y-intercepts are $\left(0, -d - \dfrac{b\sqrt{a^2-c^2}}{a}\right)$ and $\left(0, -d + \dfrac{b\sqrt{a^2-c^2}}{a}\right)$.

If $|a| = |c|$, then the (only) y-intercept is $(0, -d)$. (Either the leftmost or rightmost point of the ellipse just touches the y-axis.)

And if $|a| < |c|$, then there are no y-intercepts (the ellipse doesn't touch the y-axis).

Similarly, to find the x-intercepts, plug in $y = 0$:

$$\frac{(x+c)^2}{a^2} + \frac{(0+d)^2}{b^2} = 1 \iff \frac{(x+c)^2}{a^2} = 1 - \frac{d^2}{b^2} = \frac{b^2-d^2}{b^2}$$

$$\iff \frac{x+c}{a} = \frac{\pm\sqrt{b^2-d^2}}{b} \iff x = -c \pm \frac{a\sqrt{b^2-d^2}}{b}.$$

So, if $|b| > |d|$, then the x-intercepts are $\left(-c - \dfrac{a\sqrt{b^2-d^2}}{b}, 0\right)$ and $\left(-c + \dfrac{a\sqrt{b^2-d^2}}{b}, 0\right)$.

If $|b| = |d|$, then the (only) x-intercept is $(0, -d)$. (Either the topmost or bottommost point of the ellipse just touches the x-axis.)

And if $|b| < |d|$, then there are no x-intercepts (the ellipse doesn't touch the x-axis).

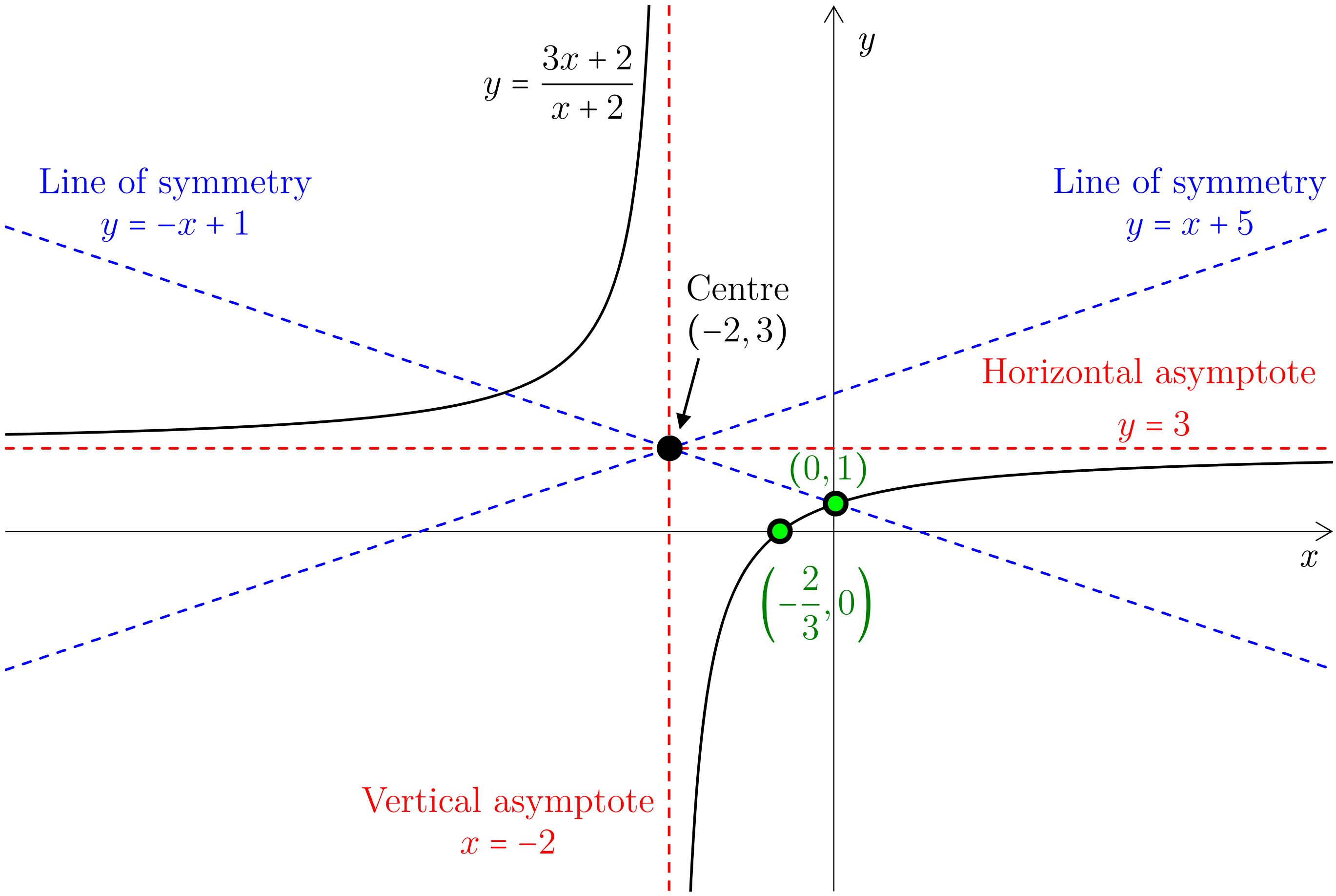
A194(a) Do the long division: $\quad y = \dfrac{3x+2}{x+2} = 3 - \dfrac{4}{x+2}$.

1. There are **two branches**—one on the top-left and another on the bottom-right.
2. **Intercepts.** Plug in $x = 0$ to get $y = 2/2 = 1$. So, the y-intercept is $(0,1)$.

   Plug in $y = 0$ to get $3x + 2 = 0$ or $x = -2/3$. So, the x-intercept is $(-2/3, 0)$.
3. There are **no turning points**.
4. **Asymptotes**. The value of x that makes the denominator 0 is -2—hence, the vertical asymptote is $x = -2$.

   The quotient in the long division is 3—hence, the horizontal asymptote is $y = 3$.

   (Note that since the two asymptotes $x = -2$ and $y = 3$ are perpendicular, this is a **rectangular hyperbola**.)
5. The hyperbola's **centre** (the point at which the two asymptotes intersect) is $(-2, 3)$. (These coordinates are simply given by the vertical and horizontal asymptotes.)
6. The **two lines of symmetry** may be written as $y = x + \alpha$ and $y = -x + \beta$ and pass through the centre $(-2, 3)$. Plugging in the numbers, we find that $\alpha = 5$ and $\beta = 1$. Thus, the two lines of symmetry are $y = x + 5$ and $y = -x + 1$.



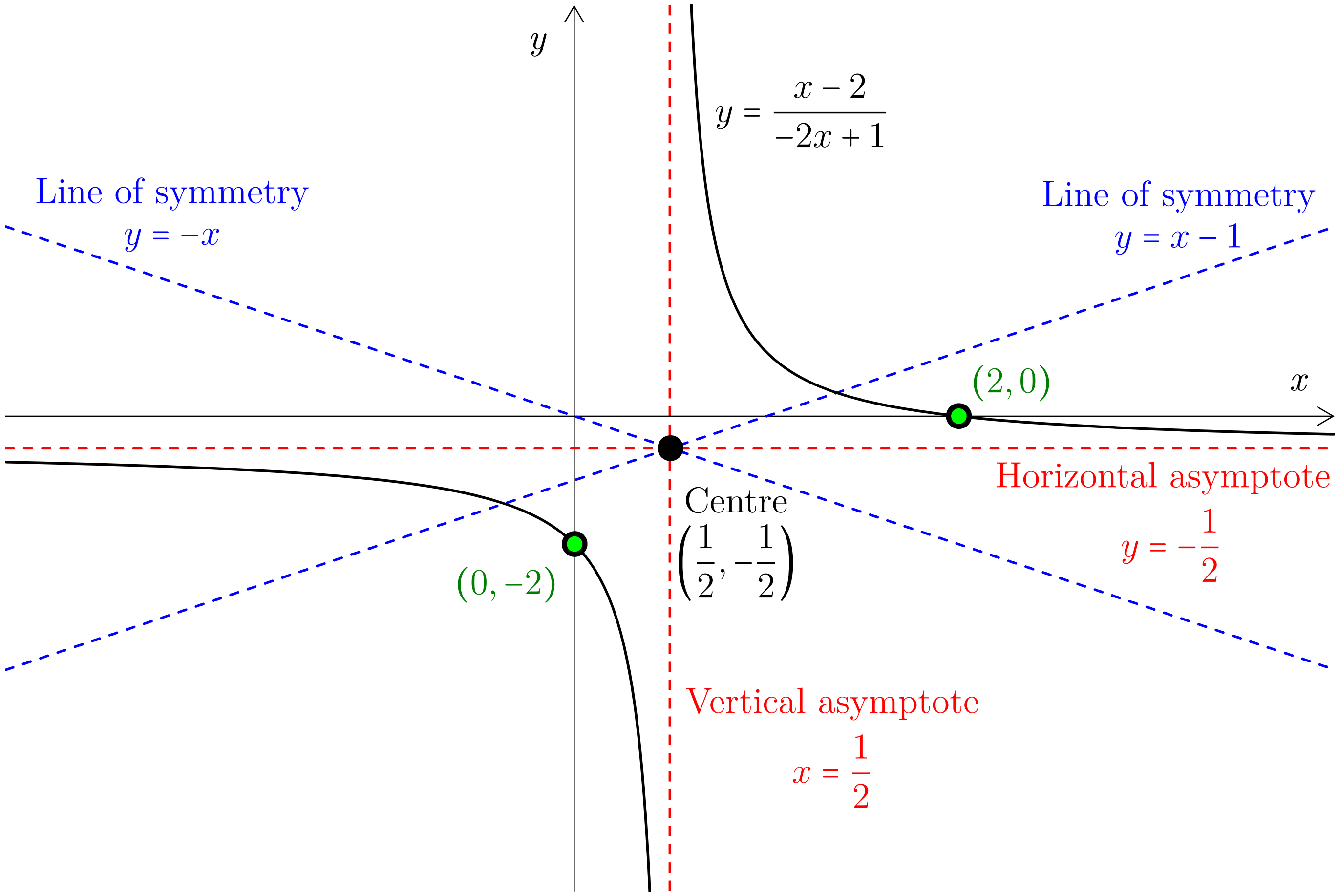
A194(b) Do the long division: $y = \frac{x-2}{-2x+1} = -\frac{1}{2} - \frac{3/2}{-2x+1}$.

1. There are **two branches**—one on the top-right and another on the bottom-left.
2. **Intercepts.** Plug in $x = 0$ to get $y = -2/1 = -2$. So, the y-intercept is $(0, -2)$.

   Plug in $y = 0$ to get $x - 2 = 0$ or $x = 2$. So, the x-intercept is $(2, 0)$.
3. There are **no turning points**.
4. **Asymptotes**. The value of x that makes the denominator 0 is $1/2$—hence, the vertical asymptote is $x = 1/2$.

   The quotient in the long division is $-1/2$—hence, the horizontal asymptote is $y = -1/2$.

   (Note that since the two asymptotes $x = 1/2$ and $y = -1/2$ are perpendicular, this is a **rectangular hyperbola**.)
5. The hyperbola's **centre** (the point at which the two asymptotes intersect) is $(1/2, -1/2)$. (These coordinates are simply given by the vertical and horizontal asymptotes.)
6. The **two lines of symmetry** may be written as $y = x + \alpha$ and $y = -x + \beta$ and pass through the centre $(1/2, -1/2)$. Plugging in the numbers, we find that $\alpha = -1$ and $\beta = 0$. Thus, the two lines of symmetry are $y = x - 1$ and $y = -x$.



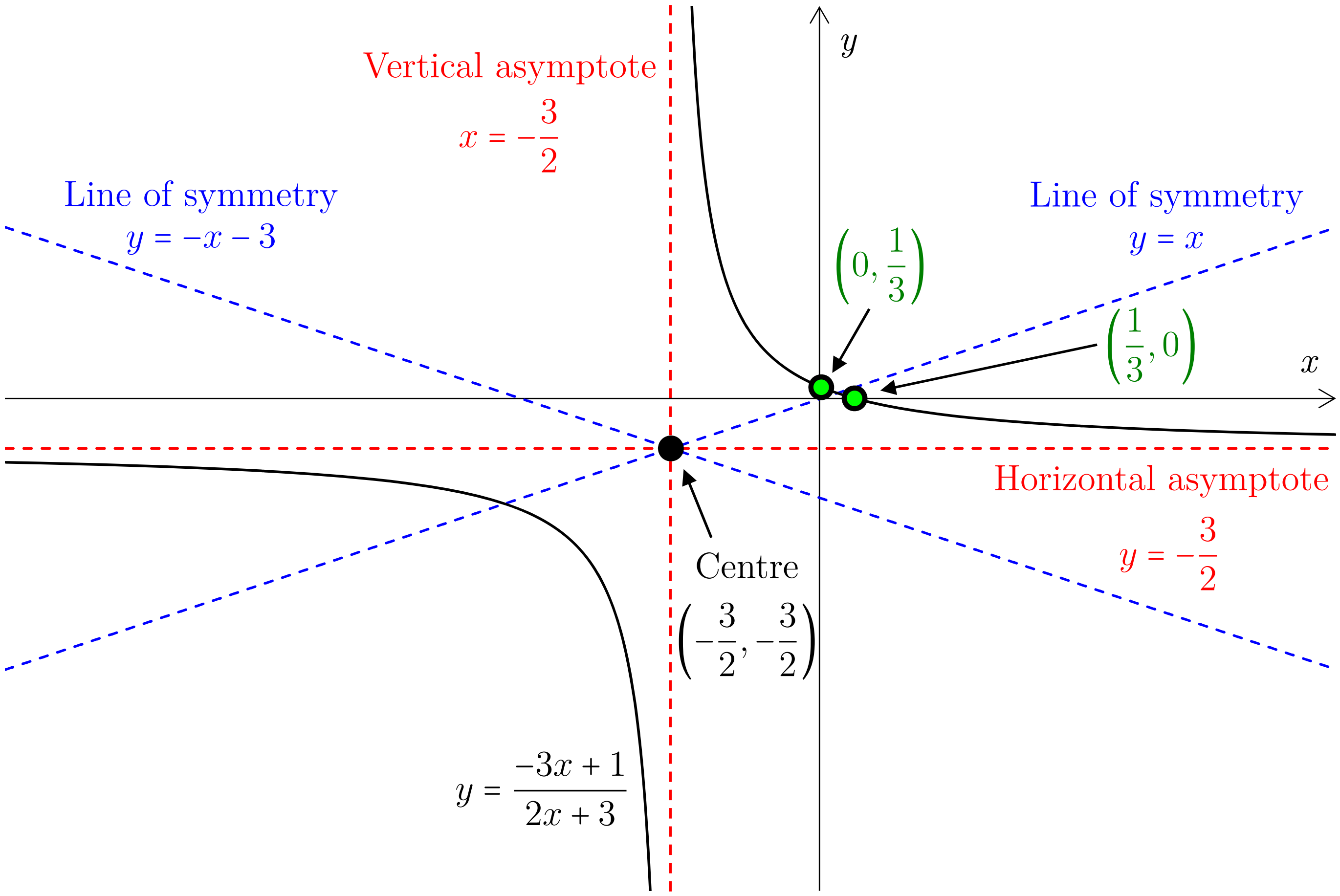
A194(c) Do the long division: $y = \dfrac{-3x+1}{2x+3} = -\dfrac{3}{2} + \dfrac{11/2}{2x+3}$.

1. There are **two branches**—one on the top-right and another on the bottom-left.
2. **Intercepts.** Plug in $x = 0$ to get $y = 1/3$. So, the y-intercept is $(0, 1/3)$.

   Plug in $y = 0$ to get $-3x + 1 = 0$ or $x = 1/3$. So, the x-intercept is $(1/3, 0)$.
3. There are **no turning points**.
4. **Asymptotes**. The value of x that makes the denominator 0 is $-3/2$—hence, the vertical asymptote is $x = -3/2$.

   The quotient in the long division is $-3/2$—hence, the horizontal asymptote is $y = -3/2$.

   (Note that since the two asymptotes $x = -3/2$ and $y = -3/2$ are perpendicular, this is a **rectangular hyperbola**.)
5. The hyperbola's **centre** (the point at which the two asymptotes intersect) is $(-3/2, -3/2)$. (These coordinates are simply given by the vertical and horizontal asymptotes.)
6. The **two lines of symmetry** may be written as $y = x+\alpha$ and $y = -x+\beta$ and pass through the centre $(-3/2, -3/2)$. Plugging in the numbers, we find that $\alpha = 0$ and $\beta = -3$. Thus, the two lines of symmetry are $y = x$ and $y = -x - 3$.



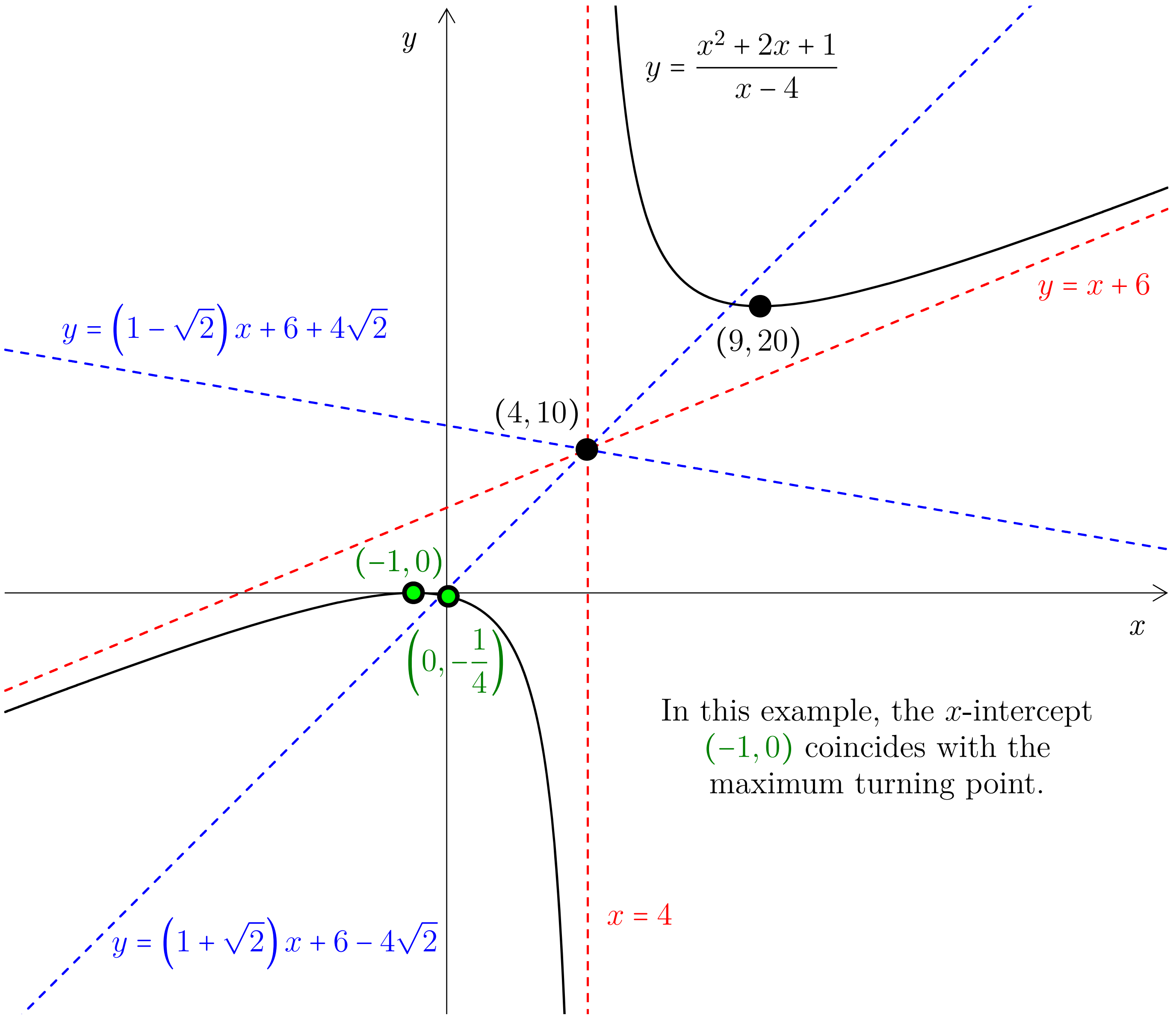
A195(a) Do the long division: $\dfrac{x^2+2x+1}{x-4} = x+6+\dfrac{25}{x-4}$.

Intercepts. Plug in $x=0$ to get $y=1/(-4)=-1/4$. Thus, the y-intercept is $(0,-1/4)$. Plug in $y=0$ to get $x^2+2x+1=0$, an equation which has one (real) solution $x=-1$. Thus, the (only) x-intercept is $(-1,0)$.

Asymptotes. The vertical asymptote $x=4$ is given by the value of x for which $x-4=0$. The oblique asymptote $y=x+6$ is given by the quotient in the long division.

The **centre's** x-coordinate is given by the vertical asymptote $x=4$. For its y-coordinate, plug $x=4$ into the oblique asymptote to get $y=4+6=10$. Hence, the centre is $(4,10)$.

You should be able to sketch the **two lines of symmetry** and the **two turning points**.[649] (The two lines of symmetry run through the centre and bisect an angle formed by the two asymptotes. And there is one turning point for each branch.)



[649]To find the turning points, write $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(x+6+\dfrac{25}{x-4}\right)=1-25(x-4)^{-2}\overset{!}{=}0 \iff (x-4)^2=25$. So $x=-1,9$. The corresponding y-values are $-1+6+25/(-1-4)=0$ and $9+6+25/(9-4)=20$. Thus, the two turning points are $(-1,0)$ and $(9,20)$. (If necessary, we can also show that these are respectively the strict local maximum and minimum.)

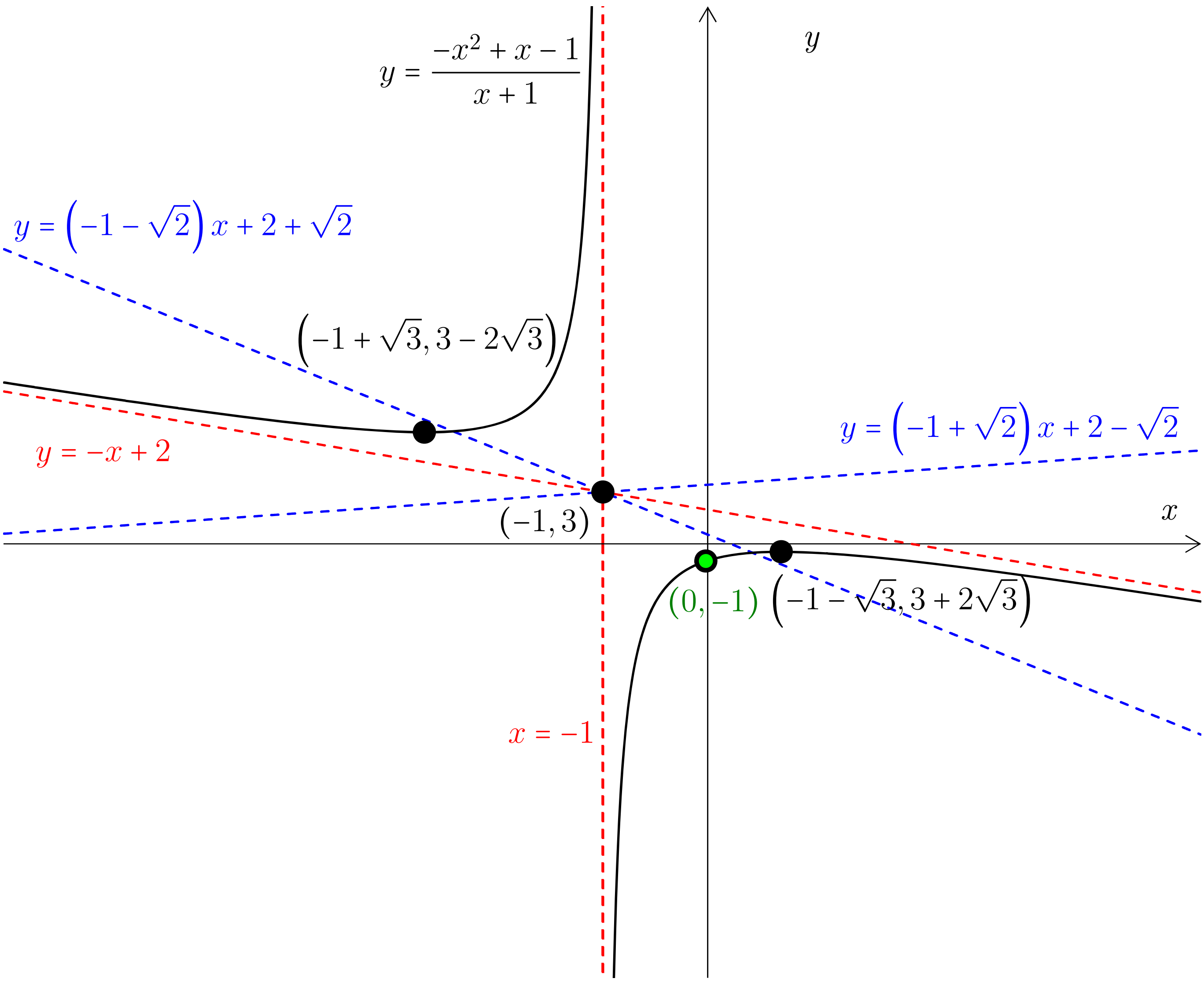
A195(b) Do the long division: $\dfrac{-x^2+x-1}{x+1}=-x+2-\dfrac{3}{x+1}$.

Intercepts. Plug in $x=0$ to get $y=-1/1=-1$. Thus, the y-intercept is $(0,-1)$. Plug in $y=0$ to get $-x^2+x-1=0$, an equation for which there are no (real) solutions. Thus, there are no x-intercepts.

Asymptotes. The vertical asymptote $x=-1$ is given by the value of x for which $x+1=0$. The oblique asymptote $y=-x+2$ is given by the quotient in the long division.

The **centre's** x-coordinate is given by the vertical asymptote $x=-1$. For its y-coordinate, plug $x=-1$ into the oblique asymptote to get $y=-(-1)+2=3$. Hence, the centre is $(-1,3)$.

You should be able to sketch the **two lines of symmetry** and the **two turning points**.[650]



[650]To find the turning points, write $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(-x+2-\dfrac{3}{x+1}\right)=-1+3(x+1)^{-2}\stackrel{!}{=}0\iff(x+1)^2=3$.
So $x=-1\pm\sqrt{3}$. The corresponding y-values are $1\mp\sqrt{3}+2-3/\left(-1\pm\sqrt{3}+1\right)=3\mp2\sqrt{3}$. Thus, the two turning points are $\left(-1\pm\sqrt{3},3\mp2\sqrt{3}\right)$. (If necessary, we can also show that these are respectively the strict local maximum and minimum.)

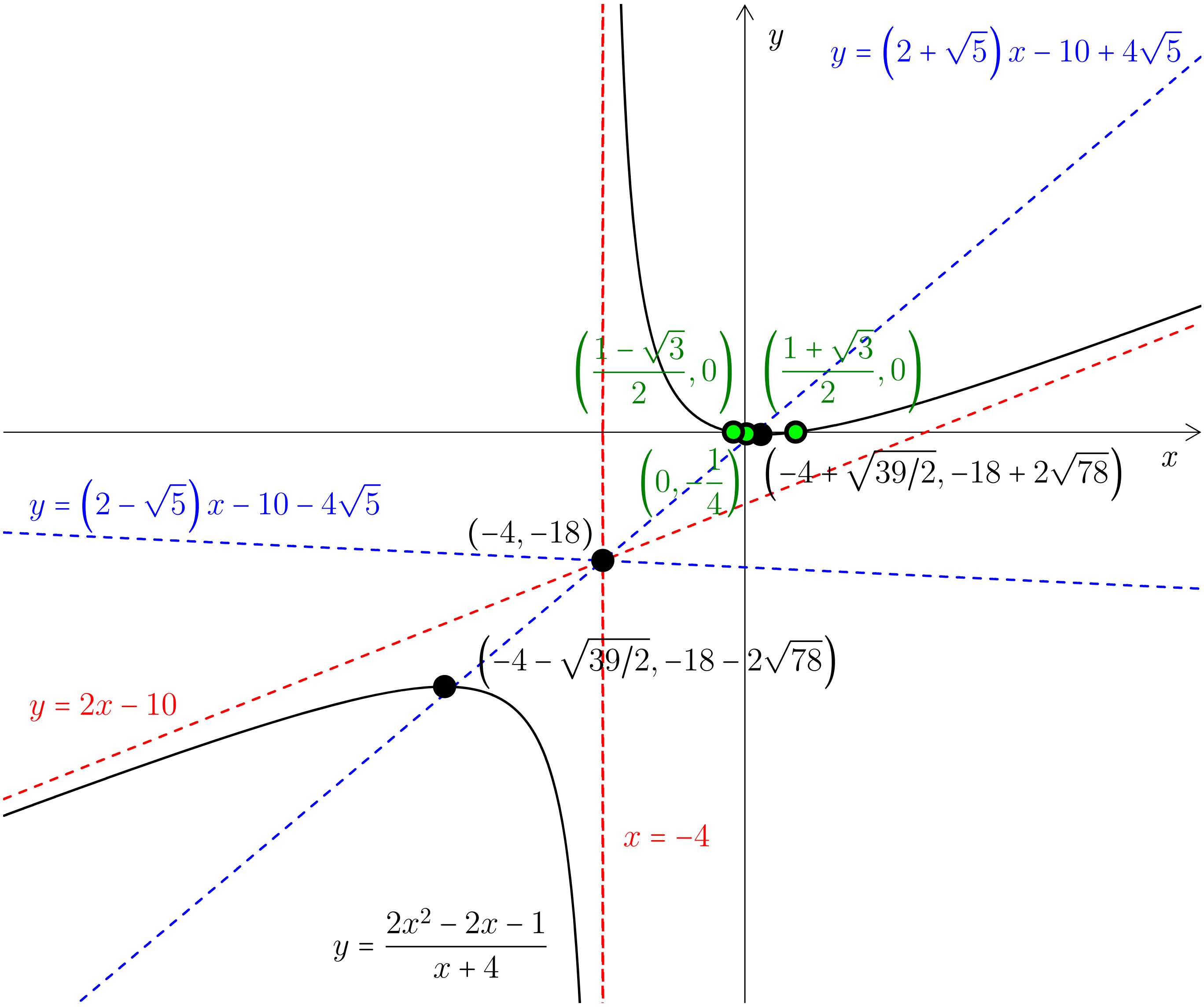
A195(c) Do the long division: $\dfrac{2x^2-2x-1}{x+4} = 2x-10+\dfrac{39}{x+4}$.

Intercepts. Plug in $x=0$ to get $y=-1/4$. Thus, the y-intercept is $(0,-1/4)$. Plug in $y=0$ to get $2x^2-2x-1=0$, an equation for which there are two (real) solutions: $x=\left(2\pm\sqrt{12}\right)/4=\left(1\pm\sqrt{3}\right)/2$. Thus, there are two x-intercepts: $\left(\left(1\pm\sqrt{3}\right)/2,0\right)$.

Asymptotes. The vertical asymptote $x=-4$ is given by the value of x for which $x+4=0$. The oblique asymptote $y=2x-10$ is given by the quotient in the long division.

The **centre's** x-coordinate is given by the vertical asymptote $x=-4$. For its y-coordinate, plug $x=-4$ into the oblique asymptote to get $y=2(-4)-10=-18$. Hence, the centre is $(-4,-18)$.

You should be able to sketch the **two lines of symmetry** and the **two turning points**.[651]



[651]To find the turning points, write $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(2x-10+\dfrac{39}{x+4}\right)=2-39(x+4)^{-2}\overset{!}{=}0 \iff (x+4)^2=39/2$. So $x=-4\pm\sqrt{39/2}$. The corresponding y-values are $2\left(-4\pm\sqrt{39/2}\right)-10+39/\left(-4\pm\sqrt{39/2}+4\right)=-18\pm 2\sqrt{78}$. Thus, the two turning points are $\left(-4\pm\sqrt{39/2},-18\pm2\sqrt{78}\right)$. (If necessary, we can also show that these are respectively the strict local maximum and minimum.)

A349(a) As stated in the above example, at time $t=0$, particle P is at $(x,y)=(\cos 0,\sin 0)=(1,0)$. In contrast, particle Q is at $(x,y)=(\sin 0,\cos 0)=(0,1)$.

(b) At $t=0$, Q is at $(x,y)=(\sin 0,\cos 0)=(0,1)$. A little after $t=0$, $x=\sin t$ will have grown a little while $y=\cos t$ will have shrunk a little. That is, the particle Q will have moved a little to the right and a little to the south. Thus, Q travels **clockwise**.

(c) Every 2π s, each particle travels one full circle. Therefore, at $t=664\pi$, each particle will be at its starting point. And π s later, each particle will have travelled an additional half-circle. Thus, at $t=665\pi$, particle P will be at $(x,y)=(\cos\pi,\sin\pi)=(-1,0)$ (at the left of the circle), while particle Q will be at $(x,y)=(\sin\pi,\cos\pi)=(0,-1)$ (at the bottom of the circle).

(d) The two particles are at the exact same position whenever $(\cos t,\sin t)=(\sin t,\cos t)$.

Thus, they are at the exact same position whenever $\cos t=\sin t$. By inspecting the graphs of cos and sin (see e.g. p. 372), we see that this occurs at $t=(k+1/4)\pi$, for every $k\in\mathbb{Z}_0^+$.

(e) For the particle Q, we have

$$v_x=\frac{\mathrm{d}x}{\mathrm{d}t}=\cos t,\qquad v_y=\frac{\mathrm{d}y}{\mathrm{d}t}=-\sin t,\qquad a_x=\frac{\mathrm{d}v_x}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\sin t,\qquad a_y=\frac{\mathrm{d}v_y}{\mathrm{d}t}=\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=-\cos t.$$

A350(a) For the particle R, we have

$$v_x=\frac{\mathrm{d}x}{\mathrm{d}t}=-a\sin t,\quad v_y=\frac{\mathrm{d}y}{\mathrm{d}t}=b\cos t,\quad a_x=\frac{\mathrm{d}v_x}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-a\cos t,\quad a_y=\frac{\mathrm{d}v_y}{\mathrm{d}t}=\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=-b\sin t.$$

(b)(i) At $t=\pi/4$,

$$\begin{aligned}(x,y)&=(a\cos(\pi/4),b\sin(\pi/4))=\left(a\sqrt{2}/2,b\sqrt{2}/2\right),\\(v_x,v_y)&=(-a\sin(\pi/4),b\cos(\pi/4))=\left(-a\sqrt{2}/2,b\sqrt{2}/2\right),\\(a_x,a_y)&=(-a\cos(\pi/4),-b\sin(\pi/4))=\left(-a\sqrt{2}/2,-b\sqrt{2}/2\right).\end{aligned}$$

The particle R starts at $(a,0)$ and travels anticlockwise. At $t=\pi/4$, R has completed one-eighth of the full revolution and is now at the top-right of the ellipse; it is travelling *left*wards at $a\sqrt{2}/2\,\mathrm{m\,s^{-1}}$ and upwards at $b\sqrt{2}/2\,\mathrm{m\,s^{-1}}$; and it is accelerating *left*wards at $a\sqrt{2}/2\,\mathrm{m\,s^{-2}}$ and *down*wards at $b\sqrt{2}/2\,\mathrm{m\,s^{-2}}$.

(b)(ii) At $t=\pi/2$,

$$\begin{aligned}(x,y)&=(a\cos(\pi/2),b\sin(\pi/2))=(0,b),\\(v_x,v_y)&=(-a\sin(\pi/2),b\cos(\pi/2))=(-a,0),\\(a_x,a_y)&=(-a\cos(\pi/2),-b\sin(\pi/2))=(0,-b).\end{aligned}$$

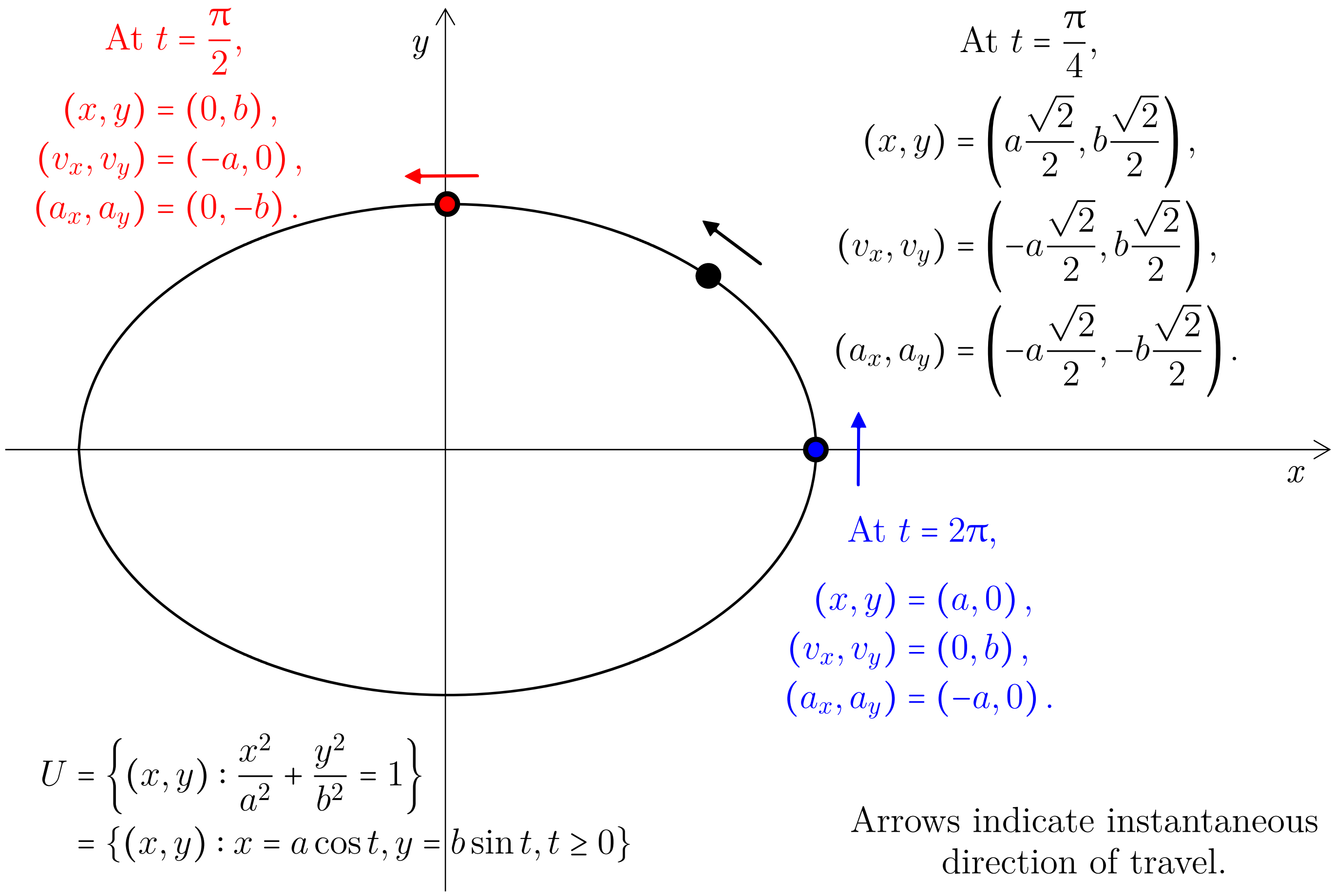
At $t=\pi/2$, R has completed one-quarter of the full revolution and is now at the top of the ellipse; it is travelling *left*wards at $a\,\mathrm{m\,s^{-1}}$ (and not upwards at all); and it is accelerating *down*wards at $b\,\mathrm{m\,s^{-2}}$ (and not rightwards at all).

(*Answer continues below ...*)

(... *Answer continued from above.*)

A350(b)(iii) At $t = 2\pi$,
$$(x, y) = (a\cos 2\pi, b\sin 2\pi) = (a, 0),$$
$$(v_x, v_y) = (-a\sin 2\pi, b\cos 2\pi) = (0, b),$$
$$(a_x, a_y) = (-a\cos 2\pi, -b\sin 2\pi) = (-a, 0).$$

At $t = 2\pi$, R has completed one full revolution and is back at its starting position; it is travelling upwards at $b\,\mathrm{m\,s^{-1}}$ (and not rightwards at all); and it is accelerating *left*wards at $a\,\mathrm{m\,s^{-2}}$ (and not upwards at all).



(c) At $t = 0$, $(x, y) = (a\cos 0, b\sin 0) = (a, 0)$ and $(v_x, v_y) = (-a\sin 0, b\cos 0) = (0, b)$. Hence,

(i) If $a, b < 0$, R starts at the **left** of the ellipse. It also starts by moving *down*wards and is thus moving **anticlockwise**.

(ii) If $a > 0$, $b < 0$, R starts at the **right** of the ellipse. It also starts by moving *down*wards and is thus moving **clockwise**.

(ii) If $a < 0$, $b > 0$, R starts at the **left** of the ellipse. It also starts by moving upwards and is thus moving **clockwise**.

A197(a) An instant after $t = 1.5\pi$, the particle magically reappears "near" "bottom-right infinity" $(\infty, -\infty)$.

(b) During $t \in (1.5\pi, 2.5\pi)$, the particle moves upwards along the right branch of the hyperbola. At $t = 2\pi$, it is back to its starting position $(1, 0)$. And as $t \to 2.5\pi$, it "flies off" towards "top-right infinity" (∞, ∞).

A351(a) $x = \tan t$, $y = \sec t \implies y^2 - x^2 = 1$.

(b) At $t = 0$, $(x, y) = (\tan 0, \sec 0) = (0, 1)$—the particle B is at the midpoint of the top branch of the hyperbola.

(c) $v_x = \mathrm{d}x/\mathrm{d}t = \sec^2 t$ is always positive and so the particle is always moving **rightwards**.

(d)(i) At $t = 0$, B starts at the midpoint of the top branch of the hyperbola. During $t \in [0, 0.5\pi)$,B travels rightwards along the right portion of the top branch. As $t \to 0.5\pi$, B "flies off" towards the "top-right" infinity; its position, velocity, and acceleration in both the x- and y-directions approach ∞.

(d)(ii) An instant after $t = 0.5\pi$, B magically reappears "near" "bottom-left" infinity. During $t \in (0.5\pi, 1.5\pi)$, it travels rightwards, along the **bottom** branch of the hyperbola.

Specifically, during $t \in (0.5\pi, \pi)$, it is on the left portion of the bottom branch. At $t = \pi$, it is at the midpoint of the bottom branch. And during $t \in (\pi, 1.5\pi)$, it is on the right portion of the bottom branch.

As $t \to 1.5\pi$, B "flies off" towards the "bottom-right" infinity; its position, velocity, and acceleration in the x-direction approach ∞; and in the y-direction, they approach $-\infty$.

(d)(iii) An instant after $t = 1.5\pi$, B magically reappears "near" "top-left" infinity. During $t \in (1.5\pi, 2.5\pi)$, it travels rightwards, along the **top** branch of the hyperbola.

Specifically, during $t \in (1.5\pi, 2\pi)$, it is on the left portion of the top branch. At $t = 2\pi$, it is back to its starting position—the midpoint of the top branch. And during $t \in (2\pi, 2.5\pi)$, it is on the right portion of the bottom branch.

As $t \to 2.5\pi$, B again "flies off" towards the "top-right" infinity as it did when t approached 0.5π; its position, velocity, and acceleration in both the x- and y-directions approach ∞.

(e) At $t = 0$, B is at the midpoint of the top branch—hence, B_b.

Since $1 \in [0, 0.5\pi) \approx [0, 1.57)$, at $t = 1$, B must be on the right portion of the top branch—hence, B_c.

Position	B_a	B_b	B_c	B_d	B_e	B_f
Time t	5	0	1	2	3	4

Since $2, 3 \in (0.5\pi, \pi) \approx (1.57, 3.14)$, at $t = 2$ and $t = 3$, B must be on the left portion of the bottom branch. Since B is "near" "bottom-left" infinity an instant after $t = 0.5\pi$ and $t = 2$ is earlier than $t = 3$, it must be that $t = 2$ corresponds to B_d and $t = 3$ corresponds to B_e.

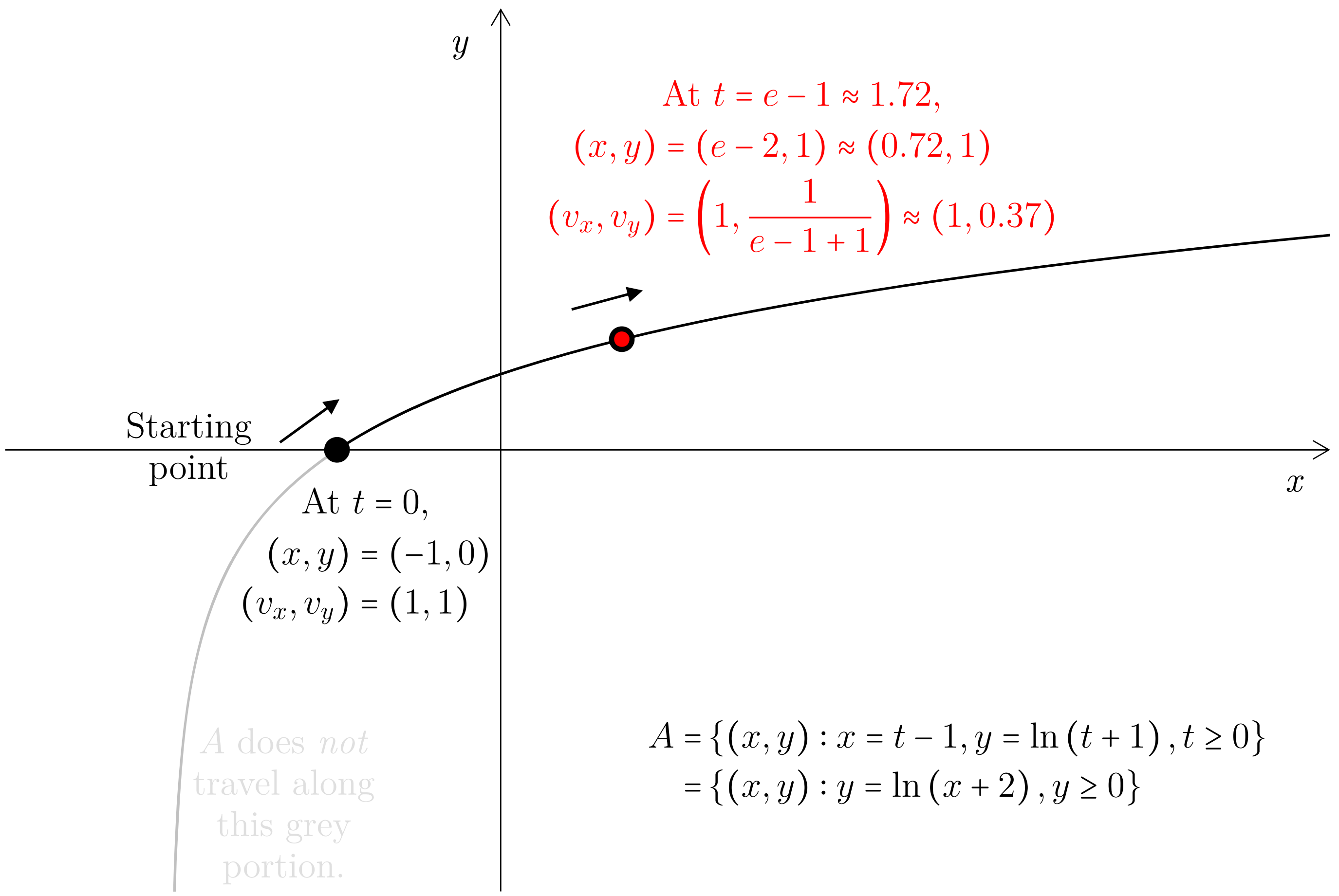
Since $4 \in (\pi, 1.5\pi) \approx (3.14, 4.71)$, at $t = 4$, B must be on the right portion of the bottom branch—hence, B_f.

Since $5 \in (1.5\pi, 2\pi) \approx (4.71, 6.28)$, at $t = 5$, B must be on the left portion of the top branch—hence, B_a.

A198(a)(i) Rewrite $x = t - 1$ as $t = x + 1$ and plug this into $y = \ln(t+1)$ to get $y = \ln(x+2)$. Noting that $t \geq 0 \iff t + 1 \geq 1 \iff y = \ln(t+1) \geq 0$, we can rewrite the set as:

$$A = \{(x, y) : y = \ln(x+2), y \geq 0\}.$$

(ii) The graph of $y = \ln(x+2)$ is simply the graph of $y = \ln x$ shifted leftwards by 2 units. Note the constraint $y \geq 0$—the particle A travels only along the black graph and does not travel along the grey portion.



(iii) As usual, compute $v_x = \dfrac{dx}{dt} = 1$ and $v_y = \dfrac{dy}{dt} = \dfrac{1}{t+1}$.

At $t = 0$, A starts at the position $(x, y) = (0 - 1, \ln(0+1)) = (-1, 0)$, and is moving rightwards $1\,\mathrm{m\,s^{-1}}$ and upwards at $1/(0+1) = 1\,\mathrm{m\,s^{-1}}$.

A's rightwards velocity stays fixed at $1\,\mathrm{m\,s^{-1}}$, while its upwards velocity decreases towards zero. As time progresses, A travels steadily towards "top-right infinity".

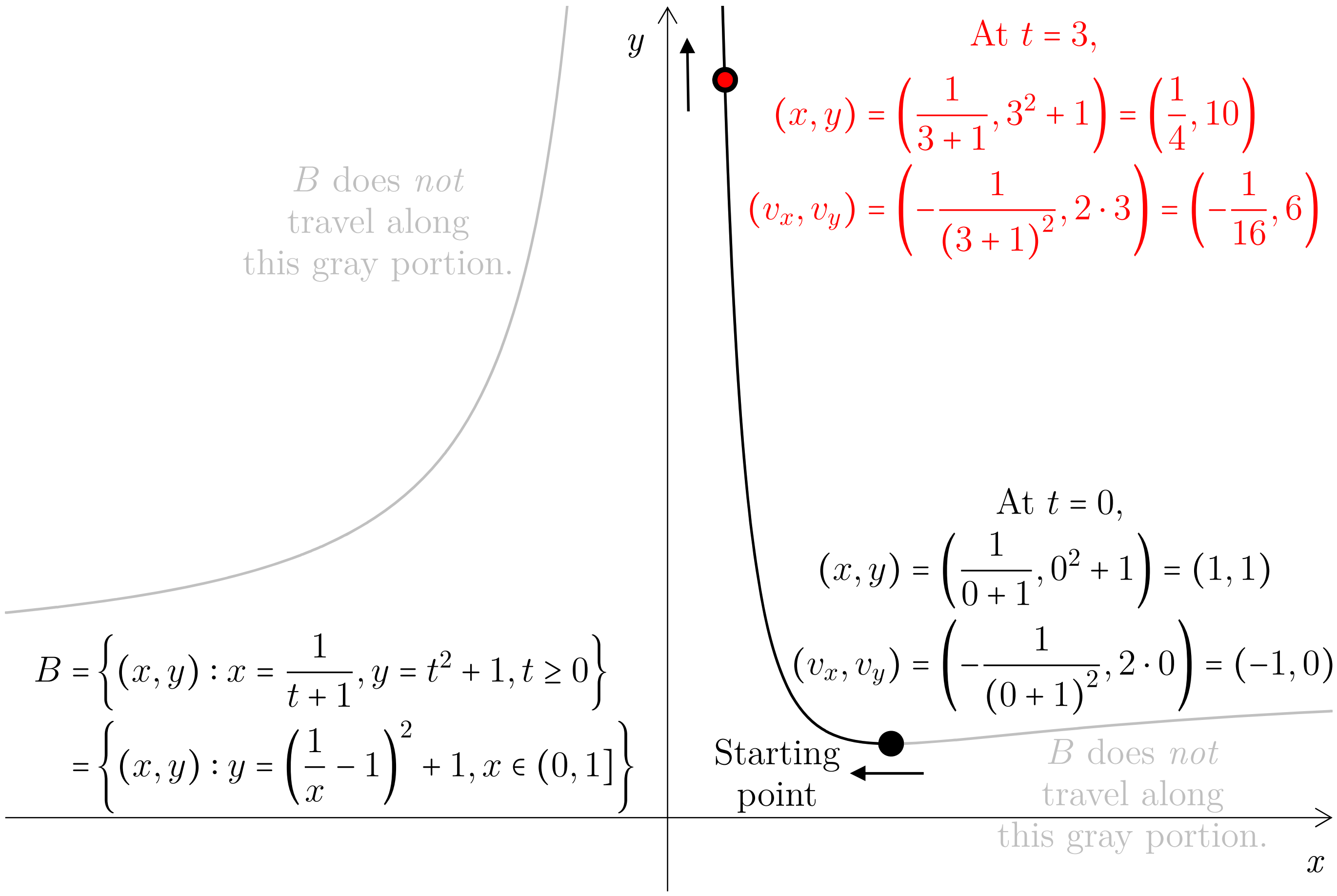
A198(b)(i) Rewrite $x = 1/(t+1)$ as $t = 1/x - 1$. Plug into $y = t^2 + 1$ to get $y = (1/x - 1)^2 + 1$.

Noting that $t \geq 0 \iff t+1 \geq 1 \iff x = \dfrac{1}{t+1} \in (0,1]$, we can rewrite the set as:

$$B = \left\{(x,y) : y = \left(\frac{1}{x} - 1\right)^2 + 1, x \in (0,1]\right\}.$$

(ii) Using your graphing calculator, we see that the complete graph of $y = (1/x - 1)^2 + 1$ has two branches.

However, we have the constraint $x \in (0,1]$. And so, particle B travels only along the black graph and does not travel along the grey portion.



(iii) As usual, compute $v_x = \dfrac{dx}{dt} = -\dfrac{1}{(t+1)^2}$ and $v_y = \dfrac{dy}{dt} = 2t$.

At $t = 0$, B starts at the position $(x,y) = \left(1/(0+1), 0^2+1\right) = (1,1)$, and is moving *left*wards at $1/(0+1)^2 = 1\,\text{m s}^{-1}$ and is upwards at $2 \cdot 0 = 0\,\text{m s}^{-1}$. That is, B is initially not moving in the y-direction.

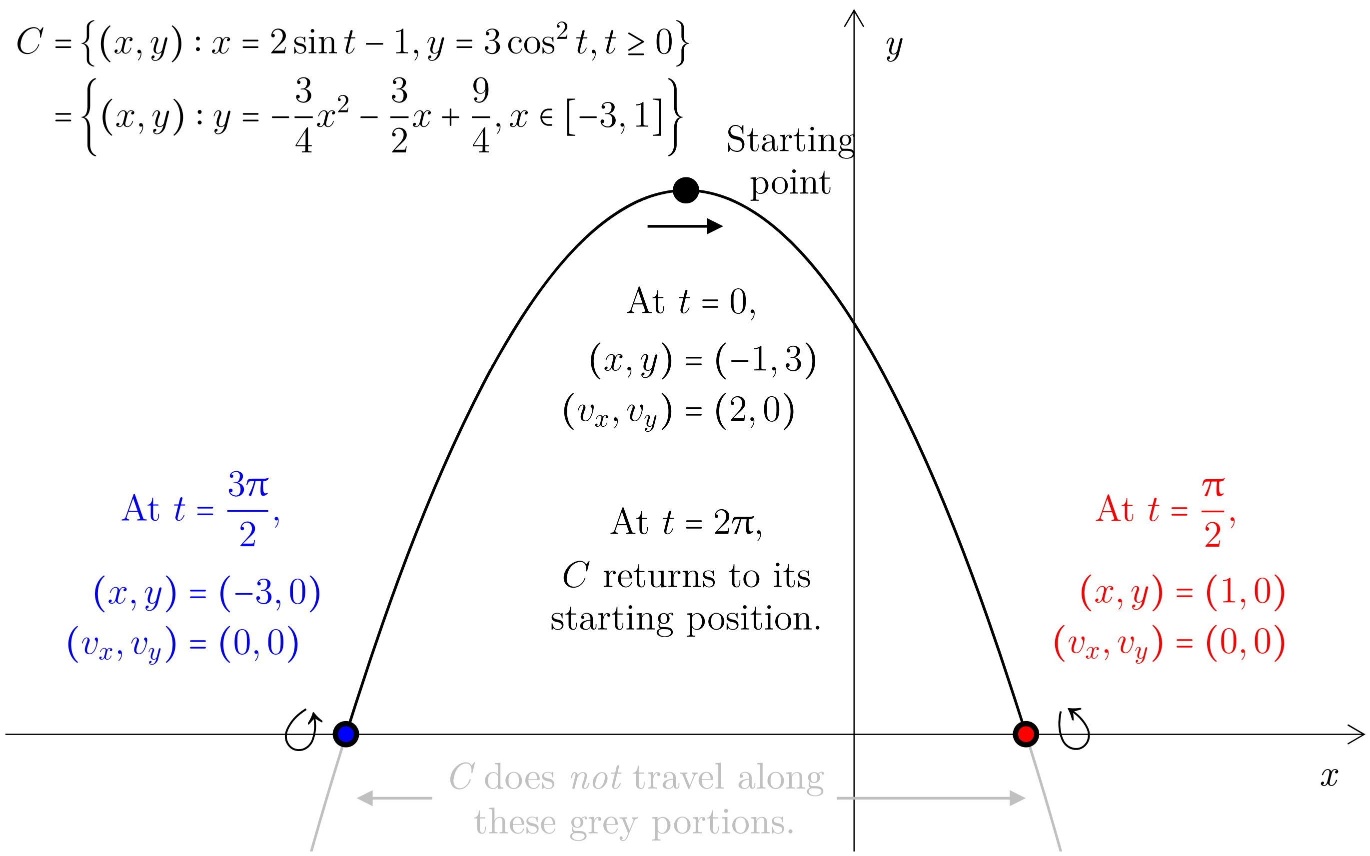
As time progresses, B move leftwards and upwards. Its leftwards velocity decreases towards zero, while its upwards velocity increases towards infinity.

A198(c)(i) Rewrite $x = 2\sin t - 1$ as $(x+1)/2 = \sin t$ and $y = 3\cos^2 t$ as $y/3 = \cos^2 t$. By the identity $\sin^2 t + \cos^2 t = 1$, we have $[(x+1)/2]^2 + (y/3)^2 = 1$, or,

$$y = 3\left[1 - \left(\frac{x+1}{2}\right)^2\right] = 3\left[1 - \frac{x^2+2x+1}{4}\right] = -\frac{3}{4}x^2 - \frac{3}{2}x + \frac{9}{4}.$$

Noting that $t \geq 0 \implies \sin t \in [-1,1] \iff x = 2\sin t - 1 \in [-3,1]$, we may rewrite the set as $C = \left\{(x,y) : y = -0.75x^2 - 1.5x + 2.25, x \in [-3,1]\right\}$.

(ii) The graph of $y = -0.75x^2 - 1.5x + 2.25$ is simply a ∩-shaped quadratic, with turning point at $x = -1$ and roots $x = -3, 1$. But note the constraint $x \in [-3,1]$—the particle C travels only along the black graph and not along the grey portion.



(iii) As usual, compute $v_x = \frac{dx}{dt} = 2\cos t$ and $v_y = \frac{dy}{dt} \overset{\text{Ch.}}{=} -6\cos t \sin t$.

At $t = 0$, C starts at $\left(2\sin 0 - 1, 3\cos^2 0\right) = (-1,3)$ (the maximum point of the parabola) and has velocity $(v_x, v_y) = (2\cos 0, -6\sin 0 \cos 0) = (2,0)$. That is, it is moving rightwards at $2\,\mathrm{m\,s^{-1}}$ (and not moving in the y-direction).

During $t \in (0, 0.5\pi)$, it moves rightwards along the parabola. At $t = 0.5\pi$, it is at the rightmost point of the black graph and $(v_x, v_y) = (0,0)$.

It then does a U-turn—during $t \in (0.5\pi, 1.5\pi)$, it moves leftwards along the parabola. At $t = \pi$, it is again at the maximum point of the parabola. And at $t = 1.5\pi$, it is at the leftmost point of the constrained parabola and again $(v_x, v_y) = (0,0)$.

It then does a U-turn—during $t \in (1.5\pi, 2\pi)$, it moves rightwards along the parabola. And at $t = 2\pi$, it is again at the maximum point of the parabola.

The particle has completed one period and will during $t \in [2\pi, 4\pi]$ repeat exactly the same movement made during $t \in [0, 2\pi]$. And so on.

150. Part II Answers (Sequences and Series)

150.1. Ch. 46 Answers (Sequences)

A199(a) Define $a : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \to \mathbb{R}$ by $a(n) = n^2$.

(b) Define $b : \{1, 2, 3, \dots, 100\} \to \mathbb{R}$ by $b(n) = 3n - 1$.

(c) Define $c : \{1, 2, 3, 4, 5, 6, 7\} \to \mathbb{R}$ by $c(n) = n^3$.

(d) Define $d : \mathbb{Z}^+ \to \mathbb{R}$ by $d(n) = 2n$ for n odd and $d(n) = 3n$ for n even.

(e) There is no obvious pattern here. So simply define $e : \{1, 2, 3\} \to \mathbb{R}$ by $e(1) = 5$, $e(2) = 0$, and $e(3) = 99$.

(f) Define $f : \mathbb{Z}^+ \to \mathbb{R}$ by $f(n) = 1 \times 2 \times \cdots \times n = n!$.

(g) Observe that the 1st, 3rd, 6th, 10th, 15th, and 21st terms are 1, while all the other terms are 0. Thus, define $g : \mathbb{Z}^+ \to \mathbb{R}$ by $g(n) = 1$ if n is a triangular number (i.e. $1, 3, 6, \dots$) and $g(n) = 0$ otherwise.

A200(a) $(c_n) = (-1, -3, -5, -7, -9, \dots)$.

(b) $(d_n) = (1, 8, 27, 64, 125, \dots)$.

(c) $(c_n + d_n) = (0, 5, 22, 57, 116, \dots)$.

(d) $(c_n - d_n) = (-2, -11, -32, -71, -134, \dots)$.

(e) $(c_n d_n) = (-1, -24, -135, -448, -1\,125, \dots)$.

(f) $\left(\dfrac{c_n}{d_n}\right) = \left(-1, -\dfrac{3}{8}, -\dfrac{5}{27}, -\dfrac{7}{64}, -\dfrac{9}{125}, \dots\right)$.

(g) $(kc_n) = (-2, -6, -10, -14, -18, \dots)$.

(h) $(kd_n) = (2, 16, 54, 128, 250, \dots)$.

150.2. Ch. 47 Answers (Series)

(*This chapter had no exercises.*)

150.3. Ch. 48 Answers (Summation Notation Σ)

A201(a) $\sum_{n=1}^{7} n! = 1 + 2 + 6 + 24 + 120 + 720 + 5\,040 = 5\,913.$

(b) $\sum_{n=1}^{8} (3n-1) = 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 = 100.$

(c) $\sum_{n=1}^{7} \frac{n}{2} = \frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + 3 + \frac{7}{2} = 14.$ **(d)** $\sum_{n=1}^{6} (9-n) = 8 + 7 + 6 + 5 + 4 + 3 = 33.$

A202(a) $\sum_{n=0}^{6} (n+1)! = 1 + 2 + 6 + 24 + 120 + 720 + 5\,040 = 5\,913.$

(b) $\sum_{n=0}^{7} (3n+2) = 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 = 100.$

(c) $\sum_{n=0}^{6} \left(\frac{n}{2} + \frac{1}{2}\right) = \frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + 3 + \frac{7}{2} = 14.$ **(d)** $\sum_{n=0}^{5} (8-n) = 8 + 7 + 6 + 5 + 4 + 3 = 33.$

A203(a)

$$\begin{aligned}\sum_{i=1}^{4} (2-i)^i &= (2-1)^1 + (2-2)^2 + (2-3)^3 + (2-4)^4 \\ &= 1^1 + 0^2 + (-1)^3 + (-2)^4 = 1 - 1 + 16 = 16.\end{aligned}$$

(b) $\sum_{\star=16}^{17} (4\star + 5) = (4 \cdot 16 + 5) + (4 \cdot 17 + 5) = 69 + 73 = 142.$

(c) $\sum_{x=31}^{33} (x-3) = (31-3) + (32-3) + (33-3) = 28 + 29 + 30 = 87.$

A204(a) $\sum_{n=1}^{\infty} n! = \sum n! = 1 + 2 + 6 + 24 + 120 + 720 + 5\,040 + \dots$

(b) $\sum_{n=1}^{\infty} (3n-1) = \sum (3n-1) = 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + \dots$

(c) $\sum_{n=1}^{\infty} \frac{n}{2} = \sum \frac{n}{2} = \frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + 3 + \frac{7}{2} + \dots$

(d) $\sum_{n=1}^{\infty} (9-n) = \sum (9-n) = 8 + 7 + 6 + 5 + 4 + 3 + \dots$

A205(a) $\sum_{n=0}^{\infty} (n+1)! = 1 + 2 + 6 + 24 + 120 + 720 + 5\,040 + \dots$

(b) $\sum_{n=0}^{\infty} (3n+2) = 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + \dots$

(c) $\sum_{n=0}^{\infty} \left(\frac{n}{2} + \frac{1}{2}\right) = \frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + 3 + \frac{7}{2} + \dots$ **(d)** $\sum_{n=0}^{\infty} (8-n) = 8 + 7 + 6 + 5 + 4 + 3 + \dots$

150.4. Ch. 49 Answers (Arithmetic Sequences and Series)

A206(a) $a_1 = 2$, $a_k = 997$, and $d = 5$. So, $k = (997 - 2)/5 + 1 = 200$. Thus, the sum is

$$(a_1 + a_k)\frac{k}{2} = (2 + 997)\frac{200}{2} = 99\,900.$$

(b) $b_1 = 3$, $b_k = 1\,703$, and $d = 17$. So, $k = (1703 - 3)/17 + 1 = 101$ terms. Thus, the sum is

$$(b_1 + b_k)\frac{k}{2} = (3 + 1703)\frac{101}{2} = 86\,153.$$

(c) $c_1 = 81$, $c_k = 8\,081$, and $d = 5$. So, $k = (8\,081 - 81)/8 + 1 = 1\,001$ terms. Thus, the sum is

$$(c_1 + c_k)\frac{k}{2} = (81 + 8\,081)\frac{1\,001}{2} = 4\,085\,081.$$

150.5. Ch. 50 Answers (Geometric Sequences and Series)

A207(a) $a_1 = 7$, $a_k = 896$, and $r = 2$. By Corollary 20, the sum of this series is

$$\frac{a_1 - ra_k}{1 - r} = \frac{7 - 2 \cdot 896}{1 - 2} = 1\,785.$$

(b) $b_1 = 20$, $b_k = 5/8$, and $r = 1/2$. By Corollary 20, the sum of this series is

$$\frac{b_1 - rb_k}{1 - r} = \frac{20 - \frac{1}{2} \cdot \frac{5}{8}}{1 - \frac{1}{2}} = 2\left(20 - \frac{1}{2} \cdot \frac{5}{8}\right) = 40 - \frac{5}{8} = 39\frac{3}{8}.$$

(c) $c_1 = 1$, $c_k = 1/243$, and $r = 1/3$. By Corollary 20, the sum of this series is

$$\frac{c_1 - rc_k}{1 - r} = \frac{1 - \frac{1}{3} \cdot \frac{1}{243}}{1 - \frac{1}{3}} = \frac{3}{2}\left(1 - \frac{1}{729}\right) = \frac{364}{243}.$$

A208(a) $a_1 = 6$ and $r = 3/4$. Thus, the sum of this series is $6/(1 - 3/4) = 24$.

(b) $b_1 = 20$ and $r = 1/2$. Thus, the sum of this series is $20/(1 - 1/2) = 40$.

(c) $c_1 = 1$ and $r = 1/3$. Thus, the sum of this series is $1/(1 - 1/3) = 3/2$.

A209(a) $\sum_{n=5}^{100} 1 = \sum_{n=1}^{100} 1 - \sum_{n=1}^{4} 1 = 100 - 4 = 96.$

(b) $\sum_{n=5}^{100} n = \sum_{n=1}^{100} n - \sum_{n=1}^{4} n = 5\,050 - 10 = 5\,040.$

(c) $\sum_{n=5}^{100} (n+1) = \sum_{n=5}^{100} n + \sum_{n=5}^{100} 1 = 5\,040 + 96 = 5\,136.$

(d) $\sum_{n=5}^{100} (3n+2) = 3\sum_{n=5}^{100} n + 2\sum_{n=5}^{100} 1 = 3 \cdot 5\,040 + 2 \cdot 96 = 15\,312.$

(e) $\sum_{n=5}^{100} nx = x\sum_{n=5}^{100} n = 5\,040x.$

A210(a) Let
$$S_5 = 1 + 2x + 3x^2 + 4x^3 + 5x^4.$$

Then,
$$xS_5 = x + 2x^2 + 3x^3 + 4x^4 + 5x^5.$$

And so,
$$S_5 - xS_5 = (1-x)\,S_5 = 1 + x + x^2 + x^3 + x^4 - 5x^5 = \frac{1-x^5}{1-x} - 5x^5$$
$$= \frac{1 - x^5 - 5x^5 + 5x^6}{1-x} = \frac{1 - 6x^5 + 5x^6}{1-x}.$$

Thus,
$$S_5 = \frac{1 - 6x^5 + 5x^6}{(1-x)^2}.$$

(b) Let
$$S_k = 1 + 2x + 3x^2 + 4x^3 + \cdots + kx^{k-1}.$$

Then,
$$xS_k = x + 2x^2 + 3x^3 + 4x^4 + \cdots + kx^k.$$

And so,
$$S_k - xS_k = (1-x)\,S_k = 1 + x + x^2 + x^3 + \cdots + x^{k-1} - kx^k = \frac{1-x^k}{1-x} - kx^k$$
$$= \frac{1 - x^k - kx^k + kx^{k+1}}{1-x} = \frac{1 - (k+1)\,x^k + kx^{k+1}}{1-x}.$$

Thus,
$$S_k = \frac{1 - (k+1)\,x^k + kx^{k+1}}{(1-x)^2}.$$

150.7. Ch. 52 Answers (Method of Differences)

A211(a) Observe that $3 = 4 - 1 = 2^2 - 1$, $8 = 9 - 1 = 3^2 - 1$, $15 = 4^2 - 1$, etc.

Hence, the nth term is $$\frac{1}{(n+1)^2 - 1}.$$

And, $$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots + \frac{1}{999\,999} = \sum_{n=1}^{999} \frac{1}{(n+1)^2 - 1}.$$

Take the nth term, factorise its denominator, then do the partial fractions decomposition:

$$\frac{1}{(n+1)^2 - 1} = \frac{1}{(n+1-1)(n+1+1)} = \frac{1}{n(n+2)}$$

$$= \frac{A}{n} + \frac{B}{n+2} = \frac{A(n+2) + Bn}{n(n+2)} = \frac{(A+B)n + 2A}{n(n+2)}.$$

Comparing coefficients, $A + B = 0$ and $2A = 1$. Hence, $A = 1/2$ and $B = -1/2$ and

$$\frac{1}{(n+1)^2 - 1} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

Thus, $$\sum_{n=1}^{999} \frac{1}{(n+1)^2 - 1}$$

$$= \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots + \frac{1}{999\,999}$$

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{1}{997} - \frac{1}{999} + \frac{1}{998} - \frac{1}{1\,000} + \frac{1}{999} - \frac{1}{1\,001}\right).$$

Observe that all the terms with denominators 3 through 999 will be cancelled out.

Thus, $$\sum_{n=1}^{999} \frac{1}{(n+1)^2 - 1} = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{1\,000} - \frac{1}{1\,001}\right) = \frac{3}{4} - \frac{1}{2\,000} - \frac{1}{2\,002} = 0.749\ldots$$

More generally, $$\sum_{n=1}^{k} \frac{1}{(n+1)^2 - 1} = \frac{3}{4} - \frac{1}{2(k+1)} - \frac{1}{2(k+2)}.$$

Hence, $$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \ldots = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{(n+1)^2 - 1}$$

$$= \lim_{k\to\infty}\left(\frac{3}{4} - \frac{1}{2(k+1)} - \frac{1}{2(k+2)}\right) = \frac{3}{4}.$$

(b) The nth term is simply: $\lg \dfrac{n}{n+1}$ or $\lg n - \lg(n+1)$.

Thus,
$$\sum_{n=1}^{999} \lg \frac{n}{n+1} = \lg\frac{1}{2} + \lg\frac{2}{3} + \lg\frac{3}{4} + \cdots + \lg\frac{999}{1\,000}$$
$$= \lg 1 - \lg 2 + \lg 2 - \lg 3 + \lg 3 - \lg 4 + \cdots + \lg 999 - \lg 1\,000$$
$$= \lg 1 - \lg 1\,000 = 0 - 3 = -3.$$

More generally:
$$\sum_{n=1}^{k} \lg \frac{n}{n+1} = \lg 1 - \lg(k+1) = -\lg(k+1)$$

Hence,
$$\lg\frac{1}{2} + \lg\frac{2}{3} + \lg\frac{3}{4} + \cdots = \lim_{k\to\infty} \sum_{n=1}^{k} \lg\frac{n}{n+1} = \lim_{k\to\infty}\left(-\lg(k+1)\right) = -\infty.$$

That is, the infinite series diverges.

(c) The nth term is $\dfrac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$. Rationalise the surds:

$$\frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{(n+1)\sqrt{n} - n\sqrt{n+1}} = \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{(n+1)^2 n - n^2(n+1)}$$
$$= \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{n^3 + 2n^2 + n - (n^3 + n^2)} = \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{n^2 + n} = \frac{\sqrt{n}}{n} - \frac{\sqrt{n+1}}{n+1}.$$

Thus,
$$\sum_{n=1}^{99} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} = \frac{1}{2\sqrt{1} + 1\sqrt{2}} + \frac{1}{3\sqrt{2} + 2\sqrt{3}} + \cdots + \frac{1}{100\sqrt{99} + 99\sqrt{100}}$$
$$= \frac{\sqrt{1}}{1} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3} + \cdots + \frac{\sqrt{99}}{99} - \frac{\sqrt{100}}{100}$$
$$= \frac{\sqrt{1}}{1} - \frac{\sqrt{100}}{100} = 1 - \frac{10}{100} = 0.9.$$

More generally:
$$\sum_{n=1}^{k} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} = 1 - \frac{\sqrt{k+1}}{k} = 1 - \frac{1}{\sqrt{k+1}}.$$

Hence,
$$\lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} = \lim_{k\to\infty}\left(1 - \frac{1}{\sqrt{k+1}}\right) = 1$$

(d) First, observe that $(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$. Hence,

$$\begin{aligned}\sum_{n=1}^{k}\left[(n+1)^4 - n^4\right] &= \sum_{n=1}^{k}\left(4n^3 + 6n^2 + 4n + 1\right) \\ &= 4\sum_{n=1}^{k} n^3 + 6\sum_{i=1}^{k} n^2 + 4\sum_{i=1}^{k} n + \sum_{i=1}^{k} 1 \\ &= 4\sum_{n=1}^{k} n^3 + k(k+1)(2k+1) + 2k(k+1) + k \\ &\overset{1}{=} 4\sum_{n=1}^{k} n^3 + 2k^3 + 5k^2 + 4k.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\sum_{n=1}^{k}\left[(n+1)^4 - n^4\right] &= 2^4 - 1^4 + 3^4 - 2^4 + 4^4 - 3^4 + \cdots + (k+1)^4 - k^4 \\ &= (k+1)^4 - 1^4 \overset{2}{=} k^4 + 4k^3 + 6k^2 + 4k.\end{aligned}$$

Putting $\overset{1}{=}$ and $\overset{2}{=}$ together, we have

$$4\sum_{n=1}^{k} n^3 + k(k+1)(2k+3) + k = k^4 + 4k^3 + 6k^2 + 4k$$

$$\iff \quad \sum_{n=1}^{k} n^3 = \frac{k^4 + 4k^3 + 6k^2 + 4k - \left(2k^3 + 5k^2 + 4k\right)}{4} = \frac{k^4 + 2k^3 + k^2}{4} = \frac{k^2(k+1)^2}{4}.$$

And so, we have in particular:

$$\sum_{n=1}^{100} n^3 = \frac{100^2(100+1)^2}{4} = \frac{10\,000\,(10\,201)}{4} = 25\,502\,500$$

The corresponding infinite series diverges. That is,

$$1^3 + 2^3 + 3^3 + \cdots = \lim_{k\to\infty}\sum_{i=1}^{k} n^3 = \lim_{k\to\infty}\frac{k^2(k+1)^2}{4} = \infty.$$

151. Part III Answers (Vectors)

151.1. Ch. 53 Answers (Introduction to Vectors)

A212.	Tail	Head	Length	This vector carries us ...
$\overrightarrow{AG} = (4,7) = \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \mathbf{a}$	A	G	$\sqrt{65}$	4 unit(s) E and 7 unit(s) N
$\overrightarrow{BA} = (-3,-4) = \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \mathbf{b}$	B	A	5	3 " W " 4 " S
$\overrightarrow{BG} = (1,3) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \mathbf{c}$	B	G	$\sqrt{10}$	1 " E " 3 " N
$\overrightarrow{GA} = (-4,-7) = \begin{pmatrix} -4 \\ -7 \end{pmatrix} = \mathbf{d}$	G	A	$\sqrt{65}$	4 " W " 7 " S
$\overrightarrow{GB} = (-1,-3) = \begin{pmatrix} -1 \\ -3 \end{pmatrix} = \mathbf{e}$	G	B	$\sqrt{10}$	1 " W " 3 " S

A213. Here is one possible counterexample (yours may be different but still correct).

Let $\mathbf{u} = (1,0)$ and $\mathbf{v} = (0,1)$. Then $|\mathbf{u}| = |(1,0)| = \sqrt{1^2+0^2} = 1$ and $|\mathbf{v}| = |(0,1)| = \sqrt{0^2+1^2} = 1$, so that $|\mathbf{u}| + |\mathbf{v}| = 1+1 = 2$. However, $\mathbf{u}+\mathbf{v} = (1,1)$, so that $|\mathbf{u}+\mathbf{v}| = |(1,1)| = \sqrt{1^2+1^2} = \sqrt{2}$. Hence, $|\mathbf{u}+\mathbf{v}| \neq |\mathbf{u}| + |\mathbf{v}|$.

(As we'll learn later, the correct assertion is this: $|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$, with $|\mathbf{u}+\mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ point in the same direction.)

A214. Given the vector $(4,-3)$:

(a) If its tail is $(0,0)$, then its head is $(0,0)+(4,-3) = (4,-3)$.

(b) If its head is $(0,0)$, then its tail is $(0,0)-(4,-3) = (-4,3)$.

(c) If its tail is $(5,2)$, then its head is $(5,2)+(4,-3) = (9,-1)$.

(d) If its head is $(5,2)$, then its tail is $(5,2)-(4,-3) = (1,5)$.

A215. $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$, $\overrightarrow{DC} + \overrightarrow{CA} = \overrightarrow{DA}$, $\overrightarrow{BD} + \overrightarrow{DA} = \overrightarrow{BA}$, $\overrightarrow{AD} - \overrightarrow{CD} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$, $-\overrightarrow{DC} - \overrightarrow{BD} = \overrightarrow{CD} + \overrightarrow{DB} = \overrightarrow{CB}$, and $\overrightarrow{BD} + \overrightarrow{DB} = \overrightarrow{BB} = (0,0) = \mathbf{0}$.

A216(a) $c\mathbf{v} = (cv_1, cv_2)$.

(b) $|c\mathbf{v}| = |(cv_1,cv_2)| = \sqrt{(cv_1)^2 + (cv_2)^2} = \sqrt{c^2v_1^2 + c^2v_2^2} = \sqrt{c^2\left(v_1^2+v_2^2\right)} = |c|\sqrt{v_1^2+v_2^2} = |c|\,|\mathbf{v}|$.

A217(a) $\overrightarrow{AB} = (1,3)$, $\overrightarrow{AC} = (4,2)$, $\overrightarrow{BC} = (3,-1)$.

$2\overrightarrow{AB} = (2,6)$, $3\overrightarrow{AC} = (12,6)$, $4\overrightarrow{BC} = (12,-4)$.

(b) $\left|2\overrightarrow{AB}\right| = \sqrt{2^2+6^2} = \sqrt{40} = 2\sqrt{10}$, $\left|\overrightarrow{AB}\right| = \sqrt{1^2+3^2} = \sqrt{10}$, and so $\left|2\overrightarrow{AB}\right| = 2\left|\overrightarrow{AB}\right|$.

$\left|3\overrightarrow{AC}\right| = \sqrt{12^2+6^2} = \sqrt{180} = 3\sqrt{20}$, $\left|\overrightarrow{AC}\right| = \sqrt{4^2+2^2} = \sqrt{20}$, and so $\left|3\overrightarrow{AC}\right| = 3\left|\overrightarrow{AC}\right|$.

$\left|4\overrightarrow{BC}\right| = \sqrt{12^2+(-4)^2} = \sqrt{160} = 4\sqrt{10}$, $\left|\overrightarrow{BC}\right| = \sqrt{3^2+(-1)^2} = \sqrt{10}$, and so $\left|4\overrightarrow{BC}\right| = 4\left|\overrightarrow{BC}\right|$.

A218. The vectors $\mathbf{b}$ and $\mathbf{c}$ point in the exact opposite directions because $\mathbf{c} = -3\mathbf{b}$. The vectors $\mathbf{b}$ and $\mathbf{d}$ point in different directions because $\mathbf{b} \neq k\mathbf{d}$ for any k.

A219. By Facts 121 and 122, $|c\hat{\mathbf{v}}| = |c|\,|\hat{\mathbf{v}}| = |c| \cdot 1 = |c|$.

A220. The unit vectors of $\overrightarrow{AB} = (1,3)$, $\overrightarrow{AC} = (4,2)$, and $\overrightarrow{BC} = (3,-1)$ are, respectively,

$$\begin{aligned}
\hat{\overrightarrow{AB}} &= \frac{(1,3)}{\sqrt{1^2+3^2}} &= \frac{1}{\sqrt{10}}(1,3) &= \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right), \\
\hat{\overrightarrow{AC}} &= \frac{(4,2)}{\sqrt{4^2+2^2}} &= \frac{1}{\sqrt{20}}(4,2) &= \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \\
\hat{\overrightarrow{BC}} &= \frac{(3,-1)}{\sqrt{3^2+(-1)^2}} &= \frac{1}{\sqrt{10}}(3,-1) &= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right).
\end{aligned}$$

Of course, the above are also the unit vectors of $2\overrightarrow{AB}$, $3\overrightarrow{AC}$, and $4\overrightarrow{BC}$, respectively.

A221. For $\mathbf{v} = (3,2)$, first write $3 \overset{1}{=} 1\alpha + 3\beta$ and $2 \overset{2}{=} 2\alpha + 4\beta$.

$\overset{2}{=}$ minus $2\times \overset{1}{=}$ yields $-2\beta = -4$ or $\beta = 2$ and hence $\alpha = -3$. Thus,

$$\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 4 \end{pmatrix} = -3\mathbf{a} + 2\mathbf{b}.$$

For $\mathbf{w} = (-1,0)$, first write $-1 \overset{1}{=} 1\alpha + 3\beta$ and $0 \overset{2}{=} 2\alpha + 4\beta$.

$\overset{2}{=}$ minus $2\times \overset{1}{=}$ yields $-2\beta = 2$ or $\beta = -1$ and hence $\alpha = 2$. Thus,

$$\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 2\mathbf{a} - \mathbf{b}.$$

A222. Since $\mathbf{a} \nparallel \mathbf{b}$, any vector can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

For $\mathbf{i} = (1, 0)$, first write $\alpha + 7\beta \overset{1}{=} 1$ and $3\alpha + 5\beta \overset{2}{=} 0$.

$\overset{2}{=}$ minus $3\times \overset{1}{=}$ yields $-16\beta = -3$ or $\beta = 3/16$ and hence $\alpha = -5/16$. Thus,

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{5}{16}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{3}{16}\begin{pmatrix} 7 \\ 5 \end{pmatrix}.$$

For $\mathbf{j} = (1, 0)$, first write $\alpha + 7\beta \overset{1}{=} 0$ and $3\alpha + 5\beta \overset{2}{=} 1$.

$\overset{2}{=}$ minus $3\times \overset{1}{=}$ yields $-16\beta = 1$ or $\beta = -1/16$ and hence $\alpha = 7/16$. Thus,

$$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{7}{16}\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{1}{16}\begin{pmatrix} 7 \\ 5 \end{pmatrix}.$$

For $\mathbf{d} = (1, 1)$, first write $\alpha + 7\beta \overset{1}{=} 1$ and $3\alpha + 5\beta \overset{2}{=} 1$.

$\overset{2}{=}$ minus $3\times \overset{1}{=}$ yields $-16\beta = -2$ or $\beta = 1/8$ and hence $\alpha = 1/8$. Thus,

$$\mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 7 \\ 5 \end{pmatrix}.$$

A223. The position vectors of P, Q, and R are

$$\begin{aligned}
\mathbf{p} &= \frac{6\mathbf{a} + 5\mathbf{b}}{5 + 6} = \frac{6}{11}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{5}{11}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 21 \\ 32 \end{pmatrix}, \\
\mathbf{q} &= \frac{\mathbf{a} + 5\mathbf{b}}{5 + 1} = \frac{1}{6}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \frac{5}{6}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 11 \\ 19 \end{pmatrix}, \\
\mathbf{r} &= \frac{3\mathbf{a} + 2\mathbf{b}}{2 + 3} = \frac{3}{5}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ -2 \end{pmatrix}.
\end{aligned}$$

Thus, $\quad P = \left(\frac{21}{11}, \frac{32}{11}\right), \quad Q = \left(\frac{11}{6}, \frac{19}{6}\right), \quad$ and $\quad R = \left(\frac{3}{5}, -\frac{2}{5}\right)$.

A224(a) $(1,3)$ (or any scalar multiple thereof). **(b)** $(1,-2/7)$ (ditto). **(c)** $(1,0)$ (ditto). **(d)** $(0,1)$ (ditto).

A225(a) The line described by $-5x+y+1=0$ contains the point $(0,-1)$ and has direction vector $(1,5)$. Thus, it can also be described by $\mathbf{r}=(0,-1)+\lambda(1,5)$ $(\lambda\in\mathbb{R})$. $\lambda=-1$, $\lambda=0$, and $\lambda=1$ produce the points $(-1,-6)$, $(0,-1)$, and $(1,4)$.

(b) The line described by $x-2y-1=0$ contains the point $(1,0)$ and has direction vector $(2,1)$. Thus, it can also be described by $\mathbf{r}=(1,0)+\lambda(2,1)$ $(\lambda\in\mathbb{R})$. $\lambda=-1$, $\lambda=0$, and $\lambda=1$ produce the points $(-1,-1)$, $(1,0)$, and $(3,1)$.

(c) The line described by $y-4=0$ contains the point $(0,4)$ and has direction vector $(1,0)$. Thus, it can also be described by $\mathbf{r}=(0,4)+\lambda(1,0)$ $(\lambda\in\mathbb{R})$. $\lambda=-1$, $\lambda=0$, and $\lambda=1$ produce the points $(-1,4)$, $(0,4)$, and $(1,4)$.

(d) The line described by $x-4=0$ contains the point $(4,0)$ and has direction vector $(0,1)$. Thus, it can also be described by $\mathbf{r}=(4,0)+\lambda(0,1)$ $(\lambda\in\mathbb{R})$. $\lambda=-1$, $\lambda=0$, and $\lambda=1$ produce the points $(4,-1)$, $(4,0)$, and $(4,1)$.

A226. If $\mathbf{v}=\mathbf{0}$, then the "line" would not be a line, but the single point P:

$$\{R:\mathbf{r}=\mathbf{p}+\lambda\mathbf{v}\ \ (\lambda\in\mathbb{R})\}=\{R:\mathbf{r}=\mathbf{p}\}=\{P\}.$$

And so, we impose the restriction that $\mathbf{v}\neq\mathbf{0}$ to rule out the above trivial (or degenerate) case.

A227(a) Write out $x\overset{1}{=}-1+\lambda$ and $y\overset{2}{=}3-2\lambda$.

Then $\overset{2}{=}$ plus $2\times\overset{1}{=}$ yields: $y+2x=1$ or $y=-2x+1$.

(b) Write out $x\overset{1}{=}5+7\lambda$ and $y\overset{2}{=}6+8\lambda$.

Then $7\times\overset{2}{=}$ minus $8\times\overset{1}{=}$ yields: $7y-8x=2$ or $y=\dfrac{8}{7}x+\dfrac{2}{7}$.

(c) Write out $x\overset{1}{=}3\lambda$ and $y\overset{2}{=}-3$.

This is a horizontal line. We can discard $\overset{1}{=}$ and be left with the single equation $y\overset{2}{=}-3$.

(d) Write out $x\overset{1}{=}1$ and $y\overset{2}{=}1+2\lambda$.

This is a vertical line. We can discard $\overset{2}{=}$ and be left with the single equation $x\overset{1}{=}1$.

A228(a) $\dfrac{x-(-1)}{1}=\dfrac{y-3}{-2}$ or $y=-2x+1$.

(b) $\dfrac{x-5}{7}=\dfrac{y-6}{8}$ or $y=\dfrac{8}{7}x-\dfrac{2}{7}$.

(c) $y=-3$.

(d) $x=1$.

151.3. Ch. 55 Answers (The Scalar Product)

A229.

$$\mathbf{v}\cdot\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 8 \\ 7 \end{pmatrix} = 16 + 7 = 23,$$

$$\mathbf{w}\cdot\mathbf{x} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 8 \\ 7 \end{pmatrix} = -32 + 0 = -32,$$

Since the scalar product is commutative, $\mathbf{w}\cdot\mathbf{v} = -8$, $\mathbf{x}\cdot\mathbf{v} = 23$, and $\mathbf{x}\cdot\mathbf{w} = -32$.

Since the scalar product is distributive, $\mathbf{w}\cdot(\mathbf{x}+\mathbf{v}) = \mathbf{w}\cdot\mathbf{x} + \mathbf{w}\cdot\mathbf{v} = -32 - 8 = -40$,

Also, $(2\mathbf{v})\cdot\mathbf{x} = 2(\mathbf{v}\cdot\mathbf{x}) = 2\cdot 23 = 46$ and $\mathbf{w}\cdot(2\mathbf{x}) = (2\mathbf{x})\cdot\mathbf{w} = 2(\mathbf{x}\cdot\mathbf{w}) = 2(-32) = -64$.

A231.

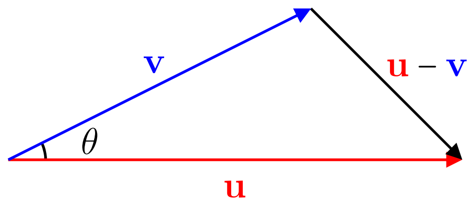
$$|\mathbf{a}| = |(-2,3)| = \sqrt{(-2)^2 + 3^2} = \sqrt{(-2,3)\cdot(-2,3)} = \sqrt{\mathbf{a}\cdot\mathbf{a}}.$$

$$|\mathbf{b}| = |(7,1)| = \sqrt{7^2 + 1^2} = \sqrt{(7,1)\cdot(7,1)} = \sqrt{\mathbf{b}\cdot\mathbf{b}}.$$

$$|\mathbf{c}| = |(5,-4)| = \sqrt{5^2 + (-4)^2} = \sqrt{(5,-4)\cdot(5,-4)} = \sqrt{\mathbf{b}\cdot\mathbf{b}}.$$

151.4. Ch. 56 Answers (The Angle Between Two Vectors)

A232(a) The third side of the triangle corresponds to the vector $\mathbf{u}-\mathbf{v}$.



(b) $|\mathbf{u}|$, $|\mathbf{v}|$, and $|\mathbf{u}-\mathbf{v}|$.

(c) $|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,|\mathbf{u}|\,|\mathbf{v}|\cos\theta$.

(d) We'll actually use distributivity twice:

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}-\mathbf{v})\cdot\mathbf{u} + (\mathbf{u}-\mathbf{v})\cdot(-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v}.\end{aligned}$$

(e)

$$\begin{aligned} & |\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,|\mathbf{u}|\,|\mathbf{v}|\cos\theta && \text{(By } (c)\text{)} \\ \iff\; & (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\,|\mathbf{u}|\,|\mathbf{v}|\cos\theta && \text{(By Fact 131)} \\ \iff\; & \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\,|\mathbf{u}|\,|\mathbf{v}|\cos\theta && \text{(By } (d)\text{)} \\ \iff\; & -2\mathbf{u}\cdot\mathbf{v} = -2\,|\mathbf{u}|\,|\mathbf{v}|\cos\theta \\ \iff\; & \mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta \\ \iff\; & \theta = \cos^{-1}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}. \end{aligned}$$

A233(a) The angle between $\mathbf{u} = (2,0)$ and $\mathbf{v} = (0,17)$ is

$$\theta = \cos^{-1}\frac{(2,0)\cdot(0,17)}{|(2,0)|\,|(0,17)|} = \cos^{-1}\frac{2\cdot 0 + 0\cdot 17}{\sqrt{2^2+0^2}\sqrt{0^2+17^2}} = \cos^{-1} 0 = \frac{\pi}{2}.$$

θ is right. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular and point in different directions.

(b) The angle between $\mathbf{u} = (5,0)$ and $\mathbf{v} = (-3,0)$ is

$$\theta = \cos^{-1}\frac{(5,0)\cdot(-3,0)}{|(5,0)|\,|(-3,0)|} = \frac{5\cdot(-3)+0\cdot 0}{\sqrt{5^2+0^2}\sqrt{(-3)+0^2}} = \cos^{-1}\frac{-15}{5\cdot 3} = \cos^{-1}(-1) = \pi.$$

θ is straight. $\mathbf{u}$ and $\mathbf{v}$ are parallel and point in exact opposite directions.

(c) The angle between $\mathbf{u} = (1,0)$ and $\mathbf{v} = \left(1,\sqrt{3}\right)$ is

$$\theta = \cos^{-1}\frac{(1,0)\cdot\left(1,\sqrt{3}\right)}{|(1,0)|\left|\left(1,\sqrt{3}\right)\right|} = \cos^{-1}\frac{1\cdot 1 + 0\cdot\sqrt{3}}{\sqrt{1^2+0^2}\sqrt{1^2+\left(\sqrt{3}\right)^2}} = \cos^{-1}\frac{1}{1\cdot\sqrt{4}} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}.$$

θ is acute. $\mathbf{u}$ and $\mathbf{v}$ are neither parallel nor perpendicular and point in different directions.

(d) The angle between $\mathbf{u} = (2,-3)$ and $\mathbf{v} = (1,2)$ is

$$\theta = \cos^{-1}\frac{(2,-3)\cdot(1,2)}{|(2,-3)|\,|(1,2)|} = \cos^{-1}\frac{2\cdot 1 + (-3)\cdot 2}{\sqrt{2^2+(-3)^2}\sqrt{1^2+2^2}} = \cos^{-1}\frac{-4}{\sqrt{13}\cdot\sqrt{5}} \approx 2.0899.$$

θ is obtuse. $\mathbf{u}$ and $\mathbf{v}$ are neither parallel nor perpendicular and point in different directions.

A234.

$$\begin{aligned}
|\mathbf{u}+\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) && \text{(Fact 131)}\\
&= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} && \text{(Distributivity)}\\
&= \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} && \text{(Commutativity)}\\
&= |\mathbf{u}|^2+2\mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2 && \text{(Fact 131 again)}\\
&= |\mathbf{u}|^2+0+|\mathbf{v}|^2 && (\mathbf{u}\cdot\mathbf{v}=0 \text{ because } \mathbf{u}\perp\mathbf{v})\\
&= |\mathbf{u}|^2+|\mathbf{v}|^2.
\end{aligned}$$

A235.

$$\begin{aligned}
|\mathbf{u}+\mathbf{v}|^2 &= |\mathbf{u}|^2+2\mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2\\
&\le |\mathbf{u}|^2+2\,|\mathbf{u}|\,|\mathbf{v}|+|\mathbf{v}|^2 && \text{(Cauchy's Inequality)}\\
&= (|\mathbf{u}|+|\mathbf{v}|)^2. && \text{(Complete the square)}
\end{aligned}$$

We've just shown that $|\mathbf{u}+\mathbf{v}|^2 \le (|\mathbf{u}|+|\mathbf{v}|)^2$. And so, taking square roots, we also have $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}|+|\mathbf{v}|$.

A236.

	(a) $(1,3)$	**(b)** $(4,2)$	**(c)** $(-1,2)$
x**-direction cosine**	$\dfrac{1}{\sqrt{1^2+3^2}}=\dfrac{1}{\sqrt{10}}$	$\dfrac{4}{\sqrt{4^2+2^2}}=\dfrac{4}{\sqrt{20}}=\dfrac{2}{\sqrt{5}}$	$\dfrac{-1}{\sqrt{(-1)^2+2^2}}=\dfrac{-1}{\sqrt{5}}$
y**-direction cosine**	$\dfrac{3}{\sqrt{1^2+3^2}}=\dfrac{3}{\sqrt{10}}$	$\dfrac{2}{\sqrt{4^2+2^2}}=\dfrac{2}{\sqrt{20}}=\dfrac{1}{\sqrt{5}}$	$\dfrac{2}{\sqrt{(-1)^2+2^2}}=\dfrac{2}{\sqrt{5}}$
Unit vector	$\left(\dfrac{1}{\sqrt{10}},\dfrac{3}{\sqrt{10}}\right)$	$\left(\dfrac{2}{\sqrt{5}},\dfrac{1}{\sqrt{5}}\right)$	$\left(\dfrac{-1}{\sqrt{5}},\dfrac{2}{\sqrt{5}}\right)$

151.5. Ch. 57 Answers (The Angle Between Two Lines)

A237(a) $$\cos^{-1}\frac{|(-1,1)\cdot(2,-3)|}{|(-1,1)||(2,-3)|}=\cos^{-1}\frac{|-5|}{\sqrt{2}\sqrt{13}}=\cos^{-1}\frac{5}{\sqrt{26}}\approx 0.197$$ ✿

(b) $$\cos^{-1}\frac{|(1,5)\cdot(8,1)|}{|(1,5)||(8,1)|}=\cos^{-1}\frac{|13|}{\sqrt{26}\sqrt{65}}=\cos^{-1}\frac{1}{\sqrt{10}}\approx 1.249$$ ✿

(c) $$\cos^{-1}\frac{|(2,6)\cdot(3,2)|}{|(2,6)||(3,2)|}=\cos^{-1}\frac{|18|}{\sqrt{40}\sqrt{13}}=\cos^{-1}\frac{9}{\sqrt{130}}\approx 0.661$$ ✿

151.6. Ch. 58 Answers (Vectors vs Scalars)

(*This chapter had no exercises.*)

151.7. Ch. 59 Answers (Projection Vectors)

A238(a) Since $(33,33)\parallel(1,1)$, the projection of $(1,0)$ on $(33,33)$ is equal to the projection of $(1,0)$ on $(1,1)$. Hence,

$$\left|\text{proj}_{(33,33)}(1,0)\right|=\left|\text{proj}_{(1,1)}(1,0)\right|=\left|\frac{(1,0)\cdot(1,1)}{|(1,1)|}\right|=\frac{1\cdot 1+0\cdot 1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}.$$ ✿

(b) The length of the projection of $(33,33)$ on $(1,0)$ is

$$\left|\text{proj}_{(1,0)}(33,33)\right|=\left|(33,33)\cdot\widehat{(1,0)}\right|=\left|\frac{(33,33)\cdot(1,0)}{|(1,0)|}\right|=\frac{33+0}{1}=33.$$ ✿

(c) We just showed that $\quad\left|\text{proj}_{(33,33)}(1,0)\right|\neq\left|\text{proj}_{(1,0)}(33,33)\right|$.

Therefore, the given statement is false. In general, the projection of **a** on **b** is not the same as the projection of **b** on **a**.

151.8. Ch. 60 Answers (Collinearity)

A239(a) The unique line that contains both $A = (3,1)$ and $B = (1,6)$ is

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} = (3,1) + \lambda(-2,5) \qquad (\lambda \in \mathbb{R}).$$

If this line also contains C, then there exists $\hat{\lambda}$ such that

$$C = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \hat{\lambda}\begin{pmatrix} -2 \\ 5 \end{pmatrix} \qquad \text{or} \qquad \begin{aligned} 0 &\overset{1}{=} 3 - 2\hat{\lambda}, \\ -1 &\overset{2}{=} 1 + 5\hat{\lambda}. \end{aligned}$$

From $\overset{1}{=}$, we have $\hat{\lambda} = 1.5$. But this contradicts $\overset{2}{=}$. So, there is no solution to the above vector equation (or system of two equations), meaning our line does **not** contain C. Thus, A, B, and C are **not** collinear.

(b) The unique line that contains both $A = (1,2)$ and $B = (0,0)$ is

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} = (1,2) + \lambda(-1,-2) \qquad (\lambda \in \mathbb{R}).$$

If this line also contains C, then there exists $\hat{\lambda}$ such that

$$C = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \hat{\lambda}\begin{pmatrix} -1 \\ -2 \end{pmatrix} \qquad \text{or} \qquad \begin{aligned} 3 &\overset{1}{=} 1 - 1\hat{\lambda}, \\ 6 &\overset{2}{=} 2 - 2\hat{\lambda}. \end{aligned}$$

Observe that $\hat{\lambda} = -2$ solves the above vector equation (or system of two equations). Hence, our line does indeed contain the point C (it corresponds to $\hat{\lambda} = -2$). Thus, A, B, and C are collinear.

A240(a) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = a_1(b_2 + c_2) - a_2(b_1 + c_1) = a_1b_2 - a_2b_1 + a_1c_2 - a_2c_1 = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

(b) $\mathbf{a} \times \mathbf{b} = a_1b_2 - a_2b_1 = -(a_2b_1 - a_1b_2) = -(b_1a_2 - b_2a_1) = -\mathbf{b} \times \mathbf{a}$.

(c) $\mathbf{a} \times \mathbf{a} = a_1a_2 - a_2a_1 = 0$.

A241. $\mathbf{a} \times \mathbf{b} = (1, -2) \times (3, 0) = 1 \cdot 0 - (-2) \cdot 3 = 0 + 6 = 6$.

$\mathbf{a} \times \mathbf{c} = (1, -2) \times (4, 1) = 1 \cdot 1 - (-2) \cdot 4 = 1 + 8 = 9$.

$\mathbf{b} \times \mathbf{c} = (3, 0) \times (4, 1) = 3 \cdot 1 - 0 \cdot 4 = 3 - 0 = 3$.

By anti-commutativity, $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} = -6$, $\mathbf{c} \times \mathbf{a} = -\mathbf{a} \times \mathbf{c} = -9$, and $\mathbf{c} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = -3$.

By distributivity, $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = 6 + 9 = 15$.

A242(a) $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$, $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2}$, $|\mathbf{a} \times \mathbf{b}| = |a_1b_2 - a_2b_1|$, and

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}.$$

(b) If $\theta \in [0, \pi]$, then $\sin\theta \geq 0$.

(c) The identity is $\sin^2\theta + \cos^2\theta = 1$ (Fact 71). Rearranging,

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}.$$

But by **(b)**, $\sin\theta \geq 0$—so, we can discard the negative value. Hence,

$$\sin\theta = \sqrt{1 - \cos^2\theta}.$$

(d)
$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{(a_1b_1 + a_2b_2)^2}{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}}.$$

(e)
$$\begin{aligned}
&\left(a_1^2 + a_2^2\right)\left(b_1^2 + b_2^2\right) - (a_1b_1 + a_2b_2)^2 \\
&= a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2 - \left(a_1^2b_1^2 + a_2^2b_2^2 + 2a_1a_2b_1b_2\right) \\
&= a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2 = (a_1b_2 - a_2b_1)^2.
\end{aligned}$$

(f)
$$\begin{aligned}
|\mathbf{a}|\,|\mathbf{b}|\sin\theta &= \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}\sqrt{1 - \frac{(a_1b_1 + a_2b_2)^2}{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}} \\
&= \sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2} \\
&\overset{(e)}{=} \sqrt{(a_1b_2 - a_2b_1)^2} = |a_1b_2 - a_2b_1| = |\mathbf{a} \times \mathbf{b}|.
\end{aligned}$$

151.10. Ch. 62 Answers (The Foot of the Perpendicular)

A243. $\overrightarrow{PA} = A - P = (-1,0) - (2,-3) = (-3,3)$, $\overrightarrow{PB} = B - P = (3,2) - (2,-3) = (1,5)$.

$$\text{proj}_{\mathbf{v}}\overrightarrow{PA} = \text{proj}_{(5,1)}(-3,3) = \frac{(-3,3)\cdot(5,1)}{5^2+1^2}\begin{pmatrix}5\\1\end{pmatrix} = \frac{-15+3}{26}\begin{pmatrix}5\\1\end{pmatrix} = -\frac{6}{13}\begin{pmatrix}5\\1\end{pmatrix},$$

$$\text{proj}_{\mathbf{v}}\overrightarrow{PB} = \text{proj}_{(5,1)}(1,5) = \frac{(1,5)\cdot(5,1)}{5^2+1^2}\begin{pmatrix}5\\1\end{pmatrix} = \frac{5+5}{26}\begin{pmatrix}5\\1\end{pmatrix} = \frac{5}{13}\begin{pmatrix}5\\1\end{pmatrix}.$$

By Fact 150 then, the feet of the perpendiculars from A and B to the line are

$$P + \text{proj}_{\mathbf{v}}\overrightarrow{PA} = \begin{pmatrix}2\\-3\end{pmatrix} - \frac{6}{13}\begin{pmatrix}5\\1\end{pmatrix} = \frac{1}{13}\begin{pmatrix}-45\\-9\end{pmatrix} = -\frac{9}{13}\begin{pmatrix}5\\1\end{pmatrix},$$

$$P + \text{proj}_{\mathbf{v}}\overrightarrow{PB} = \begin{pmatrix}2\\-3\end{pmatrix} + \frac{5}{13}\begin{pmatrix}5\\1\end{pmatrix} = \frac{1}{13}\begin{pmatrix}51\\-34\end{pmatrix} = \frac{17}{13}\begin{pmatrix}3\\-2\end{pmatrix}.$$

A244**(a)** Let $P = (8, 3)$ and $\mathbf{v} = (9, 3)$.

Method 1 (Formula Method). First compute $\overrightarrow{PA} = (7, 3) - (8, 3) = (-1, 0)$ and

$$\operatorname{proj}_{\mathbf{v}} \overrightarrow{PA} = \operatorname{proj}_{(9,3)} (-1, 0) = \frac{(-1, 0) \cdot (9, 3)}{9^2 + 3^2} \begin{pmatrix} 9 \\ 3 \end{pmatrix} = \frac{-9 + 0}{90} \begin{pmatrix} 9 \\ 3 \end{pmatrix} = -\frac{1}{10} \begin{pmatrix} 9 \\ 3 \end{pmatrix}.$$

So, $$B = P + \operatorname{proj}_{\mathbf{v}} \overrightarrow{PA} = (8, 3) - 0.1 (9, 3) = (7.1, 2.7).$$

And the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right| = \left|(-1, 0) \times \frac{(9, 3)}{\sqrt{9^2 + 3^2}}\right| = \left|\frac{(-1) \cdot 3 - 0 \cdot 9}{\sqrt{90}}\right| = \frac{3}{\sqrt{90}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}.$$

Method 2 (Perpendicular Method). Let $B = (8, 3) + \tilde{\lambda} (9, 3)$. Then,

$$\overrightarrow{AB} = B - A = (8, 3) + \tilde{\lambda} (9, 3) - (7, 3) = \left(9\tilde{\lambda} + 1, 3\tilde{\lambda}\right).$$

Since B is the foot of the perpendicular, we have $\overrightarrow{AB} \perp l$ or $\overrightarrow{AB} \perp \mathbf{v}$ or $\overrightarrow{AB} \cdot (9, 3) = 0$:

$$0 = \overrightarrow{AB} \cdot (9, 3) = \left(9\tilde{\lambda} + 1, 3\tilde{\lambda}\right) \cdot (9, 3) = 9\left(9\tilde{\lambda} + 1\right) + 3\left(3\tilde{\lambda}\right) = 90\tilde{\lambda} + 9.$$

Rearranging, $\tilde{\lambda} = -9/90 = -1/10 = -0.1$ and so,

$$B = (8, 3) + \tilde{\lambda} (9, 3) = (8, 3) - 0.1 (9, 3) = (7.1, 2.7).$$

And the distance between A and l is

$$\left|\overrightarrow{AB}\right| = |B - A| = |(7.1, 2.7) - (7, 3)| = |(0.1, -0.3)| = 0.1 |(1, -3)| = 0.1\sqrt{10}.$$

Method 3 (Calculus Method). Let R be a generic point on l. Then,

$$\overrightarrow{AR} = (9\lambda + 1, 3\lambda) \qquad \text{and} \qquad \left|\overrightarrow{AR}\right| = \sqrt{(9\lambda + 1)^2 + (3\lambda)^2} = \sqrt{90\lambda^2 + 18\lambda + 2}.$$

Differentiate: $$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(90\lambda^2 + 18\lambda + 2\right) = 180\lambda + 18.$$

FOC: $$(180\lambda + 18)\big|_{\lambda = \tilde{\lambda}} = 0 \qquad \text{or} \qquad \tilde{\lambda} = -\frac{18}{180} = -\frac{1}{10}.$$

Alternatively, we could simply have used "$-b/2a$": $\tilde{\lambda} = -\frac{18}{2 \cdot 90} = -\frac{18}{180} = -\frac{1}{10}$.

And now, we can find B and $\left|\overrightarrow{AB}\right|$ as we did in Method 2.

A244(b) Let $P = (4,4)$ and $\mathbf{v} = (6,11) - (4,4) = (2,7)$.

Method 1 (Formula Method). First compute $\overrightarrow{PA} = (8,0) - (4,4) = (4,-4)$ and

$$\text{proj}_{\mathbf{v}}\overrightarrow{PA} = \text{proj}_{(2,7)}(4,-4) = \frac{(4,-4)\cdot(2,7)}{2^2+7^2}\begin{pmatrix}2\\7\end{pmatrix} = \frac{8-28}{53}\begin{pmatrix}2\\7\end{pmatrix} = -\frac{20}{53}\begin{pmatrix}2\\7\end{pmatrix}.$$

So, $$B = P + \text{proj}_{\mathbf{v}}\overrightarrow{PA} = (4,4) - \frac{20}{53}(2,7) = \frac{1}{53}(172,72).$$

And the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{PA}\times\hat{\mathbf{v}}\right| = \left|(4,-4)\times\frac{(2,7)}{\sqrt{2^2+7^2}}\right| = \left|\frac{4\cdot 7-(-4)\cdot 2}{\sqrt{53}}\right| = \frac{36}{\sqrt{53}}.$$

Method 2 (Perpendicular Method). Let $B = (8,3) + \tilde{\lambda}(9,3)$. Then,

$$\overrightarrow{AB} = B - A = (4,4) + \tilde{\lambda}(2,7) - (8,0) = \left(2\tilde{\lambda}-4, 7\tilde{\lambda}+4\right).$$

Since B is the foot of the perpendicular, we have $\overrightarrow{AB}\perp l$ or $\overrightarrow{AB}\perp\mathbf{v}$ or $\overrightarrow{AB}\cdot(2,7) = 0$:

$$0 = \overrightarrow{AB}\cdot(2,7) = \left(2\tilde{\lambda}-4, 7\tilde{\lambda}+4\right)\cdot(2,7) = 2\left(2\tilde{\lambda}-4\right) + 7\left(7\tilde{\lambda}+4\right) = 53\tilde{\lambda}+20.$$

Rearranging, $\tilde{\lambda} = -20/53$ and so,

$$B = (4,4) + \tilde{\lambda}(2,7) = (4,4) - \frac{20}{53}(2,7) = \frac{1}{53}(172,72).$$

And the distance between A and l is

$$\left|\overrightarrow{AB}\right| = |B-A| = \left|\frac{1}{53}(172,72) - (8,0)\right| = \left|\frac{1}{53}(-252,72)\right| = \frac{36}{53}|(-7,2)| = \frac{36\sqrt{53}}{53} = \frac{36}{\sqrt{53}}.$$

Method 3 (or the **Calculus Method**). Let R be a generic point on l. Then,

$$\overrightarrow{AR} = (2\lambda-4, 7\lambda+4) \qquad \text{and} \qquad \left|\overrightarrow{AR}\right| = \sqrt{(2\lambda-4)^2+(7\lambda+4)^2} = \sqrt{53\lambda^2+40\lambda+32}.$$

Differentiate: $$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(53\lambda^2+40\lambda+32\right) = 106\lambda+40.$$

FOC: $$(106\lambda+40)\big|_{\lambda=\tilde{\lambda}} = 0 \qquad \text{or} \qquad \tilde{\lambda} = -\frac{40}{106} = -\frac{20}{53}.$$

Alternatively, we could simply have used "$-b/2a$": $\tilde{\lambda} = -\dfrac{40}{2\cdot 53} = -\dfrac{20}{53}$.

And now, we can find B and $\left|\overrightarrow{AB}\right|$ as we did in Method 2.

A244(c) Let $P = (8,4)$ and $\mathbf{v} = (5,6)$.

Method 1 (Formula Method). First compute $\overrightarrow{PA} = (8,5) - (8,4) = (0,1)$ and

$$\text{proj}_{\mathbf{v}} \overrightarrow{PA} = \text{proj}_{(5,6)} (0,1) = \frac{(0,1) \cdot (5,6)}{5^2 + 6^2} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{0+6}{61} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{6}{61} \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

So, $$B = P + \text{proj}_{\mathbf{v}} \overrightarrow{PA} = (8,4) + \frac{6}{61}(5,6) = \frac{1}{61}(518, 280).$$

And the distance between A and l is

$$\left| \overrightarrow{PA} \times \hat{\mathbf{v}} \right| = \left| (0,1) \times \frac{(5,6)}{\sqrt{5^2+6^2}} \right| = \left| \frac{0-5}{\sqrt{61}} \right| = \frac{5}{\sqrt{61}}.$$

Method 2 (Perpendicular Method). Let $B = (8,4) + \tilde{\lambda}(5,6)$. Then,

$$\overrightarrow{AB} = B - A = (8,4) + \tilde{\lambda}(5,6) - (8,5) = \left(5\tilde{\lambda}, 6\tilde{\lambda} - 1\right).$$

Since B is the foot of the perpendicular, we have $\overrightarrow{AB} \perp l$ or $\overrightarrow{AB} \perp \mathbf{v}$ or $\overrightarrow{AB} \cdot (5,6) = 0$:

$$0 = \overrightarrow{AB} \cdot (5,6) = \left(5\tilde{\lambda}, 6\tilde{\lambda} - 1\right) \cdot (5,6) = 5\left(5\tilde{\lambda}\right) + 6\left(6\tilde{\lambda} - 1\right) = 61\tilde{\lambda} - 6.$$

Rearranging, $\tilde{\lambda} = 6/61$ and so,

$$B = (8,4) + \tilde{\lambda}(5,6) = (8,4) + \frac{6}{61}(5,6) = \frac{1}{61}(518, 280).$$

And the distance between A and l is

$$\left|\overrightarrow{AB}\right| = |B - A| = \left| \frac{1}{61}(518, 280) - (8,5) \right| = \left| \frac{1}{61}(30, -25) \right| = \frac{5}{61} |(6,-5)| = \frac{5\sqrt{61}}{61}.$$

Method 3 (or the **Calculus Method**). Let R be a generic point on l. Then,

$$\overrightarrow{AR} = (5\lambda, 6\lambda - 1) \qquad \text{and} \qquad \left|\overrightarrow{AR}\right| = \sqrt{(5\lambda)^2 + (6\lambda - 1)^2} = \sqrt{61\lambda^2 + 12\lambda + 1}.$$

Differentiate: $$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(61\lambda^2 + 12\lambda + 1\right) = 122\lambda + 12.$$

FOC: $$(122\lambda + 12.)\big|_{\lambda = \tilde{\lambda}} = 0 \qquad \text{or} \qquad \tilde{\lambda} = -\frac{12}{122} = -\frac{6}{61}.$$

Alternatively, we could simply have used "$-b/2a$": $$\tilde{\lambda} = -\frac{12}{2 \cdot 61} = -\frac{6}{61}.$$

And now, we can find B and $\left|\overrightarrow{AB}\right|$ as we did in Method 2.

151.11. Ch. 63 Answers (Three-Dimensional Space)

(*This chapter had no exercises.*)

151.12. Ch. 64 Answers (Vectors in 3D)

A245(a) $\overrightarrow{AB} = (0-2, 1-5, 1-8) = (-2, -4, -7)$.

(b) $\overrightarrow{OA} = (2, 5, 8)$ and $\overrightarrow{OB} = (0, 1, 1)$. **(c)** $\overrightarrow{AA} = \mathbf{0} = (0, 0, 0)$.

A246. $\overrightarrow{AB} = (-2, -4, -7)$. So, $\left|\overrightarrow{AB}\right| = |(-2, -4, -7)| = \sqrt{(-2)^2 + (-4)^2 + (-7)^2} = \sqrt{69}$.

A247(a) $A + B$ is undefined.

(b) $A - B$ is the vector $\overrightarrow{BA} = (1 - (-1), 2 - 0, 3 - 7) = (2, 2, -4)$.

(c) $B + C$ is undefined. Hence, $A + (B + C)$ is also undefined.

(d) $B - C$ is the vector $\overrightarrow{CB} = (-1 - 5, 0 - (-2), 7 - 3) = (-6, 2, 4)$. And so, $A + (B - C)$ is a well-defined vector, namely $A + (B - C) = A + \overrightarrow{CB} = (1, 2, 3) + (-6, 2, 4) = (-5, 4, 7)$.

A248(a) $\mathbf{u} + \mathbf{v} = (1, 2, 3) + (-1, 0, 7) = (0, 2, 10)$.

(b) $\mathbf{u} - \mathbf{v} = (1, 2, 3) - (-1, 0, 7) = (2, 2, -4)$.

(c) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \mathbf{u} + \mathbf{v} + \mathbf{w} = (0, 2, 10) + (5, -2, 3) = (5, 0, 13)$.

(d) $\mathbf{u} + (\mathbf{v} - \mathbf{w}) = \mathbf{u} + \mathbf{v} - \mathbf{w} = (0, 2, 10) - (5, -2, 3) = (-5, 4, 7)$.

A249. $\overrightarrow{AB} = B - A = (3, 6, -5) - (5, -1, 0) = (-2, 7, -5)$.

$\overrightarrow{AC} = C - A = (2, 2, 3) - (5, -1, 0) = (-3, 3, 3)$.

$\overrightarrow{BC} = C - B = (2, 2, 3) - (3, 6, -5) = (-1, -4, 8)$.

$\overrightarrow{AB} - \overrightarrow{AC} = (-2, 7, -5) - (-3, 3, 3) = (1, 4, -8) = -\overrightarrow{BC} = \overrightarrow{CB}$. ✓

$\overrightarrow{AB} + \overrightarrow{BC} = (-2, 7, -5) + (-1, -4, 8) = (-3, 3, 3) = \overrightarrow{AC}$. ✓

A250(a) $\mathbf{v}$ and $\mathbf{w}$ point in the exact opposite directions because $\mathbf{v} = -1.5\mathbf{w}$. They are thus also parallel—$\mathbf{v} \parallel \mathbf{w}$.

(b) $\mathbf{v}$ and $\mathbf{x}$ point in different directions because $\mathbf{x} \neq k\mathbf{v}$ for all $k \in \mathbb{R}$.They are thus also non-parallel—$\mathbf{v} \nparallel \mathbf{x}$.

(c) $\mathbf{w}$ and $\mathbf{x}$ point in different directions because $\mathbf{x} \neq k\mathbf{w}$ for all $k \in \mathbb{R}$.They are thus also non-parallel—$\mathbf{w} \nparallel \mathbf{x}$.

(d) By our definitions (which covered only non-zero vectors), the zero vector $\mathbf{0}$ does not point in the same, exact opposite, or different direction as any other vector (including, in particular, $\mathbf{u}$).

Also, it is neither parallel nor non-parallel to any other vector (including, in particular, $\mathbf{u}$).

A251(a) $|\mathbf{a}| = |(1,2,3)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$. And $\hat{\mathbf{a}} = \dfrac{(1,2,3)}{\sqrt{14}}$.

(b) $|\mathbf{b}| = |(4,5,6)| = \sqrt{4^2+5^2+6^2} = \sqrt{77}$. And $\hat{\mathbf{b}} = \dfrac{(4,5,6)}{\sqrt{77}}$.

(c) $|\mathbf{a}-\mathbf{b}| = |(-3,-3,-3)| = \sqrt{(-3)^2+(-3)^2+(-3)^2} = \sqrt{27} = 3\sqrt{3}$. And,

$$\widehat{\mathbf{a}-\mathbf{b}} = \frac{(-3,-3,-3)}{3\sqrt{3}} = -\frac{(1,1,1)}{\sqrt{3}}.$$

(d) $|2\mathbf{a}| = 2\sqrt{14}$. And $\widehat{2\mathbf{a}} = \hat{\mathbf{a}} = \dfrac{(1,2,3)}{\sqrt{14}}$.

(e) $|3\mathbf{b}| = 3\sqrt{77}$. And $\widehat{3\mathbf{b}} = \hat{\mathbf{b}} = \dfrac{(4,5,6)}{\sqrt{77}}$.

(f) $|-4(\mathbf{a}-\mathbf{b})| = 12\sqrt{3}$. And $-4\widehat{(\mathbf{a}-\mathbf{b})} = -\widehat{\mathbf{a}-\mathbf{b}} = \dfrac{(1,1,1)}{\sqrt{3}}$.

A252. $\mathbf{v} = (9,0,-1) = 9\mathbf{i}-\mathbf{k}$ and $\mathbf{w} = (-7,3,5) = -7\mathbf{i}+3\mathbf{j}+5\mathbf{k}$.

A253. By the Ratio Theorem, the point P that divides the line segment AB in the ratio $2:3$ has position vector:

$$\mathbf{p} = \frac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu} = \frac{3(1,2,3)+2(4,5,6)}{2+3} = \frac{1}{5}\begin{pmatrix} 11 \\ 16 \\ 21 \end{pmatrix}.$$

Hence, the point is $P = \dfrac{1}{5}(11,16,21)$.

A254(a) Informally, a vector is an “arrow” with two properties: **direction** and **length**.

(b) A point and a vector are entirely different objects and should not be confused. Nonetheless, each can be described by **an ordered triple of real numbers**.

(c) Let $A = (a_1,a_2,a_3)$ be a point and $\mathbf{a} = (a_1,a_2,a_3)$ be a vector. We say that $\mathbf{a}$ is A's **position vector**.

(d) The vector $\mathbf{a} = (a_1,a_2,a_3)$ carries us from the **origin** to the point $A = (a_1,a_2,a_3)$.

A255. $\overrightarrow{OA} = A - O = (a_1,a_2,a_3) = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k} = \vec{a}$.

A256. $A+B$ is undefined.

$A+\overrightarrow{OB}$ is the **point** $(a_1+b_1, a_2+b_2, a_3+b_3)$.

$\overrightarrow{OA}+\overrightarrow{OB}$ is the **vector** $(a_1+b_1, a_2+b_2, a_3+b_3)$.

$\overrightarrow{OA}-\overrightarrow{OB}$ is the **vector** $\overrightarrow{BA} = (a_1-b_1, a_2-b_2, a_3-b_3)$.

$\overrightarrow{OA}-\overrightarrow{BA}$ is the **vector** $\overrightarrow{OB} = (b_1,b_2,b_3)$.

151.13. Ch. 65 Answers (The Scalar Product in 3D)

A257. $(1,2,3)\cdot(4,5,6) = 4+10+18 = 32$.

$(-2,4,-6)\cdot(1,-2,3) = -2-8-18 = -28$.

A258(a) The angle between the vectors $(1,2,3)$ and $(4,5,6)$ is

$$\cos^{-1}\frac{(1,2,3)\cdot(4,5,6)}{|(1,2,3)||(4,5,6)|} = \cos^{-1}\frac{32}{\sqrt{14}\sqrt{77}} \approx 0.226.$$

Thus, the vectors $(1,2,3)$ and $(4,5,6)$ are neither parallel nor perpendicular. Instead, they point in different directions.

(b) The angle between the vectors $(-2,4,-6)$ and $(1,-2,3)$ is

$$\cos^{-1}\frac{(-2,4,-6)\cdot(1,-2,3)}{|(-2,4,6)||(1,-2,3)|} = \cos^{-1}\frac{-28}{\sqrt{56}\sqrt{14}} = \cos^{-1}\frac{-28}{28} = \cos^{-1}\frac{-28}{28} = \pi.$$

Thus, the vectors $(-2,4,-6)$ and $(1,-2,3)$ are parallel (and point in exact opposite directions). Actually, we could also have arrived at this conclusion by observing that $\mathbf{u} = -2\mathbf{v}$.

A259(a) The vector $(1,3,-2)$ has length $\sqrt{1^2+3^2+(-2)^2} = \sqrt{14}$.

Hence, its unit vector is $(1,3,-2)/\sqrt{14}$.

Its x-, y-, and z-direction cosines are $1/\sqrt{14}$, $3/\sqrt{14}$, and $-2/\sqrt{14}$.

And the angles it makes with the positive x-, y-, and z-axes are

$$\cos^{-1}\frac{1}{\sqrt{14}} \approx 1.300, \qquad \cos^{-1}\frac{3}{\sqrt{14}} \approx 0.641, \quad \text{and} \quad \cos^{-1}\frac{-2}{\sqrt{14}} \approx 2.135.$$

(b) The vector $(4,2,-3)$ has length $\sqrt{4^2+2^2+(-3)^2} = \sqrt{29}$.

Hence, its unit vector is $(4,2,-3)/\sqrt{29}$.

Its x-, y-, and z-direction cosines are $4/\sqrt{29}$, $2/\sqrt{29}$, and $-3/\sqrt{29}$.

And the angles it makes with the positive x-, y-, and z-axes are

$$\cos^{-1}\frac{4}{\sqrt{29}} \approx 0.734, \qquad \cos^{-1}\frac{2}{\sqrt{29}} \approx 1.190, \quad \text{and} \quad \cos^{-1}\frac{-3}{\sqrt{29}} \approx 2.162.$$

(c) The vector $(-1,2,-4)$ has length $\sqrt{(-1)^2+2^2+(-4)^2} = \sqrt{21}$.

Hence, its unit vector is $(-1,2,-4)/\sqrt{21}$.

Its x-, y-, and z-direction cosines are $-1/\sqrt{21}$, $2/\sqrt{21}$, and $-4/\sqrt{21}$.

And the angles it makes with the positive x-, y-, and z-axes are

$$\cos^{-1}\frac{-1}{\sqrt{21}} \approx 1.791, \qquad \cos^{-1}\frac{2}{\sqrt{21}} \approx 1.119, \quad \text{and} \quad \cos^{-1}\frac{-4}{\sqrt{21}} \approx 2.362.$$

151.14. Ch. 66 Answers (The Proj. and Rej. Vectors in 3D)

A260. $\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}} = \dfrac{(2,5,-1)\cdot(1,1,4)}{1^2+1^2+4^2}(1,1,4) = \dfrac{2+5-4}{18}(1,1,4) = \dfrac{1}{6}(1,1,4)$.

$\text{rej}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a} = (2,5,-1) - \dfrac{1}{6}(1,1,4) = \dfrac{1}{6}(11,29,-10)$.

$$\text{proj}_{\mathbf{c}}\mathbf{a} = (\mathbf{a}\cdot\hat{\mathbf{c}})\,\hat{\mathbf{c}} = \frac{(2,5,-1)\cdot(-2,-2,1)}{(-2)^2+(-2)^2+1^2}\begin{pmatrix}-2\\-2\\1\end{pmatrix}$$

$$= \frac{-4-10-1}{9}\begin{pmatrix}-2\\-2\\1\end{pmatrix} = -\frac{15}{9}\begin{pmatrix}-2\\-2\\1\end{pmatrix} = \frac{5}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}$$

$\text{rej}_{\mathbf{c}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{c}}\mathbf{a} = (2,5,-1) - \dfrac{5}{3}(2,2,-1) = \dfrac{1}{3}(-4,5,2)$.

$(\text{rej}_{\mathbf{b}}\mathbf{a})\cdot\mathbf{b} = \dfrac{1}{6}(11,29,-10)\cdot(1,1,4) = \dfrac{1}{6}(11+29-40) = 0$. ✓

$(\text{rej}_{\mathbf{c}}\mathbf{a})\cdot\mathbf{c} = \dfrac{1}{3}(-4,5,2)\cdot(-2,-2,1) = \dfrac{1}{3}(8-10+2) = 0$. ✓

A261. $\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}} = \dfrac{(1,2,3)\cdot(4,5,6)}{4^2+5^2+6^2}\begin{pmatrix}4\\5\\6\end{pmatrix} = \dfrac{4+10+18}{77}\begin{pmatrix}4\\5\\6\end{pmatrix} = \dfrac{32}{77}\begin{pmatrix}4\\5\\6\end{pmatrix}$.

$|\text{proj}_{\mathbf{b}}\mathbf{a}| = \dfrac{32}{77}\sqrt{77} = \dfrac{32}{\sqrt{77}}$.

$\text{rej}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a} = (1,2,3) - \dfrac{32}{77}(4,5,6) = \dfrac{1}{77}(-51,-6,39) = -\dfrac{3}{77}(17,2,-13)$.

$|\text{rej}_{\mathbf{b}}\mathbf{a}| = \dfrac{3}{77}\left[17^2+2^2+(-13)^2\right] = \dfrac{3}{77}\sqrt{462} = \dfrac{3}{77}\sqrt{6\cdot 77} = 3\sqrt{\dfrac{6}{77}}$.

$\text{proj}_{\mathbf{b}}\mathbf{a} = \dfrac{32}{77}\mathbf{b}$ and so $\text{proj}_{\mathbf{b}}\mathbf{a} \parallel \mathbf{b}$. Also, $\text{proj}_{\mathbf{b}}\mathbf{a}$ points in the same direction as $\mathbf{b}$.

$(\text{rej}_{\mathbf{b}}\mathbf{a})\cdot\mathbf{b} = -\dfrac{3}{77}(17,2,-13)\cdot(4,5,6) = -\dfrac{3}{77}[17\cdot 4 + 2\cdot 5 + (-13)\cdot 6] = 0$ and so $\text{rej}_{\mathbf{b}}\mathbf{a} \perp \mathbf{b}$.

151.15. Ch. 67 Answers (Lines in 3D)

A262(a) $\dfrac{x+1}{3}=\dfrac{y-1}{-2}=\dfrac{z-1}{1}$. This line is not perpendicular to any of the axes.

(b) $\dfrac{x-5}{7}=\dfrac{y-6}{8}=\dfrac{z-1}{1}$. This line is not perpendicular to any of the axes.

(c) $\dfrac{x}{3}=\dfrac{z-1}{1}$ and $y=-3$. This line is perpendicular to the y-axis.

(d) $y=z=9$. This line is perpendicular to the y- and z-axes.

(e) $\dfrac{x}{4}=\dfrac{y}{8}=\dfrac{z}{5}$. This line is not perpendicular to any of the axes.

(f) $x=1$ and $z=5$. This line is perpendicular to the x- and z-axes.

A263(a) Rewrite the given equations: $\dfrac{x-2/7}{5/7}=\dfrac{y-50/3}{70/3}=\dfrac{z}{7/8}$.

So, this line may be described by $\mathbf{r}=(2/7,50/3,0)+\lambda\,(5/7,70/3,7/8)\qquad(\lambda\in\mathbb{R})$.

It is not perpendicular to any of the axes.

(b) Rewrite the given equations: $\dfrac{x}{1/2}=\dfrac{y}{1/3}=\dfrac{z}{1/5}$.

So, this line may be described by $\mathbf{r}=(0,0,0)+\lambda\,(1/2,1/3,1/5)\qquad(\lambda\in\mathbb{R})$.

It is not perpendicular to any of the axes.

(c) Rewrite the given equations: $\dfrac{x-4/17}{1/17}=\dfrac{y-1/3}{2/3}=\dfrac{z}{1/3}$.

So, this line may be described by $\mathbf{r}=(4/17,1/3,0)+\lambda\,(1/17,2/3,1/3)\qquad(\lambda\in\mathbb{R})$.

It is not perpendicular to any of the axes.

(d) Rewrite the given equations: $y=\dfrac{11}{3}$ and $\dfrac{x-3}{2}=\dfrac{z-2/5}{7/5}$.

So, this line may be described by $\mathbf{r}=(3,11/3,2/5)+\lambda\,(2,0,7/5)\qquad(\lambda\in\mathbb{R})$.

It is perpendicular to the y-axis.

(e) The free variable is y. So, this line may be described by

$$\mathbf{r}=(65,0,1/2)+\lambda\,(0,1,0)\qquad(\lambda\in\mathbb{R}).$$

It is perpendicular to the x- and z-axes.

(f) Rewrite the given equations: $x=-\dfrac{5}{13}$ and $\dfrac{y-(-2)}{5}=\dfrac{z}{1}$.

So, this line may be described by $\mathbf{r}=\left(-\dfrac{5}{13},-2,0\right)+\lambda\,(0,5,1)\qquad(\lambda\in\mathbb{R})$.

It is perpendicular to the x-axis.

A264(a) If the two lines intersect, then there are real numbers $\hat{\lambda}$ and $\hat{\mu}$ numbers such that

$$\hat{\lambda} \stackrel{1}{=} 1, \qquad 1 - \hat{\lambda} \stackrel{2}{=} 3, \quad \text{and} \quad 1 + \hat{\lambda} \stackrel{3}{=} 3 + 2\hat{\mu}.$$

$\stackrel{1}{=}$ immediately contradicts $\stackrel{2}{=}$. Hence, the two lines **do not intersect** and are **not identical**. The **angle between them** is

$$\cos^{-1} \frac{|(1,-1,1)\cdot(0,0,2)|}{|(1,-1,1)|\,|(0,0,2)|} = \cos^{-1} \frac{|0+0+2|}{\sqrt{1^2+(-1)^2+1^2}\sqrt{0^2+0^2+2^2}} = \cos^{-1} \frac{2}{\sqrt{3}\cdot 2} \approx 0.955.$$

The two lines are **neither parallel nor perpendicular**. And since they do not intersect either, they are **skew**.

(b) If the two lines intersect, then there are real numbers $\hat{\lambda}$ and $\hat{\mu}$ numbers such that

$$-1 \stackrel{1}{=} 8\hat{\mu}, \qquad 2 + \hat{\lambda} \stackrel{2}{=} -3\hat{\mu}, \quad \text{and} \quad 3 \stackrel{3}{=} 5\hat{\mu}.$$

$\stackrel{1}{=}$ immediately contradicts $\stackrel{3}{=}$. Hence, the two lines **do not intersect** and are **not identical**. The **angle between them** is

$$\cos^{-1} \frac{|(0,-1,0)\cdot(8,-3,5)|}{|(0,-1,0)|\,|(8,-3,5)|} = \cos^{-1} \frac{|0+3+0|}{\sqrt{0^2+1^2+0^2}\sqrt{8^2+(-3)^2+5^2}} = \cos^{-1} \frac{3}{1\cdot\sqrt{98}} \approx 1.263.$$

The two lines are **neither parallel nor perpendicular**. And since they do not intersect either, they are **skew**.

(c) If the two lines intersect, then there are real numbers $\hat{\lambda}$ and $\hat{\mu}$ numbers such that

$$7 + 8\hat{\lambda} \stackrel{1}{=} 9 + 3\hat{\mu}, \qquad 3 + 3\hat{\lambda} \stackrel{2}{=} 3 - 4\hat{\mu}, \quad \text{and} \quad 4 + 4\hat{\lambda} \stackrel{3}{=} 7 - 3\hat{\mu}.$$

$\stackrel{1}{=}$ plus $\stackrel{3}{=}$ yields $11 + 12\hat{\lambda} = 16$ or $\hat{\lambda} = 5/12$. And now from $\stackrel{3}{=}$, we have $\hat{\mu} = 4/9$. But these values of $\hat{\lambda}$ and $\hat{\mu}$ contradict $\stackrel{2}{=}$. Hence, the two lines **do not intersect** and are **not identical**. The **angle between them** is

$$\cos^{-1} \frac{|(8,3,4)\cdot(3,-4,-3)|}{|(8,3,4)|\,|(3,-4,-3)|} = \cos^{-1} \frac{|24-12-12|}{|(8,3,4)|\,|(3,-4,-3)|} = \cos^{-1} 0 = \frac{\pi}{2}.$$

The two lines are **perpendicular** (and hence not parallel). And since they do not intersect either, they are **skew**.

(d) The two lines have parallel direction vectors because $(-3,-6,-3) = -3(1,2,1)$. Hence, the two lines are **parallel** (and thus neither perpendicular nor skew). Since they are parallel, **the angle between them is zero**.

The first line does not contain the point $(1,0,0)$—the only point on the first line with x-coordinate 1 is $(1,2,3)$ (plug in $\lambda = 1$). Hence, the two lines are not identical. Since the two lines are parallel and distinct, they do not intersect at all.

A265(a) The unique line that contains both A and B is

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} \qquad (\lambda \in \mathbb{R}).$$

If the above line also contains C, then there exists $\hat{\lambda}$ such that

$$C = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} 0 &\overset{1}{=} 3 - 2\hat{\lambda}, \\ -1 &\overset{2}{=} 1 + 5\hat{\lambda}, \\ 0 &\overset{3}{=} 2 + 3\hat{\lambda}. \end{aligned}$$

From $\overset{1}{=}$, we have $\hat{\lambda} = 1.5$. But this contradicts $\overset{2}{=}$. This contradiction means that there is no solution to the above vector equation (or system of three equations).

Thus, our line does **not** contain C. We conclude that A, B, and C are **not** collinear.

(b) The unique line that contains both A and B is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \qquad (\lambda \in \mathbb{R}).$$

If the above line also contains C, then there exists $\hat{\lambda}$ such that

$$C = \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \hat{\lambda} \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} 3 &\overset{1}{=} 1 - 1\hat{\lambda}, \\ 6 &\overset{2}{=} 2 - 2\hat{\lambda}, \\ 10 &\overset{3}{=} 4 - 3\hat{\lambda}. \end{aligned}$$

As you can verify, $\hat{\lambda} = -2$ solves the above vector equation (or system of three equations).

Thus, our line contains C. We conclude that A, B, and C are collinear.

151.16. Ch. 68 Answers (The Vector Product in 3D)

A266(a) Given $\mathbf{u} = (0, 1, 2)$ and $\mathbf{v} = (3, 4, 5)$, we have

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \times 5 - 2 \times 4 \\ 2 \times 3 - 0 \times 5 \\ 0 \times 4 - 1 \times 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}.$$

We next verify that $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}, \mathbf{v}$:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (-3, 6, -3) \cdot (0, 1, 2) = 0 + 6 - 6 = 0, \quad ✓$$
$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (-3, 6, -3) \cdot (3, 4, 5) = -9 + 24 - 15 = 0. \quad ✓$$

(b) Given $\mathbf{w} = (-1, -2, -3)$ and $\mathbf{x} = (1, 0, 5)$, we have

$$\mathbf{w} \times \mathbf{x} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} (-2) \times 5 - (-3) \times 0 \\ (-3) \times 1 - (-1) \times 5 \\ (-1) \times 0 - (-2) \times 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 2 \\ 2 \end{pmatrix}.$$

We next verify that $\mathbf{w} \times \mathbf{x} \perp \mathbf{w}, \mathbf{x}$:

$$(\mathbf{w} \times \mathbf{x}) \cdot \mathbf{w} = (-10, 2, 2) \cdot (-1, -2, -3) = 10 - 4 - 6 = 0, \quad ✓$$
$$(\mathbf{w} \times \mathbf{x}) \cdot \mathbf{x} = (-10, 2, 2) \cdot (1, 0, 5) = -10 + 0 + 10 = 0. \quad ✓$$

(c)

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= (a_2b_3 - a_3b_2)\, a_1 + (a_3b_1 - a_1b_3)\, a_2 + (a_1b_2 - a_2b_1)\, a_3 \\ &= a_1a_2b_3 - a_1b_2a_3 + b_1a_2a_3 - a_1a_2b_3 + a_1b_2a_3 - b_1a_2a_3 = 0.\end{aligned}$$

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= (a_2b_3 - a_3b_2)\, b_1 + (a_3b_1 - a_1b_3)\, b_2 + (a_1b_2 - a_2b_1)\, b_3 \\ &= b_1a_2b_3 - b_1b_2a_3 + b_1b_2a_3 - a_1b_2b_3 + a_1b_2b_3 - b_1a_2b_3 = 0.\end{aligned}$$

A267(a) $$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1+c_1 \\ b_2+c_2 \\ b_3+c_3 \end{pmatrix} = \begin{pmatrix} a_2(b_3+c_3) - a_3(b_2+c_2) \\ a_3(b_1+c_1) - a_1(b_3+c_3) \\ a_1(b_2+c_2) - a_2(b_1+c_1) \end{pmatrix}$$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 + a_2c_3 - a_3c_2 \\ a_3b_1 - a_1b_3 + a_3c_1 - a_1c_3 \\ a_1b_2 - a_2b_1 + a_1c_2 - a_2c_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} + \begin{pmatrix} a_2c_3 - a_3c_2 \\ a_3c_1 - a_1c_3 \\ a_1c_2 - a_2c_1 \end{pmatrix} = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}.$$

(b) In Example 907, we already showed that $(1,2,3)\times(4,5,6) = (-3,6,-3)$.

We now have $$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3\cdot 5 - 2\cdot 6 \\ 1\cdot 6 - 3\cdot 4 \\ 2\cdot 4 - 1\cdot 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}.$$

So that indeed: $$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

To prove that $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$, simply write them out:

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b}\times\mathbf{a} = \begin{pmatrix} b_2a_3 - b_3a_2 \\ b_3a_1 - b_1a_3 \\ b_1a_2 - b_2a_1 \end{pmatrix}.$$

(c) $(a_1,a_2,a_3)\times(a_1,a_2,a_3) = (a_2a_3 - a_3a_2, a_3a_1 - a_1a_3, a_1a_2 - a_2a_1) = (0,0,0) = \mathbf{0}$.

(d) Let $\mathbf{a} = (a_1,a_2,a_3)$ and $\mathbf{b} = (b_1,b_2,b_3)$. Then $d\mathbf{a} = (da_1, da_2, da_3)$ and

$$(d\mathbf{a})\times\mathbf{b} = \begin{pmatrix} da_2b_3 - da_3b_2 \\ da_3b_1 - da_1b_3 \\ da_1b_2 - da_2b_1 \end{pmatrix} = d\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = d(\mathbf{a}\times\mathbf{b}),$$

$$\mathbf{a}\times(d\mathbf{b}) = \begin{pmatrix} a_2db_3 - a_3db_2 \\ a_3db_1 - a_1db_3 \\ a_1db_2 - a_2db_1 \end{pmatrix} = d\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = d(\mathbf{a}\times\mathbf{b}).$$

A268(a) $$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \qquad |\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2},$$

$$|\mathbf{a} \times \mathbf{b}| \stackrel{\star}{=} \sqrt{\left(a_2 b_3 - a_3 b_2\right)^2 + \left(a_3 b_1 - a_1 b_3\right)^2 + \left(a_1 b_2 - a_2 b_1\right)^2},$$

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

(b) Since $\theta \in [0, \pi]$, it must be that $\sin\theta$ is non-negative, i.e. $\sin\theta \geq 0$.

(c) The identity is $\sin^2\theta + \cos^2\theta = 1$. Rearranging, we have $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$. Since $\sin\theta \geq 0$, we can discard the negative value. Altogether,

$$\sin\theta = \sqrt{1 - \cos^2\theta}.$$

(d) $$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\right)^2} = \sqrt{1 - \frac{\left(a_1 b_1 + a_2 b_2 + a_3 b_3\right)^2}{\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)}}.$$

(e) As per the hint, fully expand each of LHS and RHS:

$$\begin{aligned}
\text{LHS} &= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - \left(a_1 b_1 + a_2 b_2 + a_3 b_3\right)^2 \\
&= a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + a_3^2 b_3^2 \\
&\quad - \left(a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + 2a_1 a_2 b_1 b_2 + 2a_1 a_3 b_1 b_3 + 2a_2 a_3 b_2 b_3\right) \\
&= a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 - \left(2a_1 a_2 b_1 b_2 + 2a_1 a_3 b_1 b_3 + 2a_2 a_3 b_2 b_3\right). \\
\text{RHS} &= \left(a_2 b_3 - a_3 b_2\right)^2 + \left(a_3 b_1 - a_1 b_3\right)^2 + \left(a_1 b_2 - a_2 b_1\right)^2 \\
&= a_2^2 b_3^2 + a_3^2 b_2^2 - 2a_2 a_3 b_2 b_3 + a_1^2 b_3^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3 + a_2^2 b_1^2 + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2.
\end{aligned}$$

"Clearly", LHS = RHS.

(f) $$\begin{aligned}
|\mathbf{a}|\,|\mathbf{b}|\sin\theta &= \sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)}\sqrt{1 - \frac{\left(a_1 b_1 + a_2 b_2 + a_3 b_3\right)^2}{\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)}} \\
&= \sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - \left(a_1 b_1 + a_2 b_2 + a_3 b_3\right)^2} \\
&\stackrel{\text{(e)}}{=} \sqrt{\left(a_2 b_3 - a_3 b_2\right)^2 + \left(a_3 b_1 - a_1 b_3\right)^2 + \left(a_1 b_2 - a_2 b_1\right)^2} \stackrel{\star}{=} |\mathbf{a} \times \mathbf{b}|.
\end{aligned}$$

151.17. Ch. 69 Answers: The Distance Between a Point and a Line

A269(a) Let $P = (8,3,4)$ and $\mathbf{v} = (9,3,7)$. Let B be the foot of the perpendicular from A to l. **Method 1 (Formula Method).** First, $\overrightarrow{PA} = (7,3,4) - (8,3,4) = (-1,0,0)$. So,

$$\overrightarrow{PB} = \text{proj}_{\mathbf{v}}\overrightarrow{PA} = \text{proj}_{(9,3,7)}(-1,0,0) = -\frac{9}{139}(9,3,7).$$

By Fact 150: $B = P + \text{proj}_{\mathbf{v}}\overrightarrow{PA} = (8,3,4) - \frac{9}{139}(9,3,7) = \frac{1}{139}(1\,031,390,493)$.

And by Corollary 28, the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right| = \left|(-1,0,0) \times \frac{(9,3,7)}{\sqrt{9^2+3^2+7^2}}\right| = \left|\frac{(0,7,-3)}{\sqrt{139}}\right| = \sqrt{\frac{58}{139}}.$$

Method 2 (Perpendicular Method). Let $B = (8,3,4) + \tilde{\lambda}(9,3,7)$. Write down $\overrightarrow{AB}$:

$$\overrightarrow{AB} = B - A = (8,3,4) + \tilde{\lambda}(9,3,7) - (7,3,4) = \left(9\tilde{\lambda}+1, 3\tilde{\lambda}, 7\tilde{\lambda}\right).$$

Since $\overrightarrow{AB} \perp l$, we have $\overrightarrow{AB} \perp \mathbf{v}$ or,

$$0 = \overrightarrow{AB} \cdot (9,3,7) = \left(9\tilde{\lambda}+1, 3\tilde{\lambda}, 7\tilde{\lambda}\right) \cdot (9,3,7) = 9\left(9\tilde{\lambda}+1\right) + 3\left(3\tilde{\lambda}\right) + 7\left(7\tilde{\lambda}\right) = 139\tilde{\lambda} + 9.$$

Rearranging, $\tilde{\lambda} = -9/139$.

And now, $$B = (8,3,4) - \frac{9}{139}(9,3,7) = \frac{1}{139}(1\,031,390,493).$$

Lovely—this is the same as what we found in Method 1. And now, the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \sqrt{\left(9\tilde{\lambda}+1\right)^2 + \left(3\tilde{\lambda}\right)^2 + \left(7\tilde{\lambda}\right)^2} = \sqrt{139\tilde{\lambda}^2 + 18\tilde{\lambda} + 1} = \sqrt{58/139}.$$

Method 3 (or the **Calculus Method**). Let R be a generic point on l, so that $\overrightarrow{AR} = (9\lambda+1, 3\lambda, 7\lambda+4)$ and the distance between A and R is

$$\left|\overrightarrow{AR}\right| = \sqrt{139\lambda^2 + 18\lambda + 1}.$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(139\lambda^2 + 18\lambda + 1\right) = 278\lambda + 18.$$

FOC: $$(278\lambda + 18)\Big|_{\lambda=\tilde{\lambda}} = 0 \quad \text{or} \quad \tilde{\lambda} = -\frac{18}{278} = -\frac{9}{139}.$$

Lovely—this is also what we found in Method 2. We can now find B and $\left|\overrightarrow{AB}\right|$ (omitted).

Alternatively, we could simply have used "$-b/2a$": $\tilde{\lambda} = -18/(2 \cdot 139) = -9/139$.

A269(b) Let $P = (4,4,3)$ and $\mathbf{v} = (6,11,5) - (4,4,3) = (2,7,2)$. Let B be the foot of the perpendicular from A to l.

Method 1 (Formula Method). First, $\overrightarrow{PA} = (8,0,2) - (4,4,3) = (4,-4,-1)$. So,

$$\overrightarrow{PB} = \operatorname{proj}_{\mathbf{v}} \overrightarrow{PA} = \operatorname{proj}_{(2,7,2)} (4,-4,-1) = -\frac{22}{57}(2,7,2).$$

By Fact 150: $\quad B = P + \operatorname{proj}_{\mathbf{v}} \overrightarrow{PA} = (4,4,3) - \dfrac{22}{57}(2,7,2) = \dfrac{1}{57}(184,74,127).$

And by Corollary 28, the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right| = \left|(4,-4,-1) \times \frac{(2,7,2)}{\sqrt{2^2+7^2+2^2}}\right| = \left|\frac{(-1,-10,36)}{\sqrt{57}}\right| = \sqrt{\frac{1\,397}{57}}.$$

Method 2 (Perpendicular Method). Let $B = (4,4,3) + \tilde{\lambda}(2,7,2)$. Write down $\overrightarrow{AB}$:

$$\overrightarrow{AB} = B - A = (4,4,3) + \tilde{\lambda}(2,7,2) - (8,0,2) = \left(2\tilde{\lambda} - 4, 7\tilde{\lambda} + 4, 2\tilde{\lambda} + 1\right).$$

Since $\overrightarrow{AB} \perp l$, we have $\overrightarrow{AB} \perp \mathbf{v}$ or,

$$0 = \overrightarrow{AB} \cdot (2,7,2) = \left(2\tilde{\lambda} - 4, 7\tilde{\lambda} + 4, 2\tilde{\lambda} + 1\right) \cdot (2,7,2) = 57\tilde{\lambda} + 22.$$

Rearranging, $\tilde{\lambda} = -22/57$.

And now, $\quad B = (4,4,3) - \dfrac{22}{57}(2,7,2) = \dfrac{1}{57}(184,74,127).$

Lovely—this is also what we found in Method 1. And now, the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \sqrt{\left(2\tilde{\lambda}-4\right)^2 + \left(7\tilde{\lambda}+4\right)^2 + \left(2\tilde{\lambda}+1\right)^2} = \sqrt{57\tilde{\lambda}^2 + 44\tilde{\lambda} + 33} = \sqrt{\frac{1397}{57}}.$$

Method 3 (or the **Calculus Method**). Let R be a generic point on l, so that $\overrightarrow{AR} = (2\lambda - 4, 7\lambda + 4, 2\lambda + 1)$ and the distance between A and R is

$$\left|\overrightarrow{AR}\right| = \sqrt{57\lambda^2 + 44\lambda + 33}.$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(57\lambda^2 + 44\lambda + 33\right) = 114\lambda + 44.$$

FOC: $\quad (114\lambda + 44)\big|_{\lambda = \tilde{\lambda}} = 0 \quad$ or $\quad \tilde{\lambda} = -\dfrac{44}{114} = -\dfrac{22}{57}.$

Lovely—this is also what we found in Method 2. We can now find B and $\left|\overrightarrow{AB}\right|$ (omitted).

Alternatively, we could simply have used "$-b/2a$": $\tilde{\lambda} = -44/(2 \cdot 57) = -22/57$.

A269(c) Let $P = (8,4,5)$ and $\mathbf{v} = (5,6,0)$. Let B be the foot of the perpendicular from A to l. **Method 1 (Formula Method).** First, $\overrightarrow{PA} = (8,5,9) - (8,4,5) = (0,1,4)$. So,

$$\overrightarrow{PB} = \text{proj}_{\mathbf{v}}\overrightarrow{PA} = \text{proj}_{(5,6,0)}(0,1,4) = \frac{6}{61}(5,6,0).$$

By Fact 150: $\quad B = P + \text{proj}_{\mathbf{v}}\overrightarrow{PA} = (8,4,5) + \dfrac{6}{61}(5,6,0) = \dfrac{1}{61}(518,280,305).$

And by Corollary 28, the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{PA} \times \hat{\mathbf{v}}\right| = \left|(0,1,4) \times \frac{(5,6,0)}{\sqrt{5^2+6^2+0^2}}\right| = \left|\frac{(-24,20,-5)}{\sqrt{61}}\right| = \sqrt{\frac{1\,001}{61}}.$$

Method 2 (Perpendicular Method). Let $B = (8,4,5) + \tilde{\lambda}(5,6,0)$. Write down $\overrightarrow{AB}$:

$$\overrightarrow{AB} = B - A = (8,4,5) + \tilde{\lambda}(5,6,0) - (8,5,9) = \left(5\tilde{\lambda}, 6\tilde{\lambda} - 1, -4\right).$$

Since $\overrightarrow{AB} \perp l$, we have $\overrightarrow{AB} \perp \mathbf{v}$ or,

$$0 = \overrightarrow{AB} \cdot (5,6,0) = \left(5\tilde{\lambda}, 6\tilde{\lambda} - 1, -4\right) \cdot (5,6,0) = 5\left(5\tilde{\lambda}\right) + 6\left(6\tilde{\lambda} - 1\right) + 0(-4) = 61\tilde{\lambda} - 6.$$

Rearranging, $\tilde{\lambda} = 6/61$.

And now, $\quad B = (8,4,5) + \dfrac{6}{61}(5,6,0) = \dfrac{1}{61}(518,280,305).$

Lovely—this is also what we found in Method 1. And now, the distance between A and l is

$$\left|\overrightarrow{AB}\right| = \sqrt{\left(5\tilde{\lambda}\right)^2 + \left(6\tilde{\lambda} - 1\right)^2 + (-4)^2} = \sqrt{61\tilde{\lambda}^2 - 12\tilde{\lambda} + 17} = \sqrt{\frac{1\,001}{61}}.$$

Method 3 (or the **Calculus Method**). Let R be a generic point on l, so that $\overrightarrow{AR} = (5\lambda, 6\lambda - 1, -4)$ and the distance between A and R is

$$\left|\overrightarrow{AR}\right| = \sqrt{61\lambda^2 - 12\tilde{\lambda} + 17}.$$

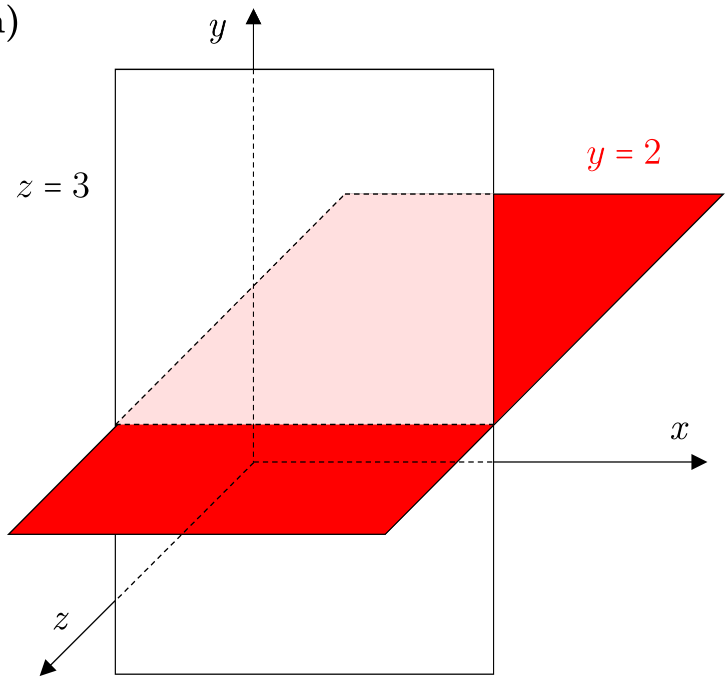
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(61\lambda^2 - 12\tilde{\lambda} + 17\right) = 122\lambda - 12.$$

FOC: $\quad (122\lambda - 12)\big|_{\lambda=\tilde{\lambda}} = 0 \quad \text{or} \quad \tilde{\lambda} = \dfrac{12}{122} = \dfrac{6}{61}.$

Lovely—this is also what we found in Method 2. We can now find B and $\left|\overrightarrow{AB}\right|$ (omitted).

Alternatively, we could simply have used "$-b/2a$": $\tilde{\lambda} = -(-12)/(2 \cdot 61) = 6/61$.

A270(a)



(b) $(0, 2, 0)$ and $(0, 2, 1)$.
(c) $(0, 0, 3)$ and $(0, 1, 3)$.
(d) $(0, 2, 3)$ and $(1, 2, 3)$.

151.19. Ch. 71 Answers (Planes: Formally Defined in Vector Form)

A271. B is on the plane q, but A and C are not:

$$\begin{array}{lllll}
\overrightarrow{OA}\cdot(-5,7,3) = (5,-3,1)\cdot(-5,7,3) & = -25-21+3 = & -43 \neq -1. & & ✗ \\
\overrightarrow{OB}\cdot(-5,7,3) = (1,-2,6)\cdot(-5,7,3) & = -5-14+18 = & -1 = -1. & & ✓ \\
\overrightarrow{OC}\cdot(-5,7,3) = (-2,2,-3)\cdot(-5,7,3) & = 10+14-9 = & 15 \neq -1. & & ✗
\end{array}$$

A272. No, l contains the point $(7,3,1)$, which isn't on q because $(7,3,1)\cdot(4,-3,2) \neq -10$.

A273. $\mathbf{a} = (2,-2,2)$ and $\mathbf{c} = \left(-\sqrt{2},\sqrt{2},-\sqrt{2}\right)$ are parallel to $(1,-1,1)$, while $\mathbf{b} = (2,2,-2)$ is not. Hence, $\mathbf{a}$ and $\mathbf{c}$ are normal to q, but not $\mathbf{b}$.

A274(a) $\overrightarrow{AB} = B - A = (-2,3,0)-(1,-1,2) = (-3,4,-2)$ and $\overrightarrow{AC} = C - A = (0,-1,1) - (1,-1,2) = (-1,0,-1)$. And so, a normal vector of q is

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (-3,4,-2)\times(-1,0,-1) = (-4,-1,4)$$

(b) $\mathbf{n}\cdot\overrightarrow{OA} = (-4,-1,4)\cdot(1,-1,2) = -4+1+8 = 5$. So, q may be described by $\mathbf{r}\cdot(-4,-1,4) = 5$.

(c) Another normal vector of q is $\mathbf{m} = 2\mathbf{n} = 2\,(-4,-1,4) = (-8,-2,8)$

(d) Hence, the plane q may also be described by $\mathbf{r}\cdot(-8,-28) = 10$.

A275. Each of the plane q and the line l is a set of points. So, in particular, l is not a point. Similarly, the vector $\mathbf{v}$ is not a point.

Hence, each of **(a)**, **(d)**, **(e)**, and **(h) (ii)** is only ever strictly speaking incorrect, but nonetheless written by us anyway (because we're sloppy and lazy).

In contrast, each of **(b)**, **(c)**, **(f)**, and **(g) (ii)** could be perfectly correct.

A276. Only $\mathbf{b}$ is perpendicular to q's normal vector (see below). Hence, only $\mathbf{b}$ is on q.

$$\begin{array}{llll}
\mathbf{a}\cdot(8,-2,1) = (3,7,-5)\cdot(8,-2,1) = 24-14-5 = & 5 & & ✗ \\
\mathbf{b}\cdot(8,-2,1) = (1,6,4)\cdot(8,-2,1) = 8-12+4 = & 0 & & ✓ \\
\mathbf{c}\cdot(8,-2,1) = (3,10,1)\cdot(8,-2,1) = 24-20+1 = & 5 & & ✗
\end{array}$$

A277. Method 1. First find $B = A + \overrightarrow{AB} = (1,4,-1)+(7,3,-2) = (8,7,-3)$. Then show that B does **not** satisfy the plane's vector equation and is thus **not** on q:

$$\overrightarrow{OB}\cdot(7,-1,3) = (8,7,-3)\cdot(7,-1,3) = 56-7-9 \neq 19. \qquad ✗$$

Method 2. Simply check if $\overrightarrow{AB} \perp (7,-1,3)$:

$$\overrightarrow{AB}\cdot(7,-1,3) = (7,3,-2)\cdot(7,-1,3) = 49-3-6 \neq 0. \qquad ✗$$

So no, $\overrightarrow{AB} \not\perp (7,-1,3)$. Hence, by Fact 163, $B \notin q$.

A278. Since $\mathbf{n} \not\perp \mathbf{a},\mathbf{b}$, we know for sure that $\mathbf{n}$ cannot be a normal vector of q.

It is true that $\mathbf{m} \perp \mathbf{a},\mathbf{b}$. However, observe that $\mathbf{a} \parallel \mathbf{b}$ and so Corollary 31 does not apply. That is, we are unable to conclude that $\mathbf{m} \perp q$. The answer is thus, "$\mathbf{m}$ may be a normal vector of q, but we don't know for sure."

151.20. Ch. 72 Answers (Planes in Cartesian Form)

A279(a) $x+2y+3z=17$ doesn't contain the origin.

(b) $-x-2z=0$ contains the origin.

(c) $-2y+5z=-3$ doesn't contain the origin.

A280(a) Rewrite $x+5=17y+z$ as $x-17y-z=-5$. Reading off, the plane may also be described by $\mathbf{r}\cdot(1,-17,-1)=-15$ and doesn't contain the origin.

(b) Rewrite $y+1=0$ as $0x+y+0z=-1$. Reading off, the plane may also be described by $\mathbf{r}\cdot(0,1,0)=-1$ and doesn't contain the origin.

(c) Rewrite $x+z=y-2$ as $x-y+z=-2$. Reading off, the plane may also be described by $\mathbf{r}\cdot(1,-1,1)=-2$ and doesn't contain the origin.

A281(a) The plane described by $\mathbf{r}\cdot(0,0,1)=32$ or $z=32$ contains the points $(0,0,32)$, $(1,0,32)$, and $(0,1,32)$. It does not contain the points $(1,0,0)$, $(0,1,0)$, or $(0,0,1)$.

(b) The plane described by $\mathbf{r}\cdot(5,3,1)=-2$ or $5x+3y+z=-2$ contains the points $(-1,1,0)$, $(0,0,-2)$, and $(0,-1,1)$. It does not contain the points $(1,0,0)$, $(0,1,0)$, or $(0,0,1)$.

(c) The plane described by $\mathbf{r}\cdot(1,-2,3)=0$ or $x-2y+3z=0$ contains the points $(0,0,0)$, $(2,1,0)$, and $(-3,0,1)$. It does not contain the points $(1,0,0)$, $(0,1,0)$, or $(0,0,1)$.

A282(a) $(2,1,0)$, $(3,0,-1)$, and $(0,3,2)$. **(c)** $(4,0,-1)$, $(4,1,-1)$, and $(4,2,-1)$.

(b) $(3,-5,0)$, $(1,0,-5)$, and $(0,1,-3)$. **(d)** $(1,0,0)$, $(0,0,1)$, and $(1,0,1)$

151.21. Ch. 73 Answers (Planes in Parametric Form)

A283(a)(i) $$x + 2y + 3z = 4.$$

(ii) Since $\mathbf{u} \cdot (1, 2, 3) = (2, -1, 0) \cdot (1, 2, 3) = 0$ and $\mathbf{v} \cdot (1, 2, 3) = (-3, 0, 1) \cdot (1, 2, 3) = 0$, both $\mathbf{u}$ and $\mathbf{v}$ are on the plane. Also, $\mathbf{u} \not\parallel \mathbf{v}$ because they aren't scalar multiples of each other.

(iii) Since $\mathbf{w} \cdot (1, 2, 3) = (-1, -1, 1) \cdot (1, 2, 3) = 0$, the vector $\mathbf{w}$ is on the plane.

The vectors $\mathbf{u}$ and $\mathbf{v}$ are on the same plane and $\mathbf{u} \not\parallel \mathbf{v}$. Hence, by Theorem 19, we should be able to write $\mathbf{w}$ as the LC of $\mathbf{u}$ and $\mathbf{v}$, as indeed we can:

$$\mathbf{w} = (-1, -1, 1) = (2, -1, 0) + (-3, 0, 1) = \mathbf{u} + \mathbf{v}.$$

(iv) Since $(1, 1, 1) \cdot (1, 2, 3) \neq 0$, we have $(1, 1, 1) \not\perp (1, 2, 3)$ and so $(1, 1, 1)$ is **not** on the plane. Hence, by Theorem 19, it **cannot** be written as a LC of $\mathbf{u}$ and $\mathbf{v}$.

(b)(i) $$x - z = 0.$$

(ii) Since $\mathbf{u} \cdot (1, 0, -1) = (0, 1, 0) \cdot (1, 0, -1) = 0$ and $\mathbf{v} \cdot (1, 0, -1) = (1, 0, 1) \cdot (1, 0, -1) = 0$, both $\mathbf{u}$ and $\mathbf{v}$ are on the plane. Also, $\mathbf{u} \not\parallel \mathbf{v}$ because they aren't scalar multiples of each other.

(iii) Since $\mathbf{w} \cdot (1, 0, -1) = (1, -1, 1) \cdot (1, 0, -1) = 0$, the vector $\mathbf{w}$ is on the plane.

The vectors $\mathbf{u}$ and $\mathbf{v}$ are on the same plane and $\mathbf{u} \not\parallel \mathbf{v}$. Hence, by Theorem 19, we should be able to write $\mathbf{w}$ as the LC of $\mathbf{u}$ and $\mathbf{v}$, as indeed we can:

$$\mathbf{w} = (1, -1, 1) = (1, 0, 1) - (0, 1, 0) = \mathbf{v} - \mathbf{u}.$$

(iv) Since $(1, 1, 1) \cdot (1, 0, -1) = 0$, we have $(1, 1, 1) \perp (1, 0, -1)$ and so $(1, 1, 1)$ **is** on the plane. And hence, by Theorem 19, it **can** be written as a LC of $\mathbf{u}$ and $\mathbf{v}$:

$$(1, 1, 1) = (1, 0, 1) + (0, 1, 0) = \mathbf{u} + \mathbf{v}.$$

(c)(i) $$9x + y + z = -5.$$

(ii) Since $\mathbf{u} \cdot (9, 1, 1) = (0, 1, -1) \cdot (9, 1, 1) = 0$ and $\mathbf{v} \cdot (1, 0, -1) = (-1, 9, 0) \cdot (9, 1, 1) = 0$, both $\mathbf{u}$ and $\mathbf{v}$ are on the plane. Also, $\mathbf{u} \not\parallel \mathbf{v}$ because they aren't scalar multiples of each other.

(iii) Since $\mathbf{w} \cdot (9, 1, 1) = (-1, 10, -1) \cdot (9, 1, 1) = 0$, the vector $\mathbf{w}$ is on the plane.

The vectors $\mathbf{u}$ and $\mathbf{v}$ are on the same plane and $\mathbf{u} \not\parallel \mathbf{v}$. Hence, by Theorem 19, awe should be able to write $\mathbf{w}$ as the LC of $\mathbf{u}$ and $\mathbf{v}$, as indeed we can:

$$\mathbf{w} = (-1, 10, -1) = (0, 1, -1) + (-1, 9, 0) = \mathbf{u} + \mathbf{v}.$$

(iv) Since $(1, 1, 1) \cdot (9, 1, 1) \neq 0$, we have $(1, 1, 1) \not\perp (9, 1, 1)$ and so $(1, 1, 1)$ is **not** on the plane. Hence, by Theorem 19, it **cannot** be written as a LC of $\mathbf{u}$ and $\mathbf{v}$.

A284(a) The plane $\mathbf{r}\cdot(-1,2,5)=5$ has cartesian equation: $-x+2y+5z=5$.

It contains the point $(-5,0,0)$ and the non-parallel vectors $(2,1,0)$ and $(5,0,1)$. And so, it may be described by the following parametric equation:

$$\mathbf{r}=(-5,0,0)+\lambda(2,1,0)+\mu(5,0,1)=(-5+2\lambda+5\mu,\lambda,\mu)\qquad(\lambda,\mu\in\mathbb{R}).$$

(b) The plane $\mathbf{r}\cdot(0,0,1)=0$ has cartesian equation: $z=0$.

It contains the point $(0,0,0)$ and the non-parallel vectors $(1,0,0)$ and $(0,1,0)$. And so, it may be described by the following parametric equation:

$$\mathbf{r}=(0,0,0)+\lambda(1,0,0)+\mu(0,1,0)=(\lambda,\mu,0)\qquad(\lambda,\mu\in\mathbb{R}).$$

(c) The plane $\mathbf{r}\cdot(1,-3,5)=-2$ has cartesian equation: $x-3y+5z=-2$.

It contains the point $(-2,0,0)$ and the non-parallel vectors $(3,1,0)$ and $(0,5,3)$. And so, it may be described by the following parametric equation:

$$\mathbf{r}=(-2,0,0)+\lambda(3,1,0)+\mu(0,5,3)=(-2+3\lambda,\lambda+5\mu,3\mu)\qquad(\lambda,\mu\in\mathbb{R}).$$

A285(a) This plane contains the vectors $(4,5,6)$ and $(7,8,9)$. And so, a normal vector is

$$(4,5,6)\times(7,8,9)=(-3,6,-3).$$

Since $(-3,6,-3)\parallel(1,-2,1)$, a normal vector of the plane is $(1,-2,1)$.

The plane contains the point $(1,2,3)$. Since $(1,2,3)\cdot(1,-2,1)=1-4+3=0$, the plane may be described in vector and cartesian forms by $\mathbf{r}\cdot(1,-2,1)=0$ and $x-2y+z=0$.

(b) First, rewrite the given parametric equation as:

$$\mathbf{r}=(\lambda-\mu,4\lambda+5,0)=(0,5,0)+\lambda(1,4,0)+\mu(-1,0,0)\qquad(\lambda,\mu\in\mathbb{R}).$$

Thus, this plane contains the vectors $(1,4,0)$ and $(-1,0,0)$. And so, a normal vector is

$$(1,4,0)\times(-1,0,0)=(0,0,4).$$

Since $(0,0,4)\parallel(0,0,1)$, a normal vector of the plane is $(0,0,1)$.

The plane contains the point $(0,5,0)$. Since $(0,5,0)\cdot(0,0,1)=0+0+0=0$, the plane may be described in vector and cartesian forms by $\mathbf{r}\cdot(0,0,1)=0$ and $z=0$.

(c) First, rewrite the given parametric equation as:

$$\mathbf{r}=(1+\mu,1+\lambda,\lambda+\mu)=(1,1,0)+\lambda(0,1,1)+\mu(1,01,1)\qquad(\lambda,\mu\in\mathbb{R}).$$

Thus, this plane contains the vectors $(0,1,1)$ and $(1,0,1)$. And so, a normal vector is

$$(0,1,1)\times(1,0,1)=(1,1,-1).$$

The plane contains the point $(1,1,0)$. Since $(1,1,0)\cdot(1,1,-1)=1+1+0=2$, the plane may be described in vector and cartesian forms by $\mathbf{r}\cdot(1,1,-1)=2$ and $x+y-z=2$.

A286(a) The plane that contains the points $(7,3,4)$, $(8,3,4)$, and $(9,3,7)$ also contains the vectors $(8,3,4)-(7,3,4)=(1,0,0)$ and $(9,3,7)-(7,3,4)=(2,0,3)$.

Since $(1,0,0) \nparallel (2,0,3)$, the plane may be described in parametric form by

$$\mathbf{r}=\begin{pmatrix}7\\3\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}+\mu\begin{pmatrix}2\\0\\3\end{pmatrix}=\begin{pmatrix}7+\lambda+2\mu\\3\\4+3\mu\end{pmatrix} \qquad (\lambda,\mu\in\mathbb{R}).$$

It has normal vector $(1,0,0)\times(2,0,3)=(0,-3,0)$.

It thus also has normal vector $(0,1,0)$. Compute $(7,3,4)\cdot(0,1,0)=0+3+0=3$. So, this plane may be described in vector or cartesian form by

$$\mathbf{r}\cdot(0,1,0)=3 \qquad \text{or} \qquad y=3.$$

(b) The plane that contains the points $(8,0,2)$, $(4,4,3)$, and $(2,7,2)$ also contains the vectors $(4,4,3)-(8,0,2)=(-4,4,1)$ and $(2,7,2)-(8,0,2)=(-6,7,0)$.

Since $(-4,4,1) \nparallel (-6,7,0)$, the plane may be described in parametric form by

$$\mathbf{r}=\begin{pmatrix}8\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}-4\\4\\1\end{pmatrix}+\mu\begin{pmatrix}-6\\7\\0\end{pmatrix}=\begin{pmatrix}8-4\lambda-6\mu\\0+4\lambda+7\mu\\2+\lambda\end{pmatrix} \qquad (\lambda,\mu\in\mathbb{R}).$$

This plane has normal vector $(-4,4,1)\times(-6,7,0)=(-7,-6,-4)$.

It thus also has normal vector $(7,6,4)$. Compute $(8,0,2)\cdot(7,6,4)=56+0+8=64$. So, this plane may be described in vector or cartesian form by

$$\mathbf{r}\cdot(7,6,4)=64 \quad \text{or} \quad 7x+6y+4z=64.$$

(c) The plane that contains the points $(8,5,9)$, $(8,4,5)$, and $(5,6,0)$ also contains the vectors $(8,5,9)-(8,4,5)=(0,1,4)$ and $(8,5,9)-(5,6,0)=(3,-1,9)$.

Since $(0,1,4) \nparallel (3,-1,9)$, the plane may be described in parametric form by

$$\mathbf{r}=\begin{pmatrix}5\\6\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\4\end{pmatrix}+\mu\begin{pmatrix}3\\-1\\9\end{pmatrix}=\begin{pmatrix}5+3\mu\\6+\lambda-\mu\\4\lambda+9\mu\end{pmatrix} \qquad (\lambda,\mu\in\mathbb{R}).$$

This plane has normal vector $(0,1,4)\times(3,-1,9)=(13,12,-3)$.

Compute $(5,6,0)\cdot(13,12,-3)=65+72+0=137$. So, this plane may be described in vector or cartesian form by

$$\mathbf{r}\cdot(13,12,-3)=137 \qquad \text{or} \qquad 13x+12y-3z=137.$$

151.22. Ch. 75 Answers (The Angle Between a Line and a Plane)

A287(a) The angle between a line with direction vector $(-1,1,0)$ and a plane with normal vector $(3,4,5)$ is

$$\sin^{-1}\frac{|(-1,1,0)\cdot(3,4,5)|}{|(-1,1,0)||(3,4,5)|}=\sin^{-1}\frac{|1|}{\sqrt{2}\sqrt{50}}=\sin^{-1}0.1\approx 0.100.$$

(b) The given line has direction vector $(-1,4,9)-(-1,2,3)=(0,2,6)$ or $(0,1,3)$. The given plane has normal vector $(-3,1,0)\times(0,5,-3)=(-3,-9,-15)$ or $(1,3,5)$.

Hence, the angle between the line and the plane is

$$\sin^{-1}\frac{|(0,1,3)\cdot(1,3,5)|}{|(0,1,3)||(1,3,5)|}=\sin^{-1}\frac{|18|}{\sqrt{10}\sqrt{35}}\approx 1.295.$$

(c) The given line has direction vector $(0,11,11)-(-1,2,3)=(1,9,8)$. The given plane also contains the vector $(1.5,0,0)-(0,0,1.5)=(1.5,0,-1.5)$ or $(1,0,-1)$; hence, it has normal vector $(4,-1,0)\times(1,0,-1)=(1,4,1)$.

Hence, the angle between the line and the plane is

$$\sin^{-1}\frac{|(1,9,8)\cdot(1,4,1)|}{|(1,9,8)||(1,4,1)|}=\sin^{-1}\frac{|45|}{\sqrt{146}\sqrt{18}}\approx 1.071.$$

A288(a) The line l has direction vector $\mathbf{v}=(2,3,5)$ and the plane q has normal vector $\mathbf{n}=(-10,0,4)$. Now, $\mathbf{v}\cdot\mathbf{n}=(2,3,5)\cdot(-10,0,4)=-20+0+20=0$. So, $\mathbf{v}\perp\mathbf{n}$ and thus $l\parallel q$.

The point $(4,5,6)$ is on l but not on q because $(4,5,6)\cdot(-10,0,4)=-40+0+24=-16\neq-26$. Hence, l and q do not intersect at all.

(b) The line l has direction vector $\mathbf{v}=(5,5,6)-(3,2,1)=(2,3,5)$.

The plane q has normal vector $\mathbf{n}=(2,0,5)\times(2,1,5)=(-5,0,2)$.

Since $\mathbf{v}\cdot\mathbf{n}=(2,3,5)\cdot(-5,0,2)=-10+0+10=0$, we have $\mathbf{v}\perp\mathbf{n}$ and thus $l\parallel q$.

Compute $(3,0,1)\cdot(-5,0,2)=-15+0+2=-13$. So, q has vector equation $\mathbf{r}\cdot(-5,0,2)=-13$.

The point $(3,2,1)$ is on l and is also on q because $(3,2,1)\cdot(-5,0,2)=-15+0+2=-13$. Hence, the line lies entirely on the plane.

(c) The line l has direction vector $\mathbf{v}=(6,8,11)-(4,5,6)=(2,3,5)$.

The plane q contains the vector $(2,1,-2)-(2,0,-2)=(0,1,0)$ and thus has normal vector $\mathbf{n}=(0,1,0)\times(3,0,10)=(10,0,-3)$.

Since $\mathbf{v}\cdot\mathbf{n}=(2,3,5)\cdot(10,0,-3)=20+0-15=5\neq 0$, , we have $\mathbf{v}\not\perp\mathbf{n}$ and thus $l\nparallel q$. Hence, l and q share exactly one intersection point.

Compute $(2,0,-2)\cdot(10,0,-3)=20+0+6=26$. So, q has vector equation $\mathbf{r}\cdot(10,0,-3)=26$.

To find the intersection point, plug a generic point of l into q's vector equation:

$$\left[(4,5,6)+\hat{\lambda}(2,3,5)\right]\cdot(10,0,-3)=26 \quad\iff\quad 22+5\hat{\lambda}=26 \quad\iff\quad \lambda=0.8.$$

Thus, l and q intersect at: $(4,5,6)+\hat{\lambda}(2,3,5)=(4,5,6)+0.8(2,3,5)=(5.6,7.4,10)$.

151.23. Ch. 76 Answers (The Angle Between Two Planes)

A289(a) The angle between planes with normal vectors $(-1,-2,-3)$ and $(3,4,5)$ is

$$\theta = \cos^{-1}\frac{|(-1,-2,-3)\cdot(3,4,5)|}{|(-1,-2,-3)|\,|(3,4,5)|} = \cos^{-1}\frac{|-26|}{\sqrt{14}\sqrt{50}} \approx 0.186.$$

(b) The first plane has normal vector $(1,-1,0)\times(3,5,-1) = (1,1,8)$.

The second plane has normal vector $(0,1,0)\times(10,2,3) = (3,0,-10)$.

And so, the angle between the two planes is

$$\cos^{-1}\frac{|(1,1,8)\cdot(3,0,-10)|}{|(1,1,8)|\,|(3,0,-10)|} = \cos^{-1}\frac{|-77|}{\sqrt{66}\sqrt{109}} \approx 0.433.$$

(c) The first plane contains vectors $(3,0,0)-(1,1,0) = (2,-1,0)$ and $(3,0,0)-(0,0,1) = (3,0,-1)$. And so, it has normal vector $(2,-1,0)\times(3,0,-1) = (1,2,3)$.

The second plane contains vectors $(1,0,-1)-(1,-1,0) = (0,1,-1)$ and $(0,3,1)-(1,-1,0) = (-1,4,1)$. And so, it has normal vector $(0,1,-1)\times(-1,4,1) = (5,1,1)$.

Thus, the angle between the two planes is

$$\theta = \cos^{-1}\frac{|(1,2,3)\cdot(5,1,1)|}{|(1,2,3)|\,|(5,1,1)|} = \cos^{-1}\frac{|10|}{\sqrt{14}\sqrt{27}} \approx 1.031.$$

A290(a) Since $(4,9,3) \nparallel (1,1,2)$, q_1 and q_2 are **not parallel** and intersect along a line with direction vector $(4,9,3)\times(1,1,2) = (15,-5,-5)$ or $(-3,1,1)$.

To find an intersection point, plug $x=0$ into their cartesian equations to get $9y+3z \stackrel{1}{=} 61$ and $y+2z \stackrel{2}{=} 19$. $\stackrel{1}{=}$ minus $9\times \stackrel{2}{=}$ yields $-15z = -110$ or $z = 22/3$. And so, $y = 13/3$. Thus, their **intersection line** has vector equation $\mathbf{r} = (0, {}^{13}/_{3}, {}^{22}/_{3}) + \lambda(-3,1,1)$ $(\lambda \in \mathbb{R})$.

(b) q_1 has normal vector $(1,-1,0)\times(1,-1,1) = (-1,-1,0)$ or $(1,1,0)$.

q_2 has normal vector $(6,-1,0)\times(8,0,-1) = (1,6,8)$.

Since $(1,1,0) \nparallel (1,6,8)$, q_1 and q_2 are **not parallel** and intersect along a line with direction vector $(1,1,0)\times(1,6,8) = (8,-8,5)$.

Compute $(1,3,-2)\cdot(1,1,0) = 1+3+0 = 4$ and $(2,3,5)\cdot(1,6,8) = 2+18+40 = 60$. Hence, q_1 and q_2 have cartesian equations $x+y=4$ and $x+6y+8z=60$.

To find an intersection point, plug in $x=0$ to get $y \stackrel{1}{=} 4$ and $6y+8z \stackrel{2}{=} 60$. So, $y=4$ and $z=4.5$. Thus, their **intersection line** has vector equation $\mathbf{r} = (0,4,4.5)+\lambda(8,-8,5)$ $(\lambda \in \mathbb{R})$.

A290(c) q_1 contains the vectors $(7,7,0)-(1,1,6)=(6,6,-6)$ or $(1,1,-1)$ and $(5,3,3)-(1,1,6)=(4,2,-3)$. And so, it has normal vector $(1,1,-1)\times(4,2,-3)=(-1,-1,-2)$ or $(1,1,2)$.

q_2 contains the vectors $(7,3,1)-(5,5,1)=(2,-2,0)$ or $(1,-1,0)$ and $(5,5,1)-(3,5,2)=(2,0,-1)$. And so, it has normal vector $(1,-1,0)\times(2,0,-1)=(1,1,2)$.

Clearly, q_1 and q_2 are **parallel**.

To check if they intersect at all, we'll pick any point on q_1—say $(7,7,0)$—and check if it's on q_2. Compute $(5,5,1)\cdot(1,1,2)=5+5+2=12$—hence, q_2 has vector equation $\mathbf{r}\cdot(1,1,2)=12$. Since $(7,7,0)\cdot(1,1,2)=7+7+0=14\neq 12$, the point $(7,7,0)$ is **not** on q_2. Thus, q_1 and q_2 **do not intersect at all**.

(d) q_1 contains the vectors $(5,3,2)-(1,5,3)=(4,-2,-1)$ and $(10,0,1)-(5,3,2)=(5,-3,-1)$. And so, it has normal vector $(4,-2,-1)\times(5,-3,-1)=(-1,-1,-2)$ or $(1,1,2)$.

q_2 contains the vectors $(8,8,-2)-(5,-1,4)=(3,9,-6)$ or $(1,3,-2)$ and $(8,8,-2)-(3,5,2)=(5,3,-4)$. And so, it has normal vector $(1,3,-2)\times(5,3,-4)=(-6,-6,-12)$ or $(1,1,2)$.

Clearly, the two planes are **parallel**.

To check if they intersect at all, we'll pick any point on q_1—say $(10,0,1)$—and check if it's on q_2. Compute $(3,5,2)\cdot(1,1,2)=3+5+4=12$—hence, q_2 has vector equation $\mathbf{r}\cdot(1,1,2)=12$. Since $(10,0,1)\cdot(1,1,2)=10+0+2=12$, the point $(10,0,1)$ **is** on the second plane. Since q_1 and q_2 are parallel and share at least one intersection point, they must be **identical**.

(e) Since $(7,1,1)\nparallel(1,1,2)$, q_1 and q_2 are **not parallel** and intersect along a line with direction vector $(7,1,1)\times(1,1,2)=(1,-13,6)$.

To find an intersection point, plug $x=0$ into their cartesian equations to get $y+z\overset{1}{=}42$ and $y+2z\overset{2}{=}6$. $\overset{2}{=}$ minus $\overset{1}{=}$ yields $z=-36$. And so, $y=78$. Thus, their **intersection line** has vector equation $\mathbf{r}=(0,78,-36)+\lambda(1,-13,6)$ $(\lambda\in\mathbb{R})$.

(f) Since $(0,1,3)\nparallel(-1,1,3)$, q_1 and q_2 are **not parallel** and intersect along a line with direction vector $(0,1,3)\times(-1,1,3)=(0,-3,1)$.

Observe that if here we try plugging $x=0$ into their cartesian equations, then we get $y+3z\overset{1}{=}0$ and $y+3z\overset{2}{=}2$, which are contradictory. This contradiction tells us that the two planes have no intersection point with x-coordinate 0.

So, let's instead try plugging in $y=0$ to get $3z\overset{3}{=}0$ and $-x+3z\overset{4}{=}2$. Solving, $z=0$ and $x=-2$. Thus, their **intersection line** has vector equation $\mathbf{r}=(-2,0,0)+\lambda(0,-3,1)$ $(\lambda\in\mathbb{R})$.

151.24. Ch. 77 Answers (Point-Plane Foot and Distance)

A291(a) Write $B = A + k\mathbf{n} = (7,3,4) + k(9,3,7)$. Since $B \in q$, we have

$$\overrightarrow{OB} \cdot (9,3,7) = 109 \quad \text{or} \quad [(7,3,4) + k(9,3,7)] \cdot (9,3,7) = 109 \quad \text{or} \quad 100 + 139k = 109.$$

So, $k = 9/139$ and $B = A + k\mathbf{n} = (7,3,4) + \dfrac{9}{139}(9,3,7) = \dfrac{1}{139}(1\,054, 444, 619)$.

And the distance between A and q is

$$\left|\overrightarrow{AB}\right| = |k\mathbf{n}| = |k|\,|\mathbf{n}| = \left|\frac{9}{139}\right| |(9,3,7)| = \frac{9}{139}\sqrt{139} = \frac{9}{\sqrt{139}}.$$

(b) Write $B = A + k\mathbf{n} = (8,0,2) + k(2,7,2)$. Since $B \in q$, we have

$$\overrightarrow{OB} \cdot (2,7,2) = 42 \quad \text{or} \quad [(8,0,2) + k(2,7,2)] \cdot (2,7,2) = 42 \quad \text{or} \quad 20 + 57k = 42.$$

So, $k = 22/57$ and $B = A + k\mathbf{n} = (8,0,2) + \dfrac{22}{57}(2,7,2) = \dfrac{1}{57}(500, 154, 158)$.

And the distance between A and q is

$$\left|\overrightarrow{AB}\right| = |k\mathbf{n}| = |k|\,|\mathbf{n}| = \left|\frac{22}{57}\right| |(2,7,2)| = \frac{22}{57}\sqrt{57} = \frac{22}{\sqrt{57}}.$$

(c) Write $B = A + k\mathbf{n} = (8,5,9) + k(5,6,0)$. Since $B \in q$, we have

$$\overrightarrow{OB} \cdot (5,6,0) = 64 \quad \text{or} \quad [(8,5,9) + k(5,6,0)] \cdot (5,6,0) = 64 \quad \text{or} \quad 70 + 61k = 64.$$

So, $k = -6/61$ and $B = A + k\mathbf{n} = A\mathbf{n} = (8,5,9) - \dfrac{6}{61}(5,6,0) = \dfrac{1}{61}(458, 269, 549)$.

And the distance between A and q is

$$\left|\overrightarrow{AB}\right| = |k\mathbf{n}| = |k|\,|\mathbf{n}| = \left|-\frac{6}{61}\right| |(5,6,0)| = \frac{6}{61}\sqrt{61} = \frac{6}{\sqrt{61}}.$$

A292. Since B is the foot of the perpendicular, we have $AB \perp q$ or $\overrightarrow{AB} \parallel \mathbf{n}$ or for some $k \neq 0$, $k\mathbf{n} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

So, **(b)** $\left|\overrightarrow{AB}\right| = |k\mathbf{n}| = |k|\,|\mathbf{n}|$. Also, $\overrightarrow{OB} = \overrightarrow{OA} + k\mathbf{n}$ or **(a)** $B = A + k\mathbf{n}$.

We now show that k is as claimed in Fact 178. Since $B \in q$,

$$\overrightarrow{OB} \cdot \mathbf{n} = d \qquad \text{or} \qquad \left(\overrightarrow{OA} + k\mathbf{n}\right) \cdot \mathbf{n} = d \qquad \text{or} \qquad \overrightarrow{OA} \cdot \mathbf{n} + k\mathbf{n} \cdot \mathbf{n} = d.$$

Rearranging,
$$k = \frac{d - \overrightarrow{OA} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} = \frac{d - \overrightarrow{OA} \cdot \mathbf{n}}{|\mathbf{n}|^2}.$$
✓

A293. By Fact 178, the distance between the origin O and the given plane is

$$\frac{\left|d - \overrightarrow{OO} \cdot \mathbf{n}\right|}{|\mathbf{n}|} = \frac{|d|}{|\mathbf{n}|}.$$

And hence, if $|\mathbf{n}| = 1$, then the distance between the plane and the origin is simply $|d|$.

A294(a) Perpendicular Method. Let A and B be the feet of the perpendiculars from S and T to q. Write $A = S + k\mathbf{n}$ and $B = T + l\mathbf{n}$.

Since $A, B \in q$, we have

$$\begin{array}{lclcl} \overrightarrow{OA} \cdot (5,-3,1) = 0 & \text{or} & [(-1,0,7) + k(5,-3,1)] \cdot (5,-3,1) & \text{or} & 2 + 35k = 0, \\ \overrightarrow{OB} \cdot (5,-3,1) = 0 & \text{or} & [(3,2,1) + l(5,-3,1)] \cdot (5,-3,1) = 0 & \text{or} & 10 + 35l = 0. \end{array}$$

Solving, we have $k = -2/35$ and $l = -10/35 = -2/7$. Thus,

$$\begin{aligned} A &= S + k\mathbf{n} = (-1,0,7) - -\frac{2}{35}(5,-3,1) = \frac{1}{35}(-45,6,243), \\ B &= T + l\mathbf{n} = (3,2,1) - -\frac{2}{7}(5,-3,1) = \frac{1}{7}(11,20,5). \end{aligned}$$

Formula Method. First compute $|\mathbf{n}| = \sqrt{5^2 + (-3)^2 + 1^2} = \sqrt{35}$. Then compute

$$k = \frac{d - \overrightarrow{OS} \cdot \mathbf{n}}{|\mathbf{n}|^2} = \frac{0 - (-1,0,7) \cdot (5,-3,1)}{35} = \frac{0 - (-5+0+7)}{35} = -\frac{2}{35}.$$

$$l = \frac{d - \overrightarrow{OT} \cdot \mathbf{n}}{|\mathbf{n}|^2} = \frac{0 - (3,2,1) \cdot (5,-3,1)}{35} = \frac{0 - (15-6+1)}{35} = -\frac{2}{7}.$$

So, the feet of the perpendiculars from S and T to q are, respectively

$$\begin{aligned} S + k\mathbf{n} &= (-1,0,7) - -\frac{2}{35}(5,-3,1) = \frac{1}{35}(-45,6,243), \\ T + l\mathbf{n} &= (3,2,1) - -\frac{2}{7}(5,-3,1) = \frac{1}{7}(11,20,5). \end{aligned}$$

(b) The distances between q and the points S and T are, respectively

$$|k|\,|\mathbf{n}| = \frac{2}{35} \cdot \sqrt{35} = \frac{2}{\sqrt{35}}.$$

$$|l|\,|\mathbf{n}| = \frac{2}{7} \cdot \sqrt{35} = \frac{2\sqrt{5}}{\sqrt{7}}.$$

(c) Since the origin is on q, the distance between the origin and q is 0.

A295. The line AB is described by $\mathbf{r} = (1,0,0) + \lambda(1,-1,0)$ ($\lambda \in \mathbb{R}$) and does not contain C or D. Hence, A, B, and C are not collinear and neither are A, B, and D.

The line CD is described by $\mathbf{r} = (0,0,1) + \lambda(1,1,-2)$ ($\lambda \in \mathbb{R}$) and does not contain A or B. Hence, C, D, and A are not collinear and neither are C, D, and B.

A296(a) Let q be the plane that contains A, B, and C. The non-parallel vectors $\overrightarrow{BA} = (3,2,4)$ and $\overrightarrow{BC} = (5,8,4)$ are on q.

Method 1 (Vector Form). q has normal vector $\overrightarrow{BA} \times \overrightarrow{BC} = (3,2,4) \times (5,8,4) = (-24,8,14)$ or $(-12,4,7)$.

Compute $\overrightarrow{OA} \cdot (-12,4,7) = (0,1,5) \cdot (-12,4,7) = 0 + 4 + 35 = 39$. Hence, q may be described by $\mathbf{r} \cdot (-12,4,7) = 39$.

Now check if $D \in q$: $\overrightarrow{OD} \cdot (-12,4,7) = (6,6,1) \cdot (-12,4,7) = -72 + 24 + 7 \neq 39$. Nope, it isn't. So the four points aren't coplanar.

Method 2 (Parametric Form). The plane q may be described in parametric form as:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\lambda + 5\mu \\ 1 + 2\lambda + 8\mu \\ 5 + 4\lambda + 4\mu \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

Now check if $D \in q$:

$$\begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\lambda + 5\mu \\ 1 + 2\lambda + 8\mu \\ 5 + 4\lambda + 4\mu \end{pmatrix} \qquad \text{or} \qquad \begin{aligned} 6 &\stackrel{1}{=} 3\lambda + 5\mu, \\ 6 &\stackrel{2}{=} 1 + 2\lambda + 8\mu, \\ 1 &\stackrel{3}{=} 5 + 4\lambda + 4\mu. \end{aligned}$$

$2\times \stackrel{2}{=}$ minus $\stackrel{3}{=}$ yields $11 = 12\mu - 3$ or $\mu \stackrel{4}{=} 7/6$. $1.5\times \stackrel{2}{=}$ minus $\stackrel{1}{=}$ yields $3 = 1.5 + 7\mu$ or $\mu \stackrel{5}{=} 3/14$, which contradicts $\stackrel{4}{=}$. So, $D \notin q$ and the four points are not coplanar.

(b) Let q be the plane that contains A, B, and C. The non-parallel vectors $\overrightarrow{AB} = (1,-3,5)$ and $\overrightarrow{BC} = (6,1,-2)$ are on q.

Method 1 (Vector Form). q has normal vector $\overrightarrow{AB} \times \overrightarrow{BC} = (1,-3,5) \times (6,1,-2) = (1,32,19)$.

Since q contains the origin, it may be described by $\mathbf{r} \cdot (1,32,19) = 0$.

Now check if $D \in q$: $\overrightarrow{OD} \cdot (1,32,19) = (4,7,-12) \cdot (1,32,19) = 4 + 224 - 228 = 0$. Yup, it is. So $D \in q$ and the four points are coplanar.

Method 2 (Parametric Form). The plane q may be described in parametric form as:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda + 6\mu \\ -3\lambda + \mu \\ 5\lambda - 2\mu \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{R}).$$

Now check if $D \in q$:

$$\begin{pmatrix} 4 \\ 7 \\ -12 \end{pmatrix} = \begin{pmatrix} \lambda + 6\mu \\ -3\lambda + \mu \\ 5\lambda - 2\mu \end{pmatrix} \qquad \text{or} \qquad \begin{aligned} 4 &\overset{1}{=} \lambda + 6\mu, \\ 7 &\overset{2}{=} -3\lambda + \mu, \\ -12 &\overset{3}{=} 5\lambda - 2\mu. \end{aligned}$$

$2\times \overset{2}{=}$ plus $\overset{3}{=}$ yields $2 = -\lambda$ or $\lambda = -2$ and hence $\mu = 1$. These values of λ and μ also satisfy $\overset{1}{=}$. So, $D \in q$ and the four points are indeed coplanar.

(c) The line AB may be described by $\mathbf{r} = (0,1,2) + \lambda(1,1,1)$ ($\lambda \in \mathbb{R}$). By picking $\lambda = 2$, we observe that AB also contains the point $C = (2,3,4)$. Hence, A, B, and C are collinear. And so by Fact 179, the four points are coplanar.

The plane containing them contains the non-parallel vectors $\overrightarrow{AB} = (1,1,1)$ and $\overrightarrow{AD} = (19,-1,-7)$. Hence, it may be described by

$$\mathbf{r} = (0,1,2) + \lambda(1,1,1) + \mu(19,-1,-7) \qquad (\lambda, \mu \in \mathbb{R}).$$

A297(a) Since $(3,2,1) \nparallel (5,6,7)$, the two lines are **not parallel**. Now write

$$\begin{pmatrix} 8 \\ 1 \\ 5 \end{pmatrix} + \hat{\lambda}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \hat{\mu}\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \qquad \text{or} \qquad \begin{aligned} 8 + 3\hat{\lambda} &\overset{1}{=} 1 + 5\hat{\mu}, \\ 1 + 2\hat{\lambda} &\overset{2}{=} 2 + 6\hat{\mu}, \\ 5 + \hat{\lambda} &\overset{3}{=} 3 + 7\hat{\mu}. \end{aligned}$$

$2\times \overset{2}{=}$ minus $\left(\overset{1}{=} + \overset{3}{=}\right)$ yields $-11 = 0$, a contradiction. So, the two lines **do not intersect**. Thus, the two lines are skew and **not coplanar**.

(b) Since $(3,9,0) \parallel (1,3,0)$, the two lines are **parallel**. They are also distinct because, for example, the point $(1,1,1)$ is on the second line but not on the first.

Thus, they are **coplanar** and **do not intersect**. Compute $(0,0,6) - (1,1,1) = (-1,-1,5)$. So, the (unique) **plane** that contains both lines is $\mathbf{r} = (0,0,6) + \lambda(1,3,0) + \mu(-1,-1,5)$ $(\lambda, \mu \in \mathbb{R})$.

(c) Since $(1,0,1) \nparallel (0,1,1)$, the two lines are **not parallel**. Now write

$$\begin{pmatrix} 6 \\ 5 \\ 5 \end{pmatrix} + \hat{\lambda}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 6 \end{pmatrix} + \hat{\mu}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \text{or} \qquad \begin{aligned} 6 + \hat{\lambda} &\overset{1}{=} 9, \\ 5 &\overset{2}{=} 3 + \hat{\mu}, \\ 5 + \hat{\lambda} &\overset{3}{=} 6 + \hat{\mu}. \end{aligned}$$

From $\overset{1}{=}$, $\hat{\lambda} = 3$. From $\overset{2}{=}$, $\hat{\mu} = 2$. These values of $\hat{\lambda}$ and $\hat{\mu}$ satisfy $\overset{3}{=}$. Plugging these back in, the two lines **intersect** at $(6,5,5) + 3(1,0,1) = (9,3,6) + 2(0,1,1) = (9,5,8)$.

Thus, the two lines are **coplanar** and the (unique) **plane** that contains them is $\mathbf{r} = (6,5,5) + \lambda(1,0,1) + \mu(0,1,1)$ $(\lambda, \mu \in \mathbb{R})$.

(d) Since $(-5,0,1) \parallel (10,0,2)$, the two lines are **parallel**. Indeed, they are **identical** because the point $(9,3,6)$ which is on the second line is also on the first (to see this, plug $\lambda = -2$ into the first line's vector equation).

Since they are identical, they **intersect** at every point along either line.

They are **coplanar** and there are *infinitely many planes that contain both lines.*

152. Part IV Answers (Complex Numbers)

152.1. Ch. 79 Answers (Complex Numbers: Introduction)

A298(a) True because **(i)** every integer and rational non-integer is rational; *and* **(ii)** every rational is either an integers or rational non-integer.

(b) True because nothing is both rational and irrational.

(c) False because $0 \in \mathbb{R}$, so $0 \notin \mathbb{C} \setminus \mathbb{R} = \{x \in \mathbb{C} \text{ AND } x \in \mathbb{R}\}$.

(d) True because nothing is both imaginary and impure.

(e) False because for example 1 is an integer but isn't imaginary.

(f) False because $\mathbb{Z} \cap \{\text{Imaginary numbers}\} = \{0\}$.

(g) False because $(\{\text{Imaginary numbers}\} \cup \{\text{Impure numbers}\}) \cap \mathbb{R} = \{0\}$.

A79.

	-2.5	$1 + \mathrm{i}\pi$	$1 + \pi$	$1 - \sqrt{6}$	$1 - 8\mathrm{i}$	$\sqrt{5} + 0\mathrm{i}$	$-200\mathrm{i}$
Complex	✓	✓	✓	✓	✓	✓	✓
Imaginary							✓
Impure		✓			✓		
Real	✓		✓	✓		✓	
An integer							
Positive			✓			✓	
Negative	✓			✓			
Rational	✓						
Irrational			✓	✓		✓	

A299. Rationalise any denominators with surds and write out the sine or cosine values:

$$a = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}, \qquad b = \frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}, \qquad c = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}, \qquad d = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}.$$

Comparing the real and imaginary parts, we see that only $c = d$.

A300(a) $z = (33, 33e)$. **(b)** $w = \left(237 + \pi, 3 - \sqrt{2}\right)$. **(c)** $\omega = (p, q)$.

152.2. Ch. 80 Answers (Some Arithmetic of Complex Numbers)

A301(a) $z+w=(-5+2\mathrm{i})+(7+3\mathrm{i})=2+5\mathrm{i}$, $z-w=(-5+2\mathrm{i})-(7+3\mathrm{i})=-12-\mathrm{i}$.

(b) $z+w=(3-\mathrm{i})+(11+2\mathrm{i})=14+\mathrm{i}$, $z-w=(3-\mathrm{i})-(11+2\mathrm{i})=-8-3\mathrm{i}$.

(c) $z+w=(1+2\mathrm{i})+\left(3-\sqrt{2}\mathrm{i}\right)=4+\left(2-\sqrt{2}\right)\mathrm{i}$, $z-w=(1+2\mathrm{i})-\left(3-\sqrt{2}\mathrm{i}\right)=-2+\left(2+\sqrt{2}\right)\mathrm{i}$.

A302(a) $zw=(-5+2\mathrm{i})(7+3\mathrm{i})=-35-15\mathrm{i}+14\mathrm{i}+6\mathrm{i}^2=-41-\mathrm{i}$.

$z^2=(-5+2\mathrm{i})^2=25+2(-5)(2\mathrm{i})-4=21-20\mathrm{i}$.

$z^3=(-5+2\mathrm{i})^2(-5+2\mathrm{i})=(21-20\mathrm{i})(-5+2\mathrm{i})=-105+42\mathrm{i}+100\mathrm{i}+40=-65+142\mathrm{i}$.

(b) $zw=(3-\mathrm{i})(11+2\mathrm{i})=33+6\mathrm{i}-11\mathrm{i}-2\mathrm{i}^2=35-5\mathrm{i}$.

$z^2=(3-\mathrm{i})^2=9+2(3)(-\mathrm{i})-1=8-6\mathrm{i}$.

$z^3=(3-\mathrm{i})^2(3-\mathrm{i})=(8-6\mathrm{i})(3-\mathrm{i})=24-8\mathrm{i}-18\mathrm{i}-6=18-26\mathrm{i}$.

(c) $zw=(1+2\mathrm{i})\left(3-\sqrt{2}\mathrm{i}\right)=3-\sqrt{2}\mathrm{i}+6\mathrm{i}-2\sqrt{2}\mathrm{i}^2=3+2\sqrt{2}+\left(6-\sqrt{2}\right)\mathrm{i}$.

$z^2=(1+2\mathrm{i})^2=1+2(1)(2\mathrm{i})-4=-3+4\mathrm{i}$.

$z^3=(1+2\mathrm{i})^2(1+2\mathrm{i})=(-3+4\mathrm{i})(1+2\mathrm{i})=-3-6\mathrm{i}+4\mathrm{i}-8=-11-2\mathrm{i}$.

A303. $zw = (a + \mathrm{i}b)(c + \mathrm{i}d) = ac + \mathrm{i}ad + \mathrm{i}bc + \mathrm{i}^2 bd = (ac - bd) + \mathrm{i}(ad + bc)$.

A304. $(2 + \mathrm{i})^2 = 4 + 2(2)(\mathrm{i}) - 1 = 3 + 4\mathrm{i}$.

$(2 + \mathrm{i})^3 = (2 + \mathrm{i})(2 + \mathrm{i})^2 = (2 + \mathrm{i})(3 + 4\mathrm{i}) = 6 + 8\mathrm{i} + 3\mathrm{i} - 4 = 2 + 11\mathrm{i}$.

Hence,
$$\begin{aligned} az^3 + bz^2 + 3z - 1 &= (2 + \mathrm{i})^3 a + (2 + \mathrm{i})^2 b + 3(2 + \mathrm{i}) - 1 \\ &= (2 + 11\mathrm{i})a + (3 + 4\mathrm{i})b + 3(2 + \mathrm{i}) - 1 \\ &= 2a + 3b + 5 + \mathrm{i}(11a + 4b + 3). \end{aligned}$$

Two complex numbers are equal if and only if their real and imaginary parts are equal. So,

$$2a + 3b + 5 \stackrel{1}{=} 0 \qquad \text{and} \qquad 11a + 4b + 3 \stackrel{2}{=} 0.$$

Take $3\times \stackrel{2}{=}$ minus $4\times \stackrel{1}{=}$: $3(11a + 4b + 3) - 4(2a + 3b + 5) = 25a - 11 = 0$.

So,
$$a = \frac{11}{25} = 0.44 \qquad \text{and} \qquad b = -\frac{5.88}{3} = -1.96.$$

A305(a) $z^* = -5 - 2\mathrm{i}$. So $\dfrac{1}{z} = \dfrac{1}{5^2 + 2^2} z^* = \dfrac{1}{29}(-5 - 2\mathrm{i}) = \dfrac{1}{29}(-5, -2) = -\dfrac{5}{29} - \dfrac{2}{29}\mathrm{i}$.

(b) $w^* = 3 + \mathrm{i}$. So $\dfrac{1}{w} = \dfrac{1}{3^2 + 1^2} w^* = \dfrac{1}{10}(3 + \mathrm{i}) = \dfrac{1}{10}(3, 1) = 0.3 + 0.1\mathrm{i}$.

(c) $\omega^* = 1 - 2\mathrm{i}$. So $\dfrac{1}{\omega} = \dfrac{1}{1^2 + 2^2} \omega^* = \dfrac{1}{5}(1 - 2\mathrm{i}) = \dfrac{1}{5}(1, -2) = 0.2 - 0.4\mathrm{i}$.

A306(a) $zz^* = (a + \mathrm{i}b)(a - \mathrm{i}b) = a^2 - (\mathrm{i}b)^2 = a^2 - \mathrm{i}^2 b^2 = a^2 + b^2$.

(b) $\dfrac{1}{z} = \dfrac{1}{z}\dfrac{z^*}{z^*} = \dfrac{z^*}{zz^*} \stackrel{(a)}{=} \dfrac{z^*}{a^2 + b^2}$.

A307(a) $\dfrac{z}{w} = \dfrac{zw^*}{0^2 + 1^2} = \dfrac{(1 + 3\mathrm{i})\,\mathrm{i}}{1} = -3 + \mathrm{i}$.

(b) $\dfrac{z}{w} = \dfrac{zw^*}{1^2 + 1^2} = \dfrac{(2 - 3\mathrm{i})(1 - \mathrm{i})}{2} = \dfrac{2 - 2\mathrm{i} - 3\mathrm{i} - 3}{2} = \dfrac{-1 - 5\mathrm{i}}{2} = -0.5 - 2.5\mathrm{i}$.

(c) $\dfrac{z}{w} = \dfrac{zw^*}{3^2 + \left(-\sqrt{2}\right)^2} = \dfrac{\left(\sqrt{2} - \pi i\right)\left(3 + \sqrt{2}\mathrm{i}\right)}{11} = \dfrac{3\sqrt{2} + 2\mathrm{i} - 3\mathrm{i}\pi + \sqrt{2}\pi}{11} = \dfrac{3 + \pi}{11}\sqrt{2} + \dfrac{2 - 3\pi}{11}\mathrm{i}$.

(d) $\dfrac{z}{w} = \dfrac{zw^*}{0^2 + 1^2} = \dfrac{(11 + 2\mathrm{i})(-\mathrm{i})}{1} = 2 - 11\mathrm{i}$.

(e) $\dfrac{z}{w} = \dfrac{zw^*}{2^2 + 1^2} = \dfrac{(-3)(2 - \mathrm{i})}{5} = \dfrac{-6 + 3\mathrm{i}}{5} = -1.2 + 0.6\mathrm{i}$.

(f) $\dfrac{z}{w} = \dfrac{zw^*}{5^2 + 1^2} = \dfrac{(7 - 2i)(5 - \mathrm{i})}{26} = \dfrac{35 - 7\mathrm{i} - 10\mathrm{i} - 2}{26} = \dfrac{33}{26} - \dfrac{17}{26}\mathrm{i}$.

152.3. Ch. 81 Answers (Solving Polynomial Equations)

A308(a) $x = \left(-1 \pm \sqrt{-3}\right)/2 = -1/2 \pm \sqrt{3}\mathrm{i}/2.$

(b) $x = \left(-2 \pm \sqrt{-4}\right)/2 = -1 \pm \mathrm{i}.$

(c) $x = \left(-3 \pm \sqrt{-3}\right)/6 = -1/2 \pm \sqrt{3}\mathrm{i}/6.$

A309. $\left(-1 \pm \sqrt{3}\mathrm{i}\right)^3 = -1 \pm 3(-1)^2\sqrt{3}\mathrm{i} + 3(-1)\left(\sqrt{3}\mathrm{i}\right)^2 \mp 3\sqrt{3}\mathrm{i} = -1 \pm 3\sqrt{3}\mathrm{i} + 9 \mp 3\sqrt{3}\mathrm{i} = 8.$ ✓

A310. Since $(-4)^3 + 64 = 0$, one root is -4 and $x+4$ is a factor for x^3+64. To find the other two roots, write

$$x^3 + 64 = (x+4)\left(ax^2 + bx + c\right) = ax^3 + (b+4a)x^2 + ?x + 4c.$$

Comparing coefficients, $\quad x^3 + 64 = (x+4)\left(x^2 - 4x + 16\right).$

We can now find the other two roots using the usual quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{4^2 - 4(1)(16)}}{2 \cdot 1} = 2 \pm \sqrt{4 - 16} = 2 \pm \sqrt{12}\mathrm{i} = 2 \pm 2\sqrt{3}\mathrm{i}.$$

Thus, the three roots of $x^3 + 64 = 0$ are -4, $2 + 2\sqrt{3}\mathrm{i}$, and $2 - 2\sqrt{3}\mathrm{i}$.

A311(a) Since 1 is a root, write

$$x^3 + x^2 - 2 = (x-1)\left(ax^2 + bx + c\right) = ax^3 + (b-a)x^2 + ?x - c.$$

Comparing coefficients, $\quad x^3 + x^2 - 2 = (x-1)\left(x^2 + 2x + 2\right).$

We can now find the other two roots using the usual quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2 \cdot 1} = -1 \pm \sqrt{1-2} = -1 \pm \mathrm{i}.$$

Thus, the three roots of $x^3 + x^2 - 2 = 0$ are 1 and $-1 \pm \mathrm{i}$.

(b) Since 1 is a root, write

$$x^4 - x^2 - 2x + 2 = (x-1)\left(ax^3 + bx^2 + cx + d\right) = ax^4 + (b-a)x^3 + (c-b)x^2 + ?x - d.$$

Comparing coefficients, $\quad x^4 - x^2 - 2x + 2 = (x-1)\left(x^3 + x^2 - 2\right).$

But in **(a)**, we already worked out the three roots of $x^3 + x^2 - 2 = 0$. Thus, the four roots of $x^4 - x^2 - 2x + 2 = 0$ are 1, 1 (repeated), and $-1 \pm \mathrm{i}$.

A312. Both equations have real coefficients and so the Complex Conjugate Root Theorem applies. That is, since $2-3\text{i}$ solves both equations, so too does $2+3\text{i}$.

Compute $\quad [x-(2-3\text{i})][x-(2+3\text{i})] = (x-2)^2-(3\text{i})^2 = x^2-4x+13.$

(a) $x^4-6x^3+18x^2-14x-39 = \left(x^2-4x+13\right)\left(ax^2+bx+c\right) = ax^4+(b-4a)x^3+?x^2+?x+13c.$

Comparing coefficients, $\quad ax^2+bx+c = x^2-2x-3.$

By the quadratic formula or otherwise, we have $x^2-2x-3=(x+1)(x-3)$.

Thus, the four roots are $2-3\text{i}$, $2+3\text{i}$, -1, and 3.

(b) $-2x^4+21x^3-93x^2+229x-195 = ax^4+(b-4a)x^3+?x^2+?x+13c.$

Comparing coefficients, $\quad ax^2+bx+c = -2x^2+13x-15.$

We can now find the other two roots using the usual quadratic formula:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-13\pm\sqrt{13^2-4(-2)(-15)}}{2(-2)} = \frac{13\mp\sqrt{49}}{4} = \frac{13\mp 7}{4} = 1.5, 5.$$

Thus, the four roots are $2-3\text{i}$, $2+3\text{i}$, 1.5, and 5.

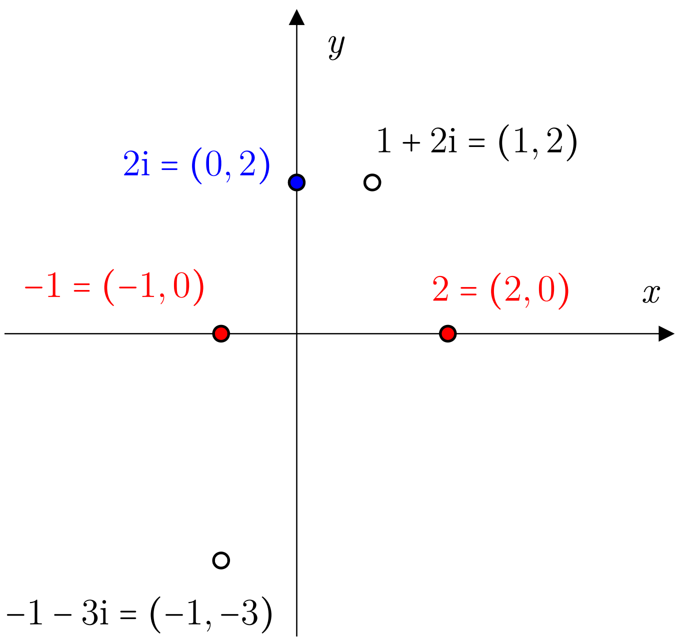
A313. If $1-\text{i}$ solves $x^2+px+q=0$, then by Theorem 21, so too does $1+\text{i}$. And so,

$$x^2+px+q = [x-(1-\text{i})][x-(1+\text{i})] = (x-1)^2-\text{i}^2 = x^2-2x+2.$$

Hence, $p=-2$ and $q=2$.

152.4. Ch. 82 Answers (The Argand Diagram)

A314.

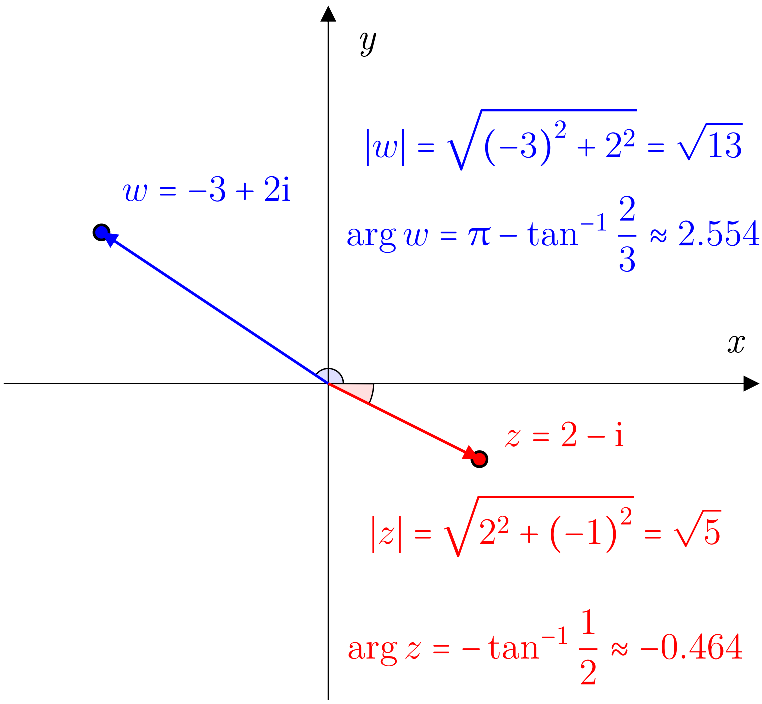


152.5. Ch. 83 Answers (Complex Numbers in Polar Form)

A315. $|2| = 2$, $|-1| = 1$, $|2\mathrm{i}| = 2$, $|1+2\mathrm{i}| = \sqrt{1^2+2^2} = \sqrt{5}$, $|-1-3\mathrm{i}| = \sqrt{(-1)^2+(-3)^2} = \sqrt{10}$.

A316. Refer to figure on the previous page. We have $\arg 2 = 0$, $\arg(-1) = \pi$, $\arg(2\mathrm{i}) = \pi/2$ and $\arg(1+2\mathrm{i}) = \tan^{-1}(2/1) \approx 1.107$. For $-1-3\mathrm{i}$, observe that $\theta = \tan^{-1}(3/1)$. Thus, $\arg(-1-3\mathrm{i}) = \theta - \pi = \tan^{-1}(3/1) - \pi \approx -1.893$.

A317.



A318. $\arg 2 = \cos^{-1}\frac{2}{2} = 0$, $\quad \arg(-1) = \cos^{-1}\frac{-1}{1} = \pi$, $\quad \arg(2\mathrm{i}) = \cos^{-1}\frac{0}{2} = \pi/2$.

$\arg(1+2\mathrm{i}) = \cos^{-1}\frac{1}{\sqrt{5}} \approx 1.107$, $\quad \arg(-1-3\mathrm{i}) = -\cos^{-1}\frac{-1}{\sqrt{10}} \approx -1.893$.

$\arg z = \arg(2-\mathrm{i}) = -\cos^{-1}\frac{2}{\sqrt{5}} \approx -0.464$, $\quad \arg w = \arg(-3+2\mathrm{i}) = \cos^{-1}\frac{-3}{\sqrt{13}} \approx 2.554$.

A319. Using the moduli and arguments found in the above answers, we have

$$2 = 2(\cos 0 + \mathrm{i}\sin 0), \qquad -1 = 1(\cos\pi + \mathrm{i}\sin\pi), \qquad 2\mathrm{i} = 2\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right),$$

$$1+2\mathrm{i} \approx \sqrt{5}(\cos 1.107 + \mathrm{i}\sin 1.107), \qquad -1-3\mathrm{i} \approx \sqrt{10}(\cos -1.893 + \mathrm{i}\sin -1.893),$$

$$2-\mathrm{i} \approx \sqrt{5}(\cos -0.464 + \mathrm{i}\sin -0.464), \qquad -3+2\mathrm{i} \approx \sqrt{13}(\cos 2.554 + \mathrm{i}\sin 2.554).$$

152.6. Ch. 84 Answers (Complex Numbers in Exponential Form)

A320. Using the moduli and arguments found in the above answers, we have

$$2 = 2e^{0\mathrm{i}} = 2e^{0\mathrm{i}} = 2, \qquad -1 = e^{\mathrm{i}\pi},\ 2\mathrm{i} = 2e^{\mathrm{i}\pi/2}, \qquad 1+2\mathrm{i} \approx \sqrt{5}e^{1.107\mathrm{i}}, \qquad -1-3\mathrm{i} \approx \sqrt{10}e^{-1.893\mathrm{i}},$$

$$w = 2-\mathrm{i} \approx \sqrt{5}e^{-0.464\mathrm{i}}, \qquad z = -3+2\mathrm{i} \approx \sqrt{13}e^{2.554\mathrm{i}}.$$

152.7. Ch. 85 Answers (More Arithmetic of Complex Numbers)

A321(a) $|z| = |1| = 1$, $\arg z = 0$, $|w| = |-3| = 3$, and $\arg w = \pi$.

Hence, $|zw| = |z||w| = 3$ and $\arg(zw) = \arg z + \arg w + 2k\pi = 0 + \pi + 0 = \pi$.

Thus, $zw = 3(\cos\pi + \mathrm{i}\sin\pi) = 3\mathrm{e}^{\mathrm{i}\pi} = -3$.

Next, $|-2zw| = 2|zw| = 2\cdot 3 = 6$ and $\arg(-2zw) = \arg(zw) - \pi = 0$.

Thus, $-2zw = 6(\cos 0 + \mathrm{i}\sin 0) = 6\mathrm{e}^{\mathrm{i}\pi} = 6$.

(b) $|z| = |2\mathrm{i}| = 2$, $\arg z = \pi/2$, $|w| = |1+2\mathrm{i}| = \sqrt{5}$, and $\arg w = \cos^{-1}\left(1/\sqrt{5}\right) \approx 1.107$.

Hence, $|zw| = |z||w| = 2\sqrt{5}$ and $\arg(zw) = \arg z + \arg w + 2k\pi = \pi/2 + 1.107 + 0 \approx 2.678$.

Thus, $zw = 2\sqrt{5}(\cos 2.678 + \mathrm{i}\sin 2.678) = 2\sqrt{5}\mathrm{e}^{2.678\mathrm{i}} \approx -4 + 2\mathrm{i}$.

Next, $|-2zw| = 2|zw| = 2\cdot 2\sqrt{5} = 4\sqrt{5}$ and $\arg(-2zw) = \arg(zw) - \pi \approx 2.678 - \pi \approx -0.464$.

Thus, $-2zw = 4\sqrt{5}(\cos -0.464 + \mathrm{i}\sin -0.464) = 4\sqrt{5}\mathrm{e}^{-0.464\mathrm{i}} \approx 8 - 4\mathrm{i}$.

(c) $|z| = \sqrt{10}$, $\arg z = -\cos^{-1}\left(-1/\sqrt{10}\right) \approx -1.893$, $|w| = 5$, and $\arg w = \cos^{-1}(3/5) \approx 0.927$.

Hence, $|zw| = |z||w| = 5\sqrt{10}$ and $\arg(zw) = \arg z + \arg w + 2k\pi = -1.893 + 0.927 + 0 \approx -0.965$.

Thus, $zw = 5\sqrt{10}(\cos -0.965 + \mathrm{i}\sin -0.965) = 5\sqrt{10}\mathrm{e}^{-0.965\mathrm{i}} \approx 9 - 13\mathrm{i}$.

Next, $|-2zw| = 2|zw| = 2\cdot 5\sqrt{10} = 10\sqrt{10}$ and $\arg(-2zw) = \arg(zw) + \pi \approx -0.965 + \pi \approx 2.177$.

Thus, $-2zw = 10\sqrt{10}(\cos 2.177 + \mathrm{i}\sin 2.177) = 10\sqrt{10}\mathrm{e}^{2.177\mathrm{i}} \approx -18 + 26\mathrm{i}$.

(d) $|z| = \sqrt{29}$, $\arg z = \cos^{-1}\left(-2/\sqrt{29}\right) \approx 1.951$, $|w| = 1$, $\arg w = \pi/2$.

Hence, $|zw| = |z||w| = \sqrt{29}$ and $\arg(zw) = \arg z + \arg w + 2k\pi = 1.951 + \pi/2 - 2\pi \approx -2.761$.

Thus, $zw = \sqrt{29}(\cos -2.761 + \mathrm{i}\sin -2.761) = \sqrt{29}\mathrm{e}^{-2.761\mathrm{i}} \approx -5 - 2\mathrm{i}$.

Next, $|-2zw| = 2|zw| = 2\sqrt{29}$ and $\arg(-2zw) = \arg(zw) + \pi \approx -2.761 + \pi \approx 0.381$.

Thus, $-2zw = 2\sqrt{29}(\cos 0.381 + \mathrm{i}\sin 0.381) = 2\sqrt{29}\mathrm{e}^{0.381\mathrm{i}} \approx 10 + 4\mathrm{i}$.

(e) $|z| = \sqrt{2}$, $\arg z = -\cos^{-1}\left(-1/\sqrt{2}\right) \approx -2.356$, $|w| = \sqrt{5}$, and $\arg w = -\cos^{-1}\left(-1/\sqrt{5}\right) \approx -2.034$.

Hence, $|zw| = |z||w| = \sqrt{10}$ and $\arg(zw) = \arg z + \arg w + 2k\pi = -2.356 - 2.034 + 2\pi \approx 1.893$.

Thus, $zw = \sqrt{10}(\cos 1.893 + \mathrm{i}\sin 1.893) = \sqrt{10}\mathrm{e}^{1.893\mathrm{i}} \approx -1 + 3\mathrm{i}$.

Next, $|-2zw| = 2|zw| = 2\sqrt{10}$ and $\arg(-2zw) = \arg(zw) - \pi \approx 1.893 - \pi \approx -1.249$.

Thus, $-2zw = 2\sqrt{10}(\cos -1.249 + \mathrm{i}\sin -1.249) = 2\sqrt{10}\mathrm{e}^{-1.249\mathrm{i}} \approx 2 - 6\mathrm{i}$.

(f) $|z| = \sqrt{34}$, $\arg z = -\cos^{-1}\left(-5/\sqrt{34}\right) \approx -2.601$, $|w| = \sqrt{26}$, and $\arg w = -\cos^{-1}\left(5/\sqrt{26}\right) \approx -0.197$.

Hence, $|zw| = |z||w| = \sqrt{34}\sqrt{26}$ and $\arg(zw) = \arg z + \arg w + 2k\pi \approx -2.601 - 0.197 + 0 \approx -2.799$.

Thus, $zw = \sqrt{34}\sqrt{26}(\cos -2.799 + \mathrm{i}\sin -2.799) = \sqrt{34}\sqrt{26}\mathrm{e}^{-2.799\mathrm{i}} \approx -28 - 10\mathrm{i}$.

Next, $|-2zw| = 2|zw| = 2\sqrt{34}\sqrt{26}$ and $\arg(-2zw) = \arg(zw) + \pi \approx -2.799 + \pi \approx 0.343$.

Thus, $-2zw = 2\sqrt{34}\sqrt{26}(\cos 0.343 + \mathrm{i}\sin 0.343) = 2\sqrt{34}\sqrt{26}\mathrm{e}^{0.343\mathrm{i}} \approx 56 + 20\mathrm{i}$.

A322(a) $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$.

(b)
$$\begin{aligned} zw &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs(\cos\theta\cos\phi + i\sin\theta\cos\phi + i\cos\theta\sin\phi - \sin\theta\sin\phi) \\ &= rs[\cos(\theta+\phi) + i\sin(\theta+\phi)]. \end{aligned}$$

(c)
$$\begin{aligned} |zw| &= \sqrt{[rs\cos(\theta+\phi)]^2 + [rs\sin(\theta+\phi)]^2} \\ &= rs\sqrt{\cos^2(\theta+\phi) + \sin^2(\theta+\phi)} = rs\sqrt{1} = rs = |z||w|. \qquad \checkmark \end{aligned}$$

A323(a) $|z| = 1$, $\arg z = 0$. So, $|1/z| = 1/|z| = 1$, $\arg(1/z) = -\arg z = 0$.

Thus, $$\frac{1}{z} = 1e^{0i} = \cos 0 + i\sin 0 = 1.$$

(b) $|w| = 2$, $\arg w = \pi/2$. So, $|1/w| = 1/|w| = 1/2$, $\arg(1/w) = -\arg w = -\pi/2$.

Thus, $$\frac{1}{w} = \frac{1}{2}e^{-i\pi/2} = \frac{1}{2}\left(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}\right) = -\frac{1}{2}i.$$

(c) $|z| = 17$, $\arg z = \pi$. So, $|1/z| = 1/|z| = 1/17$. Note importantly that $z < 0$, so that Fact 194(b) does not apply here. We have, simply, $\arg\frac{1}{z} = \arg z = \pi$. And,

$$\frac{1}{z} = \frac{1}{17}e^{\pi} = \frac{1}{17}(\cos\pi + i\sin\pi) = -\frac{1}{17}.$$

(d) $|w| = 8$, $\arg w = -\pi/2$. So, $|1/w| = 1/|w| = 1/8$, $\arg(1/w) = -\arg w = \pi/2$.

Thus, $$\frac{1}{w} = \frac{1}{8}e^{i\pi/2} = \frac{1}{8}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{8}i.$$

(e) $|z| = \sqrt{29}$, $\arg z = \cos^{-1}\left(-2/\sqrt{29}\right) \approx 1.951$. So, $|1/z| = 1/|z| = 1/\sqrt{29}$, $\arg(1/z) \approx -1.951$.

Thus, $$\frac{1}{z} \approx \frac{1}{\sqrt{29}}e^{-1.951i} \approx \frac{1}{\sqrt{29}}(\cos -1.951 + i\sin -1.951) \approx -0.069 + 0.172i.$$

(f) $|w| = \sqrt{2}$, $\arg w = -\cos^{-1}\left(-1/\sqrt{2}\right) = -3\pi/4$. So, $|1/w| = 1/|w| = 1/\sqrt{2}$, $\arg(1/w) = 3\pi/4$.

Thus, $$\frac{1}{w} = \frac{1}{\sqrt{2}}e^{3i\pi/4} = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\frac{1}{2} + \frac{1}{2}i.$$

(g) $|z| = \sqrt{10}$, $\arg z = -\cos^{-1}\left(1/\sqrt{10}\right) \approx -1.249$. So, $|1/z| = 1/|z| = 1/\sqrt{10}$, $\arg(1/z) \approx 1.249$.

Thus, $$\frac{1}{z} \approx \frac{1}{\sqrt{10}}e^{1.249i} \approx \frac{1}{\sqrt{10}}(\cos 1.249 + i\sin 1.249) \approx 0.1 + 0.3i.$$

(h) $|w| = 5$, $\arg w = \cos^{-1}(3/5) \approx 0.927$. So, $|1/w| = 1/|w| = 1/5$, $\arg(1/w) = -\arg w = -0.927$.

Thus, $$\frac{1}{z} \approx \frac{1}{5}e^{-0.927} \approx \frac{1}{5}(\cos -0.927 + i\sin -0.927) \approx 0.12 - 0.16i.$$

A324(a) $|z| = |1| = 1$, $\arg z = 0$, $|w| = |3| = 3$, and $\arg w = \pi$. Hence, $|z/w| = |z|/|w| = 1/3$ and $\arg(z/w) = \arg z - \arg w + 2k\pi = 0 - \pi + 2\pi = \pi$. Thus, $z/w = {}^1\!/_3 (\cos\pi + \mathrm{i}\sin\pi) = {}^1\!/_3 \mathrm{e}^{\mathrm{i}\pi} = -{}^1\!/_3$.

(b) $|z| = |2\mathrm{i}| = 2$, $\arg z = \pi/2$, $|w| = |1+2\mathrm{i}| = \sqrt{5}$, and $\arg w = \cos^{-1}\left(1/\sqrt{5}\right) \approx 1.107$. Hence, $|z/w| = |z|/|w| = 2/\sqrt{5}$ and $\arg(z/w) = \arg z - \arg w + 2k\pi = \pi/2 - 1.107 + 0 \approx 0.464$. Thus, $z/w \approx \left(2/\sqrt{5}\right)(\cos 0.464 + \mathrm{i}\sin 0.464) \approx \left(2/\sqrt{5}\right)\mathrm{e}^{0.464\mathrm{i}} \approx 0.8 + 0.4\mathrm{i}$.

$$\frac{2\mathrm{i}}{1+2\mathrm{i}} = \frac{2\mathrm{i}}{1+2\mathrm{i}}\frac{1-2\mathrm{i}}{1-2\mathrm{i}} = \frac{2\mathrm{i}+4}{1^2+2^2} = 0.8 + 0.4\mathrm{i}.$$

(c) $|z| = |-1-3\mathrm{i}| = \sqrt{10}$, $\arg z = -\cos^{-1}\left(-1/\sqrt{10}\right) \approx -1.893$, $|w| = |3+4\mathrm{i}| = 5$, and $\arg w = \cos^{-1}(3/5) \approx 0.927$. Hence, $|z/w| = |z|/|w| = \sqrt{10}/5 = \sqrt{2/5} = \sqrt{0.4}$ and $\arg(z/w) = \arg z - \arg w + 2k\pi = -1.893 - 0.927 + 0 \approx -2.820$.

Thus, $z/w \approx \sqrt{0.4}(\cos -2.820 + \mathrm{i}\sin -2.820) \approx \sqrt{0.4}\mathrm{e}^{-2.820\mathrm{i}} \approx -0.6 - 0.2\mathrm{i}$.

$$\frac{-1-3\mathrm{i}}{3+4\mathrm{i}} = \frac{-1-3\mathrm{i}}{3+4\mathrm{i}}\frac{3-4\mathrm{i}}{3-4\mathrm{i}} = \frac{-3+4\mathrm{i}-9\mathrm{i}-12}{3^2+4^2} = \frac{-15-5\mathrm{i}}{25} = -0.6 - 0.2\mathrm{i}.$$

(d) $|z| = |-2+5\mathrm{i}| = \sqrt{29}$, $\arg z = \cos^{-1}\left(-2/\sqrt{29}\right) \approx 1.951$, $|w| = |\mathrm{i}| = 1$, and $\arg w = \pi/2$. Hence, $|z/w| = |z|/|w| = \sqrt{29}$ and $\arg(z/w) = \arg z - \arg w + 2k\pi = 1.951 - \pi/2 + 0 \approx 0.381$.

Thus, $z/w \approx \sqrt{29}(\cos 0.381 + \mathrm{i}\sin 0.381) \approx \sqrt{29}\mathrm{e}^{0.381\mathrm{i}} \approx 5 + 2\mathrm{i}$.

$$\frac{-2+5\mathrm{i}}{\mathrm{i}} = \frac{-2+5\mathrm{i}}{\mathrm{i}}\frac{-\mathrm{i}}{-\mathrm{i}} = \frac{2\mathrm{i}+5}{1^2} = 5 + 2\mathrm{i}.$$

(e) $|z| = |-1-\mathrm{i}| = \sqrt{2}$, $\arg z = -\cos^{-1}\left(-1/\sqrt{2}\right) \approx -2.356$, $|w| = |-1-2\mathrm{i}| = \sqrt{5}$, and $\arg w = -\cos^{-1}\left(-1/\sqrt{5}\right) \approx -2.034$. Hence, $|z/w| = |z|/|w| = \sqrt{2/5} = \sqrt{0.4}$ and $\arg(z/w) = \arg z - \arg w + 2k\pi = -2.356 + 2.034 + 0 \approx -0.322$.

Thus, $z/w \approx \sqrt{0.4}(\cos -0.322 + \mathrm{i}\sin -0.322) \approx \sqrt{0.4}\mathrm{e}^{-0.322\mathrm{i}} \approx 0.6 - 0.2\mathrm{i}$.

$$\frac{-1-\mathrm{i}}{-1-2\mathrm{i}} = \frac{-1-\mathrm{i}}{-1-2\mathrm{i}}\frac{-1+2\mathrm{i}}{-1+2\mathrm{i}} = \frac{1-2\mathrm{i}+\mathrm{i}+2}{1^2+2^2} = \frac{3-\mathrm{i}}{5} = 0.6 - 0.2\mathrm{i}.$$

(f) $|z| = |-5-3\mathrm{i}| = \sqrt{34}$, $\arg z = -\cos^{-1}\left(-5/\sqrt{34}\right) \approx -2.601$, $|w| = |5-\mathrm{i}| = \sqrt{26}$, and $\arg w = -\cos^{-1}\left(5/\sqrt{26}\right) \approx -0.197$. Hence, $|z/w| = |z|/|w| = \sqrt{34/26} = \sqrt{17/13}$ and $\arg(z/w) = \arg z - \arg w + 2k\pi = -2.601 + 0.197 + 0 \approx -2.404$.

Thus, $z/w = \sqrt{17/13}(\cos -2.404 + \mathrm{i}\sin -2.404) = \sqrt{17/13}\mathrm{e}^{-2.404\mathrm{i}}$.

$$\frac{-5-3\mathrm{i}}{5-\mathrm{i}} = \frac{-5-3\mathrm{i}}{5-\mathrm{i}}\frac{5+\mathrm{i}}{5+\mathrm{i}} = \frac{-25-5\mathrm{i}-15\mathrm{i}+3}{5^2+1^2} = \frac{-22-20\mathrm{i}}{26} = -\frac{11}{13} - \frac{10}{13}\mathrm{i}.$$

A325. $\left|\frac{z}{w}\right| = \left|z\frac{1}{w}\right| \overset{1}{=} |z|\left|\frac{1}{w}\right| \overset{2}{=} |z|\frac{1}{|w|} = \frac{|z|}{|w|}$, where $\overset{1}{=}$ and $\overset{2}{=}$ use Facts 193 and 194.

153. Part V Answers (Calculus)

153.1. Ch. 86 Answers (Limits)

A326. $\lim_{x\to 0} i(x) = 0$, $\lim_{x\to 1} i(x) = 1$, $\lim_{x\to 2} i(x) = 2$, $\lim_{x\to 3} i(x) = 3$.

A327. $\lim_{x\to -5} f(x) = 1$, $\lim_{x\to 0} f(x)$ does not exist, and $\lim_{x\to 5} f(x) = 2$.

Figure to be
inserted here.

A328. From the graph below, $\lim_{x\to 0} h(x) = 0$ and $\lim_{x\to 0} i(x) = 2$.

Figure to be
inserted here.

And so, by Theorem 23, $\lim_{x\to 0} [kh(x)] = 7\times 0 = 0$, $\lim_{x\to 0} [h(x) + i(x)] = 0+2 = 2$, $\lim_{x\to 0} [h(x) - i(x)] = 0 - 2 = -2$, $\lim_{x\to 0} [h(x)\, i(x)] = 0 \times 2 = 0$, $\lim_{x\to 0} \dfrac{1}{i(x)} = \dfrac{1}{2}$, $\lim_{x\to 0} \dfrac{h(x)}{i(x)} = \dfrac{0}{2} - 0$, $\lim_{x\to 0} k = 7$, and $\lim_{x\to 0} x^k = 0^7 = 0$.

153.2. Ch. 87 Answers (Continuity, Revisited)

A332. By Definition **??**,

$$\tan = \frac{\sin}{\cos}, \qquad \operatorname{cosec} = \frac{1}{\sin}, \qquad \sec = \frac{1}{\cos}, \qquad \cot = \frac{\cos}{\sin}.$$

Each of tan, cosec, sec, and cot is defined as the quotient of two continuous functions. And so, by Theorem 26, each of these functions is continuous.

By Definition 99, $\tan^{-1}$ is defined as the inverse of the continuous function tan restricted to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. And so by Theorem 27, $\tan^{-1}$ is also continuous.

A331(a) $\lim_{x\to 0}(x+1) \overset{\pm}{=} \lim_{x\to 0} x + \lim_{x\to 0} 1 \overset{\text{P, C}}{=} 0 + 1 = 1.$

The cubing function is continuous at 1. And so by Fact **??**, we can "move" the limit in:

$$\lim_{x\to 0}(x+1)^3 = \left[\lim_{x\to 0}(x+1)\right]^3 = 1^3 = 1.$$

(b) $\lim_{x\to 0} x^2 \overset{\text{P}}{=} 0^2 = 0.$

The cosine function is continuous at 0. And so by Fact **??**, we can "move" the limit in:

$$\lim_{x\to 0}\left(\cos x^2\right) = \cos\left(\lim_{x\to 0} x^2\right) = \cos 0 = 1.$$

The sine function is continuous at 1. And so again by Fact **??**, we can "move" the limit in:

$$\lim_{x\to 0}\sin\left(\cos x^2\right) = \sin\left[\lim_{x\to 0}\left(\cos x^2\right)\right] = \sin 1.$$

A333(a) f is differentiable at -3, with $f'(-3) = -1$, as we now show:

$$\begin{aligned} f'(-3) = \lim_{x \to -3} \frac{f(x) - f(3)}{x - (-3)} &= \lim_{x \to -3} \frac{|x| - |-3|}{x + 3} \\ &= \lim_{x \to -3} \frac{-x - 3}{x + 3} \qquad \text{(For all } x \text{ "near" } 3,\ x < 0 \text{ and hence } |x| = -x\text{)} \\ &= \lim_{x \to -3} -1 \overset{\text{C}}{=} -1. \end{aligned}$$

(b) f is differentiable at any $a < 0$, with $f'(a) = -1$, as we now show:

$$\begin{aligned} f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{|x| - |a|}{x - a} \\ &= \lim_{x \to a} \frac{-x + a}{x - a} \qquad \text{(For all } x \text{ "near" } a < 0,\ x < 0 \text{ and hence } |x| = -x\text{)} \\ &= \lim_{x \to a} -1 \overset{\text{C}}{=} -1. \end{aligned}$$

A334(a) First simplify the difference quotient:

$$\frac{g(x) - g(-3)}{x - (-3)} = \frac{x^2 - (-3)^2}{x - (-3)} = \frac{(x-3)[x - (-3)]}{x - (-3)} = x - 3.$$

Now,

$$g'(-3) = \lim_{x \to -3} \frac{g(x) - g(-3)}{x - (-3)} = \lim_{x \to -3} (x - 3) \overset{\pm}{=} \underbrace{\lim_{x \to -3} x}_{-3} + \underbrace{\lim_{x \to -3} -3}_{-3} = -6.$$

The derivative of g at -3 exists and equals -6.

(b) First simplify the difference quotient:

$$\frac{g(x) - g(0)}{x - 0} = \frac{x^2 - 0^2}{x - 0} = \frac{(x-0)(x+0)}{x - 0} = x.$$

Now,

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} x \overset{\text{P}}{=} 0.$$

The derivative of g at 0 exists and equals 0.

(c) First simplify the difference quotient:

$$\frac{g(x) - g(a)}{x - a} = \frac{x^2 - a^2}{x - a} = \frac{(x-a)(x+a)}{x - a} = x + a.$$

Now,

$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a} = \lim_{x\to a}(x+a) \overset{\pm}{=} \underbrace{\lim_{x\to a} a}_{a} + \underbrace{\lim_{x\to a} a}_{a} = 2a.$$

The derivative of g at any $a \in \mathbb{R}$ exists and equals $2a$.

A335. Let $a \in \mathbb{R}$.

(a) Simplify the difference quotient: $\dfrac{f(x) - f(a)}{x - a} = \dfrac{7-7}{x-a} = 0$.

So,
$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} 0 \overset{C}{=} 0.$$

We've just shown that for any $a \in \mathbb{R}$, the derivative of f at a exists and equals 0.

Hence, the derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = 0$.

The derivative of f at 2 is $f'(2) = 0$.

(b) Simplify the difference quotient: $\dfrac{g(x) - g(a)}{x - a} = \dfrac{(5x+7) - (5a+7)}{x-a} = \dfrac{5x-5a}{x-a} = 5$.

So,
$$\lim_{x\to a} \frac{g(x) - g(a)}{x - a} = \lim_{x\to a} 5 \overset{C}{=} 5.$$

We've just shown that for any $a \in \mathbb{R}$, the derivative of g at a exists and equals 5.

Hence, the derivative of g is the function $g' : \mathbb{R} \to \mathbb{R}$ defined by $g'(x) = 5$.

The derivative of g at 2 is $g'(2) = 5$.

(c) Simplify the difference quotient:

$$\frac{h(x) - h(a)}{x - a} = \frac{(2x^2+5x+7) - (2a^2+5a+7)}{x-a} = \frac{2x^2 - 2a^2 + 5x - 5a}{x-a}$$

$$= 2\frac{x^2 - a^2}{x-a} + 5\frac{x-a}{x-a} = 2\frac{(x+a)(x-a)}{x-a} + 5 = 2(x+a) + 5.$$

So,[652]

$$\lim_{x\to a} \frac{h(x) - h(a)}{x - a} = \lim_{x\to a}[2(x+a)+5] \overset{\pm}{=} \overbrace{\lim_{x\to a} 2x}^{2a} + \overbrace{\lim_{x\to a} 2a}^{2a} + \overbrace{\lim_{x\to a} 5}^{5} = 4a + 5.$$

We've just shown that for any $a \in \mathbb{R}$, the derivative of h at a exists and equals $4a+5$.

Hence, the derivative of h is the function $h' : \mathbb{R} \to \mathbb{R}$ defined by $h'(x) = 4x + 5$.

[652] At $\overset{\pm}{=}$, we can apply the Sum Rule because these three limits exist:

1. $\lim_{x\to a} 2x = 2a$ (by the Power and Constant Factor Rules for Limits)
2. $\lim_{x\to a} 2a = 2a$ (by the Constnat Rule for Limits)
3. $\lim_{x\to a} 5 = 5$ (by the Constant Rule for Limits)

The derivative of h at 2 is $h'(2) = 4 \cdot 2 + 5 = 13$.

A336. Write $g(x) = \sqrt{|x|} = \sqrt{\sqrt{x^2}}$ to see that g is elementary.

Figure to be
inserted here.

Observe that the tangent line of g at 0 is vertical—hence, g is not differentiable at 0 (and is not a differentiable function).

153.4. Ch. 89 Answers (Differentiation Notation)

A183(a) The derivative of f is $\dot{f}:\mathbb{R}\to\mathbb{R}$ (Newton) or $\dfrac{\mathrm{d}f}{\mathrm{d}x}:\mathbb{R}\to\mathbb{R}$ (Leibniz) and is defined by $\dot{f}(x)=\dfrac{\mathrm{d}f}{\mathrm{d}x}=0$.

The derivative at 2 is the number $\dot{f}(2)=\left.\dfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=2}=\dfrac{\mathrm{d}f}{\mathrm{d}x}(2)=0$.

(b) The derivative of g is $\dot{g}:\mathbb{R}\to\mathbb{R}$ (Newton) or $\dfrac{\mathrm{d}g}{\mathrm{d}x}:\mathbb{R}\to\mathbb{R}$ (Leibniz) and is defined by $\dot{g}(x)=\dfrac{\mathrm{d}g}{\mathrm{d}x}=5$.

The derivative at 2 is the number $\dot{g}(2)=\left.\dfrac{\mathrm{d}g}{\mathrm{d}x}\right|_{x=2}=\dfrac{\mathrm{d}g}{\mathrm{d}x}(2)=5$.

(c) The derivative of h is $\dot{h}:\mathbb{R}\to\mathbb{R}$ (Newton) or $\dfrac{\mathrm{d}h}{\mathrm{d}x}:\mathbb{R}\to\mathbb{R}$ (Leibniz) and is defined by $\dot{h}(x)=\dfrac{\mathrm{d}h}{\mathrm{d}x}=4x+5$.

The derivative at 2 is the number $\dot{h}(2)=\left.\dfrac{\mathrm{d}h}{\mathrm{d}x}\right|_{x=2}=\dfrac{\mathrm{d}h}{\mathrm{d}x}(2)=4\cdot 2+5=13$.

A337. $T(h)$ is the function whose domain and codomain are both $\mathbb{R}$ and whose mapping rule is $x\mapsto(x+1)^2/2$. And $T(h)(2)=9/2$.

$T(i)$ is the function whose domain and codomain are both $\mathbb{R}$ and whose mapping rule is $x\mapsto(3x-1)^2/2$. And $T(i)(2)=25/2$.

$S(j)$ is the function whose domain and codomain are both $\mathbb{R}$ and whose mapping rule is $x\mapsto(x+1)^2/4$. And $S(j)(2)=9/4$.

$S(k)$ is the function whose domain and codomain are both $\mathbb{R}$ and whose mapping rule is $x\mapsto(3x-1)^2/4$. And $S(k)(2)=25/4$.

A338(a) If a differentiable function maps each x to x^3, then its derivative maps each x to $3x^2$.

(b) If a differentiable function maps each x to $\sin x^2$, then its derivative maps each x to $2x\cos x^2$.

(b) If a differentiable function maps each x to $\ln x$, then its derivative maps each x to $1/x$.

A339. The mistake is in Step 4.

This is a common mistake made by students. To find the derivative of f at 2 (a number), we must first find the derivative of f (a function)—*then* plug in 2.

The mistake here is to plug in 2 first, getting $f(2)=-2$, then differentiating the constant -2 (a meaningless operation), which of course yields 0.

A341. Let $a \in \mathbb{R}$. Simplify the difference quotient:

$$\frac{i(x)-i(a)}{x-a} = \frac{x^4-a^4}{x-a} \overset{\star}{=} \frac{(x-a)(x^3+ax^2+a^2x+a^3)}{x-a} = x^3+ax^2+a^2x+a^3.$$

So,[653]

$$i'(a) = \lim_{x\to a}\frac{i(x)-i(a)}{x-a} = \lim_{x\to a}\left(x^3+ax^2+a^2x+a^2\right) \overset{\pm}{=} \overbrace{\lim_{x\to a}x^3}^{a^3} + \overbrace{\lim_{x\to a}ax^2}^{a^3} + \overbrace{\lim_{x\to a}a^2x}^{a^3} + \overbrace{\lim_{x\to a}a^3}^{a^3} = 4a^3.$$

We've just shown that for any $a \in \mathbb{R}$, the derivative of i at a exists and equals $4a^3$.

Hence, the derivative of i is the function $i' : \mathbb{R} \to \mathbb{R}$ defined by $i'(x) = 4x^3$.

A342. Let $a \in \mathbb{R}$. Simplify the difference quotient:

$$\frac{j(x)-j(a)}{x-a} = \frac{x^c-a^c}{x-a} \overset{\circ}{=} \frac{(x-a)\left(x^{c-1}+x^{c-2}a+x^{c-3}a^2+\cdots+xa^{c-2}+a^{c-1}\right)}{x-a} = x^{c-1}+x^{c-2}a+x^{c-3}a^2+\cdots$$

So,

[654]

$$j'(a) = \lim_{x\to a}\frac{j(x)-j(a)}{x-a} = \lim_{x\to a}\left(x^{c-1}+x^{c-2}a+x^{c-3}a^2+\cdots+xa^{c-2}+a^{c-1}\right)$$

$$\overset{\pm}{=} \underbrace{\lim_{x\to a}x^{c-1}}_{a^{c-1}} + \underbrace{\lim_{x\to a}x^{c-2}a}_{a^{c-1}} + \underbrace{\lim_{x\to a}x^{c-3}a^2}_{a^{c-1}} + \cdots + \underbrace{\lim_{x\to a}xa^{c-2}}_{a^{c-1}} + \underbrace{\lim_{x\to a}a^{c-1}}_{a^{c-1}} = ca^{c-1}.$$

We've just shown that for any $a \in \mathbb{R}$, the derivative of j at a exists and equals ca^{c-1}.

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[653] At $\overset{\pm}{=}$, we can use the Sum Rule for Limits because these four limits exist:

1. $\lim_{x\to a} x^3 = a^3$ (by the Power Rule for Limits)
2. $\lim_{x\to a} ax^2 = a^3$ (by the Power and Constant Factor Rule for Limits)
3. $\lim_{x\to a} a^2x = a^3$ (ditto)
4. $\lim_{x\to a} a^3 = a^3$ (by the Constant Rule for Limits)

[654] At $\overset{\pm}{=}$, we can use the Sum Rule for Limits because these c limits exist:

1. $\lim_{x\to a} x^{c-1} = a^{c-1}$ (by the Power Rule for Limits)
2. $\lim_{x\to a} x^{c-2}a = a^{c-1}$ (by the Power and Constant Factor Rule for Limits)
3. $\lim_{x\to a} x^{c-3}a^2 = a^{c-1}$ (ditto)
4. $\vdots$
5. $\lim_{x\to a} xa^{c-2} = a^{c-1}$ (ditto)
6. $\lim_{x\to a} a^{c-1} = a^{c-1}$ (by the Constant Rule for Limits)

Hence, the derivative of j is the function $j' : \mathbb{R} \to \mathbb{R}$ defined by $j'(x) = cx^{c-1}$.

A343. Apply the $\frac{\mathrm{d}}{\mathrm{d}x}$ operator to the given equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\sin y\right)=\frac{\mathrm{d}}{\mathrm{d}x}1 \quad\Longrightarrow\quad 3x^2\sin y+x^3\cos y\frac{\mathrm{d}y}{\mathrm{d}x}\overset{1}{=}0 \quad\Longleftrightarrow\quad \frac{\mathrm{d}y}{\mathrm{d}x}\overset{2}{=}-\frac{3}{x}\tan y.$$

From the given equation, we have $y\overset{3}{=}\sin^{-1}\frac{1}{x^3}$.

Plug $\overset{3}{=}$ into $\overset{2}{=}$ to get $\frac{\mathrm{d}y}{\mathrm{d}x}\overset{2}{=}-\frac{3}{x}\tan\left(\sin^{-1}\frac{1}{x^3}\right)$.

A344. Apply the $\frac{\mathrm{d}}{\mathrm{d}x}$ operator to the given equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y\sin x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x+y}-1\right) \quad\Longrightarrow\quad \frac{\mathrm{d}y}{\mathrm{d}x}\sin x+y\cos x\overset{1}{=}\mathrm{e}^{x+y}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right).$$

By the given equation, at $x=0$, we have $0=\mathrm{e}^y-1$ or $y=0$.

So, plug $(x,y)=(0,0)$ into $\overset{1}{=}$:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,0)}\sin 0+0\cdot\cos 0=\mathrm{e}^{0+0}\left(1+\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,0)}\right) \quad\Longleftrightarrow$$
$$0+0=1\left(1+\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,0)}\right) \quad\Longleftrightarrow\quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(0,0)}=-1.$$

A345(a) Let $y=\cos^{-1}x\in[0,\pi]$, so that $x\overset{1}{=}\cos y$.

Apply the $\frac{\mathrm{d}}{\mathrm{d}x}$ operator to $\overset{1}{=}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}x=\frac{\mathrm{d}}{\mathrm{d}x}\cos y \quad\overset{\text{CR}}{\Longrightarrow}\quad 1=-\sin y\frac{\mathrm{d}y}{\mathrm{d}x} \quad\Longleftrightarrow\quad \frac{\mathrm{d}y}{\mathrm{d}x}\overset{2}{=}\frac{-1}{\sin y}, \quad \text{for } \sin y\neq 0.$$

From the identity $\sin^2 y+\cos^2 y=1$, we have $\sin y\overset{4}{=}\pm\sqrt{1-x^2}$. But since $y\in[0,\pi]$, we know that $\sin y\geq 0$. And so, we may discard the negative values and rewrite $\overset{2}{=}$ as $\sin y\overset{3}{=}\sqrt{1-x^2}$.

Now plug $\overset{3}{=}$ into $\overset{2}{=}$ to get $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x=\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1}{\sqrt{1-x^2}}$, which is defined for $x\in(-1,1)$.[655]

(b) Let $y=\tan^{-1}x$, so that $x\overset{1}{=}\tan y$.

Apply the $\frac{\mathrm{d}}{\mathrm{d}x}$ operator to $\overset{1}{=}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}x=\frac{\mathrm{d}}{\mathrm{d}x}\tan y \quad\overset{\text{CR}}{\Longrightarrow}\quad 1=\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x} \quad\Longleftrightarrow\quad \frac{\mathrm{d}y}{\mathrm{d}x}\overset{2}{=}\frac{1}{\sec^2 y}.$$

(Note that for all $y\in\mathbb{R}$, $\sec y\neq 0$.)

[655] So, $\cos^{-1}$ is *not* a differentiable function: Its domain is $[-1,1]$, but that of its derivative is $(-1,1)$.

Plug $\stackrel{1}{=}$ into the identity $\sec^2 y = 1 + \tan^2 y$ to get $\sec^2 y \stackrel{3}{=} 1 + x^2$.

Next, plug $\stackrel{3}{=}$ into $\stackrel{2}{=}$ to get $\dfrac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{1+x^2}$.

A346. Apply the $\dfrac{\mathrm{d}}{\mathrm{d}x}$ operator to the given equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 y + \sin x\right) = \frac{\mathrm{d}}{\mathrm{d}x} 0 \qquad \xRightarrow{\text{CR}} \qquad 2xy + x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + \cos x = 0 \qquad \Longrightarrow$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy + \cos x}{x^2}, \quad \text{for } x \neq 0.$$

And so, by the IFT:

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{x^2}{2xy + \cos x}, \qquad \text{for } 2xy + \cos x \neq 0.$$

A347. Because $\sqrt{1 - x^2} = \cos\left(\sin^{-1} x\right)$, as we now show:

Let $y = \sin^{-1} x$. Then $x = \sin y$ and $\sqrt{1 - x^2} = \sqrt{1 - \sin^2 y} = \cos y = \cos\left(\sin^{-1} x\right)$.

A352. Compute $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{e}^t - 3t^2$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -\sin t + 2t$.

So, by the Parametric Differentiation Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{e}^t - 3t^2}{-\sin t + 2t} \qquad \text{(for } -\sin t + 2t \neq 0\text{)}.$$

A353(a) First derivative: $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\cos x} = \mathrm{e}^{\cos x}(-\sin x) = -\mathrm{e}^{\cos x}\sin x$.

Observe that the expression $-\mathrm{e}^{\cos x}\sin x$ is defined for all $x \in \mathbb{R}$.

Hence, f is a differentiable function. Its (first) derivative may be denoted f', $\dot{f}$, or $\frac{\mathrm{d}f}{\mathrm{d}x}$; and has domain $\mathbb{R}$, codomain $\mathbb{R}$, and this mapping rule:

$$f'(x) = \dot{f}(x) = \frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = -\mathrm{e}^{\cos x}\sin x.$$

The (first) derivative of f at 1 is $f'(1) = -\mathrm{e}^{\cos 1}\sin 1$.

Second derivative: $\frac{\mathrm{d}}{\mathrm{d}x}(-\mathrm{e}^{\cos x}\sin x) = -\mathrm{e}^{\cos x}\sin x(-\sin x) - \mathrm{e}^{\cos x}\cos x = \mathrm{e}^{\cos x}\left(\sin^2 x - \cos x\right)$.

Observe that the expression $\mathrm{e}^{\cos x}\left(\sin^2 x - \cos x\right)$ is defined for all $x \in \mathbb{R}$.

Hence, f is a twice-differentiable function. Its second derivative may be denoted f'', $\ddot{f}$, or $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$; and has domain $\mathbb{R}$, codomain $\mathbb{R}$, and this mapping rule:

$$f''(x) = \mathrm{e}^{\cos x}\left(\sin^2 x - \cos x\right).$$

The second derivative of f at 1 is $f''(1) = \mathrm{e}^{\cos 1}\left(\sin^2 1 - \cos 1\right)$.

(b) First derivative: $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2-1} = \frac{2x}{2\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}$.

Observe $x^2 - 1 = 0 \iff x = \pm 1$. So, g' will not be defined at 1 and is defined only on $(1, \infty]$.

Hence, g is *not* a differentiable function. Its (first) derivative may be denoted g', $\dot{g}$, or $\frac{\mathrm{d}g}{\mathrm{d}x}$; and has domain $(1, \infty]$, codomain $\mathbb{R}$, and this mapping rule:

$$g'(x) = \dot{g}(x) = \frac{\mathrm{d}g}{\mathrm{d}x}(x) = \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{x}{\sqrt{x^2-1}}$$

The first derivative of g at 1 is undefined (or does not exist).

Second derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}} - \frac{x \cdot 2x}{2\left(x^2-1\right)^{1.5}} = \frac{x^2-1}{\left(x^2-1\right)^{1.5}} - \frac{x^2}{\left(x^2-1\right)^{1.5}} = \frac{-1}{\left(x^2-1\right)^{1.5}}.$$

Again, g'' will not be defined at 1 and is defined only on $(1, \infty]$.

Hence, g is *not* a twice-differentiable function. Its second derivative may be denoted g'', $\ddot{g}$, or $\frac{\mathrm{d}^2 g}{\mathrm{d}x^2}$; and has domain $(1, \infty]$, codomain $\mathbb{R}$, and this mapping rule:

$$g''(x) = \frac{-1}{\left(x^2-1\right)^{1.5}}.$$

The second derivative of g at 1 is undefined.

(c) First derivative: $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{\sin x^2+2}=\dfrac{(\cos x^2)\,2x}{2\sqrt{\sin x^2+2}}=\dfrac{x\cos x^2}{\sqrt{\sin x^2+2}}$.

Observe that for all $x\in\mathbb{R}$, $\sin x^2+2>0$.

Hence, h is a differentiable function. Its (first) derivative may be denoted h', $\dot{h}$, or $\dfrac{\mathrm{d}h}{\mathrm{d}x}$; and has domain $\mathbb{R}$, codomain $\mathbb{R}$, and this mapping rule:

$$h'(x)=\dot{h}(x)=\frac{\mathrm{d}h}{\mathrm{d}x}(x)=\frac{\mathrm{d}h}{\mathrm{d}x}=\frac{x\cos x^2}{\sqrt{\sin x^2+2}}.$$

The (first) derivative of h at 1 is $h'(1)=\dfrac{1\cos 1^2}{\sqrt{\sin 1^2+2}}=\dfrac{\cos 1}{\sqrt{\sin 1+2}}$.

Second derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x\cos x^2}{\sqrt{\sin x^2+2}}=\frac{\cos x^2-x(\sin x^2)\,2x}{\sqrt{\sin x^2+2}}-\frac{x\cos x^2(\cos x^2)\,2x}{2(\sin x^2+2)^{1.5}}=\frac{\cos x^2-2x^2\sin x^2}{\sqrt{\sin x^2+2}}-\frac{x^2\cos^2 x^2}{(\sin x^2+2)^{1.5}}$$

Again, for all $x\in\mathbb{R}$, $\sin x^2+2>0$.

Hence, h is a twice-differentiable function. Its second derivative may be denoted h'', $\ddot{h}$, or $\dfrac{\mathrm{d}^2h}{\mathrm{d}x^2}$; and has domain $\mathbb{R}$, codomain $\mathbb{R}$, and this mapping rule:

$$h''(x)=\frac{\cos x^2-2x^2\sin x^2}{\sqrt{\sin x^2+2}}-\frac{x^2\cos^2 x^2}{(\sin x^2+2)^{1.5}}.$$

The second derivative of h at 1 is $h''(1)=\dfrac{\cos 1^2-2\cdot 1^2\sin 1^2}{\sqrt{\sin 1^2+2}}-\dfrac{1^2\cos^2 1^2}{(\sin 1^2+2)^{1.5}}=\dfrac{\cos 1-2\sin 1}{\sqrt{\sin 1+2}}-\dfrac{\cos^2 1}{(\sin 1+2)^{1.5}}$.

A354(a) False. Twice differentiability implies differentiability. However, the converse ("differentiability implies twice differentiability") is false—in Example 1156, the function h is differentiable but not twice differentiable.

(b) True by definition.

(c) True by definition. Recall that the second derivative of f at a is defined as this limit (which is a real number if it exists):

$$\lim_{x\to a}\frac{f'(x)-f'(a)}{x-a}.$$

So, if $f'(a)$ doesn't exist, then clearly the above limit (or equivalently, the second derivative of f at a) cannot exist either.

A355. Since f is a differentiable function, by Definition 205, it is differentiable at every point in its domain. Thus, the (first) derivative f' has the same domain as f, namely D.

We are given no information about the second derivative of f. Thus, all we can say about the domain of f'' is that it must be a subset of D, the domain of f'. The domain of f''

could be as "big" as D (if f is twice differentiable) or as "small" as $\varnothing$ (if f is nowhere twice differentiable).

A356. The function g is (at least) five-times differentiable. Its first five derivatives have domain $\mathbb{R}$, codomain $\mathbb{R}$, and these mapping rules:

$$\begin{array}{lll}
\textbf{Lagrange Newton} & \textbf{Leibniz} & \\
g'(x) = \dot{g}(x) & = \dfrac{\mathrm{d}g}{\mathrm{d}x}(x) = \dfrac{\mathrm{d}g}{\mathrm{d}x} & = 4x^3 - 3x^2 + 2x - 1, \\
g''(x) = \ddot{g}(x) & = \dfrac{\mathrm{d}^2g}{\mathrm{d}x^2}(x) = \dfrac{\mathrm{d}^2g}{\mathrm{d}x^2} & = 12x^2 - 6x + 2, \\
g'''(x) = \dddot{g}(x) & = \dfrac{\mathrm{d}^3g}{\mathrm{d}x^3}(x) = \dfrac{\mathrm{d}^3g}{\mathrm{d}x^3} & = 24x - 6, \\
g^{(4)}(x) = \overset{4}{\dot{g}}(x) & = \dfrac{\mathrm{d}^4g}{\mathrm{d}x^4}(x) = \dfrac{\mathrm{d}^4g}{\mathrm{d}x^4} & = 24, \\
g^{(5)}(x) = \overset{5}{\dot{g}}(x) & = \dfrac{\mathrm{d}^5g}{\mathrm{d}x^5}(x) = \dfrac{\mathrm{d}^5g}{\mathrm{d}x^5} & = 0.
\end{array}$$

Clearly, for any integer $n \geq 5$, g is n-times differentiable and its nth derivative is defined by

$$\begin{array}{ll}
\textbf{Lagrange Newton} & \textbf{Leibniz} \\
g^{(n)}(x) = \overset{n}{\dot{g}}(x) & = \dfrac{\mathrm{d}^ng}{\mathrm{d}x^n}(x) = \dfrac{\mathrm{d}^ng}{\mathrm{d}x^n} = 0.
\end{array}$$

Evaluating each of g's derivatives at 1, we have

$$\begin{array}{lll}
\textbf{Lagrange Newton} & \textbf{Leibniz} & \\
g'(1) = \dot{g}(1) & = \dfrac{\mathrm{d}g}{\mathrm{d}x}(1) = \left.\dfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=1} & = 4 \cdot 1^3 - 3 \cdot 1^2 + 2 \cdot 1 - 1 = 2, \\
g''(1) = \ddot{g}(1) & = \dfrac{\mathrm{d}^2g}{\mathrm{d}x^2}(1) = \left.\dfrac{\mathrm{d}^2g}{\mathrm{d}x^2}\right|_{x=1} & = 12 \cdot 1^2 - 6 \cdot 1 + 2 = 8, \\
g'''(1) = \dddot{g}(1) & = \dfrac{\mathrm{d}^3g}{\mathrm{d}x^3}(1) = \left.\dfrac{\mathrm{d}^3g}{\mathrm{d}x^3}\right|_{x=1} & = 24 \cdot 1 - 6 = 18, \\
g^{(4)}(1) = \overset{4}{\dot{g}}(1) & = \dfrac{\mathrm{d}^4g}{\mathrm{d}x^4}(1) = \left.\dfrac{\mathrm{d}^4g}{\mathrm{d}x^4}\right|_{x=1} & = 24,
\end{array}$$

For each integer $n \geq 5$,

$$g^{(n)}(1) = \overset{n}{\dot{g}}(1) = \frac{\mathrm{d}^ng}{\mathrm{d}x^n}(1) = \left.\frac{\mathrm{d}^ng}{\mathrm{d}x^n}\right|_{x=1} = 0.$$

A357. The first five derivatives of h have domain $\mathbb{R}$, codomain $\mathbb{R}$, and these mapping rules:

$$h'(x) = 5x^4 - 4x^3 + 3x^2 - 2x + 1,$$

$$h''(x) = 20x^3 - 12x^2 + 6x - 2,$$

$$h'''(x) = 60x^2 - 24x + 6,$$

$$h^{(4)}(x) = 120x - 24,$$

$$h^{(5)}(x) = 120.$$

For any $n \geq 6$, the function h is n-times differentiable, with the nth derivative defined by

$$h^{(n)}(x) = 0.$$

Hence, h is smooth.

A358. The first four derivatives of cos are

$$\begin{aligned}\cos' &= -\sin,\\ \cos'' &= -\cos,\\ \cos''' &= \sin,\\ \cos^{(4)} &= \cos.\end{aligned}$$

We see then that we'll have a repeating cycle. Specifically, for each $k = 0, 1, 2, 3, \ldots$, we have

$$\begin{aligned}\cos^{(1+4k)} &= -\sin,\\ \cos^{(2+4k)} &= -\cos,\\ \cos^{(3+4k)} &= \sin,\\ \cos^{(4k)} &= \cos.\end{aligned}$$

We've just shown that sin is n-times differentiable for every positive integer n. Hence, exp is smooth.

And the 8 603rd derivative of cos is

$$\cos^{(8\,603)} = \cos^{(3+4\times 2\,150)} = \sin.$$

A359. We've already used f^n to denote the composite function $\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$. So, the parentheses help us distinguish $f^{(n)}$, the nth derivative of f, from the composite function f^n.

A360(a) The first derivative of h is the function $h' : \mathbb{R} \to \mathbb{R}$ defined by $h'(x) = 2x$. And $h'(1) = 2 \cdot 1 = 2$.

(b) The composite function $h^2 : \mathbb{R} \to \mathbb{R}$ is defined by $h^2(x) = h(h(x)) = h\left(x^2 + 1\right) = \left(x^2 + 1\right)^2 + 1 = x^4 + 2x^2 + 2$. And $h^2(1) = 1^4 + 2 \cdot 1^2 + 2 = 5$.

(c) The squared function $(h)^2 : \mathbb{R} \to \mathbb{R}$ is defined by $(h)^2(x) = \left(x^2 + 1\right)^2 = x^4 + 2x^2 + 1$. And $(h)^2(1) = 1^4 + 2 \cdot 1^2 + 1 = 4$.

(d) The function $\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 : \mathbb{R} \to \mathbb{R}$ is defined by $\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 = [h'(x)]^2 = (2x)^2 = 4x^2$. And $\left.\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2\right|_{x=1} = 4 \cdot 1^2 = 4$.

(e) The second derivative of h is the function $\frac{\mathrm{d}^2h}{\mathrm{d}x^2} : \mathbb{R} \to \mathbb{R}$ defined by $\frac{\mathrm{d}^2h}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}h'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(2x) = 2$. And $\left.\frac{\mathrm{d}^2h}{\mathrm{d}x^2}\right|_{x=1} = 2$.

(f) The first derivative of the composite function h^2 is the function $\frac{\mathrm{d}h^2}{\mathrm{d}x} : \mathbb{R} \to \mathbb{R}$ defined by

$$\frac{\mathrm{d}h^2}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}h^2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^4 + 2x^2 + 2\right) = 4x^3 + 4x.$$

And $\left.\frac{\mathrm{d}h^2}{\mathrm{d}x}\right|_{x=1} = 4 \cdot 1^3 + 4 \cdot 1 = 8$.

(g) Informally, the function $\frac{\mathrm{d}h}{\mathrm{d}x^2}$ gives us the rate of change of h with respect to x^2. By the Chain Rule,

$$\frac{\mathrm{d}h}{\mathrm{d}x^2}\underbrace{\frac{\mathrm{d}x^2}{\mathrm{d}x}}_{2x} = \underbrace{\frac{\mathrm{d}h}{\mathrm{d}x}}_{2x}.$$

Rearranging, $$\frac{\mathrm{d}h}{\mathrm{d}x^2} = \frac{\mathrm{d}h}{\mathrm{d}x} \div \frac{\mathrm{d}x^2}{\mathrm{d}x} = \frac{2x}{2x} = 1.$$

So, the function $\frac{\mathrm{d}h}{\mathrm{d}x^2} : \mathbb{R} \to \mathbb{R}$ is defined by $\frac{\mathrm{d}h}{\mathrm{d}x^2} = 1$. And $\left.\frac{\mathrm{d}h}{\mathrm{d}x^2}\right|_{x=1} = 1$.

(h) Informally, the function $\frac{\mathrm{d}h^2}{\mathrm{d}x^2}$ gives us the rate of change of the composite function h^2 with respect to x^2. By the Chain Rule:

$$\frac{\mathrm{d}h^2}{\mathrm{d}x^2}\underbrace{\frac{\mathrm{d}x^2}{\mathrm{d}x}}_{2x} = \underbrace{\frac{\mathrm{d}h^2}{\mathrm{d}x}}_{4x^3+4x}.$$

Rearranging, $$\frac{\mathrm{d}h^2}{\mathrm{d}x^2} = \frac{\mathrm{d}h^2}{\mathrm{d}x} \div \frac{\mathrm{d}x^2}{\mathrm{d}x} = \frac{4x^3 + 4x}{2x} = 2x^2 + 2.$$

Hence, the function $\frac{\mathrm{d}h^2}{\mathrm{d}x^2} : \mathbb{R} \to \mathbb{R}$ is defined by $\frac{\mathrm{d}h^2}{\mathrm{d}x^2} = 2x^2 + 2$. And $\left.\frac{\mathrm{d}h^2}{\mathrm{d}x^2}\right|_{x=1} = 2 \cdot 1^2 + 2 = 4$.

153.8. Ch. 94 Answers (The Increasing/Decreasing Test)

A361(a) The derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = \mathrm{e}^x - 1$. We have

$$f'(x) \begin{cases} < 0, & \text{for } x \in (-\infty, 0), \\ = 0, & \text{for } x = 0, \\ > 0, & \text{for } x \in (0, \infty). \end{cases}$$

Figure to be inserted here.

The function f is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$. At the point 0, f is both increasing and decreasing (but neither strictly increasing nor strictly decreasing).

These findings are consistent with Fact 208:

(a) $f'(x) \geq 0$ for all $x \in [0, \infty)$ $\overset{\text{IDT}}{\iff}$ f is increasing on $[0, \infty)$.

(b) $f'(x) > 0$ for all $x \in (0, \infty)$ $\overset{\text{IDT}}{\implies}$ f is strictly increasing on $(0, \infty)$.

(c) $f'(x) \leq 0$ for all $x \in (-\infty, 0]$ $\overset{\text{IDT}}{\iff}$ f is decreasing on $(-\infty, 0]$.

(d) $f'(x) < 0$ for all $x \in (-\infty, 0)$ $\overset{\text{IDT}}{\implies}$ f is strictly decreasing on $(-\infty, 0)$.

(b) The derivative of g is the function $g' : [0, 2\pi] \to \mathbb{R}$ defined by $g'(x) = \cos x$. We have

$$g'(x) = \cos(x) \begin{cases} > 0, & \text{for } x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right], \\ = 0, & \text{for } x = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ < 0, & \text{for } x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \end{cases}$$

Figure to be inserted here.

The function g is strictly increasing on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$ and strictly increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. At each of the two points $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, g is both increasing and decreasing (but neither strictly increasing nor strictly decreasing).

These findings are consistent with Fact 208:

(a) $g'(x) \geq 0$ for all $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ $\overset{\text{IDT}}{\Longleftrightarrow}$ g is increasing on $\left[0, \frac{\pi}{2}\right] \cup \left[\ldots \right.$

(b) $g'(x) > 0$ for all $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$ $\overset{\text{IDT}}{\Longrightarrow}$ g is strictly increasing on $\left[0, \frac{\pi}{2}\right) \cup \left(\ldots \right.$

(c) $g'(x) \leq 0$ for all $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ $\overset{\text{IDT}}{\Longleftrightarrow}$ g is decreasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

(d) $g'(x) < 0$ for all $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ $\overset{\text{IDT}}{\Longrightarrow}$ g is strictly decreasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

A362(a) The derivative of i is the function $i' : \mathbb{R} \to \mathbb{R}$ defined by $i'(x) = -3x^2$.

Figure to be
inserted here.

(b) The function i is strictly decreasing on $\mathbb{R}$ because if $a < b$, then $i(a) = -a^3 > -b^3 = i(b)$.

By Fact 46, i is also decreasing on $\mathbb{R}$.

(There is no interval on which i is increasing or strictly increasing.)

(c) The function i shows that the converse of IDT(d) is false.

153.9. Ch. 95 Answers (Determining the Nature of a Stationary Point)

A363(a)(i) The derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by $f'(x) = -3x^2 + 3 = 3\left(1 - x^2\right) = 3(1 - x)(1 + x)$.

We have $f'(a) = 0 \iff a = \pm 1$. So, f has two stationary points, ± 1.

Figure to be
inserted here.

(a)(ii) At the point -1, $f' < 0$ to the left and $f' > 0$ to the right. Hence, by the FDTE(**d**), -1 is a strict local minimum.

At the point 1, $f' > 0$ to the left and $f' < 0$ to the right. Hence, by the FDTE(**b**), 1 is a strict local maximum.

(a)(iii) The function f has two turning points, ± 1 (these are the only points that are both stationary points and strict local extrema).

(b)(i) The derivative of g is the function $g' : [0, 2\pi] \to \mathbb{R}$ defined by $g'(x) = \cos x$.

We have $g'(a) = 0 \iff a = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$. So, g has two stationary points, $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$.

Figure to be
inserted here.

(b)(ii) At the point $\dfrac{\pi}{2}$, $g' > 0$ to the left and $g' < 0$ to the right. Hence, by the FDTE(**d**), $\dfrac{\pi}{2}$ is a strict local maximum.

At the point $\dfrac{3\pi}{2}$, $g' > 0$ to the left and $g' > 0$ to the right. Hence, by the FDTE(**d**), $\dfrac{3\pi}{2}$ is a strict local minimum.

(b)(iii) The function g has two turning points, $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$ (these are the only points that are both stationary points and strict local extrema).

(c)(i) The derivative of h is the function $h' : \mathbb{R} \to \mathbb{R}$ defined by $h'(x) = 2xe^{x^2}$.

We have $h'(a) = 0 \iff a = 0$. So, h has one stationary point, 0.

Figure to be inserted here.

(c)(ii) At the point 0, $h' < 0$ to the left and $h' > 0$ to the right. Hence, by the FDTE(**d**), 0 is a strict local minimum.

(c)(iii) The function h has one turning point, 0 (this is the only point that is both a stationary point and a strict local extremum).

A366(a)(i) The derivative of f is the function $f' : \mathbb{R} \to \mathbb{R}$ defined by

$$f'(x) = 8x^7 + 2x^6 - 6x^5 = x^5\left(8x^2 + 2x - 6\right) = x^5(8x - 6)(x + 1).$$

The stationary points of f are given by

$$f'(a) = 0 \qquad \Longleftrightarrow \qquad a^5(8a - 6)(a + 1) = 0 \qquad \Longleftrightarrow \qquad a = 0, \frac{3}{4}, -1.$$

So, f has three stationary points: 0, $\frac{3}{4}$, and -1.

Figure to be inserted here.

(a)(ii) The second derivative of f is the function $f'' : \mathbb{R} \to \mathbb{R}$ defined by $f''(x) = 56x^6 + 12x^5 - 30x^4 = 2x^4\left(28x^2 + 6x - 15\right)$.

Evaluating f'' at each of f's three stationary points, we have

$$f''(0) = 0,$$
$$f''\left(\frac{3}{4}\right) = 2 \cdot \frac{3}{4}^4 \left[28 \cdot \left(\frac{3}{4}\right)^2 + 6 \cdot \frac{3}{4} - 15\right] > 0,$$
$$f''(-1) = 2 \cdot (-1)^4 \left[28 \cdot (-1)^2 + 6 \cdot (-1) - 15\right] > 0.$$

Hence, by the SDTE, $\frac{3}{4}$ and -1 are strict local minima of f.

Unfortunately, the SDTE is inconclusive about 0.

It turns out that 0 is a strict local maximum of f. We can justify this by simply pointing to the graph (this will suffice for your A-Level exams).[656]

[656] It is also possible to rigorously show that 0 is a strict local maximum of f, as we do now:

(a)(iii) The function f has three turning points (points that are both stationary and strict local extrema): -1, 0, and $\dfrac{3}{4}$.

(b)(i) The derivative of g is the function $g' : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ defined by

$$g'(x) = \sec^2 x.$$

The stationary points of g are given by

$$g'(a) = 0 \qquad \text{or} \qquad \sec^2 a \stackrel{1}{=} 0.$$

But $\stackrel{1}{=}$ is false for all $a \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Hence, g has no stationary points.

Figure to be
inserted here.

(b)(ii) Not applicable, since g has no stationary points.

(b)(iii) Since g has no stationary points, it has no turning points either.

(c)(i) The derivative of h is the function $h' : [0, 2\pi] \to \mathbb{R}$ defined by $h'(x) = \cos x - \sin x$. We have

$$h'(a) = 0 \qquad \Longleftrightarrow \qquad \cos a - \sin a = 0 \qquad \Longleftrightarrow \qquad \cos a = \sin a \qquad \Longleftrightarrow \qquad a = \frac{\pi}{4}, \frac{5\pi}{4}.$$

So, h has two stationary points, $\dfrac{\pi}{4}$ and $\dfrac{5\pi}{4}$.

Figure to be
inserted here.

First, observe that $f(0) = 0$.

Next, observe that if $x \in \left(-\dfrac{1}{10}, 0\right) \cup \left(0, \dfrac{1}{10}\right)$, then $x^6 > 10x^7 > 100x^8$, so that

$$f(x) = x^8 + \frac{2}{7}x^7 - x^6 < x^8 + \frac{2}{7}x^7 - 10x^7 = x^8 - \frac{68}{70}x^7 < x^8 - \frac{68}{7}x^8 = -\frac{61}{7}x^8 < 0 = f(0).$$

We've just shown that for all x that are "near" 0, we have $f(x) < f(0)$. Hence, 0 is a strict local maximum.

(c)(ii) The second derivative of h is the function $h'' : [0, 2\pi] \to \mathbb{R}$ defined by $h''(x) = -\sin x - \cos x$.

Evaluating h'' at each of h's two stationary points, we have

$$h''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} < 0,$$

$$h''\left(\frac{5\pi}{4}\right) = -\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4} = -\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2} > 0.$$

By the SDTE, $\frac{\pi}{4}$ is a strict local maximum, while $\frac{5\pi}{4}$ is a strict local minimum.

(c)(iii) The function h has two turning points (points that are both stationary and strict local extrema): $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

153.10. Ch. 96 Answers (Concavity)

153.11. Ch. 97 Answers (Inflexion Points)

153.12. Ch. 98 Answers (A Summary of Chapters 94, 95, 96, and 97)

153.13. Ch. 99 Answers (More Techniques of Differentiation)

A380. At $t = 0$, $(x, y) = (0, 0)$. At $t = 1$, $(x, y) = (2, 0)$.

By implicit differentiation, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{4t^3 - 1}{5t^4 + 1}$.

Hence, l_1 has gradient $\left.\dfrac{4t^3 - 1}{5t^4 + 1}\right|_{t=0} = -1$ and is described by $y = -x$.

And l_2 has gradient $\left.\dfrac{4t^3 - 1}{5t^4 + 1}\right|_{t=1} = \dfrac{3}{6} = \dfrac{1}{2}$ and is described by $y = 0 + \dfrac{1}{2}(x - 2) = \dfrac{x}{2} - 1$.

The intersection point of l_1 and l_2 is given by this system of equations:

$$y \overset{1}{=} -x \qquad \text{and} \qquad y \overset{2}{=} \frac{x}{2} - 1.$$

Solving, $-x = x/2 - 1$ or $x \overset{3}{=} 2/3$. Plugging $\overset{3}{=}$ back into either $\overset{1}{=}$ or $\overset{2}{=}$, we have $y = -2/3$. Thus, the intersection point of l_1 and l_2 is $(2/3, -2/3)$.

A381. The base of a cone is a circle. You will doubtless recall the formula for the area of a circle:

$$A = \pi r^2.$$

Differentiate with respect to t:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Plug in $\overset{5}{=}$ and $\overset{6}{=}$ (from the last example):

$$\left.\frac{dA}{dt}\right|_{t=2} = 2\pi r\bigg|_{t=2} \left.\frac{dr}{dt}\right|_{t=2} = 2\pi \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{2\pi^{1/3} \cdot 3^{2/3}} = \left(\frac{\pi}{3}\right)^{1/3} \approx 1.02.$$

The base area A is increasing at a rate of $1.02\,\mathrm{m^2\,s^{-1}}$.

A382(a) $V = \dfrac{1}{3}\pi r^2 h = 1$. Rearranging,

$$r = \sqrt{\frac{3}{\pi h}}.$$

(b) By Pythagoras' Theorem, $l = \sqrt{r^2 + h^2} = \sqrt{\dfrac{3}{\pi h} + h^2}$.

(c) $S = \pi r l = \pi \sqrt{\dfrac{3}{\pi h}} \sqrt{\dfrac{3}{\pi h} + h^2} = \pi \sqrt{\dfrac{9}{\pi^2 h^2} + \dfrac{3h}{\pi}} = \sqrt{\dfrac{9}{h^2} + 3\pi h}$.

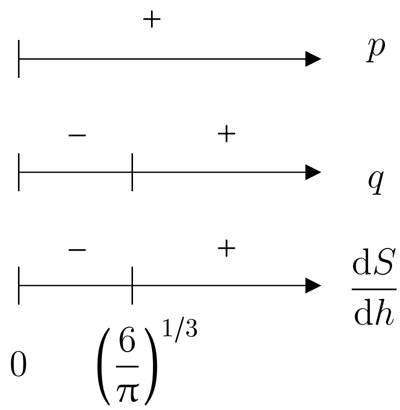
(d) Compute

$$\frac{dS}{dh} = \overbrace{\frac{1}{2\sqrt{\frac{9}{h^2} + 3\pi h}}}^{p} \Bigg(\overbrace{-\frac{18}{h^3} + 3\pi}^{q}\Bigg).$$

$$\frac{dS}{dh} = 0 \iff q = -\frac{18}{h^3} + 3\pi = 0 \iff \bar{h} = \left(\frac{6}{\pi}\right)^{1/3}.$$

Hence, the only stationary point of S (with respect to h) is at $\bar{h} = \left(\frac{6}{\pi}\right)^{1/3} \approx 1.24$.

(e) Sign diagrams for p, q, and $\frac{dS}{dh} = pq$:[657]



By the First Derivative Test for Extrema (FDTE), S attains its strict local minimum at $\bar{h}$.

By the Increasing/Decreasing Test (IDT), S is strictly decreasing on $\left(0, \bar{h}\right)$ and strictly increasing on $\left(\bar{h}, \infty\right)$.

Hence, S attains its strict global minimum at $\bar{h} = \left(\frac{6}{\pi}\right)^{1/3}$.

[657] Explanation: $p > 0$ for all $h \in (0, \infty)$.

In **(d)**, we found that $q = 0 \iff h = \left(\frac{6}{\pi}\right)^{1/3}$. Now observe also that if $h < \left(\frac{6}{\pi}\right)^{1/3}$, then $q < 0$. And if $h > \left(\frac{6}{\pi}\right)^{1/3}$, then $q > 0$.

Since $\frac{dS}{dh} = pq$ and $p > 0$ for all h, the sign diagram for $\frac{dS}{dh}$ is the same as that for q.

153.14. Ch. 100 Answers (More Fun with Your TI84)

153.15. Ch. 101 Answers (Power Series)

A390(a) We use the Four-Step Maclaurin Recipe:

1. The derivatives of f are defined by

$$\begin{aligned} f'(x) &= n(1+x)^{n-1}, \\ f''(x) &= n(n-1)(1+x)^{n-2}, \\ f^{(3)}(x) &= n(n-1)(n-2)(1+x)^{n-3}, \\ f^{(4)}(x) &= n(n-1)(n-2)(n-3)(1+x)^{n-4}, \\ &\vdots \\ f^{(r)}(x) &= n(n-1)\dots(n-r+1)(1+x)^{n-r}. \end{aligned}$$

2. Evaluate each of f, f', f'', f''', etc. at 0:

$$\begin{aligned} f(0) &= (1+0)^n = 1, \\ f'(0) &= n(1+0)^{n-1} = n, \\ f''(0) &= n(n-1)(1+0)^{n-2} = n(n-1), \\ f^{(3)}(0) &= n(n-1)(n-2)(1+0)^{n-3} = n(n-1)(n-2), \\ f^{(4)}(0) &= n(n-1)(n-2)(n-3)(1+0)^{n-4} = n(n-1)(n-2)(n-3), \\ &\vdots \\ f^{(r)}(0) &= n(n-1)\dots(n-r+1)(1+0)^{n-r} = n(n-1)\dots(n-r+1). \end{aligned}$$

3. The rth Maclaurin coefficient of f is

$$c_r = \frac{f^{(r)}(0)}{r!} = \frac{n(n-1)\dots(n-r+1)}{r!}.$$

4. The Maclaurin series of f is

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots$$

(b) We use the Four-Step Maclaurin Recipe:

1. The first five derivatives of cos are defined by

$$\cos' x = -\sin x, \qquad \cos'' x = -\cos x, \qquad \cos^{(3)} x = \sin x, \qquad \cos^{(4)} x = \cos x, \qquad \cos^{(5)} x = -\sin x.$$

We observe a cycle after every four derivatives. And so,[658]

$$\cos^{(n)} x = \begin{cases} \cos x, & \text{for } n = 0, 4, 8, \dots, \\ -\sin x, & \text{for } n = 1, 5, 9, \dots, \\ -\cos x, & \text{for } n = 2, 6, 10, \dots, \\ \sin x, & \text{for } n = 3, 7, 11, \dots. \end{cases}$$

[658] Actually, we already did this in Exercise 358.

2. Evaluate each of cos, cos′, cos″, cos‴, etc. at 0:

$$\cos^{(n)} 0 = \begin{cases} 1, & \text{for } n = 0, 4, 8, \ldots, \\ 0, & \text{for } n = 1, 5, 9, \ldots, \\ -1, & \text{for } n = 2, 6, 10, \ldots, \\ 0, & \text{for } n = 3, 7, 11, \ldots. \end{cases}$$

3. The nth Maclaurin coefficient of cos is

$$c_n = \frac{\cos^{(n)}(0)}{n!} = \begin{cases} 1/n!, & \text{for } n = 0, 4, 8, \ldots, \\ 0/n! = 0, & \text{for } n = 1, 5, 9, \ldots, \\ -1/n!, & \text{for } n = 2, 6, 10, \ldots, \\ 0/n! = 0, & \text{for } n = 3, 7, 11, \ldots. \end{cases}$$

4. The Maclaurin series of cos is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

(c) We use the Four-Step Maclaurin Recipe:

1. The derivatives of g are defined by

$$\begin{aligned} g'(x) &= \frac{1}{1+x}, \\ g''(x) &= \frac{-1}{(1+x)^2}, \\ g^{(3)}(x) &= \frac{2 \cdot 1}{(1+x)^3}, \\ g^{(4)}(x) &= \frac{-3 \cdot 2 \cdot 1}{(1+x)^4}, \\ &\vdots \\ g^{(n)}(x) &= \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}. \end{aligned}$$

2. Evaluate each of g, g', g'', g''', etc. at 0:

$$\begin{aligned}
g(0) &= \ln(1+0) = 0, \\
g'(0) &= \frac{1}{1+0} = 1, \\
g''(0) &= \frac{-1}{(1+0)^2} = -1, \\
g^{(3)}(0) &= \frac{2 \cdot 1}{(1+0)^3} = 2, \\
g^{(4)}(0) &= \frac{-3 \cdot 2 \cdot 1}{(1+0)^4} = -3! \\
&\vdots \\
g^{(n)}(0) &= \frac{(-1)^{n-1}(n-1)!}{(1+0)^n} = (-1)^{n-1}(n-1)!
\end{aligned}$$

3. The nth Maclaurin coefficient of g is

$$c_n = \frac{g^{(n)}(0)}{n!} = \frac{(-1)^{n-1}(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}.$$

4. The Maclaurin series of g is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

A392. In Example 1281, we showed that for any $n \in \mathbb{R}$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \text{ for all } x \in (-1,1)$$

And so, in particular, for $n = -1$, we have

$$(1+x)^{-2} = \frac{1}{(1+x)^2} = 1 + 2x + \frac{2(2-1)}{2!}x^2 + \frac{2(2-1)(2-2)}{3!}x^3 + \dots = 1 + 2x + x^2 + 0x^3 \text{ for all } x \in (-1,1)$$

That is, f can be represented by the power series just given.

A394(a) The Maclaurin series of exp is $M(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

So, the first four Maclaurin polynomials are

$M_0(x) = 1$

$M_1(x) = 1 + x$

$M_2(x) = 1 + x + \frac{x^2}{2!}$

$M_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

Figure to be
inserted here.

(b) The Maclaurin series of f is $M(x) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$

So, the first four Maclaurin polynomials are

$M_0(x) = 1$

$M_1(x) = 1 + nx$

$M_2(x) = 1 + nx + \frac{n(n-1)}{2!}x^2$

$M_3(x) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)x^3}{3!}$

Figure to be
inserted here.

(c) The Maclaurin series of cos is $M(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

So, the first four Maclaurin polynomials are

$M_0(x) = 1$

$M_1(x) = 1$

$M_2(x) = 1 - \frac{x^2}{2!}$

$M_3(x) = 1 - \frac{x^2}{2!}$

Figure to be
inserted here.

The small-angle approximation for cosine is $\cos x \approx 1 - \dfrac{x^2}{2!}$.

A395(a)(i) $\sin 2x = 2x - \dfrac{(2x)^3}{3!} + \dfrac{(2x)^5}{5!} - \ldots$ for all $2x \in \mathbb{R}$ or $x \in \mathbb{R}$.

(a)(ii) $\sin\left(x^2+1\right) = x^2 + 1 - \dfrac{\left(x^2+1\right)^3}{3!} + \dfrac{\left(x^2+1\right)^5}{5!} - \ldots$ for all $x^2+1 \in \mathbb{R}$ or $x \in \mathbb{R}$.

(a)(iii) $\sin(\sin x) = \sin x - \dfrac{\sin^3 x}{3!} + \dfrac{\sin^5 x}{5!} - \ldots$ for all $\sin x \in \mathbb{R}$ or $x \in \mathbb{R}$.

(a)(iv) $\sin\dfrac{x^3}{5} = \dfrac{x^3}{5} - \dfrac{\left(x^3/5\right)^3}{3!} + \dfrac{\left(x^3/5\right)^5}{5!} - \ldots$ for all $\dfrac{x^3}{5} \in \mathbb{R}$ or $x \in \mathbb{R}$.

(b)(i) $(1+2x)^n = 1 + 2nx + \dfrac{n(n-1)}{2!}(2x)^2 + \dfrac{n(n-1)(n-2)}{3!}(2x)^3 + \ldots$ for all $2x \in (-1,1)$ or $x \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

(b)(ii) $\left(1+x^2+1\right)^n = \left(2+x^2\right)^n \overset{1}{=} 1+n\left(x^2+1\right)+\dfrac{n(n-1)}{2!}\left(x^2+1\right)^2+\dfrac{n(n-1)(n-2)}{3!}\left(x^2+1\right)^3+\ldots$ for all $x^2+1 \in (-1,1)$ or $x^2 \in (-2,0)$ or $x \in \varnothing$. The interval of convergence is empty. That is, $\overset{1}{=}$ holds for **no** values of x.

(b)(iii) $(1+\sin x)^n \overset{2}{=} 1 + n\sin x + \dfrac{n(n-1)}{2!}\sin^2 x + \dfrac{n(n-1)(n-2)}{3!}\sin^3 x + \ldots$ for all $\sin x \in (-1,1)$ or $x \neq k\pi/2$ for any integer k. That is, $\overset{2}{=}$ holds for all real numbers x except those that can be written in the form $k\pi/2$ for some integer k.

(b)(iv) $\left(1+\dfrac{x^3}{5}\right)^n = 1+n\dfrac{x^3}{5}+\dfrac{n(n-1)}{2!}\left(\dfrac{x^3}{5}\right)^2+\dfrac{n(n-1)(n-2)}{3!}\left(\dfrac{x^3}{5}\right)^3+\ldots$ for all $\dfrac{x^3}{5} \in (-1,1)$ or $x^3 \in (-5,5)$ or $x \in \left(-\sqrt[3]{5}, \sqrt[3]{5}\right)$.

(c)(i) $\cos 2x = 1 - \dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!} - \ldots$ for all $2x \in \mathbb{R}$ or $x \in \mathbb{R}$.

(c)(ii) $\cos\left(x^2+1\right) = 1 - \dfrac{\left(x^2+1\right)^2}{2!} + \dfrac{\left(x^2+1\right)^4}{54} - \ldots$ for all $x^2+1 \in \mathbb{R}$ or $x \in \mathbb{R}$.

(c)(iii) $\cos(\sin x) = 1 - \dfrac{\sin^2 x}{2!} + \dfrac{\sin^4 x}{4!} - \ldots$ for all $\sin x \in \mathbb{R}$ or $x \in \mathbb{R}$.

(c)(iv) $\cos\dfrac{x^3}{5} = 1 - \dfrac{\left(x^3/5\right)^2}{2!} + \dfrac{\left(x^3/5\right)^4}{\$!} - \ldots$ for all $\dfrac{x^3}{5} \in \mathbb{R}$ or $x \in \mathbb{R}$.

A396(a) From the fifth "standard" Maclaurin series,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots, \qquad \text{for all } x \in (-1,1].$$

Substituting "x" with "$x-1$" to get

$$\ln(1+x-1) = \ln x = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \text{ for all } x+1 \in (-1,1] \text{ or } x \in (0,2].$$

We've just shown that f can be represented by the power series $x-1-\dfrac{(x-1)^2}{2}+\dfrac{(x-1)^3}{3}+\ldots$.

That is, $$f(x) = \ln x = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \text{ for all } x \in (0,2] = \text{Domain } f.$$

A397. From the first "standard" Maclaurin series,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots, \qquad \text{for all } x \in (-1,1).$$

Substitute "x" with "$2x^2$" to get

$$\frac{1}{1+2x^2} = 1 - 2x^2 + \left(2x^2\right)^2 - \left(2x^2\right)^3 + \dots = 1 - 2x^2 + 4x^4 - 8x^6 + \dots,$$

for all $2x^2 \in (-1,1)$ or $x^2 \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ or $x \in \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Hence, g has domain $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and can be represented by the above power series.

A398(a) We know that sin and exp can be represented by power series. So, by Theorem 57, $f = \sin \cdot \exp$ can also be represented by a power series.

(b) From the "standard" series, we know that sin and exp can be represented by

$$x - \frac{x^3}{3!} + \dots \qquad \text{and} \qquad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \text{ respectively.}$$

Write $$\left(x - \frac{x^3}{3!} + \dots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots.$$

By observation, $c_0 = 0$, $c_1 = 1$, $c_2 = 1$, and $c_3 = -1/6 + 1/2 = 1/3$.

Hence, f can be represented by $x + x^2 + x^3/3 + \dots$.

(c) In **(a)**, we reasoned that f can be represented by a power series. By Theorem 38 then, f can be represented by its Maclaurin series, which we now find using, as usual, the Four-Step Maclaurin Recipe:

1. The first three derivatives of f are defined by

$$\begin{aligned} f'(x) &= \cos x \exp x + \sin x \exp x = \exp x (\sin x + \cos x), \\ f''(x) &= \exp x (\sin x + \cos x) + \exp x (\cos x - \sin x) = 2 \exp x \cos x, \\ f'''(x) &= 2 \exp x (\cos x - \sin x), \end{aligned}$$

2. $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$, and $f'''(0) = 2$.
3. The first four Maclaurin coefficients of f are 0, 1, $2/2 = 1$, and $2/6 = 1/3$.
4. Hence, the Maclaurin series of f is $x + x^2 + x^3/3 + \dots$.

(d) Yes.

A399(a) Let $f : (-1,1) \to \mathbb{R}$ be the function defined by $f(x) = \ln(1+x)$. We know that cos and f can be represented by power series. So, by Theorem 57, $g = \cos \cdot f$ can also be represented by a power series.

(b) From the standard series, we know that sin and exp can be represented by $1 - \dfrac{x^2}{2!} + \dots$ and $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dots$.

Write $$\left(1 - \frac{x^2}{2!} + \dots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots.$$

By observation, $c_0 = 0$, $c_1 = 1$, $c_2 = -1/2$, and $c_3 = -1/2 + 1/3 = -1/6$.

Hence, g can be represented by $x - x^2/2 - x^3/6 + \dots$.

(c) In **(a)**, we reasoned that g can be represented by a power series. By Theorem 38 then, g can be represented by its Maclaurin series, which we now find, as usual, the Four-Step Maclaurin Recipe:

1. **Step 1.** The first three derivatives of g are defined by

$$g'(x) = -\sin x \ln(1+x) + \frac{\cos x}{1+x},$$
$$g''(x) = -\cos x \ln(1+x) - \frac{\sin x}{1+x} - \frac{\sin x}{1+x} - \frac{\cos x}{(1+x)^2},$$
$$g'''(x) = \sin x \ln(1+x) - \frac{\cos x}{1+x} - 2\left(\frac{\cos x}{1+x} - \frac{\sin x}{(1+x)^2}\right) + \frac{\sin x}{(1+x)^2} + \frac{2\cos x}{(1+x)^3}.$$

2. **Step 2.** $g(0) = 0$, $g'(0) = 1$, $g''(0) = -1$, and $g'''(0) = -1$.
3. **Step 3.** The first four Maclaurin coefficients of g are 0, 1, $-1/2$, and $-1/6 = 1/3$.
4. **Step 4.** Hence, the Maclaurin series of g is $x - x^2/2 - x^3/6 + \dots$.

(d) Yes.

A400. As usual, we use the Four-Step Maclaurin Recipe:

1. The first three derivatives of h are defined by

$$h'(x) = \frac{(-2)(-1)}{(1-x)^3}\frac{1}{1+2x^2} + \frac{1}{(1-x)^2}\frac{-1\cdot 4x}{(1+2x^2)^2} = h(x)\left(\frac{2}{1-x} - \frac{4x}{1+2x^2}\right),$$

$$h''(x) = h'(x)\left(\frac{2}{1-x} - \frac{4x}{1+2x^2}\right) + h(x)\left[\frac{2}{(1-x)^2} + \frac{16x^2}{(1+2x^2)^2} - \frac{4}{1+2x^2}\right] = \frac{[h'(x)]^2}{h(x)} + h(x)\Big[$$

$$h'''(x) = \frac{2h(x)h'(x)h''(x) - [h'(x)]^2 h'(x)}{[h(x)]^2} + h'(x)\left[\frac{2}{(1-x)^2} + \frac{16x^2}{(1+2x^2)^2} - \frac{4}{1+2x^2}\right] + h(x$$

2. Evaluate each of h, h', h'', and h''' at 0:

$$h(0) = \frac{1}{(1-0)^2}\frac{1}{1+2\cdot 0^2} = 1\cdot 1 = 1,$$

$$h'(0) = h(0)\left(\frac{2}{1-0} - \frac{4\cdot 0}{1+2\cdot 0^2}\right) = 1\cdot(2-0) = 2,$$

$$h''(0) = \frac{[h'(0)]^2}{h(0)} + h(0)\left[\frac{2}{(1-0)^2} + \frac{16\cdot 0^2}{(1+2\cdot 0^2)^2} - \frac{4}{1+2\cdot 0^2}\right] = \frac{2^2}{1} + 1[2+0-4] = 2,$$

$$h'''(0) = \frac{2h(0)h'(0)h''(0) - [h'(0)]^2 h'(0)}{[h(0)]^2} + h'(0)\left[\frac{2}{(1-0)^2} + \frac{16\cdot 0^2}{(1+2\cdot 0^2)^2} - \frac{4}{1+2\cdot 0^2}\right] + h(\text{(}$$

$$= \frac{2\cdot 1\cdot 2\cdot 2 - 2^2\cdot 2}{1} + 2\cdot[2+0-4] + 1[4-0+0] = 0-4+4 = 0.$$

3. The 0th, 1st, 2nd, and 3rd Maclaurin coefficients of h are

$$m_0 = \frac{h(0)}{0!} = \frac{1}{1} = 1, \qquad m_2 = \frac{h''(0)}{2!} = \frac{2}{2} = 1,$$

$$m_1 = \frac{h'(0)}{1!} = \frac{2}{1} = 2, \qquad m_3 = \frac{h'''(0)}{3!} = \frac{0}{6} = 0.$$

4. The Maclaurin series of h is

$$M(x) = 1 + 2x + x^2 + 0x^3 + \ldots$$

Conclude: By Theorem 38, h can be represented by the power series $1 + 2x + x^2 + 0x^3 + \ldots$.

That is, $$h(x) = \frac{1}{(1-x)^2}\frac{1}{1+2x^2} = 1 + 2x + x^2 + 0x^3 + \ldots \text{ for all } x \in \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \text{Domain } h.$$

This finding is the same as the conclusion in Example 1298.

A401. We know that sin and cos can be represented by power series. So, by Theorem 57, $f = \sin\cdot\cos$ can also be represented by a power series. Hence, by Theorem 38, f can be represented by its Maclaurin series, which we now find:

Recall that $\sin 2x = 2\sin x\cos x$. So, $f(x) = \frac{1}{2}\sin 2x$. Four-Step Maclaurin Recipe:

1. The first three derivatives of f are defined by $f'(x) = \cos 2x$, $f''(x) = -2\sin 2x$, and $f'''(x) = -4\cos 2x$.
2. We have $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -4$.
3. The first four Maclaurin coefficients of f are 0, 1, 0, and $-4/6 = -2/3$.
4. The Maclaurin series of f is $x - 2x^3/3 + \ldots$.

That is, f can be represented by $x - 2x^3/3 + \ldots$.

Our finding agrees with the conclusion in Example 1299.

A402. As usual, we use the Four-Step Maclaurin Recipe:

1. The first four derivatives of tan are defined by

$$\begin{aligned}
\tan' x &= \sec^2 x, \\
\tan'' x &= 2\sec^2 x \tan x, \\
\tan''' x &= 4\sec^2 x \tan^2 x + 2\sec^4 x, \\
\tan^{(4)} x &= 8\sec^2 x \tan^3 x + 8\sec^4 x \tan x + 8\sec^4 x \tan x = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x. \\
\tan^{(5)} x &= 16\sec^2 x \tan^4 x + 24\sec^4 x \tan^2 x + 64\sec^4 x \tan^2 x + 16\sec^6 x.
\end{aligned}$$

2. Observe that $\sec 0 = 1$ and $\tan 0 = 0$. So,

$$\begin{aligned}
\tan 0 &= 0, & \tan''' 0 &= 2, \\
\tan' 0 &= 1, & \tan^{(4)} 0 &= 0, \\
\tan'' 0 &= 0, & \tan^{(5)} 0 &= 16.
\end{aligned}$$

3. The first six Maclaurin coefficients of tan are 0, 1, 0, $2/3! = 1/3$, 0, and $16/5! = 2/15$.
4. The Maclaurin series of tan is

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

The small-angle approximation for tan is $\tan x \approx x$.

A404(a) Three antiderivatives of f are the functions $g, h, i : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \frac{1}{2}x^2 - 3x, \qquad h(x) = \frac{1}{2}x^2 - 3x + 1, \qquad i(x) = \frac{1}{2}x^2 - 3x + 2.$$

(Your answers may be different.)

(b) There exist constants $C_1, C_2, C_3 \in \mathbb{R}$ such that

$$A(x) = g(x) + C_1 = h(x) + C_2 = i(x) + C_3, \qquad \text{for every } x \in \mathbb{R}.$$

A405(a) $\dfrac{d}{dx}g(x) = \dfrac{d}{dx}(-\cos 4x) = 4\sin 4x = f(x)$.

$\dfrac{d}{dx}h(x) = 8\left(2\sin x\cos^3 x - 2\sin^3 x\cos x\right) = 16\sin x\cos x\left(\cos^2 x - \sin^2 x\right) = 8\sin 2x\cos 2x = 4\sin 4x = f(x)$.

Hence, each of g and h has derivative f. Equivalently, each of g and h is an antiderivative of f.

(b) Although the functions g and h seem very different, they actually differ by only a constant (namely 1), as we now show:

$$\begin{aligned} h(x) &= 8\sin^2 x\cos^2 x = 2(2\sin x\cos x)(2\sin x\cos x) = 2\sin^2 2x \\ &= 1 - \left(1 - 2\sin^2 2x\right) = 1 - \cos 4x = g(x) + 1. \end{aligned}$$

So no, this is consistent with and does not contradict Fact 214.

A407(a) $\dfrac{d}{dx}kx = k$ (Power and Constant Factor Rules).

(b) $\dfrac{d}{dx}\dfrac{x^{k+1}}{k+1} = (k+1)\dfrac{x^k}{k+1} = x^k$ (Power and Constant Factor Rules).

(d) $\dfrac{d}{dx}\exp x = \exp x$ (Exponential Rule).

(e) $\dfrac{d}{dx}\cos x = \sin x$ (Cosine Rule).

(f) $\dfrac{d}{dx}\sin x = \cos x$ (Sine Rule).

(g) $\dfrac{d}{dx}(F \pm G) = \dfrac{d}{dx}F \pm \dfrac{d}{dx}G = f \pm g$ (Sum and Difference Rules).

(h) $\dfrac{d}{dx}kF = k\dfrac{d}{dx}F = kf$ (Constant Factor Rule).

(i) $\dfrac{d}{dx}\dfrac{1}{a}F(ax+b) = \dfrac{1}{a}\dfrac{d}{dx}[F(ax+b)] = \dfrac{1}{a}f(ax+b)\cdot a = f(ax+b)$ (Constant Factor and Chain Rules).

A408. Consider an antidifferentiable function defined on an interval.

(b) If that function is defined by $x \mapsto x^k$,[659] then its antiderivatives are *exactly* those

[659] Also assuming $k \neq -1$ and the function's domain does not contain 0 if $k < 0$.

functions defined by $x \mapsto x^{k+1}/(k+1)+C$ (for $C \in \mathbb{R}$).

(c) If that function is defined by $x \mapsto 1/x$, then its antiderivatives are *exactly* those functions defined by $x \mapsto \ln|x|+C$ (for $C \in \mathbb{R}$).

(d) If that function is defined by $x \mapsto \exp x$, then its antiderivatives are *exactly* those functions defined by $x \mapsto \exp x + C$ (for $C \in \mathbb{R}$).

(e) If that function is defined by $x \mapsto \sin x$, then its antiderivatives are *exactly* those functions defined by $x \mapsto -\cos x + C$ (for $C \in \mathbb{R}$).

(f) If that function is defined by $x \mapsto \cos x$, then its antiderivatives are *exactly* those functions defined by $x \mapsto \sin x + C$ (for $C \in \mathbb{R}$).

(g) If that function is defined by $x \mapsto (f \pm g)(x)$, then its antiderivatives are *exactly* those functions defined by $x \mapsto (F \pm G)(x) + C$ (for $C \in \mathbb{R}$).

(h) If that function is defined by $x \mapsto kf(x)$, then its antiderivatives are *exactly* those functions defined by $x \mapsto kF(x) + C$ (for $C \in \mathbb{R}$).

(i) If that function is defined by $x \mapsto f(ax+b)$, then its antiderivatives are *exactly* those functions defined by $x \mapsto \dfrac{1}{a}F(ax+b)+C$ (for $C \in \mathbb{R}$).

A409(a) $\displaystyle\int ax+b\,\mathrm{d}x = \frac{1}{2}ax^2+bx+C.$

(b) $\displaystyle\int ax^2+bx+c\,\mathrm{d}x = \frac{1}{3}ax^3+\frac{1}{2}bx^2+cx+C.$

(c) $\displaystyle\int ax^3+bx^2+cx+d\,\mathrm{d}x = \frac{1}{4}ax^4+\frac{1}{3}bx^3+\frac{1}{2}cx^2+d+C.$

(d) $\displaystyle\int (ax+b)^c\,\mathrm{d}x = \frac{1}{a(c+1)}(ax+b)^{c+1}+C.$

(e) $\displaystyle\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b|+C.$

(f) $\displaystyle\int a\sin(bx+c)+d\,\mathrm{d}x = -\frac{a}{b}\cos(bx+c)+dx.$

(g) $\displaystyle\int a\exp(bx+c)+d\,\mathrm{d}x = \frac{a}{b}\exp(bx+c)+dx.$

(h) $\displaystyle\int a\cos bx+c+\frac{1}{dx}\,\mathrm{d}x = \frac{a}{b}\sin bx+cx+\frac{1}{d}\ln|x|+C.$

A412. The heights of our eight rectangles are

$$\begin{aligned} f(1) &= 1^2+1=2, & f(9) &= 9^2+1=82, \\ f(3) &= 3^2+1=10, & f(11) &= 11^2+1=122, \\ f(5) &= 5^2+1=26, & f(13) &= 13^2+1=170, \\ f(7) &= 7^2+1=50, & f(15) &= 15^2+1=226. \end{aligned}$$

The areas of our eight rectangles are

$$\begin{aligned} A_{81} &= 2\times f(1)=4, & A_{85} &= 2\times f(9)=164, \\ A_{82} &= 2\times f(3)=20, & A_{86} &= 2\times f(11)=244, \\ A_{83} &= 2\times f(5)=52, & A_{87} &= 2\times f(13)=340, \\ A_{84} &= 2\times f(7)=100, & A_{88} &= 2\times f(15)=452. \end{aligned}$$

Figure to be
inserted here.

The total area of our eight rectangles is

$$A_8=\sum_{i=1}^{8} A_{8i}=4+10+26+50+82+122+170+226=1\,376.$$

The approximation error is $e_3=\left|\int_0^{16} f - A_8\right| = \left|1\,381\frac{1}{3}-1\,376\right| = 5\frac{1}{3}$.
Conclude: This fourth approximation is an improvement over the previous three approximations.

A413(a) By the Constant Rule, $\int_a^b h=(b-a)\,d$ and $\int_a^b i=(b-a)\,e$.

(b) By Comparison Rule I, we have $\int_a^b h \le \int_a^b f \le \int_a^b i$.

(c) Hence, $(b-a)\,d \le \int_a^b f \le (b-a)\,e$.

A414. Following the hint, we define $h:[a,b]\to\mathbb{R}$ by $h(x)=0$. By the Constant Rule, $\int_a^b h=0$. Since $f\ge h=0$ on $[a,b]$, by Comparison Rule I, $\int_a^b f \ge \int_a^b h=0$.

A415. The Constant Factor Rule of Antidifferentiation says that if a function f has antiderivative $\int f$, then the function kf will have antiderivative $k\int f$.

The Constant Factor Rule of Integration says instead that if the area under the graph of f between a and b is S, then the area under the graph of kf between a and b is kS.

These two Rules are completely different. Once again, here we may repeat our warning about the difference between antidifferentiation and integration: *A priori*, these two Constant Factor Rules have absolutely no relationship with each other. One is concerned with antiderivatives, while the other is concerned with finding the area under a curve. That they are in fact related is established only in the next chapter by the two FTCs.

(Similar remarks also apply to the Sum and Difference Rules of Antidifferentiation vs Integration.)

153.19. Ch. 105 Answers (The Fundamental Theorems of Calculus)

A417(a) A function. Specifically, $h : [-2, 5] \to \mathbb{R}$ is the function defined by $h(x) = \int_2^x f$.

(b) $h(3) = \int_2^3 f = 6$, $h(5) = \int_2^5 f = 24$, $h(0) = \int_2^0 f = -\int_0^2 f = -6$, and $h(-2) = \int_2^{-2} f = -\int_{-2}^2 f = -8$.

(c) A real number. Specifically, $i = \int_2^3 f = h(3) = 6$.

(d) This is a trick question. Since i is a real number, "i evaluated at 3" (or 5 or 0 or −2) is meaningless.

Figure to be
inserted here.

A418(a) A function. Specifically, $g : [a, b] \to \mathbb{R}$ is the function defined by $g(x) = \int_c^x f$.

(b) The FTC1 says that $g' = f$.

(c) Hence, we have shown that f has an antiderivative, namely g.

A419(a) The FTC1 says that h is an antiderivative of f.

(b) Since g is also an antiderivative of f, by Corollary 47, there exists $C \in \mathbb{R}$ such that $h(x) = g(x) + C$ for all $x \in [a, b]$.

(c) $g(b) - g(a) = [g(b) + C] - [g(a) + C] = h(b) - h(a) = \int_a^b f - \int_a^a f = \int_a^b f - 0 = \int_a^b f$.

A423(a) $\displaystyle\int \frac{1}{4x^2-4x+1}\,dx = \int \frac{1}{(2x-1)^2}\,dx = -\frac{1}{2}\frac{1}{2x-1}+C = \frac{1}{2-4x}+C.$

(b) $\displaystyle\int \frac{1}{9x^2+30x+25}\,dx = \int \frac{1}{(3x+5)^2}\,dx = -\frac{1}{3}\frac{1}{3x+5}+C = -\frac{1}{9x+15}+C.$

A424(a) Observe that $5x^2-2x-3=(5x+3)(x-1)$. So write

$$\frac{1}{5x^2-2x-3}=\frac{A}{5x+3}+\frac{B}{x-1}=\frac{A(x-1)+B(5x+3)}{(5x+3)(x-1)}=\frac{(A+5B)x+3B-A}{5x^2-2x-3}.$$

Comparing coefficients, $A+5B\overset{1}{=}0$ and $3B-A\overset{2}{=}1$. From $\overset{1}{=}+\overset{2}{=}$, we get $8B=1$ or $B\overset{3}{=}1/8$. Plug $\overset{3}{=}$ into $\overset{1}{=}$ to get $A=-5/8$. Hence,

$$\begin{aligned}
\int \frac{1}{5x^2-2x-3}\,dx &= \int \frac{-5/8}{5x+3}+\frac{1/8}{x-1}\,dx \\
&= \int \frac{-5/8}{5x+3}\,dx+\int \frac{1/8}{x-1}\,dx && \text{(Sum Rule)}\\
&= -\frac{5}{8}\int \frac{1}{5x+3}\,dx+\frac{1}{8}\int \frac{1}{x-1}\,dx && \text{(Constant Factor Rule)}\\
&= -\frac{5}{8}\frac{1}{5}\ln|5x+3|+\frac{1}{8}\ln|x-1|+C && \text{(Reciprocal and LPC Rules)}\\
&= \frac{1}{8}\left(-\ln|5x+3|+\ln|x-1|\right)+C && \text{(Factorise)}\\
&= \frac{1}{8}\ln\frac{|x-1|}{|5x+3|}+C && \text{(Law of Logarithm)}\\
&= \frac{1}{8}\ln\left|\frac{x-1}{5x+3}\right|+C. && \text{(Fact 10)}
\end{aligned}$$

(b) Observe that $x^2-a^2=(x+a)(x-a)$. So write

$$\frac{1}{x^2-a^2}=\frac{A}{x+a}+\frac{B}{x-a}=\frac{A(x-a)+B(x+a)}{(x+a)(x-a)}=\frac{(A+B)x+(B-A)a}{x^2-a^2}.$$

Comparing coefficients, $A+B\overset{1}{=}0$ and $(B-A)a=1$ or $B-A\overset{2}{=}1/a$. From $\overset{1}{=}+\overset{2}{=}$, we get $2B=1/a$ or $B\overset{3}{=}1/(2a)$. Plug $\overset{3}{=}$ into $\overset{1}{=}$ to get $A=-1/(2a)$. Hence,

$$\begin{aligned}
\int \frac{1}{x^2-a^2}\,\mathrm{d}x &= \int \frac{-1}{2a(x+a)} + \frac{1}{2a(x-a)}\,\mathrm{d}x \\
&= \int \frac{-1}{2a(x+a)}\,\mathrm{d}x + \int \frac{1}{2a(x-a)}\,\mathrm{d}x && \text{(Sum Rule)} \\
&= -\frac{1}{2a}\int \frac{1}{x+a}\,\mathrm{d}x + \frac{1}{2a}\int \frac{1}{x-a}\,\mathrm{d}x && \text{(Constant Factor Rule)} \\
&= -\frac{1}{2a}\ln|x+a| + \frac{1}{2a}\ln|x-a| + C && \text{(Reciprocal Rule)} \\
&= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C && \text{(Factorise)} \\
&= \frac{1}{2a}\ln\frac{|x-a|}{|x+a|} + C && \text{(Law of Logarithm)} \\
&= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C. && \text{(Fact 10)}
\end{aligned}$$

(c) Instead of slaving through all the steps again, we can simply make use of **(b)**:

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = -\int \frac{1}{x^2-a^2}\,\mathrm{d}x \overset{\text{(b)}}{=} -\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| - C = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + \hat{C}.$$

Note that in the last step, we replaced "$-C$" with "$+\hat{C}$", where $\hat{C} = -C$.

A425(a) $\int \frac{7x+2}{4x^2-4x+1}\,\mathrm{d}x = \int \frac{7x+2}{(2x-1)^2}\,\mathrm{d}x = \int \frac{7}{2}\frac{2x-1}{(2x-1)^2} + \frac{11/2}{(2x-1)^2}\,\mathrm{d}x = \frac{7}{2}\int \frac{1}{2x-1}\,\mathrm{d}x + \frac{11}{2}\int \frac{1}{(2x-1)^2}\,\mathrm{d}x = \frac{7}{2}\frac{1}{2}\ln|2x-1| + \frac{11}{2}\left(-\frac{1}{2}\right)\frac{1}{2x-1} + C = \frac{7}{4}\ln|2x-1| + \frac{11}{4-8x} + C.$

(b) In Example 1358, we already showed that

$$\int \frac{1}{x^2+x-6}\,\mathrm{d}x = \int \frac{1}{(x+3)(x-2)}\,\mathrm{d}x \varnothing 1 \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + C.$$

So,

$$\int \frac{7x+2}{x^2+x-6}\,\mathrm{d}x = \int \frac{7x+2}{(x+3)(x-2)}\,\mathrm{d}x = \int 7\frac{x+3}{(x+3)(x-2)} - \frac{19}{(x+3)(x-2)}\,\mathrm{d}x = 7\int \frac{1}{x-2}\,\mathrm{d}x$$

$$\varnothing 1 7\ln|x-2| - \frac{19}{5}\ln\left|\frac{x-2}{x+3}\right| + C = \frac{1}{5}\left(16\ln|x-2| + 19\ln|x+3|\right) + C.$$

Note that the last step is nice but not necessary (you should however be perfectly capable of doing this bit of algebra).

(c) Below, $\varnothing 2$ uses our answer from Exercise 424(a).

$$\int \frac{7x+2}{5x^2-2x-3}\,\mathrm{d}x = \int \frac{7x+2}{(5x+3)(x-1)}\,\mathrm{d}x = \int 7\frac{x-1}{(5x+3)(x-1)} + \frac{9}{(5x+3)(x-1)}\,\mathrm{d}x = 7\int \frac{}{5x}$$

$$\varnothing 2\frac{7}{5}\ln|5x+3| + \frac{9}{8}\ln\left|\frac{x-1}{5x+3}\right| + C = \frac{11}{40}\ln|5x+3| + \frac{9}{8}\ln|x-1| + C.$$

Ditto the last sentence in **(b)**.

A426(e) If $\sec x > 0$, then $\frac{\mathrm{d}}{\mathrm{d}x}\ln|\sec x| = \frac{\mathrm{d}}{\mathrm{d}x}\ln\sec x = \frac{\sec x \tan x}{\sec x} = \tan x$.

If $\sec x < 0$, then $\frac{\mathrm{d}}{\mathrm{d}x}\ln|\sec x| = \frac{\mathrm{d}}{\mathrm{d}x}\ln(-\sec x) = \frac{-\sec x \tan x}{-\sec x} = \tan x$.

(Note: The requirement that x is not an odd multiple of $\pi/2$ ensures that $\sec x$ and $\tan x$ are defined.)

(f) If $\sin x > 0$, then $\frac{\mathrm{d}}{\mathrm{d}x}\ln|\sin x| = \frac{\mathrm{d}}{\mathrm{d}x}\ln\sin x = \frac{\cos x}{\sin x} = \cot x$.

If $\sin x < 0$, then $\frac{\mathrm{d}}{\mathrm{d}x}\ln|\sin x| = \frac{\mathrm{d}}{\mathrm{d}x}\ln(-\sin x) = \frac{-\cos x}{-\sin x} = \cot x$.

(Note: The requirement that x not a multiple of π ensures that $\sin x \neq 0$ and $\ln|\sin x|$ and $\cot x$ are defined.)

(g) If $\operatorname{cosec} x + \cot x > 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\ln|\operatorname{cosec} x + \cot x|) = \frac{\mathrm{d}}{\mathrm{d}x}(-\ln|\operatorname{cosec} x + \cot x|) = -\frac{-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x}$$

$$= \frac{\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x} = \frac{\operatorname{cosec} x(\cot x + \operatorname{cosec} x)}{\operatorname{cosec} x + \cot x} = \operatorname{cosec} x.$$

If $\operatorname{cosec} x + \cot x < 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\ln|\operatorname{cosec} x + \cot x|) = \frac{\mathrm{d}}{\mathrm{d}x}[-\ln(-\operatorname{cosec} x - \cot x)] = -\frac{\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{-\operatorname{cosec} x - \cot x}$$

$$= \frac{\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x} = \frac{\operatorname{cosec} x(\cot x + \operatorname{cosec} x)}{\operatorname{cosec} x + \cot x} = \operatorname{cosec} x.$$

(Note: The requirement that x not a multiple of π ensures that $\sin x \neq 0$ and $\operatorname{cosec} x + \cot x \neq 0$; and that $\operatorname{cosec} x$, $\cot x$, and $\ln|\operatorname{cosec} x + \cot x|$ are defined.)

(h) If $\sec x + \tan x > 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln|\sec x + \tan x| = \frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x + \tan x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x.$$

If $\sec x + \tan x < 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln|\sec x+\tan x| = \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(-\sec x-\tan x\right) = \frac{-\sec x\tan x-\sec^2 x}{-\mathrm{cosec}x-\cot x}$$

$$= \frac{\sec x\left(\tan x+\sec x\right)}{\sec x+\tan x} = \sec x.$$

(Note: The requirement that x not an odd multiple of $\pi/2$ ensures that $\cos x \neq 0$ and $\sec x + \tan x \neq 0$; and that $\sec x$, $\tan x$, and $\ln|\sec x + \tan x|$ are defined)

A428(a) Complete the square:

$$-3x^2+x+6 = 3\left[-x^2+\frac{1}{3}x+2\right] = 3\left[\frac{73}{36}-\left(x-\frac{1}{6}\right)^2\right] = 3\left[\left(\frac{\sqrt{73}}{6}\right)^2-\left(x-\frac{1}{6}\right)^2\right].$$

So,
$$\int\frac{1}{\sqrt{-3x^2+x+6}}\,\mathrm{d}x = \int\frac{1}{\sqrt{3}\sqrt{\left(\frac{\sqrt{73}}{6}\right)^2-\left(x-\frac{1}{6}\right)^2}}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\frac{x-1/6}{\sqrt{73}/6}+C$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\frac{6x-1}{\sqrt{73}}+C.$$

(b) Complete the square:

$$-7x^2-x+2 = 7\left[-x^2+\frac{1}{7}x+\frac{2}{7}\right] = 7\left[\frac{57}{196}-\left(x+\frac{1}{14}\right)^2\right] = 7\left[\left(\frac{\sqrt{57}}{14}\right)^2-\left(x+\frac{1}{14}\right)^2\right].$$

So,
$$\int\frac{1}{\sqrt{-7x^2-x+2}}\,\mathrm{d}x = \int\frac{1}{\sqrt{7}\sqrt{\left(\frac{\sqrt{57}}{14}\right)^2-\left(x+\frac{1}{14}\right)^2}}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{7}}\sin^{-1}\frac{x+1/14}{\sqrt{57}/14}+C$$

$$= \frac{1}{\sqrt{7}}\sin^{-1}\frac{14x+1}{\sqrt{57}}+C.$$

A429(b) Use the identity $\cos 2x = 2\cos^2 x - 1$:

$$\int\cos^2 x\,\mathrm{d}x = \int\frac{1+\cos 2x}{2}\,\mathrm{d}x = \frac{1}{2}x+\frac{\sin 2x}{4}+C.$$

(c) Use **(i)** the identity $\tan^2 x = \sec^2 x - 1$; and **(ii)** $\frac{\mathrm{d}}{\mathrm{d}x}\tan = \sec^2$ or $\int\sec^2 = \tan$:

$$\int\tan^2 x\,\mathrm{d}x = \int\sec^2 x - 1\,\mathrm{d}x = \tan x - x + C.$$

(e) As stated on List MF26 (p. 3),

$$\cos P - \cos Q \varnothing 1 - 2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$

So, let $$mx = \frac{P+Q}{2} \qquad \text{and} \qquad nx = \frac{P-Q}{2}.$$

Hence, $$P \varnothing 2\,(m+n)\,x \qquad \text{and} \qquad Q \varnothing 3\,(m-n)\,x.$$

Plug in $\varnothing 1$, $\varnothing 2$, and $\varnothing 3$:

$$\begin{aligned}\int \sin mx \sin nx \, dx &= \int -\frac{1}{2}\left[\cos\left(m+n\right)x - \cos\left(m-n\right)x\right] dx \\ &= \frac{1}{2}\left[\frac{\sin\left(m-n\right)x}{m-n} - \frac{\sin\left(m+n\right)x}{m+n}\right] + C.\end{aligned}$$

(f) As stated on List MF26 (p. 3),

$$\cos P + \cos Q \varnothing 12 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}.$$

So, let $$mx = \frac{P+Q}{2} \qquad \text{and} \qquad nx = \frac{P-Q}{2}.$$

Hence, $$P \varnothing 2\,(m+n)\,x \qquad \text{and} \qquad Q \varnothing 3\,(m-n)\,x.$$

Plug in $\varnothing 1$, $\varnothing 2$, and $\varnothing 3$:

$$\begin{aligned}\int \cos mx \cos nx \, dx &= \int \frac{1}{2}\left[\cos\left(m+n\right)x + \cos\left(m-n\right)x\right] dx \\ &= \frac{1}{2}\left[\frac{\sin\left(m-n\right)x}{m-n} + \frac{\sin\left(m+n\right)x}{m+n}\right] + C.\end{aligned}$$

A430(a) By dETAIL, we should choose $v' = e^x$ (and $u = x^3$), so that $v = e^x$ (and $u' = 3x^2$):

$$\int x^3 e^x \, dx \stackrel{1}{=} x^3 e^x - \int 3x^2 e^x \, dx.$$

From Example 1371, $\int x^2 e^x \, dx \stackrel{2}{=} e^x \left(x^2 - 2x + 2\right)$. Plug $\stackrel{2}{=}$ into $\stackrel{1}{=}$ to get

$$\int x^3 e^x \, dx = x^3 e^x - 3e^x \left(x^2 - 2x + 2\right) + C = e^x \left(x^3 - 3x^2 + 6x - 6\right) + C.$$

(b) We'll use IBP twice.

By dETAIL, we should first choose $v_1' = \sin x$ (and $u_1 = x^2$), so that $v_1 = -\cos x$ (and $u_1' = 2x$):

$$\int x^2 \sin x \, dx = x^2 \left(-\cos x\right) - \int 2x \cdot \left(-\cos x\right) \, dx \stackrel{1}{=} -x^2 \cos x + 2 \int x \cos x \, dx.$$

Now find $\int x \cos x$ by applying IBP a second time, this time choosing $v_2' = \cos x$ (and $u_2 = x$), so that $v_2 = \sin x$ (and $u_2' = 1$):

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx \stackrel{2}{=} x \sin x + \cos x.$$

Plugging $\varnothing 2$ into $\varnothing 1$, we get

$$\int x^2 \sin x \, dx \stackrel{1}{=} -x^2 \cos x + 2 \int x \cos x \, dx = -x^2 \cos x + 2 (x \sin x + \cos x) + C.$$

A431. The error is in Step 5—we cannot simply subtract $\int \frac{1}{x} \, dx$ from both sides. The LHS "$\int \frac{1}{x} \, dx$" may differ from the RHS "$\int \frac{1}{x} \, dx$" by up to a constant (because antiderivatives may differ by up to a constant). Indeed, in this case, they differ by 1.

See the more detailed discussions in Remark 156 and Ch. 103.6.

A432. This time, Step 5 is perfectly legitimate. Any definite integral, such as $\int_1^2 \frac{1}{x} \, dx$, is simply a number. It is therefore perfectly legitimate to subtract $\int_1^2 \frac{1}{x} \, dx$ from both sides of an equation.

This time, the error is in the last equation of Step 3:

$$\text{“}\left[1 + \int \frac{1}{x} \, dx\right]_1^2 = 1 + \int_1^2 \frac{1}{x} \, dx.\text{”} \qquad ✗$$

The above equation is false. We should instead have

$$\left[1 + \int \frac{1}{x} \, dx\right]_1^2 = [1]_1^2 + \left[\int \frac{1}{x} \, dx\right]_1^2 = 1\Big|_{x=2} - 1\Big|_{x=1} + \int_1^2 \frac{1}{x} \, dx = 1 - 1 + \int_1^2 \frac{1}{x} \, dx = \int_1^2 \frac{1}{x} \, dx.$$

153.21. Ch. 106 Answers (The Substitution Rule)

A437(a) Observe that $\underset{\cdot}{\mathrm{x}}\left(x^3-\cos x\right)=3x^2+\sin x$. So,

$$\int \frac{3x^2+\sin x}{x^3-\cos x}\,\mathrm{d}x=\ln\left|x^3-\cos x\right|+C.$$

(b) Observe that $\underset{\cdot}{\mathrm{x}}\left(\sin^2 x+1\right)=2\sin x\cos x=\sin 2x$. So,

$$\int \frac{\sin 2x}{\sin^2 x+1}\,\mathrm{d}x=\ln\left|\sin^2 x+1\right|+C=\ln\left(\sin^2 x+1\right)+C.$$

(In the very last step, we can remove the absolute value operator because $\sin^2 x+1\geq 0$.)

(c) Observe that $\underset{\cdot}{\mathrm{x}}\left(5x^2+\sin x\right)=10x+\cos x$. So,

$$\int \frac{10x+\cos x}{5x^2+\sin x}\,\mathrm{d}x=\ln\left|5x^2+\sin x\right|+C.$$

(d) Observe that $\underset{\cdot}{\mathrm{x}}\left(x^3+x^2+x+1\right)=3x^2+2x+1$. So,

$$\int \frac{3x^2+2x+1}{x^3+x^2+x+1}\,\mathrm{d}x=\ln\left|x^3+x^2+x+1\right|+C.$$

(e) Observe that $\cot x=\dfrac{\cos x}{\sin x}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x=\cos x$. So,

$$\int \cot x\,\mathrm{d}x=\int \frac{\cos x}{\sin x}\,\mathrm{d}x=\ln|\sin x|+C.$$

(f) $\displaystyle\int \sec x\,\mathrm{d}x=\int \sec x\frac{\sec x+\tan x}{\sec x+\tan x}\,\mathrm{d}x=\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\,\mathrm{d}x.$

Observe that $\underset{\cdot}{\mathrm{x}}\left(\sec x+\tan x\right)=\sec^2 x+\sec x\tan x$. So,

$$\int \sec x\,\mathrm{d}x=\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\,\mathrm{d}x=\ln|\sec x+\tan x|+C.$$

A440. As in the above example, we'll use the **(1)** Times One Trick; **(2)** IBP; and the **(3)** Substitution Rule:

(a) $\displaystyle\int \sin^{-1}x\,\mathrm{d}x\stackrel{1}{=}\int 1\cdot\sin^{-1}x\,\mathrm{d}x\stackrel{2}{=}x\sin^{-1}x-\int x\cdot\frac{1}{\sqrt{1-x^2}}\stackrel{3}{=}x\sin^{-1}x+\sqrt{1-x^2}+C.$

(b) $\displaystyle\int \cos^{-1}x\,\mathrm{d}x\stackrel{1}{=}\int 1\cdot\cos^{-1}x\,\mathrm{d}x\stackrel{2}{=}x\cos^{-1}x-\int x\cdot\frac{-1}{\sqrt{1-x^2}}\stackrel{3}{=}x\cos^{-1}x-\sqrt{1-x^2}+C.$

A441(a) Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{\mathrm{d}x}{\mathrm{d}u}\stackrel{1}{=}\sec u\tan u$.

2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,\mathrm{d}x \overset{\text{a}}{=} \int \frac{1}{\sec^2 u\sqrt{\sec^2 u - 1}}\,\mathrm{d}x$

$$= \int \frac{1}{\sec^2 u\sqrt{\tan^2 u}}\,\mathrm{d}x$$

$$= \int \frac{1}{\sec^2 u\,|\tan u|}\,\mathrm{d}x \qquad \left(\sqrt{\alpha^2} = |\alpha|\right)$$

$$= \int \frac{1}{\sec^2 u\tan u}\,\mathrm{d}x \qquad u \in \left(0, \frac{\pi}{2}\right) \implies \tan u \geq 0$$

$$\overset{1}{=} \int \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}u}\sec u}\,\mathrm{d}x$$

$$= \int \frac{1}{\sec u}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int \cos u\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

3. $\displaystyle= \int \cos u\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = \int \cos u\,\mathrm{d}u$

4. $= \sin u + C_1$

5. $\overset{\text{a}}{=} \sin\left(\sec^{-1} x\right) + C_1.$

(b) Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} \overset{2}{=} \dfrac{2}{x^3}$ or $\dfrac{\mathrm{d}x}{\mathrm{d}u} \overset{2}{=} \dfrac{x^3}{2}$.

2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,\mathrm{d}x = \int \frac{1}{x^2\sqrt{x^2-1}}\frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

$$\overset{2}{=} \int \frac{1}{x^2\sqrt{x^2-1}}\frac{x^3}{2}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{x}{2\sqrt{x^2-1}}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

$$\overset{\text{b}}{=} \int \frac{1}{2\sqrt{u}}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

3. $\displaystyle= \int \frac{1}{2\sqrt{u}}\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = \int \frac{1}{2\sqrt{u}}\,\mathrm{d}u$

4. $= \sqrt{u} + C_2$

5. $\displaystyle\overset{\text{b}}{=} \frac{\sqrt{x^2-1}}{x} + C_2.$

(c) By Corollary 47, $\sin\left(\sec^{-1} x\right)$ and $\dfrac{\sqrt{x^2-1}}{x}$ for $x \in (1, \infty)$ may differ by only a constant:

$$\sin\left(\sec^{-1} x\right) = \frac{\sqrt{x^2-1}}{x} + C_3, \qquad \text{for some } C_3 \in \mathbb{R}.$$

Following the Hint, plug in $x = 2$ to get

$$\sin\left(\sec^{-1} 2\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad \frac{\sqrt{2^2-1}}{2} + C_3 = \frac{\sqrt{3}}{2} + C_3.$$

Hence, $C_3 = 0$ and $\sin\left(\sec^{-1} x\right) = \dfrac{\sqrt{x^2-1}}{x}$.

A442(a) Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} \stackrel{1}{=} 2x$.

$$\begin{aligned}
&2. \quad \int \frac{x^3}{\left(x^2+1\right)^{3/2}}\,\mathrm{d}x \stackrel{\text{a}}{=} \int \frac{x^3}{u^{3/2}}\,\mathrm{d}x \stackrel{1}{=} \frac{1}{2}\int \frac{x^2}{u^{3/2}}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \stackrel{\text{a}}{=} \frac{1}{2}\int \frac{u-1}{u^{3/2}}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \\
&3. \quad = \frac{1}{2}\int \frac{u-1}{u^{3/2}}\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = \frac{1}{2}\int \frac{u-1}{u^{3/2}}\,\mathrm{d}u = \frac{1}{2}\int \frac{1}{u^{1/2}} - \frac{1}{u^{3/2}}\,\mathrm{d}u \\
&4. \quad = \sqrt{u} + \frac{1}{\sqrt{u}} + C_1 \\
&5. \quad \stackrel{\text{a}}{=} \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}} + C_1.
\end{aligned}$$

(b) Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{\mathrm{d}x}{\mathrm{d}u} \stackrel{2}{=} \sec^2 u$.

$$\begin{aligned}
2. \quad \int \frac{x^3}{\left(x^2+1\right)^{3/2}}\,\mathrm{d}x &\stackrel{\text{b}}{=} \int \frac{\tan^3 u}{\left(\tan^2 u+1\right)^{3/2}}\,\mathrm{d}x \\
&= \int \frac{\tan^3 u}{\left(\sec^2 u\right)^{3/2}}\,\mathrm{d}x = \int \frac{\tan^3 u}{\sec^3 u}\,\mathrm{d}x \\
&\stackrel{1}{=} \int \frac{\tan^3 u}{\sec u}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \\
3. \quad &= \int \frac{\tan^3 u}{\sec u}\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = \int \frac{\tan^3 u}{\sec u}\,\mathrm{d}u \\
&= \int \frac{\sin^3 u}{\cos^2 u}\,\mathrm{d}u = \int \frac{\sin^2 u}{\cos^2 u}\sin u\,\mathrm{d}u \\
&= \int \frac{1-\cos^2 u}{\cos^2 u}\sin u\,\mathrm{d}u = \int \frac{\sin u}{\cos^2 u} - \sin u\,\mathrm{d}u \\
4. \quad &= \frac{1}{\cos u} + \cos u + C_2 \\
5. \quad &\stackrel{\text{b}}{=} \frac{1}{\cos\left(\tan^{-1} x\right)} + \cos\left(\tan^{-1} x\right) + C_2.
\end{aligned}$$

(c) By Corollary 47, $\dfrac{1}{\sqrt{x^2+1}} + \sqrt{x^2+1}$ and $\dfrac{1}{\cos\left(\tan^{-1} x\right)} + \cos\left(\tan^{-1} x\right)$ may differ by only a constant:

$$\frac{1}{\sqrt{x^2+1}} + \sqrt{x^2+1} = \frac{1}{\cos\left(\tan^{-1} x\right)} + \cos\left(\tan^{-1} x\right) + C_3, \quad \text{for some } C_3 \in \mathbb{R}.$$

Following Hint 1, plug in $x = 0$ to get

$$\frac{1}{\sqrt{0^2+1}} + \sqrt{0^2+1} = 2 \qquad \text{and}$$
$$\frac{1}{\cos\left(\tan^{-1} 0\right)} + \cos\left(\tan^{-1} 0\right) + C_3 = \frac{1}{\cos 0} + \cos 0 + C_3 = 2 + C_3.$$

Hence, $C_3 = 0$ and $\frac{1}{\sqrt{x^2+1}} + \sqrt{x^2+1} = \frac{1}{\cos\left(\tan^{-1} x\right)} + \cos\left(\tan^{-1} x\right)$ (for all $x \in \mathbb{R}$).

153.22. Ch. 109 Answers (More Definite Integrals)

A444. The requested area is labelled A in the following figure:

Figure to be inserted here.

Method 1. $A + B + C + D = 2^{1/3} \times 2 = 2^{4/3}$.

$B + C = 1 \times 1 = 1$.

$D = \int_1^{2^{1/3}} x^3 \, \mathrm{d}x = \left[\frac{x^4}{4}\right]_1^{2^{1/3}} = \frac{2^{4/3} - 1}{4}$. Hence,

$$A = 2^{4/3} - \left(1 + \frac{2^{4/3} - 1}{4}\right) = \frac{3}{4}\left(2^{4/3} - 1\right).$$

Method 2. $y = x^3 \iff x = y^{1/3}$. So,

$$A = \int_{y=1}^{y=2} x \, \mathrm{d}y = \int_1^2 y^{1/3} \, \mathrm{d}y = \frac{3}{4}\left[y^{4/3}\right]_1^2 = \frac{3}{4}\left(2^{4/3} - 1\right).$$

A445. The points at which the curve and line intersect are given by

$$\sin x = \frac{1}{2}, \text{ for } x \in (0, \pi) \qquad \iff \qquad x = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Hence, the requested area is

$$\int_{\pi/6}^{5\pi/6} \sin x - \frac{1}{2} \, \mathrm{d}x = \left[-\cos x - \frac{x}{2}\right]_{\pi/6}^{5\pi/6}$$

$$= \left[-\left(-\frac{\sqrt{3}}{2}\right) - \frac{5\pi}{12}\right] - \left[-\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{12}\right]$$

$$= \sqrt{3} - \frac{\pi}{3}.$$

Figure to be inserted here.

A446. The two curves intersect at

$$2-x^2=x^2+1 \qquad \Longleftrightarrow \qquad 1-2x^2=0 \qquad \Longleftrightarrow \qquad x=\pm\frac{1}{\sqrt{2}}=\pm 2^{-1/2}$$

So, the requested area is

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} 2-x^2-\left(x^2+1\right)\,\mathrm{d}x=\left[x-\frac{2}{3}x^3\right]_{-1/\sqrt{2}}^{1/\sqrt{2}}=\left[\frac{1}{\sqrt{2}}-\frac{2}{3}\frac{1}{2\sqrt{2}}\right]-\left[\frac{1}{-\sqrt{2}}-\frac{2}{3}\frac{1}{-2\sqrt{2}}\right]=\frac{4}{3\sqrt{2}}.$$

Figure to be inserted here.

A447. The curve intersects the x-axis at

$$x^4-16=0 \qquad \Longleftrightarrow \qquad x=\pm 2.$$

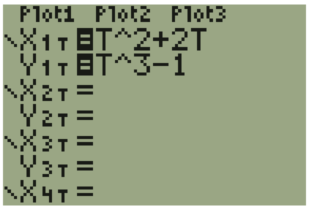
Compute $$\int_{-2}^{2} x^4-16\,\mathrm{d}x=\left[\frac{x^5}{5}-16x\right]_{-2}^{2}=\left[\frac{32}{5}-32\right]-\left[\frac{-32}{5}+32\right]=-\frac{256}{5}.$$

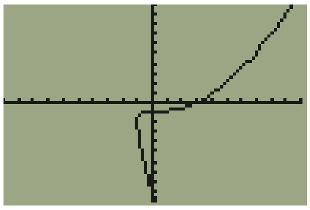
Hence the requested area is $256/5$.

Figure to be inserted here.

A448. The line $y=7$ intersects C at $7=t^3-1$ or $8=t^3$ or $t=2$.
Similarly, the line $y=26$ intersects C at $26=t^3-1$ or $27=t^3$ or $t=3$.
So the requested area is

$$
\begin{aligned}
\int_7^{26} x\,\mathrm{d}y &= \int_7^{26} t^2 + 2t\,\mathrm{d}y \\
&= \int_7^{26} \left(t^2 + 2t\right) \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}y}\,\mathrm{d}y && \text{(Times One Trick)} \\
&= \int_7^{26} \left(t^2 + 2t\right) 3t^2 \frac{\mathrm{d}t}{\mathrm{d}y}\,\mathrm{d}y && \left(\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2\right) \\
&= \int_{y=7}^{y=26} \left(t^2 + 2t\right) 3t^2 \frac{\mathrm{d}t}{\cancel{\mathrm{d}y}}\,\cancel{\mathrm{d}y} && \text{(Substitution Rule)} \\
&= \int_{t=2}^{t=3} \left(t^2 + 2t\right) 3t^2\,\mathrm{d}t && \text{(Change of limits)} \\
&= \int_2^3 3t^4 + 6t^3\,\mathrm{d}t \\
&= \left[\frac{3}{5}t^5 + \frac{3}{2}t^4\right]_2^3 = \left[\frac{729}{5} + \frac{343}{2}\right] - \left[\frac{96}{5} + \frac{48}{2}\right] = \frac{633}{5} + \frac{295}{2} = 274.1.
\end{aligned}
$$





A449. Use Fact 17(a):[660]

$$\int_0^{\pi} \pi y^2\,\mathrm{d}x = \int_0^{\pi} \pi \sin^2 x\,\mathrm{d}x = \pi\left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{\pi^2}{2}.$$

A450(a) $\int u^2 \cos u\,\mathrm{d}u = u^2 \sin u - \int 2u \sin u\,\mathrm{d}u = u^2 \sin u - 2\left[-u\cos u - \int -\cos u\,\mathrm{d}u\right] = u^2 \sin u + 2u\cos u - 2\sin u + C$.

(b) Five-Step Substitution Rule Recipe:

1. Compute $\dfrac{\mathrm{d}y}{\mathrm{d}u} \overset{1}{=} \cos u$.

[660] Again, this is not on List MF26, so this is something you'll have to either mug or know how to derive during the exam.

2. $\int \left(\sin^{-1} y\right)^2 \, dy = \int u^2 \, dy$

$$= \int u^2 \frac{dy}{du}\frac{du}{dy} \, dy \qquad \text{(Ti}$$

$$\overset{1}{=} \int u^2 \cos u \frac{du}{dy} \, dy$$

3. $= \int u^2 \cos u \frac{du}{\cancel{dy}} \cancel{dy} = \int u^2 \cos u \, du$

4. $\overset{\textbf{(a)}}{=} u^2 \sin u + 2u \cos u - 2 \sin u + C$

5. $= \left(\sin^{-1} y\right)^2 \sin\left(\sin^{-1} y\right) + 2\left(\sin^{-1} y\right)\cos\left(\sin^{-1} y\right) - 2\sin\left(\sin^{-1} y\right) + C$

$$= \left(\sin^{-1} y\right)^2 y + 2\left(\sin^{-1} y\right)\cos\left(\sin^{-1} y\right) - 2y + C$$

$$= y\left(\sin^{-1} y\right)^2 + 2\sqrt{1-y^2}\sin^{-1} y - 2y + C \qquad \text{(Le}$$

(c)

$$\int_0^1 \pi x^2 \, dy = \int_0^1 \pi \left(\sin^{-1} y\right)^2 \, dy \overset{\textbf{(b)}}{=} \pi\left[y\left(\sin^{-1} y\right)^2 + 2\sqrt{1-y^2}\sin^{-1} y - 2y\right]_0^1 = \pi\left(\frac{\pi^2}{4} - 2\right).$$

153.23. Ch. 110 Answers (Differential Equations)

A451. Use Integration by Parts (IBP) twice:

$$y = \int \exp x \sin x \, dx = \exp x \sin x - \int \exp x \cos x \, dx$$
$$= \exp x \sin x - \left(\exp x \cos x - \int \exp x \, (-\sin x) \, dx\right) = \exp x \sin x + \exp x \cos x - \int \exp x \sin x \, dx.$$

Rearranging, the general solution is $y = \exp x \dfrac{\sin x - \cos x}{2} + C$.

Plug in the initial condition to get $1 = 1 \cdot \dfrac{0-1}{2} + C$ or $C = \dfrac{3}{2}$. Hence, the particular solution is

$$y = \frac{\exp x \, (\sin x - \cos x) + 3}{2}.$$

A452. By the Inverse Function Theorem (Ch. 91.2),

$$\frac{dx}{dy} \overset{1}{=} \frac{1}{\exp y} = \exp(-y).$$

So, the general solution is

$$x \overset{2}{=} \int \exp(-y) \, dy = -\exp(-y) + C.$$

Rearranging, we find that the general solution to $\overset{1}{=}$ is also

$$\exp y = \frac{1}{C - x} \qquad \text{or} \qquad y \overset{2}{=} \ln \frac{1}{C - x} = -\ln(C - x).$$

We are now given this initial condition:

$$(x, y) \overset{3}{=} (0, 0).$$

Plugging $\overset{3}{=}$ into $\overset{2}{=}$, we get $C = 1$. Hence, the particular solution is

$$x \overset{2}{=} 1 - \exp(-y) \qquad \text{or} \qquad y \overset{2}{=} -\ln(1 - x).$$

A453. Use Integration by Parts (IBP) twice (we actually already did this in Exercise 451):

$$\frac{dy}{dx} = \int \exp x \sin x \, dx = \exp x \sin x - \int \exp x \cos x \, dx$$
$$= \exp x \sin x - \left(\exp x \cos x - \int \exp x \, (-\sin x) \, dx\right) = \exp x \sin x + \exp x \cos x - \int \exp x \sin x \, dx$$

Rearranging, $\dfrac{dy}{dx} = \displaystyle\int \exp x \sin x \, dx \overset{1}{=} \exp x \dfrac{\sin x - \cos x}{2} + C$. Next,

$$y = \int \exp x \frac{\sin x - \cos x}{2} + C \, dx = \int \exp x \frac{\sin x}{2} \, dx + \int \exp x \frac{-\cos x}{2} \, dx + \int C \, dx$$
$$\stackrel{1}{=} \exp x \frac{\sin x - \cos x}{4} - \frac{1}{2} \int \exp x \cos x \, dx + Cx. \qquad ☺$$

To find $\int \exp x \cos x \, dx$, again use IBP (but this time only once):

$$\int \exp x \cos x \, dx = \exp x \cos x - \int \exp x \, (-\sin x) \, dx \stackrel{1}{=} \exp x \cos x + \exp x \frac{\sin x - \cos x}{2} + \bar{C}$$
$$\stackrel{2}{=} \exp x \frac{\sin x + \cos x}{2} + \bar{C}.$$

Now plug $\stackrel{2}{=}$ back into ☺ to get our general solution

$$y \stackrel{3}{=} \exp x \frac{\sin x - \cos x}{4} - \exp x \frac{\sin x + \cos x}{4} - \frac{1}{2}\bar{C} + Cx = -\frac{\exp x \cos x}{2} + Cx + \hat{C}.$$

Plug the initial condition $(x, y) = (0, 1)$ into $\stackrel{3}{=}$ to get $1 = -\frac{1 \cdot 1}{2} + 0 + \hat{C}$ or $\hat{C} \stackrel{4}{=} \frac{3}{2}$.

Plug $\stackrel{4}{=}$ and the initial condition $(x, y) = \left(\frac{\pi}{2}, 0\right)$ into $\stackrel{3}{=}$ to get $0 = 0 + C\frac{\pi}{2} + \frac{3}{2}$ or $C = -\frac{3}{\pi}$.

Hence, the particular solution is $y = -\frac{\exp x \cos x}{2} + -\frac{3}{\pi}x + \frac{3}{2}$.

A454(a) $\mathbf{F} = \frac{GMm}{r^2}$.

(b)(i) $\mathbf{F} = \frac{d}{dt}(m\mathbf{v})$.[661]

(b)(ii) If m is constant, then $\frac{dm}{dt} = 0$ and

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \stackrel{\times}{=} \mathbf{v}\frac{dm}{dt} + m\frac{d\mathbf{v}}{dt} = 0 + m\mathbf{a} = m\mathbf{a}.$$

(c) By assumption, the sole force acting on the object is the Earth's gravity. Hence, the ball's rate of change of momentum *is* the force of gravity. That is,

$$m\mathbf{a} = -\frac{GMm}{r^2} \qquad \text{or} \qquad \mathbf{a} = -\frac{GM}{r^2}.$$

The negative sign in front of the force of gravity $\frac{GMm}{r2}$ is there because we've adopted the convention[662] that upwards is the positive direction—gravity acts downwards and hence in the negative direction.

(d)(i) $\int \frac{d\mathbf{v}}{dt} dr = \int \frac{dr}{dt} d\mathbf{v} = \int \mathbf{v} \, d\mathbf{v} = \frac{1}{2}\mathbf{v}^2 + C$.

(d)(ii) $\int -\frac{GM}{r^2} dr = \frac{GM}{r} + C$.

[661]We actually also did this in Exercise 188.

[662]We adopted this convention when we said that the object is shot upward with initial velocity $V > 0$.

(d)(iii) We were given the initial condition $(r, \mathbf{v}) = (R, V)$.

(d)(iv) Combining **(i)** and **(ii)**, $\frac{1}{2}\mathbf{v}^2 = \frac{GM}{r} + C$.

Plug in **(iii)** to get $\frac{1}{2}V^2 = \frac{GM}{R} + C$ or $C = \frac{1}{2}V^2 - \frac{GM}{R}$.

Hence, $\frac{1}{2}\mathbf{v}^2 = \frac{GM}{r} + \frac{1}{2}V^2 - \frac{GM}{R}$ or $\mathbf{v} = \sqrt{\frac{2GM}{r} + V^2 - \frac{2GM}{R}}$.

(e)(i) At the object's maximum height H, $\mathbf{v} = 0$.

(e)(ii) Plug **(i)** and $r = R + H$ into what we found in **(d)(iv)** to get

$$0 = \sqrt{\frac{2GM}{R+H} + V^2 - \frac{2GM}{R}} \qquad \text{or} \qquad \frac{2GM}{R+H} + V^2 - \frac{2GM}{R} = 0 \qquad \text{or}$$
$$H = \frac{1}{\frac{1}{R} - \frac{V^2}{2GM}} - R = \frac{2RGM}{2GM - RV^2} - R = \frac{RV^2}{2GM - RV^2}.$$

(f)(i) If the object keeps travelling upward forever, then it must be that $\mathbf{v} > 0$ for all r.

(f)(ii) $\sqrt{\frac{2GM}{r} + V^2 - \frac{2GM}{R}} > 0$ for all $r \iff \frac{2GM}{r} + V^2 - \frac{2GM}{R} > 0$ for all $r \implies V^2 - \frac{2GM}{R} > 0 \iff V > \sqrt{\frac{2GM}{R}}$.

Hence, the escape velocity (or minimum initial velocity for the object to travel upwards forever) is

$$V_e = \sqrt{\frac{2GM}{R}}.$$

(f)(iii) $V_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2\left(6.67 \times 10^{-11}\right)\left(5.97 \times 10^{24}\right)}{6.37 \times 10^{6}}} \approx 11\,200\,\mathrm{m\,s^{-1}}$.

153.24. Ch. 111 Answers (Revisiting Logarithms)

A455. Differentiate both sides with respect to x to get

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\frac{1}{x} = x\left(-\frac{1}{x^2}\right) = -\frac{1}{x},$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(-\ln x) = -\frac{1}{x}.$$

By Corollary 47, $\ln\frac{1}{x}$ and $-\ln x$ differ by at most a constant:

$$\ln\frac{1}{x} = -\ln x + C, \qquad \text{for some } C \in \mathbb{R}.$$

Plug in $x = 1$ to get $\ln 1 = -\ln 1 + C$ or $C = 0$. Thus, $\ln\frac{1}{x} = -\ln x$.

A456(e) $c\log_b a \overset{\star}{=} \frac{c\ln a}{\ln b} = \frac{\ln a^c}{\ln b} \overset{\star}{=} \log_b a^c$. (The second step uses Fact 223.)

(f) $\log_b\frac{1}{a} \overset{\star}{=} \frac{\ln(1/a)}{\ln b} = -\frac{\ln a}{\ln b} \overset{\star}{=} -\log_b a$. (The second step uses Fact 222(c).)

(g) $\log_b(ac) \overset{\star}{=} \frac{\ln(ac)}{\ln b} = \frac{\ln a + \ln c}{\ln b} = \frac{\ln a}{\ln b} + \frac{\ln c}{\ln b} \overset{\star}{=} \log_b a + \log_b c$. (The second step uses Fact 222(b).)

(h) is immediate from **(f)** and **(g)**. But if we'd like, we can also prove this without using **(f)** and **(g)**:

$$\log_b\frac{a}{c} \overset{\star}{=} \frac{\ln(a/c)}{\ln b} = \frac{\ln a - \ln c}{\ln b} = \frac{\ln a}{\ln b} - \frac{\ln c}{\ln b} \overset{\star}{=} \log_b a - \log_b c.$$

(The second step uses Fact 222(d).)

(i) $\frac{\log_c b}{\log_c a} \overset{\star}{=} \frac{\ln a/\ln c}{\ln b/\ln c} = \frac{\ln a}{\ln b} \overset{\star}{=} \log_b a$.

(j) $\log_{a^b} c \overset{\star}{=} \frac{\ln c}{\ln a^b} = \frac{\ln c}{b\ln a} \overset{\star}{=} \frac{1}{b}\log_a c$. (The second step uses Fact 223.)

154. Part VI Answers (Probability and Statistics)

154.1. Ch. 112 Answers (How to Count: Four Principles)

A457. Taking the green path, there are 3 ways. Taking the red path, there are 2 ways. Hence, there are $3 + 2 = 5$ ways to get from the Starting Point to the River.

A458. The tree diagram below illustrates.

Case #1. First letter is a D.

Case #1(i). Second letter is a D. Then the last two letters must both be E's. (1 permutation.)

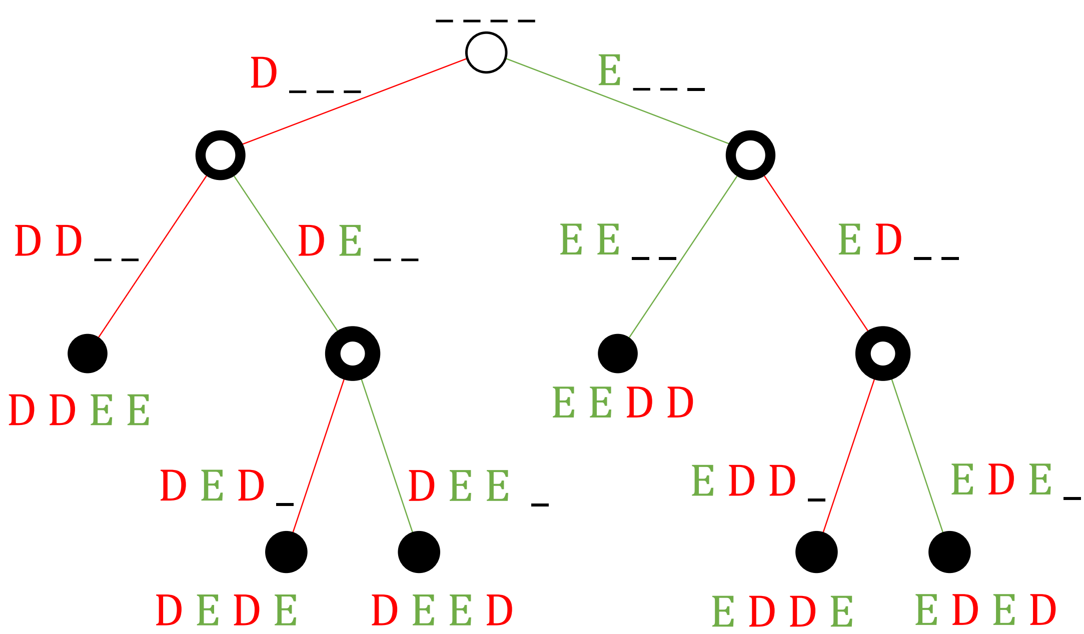
Case #1(ii). Second letter is an E. Then the last two letters must be either DE or ED. (2 permutations.)

Case #2. First letter is a E.

Case #2(i). Second letter is an E. Then the last two letters must both be D's. (1 permutation.)

Case #2(ii). Second letter is a D. Then the last two letters must be either DE or ED. (2 permutations.)

Altogether then, there are $1 + 2 + 1 + 2 = 6$ possible permutations of the letters in DEED.



A459. $3 \times 5 \times 10 = 150$.

A460. We must choose three 4D numbers. Choosing the first 4D number involves four decisions—what to put as the first, second, third and fourth digits, with the condition that no digit is repeated.

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}$$

Thus, by the MP, there are $10 \times 9 \times 8 \times 7 = 5040$ ways to choose the first 4D number.

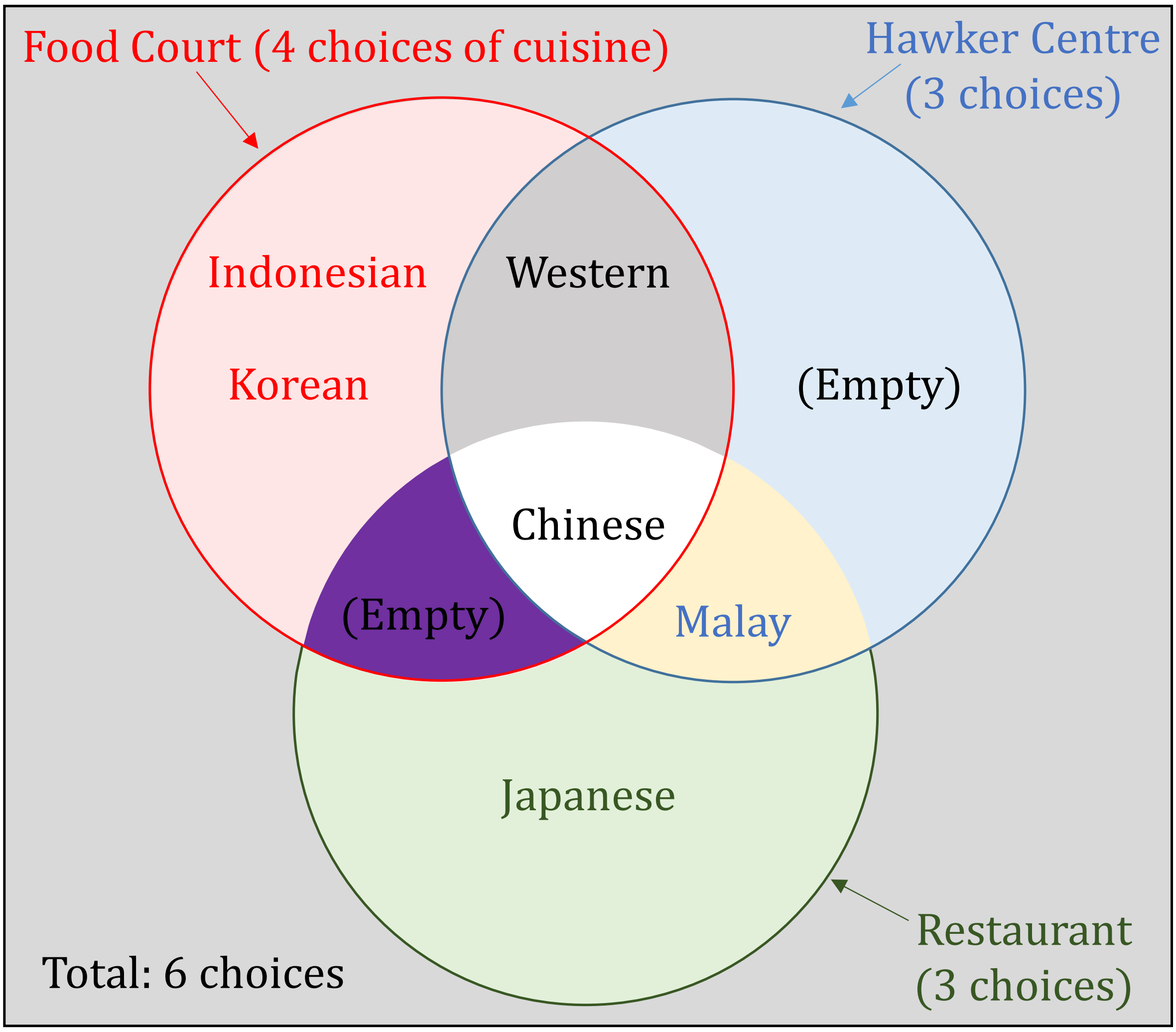
If we ignored the fact that we already chose the first 4D number, then there'd similarly be 5040 ways to choose the second 4D number (given the condition that this second 4D number does not have any repeated digits). However, there is an additional condition—namely, the second 4D number cannot be the same as the first. Thus, there are $5040 - 1 = 5039$ ways to choose the second 4D number.

By similar reasoning, we see that there are $5040 - 2 = 5038$ ways to choose the third 4D number.

Altogether then, by the MP, there are $5040 \times 5039 \times 5038 = 127,947,869,280$ ways to choose the three 4D numbers.

A461. Apply the IEP twice.

1. The food court and hawker centre share 2 types of cuisine (Chinese and Western) in common. And so together, the food court and the hawker centre have $4 + 3 - 2 = 5$ different types of cuisine.
2. Combine together the food court and the hawker centre (call this the "Low-Class Place"). The Low-Class Place has 5 types of cuisine and shares 2 types of cuisine (Chinese and Malay) with the restaurant. And so together, the Low-Class Place and restaurant have $5 + 3 - 2 = 6$ different types of cuisine (namely Chinese, Indonesian, Japanese, Korean, Malay, and Western).



A462. $10 - 3 = 7$. (Can you name them?)

154.2. Ch. 113 Answers (How to Count: Permutations)

A463. $6! = 720$, $7! = 5040$, and $8! = 40320$.

A464. $7!/(4!3!) = 35$.

A465. $9!$.

A466. The problem of choosing a president and vice-president from a committee of 11 members is equivalent to the problem of filling 2 spaces with 11 distinct objects. The answer is thus $P(11,2) = 11!/9! = 11 \times 10 = 110$.

A467. Let B and S stand for brother and sister, respectively.

(a) First consider the problem of permuting the seven letters in $BBBBSSS$, without any two B's next to each other. There is only 1 possible arrangement, namely $BSBSBSB$.

There are 4! ways to permute the brothers and 3! ways to permute the sisters.

Hence, there are in total $1 \times 4!3! = 144$ possible ways to arrange the siblings in a line, so that no two brothers are next to each other.

(b) First consider the problem of permuting the seven letters in $BBBBSSS$, without any two S's next to each other. We'll use the AP.

1. B in position #1.
   (a) B in position #2. Then the only way to fill the remaining five positions is $SBSBS$. **Total: 1 possible arrangement.**
   (b) S in position #2. Then we must have B in position #3.
      i. B in position #4. Then the only way to fill the remaining three positions is SBS. **Total: 1 possible arrangement.**
      ii. S in position #4. Then we must have B in position #5. And there are two ways to fill the remaining two positions: either BS or SB. **Total: 2 possible arrangements.**

(... A467 continued on the next page ...)

2. S in position #1. Then we must have B in position #2.

   (a) B in position #3. Then, like in 1(b), we are left with two B's and two S's to fill the remaining four positions. Hence, **Total: 3 possible arrangements.**

   (b) S in position #3. Then we must have B in position #4. There are three ways to fill the remaining three positions: SBB, BSB, and BBS. **Total: 3 possible arrangements.**

By the AP, there are $1+1+2+3+3 = 10$ possible arrangements.

Again, there are 4! ways to permute the brothers and 3! ways to permute the sisters.

Hence, there are in total $10 \times 4!3! = 1440$ possible ways to arrange the siblings in a line, so that no two sisters are next to each other.

(c) We saw that there was only 1 possible (linear) permutation of $BBBBSSS$ that satisfied the restriction, namely $BSBSBSB$.

If we now arrange the siblings in a circle, there will necessarily be two brothers next to each other.

We thus conclude: There are 0 possible ways to arrange the siblings in a circle so that no two brothers are next to each other.

(d) In part (b), we found 10 possible (linear) permutations of $BBBBSSS$ that satisfied the restriction.

Of these, 3 have sisters at the two ends: $SBSBBBS$, $SBBSBBS$, and $SBBBSBS$. If arranged in a circle, these 3 arrangements would involve two sisters next to each other. So we must deduct these 3 arrangements.

We are left with 7 possible arrangements: $BBSBSBS$, $SBBSBSB$, $BSBBSBS$, $SBSBBSB$, $BSBSBBS$, $SBSBSBB$, and $BSBSBSB$. But of course, these are simply one and the same fixed circular permutation! (This is consistent with Fact 228, which tells us to simply divide by 7.)

And now again, we must now take into account the fact that the brothers are distinct and the sisters are distinct. We conclude that there are in total $1 \times 4!3! = 144$ possible ways to arrange the siblings in a circle, so that no two sisters are next to each other.

154.3. Ch. 114 Answers (How to Count: Combinations)

A468.

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n \times (n-1) \times \cdots \times (n-k+1) \times (n-k) \times (n-k-1) \times \cdots \times 1}{k!\,(n-k) \times (n-k-1) \times \cdots \times 1}$$

$$= \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)}{k!} \qquad \text{(mass cancellation)}.$$

A469.

$$C(4,2) = \frac{4!}{2!\,(4-2)!} = \frac{4!}{2!2!} = \frac{4 \times 3}{2 \times 1} = 6,$$

$$C(6,4) = \frac{6!}{4!\,(6-4)!} = \frac{6!}{4!2!} = \frac{6 \times 5}{2 \times 1} = 15,$$

$$C(7,3) = \frac{7!}{3!\,(7-3)!} = \frac{7!}{3!4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35.$$

A470. $\begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 630.$

A471. **(a)** $C(1,0) + C(1,1) = 1 + 1 = 2 = C(2,1)$.
(b) $C(4,2) + C(4,3) = 3 + 3 = 6 = C(5,3)$.

$$(c) \quad C(17,2) + C(17,3) = \frac{17!}{2!15!} + \frac{17!}{3!14!} = \frac{17 \times 16}{2 \times 1} + \frac{17 \times 16 \times 15}{3 \times 2 \times 1}$$

$$= 17 \times 8 + 17 \times 8 \times 5 = 17 \times 8 \times 6 = \frac{18 \times 17 \times 16}{3 \times 2 \times 1}.$$

A472. $\binom{7}{0} = 1$, $\binom{7}{1} = 7$, $\binom{7}{2} = 21$, $\binom{7}{3} = 35$, $\binom{7}{4} = 35$, $\binom{7}{5} = 21$, $\binom{7}{6} = 7$, $\binom{7}{7} = 1$.

A473. Expanding, we have

$$\begin{aligned}(1+x) &= (1+x)(1+x)(1+x) \\ &= \underbrace{1\cdot 1\cdot 1}_{0\ x'\text{S}} + \underbrace{1\cdot 1\cdot x + 1\cdot x\cdot 1 + x\cdot 1\cdot 1}_{1\ x} + \underbrace{1\cdot x\cdot x + x\cdot 1\cdot x + x\cdot x\cdot 1}_{2\ x'\text{S}} + \underbrace{x\cdot x\cdot x}_{3\ x'\text{S}}.\end{aligned}$$

Consider the 6 terms on the right. There is $C(3,0) = 1$ way to choose 0 of the x's. Hence, the coefficient on x^0 is $C(3,0)$—this corresponds to the term $1\cdot 1\cdot 1$ above.

There are $C(3,1) = 3$ ways to choose 1 of the x's. Hence, the coefficient on x^1 is $C(3,1)$—this corresponds to the terms $1\cdot 1\cdot x$, $1\cdot x\cdot 1$, and $x\cdot 1\cdot 1$ above.

There are $C(3,2) = 3$ ways to choose 2 of the x's. Hence, the coefficient on x^2 is $C(3,2)$—this corresponds to the terms $1\cdot x\cdot x$, $x\cdot 1\cdot x$, and $x\cdot x\cdot 1$ above.

There is $C(3,03) = 1$ way to choose 3 of the x's. Hence, the coefficient on x^3 is $C(3,3)$—this corresponds to the term $x\cdot x\cdot x$ above.

Altogether then,

$$(1+x) = \binom{3}{0}x^0 + \binom{3}{1}x^1 + \binom{3}{2}x^2 + \binom{3}{3}x^3 = 1 + 3x + 3x^2 + x^3.$$

A474. $2^7 = 128$.

$$\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \cdots + \binom{7}{7} = 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128.$$

So indeed, $2^7 = \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \cdots + \binom{7}{7}$.

A475.

$$(3+x)^4 = \binom{4}{0}3^4x^0 + \binom{4}{1}3^3x^1 + \binom{4}{2}3^2x^2 + \binom{4}{3}3^1x^3 + \binom{4}{4}3^4x^4$$

$$= 81 + 4\cdot 27x + 6\cdot 9x^2 + 4\cdot 3x^3 + x^4 = 81 + 108x + 54x^2 + 12x^3 + x^4.$$

A476. **(a)** There are $\binom{4}{2} = 4$ ways of choosing the two Tan sons and $\binom{3}{2} = 3$ ways of choosing the two Wong daughters.

Having chosen these sons and daughters, there are only $2! = 2\times 1$ possible ways of matching them up. This is because for the first chosen Tan Son, we have 2 possible choices of brides for him. And then for the second chosen Tan Son, there is only 1 possible choice of bride left for him.

Altogether then, there are $\binom{4}{2}\binom{3}{2}\cdot 2 = 24$ ways of forming the two couples.

(b) There are $\binom{6}{5} = 6$ ways of choosing the five Lee sons and $\binom{9}{5} = 126$ ways of choosing the five Ho daughters.

Having chosen these sons and daughters, there are $5! = 5\times 4\times 3\times 2\times 1$ possible ways of matching them up. This is because for the first chosen Tan Son, we have 5 possible choices of brides for him. And then for the second chosen Tan Son, there are 4 possible choices of brides left for him. Etc.

Altogether then, there are $\binom{6}{5}\binom{9}{5}\cdot 5! = 6\cdot 126\cdot 5! = 90,720$ ways of forming the five couples.

154.4. Ch. 115 Answers (Probability: Introduction)

A477(a). (i) The appropriate sample space is

$$S = \{A\spadesuit, K\spadesuit, Q\spadesuit, \dots, 2\spadesuit, A\heartsuit, K\heartsuit, Q\heartsuit, \dots, 2\heartsuit, A\diamondsuit, K\diamondsuit, Q\diamondsuit, \dots, 2\diamondsuit, A\clubsuit, K\clubsuit, Q\clubsuit, \dots, 2\clubsuit\}.$$

(a) (ii) Since there are 52 possible outcomes, there are 2^{52} possible events. Hence, the event space contains 2^{52} elements. It is too tedious to write this out explicitly.

(a) (iii) As always, P has domain Σ and $\mathbb{R}$. We have $P(\{3\diamondsuit\}) = P(\{5\clubsuit\}) = 1/52$ and $P(\{3\diamondsuit, 5\clubsuit\}) = 2/52$. In general, given any event $A \in \Sigma$, we have

$$P(A) = \frac{|A|}{|S|} = \frac{|A|}{52}.$$

In words, given any event A, its probability $P(A)$ is simply the number of elements it contains, divided by 52. So for example, $P(\{3\diamondsuit, 5\clubsuit, A\spadesuit\}) = 3/52$, as we would expect.

(a) (iv) John might argue that since packs of poker cards usually come with Jokers, there is the possibility that we mistakenly included one or more Jokers in our deck of cards. He might thus argue that to cover this possibility, we should set our sample space to be

$$S = \{A\spadesuit, K\spadesuit, , \dots, 2\spadesuit, A\heartsuit, K\heartsuit, \dots, 2\heartsuit, A\diamondsuit, K\diamondsuit, \dots, 2\diamondsuit, A\clubsuit, K\clubsuit, \dots, 2\clubsuit, \text{Joker}\}.$$

The event space would be appropriately adjusted to contain 2^{53} elements.

The mapping rule of the probability function would be appropriately adjusted, based on John's belief of the probability of selecting a Joker. For example, if he reckons that the probability of selecting a Joker is $1/10,000$, then he might assign $P(\{\text{Joker}\}) = 1/10,000$ and for any other card C, $P(\{C\}) = 9999/(10000 \cdot 52)$. The probability of any other event $A \in \Sigma$ is as given by the Additivity Axiom.

A477(b). (i) The appropriate sample space is $S = \{HH, HT, TH, TT\}$.

(b) (ii) Since there are 4 possible outcomes, there are $2^4 = 16$ possible events. Hence, the event space contains 16 elements. It is not too tedious to write these out explicitly:

$$\Sigma = \Big\{\varnothing, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\},$$
$$\{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\},$$
$$\{HH, HT, TT\}, \{HH, TH, TT\}, \{HT, TH, TT\}, S\Big\}.$$

(b) (iii) As always, P has domain Σ and $\mathbb{R}$. We have $\mathrm{P}(\{HH\}) = \mathrm{P}(\{HT\}) = 1/4$ and $\mathrm{P}(\{HT, HT, TH\}) = 3/4$. In general, given any event $A \in \Sigma$, we have

$$\mathrm{P}(A) = \frac{|A|}{|S|} = \frac{|A|}{4}.$$

In words, given any event A, its probability $\mathrm{P}(A)$ is simply the number of elements it contains, divided by 4. So for example, $\mathrm{P}(\{TH, TT\}) = 2/4$, as we would expect.

(b) (iv) John might, as before, argue that there is the possibility that a coin lands on its edge. He might thus argue that the sample space should be

$$S = \{HH, HT, HX, TH, TT, TX, XH, XT, XX\}.$$

The event space would be appropriately adjusted to contain $2^9 = 512$ elements.

The mapping rule of the probability function would be appropriately adjusted. For example, if John believes that any given coin flip has probability 1/6000 of landing on its edge, then we might assign $\mathrm{P}(\{XX\}) = 1/6000^2$, $\mathrm{P}(\{HH\}) = (5999/6000)^2$, $\mathrm{P}(\{XH\}) = (1/6000) \cdot (5999/6000)$, etc.

A477(c). (i) The appropriate sample space contains 36 outcomes:

$$S=\left\{\begin{matrix}\text{⚀}\\\text{⚀}\end{matrix},\begin{matrix}\text{⚀}\\\text{⚁}\end{matrix},\dots,\begin{matrix}\text{⚁}\\\text{⚀}\end{matrix},\begin{matrix}\text{⚁}\\\text{⚁}\end{matrix},\dots,\begin{matrix}\text{⚅}\\\text{⚀}\end{matrix},\begin{matrix}\text{⚅}\\\text{⚁}\end{matrix},\dots,\begin{matrix}\text{⚅}\\\text{⚅}\end{matrix}\right\}.$$

(c) (ii) Since there are 36 possible outcomes, there are 2^{36} possible events. Hence, the event space contains 2^{36} elements.

(c) (iii) As always, P has domain Σ and $\mathbb{R}$. We have $\mathrm{P}\left(\left\{\begin{matrix}\text{⚀}\\\text{⚀}\end{matrix}\right\}\right)=\mathrm{P}\left(\left\{\begin{matrix}\text{⚄}\\\text{⚂}\end{matrix}\right\}\right)=\dfrac{1}{36}$ and $\mathrm{P}\left(\left\{\begin{matrix}\text{⚀}\\\text{⚀}\end{matrix},\begin{matrix}\text{⚄}\\\text{⚂}\end{matrix}\right\}\right)=\dfrac{2}{36}$. In general, given any event $A\in\Sigma$, we have

$$\mathrm{P}(A)=\frac{|A|}{|S|}=\frac{|A|}{36}.$$

In words, given any event A, its probability $\mathrm{P}(A)$ is simply the number of elements it contains, divided by 52. So for example, $\mathrm{P}\left(\left\{\begin{matrix}\text{⚀}\\\text{⚀}\end{matrix},\begin{matrix}\text{⚄}\\\text{⚂}\end{matrix},\begin{matrix}\text{⚁}\\\text{⚅}\end{matrix},\begin{matrix}\text{⚃}\\\text{⚃}\end{matrix}\right\}\right)=\dfrac{4}{36}$, as we would expect.

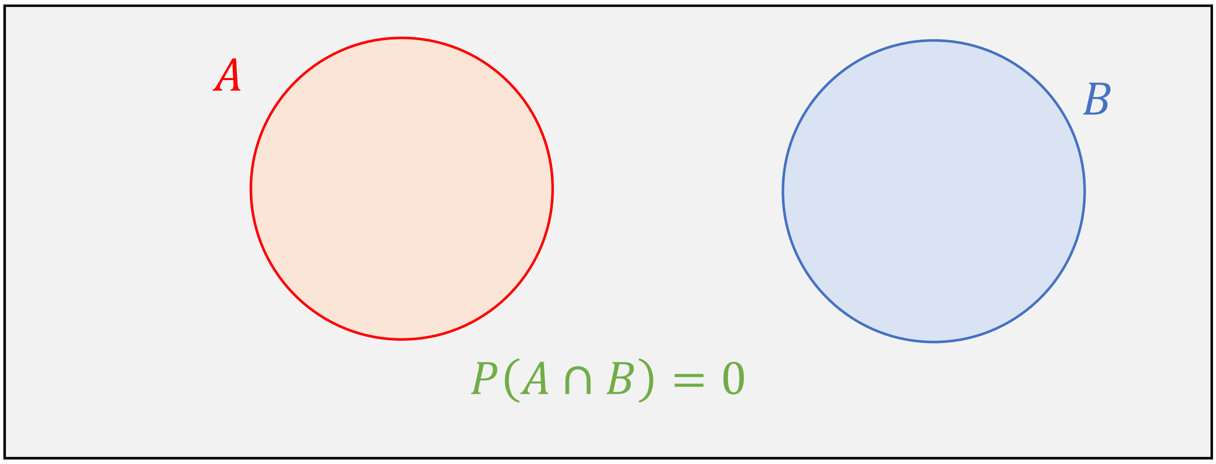
(c) (iv) John might argue that there is the possibility that a die lands on a vertex. He might thus argue that the sample space contains $7^2=49$ outcomes and should be

$$S=\left\{\begin{matrix}\text{⚀}\\\text{⚀}\end{matrix},\begin{matrix}\text{⚀}\\\text{⚁}\end{matrix},\dots,\begin{matrix}\text{⚀}\\V\end{matrix},\begin{matrix}\text{⚁}\\\text{⚀}\end{matrix},\begin{matrix}\text{⚁}\\\text{⚁}\end{matrix},\dots,\begin{matrix}\text{⚁}\\V\end{matrix},\begin{matrix}V\\\text{⚀}\end{matrix},\begin{matrix}V\\\text{⚁}\end{matrix},\dots,\begin{matrix}V\\V\end{matrix}\right\}.$$

The event space would be appropriately adjusted to contain 2^{49} elements.

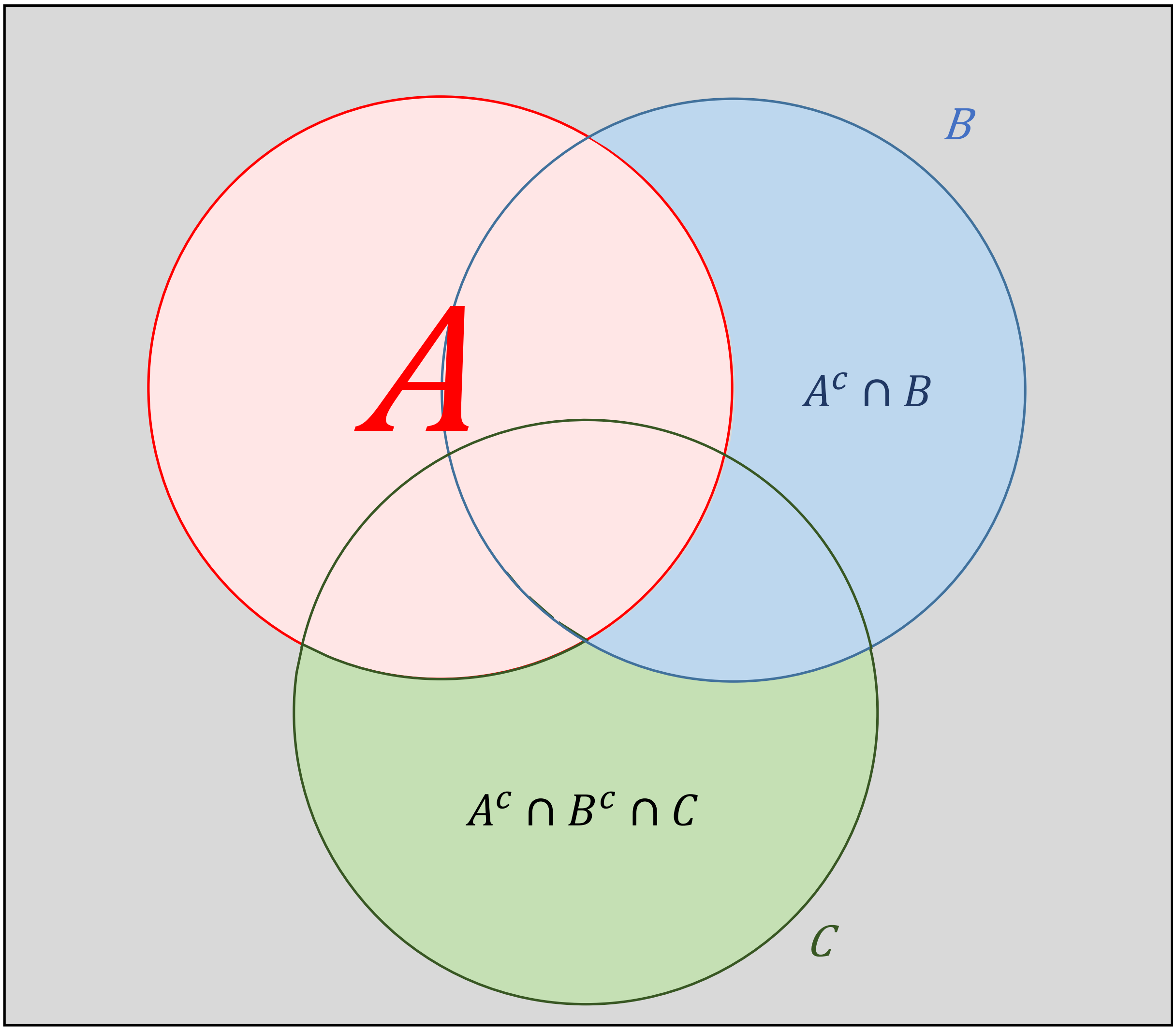
The mapping rule of the probability function would be appropriately adjusted. For example, if John believes that any given die roll has probability 1/1000000 of landing on a vertex, then we might assign $\mathrm{P}\left(\left\{\begin{matrix}V\\V\end{matrix}\right\}\right)=\dfrac{1}{1000000^2}$, $\mathrm{P}\left(\left\{\begin{matrix}\text{⚀}\\\text{⚀}\end{matrix}\right\}\right)=\left(\dfrac{999999}{1000000}\right)^2$, etc.

A478. **(a)** By definition, A and B are mutually exclusive if $A \cap B = \varnothing$. Since $\mathrm{P}(\varnothing) = 0$, the result follows.



(b) The events A, $A^c \cap B$, and $A^c \cap B^c \cap C$ are mutually exclusive. Moreover, their union is $A \cup B \cup C$. Hence, by the Additivity Axiom (applied twice),

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}\left(A^c \cap B\right) + \mathrm{P}\left(A^c \cap B^c \cap C\right).$$



154.5. Ch. 116 Answers (Conditional Probability)

A479. Let A be the event that we rolled at least one even number and B be the event that the sum of the two dice was 8. We have $\mathrm{P}(B) = 5/36$ (see Exercise 485).

And $A \cap B$ can occur if and only if the two dice were ⚁ ⚅, ⚃ ⚃, or ⚅ ⚁. Hence, $\mathrm{P}(A \cap B) = 3/36$.

Altogether then,

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{3/36}{5/36} = \frac{3}{5}.$$

A480. It may be true that

$$\mathrm{P}\left(\text{DNA match}|\text{Blood stain is not John Brown}'\text{s}\right) = \frac{1}{10,000,000}.$$

It does not however follow, except by the CPF, that

$$\mathrm{P}\left(\text{Blood stain is not John Brown}'\text{s}|\text{DNA match}\right) = \frac{1}{10,000,000}.$$

There is reason to believe that $\mathrm{P}\left(\text{Blood stain is not John Brown}'\text{s}\right)$ is much greater than $\mathrm{P}\left(\text{DNA match}\right)$ and thus that $\mathrm{P}\left(\text{Blood stain is not John Brown}'\text{s}|\text{DNA match}\right)$ is much greater than $\mathrm{P}\left(\text{DNA match}|\text{Blood stain is not John Brown}'\text{s}\right)$.

One important factor is that if the DNA database is large, then invariably we'd expect to find, purely by coincidence, a DNA match to the blood stain at the murder scene. As of May 2016, the US National DNA Index contains over the DNA profiles of over 12.3 *million* individuals. And so, even if it were true that there is only probability $\dfrac{1}{10,000,000}$ that two random individuals have a DNA match, we'd expect to find a match, simply by combing through the entire US National DNA Index!

The error here is similar to the lottery example, where we conclude (erroneously) that a lottery winner must have cheated, simply because it was so unlikely that she won.

154.6. Ch. 117 Answers (Probability: Independence)

A481. By Fact 234, A, B are independent events $\iff$ $\mathrm{P}(A|B) = \mathrm{P}(A)$. Rearranging, $\mathrm{P}(B) = \mathrm{P}(A \cap B)/\mathrm{P}(A) = \mathrm{P}(B|A)$, as desired.

A482. First, note that $\mathrm{P}(H_1) = \mathrm{P}(T_1) = \mathrm{P}(H_2) = 0.5$.

(a) $\mathrm{P}(H_1 \cap H_2) = 0.25 = 0.5 \times 0.5 = \mathrm{P}(H_1)\mathrm{P}(H_2)$, so that indeed H_1 and H_2 are independent.

(b) $\mathrm{P}(H_2 \cap T_1) = 0.25 = 0.5 \times 0.5 = \mathrm{P}(H_2)\mathrm{P}(T_1)$, so that indeed H_2 and T_1 are independent.

(c) Observe that $H_1 \cap T_1 = \varnothing$ (it is impossible that "the first coin flip is heads" AND also "the first coin flip is tails").

Hence, $\mathrm{P}(H_1 \cap T_1) = \mathrm{P}(\varnothing) = 0 \neq 0.25 = 0.5 \times 0.5 = \mathrm{P}(H_1)\mathrm{P}(T_1)$, so that indeed H_1 and T_1 are **not** independent.

A483. No, the journalist is incorrectly assuming that the probability of one family member making the NBA is independent of another family member making the NBA. But such an assumption is almost certainly false.

The same excellent genes that made Rick Barry a great basketball player, probably also helped his three sons. Not to mention that having an NBA player as your father probably helps a lot too.

The two events "family member #1 in NBA" and "family member #2 in NBA" are probably not independent. So we cannot simply multiply probabilities together.

A484. First, note that $\mathrm{P}(H_1) = \mathrm{P}(T_2) = \mathrm{P}(x) = 0.5$.

(a) $\mathrm{P}(H_1 \cap T_2) = 0.25 = 0.5 \times 0.5 = \mathrm{P}(H_1)\mathrm{P}(T_2)$, so that indeed H_1, T_2 are independent.

$\mathrm{P}(H_1 \cap X) = 0.25 = 0.5 \times 0.5 = \mathrm{P}(H_1)\mathrm{P}(X)$, so that indeed H_1, X are independent.

$\mathrm{P}(T_2 \cap X) = 0.25 = 0.5 \times 0.5 = \mathrm{P}(T_2)\mathrm{P}(X)$, so that indeed T_2, X are independent.

Altogether then, H_1, T_2, and X are indeed pairwise independent.

(b) The event $H_1 \cap T_2 \cap X$ is the same as the event $H_1 \cap T_2$. Thus, $\mathrm{P}(H_1 \cap T_2 \cap X) = \mathrm{P}(H_1 \cap T_2) = 0.25 \neq 0.5 \times 0.5 \times 0.5 = \mathrm{P}(H_1)\mathrm{P}(T_2)\mathrm{P}(x)$, so that indeed the three events are **not** independent.

154.7. Ch. 119 Answers (Random Variables: Introduction)

A485(a)

k	s such that $X(s) = k$	$\mathrm{P}(X = k)$
2	⚀⚀.	$\frac{1}{36}$
3	⚀⚁, ⚁⚀.	$\frac{2}{36}$
4	⚀⚂, ⚁⚁, ⚂⚀.	$\frac{3}{36}$
5	⚀⚃, ⚁⚂, ⚂⚁, ⚃⚀.	$\frac{4}{36}$
6	⚀⚄, ⚁⚃, ⚂⚂, ⚃⚁, ⚄⚀.	$\frac{5}{36}$
7	⚀⚅, ⚁⚄, ⚂⚃, ⚃⚂, ⚄⚁, ⚅⚀.	$\frac{6}{36}$
8	⚁⚅, ⚂⚄, ⚃⚃, ⚄⚂, ⚅⚁.	$\frac{5}{36}$
9	⚂⚅, ⚃⚄, ⚄⚃, ⚅⚂.	$\frac{4}{36}$
10	⚃⚅, ⚄⚄, ⚅⚃.	$\frac{3}{36}$
11	⚄⚅, ⚅⚄.	$\frac{2}{36}$
12	⚅⚅.	$\frac{1}{36}$

(b) E is the event $X \geq 10$.

(c) $\mathrm{P}(E) = \mathrm{P}(X \geq 10) = \mathrm{P}(X = 10) + \mathrm{P}(X = 11) + \mathrm{P}(X = 12) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$.

A486. $P\begin{pmatrix} \epsdice{2} \\ \epsdice{5} \end{pmatrix} = 5$ and $P\begin{pmatrix} \epsdice{4} \\ \epsdice{1} \end{pmatrix} = 4$.

$Q\begin{pmatrix} \epsdice{2} \\ \epsdice{5} \end{pmatrix} = 3$ and $Q\begin{pmatrix} \epsdice{4} \\ \epsdice{1} \end{pmatrix} = 3$.

$(PQ)\begin{pmatrix} \epsdice{2} \\ \epsdice{5} \end{pmatrix} = 15$ and $(PQ)\begin{pmatrix} \epsdice{4} \\ \epsdice{1} \end{pmatrix} = 12$.

A487. If $S = \{1,2,3,4,5,6\}$, then $X : S \to \mathbb{R}$ defined by $X(s) = s$ is of course a random variable. A random variable is simply any function with domain S and codomain $\mathbb{R}$; and X certainly meets these requirements.

$\mathrm{P}(X = 1) = \mathrm{P}(X = 2) = \mathrm{P}(X = 3) = \mathrm{P}(X = 4) = \mathrm{P}(X = 5) = \mathrm{P}(X = 6) = 1/6$ and $\mathrm{P}(X = K) = 0$ for any $k \neq 1,2,3,4,5,6$.

A488(a). **(i)** The sample space is

$$S = \Big\{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT,$$
$$HTTH, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT\Big\}.$$

The event space Σ is the set of all possible subsets of S and contains 2^{16} elements.

The probability function $\mathrm{P} : \Sigma \to \mathbb{R}$ is defined by $\mathrm{P}(A) = |A|/16$, for any event $A \in \Sigma$.

(a) (ii) The random variable $X : S \to \mathbb{R}$ is the function defined by

$$\begin{aligned} HTTT, THTT, TTHT, TTTH &\mapsto 1, \\ HHTT, HTHT, THHT, HTTH, THTH, TTHH &\mapsto 2, \\ HHHT, HHTH, HTHH, THHH &\mapsto 3, \\ TTTT \mapsto 0, \qquad HHHH &\mapsto 4. \end{aligned}$$

(a)**(iii)** $\mathrm{P}(X = 4) = 1/16$, $\mathrm{P}(X = 3) = 4/16$, $\mathrm{P}(X = 2) = 6/16$,
$\mathrm{P}(X = 1) = 4/16$, $\mathrm{P}(X = 0) = 1/16$, $\mathrm{P}(X = k) = 0$, for any $k \neq 0,1,2,3,4$.

A488(b). **(i)** The sample space S consists of 216 outcomes:

$$S = \left\{ \begin{matrix} ⚀ \\ ⚀ \\ ⚀ \end{matrix}, \begin{matrix} ⚀ \\ ⚀ \\ ⚁ \end{matrix}, \dots, \begin{matrix} ⚀ \\ ⚅ \\ ⚅ \end{matrix}, \begin{matrix} ⚁ \\ ⚀ \\ ⚀ \end{matrix}, \begin{matrix} ⚁ \\ ⚀ \\ ⚁ \end{matrix}, \dots, \begin{matrix} ⚁ \\ ⚅ \\ ⚅ \end{matrix}, \begin{matrix} ⚅ \\ ⚀ \\ ⚀ \end{matrix}, \begin{matrix} ⚅ \\ ⚀ \\ ⚁ \end{matrix}, \dots, \begin{matrix} ⚅ \\ ⚅ \\ ⚅ \end{matrix} \right\}$$

The event space Σ is the set of all possible subsets of S and contains 2^{216} elements.

The probability function $\mathrm{P} : \Sigma \to \mathbb{R}$ is defined by $\mathrm{P}(A) = |A|/216$, for any event $A \in \Sigma$.

(b) **(ii)** The range of X is $\{3, 4, 5, \dots, 18\}$. We now count the number of ways there are for the three dice to reach a sum of 3, to reach a sum of 4, etc. This will enable us to write down the mapping rule of the function $X : S \to \mathbb{R}$.

To get a sum of 3, the three dice must be ⚀⚀⚀ or permutations thereof. There is thus $\dfrac{3!}{3!} = 1$ possibility.

To get a sum of 4, the three dice must be ⚀⚀⚁, or permutations thereof. There are thus $\dfrac{3!}{2!} = 3$ possibilities.

To get a sum of 5, the three dice must be ⚀⚀⚂, ⚀⚁⚁, or permutations thereof. There are thus $\dfrac{3!}{2!} + \dfrac{3!}{2!} = 6$ possibilities.

To get a sum of 6, the three dice must be ⚀⚀⚃, ⚀⚁⚂, ⚁⚁⚁, or permutations thereof. There are $\dfrac{3!}{2!} + 3! + \dfrac{3!}{3!} = 10$ such possibilities.

To get a sum of 7, the three dice must be ⚀⚀⚄, ⚀⚁⚃, ⚀⚂⚂, ⚁⚁⚂, or permutations thereof. There are $\dfrac{3!}{2!} + 3! + \dfrac{3!}{2!} + \dfrac{3!}{2!} = 15$ such possibilities.

To get a sum of 8, the three dice must be ⚀⚀⚅, ⚀⚁⚄, ⚀⚂⚃, ⚁⚁⚃, ⚁⚂⚂, or permutations thereof. There are $\dfrac{3!}{2!} + 3! + 3! + \dfrac{3!}{2!} + \dfrac{3!}{2!} = 21$ such possibilities.

To get a sum of 9, the three dice must be ⚀⚁⚅, ⚀⚂⚄, ⚀⚃⚃, ⚁⚁⚄, ⚁⚂⚃, ⚂⚂⚂, or permutations thereof. There are $3! + 3! + \dfrac{3!}{2!} + \dfrac{3!}{2!} + 3! + \dfrac{3!}{3!} = 25$ such possibilities.

To get a sum of 10, the three dice must be ⚀⚂⚅, ⚀⚃⚄, ⚁⚁⚅, ⚁⚂⚄, ⚁⚃⚃, ⚂⚂⚃, or permutations thereof. There are $3! + 3! + \dfrac{3!}{2!} + 3! + \dfrac{3!}{2!} + \dfrac{3!}{2!} = 27$ such possibilities.

By symmetry, there are also 27 ways to get a sum of 11; also 25 ways to get a sum of 12, etc.

(... Answer continued on the next page ...)

(... Answer continued from the previous page ...)

So $X : S \to \mathbb{R}$ is defined by

$$X\begin{pmatrix} \text{⚀} \\ \text{⚀} \\ \text{⚀} \end{pmatrix} = 3,$$

$$X\begin{pmatrix} \text{⚀} \\ \text{⚀} \\ \text{⚁} \end{pmatrix} = X\begin{pmatrix} \text{⚀} \\ \text{⚁} \\ \text{⚀} \end{pmatrix} = X\begin{pmatrix} \text{⚁} \\ \text{⚀} \\ \text{⚀} \end{pmatrix} = 4,$$

$$X\begin{pmatrix} \text{⚀} \\ \text{⚀} \\ \text{⚂} \end{pmatrix} = X\begin{pmatrix} \text{⚀} \\ \text{⚂} \\ \text{⚀} \end{pmatrix} = X\begin{pmatrix} \text{⚂} \\ \text{⚀} \\ \text{⚀} \end{pmatrix} = X\begin{pmatrix} \text{⚀} \\ \text{⚁} \\ \text{⚁} \end{pmatrix} = X\begin{pmatrix} \text{⚁} \\ \text{⚀} \\ \text{⚁} \end{pmatrix} = X\begin{pmatrix} \text{⚁} \\ \text{⚁} \\ \text{⚀} \end{pmatrix} = 5,$$

$$\vdots$$

(*b*)**(iii)** $\mathrm{P}(X=3) = \frac{1}{216}$, $\mathrm{P}(X=4) = \frac{3}{216}$, $\mathrm{P}(X=5) = \frac{6}{216}$,

$\mathrm{P}(X=6) = \frac{10}{216}$, $\mathrm{P}(X=7) = \frac{15}{216}$, $\mathrm{P}(X=8) = \frac{21}{216}$,

$\mathrm{P}(X=9) = \frac{25}{216}$, $\mathrm{P}(X=10) = \frac{27}{216}$, $\mathrm{P}(X=11) = \frac{27}{216}$,

$\mathrm{P}(X=12) = \frac{25}{216}$, $\mathrm{P}(X=13) = \frac{21}{216}$, $\mathrm{P}(X=14) = \frac{15}{216}$,

$\mathrm{P}(X=15) = \frac{10}{216}$, $\mathrm{P}(X=16) = \frac{6}{216}$, $\mathrm{P}(X=17) = \frac{3}{216}$,

$\mathrm{P}(X=18) = \frac{1}{216}$, $\mathrm{P}(X=k) = 0$,

for any $k \notin \{3, 4, 5, \dots, 18\}$.

154.8. Ch. 120 Answers (Random Variables: Independence)

A489. No. For example, $\mathrm{P}(X=0,Y=0)=0$, but

$$\mathrm{P}(X=0)\,\mathrm{P}(Y=0)=0.5\times 0.25=0.125.$$

154.9. Ch. 121 Answers (Random Variables: Expectation)

A490. **(a)** $\mathrm{P}(X+Y=2)$ is simply the probability of 2 heads and 0 sixes OR 1 head and 1 six OR 0 heads and 2 sixes. So

$$\mathrm{P}(X+Y=2)=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{5}{6}\cdot\frac{5}{6}+\binom{2}{1}\frac{1}{2}\cdot\frac{1}{2}\binom{2}{1}\frac{5}{6}\cdot\frac{1}{6}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{6}=\frac{25}{144}+\frac{20}{144}+\frac{1}{144}=\frac{46}{144}.$$

(b) $\mathrm{P}(X+Y=3)$ is simply the probability of 2 heads and 1 six OR 1 head and 2 sixes. So

$$\mathrm{P}(X+Y=3)=\frac{1}{2}\cdot\frac{1}{2}\cdot\binom{2}{1}\frac{5}{6}\cdot\frac{1}{6}+\binom{2}{1}\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{6}=\frac{10}{144}+\frac{2}{144}=\frac{12}{144}.$$

(c) $\mathrm{P}(X+Y=4)$ is simply the probability of 2 heads and 2 sixes. So

$$\mathrm{P}(X+Y=4)=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{144}.$$

(d) $\mathbf{E}[X+Y]$

$$=\sum_{k\in\mathrm{Range}(X+Y)}\mathrm{P}(X+Y=k)\cdot k$$

$$=\mathrm{P}(X+Y=0)\cdot 0+\mathrm{P}(X+Y=1)\cdot 1+\mathrm{P}(X+Y=2)\cdot 2 \\ +\mathrm{P}(X+Y=3)\cdot 3+\mathrm{P}(X+Y=4)\cdot 4$$

$$=\frac{25}{144}\cdot 0+\frac{60}{144}\cdot 1+\frac{46}{144}\cdot 2+\frac{12}{144}\cdot 3+\frac{1}{144}\cdot 4=\frac{60+92+36+4}{144}=\frac{192}{144}=\frac{4}{3}.$$

A491(a). The range of X consists simply of the possible prizes from the "big" game. Range $(x) = \{2000, 1000, 490, 250, 60, 0\}$. (Don't forget to include 0.)

Similarly, Range $(y) = \{3000, 2000, 800, 0\}$.

$$(b) \quad \mathrm{P}(X = 2000) = \mathrm{P}(X = 1000) = \mathrm{P}(X = 490) = \frac{1}{10000},$$

$$\mathrm{P}(X = 250) = \mathrm{P}(X = 60) = \frac{10}{10000},$$

$$\mathrm{P}(X = 0) = \frac{9977}{10000},$$

$$\mathrm{P}(Y = 3000) = \mathrm{P}(Y = 2000) = \mathrm{P}(Y = 800) = \frac{1}{10000},$$

$$\mathrm{P}(Y = 0) = \frac{9997}{10000}.$$

$$(c) \quad \mathbf{E}[X] = \sum_{k \in \mathrm{Range}(x)} \mathrm{P}(X = k) \cdot k = 2000\mathrm{P}(X = 2000) + 1000\mathrm{P}(X = 1000) + \dots$$

$$\dots + 490\mathrm{P}(X = 490) + 250\mathrm{P}(X = 250) + 60\mathrm{P}(X = 60) + 0\mathrm{P}(X = 0)$$

$$= \frac{2000}{10000} + \frac{1000}{10000} + \frac{490}{10000} + \frac{250 \cdot 10}{10000} + \frac{60 \cdot 10}{10000} + \frac{9977 \cdot 0}{10000} = 0.659$$

$$\mathbf{E}[Y] = \sum_{k \in \mathrm{Range}(y)} \mathrm{P}(Y = 2000) \cdot k$$

$$= \mathrm{P}(Y = 3000) \cdot 3000 + \mathrm{P}(Y = 2000) \cdot 2000 + \mathrm{P}(Y = 800) \cdot 800 + \mathrm{P}(Y = 0) \cdot 0$$

$$= \frac{1}{10000} \cdot 3000 + \frac{1}{10000} \cdot 2000 + \frac{1}{10000} \cdot 800 + \frac{9997}{10000} \cdot 0 = 0.3 + 0.2 + 0.08 + 0 = 0.58.$$

(d) For every \$1 staked, the "big" game is expected to lose you \$0.341 and the "small" game is expected to lose you \$0.42. Thus, the "big" game is expected to lose you less money.

154.10. Ch. 122 Answers (Random Variables: Variance)

A492. In Exercise 488(b), we already found that $\mathrm{P}(Z=3)=1/216$, $\mathrm{P}(Z=4)=3/216$, ..., $\mathrm{P}(Z=18)=1/216$. By symmetry, we have $\mu=\mathrm{E}[Z]=10.5$. So,

$$\mathbf{E}\left[Z^2\right]=\frac{1}{216}\cdot 3^2+\frac{3}{216}\cdot 4^2+\cdots+\frac{1}{216}\cdot 18^2=\frac{25704}{216}=119.$$

$$\text{Hence,}\qquad \mathbf{V}[Z]=\mathbf{E}\left[Z^2\right]-\mu^2=119-10.5^2=\frac{105}{12}.$$

A493. $\mathbf{E}[Y]=\dfrac{35}{100}\times 20\ \mathbf{cm}+\dfrac{65}{100}\times 30\ \mathbf{cm}=26.5\ \mathbf{cm.}$

$$\mathbf{V}[Y]=\frac{35}{100}\times(20\text{ cm}-26.5\text{ cm})^2+\frac{65}{100}\times(30\text{ cm}-26.5\text{ cm})^2=22.75\text{ cm}^2.$$

$$\mathbf{SD}[Y]=\sqrt{\mathbf{V}[Y]}\approx 4.77\text{ cm}.$$

A494. **(a)** 2μ kg, $2\sigma^2$ kg^2.

(b) 2μ kg, $4\sigma^2$ kg^2.

(c) The mean of the total weight of the two fish is 2μ kg. However, we do not know the variance, since the weights of the two fish are not independent.

154.11. Ch. 123 Answers (The Coin-Flips Problem)

(*This chapter had no exercises.*)

154.12. Ch. 124 Answers (Bernoulli Trial and Distribution)

(*This chapter had no exercises.*)

154.13. Ch. 125 Answers (Binomial Distribution)

A495. Let $X \sim \mathrm{B}(20, 0.01)$ be the number of components in engine #1 that fail. Let $Y \sim \mathrm{B}(35, 0.005)$ be the number of components in engine #2 that fail.

The probability that engine #1 fails is

$$\begin{aligned}
\mathrm{P}(X \geq 2) = 1 - \mathrm{P}(X \leq 1) &= 1 - \mathrm{P}(X = 0) - \mathrm{P}(X = 1) \\
&= 1 - \binom{20}{0} 0.01^0 0.99^{20} - \binom{20}{1} 0.01^1 0.99^{19} \\
&\approx 0.0169.
\end{aligned}$$

The probability that engine #2 fails is

$$\begin{aligned}
\mathrm{P}(Y \geq 2) = 1 - \mathrm{P}(Y \leq 1) &= 1 - \mathrm{P}(Y = 0) - \mathrm{P}(Y = 1) \\
&= 1 - \binom{35}{0} 0.005^0 0.995^{35} - \binom{35}{1} 0.005^1 0.995^{34} \\
&\approx 0.0133.
\end{aligned}$$

Hence, the probability that both engines fail is

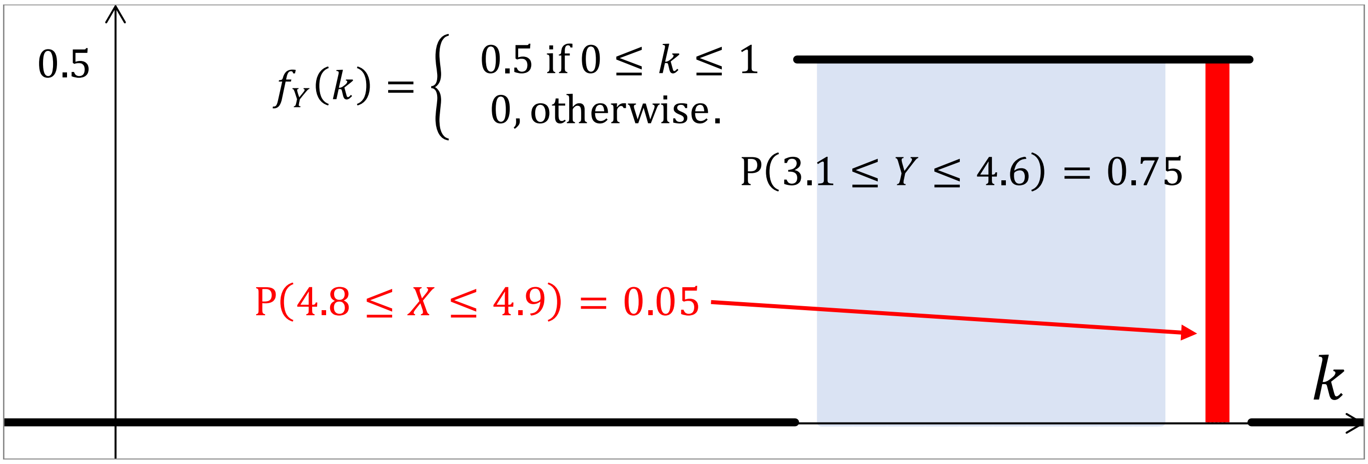
$$\mathrm{P}(X \geq 2)\,\mathrm{P}(Y \geq 2) \approx 0.00022.$$

154.14. Ch. 126 Answers (Continuous Uniform Distribution)

A496.

$$(a) \quad F_Y(k) = \begin{cases} 0, & \text{if } k < 3, \\ 0.5k, & \text{if } k \in [3,5], \\ 1, & \text{if } k > 5. \end{cases} \qquad (b) \quad f_Y(k) = \begin{cases} 0.5, & \text{if } k \in [3,5] \\ 0, & \text{otherwise.} \end{cases}$$

(c) $P(3.1 \leq Y \leq 4.6) = 0.75$ is in blue and $P(4.8 \leq Y \leq 4.9) = 0.05$ is in red.

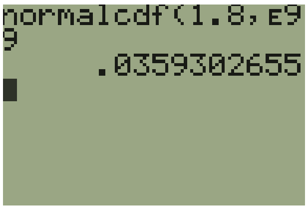


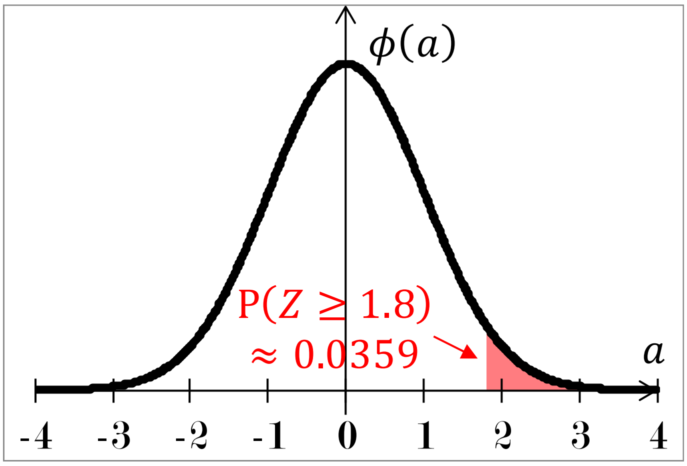
154.15. Ch. 127 Answers (Normal Distribution)

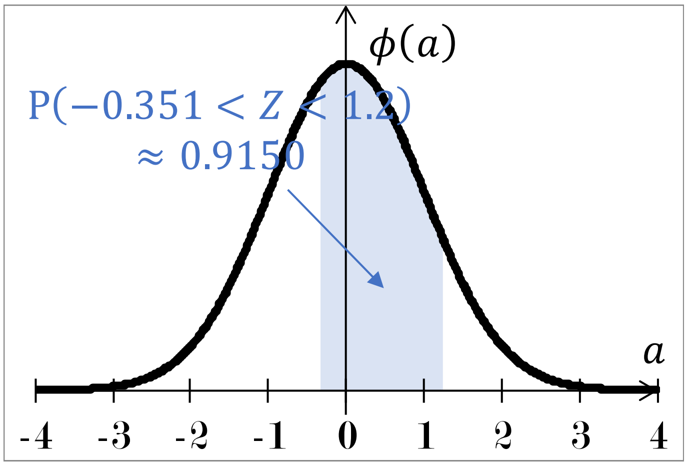
A497. **(a)** From Z-tables,

$$\mathrm{P}(Z \geq 1.8) = 1 - \mathrm{P}(Z \leq 1.8) = 1 - \Phi(1.8) \approx 1 - 0.9641 = 0.0359.$$

Graphing calculator screenshot:







(b) From Z-tables,

$$\begin{aligned}\mathrm{P}(-0.351 < Z < 1.2) = \Phi(1.2) - \Phi(-0.351) &= \Phi(1.2) - [1 - \Phi(0.351)] \\ &\approx 0.8849 - (1 - 0.6372) = 0.8849 - 0.3628 = 0.5221.\end{aligned}$$

Graphing calculator screenshot:

A498. If $\mu = 0$ and $\sigma^2 = 1$, then

$$f_X(a) = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5\left(\frac{a-\mu}{\sigma}\right)^2} = \frac{1}{1\sqrt{2\pi}} e^{-0.5\left(\frac{a-0}{1}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-0.5a^2} = \phi(a).$$

We've just shown that the PDF of $X \sim \mathrm{N}(\mu, \sigma^2)$ when $\mu = 0$ and σ^2, is the same as the PDF of the SNRV $Z \sim \mathrm{N}(0,1)$. Hence, the SNRV is indeed simply a normal random variable with mean $\mu = 0$ and variance $\sigma^2 = 1$.

A499. First observe that $\dfrac{X-\mu}{\sigma} = \dfrac{X}{\sigma} + \dfrac{-\mu}{\sigma}$. Now simply use Fact 242, with $a = \dfrac{1}{\sigma}$ and $b = \dfrac{-\mu}{\sigma}$:

$$\frac{X-\mu}{\sigma} = \frac{X}{\sigma} + \frac{-\mu}{\sigma} \sim \mathrm{N}\left(\frac{\mu}{\sigma} + \frac{-\mu}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right) = \mathrm{N}(0,1).$$

A500. Let $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$ and let f_X and F_X be the PDF and CDF of X.

1. We know from Fact 241 that $\Phi(\infty) = 1$. And by Corollary 49, $F_X(\infty) = \mathrm{P}(X \leq \infty) = \Phi(\infty)$. Thus, $F_X(\infty) = 1$.
2. $f_X(a) = \phi\left(\dfrac{a-\mu}{\sigma}\right)$. But we already know from Fact 241 that $\phi\left(\dfrac{a-\mu}{\sigma}\right) > 0$.
3. $\mathrm{E}[X] = \mathrm{E}[\sigma Z + \mu] = \sigma \mathrm{E}[Z] + \mu = \mu$.
4. We know from Fact 241 that the standard normal PDF ϕ attains a global maximum at 0. That is, $\phi(0) \geq \phi(a)$, for all $a \in \mathbb{R}$. Since $X = \sigma Z + \mu$, this is equivalent to $f_X(\sigma \cdot 0 + \mu) \geq f_X(\sigma \cdot a + \mu)$, for all $a \in \mathbb{R}$. Equivalently, $f_X(\mu) \geq f_X(b)$, for all $b \in \mathbb{R}$. That is, f_X attains a global maximum at μ.
5. $\mathrm{Var}[X] = \mathrm{Var}[\sigma Z + \mu] = \sigma^2 \mathrm{Var}[Z] = \sigma^2$.
6. We know from Fact 241 that for any $a \in \mathbb{R}$, we have $\mathrm{P}(Z \leq a) = \mathrm{P}(Z < a)$. Equivalently, for all $a \in \mathbb{R}$, $\mathrm{P}(X \leq \sigma a + \mu) = \mathrm{P}(X < \sigma a + \mu)$. Equivalently, for all $b \in \mathbb{R}$, $\mathrm{P}(X \leq b) = \mathrm{P}(X < b)$.
7. We know from Fact 241 that ϕ is symmetric about 0. Since $X = \sigma Z + \mu$, f_X must likewise be symmetric about $\sigma \cdot 0 + \mu = \mu$.
   (a) By Corollary 49, $\mathrm{P}(X \geq \mu + a) = \mathrm{P}\left(Z \geq \dfrac{a}{\sigma}\right)$. By Fact 241, $\mathrm{P}\left(Z \geq \dfrac{a}{\sigma}\right) = \mathrm{P}\left(Z \leq -\dfrac{a}{\sigma}\right)$. Now again by Corollary 49, $\mathrm{P}\left(Z \leq -\dfrac{a}{\sigma}\right) = \mathrm{P}(X \leq \mu - a)$. Altogether then, $\mathrm{P}(X \geq \mu + a) = \mathrm{P}(X \leq \mu - a)$, as desired. And of course, by definition, $\mathrm{P}(X \leq \mu - a) = F_X(\mu - a)$.
   (b) Obvious.
   (c) Obvious.
8. First use Corollary 49: $\mathrm{P}(\mu - \sigma \leq X \leq \mu + \sigma) = \mathrm{P}(-1 \leq Z \leq 1)$. Now use Fact 241: $\mathrm{P}(-1 \leq Z \leq 1) = 0.6827$.
9. First use Corollary 49: $\mathrm{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = \mathrm{P}(-2 \leq Z \leq 2)$. Now use Fact 241: $\mathrm{P}(-2 \leq Z \leq 2) = 0.9545$.
10. First use Corollary 49: $\mathrm{P}(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = \mathrm{P}(-3 \leq Z \leq 3)$. Now use Fact 241: $\mathrm{P}(-3 \leq Z \leq 3) = 0.9973$.
11. By Fact 241, ϕ has two points of inflexion, namely at ± 1. That is, ϕ changes concavity at ± 1. Since by Corollary 49, $X = \sigma Z + \mu$, f_X must likewise change concavity at $\pm 1 \cdot \sigma + \mu = \mu \pm \sigma$. That is, f_X has two points of inflexion, namely at $\mu \pm \sigma$.

A501. We are given that $X \sim \mathrm{N}(2.14, 5)$ and $Y \sim \mathrm{N}(-0.33, 2)$.

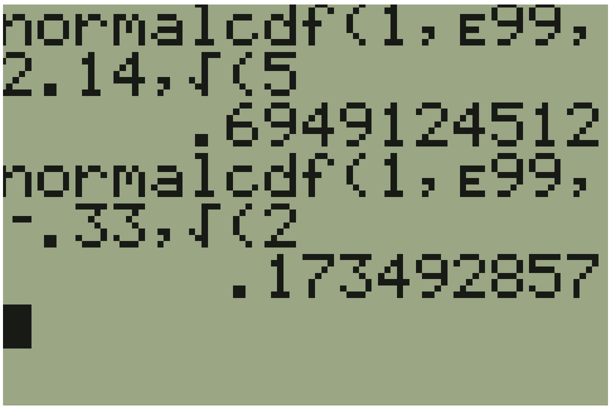
$$(a)\ \mathrm{P}(X \geq 1) = \mathrm{P}\left(Z \geq \frac{1-2.14}{\sqrt{5}}\right) \approx \mathrm{P}(Z \geq -0.5098)$$

$$= \mathrm{P}(Z \leq 0.5098) = \Phi(0.5098) \approx 0.6949.$$

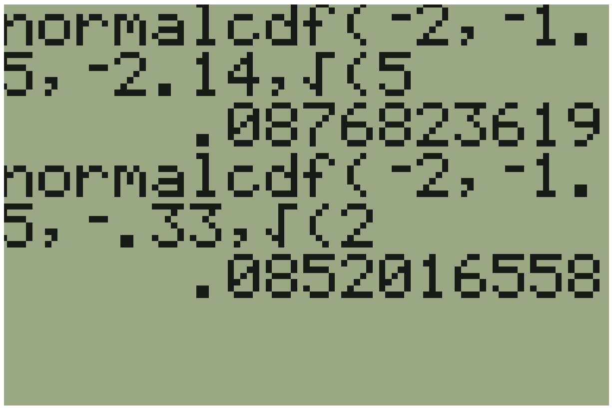
$$\mathrm{P}(Y \geq 1) = \mathrm{P}\left(Z \geq \frac{1-(-0.33)}{\sqrt{2}}\right) \approx \mathrm{P}(Z \geq 0.9405)$$

$$= 1 - \mathrm{P}(Z \leq 0.9405) = 1 - \Phi(0.9405) \approx 0.1735.$$

$\mathrm{P}(X \geq 1)$ and $\mathrm{P}(Y \geq 1)$



$\mathrm{P}(-2 \leq X \leq -1.5)$ and $\mathrm{P}(-2 \leq Y \leq -1.5)$



$$(b)\ \mathrm{P}(-2 \leq X \leq -1.5) = \mathrm{P}\left(\frac{(-2)-2.14}{\sqrt{5}} \leq Z \leq \frac{(-1.5)-2.14}{\sqrt{5}}\right)$$

$$\approx \mathrm{P}(-1.8515 \leq Z \leq -1.6279) = \mathrm{P}(1.6279 \leq Z \leq 1.8515)$$

$$= \Phi(1.8515) - \Phi(1.6279) \approx 0.9679 - 0.9482 = 0.0197.$$

$$\mathrm{P}(-2 \leq Y \leq -1.5) = \mathrm{P}\left(\frac{(-2)-(-0.33)}{\sqrt{2}} \leq Z \leq \frac{(-1.5)-(-0.33)}{\sqrt{2}}\right)$$

$$\approx \mathrm{P}(-1.1809 \leq Z \leq -0.8273) = \mathrm{P}(0.8273 \leq Z \leq 1.1809)$$

$$= \Phi(1.1809) - \Phi(0.8273) \approx 0.8812 - 0.7959 = 0.0853.$$

A502. **(a)** Let $W \sim \mathrm{N}(25000, 64000000)$ and $E \sim \mathrm{N}(200, 10000)$. Let $B = 0.002W + 0.3E$ be the total bill in a given month. Then

$$\begin{aligned} B &\sim \mathrm{N}\left(0.002 \times 25000 + 0.3 \times 200, 0.002^2 \times 64000000 + 0.3^2 \times 10000\right) \\ &= \mathrm{N}(50 + 60, 256 + 900) = \mathrm{N}(110, 1156). \end{aligned}$$

Thus, $\mathrm{P}(B > 100) \approx 0.6157$ (calculator).

(b) Let $B_1 \sim \mathrm{N}(110, 1156)$, $B_2 \sim \mathrm{N}(110, 1156)$, ..., $B_{12} \sim \mathrm{N}(110, 1156)$ be the bills in each of the 12 months.

Then the total bill in a year is $T = B_1 + B_2 + \cdots + B_{12} \sim \mathrm{N}(12 \times 110, 12 \times 1156) = \mathrm{N}(1320, 13872)$.

Thus, $\mathrm{P}(T > 1000) \approx 0.9967$ (calculator).

(c) The total bill in a given month is $B = 0.002W + xE$ and

$$B \sim \mathrm{N}\left(50 + 200x, 256 + 10000x^2\right).$$

Our goal is to find the value of x for which $\mathrm{P}(B > 100) = 0.1$. We have

$$\begin{aligned} \mathrm{P}(B > 100) &= \mathrm{P}\left(Z > \frac{100 - (50 + 200x)}{\sqrt{256 + 10000x^2}}\right) = \mathrm{P}\left(Z > \frac{50 - 200x}{\sqrt{256 + 10000x^2}}\right) \\ &= 1 - \Phi\left(\frac{50 - 200x}{\sqrt{256 + 10000x^2}}\right) = 0.1. \end{aligned}$$

From the Z-tables,

$$\Phi\left(\frac{50 - 200x}{\sqrt{256 + 10000x^2}}\right) = 0.9 \quad \iff \quad \frac{50 - 200x}{\sqrt{256 + 10000x^2}} \approx 1.2815.$$

One can rearrange, do the algebra (square both sides), and use the quadratic formula. Alternatively, one can simply use one's graphing calculator to find that $x \approx 0.084$. We conclude that the maximum value of x is approximately 0.084, in order for the probability that the total utility bill in a given month exceeds \$100 is 0.1 or less.

A503. From our earlier work, we know that each die roll has mean 3.5 and variance 35/12.

The CLT says that since $n = 30 \geq 30$ is large enough and the distribution is "nice enough" (we are assuming this), X can be approximated by the normal random variable $Y \sim \mathrm{N}(30 \times 3.5, 30 \times 35/12) = \mathrm{N}(105, 1050/12)$. Thus, using also the continuity correction, we have $\mathrm{P}(100 \leq X \leq 110) \approx \mathrm{P}(99.5 \leq Y \leq 110.5) \approx 0.4435$ (calculator).

A504. Let X be the random variable that is the sum of the weights of the $5,000$ Coco-Pops.

The CLT says that since $n = 5000 \geq 30$ is large enough and the distribution is "nice enough" (we are assuming this), X can be approximated by the normal random variable $Y \sim \mathrm{N}(5000 \times 0.1, 5000 \times 0.004) = \mathrm{N}(500, 20)$. Thus, $\mathrm{P}(X \leq 499) \approx \mathrm{P}(Y \leq 499) \approx 0.4115$ (calculator).

A505. $\bar{x} = \dfrac{3+14+2+8+8+6+0}{7} = \dfrac{41}{7}$ and

$$s^2 = \frac{\sum x_i^2 - n\bar{x}^2}{n-1} = \frac{9+156+4+64+64+36-41^2/7}{6} = \frac{155}{7}.$$

A506. **(a)** The sample mean and sample variance are

$$\bar{x} = \frac{\sum_{i=1}^n x}{n} = \frac{1885}{10} = 188.5,$$

$$s^2 = \frac{\sum_{i=1}^n x^2 - \frac{\left(\sum_{i=1}^n x\right)^2}{n}}{n-1} = \frac{378,265 - \frac{1885^2}{10}}{9} \approx 2550.$$

(b) The sample mean and sample variance are

$$\bar{x} = \frac{\sum_{i=1}^n x}{n} = \frac{\sum_{i=1}^n (x-50+50)}{n} = \frac{\sum_{i=1}^n (x-50) + \sum_{i=1}^n 50}{n} = \frac{1885+50n}{n} = \frac{1885}{n} + 50 = 238.5,$$

$$s^2 = \frac{\sum_{i=1}^n (x_i-50)^2 - \frac{\left[\sum_{i=1}^n (x_i-50)\right]^2}{n}}{n-1} = \frac{378,265 - \frac{1885^2}{10}}{9} \approx 2550.$$

A507. **(a)** Assume that the weights of the five Singaporeans sampled are independently and identically distributed. Then unbiased estimates for the population mean μ and variance σ^2 of the weights of Singaporeans are, respectively, the observed sample mean $\bar{x}$ and observed sample variance s^2:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{32+88+67+75+56}{5} = 63.6,$$
$$s^2 = \frac{\sum x_i^2 - n\bar{x}^2}{n-1} = \frac{32^2+88^2+67^2+75^2+56^2-4\times 63.6}{4} = 448.3.$$

(b) We don't know! And unless we literally gather and weigh every single Singaporean, we will never know what exactly the average weight of a Singaporean is.

All we've found in part **(a)** is an estimate (63.6 kg) for the average weight of a Singaporean. We know that on average, the estimator we uses "gets it right".

However, it could well be that we're unlucky (and got 5 unusually heavy or unusually light persons) and the estimate of 63.6 kg is thus way off.

A508.

$$\mathbf{E}\left[\bar{X}\right] = \mathbf{E}\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right] = \frac{\mathbf{E}\left[X_1 + X_2 + \cdots + X_n\right]}{n}$$
$$= \frac{\mathbf{E}\left[X_1\right] + \mathbf{E}\left[X_2\right] + \cdots + \mathbf{E}\left[X_n\right]}{n} = \frac{\mu + \mu + \cdots + \mu}{n} = \frac{n\mu}{n} = \mu.$$

We have just shown that $\mathrm{E}\left[\bar{X}\right] = \mu$. In other words, we've just shown that $\bar{X}$ is an unbiased estimator for μ.

A509. **(a)** The observed random sample is $(x_1, x_2, \dots, x_{10}) = (1,1,1,1,1,1,1,0,0,0)$. The observed sample mean and observed sample variance are

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_{10}}{n} = 0.7,$$

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_{10} - \bar{x})^2}{n-1} = \frac{7 \cdot 0.3^2 + 3 \cdot 0.7^2}{9} = 0.2\dot{3}.$$

(b) Yes, the observed sample mean $\bar{x} = 0.7$ is an unbiased estimate for the true population mean μ (i.e. the true proportion of coin flips that are heads).

And yes, the observed sample variance $s^2 = 0.2\dot{3}$ is an unbiased estimate for the true population variance σ^2.

(c) No, this is merely one observed random sample, from which we generated a single estimate ("guess")—namely $\bar{x} = 0.7$—of the true population mean μ.

All we know is that the sample mean $\bar{X}$ is an unbiased estimator for the true population mean μ. That is, the average estimate generated by $\bar{X}$ will equal μ.

However, any particular estimate $\bar{x}$ may or may not be equal to μ. Indeed, if we're unlucky, our particular estimate may be very far from the true μ.

A510. $\mathrm{Var}\left[\bar{X}\right] = \mathrm{Var}\left[\frac{1}{n}(X_1 + X_2 + \cdots + X_n)\right] = \frac{1}{n^2}\mathrm{Var}\left[X_1 + X_2 + \cdots + X_n\right] = \frac{1}{n^2}(\mathrm{Var}\left[X_1\right] + \mathrm{Va}$
$\frac{1}{n^2}\left(n\sigma^2\right) = \frac{\sigma^2}{n}.$

A511. **(a)** The population mean μ is the **number** defined by $\mu = \sum_{i=1}^{k} x_i/k$. It is the average across all population values.

(b) The population variance σ^2 is the **number** defined by $\sigma^2 = \sum_{i=1}^{k} (x_i - \mu)/k$. It measures the dispersion across the population values.

(c) The sample mean $\bar{X}$ is a **random variable** defined by $\bar{X} = \sum_{i=1}^{n} X_i/n$. It is the average of all values in a random sample.

(d) The sample variance S^2 is a **random variable** defined by $S^2 = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)/(n-1)$. It measures the dispersion across the values in a random sample.

(e) The mean of the sample mean, also called the expected value of the sample mean, is the **number** $\mathrm{E}\left[\bar{X}\right]$. The interpretation is that if we we have infinitely many observed samples of size n, calculate the observed sample mean for each, then $\mathrm{E}\left[\bar{X}\right]$ is equal to the average across the observed sample means. It can be shown that $\mathrm{E}\left[\bar{X}\right] = \mu$ and hence that the sample mean $\bar{X}$ is an unbiased estimator for the population mean μ.

(f) The variance of the sample mean is the **number** $\mathrm{Var}\left[\bar{X}\right]$. The interpretation is that if we have infinitely many observed random samples of size n, calculate the observed sample mean for each, then $\mathrm{Var}\left[\bar{X}\right]$ measures the dispersion across the observed sample means.

(g) The mean of the sample variance, also called the expected value of the sample variance, is the **number** $\mathrm{E}\left[S^2\right]$. The interpretation is that if we have infinitely many observed random samples of size n, calculate the observed sample variance for each, then $\mathrm{E}\left[S^2\right]$ is equal to the average across the observed sample variances. It can be shown that $\mathrm{E}\left[S^2\right] = \sigma^2$ and hence that the sample variance S^2 is an unbiased estimator for the population variance σ^2 .

(h) Given an observed random sample, e.g. $(x_1, x_2, x_3) = (1, 1, 0)$, we can calculate the corresponding observed sample mean as

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3} = \frac{1+1+0}{3} = \frac{2}{3}.$$

The observed sample mean is the average of all values in an observed random sample.

(i) Given an observed random sample, e.g. $(x_1, x_2, x_3) = (1, 1, 0)$, we can calculate the corresponding observed sample variance as

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{3-1} = \frac{1/9 + 1/9 + 4/9}{2} = \frac{1}{3}.$$

The observed sample variance measures the dispersion across the observed sample variances.

154.17. Ch. 131 Answers (Null Hypothesis Significance Testing)

A512. Let μ be the probability that a coin-flip is heads. The **null and alternative hypotheses** are

$$H_0 : \mu = 0.5 \quad \text{and} \quad H_A : \mu > 0.5.$$

Our **random sample** is 20 coin-flips: $(X_1, X_2, \ldots, X_{20})$, where X_i takes on the value 1 if the ith coin-flip is heads and 0 otherwise.

Our **test statistic** is the number of heads: $T = X_1 + X_2 + \cdots + X_{20}$.

In our **observed random sample** $(x_1, x_2, \ldots, x_{20})$, there are 17 heads. So the **observed test statistic** is $t = 17$.

Assuming H_0 were true, we'd have $T \sim \mathrm{B}(20, 0.5)$. Thus, the p**-value** is

$$\mathrm{P}(T \geq 17 | H_0) = \mathrm{P}(T = 17 | H_0) + \mathrm{P}(T = 18 | H_0) + \mathrm{P}(T = 19 | H_0) + \mathrm{P}(T = 20 | H_0)$$
$$= \binom{20}{17} 0.5^{17} 0.5^3 + \binom{20}{18} 0.5^{18} 0.5^2 + \binom{20}{19} 0.5^{19} 0.5^1 + \binom{20}{20} 0.5^{20} 0.5^0 \approx 0.0013.$$

Since $p \approx 0.0013 < \alpha = 0.05$, we reject H_0 at the 5% significance level.

A513. Let μ be the true long-run proportion of coin-flips that are heads. The **null and alternative hypotheses** are

$$H_0 : \mu = 0.5 \quad \text{and} \quad H_A : \mu \neq 0.5.$$

Our **random sample** is 20 coin-flips: $(X_1, X_2, \ldots, X_{20})$, where X_i takes on the value 1 if the ith coin-flip is heads and 0 otherwise.

Our **test statistic** is the number of heads: $T = X_1 + X_2 + \cdots + X_{20}$.

In our **observed random sample** $(x_1, x_2, \ldots, x_{20})$, there are 17 heads. So the **observed test statistic** is $t = 17$.

Assuming H_0 were true, we'd have $T \sim \mathrm{B}(20, 0.5)$. Thus, the p**-value** is

$$\mathrm{P}(T \geq 17, T \leq 3 | H_0) = \mathrm{P}(T = 0 | H_0) + \cdots + \mathrm{P}(T = 3 | H_0) + \mathrm{P}(T = 17 | H_0) + \cdots + \mathrm{P}(T = 20 | H_0)$$
$$= \binom{20}{0} 0.5^0 0.5^{20} + \binom{20}{1} 0.5^1 0.5^{19} + \binom{20}{17} 0.5^{17} 0.5^3 + \cdots + \binom{20}{20} 0.5^{20} 0.5^0 \approx 0.0026.$$

Since $p \approx 0.0026 < \alpha = 0.05$, we reject H_0 at the 5% significance level.

A514. Let μ be the probability that a coin-flip is heads.

(a) The competing hypotheses are $H_0 : \mu = 0.5$, $H_A : \mu > 0.5$.

The test statistic T is the number of heads (out of the 20 coin-flips).

For $t = 14$, the corresponding p-value is

$$\begin{aligned}\mathrm{P}(T \geq 14|H_0) &= \mathrm{P}(T = 14|H_0 \text{ true}) + \mathrm{P}(T = 15|H_0 \text{ true}) + \cdots + \mathrm{P}(T = 20|H_0 \text{ true}) \\ &= \binom{20}{14} 0.5^{14} 0.5^{6} + \binom{20}{15} 0.5^{15} 0.5^{5} + \cdots + \binom{20}{20} 0.5^{20} 0.5^{0} \approx 0.05766.\end{aligned}$$

For $t = 15$, the corresponding p-value is

$$\begin{aligned}\mathrm{P}(T \geq 15|H_0) &= \mathrm{P}(T = 15|H_0 \text{ true}) + \mathrm{P}(T = 15|H_0 \text{ true}) + \cdots + \mathrm{P}(T = 20|H_0 \text{ true}) \\ &= \binom{20}{15} 0.5^{14} 0.5^{6} + \binom{20}{15} 0.5^{15} 0.5^{5} + \cdots + \binom{20}{20} 0.5^{20} 0.5^{0} \approx 0.02069.\end{aligned}$$

Thus, the critical value is 15 (this is the value of t at which we are just able to reject H_0 at the $\alpha = 0.05$ significance level).

And the critical region is $\{15, 16, \ldots, 20\}$ (this is the set of values of t at which we'd be able to reject H_0 at the $\alpha = 0.05$ significance level).

(b) The competing hypotheses are $H_0 : \mu = 0.5$, $H_A : \mu \neq 0.5$.

The test statistic T is the number of heads (out of the 20 coin-flips).

For $t = 14$, the corresponding p-value is

$$\begin{aligned}&\mathrm{P}(T \geq 14, T \leq 6|H_0) = 1 - \mathrm{P}(7 \leq T \leq 13|H_0) \\ &= 1 - [\mathrm{P}(T = 7|H_0 \text{ true}) + \mathrm{P}(T = 8|H_0 \text{ true}) + \cdots + \mathrm{P}(T = 13|H_0 \text{ true})] \\ &= 1 - \left[\binom{20}{7} 0.5^{7} 0.5^{13} + \binom{20}{8} 0.5^{8} 0.5^{12} + \cdots + \binom{20}{13} 0.5^{13} 0.5^{7}\right] \approx 0.1153.\end{aligned}$$

For $t = 15$, the corresponding p-value is

$$\begin{aligned}&\mathrm{P}(T \geq 15, T \leq 5|H_0) = 1 - \mathrm{P}(6 \leq T \leq 14|H_0) \\ &= 1 - [\mathrm{P}(T = 6|H_0 \text{ true}) + \mathrm{P}(T = 7|H_0 \text{ true}) + \cdots + \mathrm{P}(T = 14|H_0 \text{ true})] \\ &= 1 - \left[\binom{20}{6} 0.5^{6} 0.5^{14} + \binom{20}{7} 0.5^{7} 0.5^{13} + \cdots + \binom{20}{14} 0.5^{13} 0.5^{7}\right] \approx 0.1153.\end{aligned}$$

Thus, the critical value is 15 and the critical region is $\{15, 16, \ldots, 20\}$.

A515. The competing hypotheses are

$$H_0 : \mu = 34,$$
$$H_A : \mu \neq 34.$$

The observed sample mean is

$$\bar{x} = \frac{35 + 35 + 31 + 32 + 33 + 34 + 31 + 34 + 35 + 34}{10} = 33.4.$$

The corresponding p-value is

$$p = \mathrm{P}\left(\bar{X} \geq 33.4, \bar{X} \leq 34.6 \middle| H_0\right) = \mathrm{P}\left(\bar{X} \geq 33.4 \middle| H_0\right) + \mathrm{P}\left(\bar{X} \leq 34.6 \middle| H_0\right)$$

$$= \mathrm{P}\left(Z \geq \frac{33.4 - 34}{\sqrt{9/10}}\right) + \mathrm{P}\left(Z \leq \frac{34.6 - 34}{\sqrt{9/10}}\right) \approx 0.5271.$$

The large p-value does not cast doubt on or provide evidence against H_0. We fail to reject H_0 at the $\alpha = 0.05$ significance level.

A516. The competing hypotheses are

$$H_0 : \mu = 34,$$
$$H_A : \mu \neq 34.$$

The observed sample mean is $\bar{x} = 33.4$.

The corresponding p-value is

$$p = \mathrm{P}\left(\bar{X} \leq 33.4, \bar{X} \geq 34.6 \middle| H_0\right) = \mathrm{P}\left(\bar{X} \leq 33.4 \middle| H_0\right) + \mathrm{P}\left(\bar{X} \geq 34.6 \middle| H_0\right)$$

$$\overset{\text{CLT}}{\approx} \mathrm{P}\left(Z \leq \frac{33.4 - 34}{\sqrt{9/100}}\right) + \mathrm{P}\left(Z \geq \frac{34.6 - 34}{\sqrt{9/100}}\right) \approx 0.04550.$$

The large p-value casts doubt on or provides evidence against H_0. We reject H_0 at the $\alpha = 0.05$ significance level.

A517. The competing hypotheses are

$$H_0 : \mu = 34,$$
$$H_A : \mu \neq 34.$$

The observed sample mean is $\bar{x} = 33.4$. And the observed sample variance is $s^2 = 11.2$.

The corresponding p-value is

$$p = \mathrm{P}\left(\bar{X} \leq 33.4, \bar{X} \geq 34.6 \middle| H_0\right) = \mathrm{P}\left(\bar{X} \leq 33.4 \middle| H_0\right) + \mathrm{P}\left(\bar{X} \geq 34.6 \middle| H_0\right)$$

$$\overset{\text{CLT}}{\approx} \mathrm{P}\left(Z \leq \frac{33.4 - 34}{\sqrt{11.2/100}}\right) + \mathrm{P}\left(Z \geq \frac{34.6 - 34}{\sqrt{11.2/100}}\right) \approx 0.07300.$$

The fairly small p-value casts some doubt on or provides some evidence against H_0. But we fail to reject H_0 at the $\alpha = 0.05$ significance level.

A518. The observed sample mean is $\bar{x} = 68$ and the observed sample variance (use Fact 244(a)) is

$$s^2 = \frac{\sum_{i=1}^n x_i^2 - \frac{\left[\sum_{i=1}^n x_i\right]^2}{n}}{n-1} = \frac{50 \times 5000 - \frac{(68 \times 50)^2}{50}}{49} \approx 383.7.$$

Let μ be the true average weight of a Singaporean. The competing hypotheses are $H_0 : \mu = 75$ and $H_A : \mu < 75$.

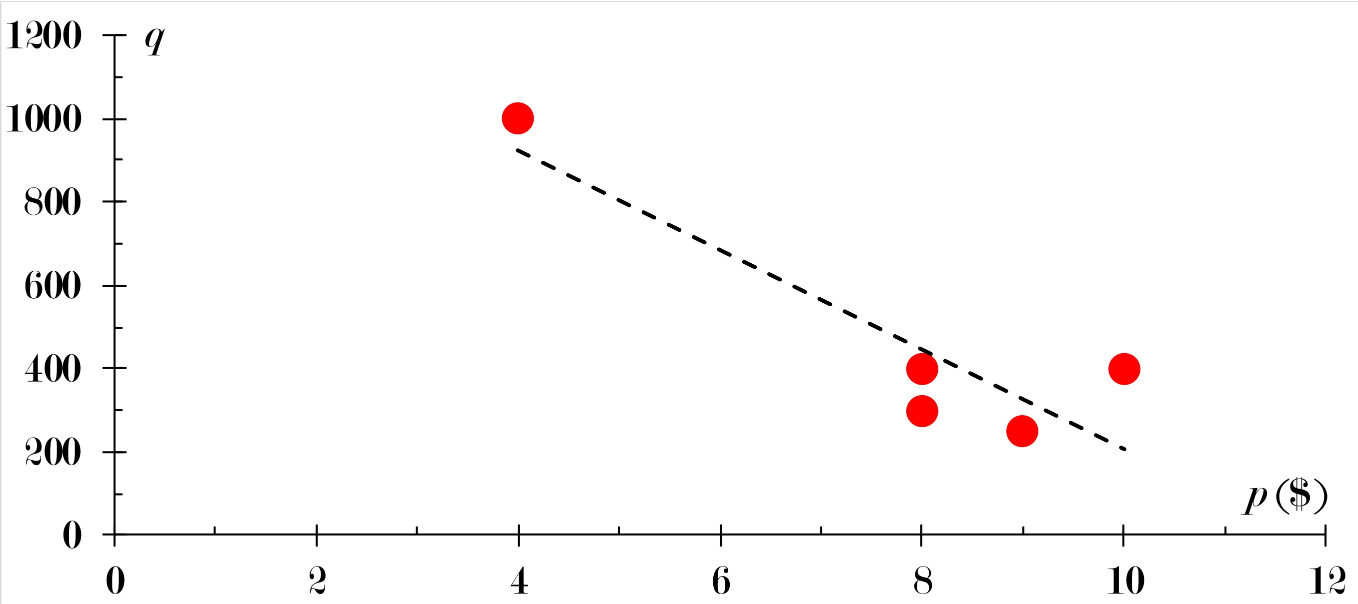
(This is a one-tailed test, because your friend's claim is that the average American is *heavier* than the average Singaporean. If the claim were instead that the average American's weight is *different* from the average Singaporean's, then we'd have a two-tailed test.)

Since the sample size $n = 50$ is "large enough", we can appeal to the CLT. The p-value is

$$p = \mathrm{P}\left(\bar{X} \leq 68 \middle| H_0\right) \overset{\text{CLT}}{\approx} \mathrm{P}\left(Z \leq \frac{68 - 75}{\sqrt{383.7/50}}\right) \approx 0.0058.$$

The small p-value casts doubt on or provides evidence against H_0. We can reject H_0 at any conventional significance level ($\alpha = 0.1$, $\alpha = 0.05$, or $\alpha = 0.01$).

A519.



A520. Compute $\bar{p} = (8+9+4+10+8)/5 = 7.8$ and $\bar{q} = (300+250+1000+400+400)/5 = 470$. Also,

$$\sum_{i=1}^{n}(p_i-\bar{p})(q_i-\bar{q}) = (8-\bar{p})(300-\bar{q}) + (9-\bar{p})(250-\bar{q}) + \cdots + (8-\bar{p})(400-\bar{q})$$
$$= (8-7.8)(300-470) + (9-7.8)(250-470) + \cdots + (8-7.8)(400-470) = -2480,$$

$$\sqrt{\sum_{i=1}^{n}(p_i-\bar{p})^2} = \sqrt{(8-\bar{p})^2 + (9-\bar{p})^2 + (4-\bar{p})^2 + (10-\bar{p})^2 + (8-\bar{p})^2}$$
$$= \sqrt{(8-7.8)^2 + (9-7.8)^2 + (4-7.8)^2 + (10-7.8)^2 + (8-7.8)^2} = \sqrt{20.8} \approx 4.56070170,$$

$$\sqrt{\sum_{i=1}^{n}(q_i-\bar{q})^2} = \sqrt{(300-\bar{q})^2 + (250-\bar{q})^2 + \cdots + (400-\bar{q})^2}$$
$$= \sqrt{(300-470)^2 + (250-470)^2 + \cdots + (400-470)^2} = \sqrt{368000} \approx 606.63003552.$$

Thus, $$r = \frac{\sum_{i=1}^{n}(p_i-\bar{p})(q_i-\bar{q})}{\sqrt{\sum_{i=1}^{n}(p_i-\bar{p})^2}\sqrt{\sum_{i=1}^{n}(q_i-\bar{q})^2}} \approx \frac{-2480,}{4.56070170 \times 606.63003552} \approx -0.8964.$$

A521. **(a)** We already computed (in the previous exercise) that $\bar{p} = 7.8$, $\bar{q} = 470$, $\sum_{i=1}^{n} (p_i - \bar{p})(q_i - \bar{q}) = -2480$ and $\sum_{i=1}^{n} (p_i - \bar{p})^2 = 20.8$. So,

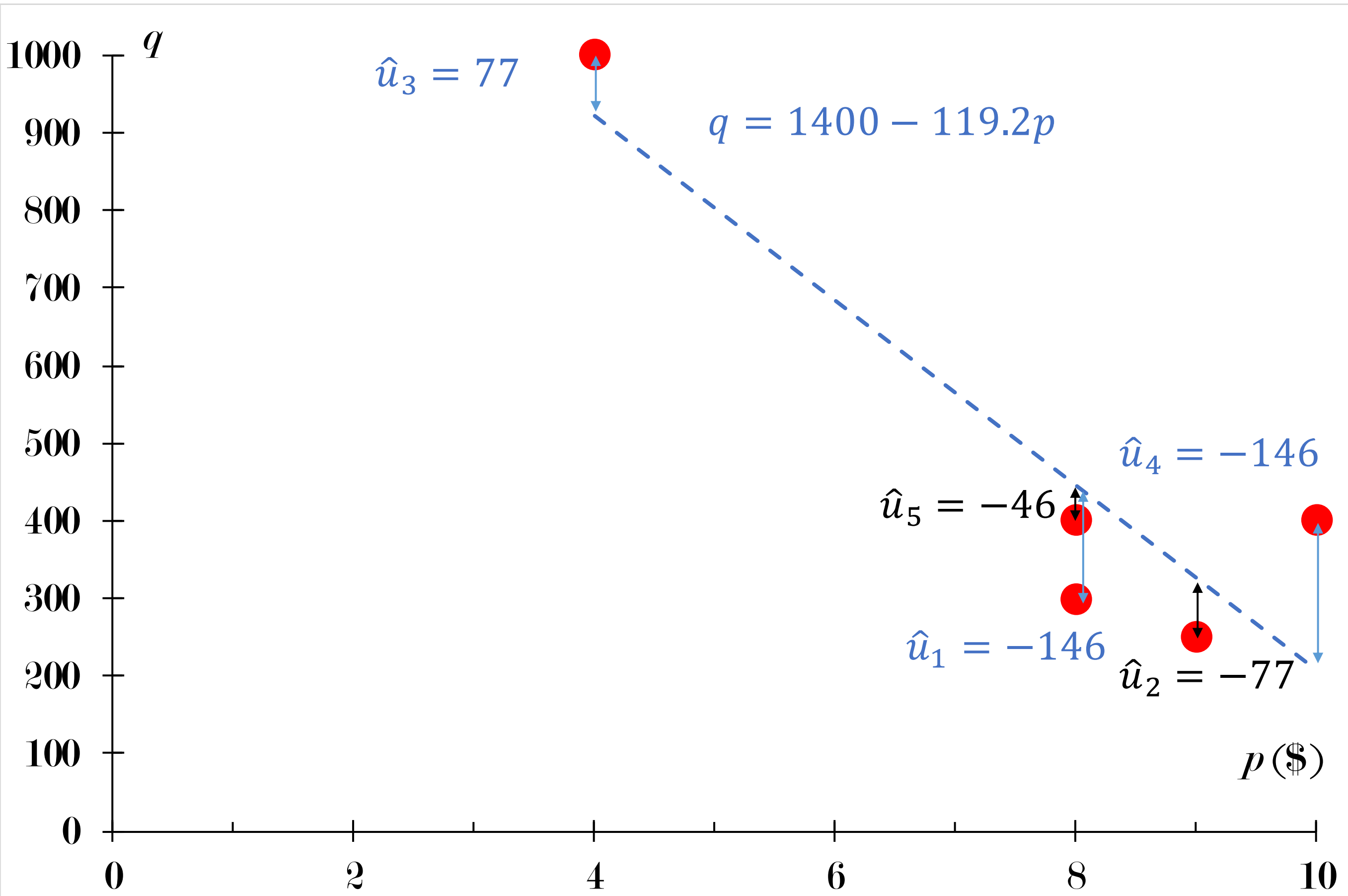
$$\hat{b} = \frac{\sum_{i=1}^{n} (p_i - \bar{p})(q_i - \bar{q})}{\sum_{i=1}^{n} (p_i - \bar{p})^2} = \frac{-2480}{20.8} \approx -119.2$$

Thus, the regression line of q on p is $q - \bar{q} = \hat{b}(p - \bar{p})$ or $q - 470 = -119.2(p - 7.8)$ or $q = 1400 - 119.2p$.

(b)

i	1	2	3	4	5
p_i (\$)	8	9	4	10	8
q_i	300	250	1000	400	400
$\hat{q}_i$	446	327	923	208	406
$\hat{u}_i = q_i - \hat{q}_i$	−146	−77	77	192	−46

(c)

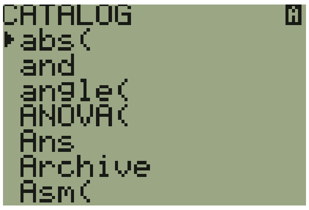


(d) The SSR is $\sum_{i=1}^{5} \hat{u}_i^2 \approx (-146)^2 + (-77)^2 + 77^2 + 192^2 + (-46)^2 = 72308$.

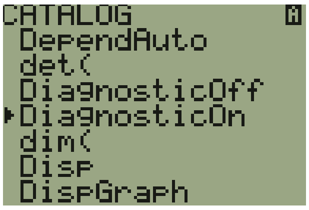
A522.

After Step 1.

After Step 2.

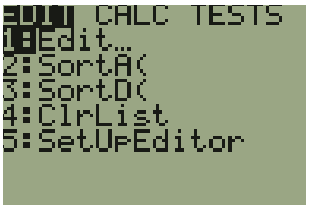


After Step 3.

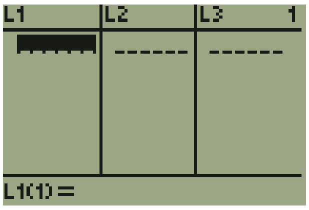


After Step 4.

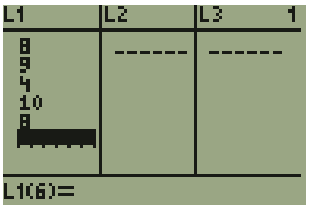
After Step 5.



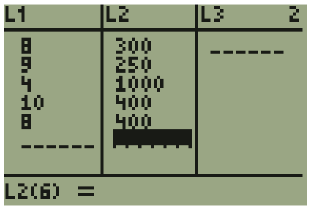
After Step 6.



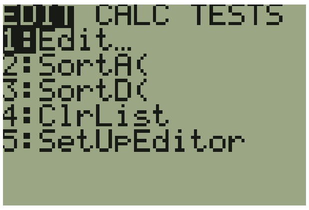
After Step 7.



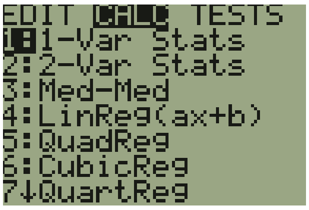
After Step 8.



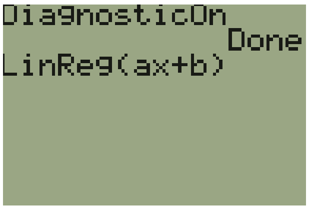
After Step 9.



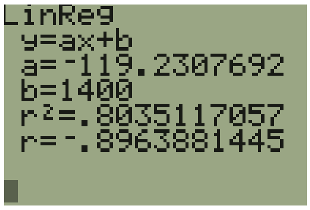
After Step 10.



After Step 11.



After Step 12.



The TI84 tells us that $r = -.8963881445$ and the regression line is $y = ax + b = -119.2307692 + 1400$. This is indeed consistent with the answers from the previous exercises.

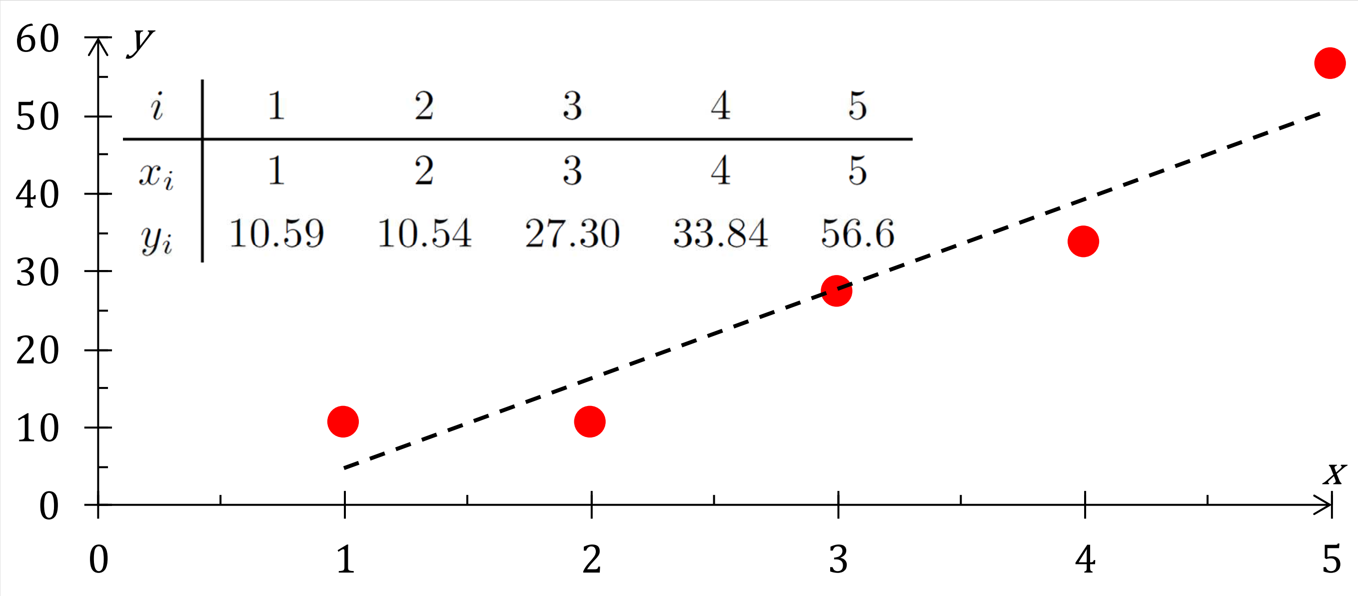
A523. In the previous exercises, we already calculated that the OLS line of best fit is $q = 1400 - 119.2p$. Thus,

(a) By interpolation, a barber who charged \$7 per haircut sold $1400 - 119.2 \times 7 \approx 566$ haircuts.

(b) By extrapolation, a barber who charged \$200 per haircut sold $1400 - 119.2 \times 200 = -22440$ haircuts. This is plainly absurd.

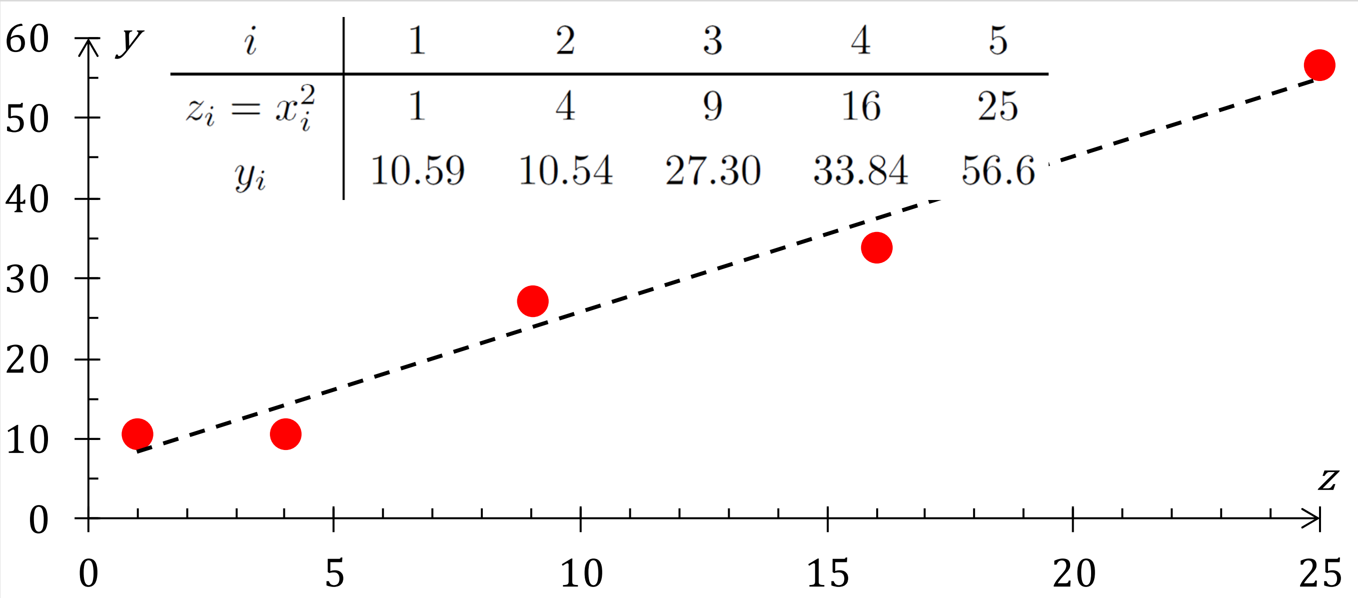
The second prediction is obviously absurd and thus obviously less reliable than the first.

A524. **(a)** $r \approx 0.954$.



i	1	2	3	4	5
x_i	1	2	3	4	5
y_i	10.59	10.54	27.30	33.84	56.6

(b) $r \approx 0.984$.



i	1	2	3	4	5
$z_i = x_i^2$	1	4	9	16	25
y_i	10.59	10.54	27.30	33.84	56.6

155. Part VII Answers (2006–19 A-Level Exams)

155.1. Ch. 133 Answers (Functions and Graphs)

A525 (9758 N2019/I/3)(i) $p\left\{(x+q)^3+r\right\}=p\left(x^3+3qx^2+3q^2x+q^3+r\right)$.

Comparing coefficients, $p=2$, $3pq=-6$ or $q=-1$, and $p\left(q^3+r\right)=-12$ or $r=-5$.

So, $f(x)=2x^3-6x^2+6x-12=2\left\{(x-1)^3-5\right\}$.

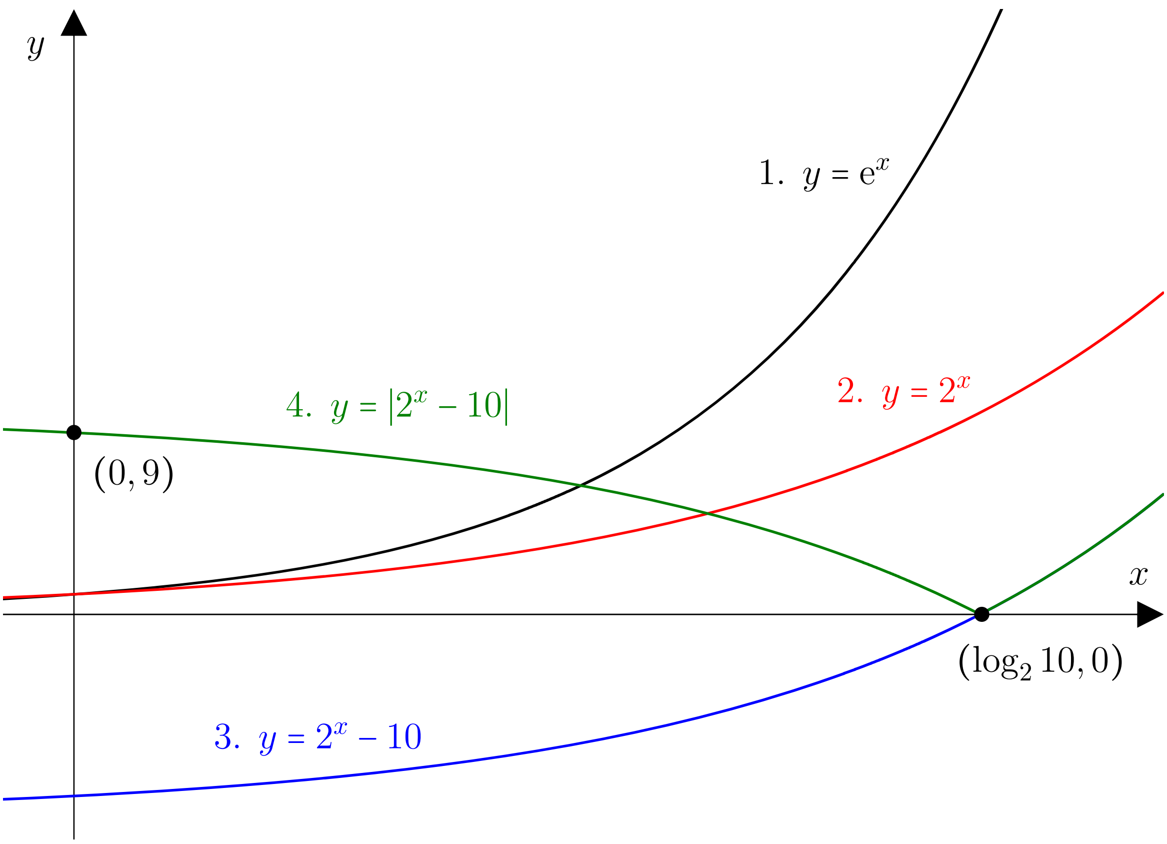
(ii) Right 1, down 5, stretch vertically by a factor of 2 (outward from the x-axis).

A526 (9758 N2019/I/4)(i) You are supposed to mindlessly use your calculator and copy. But as an exercise (and proof that we are not mindless monkeys), let's not:

1. Start with the graph of $y=e^x$.
2. The graph of $y=2^x$ is similar, just flatter.
3. Translate down 10 to get $y=2^x-10$.
4. Reflect portion below the x-axis in the x-axis to get $y=|2^x-10|$.

If $x=0$, then $y=\left|2^0-10\right|=9$. So, the only y-intercept is $(0,9)$.

If $y=0$, then $0=|2^x-10| \iff 10=2^x \iff x=\log_2 10$. So, the only x-intercept is $(\log_2 10, 0)$.



(ii) $|2^x-10|\le 6 \iff 2^x-10\in[-6,6] \iff 2^x\in[4,16] \iff \ln 2^x\in[\ln 4,\ln 16]$

$$\iff x\ln 2\in[\ln 4,\ln 16] \iff x\in\left[\frac{\ln 4}{\ln 2},\frac{\ln 16}{\ln 2}\right]=[2,4].$$

A527 (9758 N2019/I/5)(i) Write $y = f(x) = e^{2x} - 4 \in \text{Range } f = (-4, \infty)$.

Do the algebra: $y + 4 = e^{2x}$ or $x = 0.5 \ln(y + 4)$.

So, f is one-to-one and its inverse is $f^{-1} : (-4, \infty) \to \mathbb{R}$ defined by $f^{-1}(x) = 0.5 \ln(x + 4)$.

(ii) $5 = fg(x) = f(g(x)) = f(x + 2) = e^{2(x+2)} - 4 \iff 9 = e^{2(x+2)} \iff x = \ln 3 - 2$.

A528 (9758 N2019/I/7)(i) $y'(x) = e^{-x} - xe^{-x}$.

$y(1) = 1 \cdot e^{-1} = e^{-1}$, $y'(1) = e^{-1} - 1e^{-1} = 0$. So, the tangent at $x = 1$ is

$$y - y(1) = y'(1)(x - 1) \qquad \text{or} \qquad y - e^{-1} = 0 \qquad \text{or} \qquad y = e^{-1}.$$

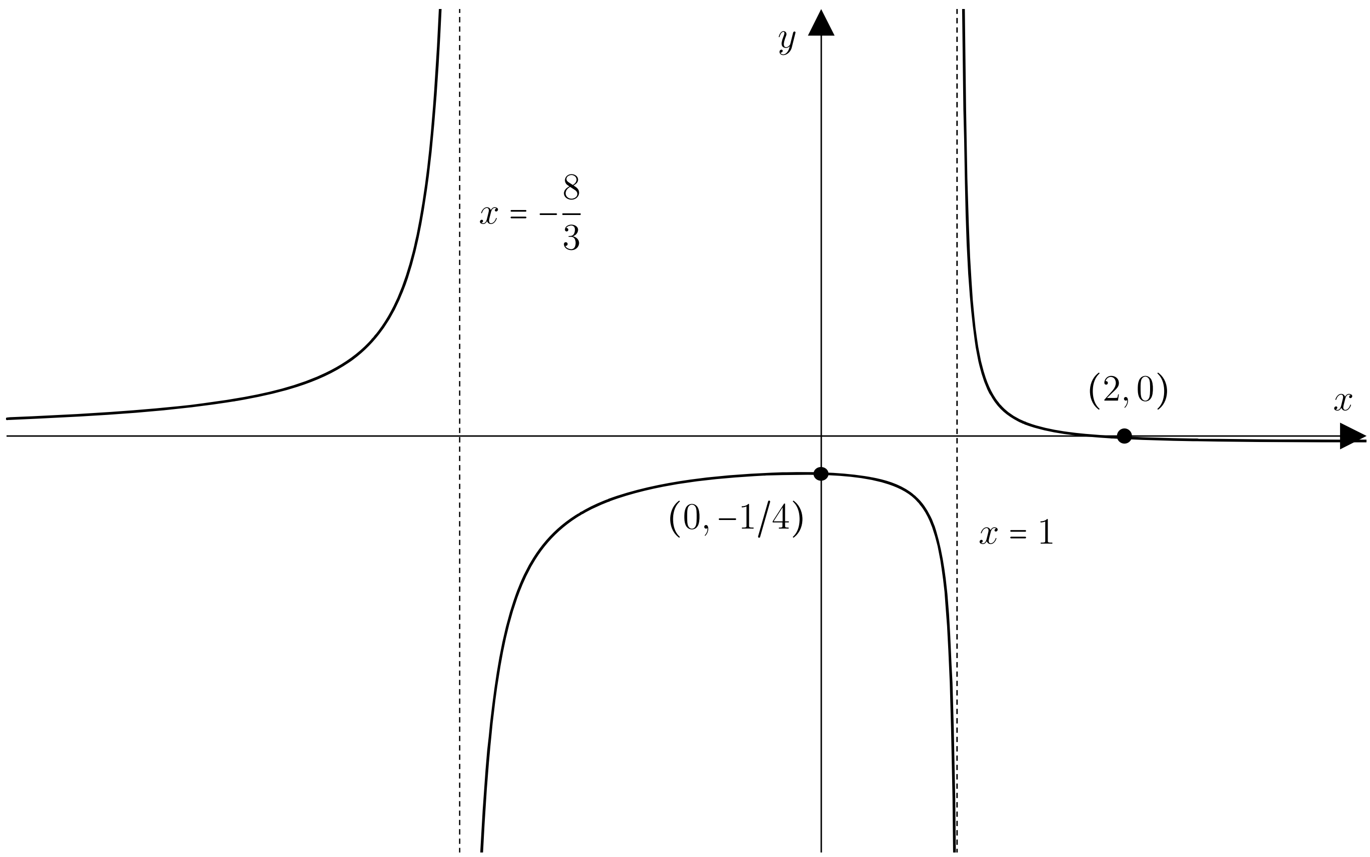
$y(-1) = -e^{-(-1)} = -e$, $y'(-1) = e^{-(-1)} - (-1)e^{-(-1)} = 2e$. So, the tangent at $x = -1$ is

$$y - y(-1) = y'(-1)[x - (-1)] \qquad \text{or} \qquad y + e = 2e(x + 1) \qquad \text{or} \qquad y = 2ex + e.$$

(ii) The first tangent line is horizontal. The second has slope $2e$.

Hence, the acute angle between these two lines is $\tan^{-1}(2e) \approx 1.39$.

A529 (9758 N2019/II/2)(i) You're supposed to just copy from your calculator:



Since $3x^2 + 5x - 8 = (3x + 8)(x - 1)$, the graph has vertical asymptotes $x = -8/3$ and $x = 1$.

As $x \to \pm\infty$, $y \to 0$. So, the graph has horizontal asymptote $y = 0$.

If $y = 0$, then $2 - x = 0$ or $x = 2$. So, the only x-intercept is $(2, 0)$. (Note: $x > 2 \implies y < 0$.)

If $x = 0$, then $y = -1/4$. So, the only y-intercept is $(0, -1/4)$.

(ii) $x \in (-\infty, -8/3) \cup (1, 2)$. **(iii)** $x \in (-8/3, 1) \cup (2, \infty)$.

A530 (9758 N2018/I/4)(a) $2x^2+3x-2=(2x-1)(x+2)=0 \iff x=-2, 0.5$.

Since $2x^2+3x-2$ is a ∪-shaped quadratic, $2x^2+3x-2 \geq 0 \iff x \leq -2$ OR $x \geq 0.5$.

For $x \leq -2$ OR $x \geq 0.5$, $\left|2x^2+3x-2\right| = 2-x \iff 2x^2+3x-2=2-x$

$$\iff \quad 0=2x^2+4x-4=x^2+2x-2 \iff x=\frac{-2\pm\sqrt{2^2-4(1)(-2)}}{2(1)}=-1\pm\sqrt{3}.$$

Check whether each of $-1\pm\sqrt{3}$ satisfies the condition "$x \leq -2$ OR $x \geq 1/2$". Both do: $-1-\sqrt{3}\approx -2.73 \leq -2$, while $-1+\sqrt{3}\approx 0.732 \geq 1/2$. So, both $-1\pm\sqrt{3}$ are roots.

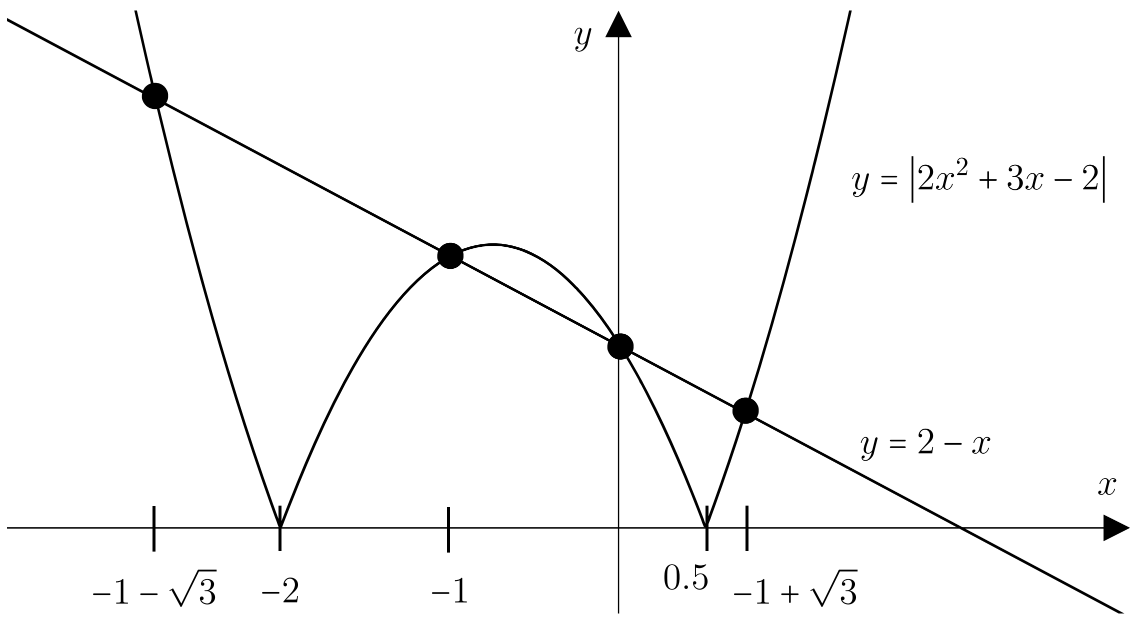
Next, for $-2<x<1/2$, we have $\left|2x^2+3x-2\right|=x \iff$

$$-2x^2-3x+2=2-x \iff 0=2x^2+2x=x^2+x=x(x+1) \iff x=0,-1.$$

Check whether each of 0 and -1 satisfies the condition "$-2<x<1/2$". Both do.

So, the given equation has four roots: $-1\pm\sqrt{3}$, 0, and -1.

(b) The inequality holds if and only if $x \in \left(-1-\sqrt{3},-1\right) \cup \left(0,-1+\sqrt{3}\right)$.



A531 (9758 N2018/I/5). We are given that for all $x \in \mathbb{R} \setminus \{b\}$,

$$ff(x)=f(f(x))=f\left(\frac{x+a}{x+b}\right)=\frac{\frac{x+a}{x+b}+a}{\frac{x+a}{x+b}+b}=\frac{x+a+a(x+b)}{x+a+b(x+b)}=\frac{(1+a)x+a+ab}{(1+b)x+a+b^2}=x$$

$$\implies \quad (1+a)x+a+ab=(1+b)x^2+ax+b^2x$$

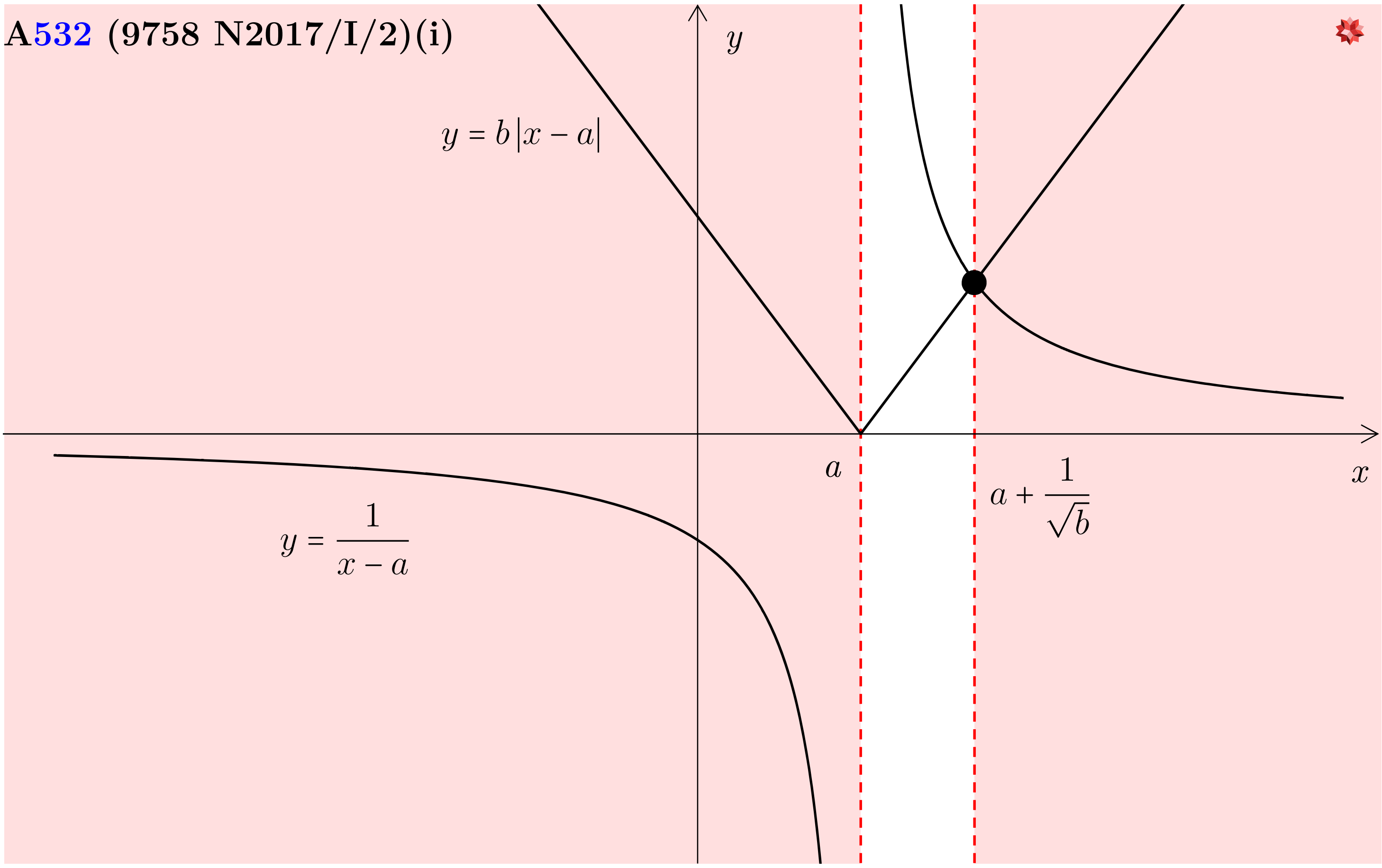
$$\iff \quad 0=(1+b)x^2+\left(b^2-1\right)x-a(b+1)=(1+b)\left[x^2+(b-1)x-a\right]$$

Since the above is true for *all* $x \in \mathbb{R} \setminus \{b\}$, we must have $b=-1$.

Since f is a rectangular hyperbola whose asymptotes are parallel to the axes, f is one-to-one and its inverse f^{-1} exists.

Since $ff(x)=x$ if and only if $f^{-1}(x)=f(x)$, we have $f^{-1}(x)=f(x)=\dfrac{x+a}{x+b}=\dfrac{x+a}{x-1}$.

A532 (9758 N2017/I/2)(i)



(ii) First find the two graphs' intersection point(s): For $x \overset{1}{\geq} a$,

$$\frac{1}{x-a} = b\,|x-a| = b\,(x-a) \iff \frac{1}{b} = (x-a)^2 \overset{2}{\iff} x-a = \sqrt{\frac{1}{b}} \iff x = a+\frac{1}{\sqrt{b}}.$$

(At $\overset{2}{\iff}$, we can discard the negative square root because $x \geq a$.)

We must check that $a+1/\sqrt{b}$ satisfies $\overset{1}{\geq}$, as indeed it does: $a+1/\sqrt{b} \geq a$.

So, for $x \overset{1}{\geq} a$, the two graphs intersect at $x = a+1/\sqrt{b}$.

Next, for $x < a$, $\qquad \dfrac{1}{x-a} = b\,|x-a| = b\,(a-x) \iff -\dfrac{1}{b} = (x-a)^2,$

which never holds since $(x-a)^2 \geq 0$, while $-1/b < 0$ (because $b > 0$). So, for $x < a$, the two graphs do not intersect.

Altogether, the two graphs intersect only at $x = a+1/\sqrt{b}$.

And now, from the graph, we see the given inequality holds if and only if

$$x \in (-\infty, a) \cup \left(a+\frac{1}{\sqrt{b}}, \infty\right).$$

A533 (9758 N2017/I/4). First, observe that $y = \dfrac{4x+9}{x+2} = 4 + \dfrac{1}{x+2}$ $(x \in \mathbb{R} \setminus \{-2\})$.

(i) Compute $y'(x) = -(x+2)^2$. So, $y'(x) < 0$ for all $x \in \mathbb{R} \setminus \{-2\}$.

(ii) We already did this above: $a = 4$ and $b = 1$. The horizontal asymptote is $y = 4$. The vertical asymptote is $x = -2$.

(iii) Right 2 (to get $y = 4 + 1/x$), then down 4 (to get $y = 1/x$). ✸

A534 (9758 N2017/I/5)(a) Let $f(x) = x^3 + ax^2 + bx + c$. By the Remainder Theorem,[663]

$$f(1) = 1 + a + b + c \overset{1}{=} 8,$$

$$f(2) = 8 + 4a + 2b + c \overset{2}{=} 12,$$

$$f(3) = 27 + 9a + 3b + c \overset{3}{=} 25.$$

Either solve this system of three equations with three unknowns using your graphing calculator or "manually", as we now do:

Taking $\overset{2}{=} - \overset{1}{=}$ yields $7 + 3a + b = 4$ or $b \overset{4}{=} -3(a+1)$.

Plug $\overset{4}{=}$ into $\overset{3}{=}$ to get $18 + c = 25$ or $c \overset{5}{=} 7$.

Next, plug $\overset{4}{=}$ and $\overset{5}{=}$ into $\overset{1}{=}$ to get $1 + a - 3(a+1) + 7 = 8$ or $5 - 2a = 8$ or $a = -1.5$.

Now from $\overset{4}{=}$, we also have $b = 1.5$. ✸

(b) From $f(x) = x^3 - 1.5x^2 + 1.5x + 7$. compute $f'(x) = 3x^2 - 3x + 1.5$. This quadratic has negative discriminant $(-3)^2 - 4(3)(1.5) = 9 - 18 = -9 < 0$ and so does not touch the x-axis. It has positive coefficient on the x^2 term and so is $\cup$-shaped. Altogether, it is everywhere above the x-axis—i.e. $f'(x) > 0$ for all $x \in \mathbb{R}$.

Which means[664] that f is everywhere strictly increasing. And so, f can only intersect the x-axis at most once—equivalently, f can have at most one (real) root. ☺

Now observe that $f(-100) < 0$ and $f(0) = 7 > 0$. Since the polynomial function f is continuous, by the Intermediate Value Theorem,[665] there exists $d \in (-100, 0)$ such that $f(d) = 0$. This shows that f has at least one (real) root. ☻

Together, ☺ and ☻ show that f has exactly one real root. Using our graphing calculator, we find that it is $x \approx -1.33$. ✸

(c) $f'(x) = 2 \iff 3x^2 - 3x + 1.5 = 2 \iff 3x^2 - 3x - 0.5 = 0.$

$$\iff x = \frac{3 \pm \sqrt{(-3)^2 - 4(3)(-0.5)}}{2 \cdot 3} = \frac{3 \pm \sqrt{15}}{6} \approx 1.15, -0.145.$$ ✸

[663] Theorem 9
[664] By the Increasing/Decreasing Test (Fact 208).
[665] Theorem 38.8.

A535 (9758 N2017/II/1)(i) Plug $x = 3/t$ and $y = 2t$ into $y = 2x$ to get $2t = 6/t$ or $t = \pm\sqrt{3}$. So, the two points are $A = \left(\sqrt{3}, 2\sqrt{3}\right)$ and $B = \left(-\sqrt{3}, -2\sqrt{3}\right)$. And

$$|AB| = \sqrt{\left[\sqrt{3} - \left(-\sqrt{3}\right)\right]^2 + \left[2\sqrt{3} - \left(-2\sqrt{3}\right)\right]^2} = \sqrt{4 \cdot 3 + 16 \cdot 3} = \sqrt{60} = 2\sqrt{15}.$$

(ii) $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = 2 \div \dfrac{-3}{t^2} = -\dfrac{2}{3}t^2$. The tangent to C at P is $y - 2p = -\dfrac{2}{3}p^2\left(x - \dfrac{3}{p}\right)$.

At D, $0 - 2p = -\dfrac{2p^2}{3}\left(x - \dfrac{3}{p}\right) \iff \dfrac{1}{p} = \dfrac{1}{3}\left(x - \dfrac{3}{p}\right) = \dfrac{1}{3}x - \dfrac{1}{p} \iff x = \dfrac{6}{p}$. So, $D = \left(\dfrac{6}{p}, 0\right)$.

At E, $y - 2p = -\left(2p^2/3\right)(0 - 3/p) \iff y = 2p + 2p^2/p = 4p$. So, $E = (0, 4p)$.

Thus, the midpoint of the line segment DE is $F = (3/p, 2p)$.

Write $x \overset{1}{=} 3/p$ and $y \overset{2}{=} 2p$. Rearrange $\overset{1}{=}$ to get $p = 3/x$—plug into $\overset{2}{=}$ to get $y = 6/x$.

A536 (9758 N2017/II/3)(a) To get from $y = f(x)$ to each equation, ...

(a)(i) Compress inwards towards the y-axis by a factor of 2. So, an x-intercept is $(a/2, 0)$ and a y-intercept $(0, b)$ is the same.

(a)(ii) Translate 1 unit right. So, an x-intercept is $(a+1, 0)$. We can't say anything about the y-intercepts.

(a)(iii) Translate 1 unit right, then compress inwards towards the y-axis by a factor of 2. So, an x-intercept is $((a+1)/2, 0)$. We cannot say anything about the y-intercepts.

(a)(iv) Reflect in the line $y = x$. So, an x-intercept is $(b, 0)$ and a y-intercept is $(0, a)$.

(b)(i) $a = 1$ is excluded because $g(1)$ would be undefined.

(b)(ii) $g^2(x) = 1 - \dfrac{1}{1 - \left(1 - \frac{1}{1-x}\right)} = 1 - \dfrac{1}{\frac{1}{1-x}} = 1 - (1 - x) = x$.

Since $g^2(x) = x$, $g^{-1}(x) = g(x) = 1 - 1/(1-x)$.

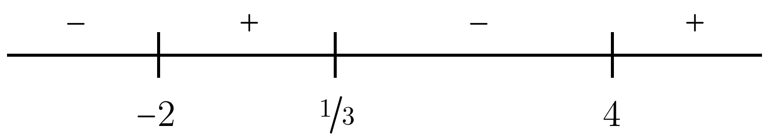
(b)(iii) $g^2(b) = g^{-1}(b) \iff b = 1 - 1/(1-b) \iff 1 - b = 1/(1-b)$

$\iff (1-b)^2 = 1 \iff 1 - b = \pm 1 \iff b = 0, 2$.

A537 (9740 N2016/I/1). $\dfrac{4x^2 + 4x - 14}{x - 4} - (x + 3) = \dfrac{4x^2 + 4x - 14 - (x^2 - x - 12)}{x - 4}$

$$= \frac{3x^2 + 5x - 2}{x - 4} = \frac{(3x-1)(x+2)}{x-4}.$$

So, the given inequality is equivalent to $\dfrac{(3x-1)(x+2)}{x-4} < 0$. Sign diagram for LHS:



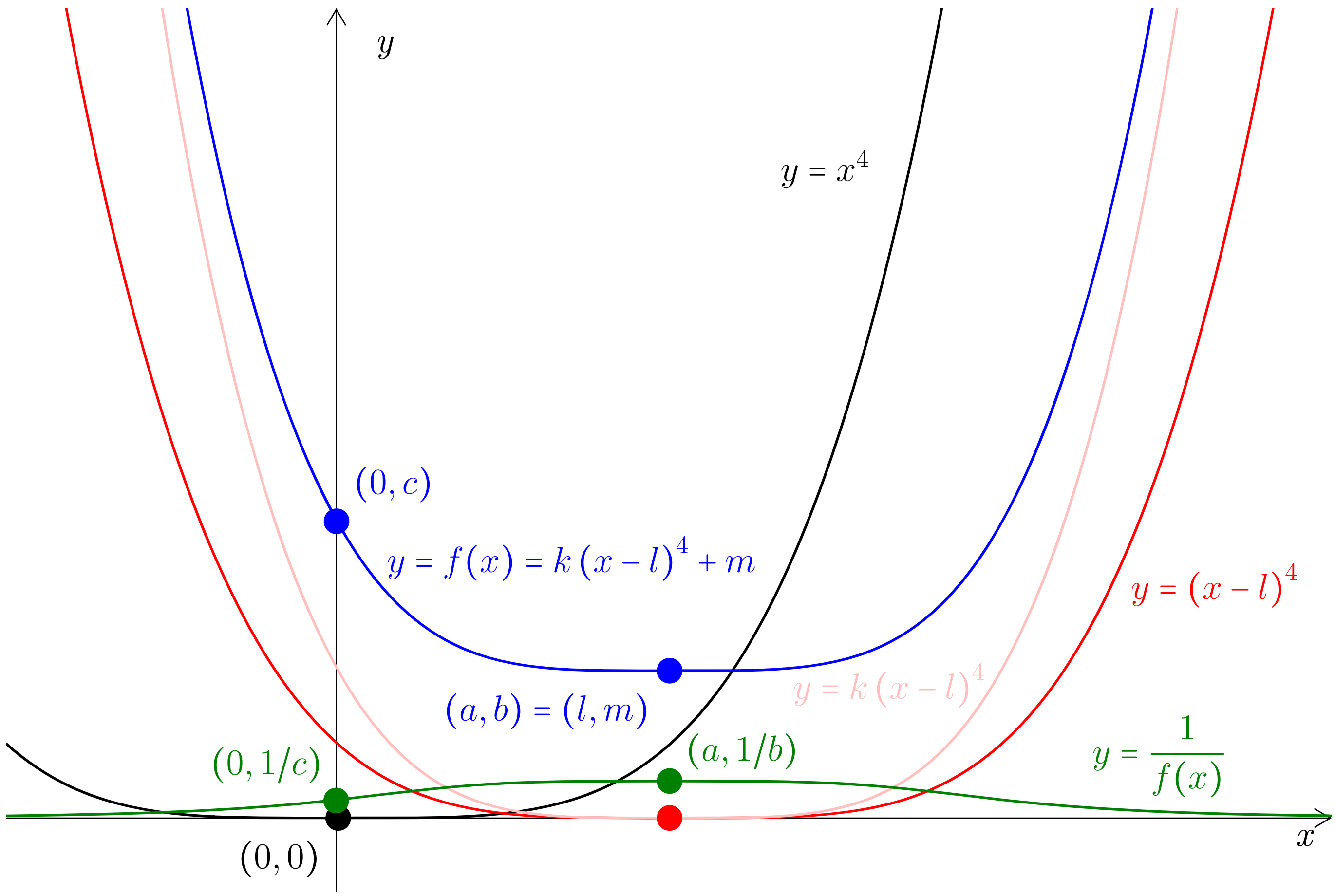
The given inequality holds if and only if $x \in (-\infty, -2) \cup (1/3, 4)$.

A538 (9740 N2016/I/3). $y = x^4$has turning point $(0, 0)$ and y-intercept $(0, 0)$.

1. Start with the graph of $y = x^4$.
2. Translate l units right to get $y = (x - l)^4$ (turning point $(l, 0)$, y-intercept $\left(0, l^4\right)$).
3. Stretch vertically (outwards from the x-axis) by a factor of k to get $y = k(x - l)^4$ (turning point $(l, 0)$, y-intercept $\left(0, kl^4\right)$).
4. Translate m units upwards to get $y = f(x) = k(x - l)^4 + m$ (turning point (l, m), y-intercept $\left(0, kl^4 + m\right)$).

So, $(l, m) = (a, b)$ and $\left(0, kl^4 + m\right) = (0, c)$.

Hence, $l = a$, $m = b$, and $k = (c - m)/l^4 = (c - b)/a^4$.



The graph of $y = 1/f(x)$ has y-intercept $(0, 1/c)$ and turning point $(a, 1/b)$.

A539 (9740 N2016/I/10)(a)(i) Write $y = f(x) = 1 + \sqrt{x} \in \text{Range } f = [1, \infty)$.

Do the algebra: $x = (y-1)^2$.

So, the inverse of f is the function $f^{-1} : [1, \infty) \to \mathbb{R}_0^+$ defined by $f^{-1}(y) = (y-1)^2$.

(a)(ii) $ff(x) = f\left(1 + \sqrt{x}\right) = 1 + \sqrt{1 + \sqrt{x}} \stackrel{1}{=} x \iff \sqrt{1 + \sqrt{x}} = x - 1 \implies 1 + \sqrt{x} = x^2 - 2x + 1 \iff \sqrt{x} = x^2 - 2x \implies x = x^4 - 4x^3 + 4x^2 \iff x\left(x^3 - 4x^2 + 4x - 1\right) = 0 \iff$ (by $\stackrel{1}{=}$, $x \neq 0$) $x^3 - 4x^2 + 4x - 1 = 0$.

(a)(iii) Let $p(x) = x^3 - 4x^2 + 4x - 1$. Since $p(1) = 0$, by the Factor Theorem, $x - 1$ is a factor of $p(x)$. So, write $x^3 - 4x^2 + 4x - 1 = (x-1)\left(ax^2 + bx + c\right)$.

Comparing coefficients, $a = 1$, $-a + b = -4$, and $-c = -1$ (so, $b = -3$ and $c = 1$).

Now, $0 = x^3 - 4x^2 + 4x - 1 = (x-1)\left(x^2 - 3x + 1\right) \iff x = 1$ OR

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2}.$$

We now check whether each of these three possible values of x satisfies $\stackrel{1}{=}$. We find that $x = 1$ and $x = \left(3 - \sqrt{5}\right)/2$ do not,[666] while $x = \left(3 + \sqrt{5}\right)/2$ does.

(a)(iv) Suppose $f(f(x)) = x$. Since $x \in \text{Range } f = \text{Domain } f$, we can apply f^{-1} to get $f^{-1}(f(f(x))) \stackrel{1}{=} f^{-1}(x)$. But by the Cancellation Law, we also have $f^{-1}(f(f(x))) \stackrel{2}{=} f(x)$. Now plug $\stackrel{1}{=}$ into $\stackrel{2}{=}$ to get $f^{-1}(x) = f(x)$.[667]

(b)(i) $g(0) = 1$, $g(1) = 1 + g(0) = 1 + 1 = 2$, $g(2) = 2 + g(1) = 2 + 2 = 4$, $g(3) = 1 + g(2) = 5$, $g(4) = 2 + g(2) = 6$, $g(5) = 1 + g(4) = 7$, $g(6) = 2 + g(3) = 7$, $g(7) = 1 + g(6) = 8$, $g(12) = 2 + g(6) = 9$.

(b)(ii) The element 7 in Codomain g is "hit" twice: $g(5) = 7$ and $g(6) = 7$. So, g is not one-to-one. Hence, g has no inverse.

A540 (9740 N2015/I/1)(i) Compute $y'(x) = -2a/x^3 + b$.

The given information yields this system of equations:

$$-2.4 \stackrel{1}{=} a/1.6^2 + 1.6b + c = a/2.56 + 1.6b + c,$$

$$3.6 \stackrel{2}{=} a/(-0.7)^2 - 0.7b + c = a/0.49 - 0.7b + c,$$

$$y'(1) = 2 \stackrel{3}{=} -2a/1^3 + b = -2a + b.$$

Solving by calculator or hand, $a \approx -3.593$, $b \approx -5.187$, $c \approx 7.303$.

(ii) $a/x^2 + bx + c = 0 \implies a + bx^3 + cx^2 = 0 \implies x \approx -0.589$ (calculator).

(iii) As $x \to \pm\infty$, $y \to bx + c$. Hence, the other asymptote is $y = bx + c$ or $y \approx 5.187x + 7.303$.

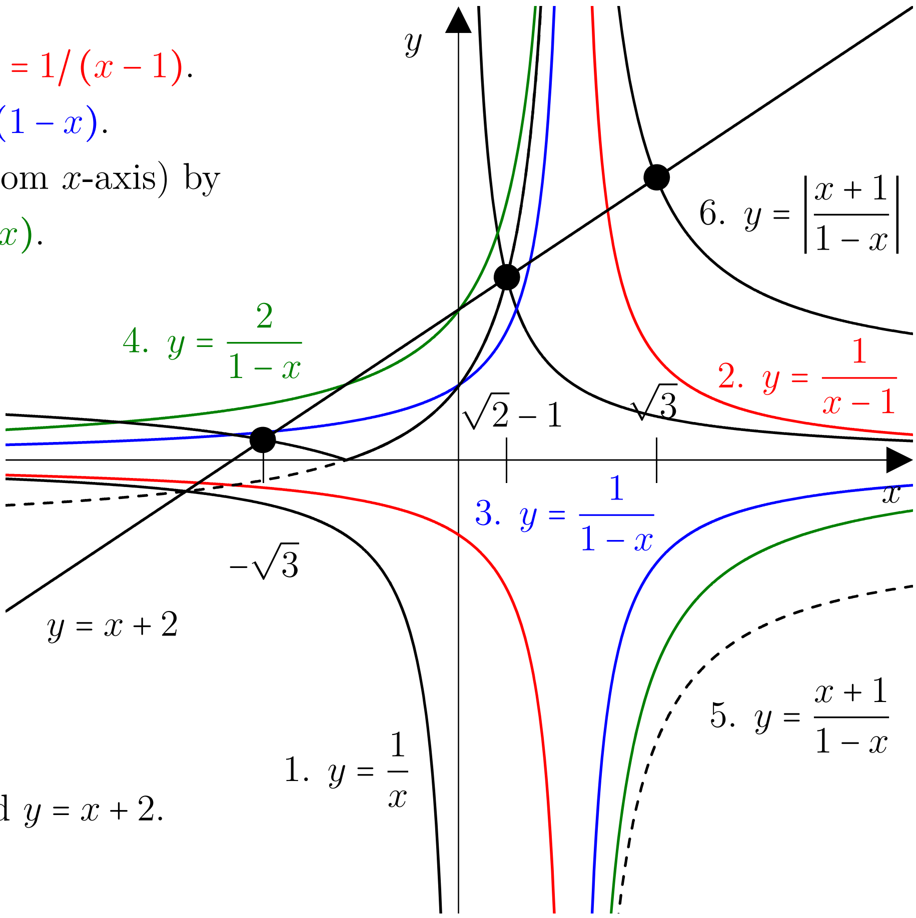
[666]These two extraneous solutions (see Ch. 42.1) were introduced at the two $\implies$'s above.

[667]This answer is largely a reproduction of the proof of Fact 56 ($\implies$).

A541 (9740 N2015/I/2)(i) You're supposed to just copy from your calculator. But here we won't. First, observe that $(x+1)/(1-x) = -1 + 2/(1-x)$.

1. Graph $y = 1/x$.
2. Translate 1 unit right to get $y = 1/(x-1)$.
3. Reflect in x-axis to get $y = 1/(1-x)$.
4. Stretch vertically (outwards from x-axis) by a factor of 2 to get $y = 2/(1-x)$.
5. Down 1 to get
   $$y = -1 + \frac{2}{1-x} = \frac{x+1}{1-x}$$
   (asymptotes $x = 1$, $y = -1$).
6. Reflect any portion of the graph below x-axis in the x-axis to get
   $$y = \left|\frac{x+1}{1-x}\right|$$
   (asymptotes $x = 1$, $y = 1$).

As instructed, we've also graphed $y = x + 2$.



(ii) We look for the intersection points of the two graphs.

First, for $(x+1)/(1-x) > 0$ or $x \in (-1,1)$, we have

$$\left|\frac{x+1}{1-x}\right| = x+2 \iff \frac{x+1}{1-x} = x+2 \iff x+1 = -x^2 - x + 2 \iff 0 = x^2 + 2x - 1$$

$$\iff \qquad x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)} = -1 \pm \sqrt{2}.$$

We can verify that $-1+\sqrt{2} \in [-1,1]$, while $-1-\sqrt{2} \notin [-1,1]$.

So, here, we've found only one intersection point: $x = -1+\sqrt{2}$.

Next, for $(x+1)/(1-x) < 0$ or $x \in \mathbb{R} \setminus [-1,1]$, we have

$$\left|\frac{x+1}{1-x}\right| = x+2 \iff -\frac{x+1}{1-x} = x+2 \iff -x-1 = -x^2 - x + 2 \iff 0 = x^2 - 3 \iff x = \pm\sqrt{3}.$$

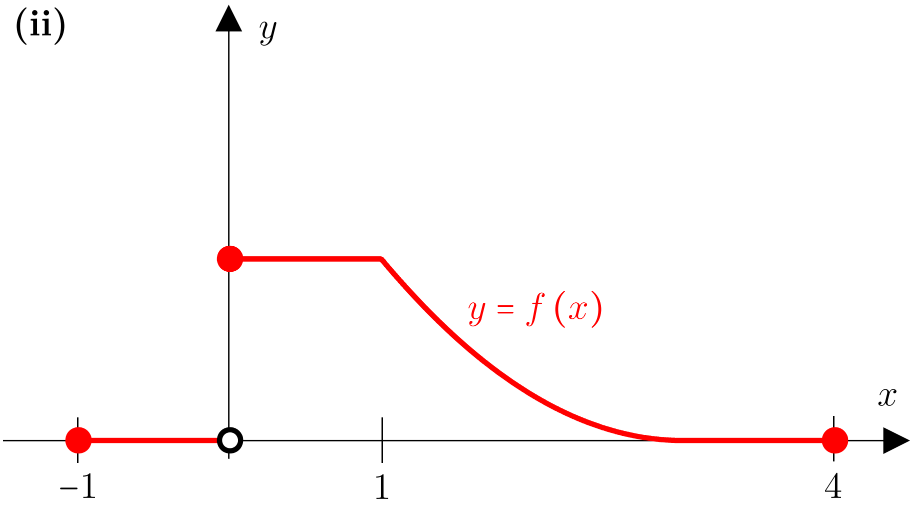
We can verify that $\pm\sqrt{3} \in \mathbb{R} \setminus [-1,1]$.

So, here, we've found two intersection points: $x = \pm\sqrt{3}$.

From the graph, the given inequality holds $\iff x \in \left(-\sqrt{3}, \sqrt{2}-1\right) \cup \left(\sqrt{3}, \infty\right)$.

A542 (9740 N2015/I/5)(i) Translate 3 units right to get $y = (x-3)^2$, then stretch vertically (outwards from the x-axis) by a factor of 0.25[668] to get $y = 0.25\,(x-3)^2$.

(ii)

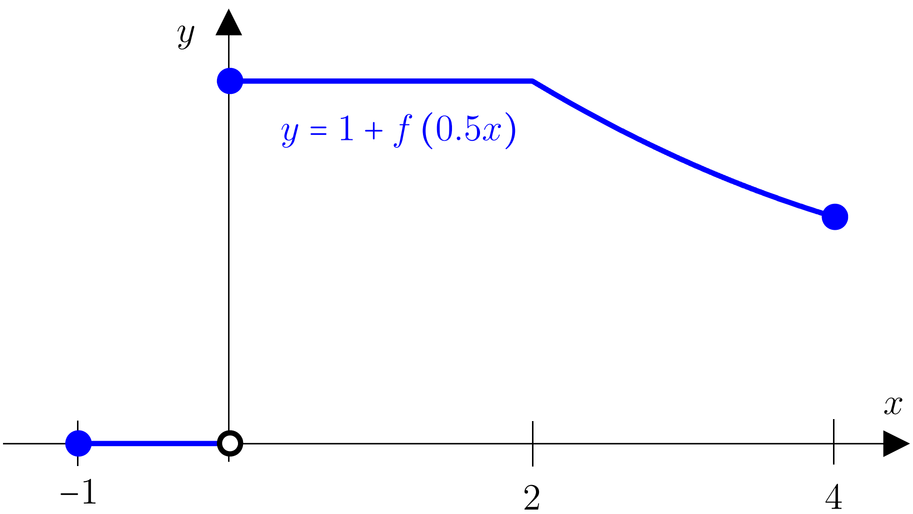


(iii) We were given
$$f(x) = \begin{cases} 1, & \text{for } 0 \le x \le 1, \\ 0.25\,(x-3)^2, & \text{for } 1 < x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

So,
$$f(0.5x) = \begin{cases} 1, & \text{for } 0 \le x \le 2, \\ 0.25\,(0.5x-3)^2, & \text{for } 2 < x \le 6, \\ 0, & \text{otherwise.} \end{cases}$$

And,
$$1 + f(0.5x) = \begin{cases} 2, & \text{for } 0 \le x \le 2, \\ 1 + 0.25\,(0.5x-3)^2, & \text{for } 2 < x \le 6, \\ 1, & \text{otherwise.} \end{cases}$$

Note that the usual method *does* also work here: First stretch horizontally (outwards from the y-axis) by a factor of 2, then translate 1 unit upward. However, we have to do these operations carefully for each of the three separate "pieces" or intervals.



[668]Or equivalently, *compress* vertically (inwards towards the x-axis) by a factor of 4.

A543 (9740 N2015/II/3)(a)(i) Write $y = f(x) = 1/\left(1-x^2\right) \in \text{Range}\, f = (0,\infty)$.

Do the algebra: $x = \pm\sqrt{1-1/y}$. Since $x > 1$, we can discard the negative value and be left with $x = \sqrt{1-1/y}$.

Hence, f is one-to-one and has an inverse.

(ii) From our above work, the inverse of f is the function $f^{-1} : (0,\infty) \to (1,\infty)$ defined by $f^{-1}(y) = \sqrt{1-1/y}$.

(b) Suppose $y \in \text{Range}\, g$. Then there exists some $x \in \mathbb{R} \setminus \{-1,1\}$ such that $y = g(x)$ or

$$y = \frac{2+x}{1-x^2} \qquad \text{or} \qquad y - yx^2 = 2+x \qquad \text{or} \qquad yx^2 + x + 2 - y \overset{1}{=} 0.$$

The quadratic equation $\overset{1}{=}$ (in x) holds for some $x \in \mathbb{R}$ if and only if the determinant is non-negative:

$$1^2 - 4y(2-y) \geq 0 \qquad \text{or} \qquad 4y^2 - 8y + 1 \overset{2}{\geq} 0.$$

The quadratic polynomial $4y^2 - 8y + 1$ is $\cup$-shaped (because its coefficient on y^2 is positive). By the quadratic formula, its roots are

$$\frac{8 \pm \sqrt{(-8)^2 - 4(4)(1)}}{2(4)} = 1 \pm \sqrt{1-\frac{1}{4}} = 1 \pm \frac{\sqrt{3}}{2}.$$

Thus, $\overset{2}{\geq}$ and therefore also $\overset{1}{=}$ hold for some $x \in \mathbb{R}$ if and only if

$$y \overset{3}{\in} \left(-\infty, 1-\frac{\sqrt{3}}{2}\right] \cup \left[1+\frac{\sqrt{3}}{2}, \infty\right).$$

But now, observe that $\overset{1}{=}$ does not hold if $x = -1$ or $x = 1$.

Hence, there exists some $x \in \mathbb{R} \setminus \{-1,1\}$ such that $y = g(x)$ if and only if $\overset{3}{\in}$ holds.

Thus, $$\text{Range}\, g = \left(-\infty, 1-\frac{\sqrt{3}}{2}\right] \cup \left[1+\frac{\sqrt{3}}{2}, \infty\right).$$

A544 (9740 N2014/I/1)(i) $f^2(x) = f\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{1-x}{-x} = \frac{x-1}{x} = 1 - \frac{1}{x}$.

Write $y = f(x) = \frac{1}{1-x} \in \text{Range}\, f = \mathbb{R} \setminus \{0,1\}$.

Do the algebra: $x = 1 - 1/y$.

So, f is one-to-one and its inverse is the function $f^{-1} : \mathbb{R} \setminus \{0,1\} \to \mathbb{R} \setminus \{0,1\}$ defined by $f^{-1}(y) = 1 - 1/y$.

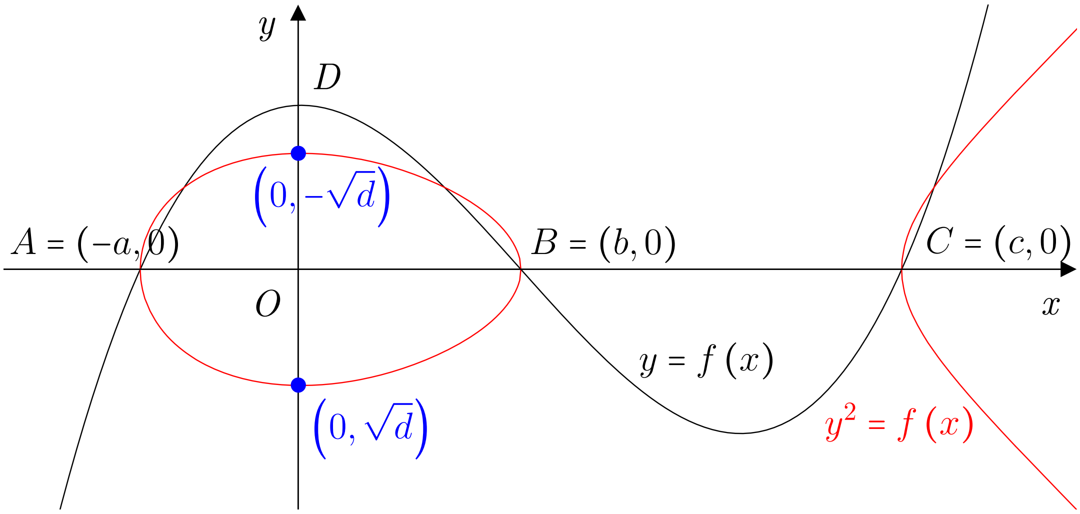
We observe also that, as was to be shown, $f^2(x) = f^{-1}(x)$ for all $x \in \mathbb{R} \setminus \{0,1\}$.

(ii) $f^3(x) = ff^2(x) = ff^{-1}(x) = x$.[669]

A545 (9740 N2014/I/4)(i) The graph of $y^2 = f(x)$ ...

[669]The last step uses an Inverse Cancellation Law—Proposition 5(b).

- Is symmetric in the x-axis.
- Is empty wherever $f(x) < 0$. So here, it is empty to the left of A and between B and C.
- Has turning points $\left(0, \sqrt{d}\right)$ and $\left(0, -\sqrt{d}\right)$ because the graph of $y = f(x)$ has turning point $(0, d)$.
- Intersects the graph of $y = f(x)$ wherever $f(x) = 1$.
- Has the same x-intercepts as $y = f(x)$, namely $A = (-a, 0)$, $B = (b, 0)$, and $C = (c, 0)$.



(ii) The tangents to the curve $y^2 = f(x)$ at the points where it crosses the x-axis are vertical.

A546 (9740 N2014/II/1)(i) $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = 6 \div 6t = \dfrac{1}{t} = 0.4$. So, $t = 2.5$.

(ii) The tangent line at $\left(3p^2, 6p\right)$ has equation $y - 6p = \dfrac{1}{p}\left(x - 3p^2\right)$. Where this line meets the y-axis, we have $x = 0$ and so $y = 6p + \dfrac{1}{p}\left(0 - 3p^2\right) = 6p - 3p = 3p$. Hence, $D = (0, 3p)$ and the mid-point of the line segment PD is $\left(1.5p^2, 4.5p\right)$.

Write $x \overset{1}{=} 1.5p^2$ and $y \overset{2}{=} 4.5p$. Rearrange $\overset{2}{=}$ to get $p \overset{3}{=} y/4.5$, then plug $\overset{3}{=}$ into $\overset{1}{=}$ to get $x = 1.5\,(y/4.5)^2 = 2y^2/27$.

A547 (9740 N2013/I/2). Rearrange the given equation to get

$$xy - y = x^2 + x + 1 \qquad \text{or} \qquad x^2 + (1 - y)\,x + y + 1 \overset{1}{=} 0.$$

Observe that $\overset{1}{=}$ is a quadratic equation (in the variable x). It holds if and only if its discriminant is non-negative:

$$(1 - y)^2 - 4\,(1)\,(y + 1) = y^2 - 6y - 3 \overset{2}{\geq} 0.$$

By the quadratic formula, $y^2 - 6y - 3 = 0$ has roots

$$y = \frac{6 \pm \sqrt{(-6)^2 - 4\,(1)\,(-3)}}{2} = 3 \pm \sqrt{9 + 3} = 3 \pm 2\sqrt{3}.$$

Since $y^2 - 6y - 3$ is a ∪-shaped quadratic (because it has positive coefficient on y^2), $\overset{2}{\geq}$ holds if and only if $y \in \left(-\infty, 3 - 2\sqrt{3}\right] \cup \left[3 + 2\sqrt{3}, \infty\right)$—so, this is the set of values that y can take.

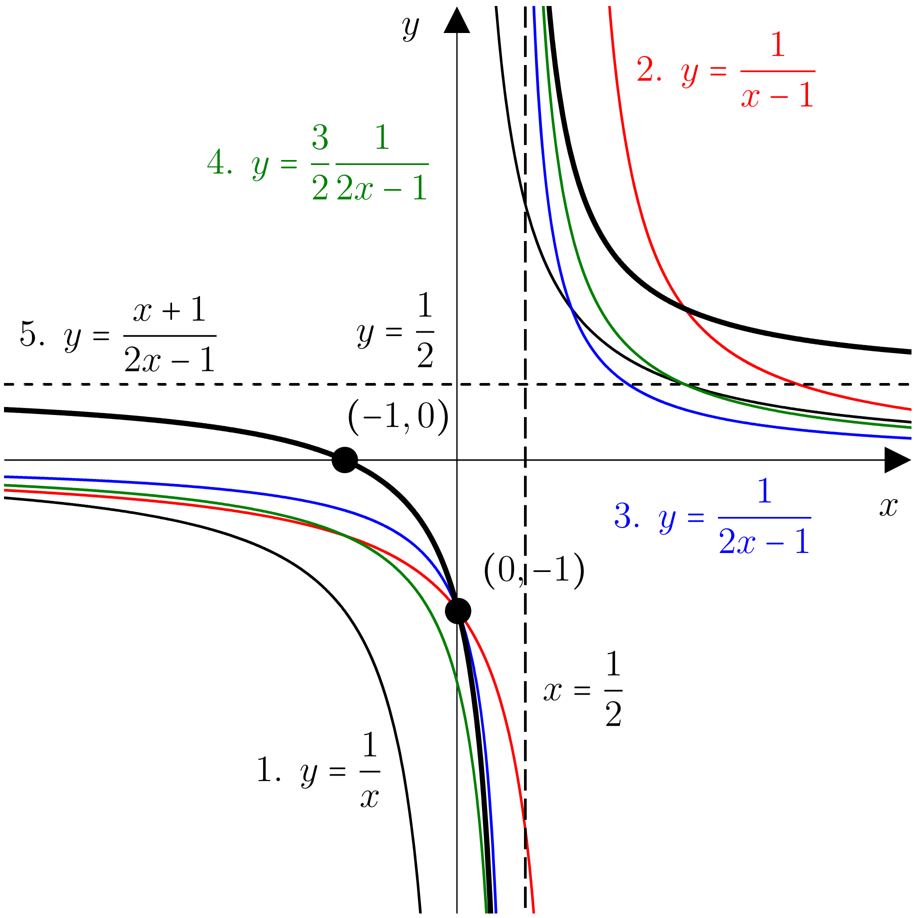
A548 (9740 N2013/I/3)(i) $y = \dfrac{x+1}{2x-1} = \dfrac{1}{2} + \dfrac{3}{2}\dfrac{1}{2x-1}$.

As usual, you can just copy from your calculator. But here as an exercise, let's see how we can get this graph as a series of transformations from $y = 1/x$:

1. Start with the graph of $y = 1/x$ (which has horizontal asymptote $y = 0$ and vertical asymptote $x = 0$).
2. Translate 1 unit right to get $y = 1/(x-1)$ (horizontal asymptote $y = 0$ and vertical asymptote $x = 1$).
3. Compress (inwards towards the y-axis) by a factor of 2 to get $y = 1/(2x-1)$ (horizontal asymptote $y = 0$ and vertical asymptote $x = 1/2$).
4. Stretch (outwards from the y-axis) by a factor of $\dfrac{3}{2}$ to get $y = \dfrac{3}{2}\dfrac{1}{2x-1}$ (horizontal asymptote $y = 0$ and vertical asymptote $x = \dfrac{1}{2}$).
5. Translate 1/2 unit up to get $y = \dfrac{1}{2} + \dfrac{3}{2}\dfrac{1}{2x-1}$ (horizontal asymptote $y = \dfrac{1}{2}$ and vertical asymptote $x = \dfrac{1}{2}$).

If $x = 0$, then $y = -1$. So, the only y-intercept is $(0, -1)$.

If $y = 0$, then $x = -1$. So, the only x-intercept is $(-1, 0)$.



(ii) $\dfrac{x+1}{2x-1} < 1 \iff 0 > \dfrac{x+1}{2x-1} - 1 = \dfrac{-x+2}{2x-1} \iff x \in (-\infty, 1/2) \cup (2, \infty)$.

A549 (9740 N2013/II/1)(i) Range $g = \mathbb{R} \nsubseteq \mathbb{R} \setminus \{1\} =$ Domain f.

(ii) Since Range $f \subseteq \mathbb{R} =$ Domain g, the composite function gf exists.

Now, $(gf)(x) = g(f(x)) = 1 - 2f(x) = 1 - 2\dfrac{2+x}{1-x} = \dfrac{-3x-3}{1-x} = 3\dfrac{x+1}{x-1} = 3 + \dfrac{6}{x-1}$.

(iii) Write $(gf)(y) = 3 + \dfrac{6}{y-1} = 5$. Solving, $y = 4$. So, $(gf)^{-1}(5) = 4$.

The above line probably sufficed on the A-Level exam. But strictly speaking, it is incomplete and thus incorrect. See footnote for details.[670]

A550 (9740 N2012/I/1). Let x, y, and z be, respectively, the costs of the under-16, 16–65, and over-65 tickets. Then the given information yields this system of equations:

$$9x + 6y + 4z \overset{1}{=} \$162.03, \qquad 7x + 5y + 3z \overset{2}{=} \$128.36, \qquad 10x + 4y + 5z \overset{3}{=} \$158.50.$$

You can use your graphing calculator to solve this system of equations. But here as an exercise, let's not:

Taking $2\times \overset{1}{=} - \overset{2}{=} - \overset{3}{=}$ yields $x + 3y = \$37.20$ or $y \overset{4}{=} \$12.40 - x/3$.

Plug $\overset{4}{=}$ into $\overset{1}{=}$ to get $7x + 4z + \$74.40 = \$162.03$ or $z \overset{5}{=} (\$87.63 - 7x)/4$.

Plug $\overset{4}{=}$ and $\overset{5}{=}$ into $\overset{3}{=}$ to get $10x + \$49.60 - 4x/3 + \$438.15/4 - 35x/4 = \$158.50$ or

$x(10 - 4/3 - 35/4) = \$158.50 - \$438.15/4 - \$49.60$ or $x = \$7.65$.

And now from $\overset{4}{=}$ and $\overset{5}{=}$, we also have $y = \$9.85$ and $z = \$8.52$.

A551 (9740 N2012/I/7)(i) Write $y = g(x) = \dfrac{x+k}{x-1} = 1 + \dfrac{k+1}{x-1} \in \text{Range}\, g = \mathbb{R} \setminus \{1\}$.

Do the algebra: $x = 1 + \dfrac{k+1}{y-1}$.

So, g is one-to-one and its inverse is the function $g^{-1} : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{1\}$ defined by $g^{-1}(x) = 1 + \dfrac{k+1}{x-1}$.

Since $g(x) = g^{-1}(x)$ for all $x \in \text{Domain}\, f = \mathbb{R} \setminus \{1\}$, g is self-inverse (by the given definition).

(ii) As $x \to 1$, $y \to \pm\infty$. So, the graph has the vertical asymptote $x = 1$.

As $x \to \pm\infty$, $y \to 1$. So, the graph has the horizontal asymptote $y = 1$.

If $x = 0$, then $y = \dfrac{0+k}{0-1} = -k$. So, the only y-intercept is $(0, -k)$.

If $y = 0$, then $0 = x + k$ or $x = -k$. So, the only x-intercept is $(-k, 0)$.

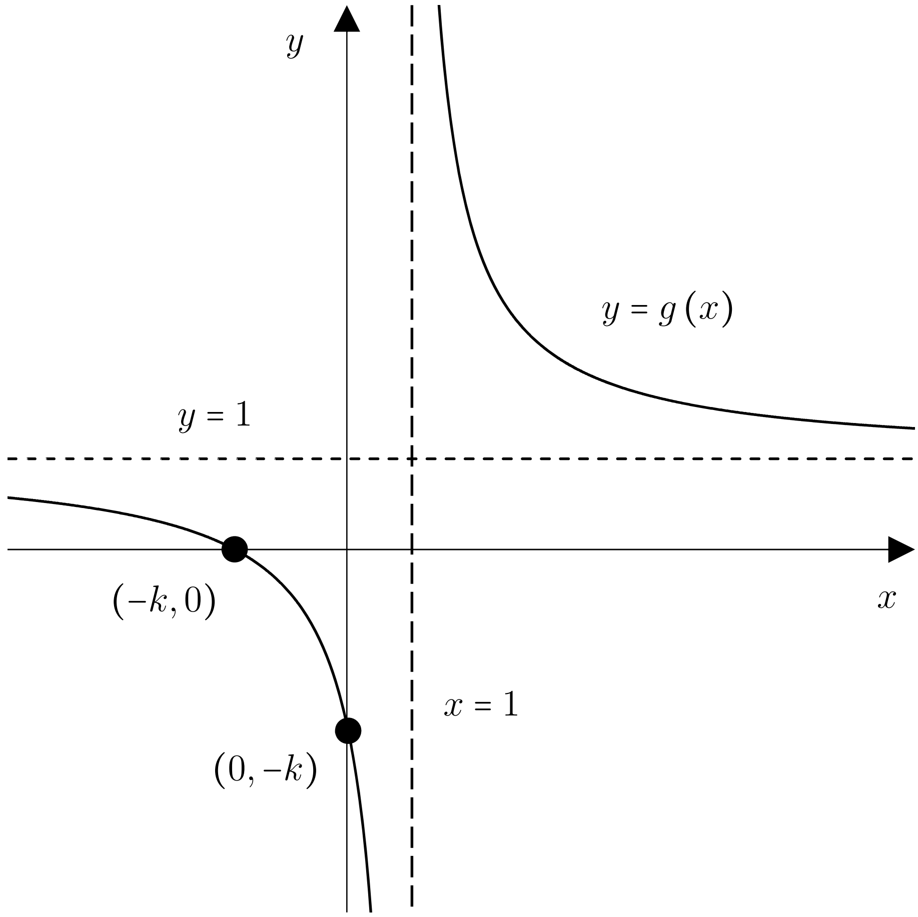
[670] In order to even write $(gf)^{-1}(5)$, we must first prove the existence of the inverse function $(gf)^{-1}$. So, strictly speaking, the above is incomplete and incorrect because we simply presume the existence of $(gf)^{-1}$ without actually proving it. A correct answer for **(iii)** is the following:

Write $y = gf(x) = 3 + \dfrac{6}{x-1} \in \text{Range}\, gf = \mathbb{R} \setminus \{3\}$.

Do the algebra: $x = 1 + 6/(y-3)$.

So, gf is one-to-one and its inverse is the function $(gf)^{-1} : \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{1\}$ defined by $(gf)^{-1}(y) = 1 + 6/(y-3)$.

Hence, $(gf)^{-1}(5) = 1 + 6/(5-3) = 4$.



(iii) Since g is self-inverse, a line of symmetry is $y = x$.

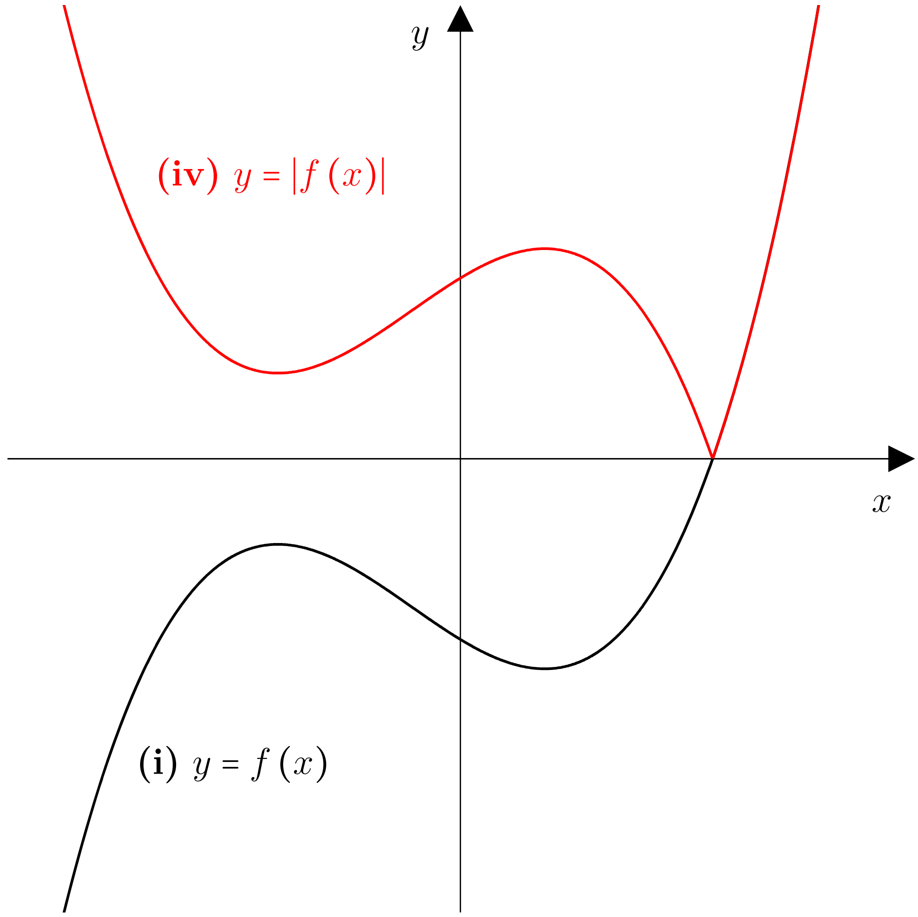
1. Start with the graph of $y = \dfrac{1}{x}$.

2. Translate 1 unit right to get $y = \dfrac{1}{x-1}$.

3. Stretch vertically (outwards from the x-axis) by a factor of $k+1$ to get $y = \dfrac{k+1}{x-1}$.

4. Translate 1 unit up to get $y = 1 + \dfrac{k+1}{x-1} = \dfrac{x+k}{x-1}$.

A552 (9740 N2012/II/3)(i) You're supposed to just graph on your calculator and copy:



(ii) $f(x) = 4 \iff x^3 + x^2 - 2x - 4 = 4 \iff x^3 + x^2 - 2x - 8 = 0$. By observation, $x = 2$ is an integer solution to this last equation.

Write $$x^3 + x^2 - 2x - 8 = (x-2)\left(ax^2 + bx + c\right).$$

Comparing coefficients on the x^3 and the constant terms, $a = 1$ and $c = 4$.

Comparing coefficients on the x^2 term, $-2a + b = 1$ or $b = 3$. Hence,

$$x^3 + x^2 - 2x - 8 = (x-2)\left(x^2 + 3x + 4\right).$$

The quadratic polynomial $x^2 + 3x + 4$ has negative determinant. Thus, there are no other real solutions to $f(x) = 4$.

(iii) The given equation is equivalent to $f(x+3) = 4$.

In **(ii)**, we showed that $f(x) = 4$ has only one real solution: $x = 2$.

Hence, the equation here in **(iii)** has solution has $x + 3 = 2$ or $x = -1$.

(iv) See above. Where $f(x) < 0$, reflect the graph in the x-axis. Where $f(x) \geq 0$, keep it unchanged.

(v) $|f(x)| = 4 \iff \left|x^3 + x^2 - 2x - 4\right| \overset{1}{=} 4$.

Suppose $x^3 + x^2 - 2x - 4 \overset{2}{\geq} 0$. Then $\overset{1}{=}$ becomes $x^3 + x^2 - 2x - 4 = 4$ or $x^3 + x^2 - 2x - 8 = 0$. We already found that this cubic equation has only one real root, namely 2. We can verify that $x = 2$ satisfies $\overset{2}{\geq}$. Hence, $x = 2$ is a solution for $\overset{1}{=}$.

Now suppose instead that $x^3 + x^2 - 2x - 4 \overset{3}{<} 0$. Then $\overset{1}{=}$ becomes $-x^3 - x^2 + 2x + 4 = 4$ or $0 = x^3 + x^2 - 2x = x\left(x^2 + x - 2\right) = x(x+2)(x-1)$.

This second cubic equation has three real roots, namely 0, −2, and 1. We can verify that all three of these values satisfy $\overset{3}{<}$. Hence, $x = 0, -2, 1$ are solutions for $\overset{1}{=}$.

Altogether, the equation $|f(x)| = 4$ has four real solutions (or roots): −2, 0, 1, and 2.

A553 (9740 N2011/I/1). The numerator $x^2 + x + 1$ is a quadratic polynomial with positive coefficient on x^2 $(1 > 0)$ and negative discriminant $(1^2 - 4(1)(1) = -3 < 0)$. So, the numerator is always positive.

The denominator $x^2 + x - 2$ is a quadratic polynomial with positive coefficient on x^2 $(1 > 0)$ and these roots:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm 3}{2} = -2, 1.$$

Hence, the denominator is negative $\iff x \in (-2, 1)$.

Altogether, the given inequality holds $\iff x \in (-2, 1)$.

A554 (9740 N2011/I/2)(i) The given information yields this system of equations:

$$a(-1.5)^2 + b(-1.5) + c = 4.5, \qquad a(2.1)^2 + b(2.1) + c = 3.2, \qquad a(3.4)^2 + b(3.4) + c = 4.1.$$

$$\text{Or,} \qquad 2.25a - 1.5b + c \overset{1}{=} 4.5, \qquad 4.41a + 2.1b + c \overset{2}{=} 3.2, \qquad 11.56a + 3.4b + c \overset{3}{=} 4.1.$$

Taking $\overset{2}{=} - \overset{1}{=}$ yields $2.16a + 3.6b \overset{4}{=} -1.3$.

Taking $\overset{3}{=} - \overset{2}{=}$ yields $7.15a + 1.3b \overset{5}{=} 0.9$.

Taking $\dfrac{36}{13} \times \overset{5}{=} - \overset{4}{=}$ yields $\dfrac{36}{13} \times 7.15a - 2.16a = \dfrac{36}{13} \times 0.9 + 1.3$ or

$$a = \left(\frac{36}{13} \times 0.9 + 1.3\right) \Big/ \left(\frac{36}{13} \times 7.15 - 2.16\right) \overset{6}{\approx} 0.215.$$

From $\overset{4}{=}$ and $\overset{6}{\approx}$, $b = (-1.3 - 2.16a)/3.6 \overset{7}{\approx} -0.490$.

From $\overset{1}{=}$, $\overset{6}{\approx}$, and $\overset{7}{\approx}$, $c = 4.5 - 2.25a + 1.5b \approx 3.281$.

(ii) f is increasing $\iff f'(x) = 2ax + b \geq 0 \iff x \geq -\dfrac{b}{2a} \approx 1.14$.

A555 (9740 N2011/II/3)(i) Write $y = f(x) = \ln(2x + 1) + 3 \in \text{Range } f = \mathbb{R}$.

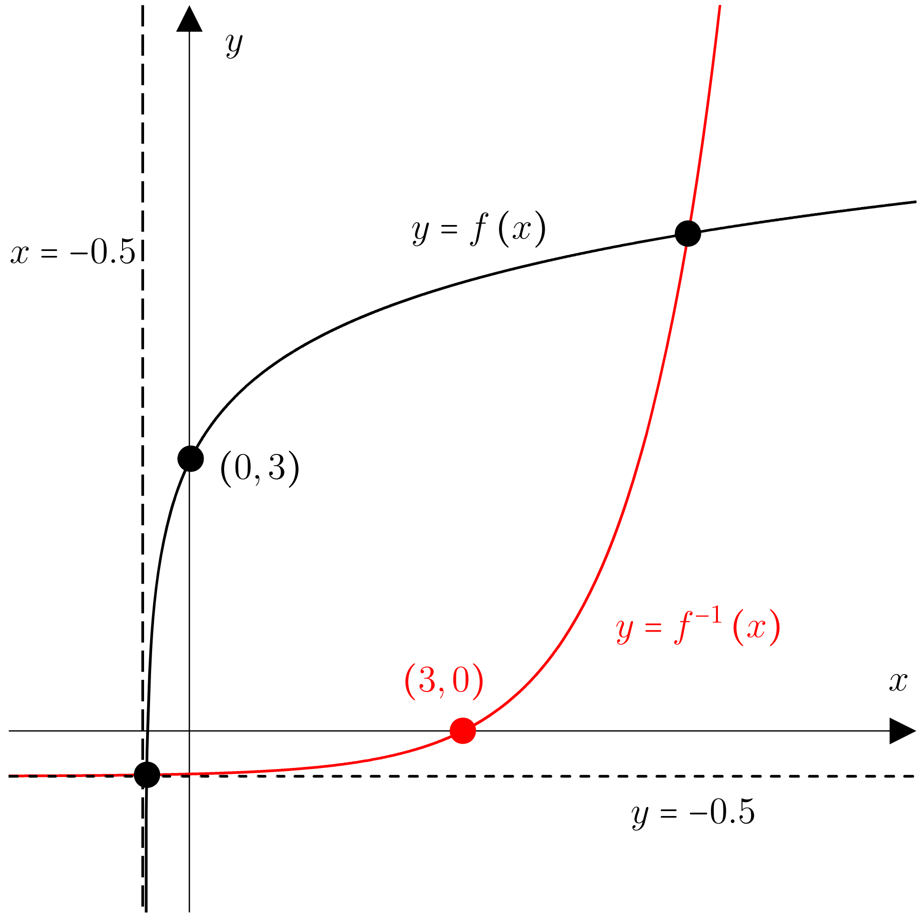
Do the algebra: $x = [\exp(y - 3) - 1]/2$.

So, f is one-to-one and its inverse is the function $f^{-1} : \mathbb{R} \to (-1/2, \infty)$ defined by $f^{-1}(x) = [\exp(x - 3) - 1]/2$.

As usual, $\text{Range } f^{-1} = \text{Codomain } f^{-1} = (-1/2, \infty)$.

(ii) For the graph of f,

- As $x \to -0.5$, $f(x) \to -\infty$. So, $x = -0.5$ is a vertical asymptote.
- If $x = 0$, then $y = \ln(2 \cdot 0 + 1) + 3 = \ln 1 + 3 = 3$. So, the only y-intercept is $(0, 3)$.
- If $y = 0$, then $0 = \ln(2x + 1) + 3$ or $-3 \ln(2x + 1)$ or $x = [\exp(-3) - 1]/2$. So, the only x-intercept is $([\exp(-3) - 1]/2, 0)$.



The graph of f^{-1} is the reflection of the graph of f in the line $y = x$. So, the graph of f^{-1} has horizontal asymptote $y = -0.5$, x-intercept $(3, 0)$, and y-intercept $(0, [\exp(-3) - 1]/2)$.

(iii) If f intersects the line $y = x$ at the point P, then f and f^{-1} also intersect at P.[671]

The points where f intersects the line $y = x$ are the points where $f(x) = x$ or $\ln(2x + 1) + 3 \overset{1}{=} x$. Hence, f and f^{-1} also intersect at the points where $\overset{1}{=}$ holds.

You're supposed to simply solve $\overset{1}{=}$ using your graphing calculator: $x \approx -0.4847, 5.482$.

> *Remark* 217. As discussed in Ch. 24.6, the converse to the first sentence of of **(iii)** is false. That is, the following statement is false:
>
> "If f intersects f^{-1} at some point P, then f also intersects the line $y = x$ at P." ✗

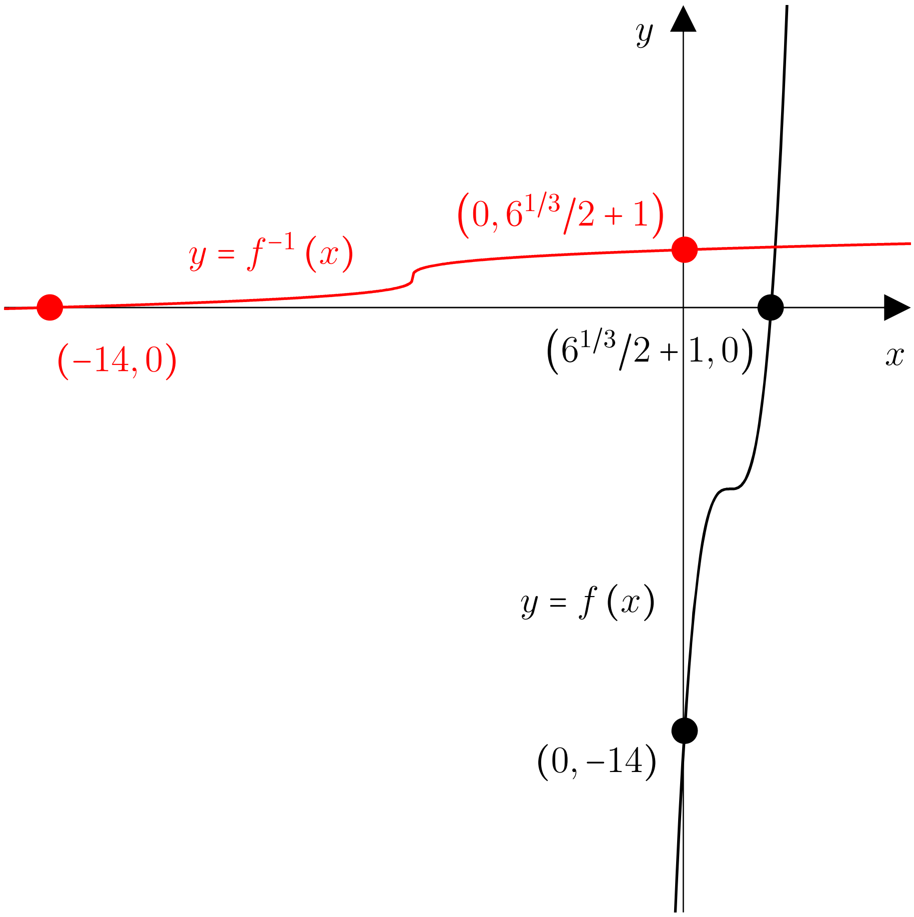
A556 (9740 N2010/I/5)(i) Starting with $y = x^3$,

1. First, "a translation of 2 units in the positive x-direction" produces $y = (x-2)^3$;
2. Next, "a stretch with scale factor 0.5 parallel to the y-axis" produces $y = (2x - 2)^3$
3. Finally, "a translation of 6 units in the negative y-direction" produces $y = (2x - 2)^3 - 6$.

If $x = 0$, then $y = (2 \cdot 0 - 2)^3 - 6 = -14$. So, the only y-intercept is $(0, -14)$.

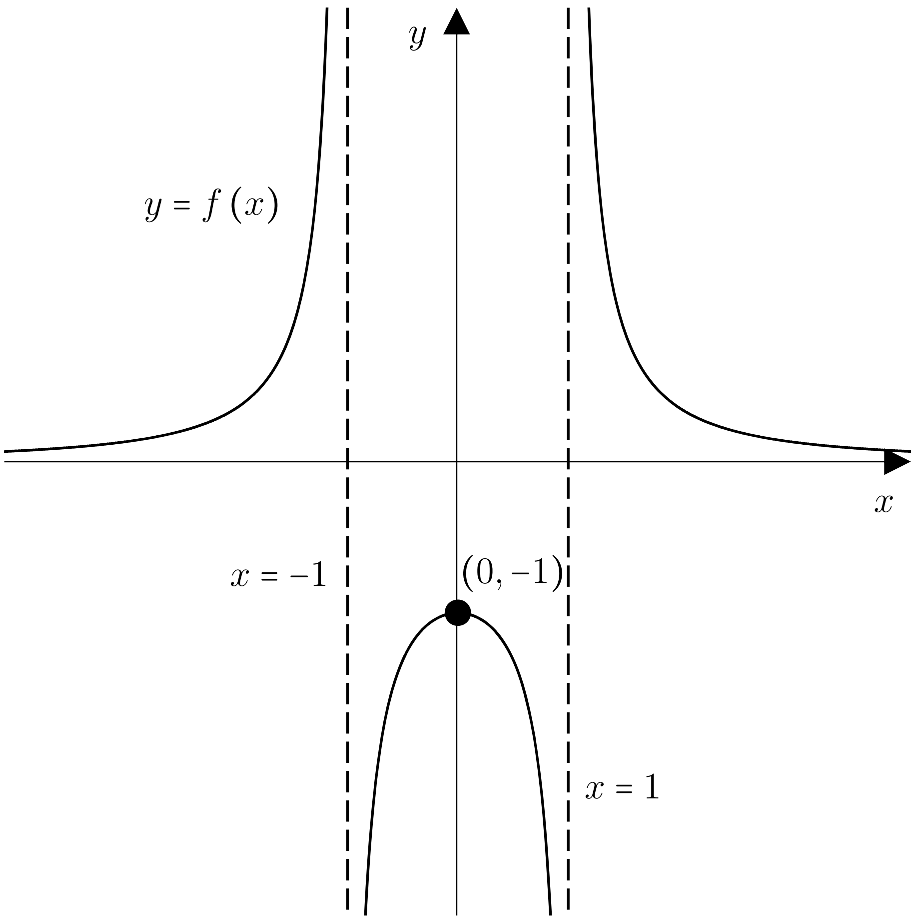
If $y = 0$, then get $0 = (2x - 2)^3 - 6$ or $x = 6^{1/3}/2 + 1$. So, the only x-intercept is $\left(6^{1/3}/2 + 1, 0\right)$.

[671] See Fact 53.



(ii) The graph of f^{-1} is the reflection of f in the line $y = x$. So, it has x-intercept $(-14, 0)$ and y-intercept $\left(0, 6^{1/3}/2 + 1\right)$.

A557 (9740 N2010/II/4)(i) You're supposed to just graph on your calculator and copy. Two vertical asymptotes are $x = \pm 1$ and a horizontal asymptote is $y = 0$.



(ii) Observe that f is symmetric in the y-axis. So, by restricting Domain f to $\mathbb{R}_0^+$, the new function produced would be one-to-one. Hence, the smallest k for which f^{-1} exists is $k=0$.

(iii) Note that Range $g = \mathbb{R}\smallsetminus\{0,-1,1\} \subseteq \mathbb{R}\smallsetminus\{\pm1\} =$ Domain f, so that the composite function fg exists.

$$fg(x) = f(g(x)) = f\left(\frac{1}{x-3}\right) = \frac{1}{\left(\frac{1}{x-3}\right)^2 - 1} = \frac{(x-3)^2}{1-(x-3)^2}$$

$$= \frac{(x-3)^2}{[1-(x-3)][1+(x-3)]} = \frac{(x-3)^2}{(4-x)(x-2)}.$$

(iv) Consider the inequality $\dfrac{(x-3)^2}{(4-x)(x-2)} \overset{1}{>} 0$

The numerator is strictly positive for all $x \in$ Domain $g = \mathbb{R}\smallsetminus\{2,3,4\}$.

Next, the denominator is a quadratic polynomial with negative coefficient on x^2 $(-1<0)$ and roots 4 and 2. Hence, the denominator is strictly positive if and only if $x \in (2,4)$.

Altogether then, $\overset{1}{>}$ holds if and only if $x \in (2,4)$.

Note though that $3 \notin$ Domain fg. Thus, the given inequality $fg(x) > 0$ holds if and only if $x \in (2,4)\smallsetminus\{3\}$.

(v) Let $y = fg(x) = \dfrac{(x-3)^2}{(4-x)(x-2)}$. Rearranging, we get

$$(x-3)^2 = y(4-x)(x-2) \qquad \text{or} \qquad x^2-6x+9 = y\left(-x^2+6x-8\right) \qquad \text{or}$$
$$(1+y)x^2 - 6(1+y)x + 9 + 8y \overset{2}{=} 0.$$

If $y=-1$, then $\overset{2}{=}$ becomes $9-8=0$, which is false. So, $y = fg(x) \neq -1$.

Next, if $y \overset{3}{\neq} -1$, then $\overset{2}{=}$ is a quadratic equation in x that holds for some $x \in \mathbb{R}$ if and only if the discriminant is non-negative, i.e.

$$0 \le [-6(1+y)]^2 - 4(1+y)(9+8y) = (1+y)(36+36y-36-32y) = (1+y)4y.$$

This last inequality in y in turn holds if and only if $y \in (-\infty,-1] \cup [0,\infty)$. By $\overset{3}{\neq}$, we must exclude -1.

Note also that Domain $fg = \mathbb{R}\smallsetminus\{2,3,4\}$. That is, x cannot take on the values 2, 3, or 4. So, we must check and see what values $\dfrac{(x-3)^2}{(4-x)(x-2)}$ take on when we plug in $x = 2,3,4$:

$$\frac{(2-3)^2}{(4-2)(2-2)} \text{ is undefined,} \qquad \frac{(3-3)^2}{(4-3)(3-2)} = 0, \text{ and} \qquad \frac{(4-3)^2}{(4-4)(4-2)} \text{ is undefined.}$$

So, we must also exclude 0 from our answer.

Altogether, Range $fg = (-\infty,-1) \cup (0,\infty)$.

A558 (9740 N2009/I/1)(i) Write $u_n = an^2 + bn + c$. The information given yields this system of equations:

$$a \cdot 1^2 + b \cdot 1 + c \stackrel{1}{=} 10, \qquad a \cdot 2^2 + b \cdot 2 + c \stackrel{2}{=} 6, \qquad a \cdot 3^2 + b \cdot 3 + c \stackrel{3}{=} 5.$$

Taking $\stackrel{1}{=} - \stackrel{2}{=}$ yields $-3a - b \stackrel{4}{=} 4$.

Taking $\stackrel{2}{=} - \stackrel{3}{=}$ yields $-5a - b \stackrel{5}{=} 1$.

Taking $\stackrel{4}{=} - \stackrel{5}{=}$ yields $2a = 3$ or $a \stackrel{6}{=} 1.5$.

From $\stackrel{5}{=}$ and $\stackrel{6}{=}$, we have $b = -5a - 1 \stackrel{7}{=} -8.5$.

From $\stackrel{1}{=}$, $\stackrel{6}{=}$, and $\stackrel{7}{=}$, we have $c = 10 - a - b = 17$.

Altogether, $u_n = 1.5n^2 - 8.5n + 17$.

(ii) $u_n = 1.5n^2 - 8.5n + 17 > 100 \iff 3n^2 - 17n - 166 \stackrel{1}{>} 0$.

The LHS is a quadratic polynomial in n with positive coefficient on n^2 $(3 > 0)$. By the quadratic formula, its roots are

$$n = \frac{17 \pm \sqrt{(-17)^2 - 4(3)(-166)}}{2(3)} = \frac{17 \pm \sqrt{289 + 1\,992}}{6} = \frac{17 \pm \sqrt{2\,281}}{6}.$$

We can discard the negative value. The positive value that remains is $\left(17 + \sqrt{2\,281}\right)/6 \approx 10.8$.

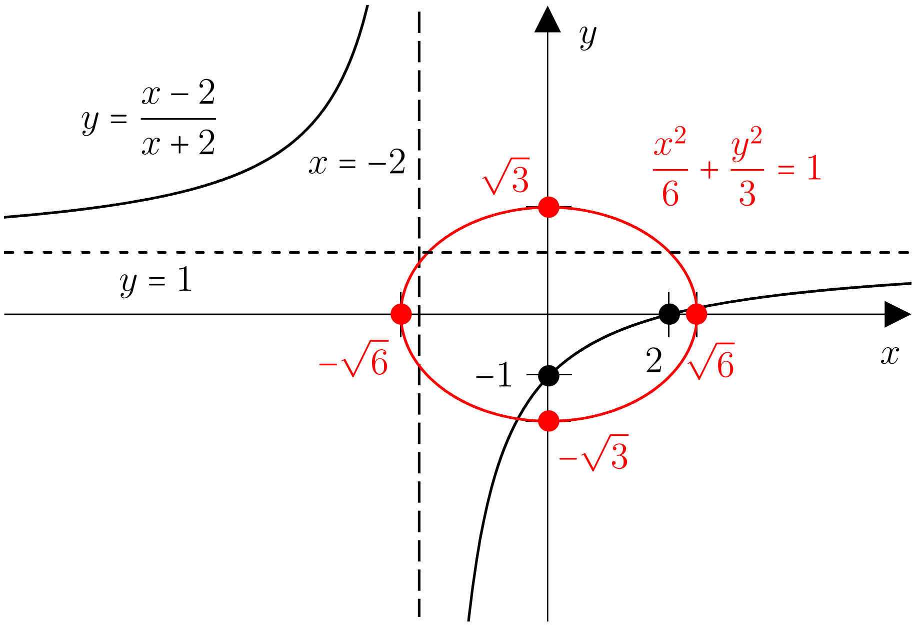
Since n must be an integer, the set of values of n for which $u_n > 100$ is $\{11, 12, 13, \ldots\}$.

A559 (9740 N2009/I/6)(i) C_1 is a rectangular hyperbola with y-intercept $(0, -1)$ and x-intercept $(2, 0)$.

Write $y = \dfrac{x-2}{x+2} = 1 - \dfrac{4}{x+2}$.

As $x \to -2$, $y \to \pm\infty$. So, $x = -2$ is a vertical asymptote.

As $x \to \pm\infty$, $y \to 1$. So $y = 1$ is a horizontal asymptote.



C_2 is an ellipse centred on the origin, with y-intercepts $\left(0, \pm\sqrt{3}\right)$, x-intercepts $\left(\pm\sqrt{6}, 0\right)$, and no asymptotes.

(ii) Plug the equation for C_1 into that for C_2 to get

$$\frac{x^2}{6}+\frac{\left(\frac{x-2}{x+2}\right)^2}{3}=1 \qquad \text{or} \qquad x^2(x+2)^2+2(x-2)^2=6(x+2)^2 \qquad \text{or}$$
$$2(x-2)^2=(x+2)^2\left(6-x^2\right).$$

(iii) $x\approx -0.515, 2.45.$

A560 (9740 N2009/II/3)(i) Write $y=f(x)=\dfrac{ax}{bx-a}=\dfrac{a}{b}+\dfrac{a^2/b}{bx-a}\in \text{Range}\, f=\mathbb{R}\setminus\{a/b\}$.

Do the algebra: $\dfrac{1}{y-a/b}=\dfrac{bx-a}{a^2/b} \iff \dfrac{a^2}{by-a}=bx-a \iff x=\dfrac{a^2/b}{by-a}+\dfrac{a}{b}$.

So, f is one-to-one and its inverse is the function $f^{-1}:\mathbb{R}\setminus\{a/b\}\to\mathbb{R}\setminus\{a/b\}$ defined by $f^{-1}(y)=\dfrac{a^2/b}{by-a}+\dfrac{a}{b}$.

Observe that $f(x)\stackrel{1}{=}f^{-1}(x)$, for all $x\in\text{Domain}\, f$.

Hence, $f^2(x)=f(f(x))\stackrel{1}{=}f\left(f^{-1}(x)\right)=x$ (the last step uses an Inverse Cancellation Law).

Range $f^2=\text{Domain}\, f=\mathbb{R}\setminus\{a/b\}$.

(ii) Range $g=\mathbb{R}\setminus\{0\}\nsubseteq\mathbb{R}\setminus\{a/b\}=\text{Domain}\, f$. So, fg does not exist.

(iii) $f^{-1}(x)=x \iff \dfrac{ax}{bx-a}=x \iff ax=x(bx-a) \iff 0=x(bx-2a) \iff x=0$ OR $x=2a/b$.

A561 (9740 N2008/I/9)(i) $f'(x)=\dfrac{(cx+d)a-(ax+b)c}{(cx+d)^2}=\dfrac{ad-bc}{(cx+d)^2}$ (Quotient Rule).

Since $ad-bc\neq 0$, $f'(x)\neq 0$ for any x. So, there are no turning points.

(ii) Suppose $ad-bc=0$. Then $f'(x)=0$ for all x, f is a constant function, and its graph is simply a horizontal line.

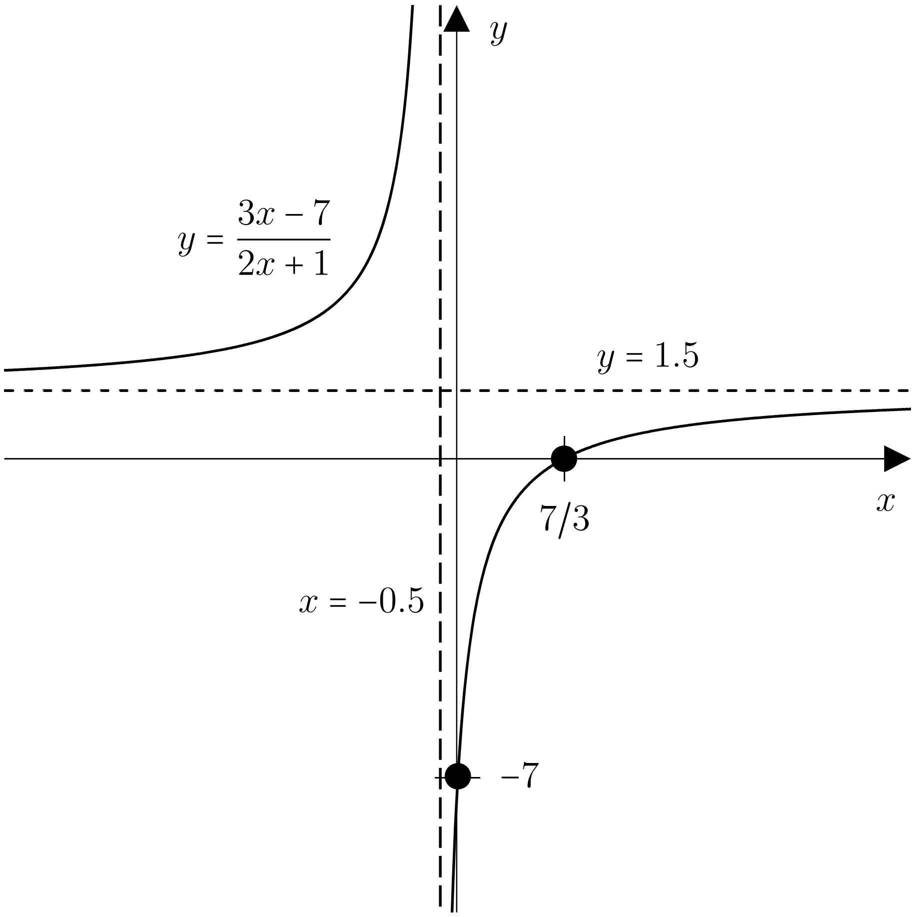
(iii) $a=3$, $b=-7$, $c=2$, and $d=1$. Since $ad-bc=3\cdot1-(-7)\cdot2=17>0$, by **(i)**, the given graph has a positive gradient at all points.

(iv)(a) This is a rectangular hyperbola with y-intercept $(0,-7)$ and x-intercept $(7/3,0)$.

Write $y=\dfrac{3x-7}{2x+1}=1.5-\dfrac{8.5}{2x+1}$.

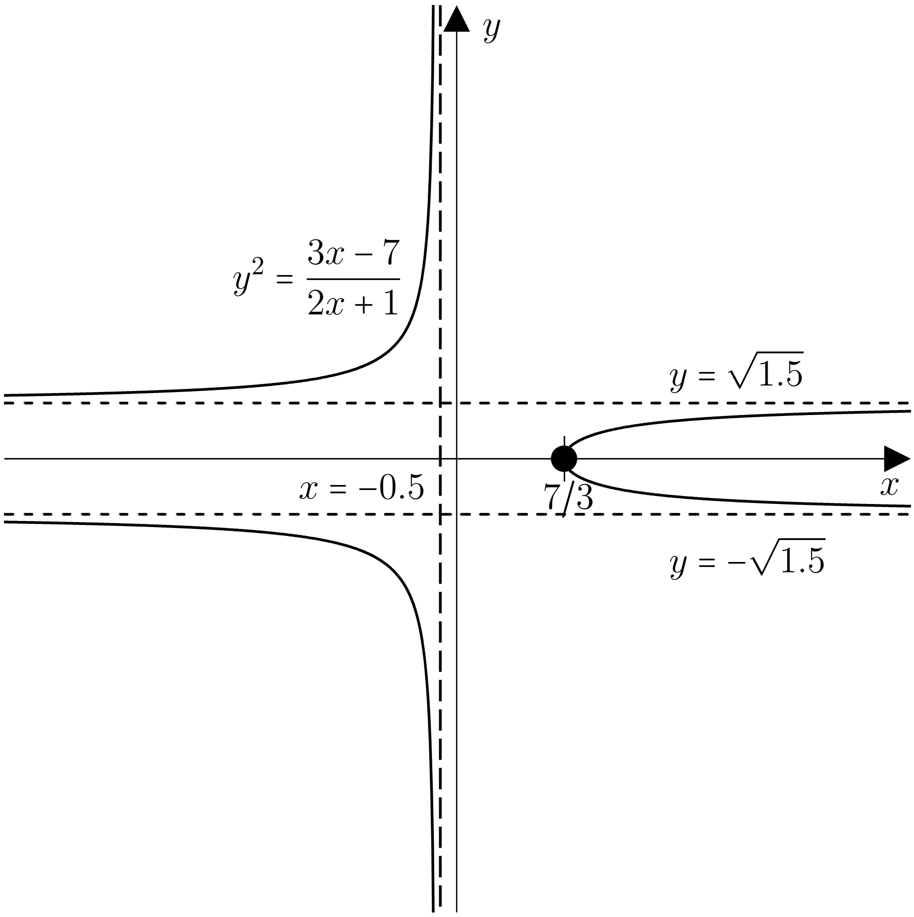
As $x\to-0.5$, $y\to\pm\infty$. So, a vertical asymptote is $x=-0.5$.

As $x\to\pm\infty$, $y\to1.5$. So, a horizontal asymptote is $y=1.5$.



(iv)(b) The graph of $y^2 = f(x)$ ...

- Is symmetric in the x-axis.
- Has the same x-intercept as $y = f(x)$, namely $(7/3, 0)$.
- Intersects the graph of $y = f(x)$ wherever $f(x) = 1$.
- Is empty wherever $f(x) < 0$. So here, it is empty between the vertical asymptote $x = -0.5$ and the x-intercept.
- Has horizontal asymptotes $y = \pm\sqrt{1.5}$.
- Has no y-intercepts



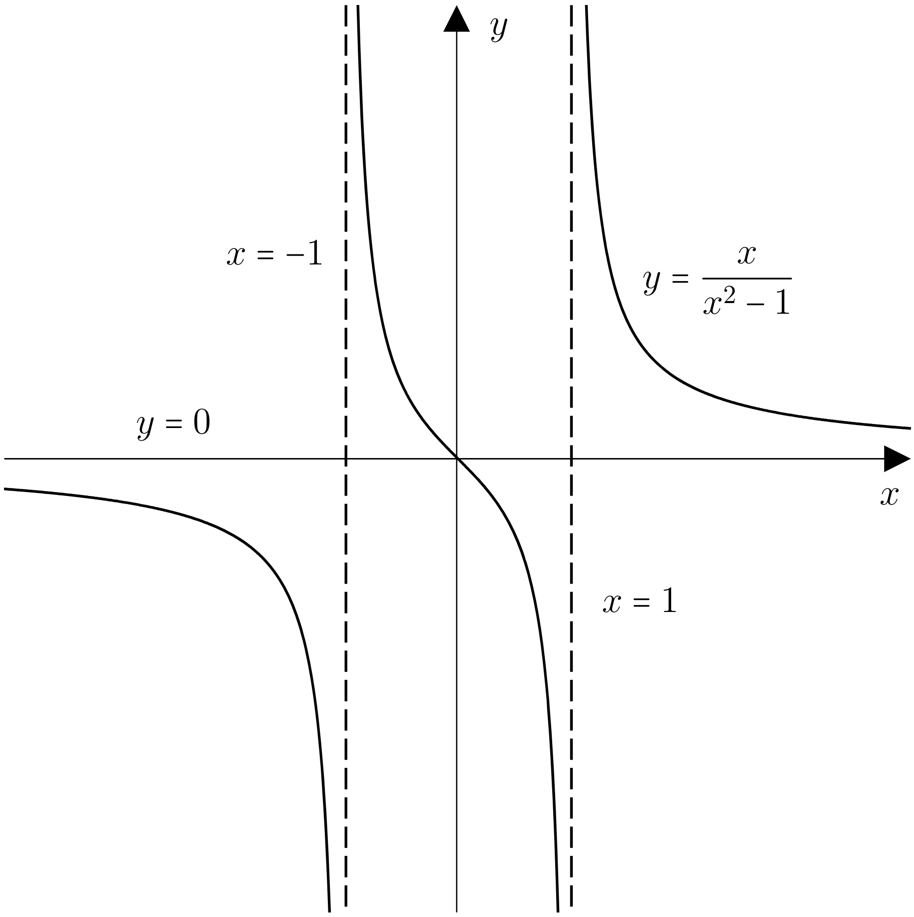
A562 (9233 N2008/I/14)(i) Write $y = \dfrac{x}{x^2-1} = \dfrac{x}{(x+1)(x-1)}$.

As $x \to \pm 1$, $y \to \pm\infty$. So, two vertical asymptotes are $x = 1$ and $x = -1$.

As $x \to \pm\infty$, $y \to 0$. So, a horizontal asymptote is $y = 0$.

If $x = 0$, then $y = 0/\left(0^2 - 1\right) = 0$. So, the only y-intercept is $(0, 0)$.

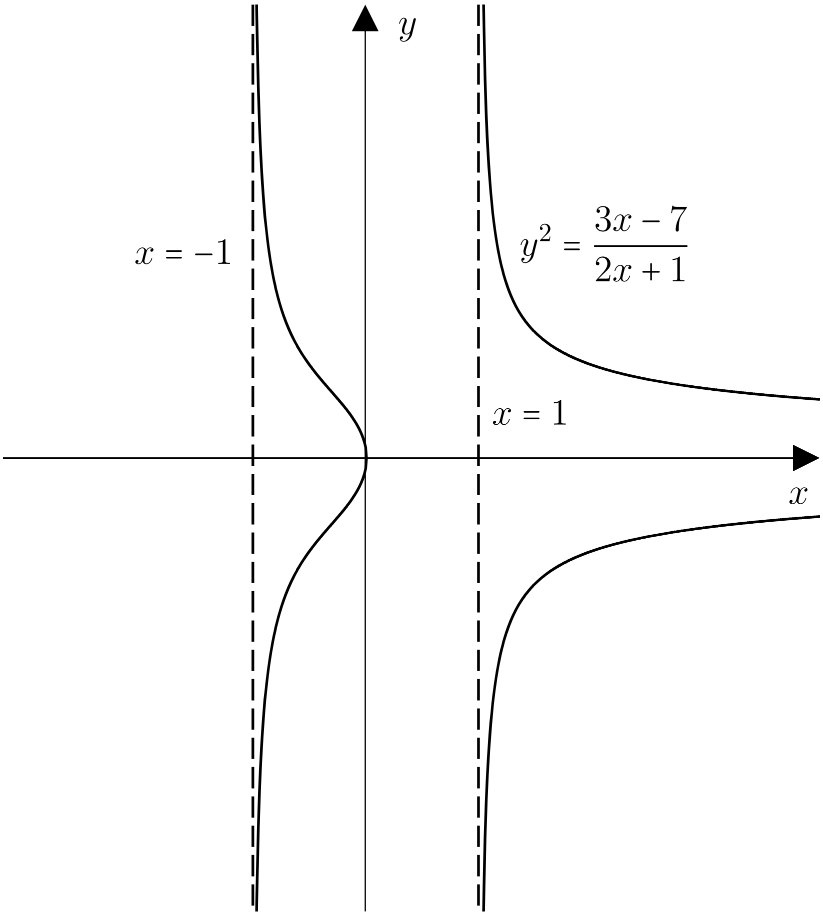
If $y = 0$, then $x = 0$. So, the only x-intercept is $(0, 0)$.



(ii) The graph of $y^2 = f(x)$ ...

- Is symmetric in the x-axis.
- Is empty wherever $f(x) < 0$. So here, it is empty to the left of $x = -1$ and between $x = 0$ and $x = 1$.
- Intersects the graph of $y = f(x)$ wherever $f(x) = 1$.
- Has the same x-intercept as $y = f(x)$, namely $(0, 0)$. Of course, this is also a y-intercept.

At the origin, the tangent to the curve $y^2 = \dfrac{x}{x^2 - 1}$ is vertical.



(iii) For $x \neq \pm 1$, $\dfrac{x}{x^2-1} = e^x \iff x = e^x\left(x^2-1\right) \iff xe^{-x} = x^2 - 1 \iff 1 + xe^{-x} = x^2$.

(iv) Try $x_1 = 1$. Then

$$x_2 = \sqrt{1 + x_1 e^{-x_1}} = \sqrt{1 + e^{-1}} \approx 1.169\,564.$$

$$x_3 = \sqrt{1 + x_2 e^{-x_2}} = \sqrt{1 + 1.169\,564 e^{-1.169\,564}} \approx 1.167\,541.$$

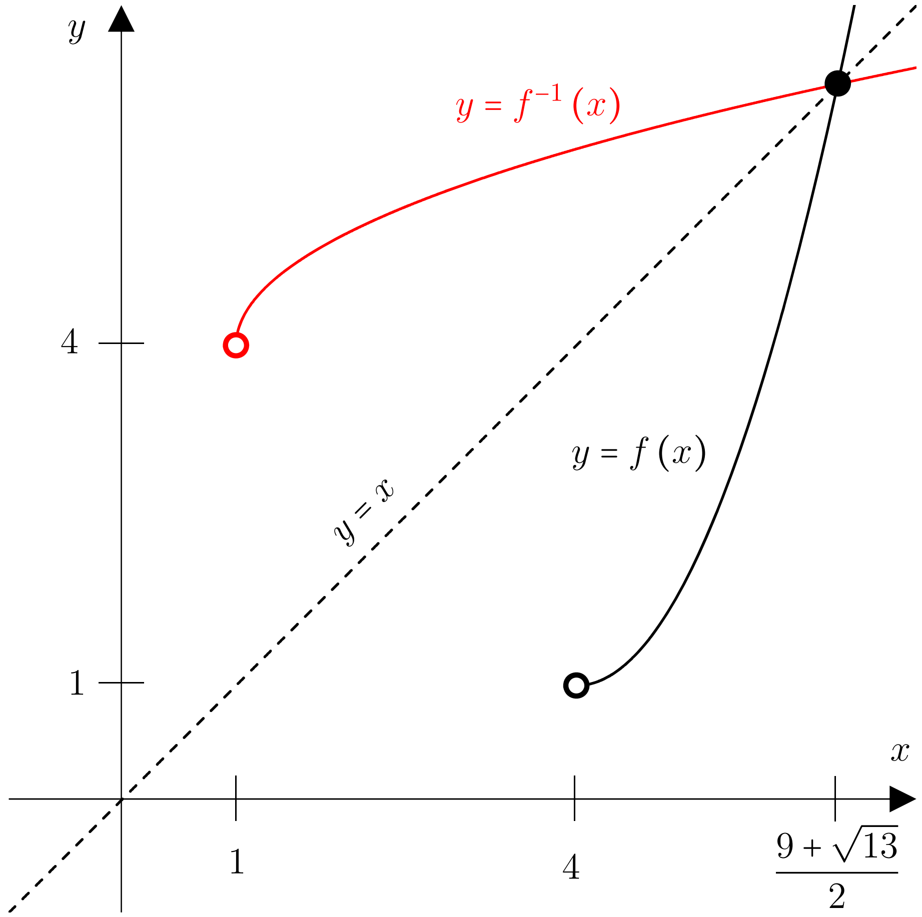
$$x_4 = \sqrt{1 + x_3 e^{-x_3}} = \sqrt{1 + 1.167\,541 e^{-1.167\,541}} \approx 1.167\,587.$$

So the positive root of $x = \sqrt{1 + xe^{-x}}$ is $x \approx 1.17$.

A563 (9740 N2008/II/4)(i) Starting with the graph of $y = x^2$,

1. Translate 4 units right to get $y = (x-4)^2$; then
2. Translate 1 unit up to get $y = (x-4)^2 + 1$.

Note that Domain $f = (4, \infty)$—and in particular, the graph of f does not include the point $(4, 1)$.



(ii) Write $y = f(x) = (x-4)^2 + 1 \in \text{Range } f = (1, \infty)$.

Do the algebra: $x = 4 \pm \sqrt{x-1}$. Since $x > 4$, we can discard $x = 4 - \sqrt{x-1}$ and be left with $x = 4 + \sqrt{x-1}$.

So, f is one-to-one and its inverse is the function $f^{-1} : (1, \infty) \to (4, \infty)$ defined by $f^{-1}(x) = \sqrt{x-1} + 4$.

(iii) See above.

(iv) Reflect (the graph of) f in the line $y = x$ to get (the graph of) f^{-1}.

(v) Any point at which f intersects the line $y = x$ is also a point at which f intersects f^{-1}.[672] And so, let us find points at which f intersects $y = x$. To do so, write

$$f(x) = x \qquad \text{or} \qquad (x-4)^2 + 1 = x \qquad \text{or} \qquad x^2 - 9x + 17 \stackrel{1}{=} 0,$$

or,
$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(17)}}{2(1)} = \frac{9 \pm \sqrt{13}}{2}.$$

Since Domain $f = (4, \infty)$, we may discard $\left(9 - \sqrt{13}\right)/2 < 4$ and be left with $\left(9 + \sqrt{13}\right)/2 > 4$ as *one* solution to $f(x) = f^{-1}(x)$.

The above answer probably sufficed for the A-Level exam. However, it is incomplete (and thus incorrect) because there may be additional points at which $f(x) = f^{-1}(x)$ but which are **not** on the line $y = x$. My guess is that those who wrote this question made the mistake of believing that the following statement is true:

[672] Fact 53 of this textbook.

"If f intersects f^{-1} at some point P, then f also intersects the line $y = x$ at P." ✗

As discussed in Ch. 24.6, the above statement is false.

It turns out that there are no other points at which $f(x) = f^{-1}(x)$. However, the above answer has not shown this. For how to show this (and hence complete and correct the above answer), see footnote.[673]

A564 (9740 N2007/I/1) $\dfrac{2x^2 - x - 19}{x^2 + 3x + 2} - 1 = \dfrac{2x^2 - x - 19 - (x^2 + 3x + 2)}{x^2 + 3x + 2} = \dfrac{x^2 - 4x - 21}{x^2 + 3x + 2}$. So,

$$1 < \frac{2x^2 - x - 19}{x^2 + 3x + 2} \iff 0 = 1 - 1 < \frac{2x^2 - x - 19}{x^2 + 3x + 2} - 1 = \frac{x^2 - 4x - 21}{x^2 + 3x + 2} = \frac{(x+3)(x-7)}{(x+1)(x+2)}.$$

Sign diagram:

$+$	$-$	$+$	$-$	$+$	$\dfrac{(x+3)(x-7)}{(x+1)(x+2)}$
	-3	-2	-1	7	

The inequality holds if and only if $x \in (-\infty, -3) \cup (-2, -1) \cup (7, \infty)$.

A565 (9740 N2007/I/2)(i) Since $\operatorname{Range} g = \mathbb{R}_0^+ \nsubseteq \mathbb{R} \setminus \{3\} = \operatorname{Domain} f$, fg does not exist.

Since $\operatorname{Range} f = \mathbb{R} \setminus \{0\} \subseteq \operatorname{Domain} g = \mathbb{R}$, gf exists. Assuming $\operatorname{Codomain} g = \mathbb{R}$, we have the composite function $gf : \mathbb{R} \setminus \{3\} \to \mathbb{R}$ defined by

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[673] We give two methods:

Method 1. $f(x) = f^{-1}(x) \iff (x-4)^2 + 1 \overset{2}{=} \sqrt{x-1} + 4 \iff x^2 - 8x + 13 = \sqrt{x-1}$

$\implies x^4 + 64x^2 + 169 - 16x^3 + 26x^2 - 208x = x - 1$

$$\iff 0 = x^4 - 16x^3 + 90x^2 - 209x + 170 = \left(x^2 - 9x + 17\right)\left(ax^2 + bx + c\right)$$

Comparing coefficients on the x^4, constant, and x^3 terms, we have $a = 1$, $c = 10$, and $b - 9a = -16$ (so, $b = -7$).

Hence, $ax^2 + bx + c = x^2 - 7x + 10 = (x-2)(x-5)$.

So, the other two possible solutions to $\overset{2}{=}$ are 2 and 5.

Since $2 \notin \operatorname{Domain} f = (4, \infty)$, we may discard 2 as a possible solution.

Next, let's check whether 5 actually solves $\overset{2}{=}$: $(5-4)^2 + 1 = 2 \neq 6 = \sqrt{5-1} + 4$. It doesn't.

Altogether, the unique solution to $f(x) = f^{-1}(x)$ is $\left(9 + \sqrt{13}\right)/2$. This completes the answer.

Method 2 (calculus). Define $g : (4, \infty) \to \mathbb{R}$ by $g(x) = f(x) - f^{-1}(x) = (x-4)^2 + 1 - \sqrt{x-1} - 4$.

Compute $g'(x) = 2(x-4) - \dfrac{1}{2\sqrt{x-1}}$. We now show that $g'(x) > 0$ for all $x \in (4, \infty)$:

$$g'(x) > 0 \iff 2(x-4) > \frac{1}{2\sqrt{x-1}} \iff 4(x-4) > \frac{1}{\sqrt{x-1}} \overset{3}{\iff} 16(x-4)^2 > \frac{1}{x-1} \overset{4}{\iff}$$
$$16(x-4)^2(x-1) > 0 \iff x \in (4, \infty).$$

At $\overset{3}{\iff}$, we use the fact that both sides of the inequality are positive (because $x > 4$).

At $\overset{4}{\iff}$, we use the fact that $x - 1 > 0$ (again because $x > 4$).

We've just shown that $g' > 0$ everywhere. So, g is strictly increasing everywhere. Hence, there exists at most one x such that $g(x) = f(x) - f^{-1}(x) = 0$ or $f(x) = f^{-1}(x)$.

We already found one such x above—namely $\left(9 + \sqrt{13}\right)/2$. So, this must be the unique solution to $f(x) = f^{-1}(x)$.

$$gf(x) = g(f(x)) = g\left(\frac{1}{x-3}\right) = \frac{1}{(x-3)^2}.$$

(ii) Write $y = f(x) = 1/(x-3) \in \text{Range } f = \mathbb{R} \setminus \{0\}$.

Do the algebra: $x = 1/y + 3$.

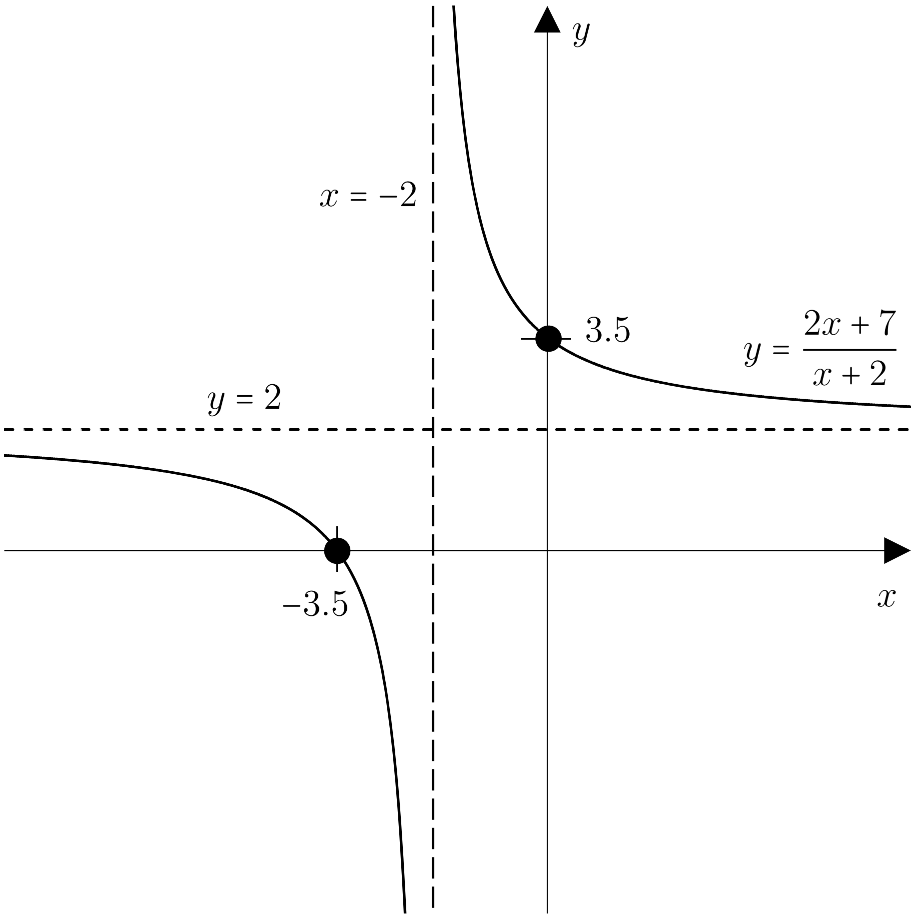
So, f is one-to-one and its inverse is the function $f^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{3\}$ defined by $f^{-1}(y) = 1/y + 3$.

A566 (9740 N2007/I/5) $y = \dfrac{2x+7}{x+2} = 2 + \dfrac{3}{x+2}$.

Starting with $y = 1/x$,

1. Translate 2 units left to get $y = 1/(x+2)$.
2. Stretch vertically (outwards from the x-axis) by a factor of 3 to get $y = 3/(x+2)$.
3. Translate 2 units up to get $y = 2 + 3/(x+2)$.

This is a rectangular hyperbola.



As $x \to -2$, $y \to \pm\infty$. So, $x = -2$ is a vertical asymptote.

As $x \to \pm\infty$, $y \to 2$. So, $y = 2$ is a horizontal asymptote.

If $x = 0$, then $y = 2 + 3/(0+2) = 3.5$. So, the only y-intercept is $(0, 3.5)$.

If $y = 0$, then $0 = 2x + 7$ or $x = -3.5$. So, the only x-intercept is $(-3.5, 0)$.

A567 (9740 N2007/II/1) Let p, m, and l be the prices (in dollars per kilogram) of the pineapples, mangoes, and lychees, respectively. Then the given table yields this system of equations:

$$1.15p + 0.6m + 0.55l \overset{1}{=} 8.28, \qquad 1.2p + 0.45m + 0.3l \overset{2}{=} 6.84, \qquad 2.15p + 0.9m + 0.65l \overset{3}{=} 13.05.$$

Taking $2\times \overset{2}{=} - \overset{3}{=}$ yields $0.25p - 0.05l = 0.63$ or or $25p - 5l = 63$ or $l \overset{4}{=} 5p - 12.6$.

Taking $4\times \overset{2}{=} -3\times \overset{1}{=}$ yields $1.35p - 0.45l = 2.52$ or $135p - 45l = 252$ or $l \overset{5}{=} 3p - 5.6$.

Putting $\overset{4}{=}$ and $\overset{5}{=}$together, $p = 3.5$. Now we can also find $l = 4.9$ and $m = 2.6$.

So, Lee Lian paid $1.3p + 0.25m + 0.5l = 4.55 + 0.65 + 2.45 = 7.65$ dollars.

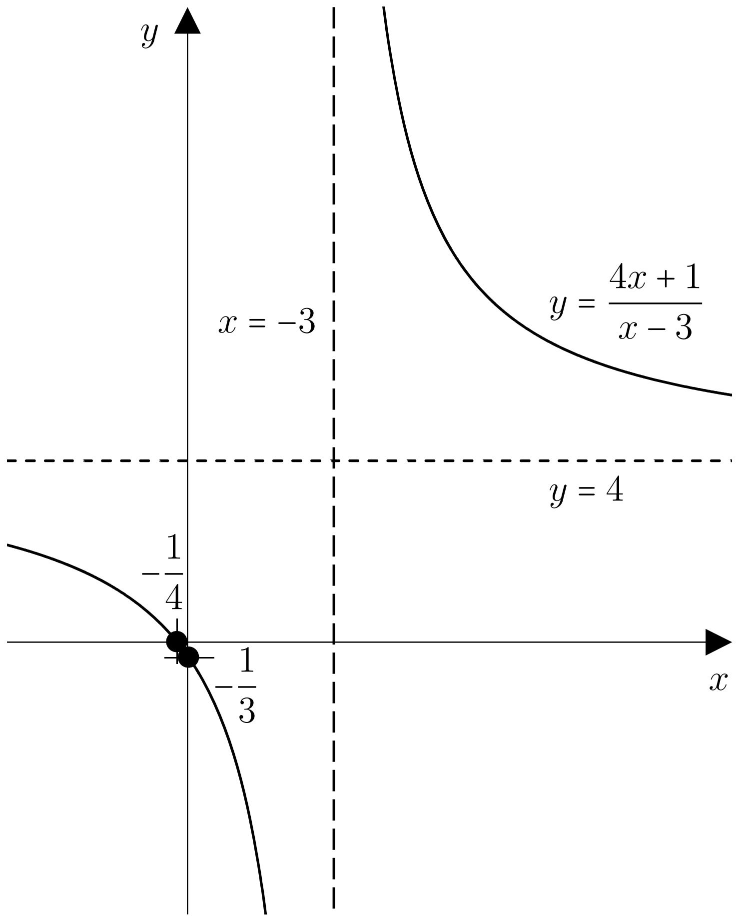
A568 (9233 N2007/II/4)(i) Write $y = \dfrac{4x+1}{x-3} = 4 + \dfrac{13}{x-3}$.

As $x \to 3$, $y \to \pm\infty$. So, $x = 3$ is a vertical asymptote.

As $x \to \pm\infty$, $y \to 4$. So, $y = 4$ is a horizontal asymptote.

(ii) If $x = 0$, then $y = -1/3$. So, the only y-intercept is $(0, -1/3)$.

If $y = 0$, then $0 = 4x + 1$ or $x = -1/4$. So, the only x-intercept is $(-1/4, 0)$.



(iii) Write $y = f(x) = 4 + 13/(x-3) \in \text{Range } f = \mathbb{R} \setminus \{4\}$.

Do the algebra: $x = 13/(y-4) + 3$.

So, f is one-to-one and its inverse is the function $f^{-1} : \mathbb{R} \setminus \{4\} \to \mathbb{R} \setminus \{3\}$ defined by $f^{-1}(y) = 13/(y-4) + 3$.

A569 (9233 N2006/I/3)(i) $\text{Range } g = \mathbb{R}^+ \subseteq \mathbb{R}^+ = \text{Domain } f$. So, fg exists and

$$fg(x) = f(g(x)) = f\left(\frac{3}{x}\right) = 5 \cdot \frac{3}{x} + 3 = \frac{15}{x} + 3.$$

$\text{Range } g = \mathbb{R}^+ \subseteq \mathbb{R}^+ = \text{Domain } g$. So, g^2 exists and

$$g^2(x) = g(g(x)) = g\left(\frac{3}{x}\right) = \frac{3}{3/x} = x.$$

Let $i = g^2$. Then $g^{35}(x) = g \circ g^{34}(x) = g \circ i^{17}(x) = g(x) = 3/x$.

(ii) $h(x) = 5f(x) + 3$.

A570 (9233 N2006/II/1) $\dfrac{x-9}{x^2-9} \le 1 \iff 0 \ge \dfrac{x-9}{x^2-9} - 1 = \dfrac{x-9-x^2+9}{x^2-9} = \dfrac{x(1-x)}{(x-3)(x+3)}$.

Sign diagram:

$$\begin{array}{ccccccccc} - & | & + & | & - & | & + & | & - \\ & -3 & & 0 & & 1 & & 3 & \end{array} \qquad \frac{x(1-x)}{(x-3)(x+3)}$$

Take care to note that $x \neq \pm 3$.

The inequality holds if and only if $x \in (-\infty, -3) \cup [0, 1] \cup (3, \infty)$.

155.2. Ch. 134 Answers (Sequences and Series)

A571 (9758 N2019/I/6)(i) Write $\dfrac{1}{4r^2-1} = \dfrac{A}{2r+1} + \dfrac{B}{2r-1} = \dfrac{2(A+B)r + B - A}{4r^2-1}$.

Comparing coefficients, $A + B = 0$ and $B - A = 1$, so $B = 1/2$ and $A = -1/2$:

$$\frac{1}{4r^2-1} = \frac{-1/2}{2r+1} + \frac{1/2}{2r-1}.$$

Hence,
$$\begin{aligned}\sum_{r=1}^{n} \frac{1}{4r^2-1} &= \sum_{r=1}^{n} \left(\frac{-1/2}{2r+1} + \frac{1/2}{2r-1} \right) \\ &= \frac{1}{2}\left[\left(-\cancel{\frac{1}{3}} + \frac{1}{1}\right) + \left(-\cancel{\frac{1}{5}} + \cancel{\frac{1}{3}}\right) + \left(-\cancel{\frac{1}{7}} + \cancel{\frac{1}{5}}\right) + \cdots + \left(-\frac{1}{2n+1} + \cancel{\frac{1}{2n-1}}\right)\right] \\ &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}.\end{aligned}$$

(ii) $\displaystyle\sum_{r=11}^{\infty} \frac{1}{4r^2-1} = \sum_{r=1}^{\infty} \frac{1}{4r^2-1} - \sum_{r=1}^{10} \frac{1}{4r^2-1} = \lim_{n\to\infty} \frac{n}{2n+1} - \frac{10}{2\cdot 10+1} = \frac{1}{2} - \frac{10}{21} = \frac{1}{42}$.

A572 (9758 N2019/I/8)(i) For the arithmetic series, the 64th term is $a + 126a$; so, the sum of the first 64 terms is $64(a + a + 126a)/2 = 32 \times 128a = 2^{12}a$.

The kth term of the geometric series is $a2^{k-1}$. So, $k = 13$.

(ii) $0 = f + fr + fr^2 + fr^3 = f\left(1 + r + r^2 + r^3\right) = f(1+r)\left(1+r^2\right)$. Since $f \neq 0$ and $r \in \mathbb{R}$, it must be that $r = -1$ and also, f can be any real number.

The sum of the first n terms is $\dfrac{f(r^n - 1)}{r-1} = \dfrac{1}{-2} f\left((-1)^n - 1\right) = \begin{cases} 0, & \text{for } n \text{ even}, \\ f, & \text{for } n \text{ odd}. \end{cases}$

(iii) Let $a < 0$ be the first term and d be the common difference. We are given that

$$a + (a+d) + (a+2d) + (a+3d) = 4a + 6d \overset{1}{=} 14 \qquad \text{and} \qquad a(a+d)(a+2d)(a+3d) \overset{2}{=} 0.$$

From $\overset{1}{=}$, $a = \dfrac{7-3d}{2}$.

From $\overset{2}{=}$, at least one of these three equations is true:

$$a \overset{3}{=} -d, \qquad a \overset{4}{=} -2d, \quad \text{or} \quad a \overset{5}{=} -3d.$$

If $\overset{3}{=}$, then $d = 7$ and $a = -7$.

If $\overset{4}{=}$, then $d = -7$ and $a = 14$, contradicting $a < 0$.

If $\overset{5}{=}$, then $d = -7/3$ and $a = 7$, contradicting $a < 0$.

Hence, $d = 7$, $a = -7$, and The 11th term of the series is $a + 10d = 63$.

A573 (9758 N2018/I/8)(i) $u_2 = 2u_1 + A \cdot 1 \iff 15 = 2 \cdot 5 + A \iff A = 5$.
$u_3 = 2u_2 + A \cdot 2 = 2 \cdot 15 + 5 \cdot 2 = 40$.

(ii) Plug $n = 1, 2, 3$ into the given equation:

$$u_1 = 5 = a\left(2^1\right) + b \cdot 1 + c \overset{1}{=} 2a + b + c,$$

$$u_2 = 15 = a\left(2^2\right) + b \cdot 2 + c \overset{2}{=} 4a + 2b + c,$$

$$u_3 = 40 = a\left(2^3\right) + b \cdot 3 + c \overset{3}{=} 8a + 3b + c.$$

First, $2 \times \overset{1}{=} - \overset{2}{=}$ yields $c \overset{4}{=} -5$. Next, $\overset{3}{=} - \overset{1}{=} - \overset{2}{=}$ yields $2a - c \overset{4}{=} 2a + 5 = 20$ or $a \overset{5}{=} 7.5$. Now plug $\overset{4}{=}$ and $\overset{5}{=}$ into $\overset{2}{=}$ to get $15 = 4 \times 7.5 + 2b - 5$ or $b = -5$.

(iii) $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \left[7.5\left(2^r\right) - 5r - 5\right] = 7.5 \sum_{r=1}^{n} 2^r - 5 \sum_{r=1}^{n} r - 5 \sum_{r=1}^{n} 1$
$= 7.5\left(2^{n+1} - 2\right) / (2-1) - 5n(n+1)/2 - 5n = 7.5 \cdot 2^{n+1} - 15 - 2.5n(n+1) - 5n$
$= 7.5 \cdot 2^{n+1} - 2.5n^2 - 7.5n - 15$.

A574 (9758 N2018/I/11)(i)(a) $\$100 \cdot 1.002^{12} \approx \$102.43$.

(i)(b) $\$100 \cdot \left(1.002^1 + 1.002^{12} + \cdots + 1.002^{12}\right) = \$100 \dfrac{1.002^{13} - 1.002}{1.002 - 1} \approx \$1\,215.71$.

(i)(c) On the last day of the nth month, the account will contain

$$\$100 \cdot \left(1.002^1 + 1.002^{12} + \cdots + 1.002^{n}\right) = \$100 \frac{1.002^{n+1} - 1.002}{1.002 - 1} \approx \begin{cases} \$2\,988.65, & \text{for } n = 29, \\ \$3\,094.82, & \text{for } n = 30. \end{cases}$$

Hence, the account will first exceed $\$3\,000$ after the last day of the 29th month—or equivalently, the first day of the 30there month, i.e. 01 June 2018.

(ii)(a) $\$\left(100 + 12b\right)$.

(ii)(b) The total at the end of 31 December 2017 is

$$\$\left[2\,400 + b\left(1 + 2 + \cdots + 24\right)\right] = \$\left(2\,400 + 300b\right) = \$2\,800.$$

So, $b = 4/3$.

(iii) On the last day of the 60th month, the totals under plan P and plan Q are

$$\$100 \frac{1.01^{61} - 1.01}{1.01 - 1} \qquad \text{and} \qquad \$\left[6\,000 + b\left(1 + 2 + \cdots + 60\right)\right] = \$\left(6\,000 + 1\,830b\right).$$

Setting these two equal and rearranging, $b = \dfrac{100\frac{1.01^{61} - 1.01}{1.01 - 1} - 6\,000}{1\,830} \approx 1.23$.

A575 (9758 N2017/I/9)(a)(i) $u_n = S_n - S_{n-1} = An^2 + Bn - \left[A(n-1)^2 + B(n-1)\right] = 2An - A + B$.

(ii) $u_{10} = 2A \cdot 10 - A + B = 19A + B \overset{1}{=} 48$ and $u_{17} = 2A \cdot 17 - A + B = 33A + B \overset{2}{=} 90$.

$\overset{2}{=}$ minus $\overset{1}{=}$ yields $14A = 42$ or $A = 3$. Now by $\overset{1}{=}$, $B = 48 - 19A = 48 - 57 = -9$.

(b) $r^2(r+1)^2 - (r-1)^2 r^2 = r^2\left[(r+1)^2 - (r-1)^2\right] = r^2\left[(r+1-(r-1))(r+1+r-1)\right] = r^2\left[2(2r)\right] = 4r^3$. So $k = 4$.

$$\sum_{r=1}^{n} 4r^3 = \sum_{r=1}^{n}\left[r^2(r+1)^2 - (r-1)^2 r^2\right]$$

$$= 1^2 \cdot 2^2 - 0^2 \cdot 1^2 + 2^2 \cdot 3^2 - 1^2 \cdot 2^2 + \cdots + n^2 \cdot (n+1)^2 - (n-1)^2 \cdot n^2.$$

$$= -0^2 \cdot 1^2 + n^2 \cdot (n+1)^2 = n^2 \cdot (n+1)^2$$

(c) We have
$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1}.$$

Thus,
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1.$$

By D'Alembert's ratio test then, the given series converges.

If you've done Part V (Calculus), you should be able to easily recognise that this is the Maclaurin series expansion for e^x. In fact, this is even printed on List MF26, p. 2:

$$e^x = \sum_{r=1}^{\infty}\frac{x^r}{r!}.$$

A576 (9758 N2017/II/2). Let the arithmetic progression be (a_i) and the geometric progression be (g_i). Let $d = a_2 - a_1$ be the common difference in the arithmetic progression.

(i) $a_1 = 3$ and $a_{13} = 3 + 12d$. So, $(3 + 3 + 12d) \times 13/2 = 156$, so $d = (2 \times 156/13 - 6)/12 = 1.5$.

(ii) The common ratio r cannot be equal to 1, because if so, $\sum_{i=1}^{13} g_i = 13g_1 = 13 \times 3 = 39 \neq 156$.

The sum of the first 13 terms is $3\left(1 - r^{13}\right)/(1-r) = 156 \iff 3 - 3r^{13} = 156 - 156r \iff r^{13} - 52r + 51 = 0$.

Use your graphing calculator to find that besides 1, the other two possible roots to this last equation are $r \approx -1.451, 1.210$.

These are thus also the two possible values of r.

(iii) We know that $g_n = 3r^{n-1}$ and $a_n = 3 + 1.5(n-1)$.

We are told that $r \approx 1.210$. We are told also that $g_n > 100a_n$.

Thus, $3 \cdot 1.210^{n-1} > 100\left[3 + 1.5(n-1)\right] = 150 + 150n$. Graph $3 \cdot 1.210^{x-1} - 150x - 150$ in your graphing calculator. You should find that there is a positive x-intercept. To the left of this x-intercept, the graph is below the x-axis and to the right, it is above.

This x-intercept is given by $x \approx 41.149$. Thus, the smallest value of n for which the inequality holds is 42.

A578 (9740 N2016/I/6)(i) Let $\mathbf{P}(k)$ be the proposition: $\sum_{r=1}^{k} r\left(r^2+1\right) = \frac{1}{4}k(k+1)\left(k^2+k+2\right)$

We first verify that $\mathbf{P}(1)$ is true:

$$\sum_{r=1}^{1} r\left(r^2+1\right) = 1\left(1^2+1\right) = 2 = \frac{1}{4}\cdot 1\cdot 2\cdot 4 = \frac{1}{4}\cdot 1(1+1)\left(1^2+1+2\right) = \frac{1}{4}. \qquad \checkmark$$

Now let k be any positive integer. Suppose $\mathbf{P}(k)$ is true. Below we show that $\mathbf{P}(k+1)$ is also true and hence, by the principle of mathematical induction, that the given proposition is also true:

$$\begin{aligned}
\sum_{r=1}^{k+1} r\left(r^2+1\right) &\stackrel{\mathbf{P}(k)}{=} \sum_{r=1}^{k} r\left(r^2+1\right) + (k+1)\left[(k+1)^2+1\right] \\
&= \frac{1}{4}k(k+1)\left(k^2+k+2\right) + (k+1)\left(k^2+2k+2\right) \\
&= (k+1)\left[\frac{1}{4}k\left(k^2+k+2\right) + \left(k^2+2k+2\right)\right] \\
&= (k+1)\left(\frac{1}{4}k^3 + \frac{5}{4}k^2 + 2.5k + 2\right) \\
&= \frac{1}{4}(k+1)\left(k^3+5k^2+10k+8\right) \\
&= \frac{1}{4}(k+1)(k+2)\left(k^2+3k+4\right) \\
&= \frac{1}{4}(k+1)(k+2)\left[(k+1)^2+(k+1)+2\right]. \qquad \checkmark
\end{aligned}$$

(ii) $u_1 = u_0 + 1^3 + 1 = 2 + 1 + 1 = 4$.

$u_2 = u_1 + 2^3 + 2 = 4 + 8 + 2 = 14$.

$u_3 = u_2 + 3^3 + 3 = 14 + 27 + 3 = 44$.

(iii) Through telescoping, we have $\sum_{r=1}^{n}\left(u_r - u_{r-1}\right) = u_n - u_0 \stackrel{1}{=} u_n - 2$.

But for $r \geq 1$, we also have $u_r - u_{r-1} = u_{r-1} + r^3 + r - u_{r-1} \stackrel{3}{=} r^3 + r$. So,

$$\sum_{r=1}^{n}\left(u_r - u_{r-1}\right) \stackrel{3}{=} \sum_{r=1}^{n}\left(r^3+r\right) = \sum_{r=1}^{n} r\left(r^2+1\right) \stackrel{\text{(i)}}{=} \frac{1}{4}k(k+1)\left(k^2+k+2\right).$$

Plugging this last equation into $\stackrel{1}{=}$, we have $u_n = \frac{1}{4}k(k+1)\left(k^2+k+2\right) + 2$.

A577 (9740 N2016/I/4). We are given

$$\begin{aligned}
a + 3d &\stackrel{1}{=} br^4, \\
a + 8d &\stackrel{2}{=} br^7, \\
a + 11d &\stackrel{3}{=} br^{14}.
\end{aligned}$$

(i) $\stackrel{3}{=}$ divided by $\stackrel{1}{=}$ yields $r^{10} = (a+11d)/(a+3d)$, while $\stackrel{2}{=}$ divided by $\stackrel{1}{=}$ yields $r^3 = (a+8d)/(a+3d)$. Now,

$$5r^{10}-8r^3+3=5\frac{a+11d}{a+3d}-8\frac{a+8d}{a+3d}+3=\frac{5a+55d-8a-64d+3a+9d}{a+3d}=0.$$

Use your graphing calculator to find that the solutions to this 10th-degree polynomial equation are $r\approx 0.74, 1$.

You can verify by the Factor Theorem that $r=1$ is indeed a root. So, given that $|r|<1$, the only possible value of r is $r\approx 0.74$.

(ii) The limit of the infinite geometric series is $b/(1-r)$. And the sum of the first n terms is $b(1-r^n)/(1-r)$.

Hence, the sum of the terms after the nth is

$$\frac{b}{1-r}-\frac{b(1-r^n)}{1-r}=\frac{b}{1-r}[1-(1-r^n)]=\frac{br^n}{1-r}\approx\frac{0.74^b b}{0.26}.$$

A579 (9740 N2015/I/8). First, note that in seconds, the required time interval is $[5\,400, 6\,300]$.

(i) The time (in seconds) taken by A to complete the 50 laps is

$$(\text{First term}+\text{Last term})\times\frac{\text{Number of terms}}{2}=(T+T+49\times 2)\times\frac{50}{2}=50T+49\times 50=50T+2450.$$

So, we need $50T+2\,450\in[5\,400,6\,300]$ or $50T\in[2\,950,3\,850]$ or $T\in[59,77]$.

(ii) The time (in seconds) taken by B to complete the 50 laps is

$$t\frac{1-r^{50}}{1-r}=t\frac{1-1.02^{50}}{1-1.02}=t\frac{1.02^{50}-1}{0.02}=50t\left(1.02^{50}-1\right).$$

So, we need $50t\left(1.02^{50}-1\right)\in[5\,400,6\,300]$ or $3\,192.267\lesssim 50t\lesssim 3\,724.311$ or $63.845\lesssim t\lesssim 74.486$.

(iii) $T=59$ and $t\approx 63.845$. So the times taken to complete the 50th lap by A and B are

$$T+49\times 2=157 \qquad \text{and} \qquad t\times 1.02^{49}\approx 168 \text{ seconds}.$$

And so the desired difference is 11 seconds.

A580 (9740 N2015/II/4)(a) Let $\mathbf{P}(k)$ be the following proposition:

$$\sum_{r=1}^{k} r(r+2)(r+5)=\frac{1}{12}k(k+1)\left(3k^2+31k+74\right).$$

We show that $\mathbf{P}(1)$ is true:

$$\sum_{r=1}^{1} r(r+2)(r+5)=1\times 3\times 6=18=\frac{1}{12}1\times 2\times 108=\frac{1}{12}1(1+1)\left(3\cdot 1^2+31\cdot 1+74\right). \qquad \checkmark$$

We next show that for all $j\in\mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$\sum_{r=1}^{j+1} r(r+2)(r+5) = \sum_{r=1}^{j} r(r+2)(r+5) + (j+1)(j+3)(j+6)$$

$$\begin{aligned}
&\overset{\mathbf{P}(j)}{=} \frac{1}{12} j(j+1)\left(3j^2+31j+74\right) + (j+1)(j+3)(j+6) \\
&= \frac{j+1}{12}\left(3j^3+31j^2+74j\right) + (j+1)\left(j^2+9j+18\right) \\
&= \frac{j+1}{12}\left(3j^3+31j^2+74j+12j^2+108j+216\right) \\
&= \frac{j+1}{12}\left(3j^3+43j^2+182j+12j^2+216\right) \\
&= \frac{j+1}{12}(j+2)\left(3(j+1)^2+31(j+1)+74\right). \qquad \checkmark
\end{aligned}$$

(b)(i) Write

$$\frac{A}{2r+1} + \frac{B}{2r+3} = \frac{(2r+3)A + (2r+1)B}{(2r+1)(2r+3)} = \frac{(2A+2B)r + 3A + B}{4r^2+8r+3}.$$

So $2A+2B \overset{1}{=} 0$ and $3A+B \overset{2}{=} 2$.

$2\times \overset{2}{=}$ minus $\overset{1}{=}$ yields $4A = 4$ or $A = 1$ and thus $B = -1$. Hence,

$$\frac{2}{4r^2+8r+3} = \frac{1}{2r+1} - \frac{1}{2r+3}.$$

(ii)

$$\begin{aligned}
\sum_{r=1}^{n} \frac{2}{4r^2+8r+3} &= \sum_{r=1}^{n}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right) \\
&= \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{9}\right) + \cdots + \left(\frac{1}{2n+1}-\frac{1}{2n+3}\right) \\
&= \frac{1}{3} - \frac{1}{2n+3}.
\end{aligned}$$

(iii) The sum to infinity is $1/3$. Hence, the difference between S_n and the sum to infinity is $\dfrac{1}{2n+3}$. Now,

$$\frac{1}{2n+3} \le 10^{-3} \iff 1\,000 \le 2n+3 \iff n \ge 498.5.$$

So the smallest such n is 499.

A581 (9740 N2014/I/6)(i) Let $\mathbf{P}(k)$ be the following proposition:

$$p_k = \frac{1}{3}\left(7-4^k\right).$$

We show that $\mathbf{P}(1)$ is true:

$$p_1 = \frac{1}{3}(7-4) = \frac{1}{3}\left(7-4^1\right). \qquad \checkmark$$

We next show that for all $j \in \mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$\begin{aligned} p_{j+1} &= 4p_j - 7 \\ &\overset{\mathbf{P}(j)}{=} \frac{4}{3}\left(7-4^j\right) - 7 \\ &= \frac{1}{3}\left(7-4^{j+1}\right). \qquad \checkmark \end{aligned}$$

(ii)

$$\begin{aligned} \sum_{r=1}^{n} p_r &= \sum_{r=1}^{n} \frac{1}{3}\left(7-4^r\right) = \frac{1}{3}\sum_{r=1}^{n}\left(7-4^r\right) = \frac{1}{3}\left(\sum_{r=1}^{n} 7 - \sum_{r=1}^{n} 4^r\right) \\ &= \frac{1}{3}\left(7n - 4\frac{1-4^n}{1-4}\right) = \frac{1}{3}\left(7n + 4\frac{1-4^n}{3}\right) = \frac{4}{9} + \frac{7n}{3} - \frac{4^{n+1}}{9}. \end{aligned}$$

(b)(i) As $n \to \infty$, we have $(n+1)! \to \infty$ and so $\frac{1}{(n+1)!} \to 0$ and thus $S_n = 1 - \frac{1}{(n+1)!} \to 1$.

(ii)

$$\begin{aligned} u_n &= S_n - S_{n-1} = 1 - \frac{1}{(n+1)!} - \left(1 - \frac{1}{n!}\right) = \frac{1}{n!} - \frac{1}{(n+1)!} \\ &= \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{n}{(n+1)!}. \end{aligned}$$

A582 (9740 N2014/II/3)(i)(a) She runs 8 m in Stage 1 and an additional 8 m in each subsequent stage. So, in Stage n, she runs $8n$ m. Altogether then, the distance run in the first 10 stages is

$$(\text{First term} + \text{Last term}) \times \frac{\text{Number of terms}}{2} = (8 + 10 \cdot 8) \times \frac{10}{2} = 440\,\text{m}.$$

(b) The distance run in the first n stages is

$$(\text{First term} + \text{Last term}) \times \frac{\text{Number of terms}}{2} = (8 + 8n) \times \frac{n}{2} = 4n + 4n^2\,\text{m}.$$

Write $4n + 4n^2 \geq 5\,000$ or $n^2 + n - 1\,250 \overset{1}{\geq} 0$. By the quadratic formula:

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1\,250)}}{2(1)} = -0.5 \pm 0.5\sqrt{5\,001} \approx 34.859, -35.859.$$

Hence, $\overset{1}{\geq}$ holds if and only if $n \lesssim -35.859$ or $n \gtrsim 34.859$.

Thus, the minimum number of stages to complete in order to have run at least 5 km is 35.

(ii) The distance run in the nth stage is $2^{n-1} \cdot 8$ m. Thus, the distance run in the first n stages is

$$\sum_{k=1}^{n}\left(2^{k-1}\cdot 8\right)=8\sum_{k=1}^{n}2^{k-1}=8\frac{2^n-1}{2-1}=2^{n+3}-8\,\mathrm{m}.$$

Let j be the largest integer such that $2^{j+3}-8<10\,000$. Since $2^{13}=8\,192$ and $2^{14}=16\,384$, we have $j+3=13$ or $j=10$.

So, the distance run after completing exactly 10 stages is $2^{13}-8=8\,184\,\mathrm{m}$.

So, at the instant at which he has run exactly 10 km or 10 000 m, he has completed 1 816 m of the 11th stage. Since Stage 11 is $2^{11-1}\cdot 8=8\,192\,\mathrm{m}$ long , at this instant, he will not even have completed half of Stage 11. Thus, at this instant, he is 1 816 m away from O and running away from O.

A583 (9740 N2013/I/7)(i) The nth piece is $p\overset{1}{=}(2/3)^{n-1}\times 128\,\mathrm{cm}$ long. Applying ln to $\overset{1}{=}$, we get

$$\begin{aligned}\ln p&=\ln\left[(2/3)^{n-1}\times 128\right]=\ln(2/3)^{n-1}+\ln 128\\&=(n-1)\ln(2/3)+\ln 2^7=(n-1)(\ln 2-\ln 3)+7\ln 2=(n+6)\ln 2+(-n+1)\ln 3.\end{aligned}$$

Thus, $A=1$, $B=6$, $C=-1$, and $D=1$.

(ii) Let S_n be the total length of string cut off after cutting off n pieces. Then,

$$S_n=\sum_{k=1}^{n}\left(\frac{2}{3}\right)^{k-1}\times 128=128\frac{1-(2/3)^n}{1-2/3}=384-384\left(\frac{2}{3}\right)^n.$$

Thus, as $n\to\infty$, $S_n=384-384\left(\frac{2}{3}\right)^n\to 384$.

(iii) Let j be the smallest integer such that $S_j>380$. Then,

$$S_j=384-384\left(\frac{2}{3}\right)^j>380\quad\text{or}\quad 4>384\left(\frac{2}{3}\right)^j\quad\text{or}\quad\left(\frac{3}{2}\right)^j>\frac{384}{4}=96\quad\text{or}$$
$$j\ln\frac{3}{2}>\ln 96\quad\text{or}\quad j>\frac{\ln 96}{\ln(3/2)}\approx 11.257.$$

Thus, the minimum number of pieces one must cut off in order for the length cut off to exceed 380 cm is $j=12$.

A584 (9740 N2013/I/9)(i) Let $\mathbf{P}(k)$ be the following proposition:

$$\sum_{r=1}^{k}r\left(2r^2+1\right)=\frac{1}{2}k(k+1)\left(k^2+k+1\right).$$

We show that $\mathbf{P}(1)$ is true:

$$\sum_{r=1}^{1}r\left(2r^2+1\right)=1\left(2\cdot 1^2+1\right)=3=\frac{1}{2}\cdot 1\cdot 2\cdot 3=\frac{1}{2}\cdot 1(1+1)\left(1^2+1+1\right).\qquad\checkmark$$

We next show that for all $j\in\mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$\begin{aligned}\sum_{r=1}^{j+1} r\left(2r^2+1\right) &= \sum_{r=1}^{j} r\left(2r^2+1\right)+(j+1)\left[2(j+1)^2+1\right] \\ &\overset{\mathbf{P}(j)}{=} \frac{1}{2}j(j+1)\left(j^2+j+1\right)+(j+1)\left(2j^2+4j+3\right) \\ &= \frac{j+1}{2}\left(j^3+j^2+j\right)+\frac{j+1}{2}\left(4j^2+8j+6\right) \\ &= \frac{j+1}{2}\left(j^3+5j^2+9j+6\right) \\ &= \frac{1}{2}(j+1)(j+2)\left(j^2+aj+b\right) \\ &= \frac{1}{2}(j+1)(j+2)\left(j^2+3j+3\right) \\ &= \frac{1}{2}(j+1)(j+2)\left[(j+1)^2+(j+1)+1\right]. \qquad \checkmark\end{aligned}$$

(ii)
$$\begin{aligned} f(r)-f(r-1) &= \left(2r^3+3r^2+r+24\right)-\left[2(r-1)^3+3(r-1)^2+(r-1)+24\right] \\ &= \left(\cancel{2r^3}+3r^2\cancel{+r}\cancel{+24}\right)-\left[2\left(\cancel{r^3}-3r^2\cancel{+3r}\cancel{-1}\right)+3\left(r^2\cancel{-2r}\cancel{+1}\right)+\left(\cancel{r}\cancel{-1}\right)\cancel{+24}\right]=6r^2.\end{aligned}$$

So, $a=6$. And hence,

$$\begin{aligned}\sum_{r=1}^{n} r^2 = \frac{1}{6}\sum_{r=1}^{n}\left[f(r)-f(r-1)\right] &= \frac{1}{6}\left[f(1)-f(0)+f(2)-f(1)+\cdots+f(n)-f(n-1)\right] \\ = \frac{f(n)-f(0)}{6} = \frac{2n^3+3n^2+n+24-24}{6} &= \frac{2n^3+3n^2+n}{6}=\frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

(iii)
$$\begin{aligned}\sum_{r=1}^{n} f(r) &= \sum_{r=1}^{n}\left(2r^3+3r^2+r+24\right)=\sum_{r=1}^{n}\left[r\left(2r^2+1\right)+3r^2+24\right] \\ &= \sum_{r=1}^{n}\left[r\left(2r^2+1\right)\right]+3\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}24 \\ &= \frac{1}{2}n(n+1)\left(n^2+n+1\right)+\frac{n(n+1)(2n+1)}{2}+24n.\end{aligned}$$

A585 (9740 N2012/I/3)(i) $u_2=\dfrac{3\cdot 2-1}{6}=\dfrac{5}{6}$ and $u_3=\dfrac{3\cdot 5/6-1}{6}=\dfrac{1}{4}$.

(ii) As $n\to\infty$, $u_{n+1}-u_n\to 0 \iff \dfrac{3u_n-1}{6}-u_n=-\dfrac{u_n}{2}-\dfrac{1}{6}\to 0 \iff u_n\to -\dfrac{1}{3}$.

(iii) Let $\mathbf{P}(k)$ be the following proposition:

$$u_k=\frac{14}{3}\left(\frac{1}{2}\right)^k-\frac{1}{3}.$$

We show that $\mathbf{P}(1)$ is true:

$$u_1 = 2 = \frac{7}{3} - \frac{1}{3} = \frac{14}{3}\left(\frac{1}{2}\right)^1 - \frac{1}{3}. \qquad \checkmark$$

We next show that for all $j \in \mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$u_{j+1} = u_j + u_{j+1} - u_j = u_j + \left(-\frac{u_j}{2} - \frac{1}{6}\right) = \frac{u_j}{2} - \frac{1}{6}$$

$$\stackrel{\mathbf{P}(j)}{=} \frac{\frac{14}{3}\left(\frac{1}{2}\right)^j - \frac{1}{3}}{2} - \frac{1}{6} = \frac{14}{3}\left(\frac{1}{2}\right)^{j+1} - \frac{1}{3}. \qquad \checkmark$$

A586 (9740 N2012/II/4)(i) On the first day of the nth month, she deposits $100 + (n-1)10 = 10n + 90$. Hence, through the first day of the nth month, her account has:

$$(\text{First term} + \text{Last term}) \times \frac{\text{Number of terms}}{2} = (100 + 10n + 90) \times \frac{n}{2} = 5n^2 + 95n.$$

Let j be the smallest positive integer such that $5j^2 + 95j > 5\,000$ or $j^2 + 19j - 1\,000 \stackrel{1}{>} 0$. By the quadratic formula, the two roots of $x^2 + 19x - 1\,000 = 0$ are

$$x = \frac{-19 \pm \sqrt{19^2 - 4(1)(-1\,000)}}{2(1)} = -\frac{19 \pm \sqrt{627}}{2} \approx -42.519, 23.519.$$

Hence, $j = 24$. Thus, her account first exceeded \$5 000 on the 24th month—that is, on December 1 2002.

(ii) Let S_n be the amount in his account on the last day of each month, after interest has been paid.

Then $S_1 = 1.005 \cdot 100$ and $S_{n+1} = 1.005(S_n + 100)$.

So, in general:

$$S_n = 1.005^n \cdot 100 + 1.005^{n-1} \cdot 100 + \cdots + 1.005 \cdot 100 = 1.005 \cdot 100 \frac{1.005^n - 1}{1.005 - 1} = 20\,100(1.005^n - 1).$$

Let j be the smallest positive integer such that $S_j = 20\,100\left(1.005^j - 1\right) > 5\,000$ or $201 \cdot 1.005^j > 251$ or,

$$1.005^j > \frac{251}{201} \qquad \text{or} \qquad j \ln 1.005 > \ln\frac{251}{201} \qquad \text{or} \qquad j > \frac{\ln(251/201)}{\ln 1.005} \approx 44.541.$$

Hence, $j = 45$. Thus, his account first exceeded \$5 000 in the 45th month—that is, in September 2004.

(iii) Let r be the interest rate. Then given r, the amount in the account on 2 December 2003 is

$$\underbrace{100(1+r)^{35}}_{\substack{\text{Jan 2001 deposit has}\\ \text{earned interest 35 times}}} + \underbrace{100(1+r)^{34}}_{\substack{\text{Feb 2001 deposit has}\\ \text{earned interest 34 times}}} + \cdots + \underbrace{100(1+r)}_{\substack{\text{Nov 2003 deposit has}\\ \text{earned interest once}}} + \underbrace{10}_{\substack{\text{Dec 2003}\\ \text{has not earn}}}$$

We want the above amount to equal 5 000. So, write $100\dfrac{(1+r)^{36}-1}{r}=5\,000$ or,

$$(1+r)^{36}-1 \stackrel{1}{=} 50r \qquad \text{or} \qquad r \approx 0.01796 = 1.796\%.$$

Note that $\stackrel{1}{=}$ is an equation we haven't learnt to solve in H2 Maths, so you'll need to use your calculator here.

A587 (9740 N2011/I/6)(i)

$$\sin\left(r+\frac{1}{2}\right)\theta-\sin\left(r-\frac{1}{2}\right)\theta=\cancel{\sin r\theta\cos\frac{1}{2}\theta}+\cos r\theta\sin\frac{1}{2}\theta-\left[\cancel{\sin r\theta\cos\frac{1}{2}\theta}-\cos r\theta\sin\frac{1}{2}\theta\right]=2\cos r\theta\,\mathrm{s}$$

(ii) Rearranging **(i)**, we have $\cos r\theta=\dfrac{\sin\left(r+\frac{1}{2}\right)\theta-\sin\left(r-\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$. And so,

$$\begin{aligned}\sum_{r=1}^{n}\cos r\theta &= \sum_{r=1}^{n}\frac{\sin\left(r+\frac{1}{2}\right)\theta-\sin\left(r-\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}\\ &=\frac{1}{2\sin\frac{1}{2}\theta}\left(\sin\frac{3}{2}\theta-\sin\frac{1}{2}\theta+\sin\frac{5}{2}\theta-\sin\frac{3}{2}\theta+\cdots+\sin\frac{2n+1}{2}\theta-\sin\frac{2n-1}{2}\theta\right)\\ &=\frac{1}{2\sin\frac{1}{2}\theta}\left(\sin\frac{2n+1}{2}\theta-\sin\frac{1}{2}\theta\right)=\frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}-\frac{1}{2}.\end{aligned}$$

(iii) Let $\mathbf{P}(k)$ be the following proposition:

$$\sum_{r=1}^{k}\sin r\theta=\frac{\cos\frac{1}{2}\theta-\cos\left(k+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}.$$

We show that $\mathbf{P}(1)$ is true:

$$\sum_{r=1}^{1}\sin r\theta=\sin\theta=\frac{2\sin\frac{1}{2}\theta\sin\theta}{2\sin\frac{1}{2}\theta}=\frac{-2\sin-\frac{1}{2}\theta\sin\theta}{2\sin\frac{1}{2}\theta}=\frac{-2\sin\frac{1/2-3/2}{2}\theta\sin\frac{1/2+3/2}{2}\theta}{2\sin\frac{1}{2}\theta}=$$

$$\frac{\cos\overbrace{\tfrac{1}{2}\theta}^{P}-\cos\overbrace{\tfrac{3}{2}\theta}^{Q}}{2\sin\frac{1}{2}\theta}, \qquad \checkmark$$

where the last step uses the last formula printed under Trigonometry in List MF26, p. 3.

We next show that for all $j\in\mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$\sum_{r=1}^{j+1} \sin r\theta = \sum_{r=1}^{j} \sin r\theta + \sin(j+1)\theta$$

$$\overset{\mathbf{P}(j)}{=} \frac{\cos\frac{1}{2}\theta - \cos\left(j+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta} + \sin(j+1)\theta$$

$$= \frac{\cos\frac{1}{2}\theta - \cos\left(j+\frac{1}{2}\right)\theta + 2\sin\frac{1}{2}\theta\sin(j+1)\theta}{2\sin\frac{1}{2}\theta}$$

$$\overset{1}{=} \frac{\cos\frac{1}{2}\theta - \cos\left(j+\frac{1}{2}\right)\theta + \cos\left(j+\frac{1}{2}\right)\theta - \cos\left(j+\frac{3}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{\cos\frac{1}{2}\theta - \cos\left(j+\frac{3}{2}\right)\theta}{2\sin\frac{1}{2}\theta}, \qquad \checkmark$$

where again $\overset{1}{=}$ uses the same trigonometric identity as before.

as desired. (Again, to get from $\overset{3}{=}$ to $\overset{4}{=}$, I used the same trigonometric identity as before.)

A588 (9740 N2011/I/9)(i) The depth drilled on the nth day is $256 - 7(n-1) = 263 - 7n$ metres and the depth drilled through the nth day is

$$D_n = (\text{First term} + \text{Last term}) \times \frac{\text{Number of terms}}{2} = (256 + 263 - 7n)\frac{n}{2} = \frac{519}{2}n - \frac{7}{2}n^2.$$

Thus, the depth drilled on the 10th day is $263 - 7 \cdot 10 = 193$ metres.

Let j be the smallest integer such that $263 - 7j < 10$ or $j > 253/7 \approx 36.1$. Then $j = 37$. So, the total depth drilled is

$$D_j = D_{37} = \frac{519}{2} \cdot 37 - \frac{7}{2} \cdot 37^2 = 4\,810 \text{ metres.}$$

(ii) Through the nth day, the depth drilled is

$$d_n = \sum_{r=1}^{n} 256\left(\frac{8}{9}\right)^{r-1} = 256\frac{1-(8/9)^n}{1-8/9} = 2\,304\left[1-\left(\frac{8}{9}\right)^n\right] \text{ metres.}$$

By "theoretical maximum", the writers of this question probably mean this:

$$\lim_{n\to\infty} \sum_{r=1}^{n} 256\left(\frac{8}{9}\right)^{r-1} = \lim_{n\to\infty} 2\,304\left[1-\left(\frac{8}{9}\right)^n\right] = 2\,304 \text{ metres.}$$

Let j be the smallest integer such that $d_j > 0.99 \cdot 2\,304$ or $2\,304\left[1-\left(\frac{8}{9}\right)^j\right] > 0.99 \cdot 2\,304$ or,

$$1 - \left(\frac{8}{9}\right)^j > 0.99 \quad \text{or} \quad 0.01 > \left(\frac{8}{9}\right)^j \quad \text{or} \quad -\ln 100 > j \ln\frac{8}{9} \quad \text{or}$$
$$j > \frac{\ln 100}{\ln(9/8)} \approx 39.1.$$

Thus, $j = 40$.

A589 (9740 N2010/I/3)(i)

$$u_n = S_n - S_{n-1} = n(2n+c) - (n-1)[2(n-1)+c]$$
$$= 2n^2 + cn - \left(2n^2 - 4n + 2 + cn - c\right) = 4n - 2 + c.$$

(ii) Since $u_n = 4n - 2 + c$ and $u_{n+1} = 4(n+1) - 2 + c = 4n + 2 + c$, we have

$$u_{n+1} = u_n + 4.$$

A590 (9740 N2010/II/2)(i) Let $\mathbf{P}(k)$ be the following proposition:

$$\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7).$$

We show that $\mathbf{P}(1)$ is true:

$$\sum_{r=1}^{1} r(r+2) = 1 \cdot 3 = 3 = \frac{1}{6} \cdot 1 \cdot 2 \cdot 9 = \frac{1}{6}1(1+1)(2 \cdot 1 + 7). \qquad \checkmark$$

We next show that for all $j \in \mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$\begin{aligned}
\sum_{r=1}^{j+1} r(r+2) &= \sum_{r=1}^{j} r(r+2) + (j+1)(j+3) \\
&\overset{\mathbf{P}(j)}{=} \frac{1}{6}j(j+1)(2j+7) + (j+1)(j+3) \\
&= \frac{j+1}{6}\left(2j^2 + 7j\right) + \frac{j+1}{6}(6j+18) \\
&= \frac{j+1}{6}\left(2j^2 + 13j + 18\right) \\
&= \frac{j+1}{6}(j+2)(2j+7) \\
&= \frac{1}{6}(j+1)(j+1+1)[2(j+1)+7]. \qquad \checkmark
\end{aligned}$$

(ii)(a) Observe that

$$\frac{1}{r(r+2)} = \frac{0.5}{r} - \frac{0.5}{r+2}.$$

Hence,

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{r(r+2)} &= \sum_{r=1}^{n} \left(\frac{0.5}{r} - \frac{0.5}{r+2}\right) \\
&= \frac{0.5}{1} - \frac{0.5}{3} + \frac{0.5}{2} - \frac{0.5}{4} + \frac{0.5}{3} - \frac{0.5}{5} + \cdots + \frac{0.5}{n-1} - \frac{0.5}{n+1} + \frac{0.5}{n} - \frac{0.5}{n+2} \\
&= \frac{0.5}{1} + \frac{0.5}{2} - \frac{0.5}{n+1} - \frac{0.5}{n+2} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}. \qquad \checkmark
\end{aligned}$$

(b) In the formula found in **(ii)(a)**, as $n \to \infty$, the second and third terms vanish. Hence, as $n \to \infty$, $\sum_{r=1}^{n} \frac{1}{r(r+2)} \to \frac{3}{4}$.

A591 (9740 N2009/I/3)(i) $\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} = \frac{n(n+1) - 2(n-1)(n+1) + (n-1)n}{(n-1)n(n+1)}$

$$= \frac{n^2 + n - 2(n^2-1) + n^2 - n}{n(n^2-1)} = \frac{-2(-1)}{n^3 - n} = \frac{2}{n^3 - n}.$$

So, $A = 2$.

(ii)

$$\sum_{r=2}^{n} \frac{1}{r^3 - r} = \frac{1}{2}\sum_{r=2}^{n}\left(\frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} + \cdots + \frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n} + \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right).$$

Observe that the terms with denominators 3 through $n-1$ are cancelled out. Hence,

$$\sum_{r=2}^{n} \frac{1}{r^3 - r} = \frac{1}{2}\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right) = \frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)}.$$

(iii) In the formula found in **(ii)**, as $n \to \infty$, the second and third terms vanish. Hence, as $n \to \infty$, $\sum_{r=2}^{n} \frac{1}{r^3 - r} \to \frac{1}{4}$.

A592 (9740 N2009/I/5)(i) Let $\mathbf{P}(k)$ be the following proposition:

$$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1).$$

We show that $\mathbf{P}(1)$ is true:

$$\sum_{r=1}^{1} r^2 = 1^2 = 1 = \frac{1}{6} \cdot 1 \cdot 2 \cdot 3 = \frac{1}{6}1(1+1)(2 \cdot 1 + 1). \quad \checkmark$$

We next show that for all $j \in \mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$\sum_{r=1}^{j+1} r^2 = \sum_{r=1}^{j} r^2 + (j+1)^2 \overset{\mathbf{P}(j)}{=} \frac{1}{6}j(j+1)(2j+1) + (j+1)^2$$

$$= \frac{j+1}{6}\left(2j^2 + j\right) + \frac{j+1}{6}(6j+6) = \frac{j+1}{6}\left(2j^2 + 7j + 6\right)$$

$$= \frac{j+1}{6}(j+2)(2j+3) = \frac{1}{6}(j+1)(j+1+1)[2(j+1)+1]. \quad \checkmark$$

(ii)

$$\sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2 = \frac{1}{6}2n(2n+1)(2 \cdot 2n + 1) - \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{n(2n+1)}{6}(8n+2) - \frac{n(2n+1)}{6}(n+1) = \frac{n(2n+1)}{6}(7n+1).$$

A593 (9740 N2009/I/8)(i) Let r be the common ratio. Then the 25th bar has length $20r^{24} = 5\,\text{cm}$ and so $r = \left(\frac{1}{4}\right)^{1/24} = 0.5^{1/12}$.

In the limit, the total length of all the bars is $\dfrac{20}{1-r} \approx 356.343\,\text{cm}$.

And so indeed, no matter how many bars there are, their total length cannot exceed 357 cm.

(ii) The total length is $L = 20\dfrac{1-r^{25}}{1-r} \approx 272.257\,\text{cm}$.

The length of the 13th bar is $20r^{12} = 20 \cdot \left(0.5^{1/12}\right)^{12} = 10\,\text{cm}$.

(iii) The total length L is

$$L = (\text{Length of 25th bar} + \text{Length of 1st bar}) \times \frac{\text{Number of terms}}{2} = (5 + 5 + 24d)\,\frac{25}{2} = 125 + 300d = 272.257.$$

So, $d \approx 0.491\,\text{cm}$ and the length of the longest bar (the 1st bar) is $5 + 24d \approx 16.781\,\text{cm}$.

A594 (9740 N2008/I/2). Let $\mathbf{P}(k)$ be the following proposition:

$$S_k = \sum_{r=1}^{k} u_r = \frac{1}{6}k(k+1)(4k+5).$$

We show that $\mathbf{P}(1)$ is true:

$$S_1 = \sum_{r=1}^{1} u_r = u_1 = 1(2 \cdot 1 + 1) = 3 = \frac{1}{6} \cdot 1 \cdot 2 \cdot 9 = \frac{1}{6} \cdot 1(1+1)(4 \cdot 1 + 5). \qquad \checkmark$$

We next show that for all $j \in \mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$S_{j+1} = \sum_{r=1}^{j} u_r + (j+1)(2j+3) \overset{\mathbf{P}(j)}{=} \frac{1}{6}j(j+1)(4j+5) + (j+1)(2j+3) = \frac{j+1}{6}\left(4j^2 + 5j\right) + \frac{j+1}{6}(1$$

$$= \frac{j+1}{6}\left(4j^2 + 17j + 18\right) = \frac{j+1}{6}(j+2)(4j+9) = \frac{1}{6}(j+1)(j+1+1)[4(j+1)+5]. \qquad \checkmark$$

A595 (9740 N2008/I/10)(i) On the first day of the nth month, she saves $10 + 3(n-1) = 7 + 3n$ dollars.

Thus, the total saved through the first day of the nth month is

$$(\text{First term} + \text{Last term}) \times \frac{\text{Number of terms}}{2} = (10 + 7 + 3n)\,\frac{n}{2} = \frac{3}{2}n^2 + \frac{17}{2}n.$$

Let j be the smallest positive integer such that $\frac{3}{2}j^2 + \frac{17}{2}j > 2\,000$ or $3j^2 + 17j - 4\,000 > 0$.
By the quadratic formula, the solution to $3x^2 + 17x - 4\,000 = 0$ is

$$x = \frac{-17 \pm \sqrt{17^2 - 4(3)(-4\,000)}}{2(3)} = \frac{-17 \pm \sqrt{48\,289}}{6} \approx -39.458, 33.791$$

Thus, $j = 34$. So, she will have saved over \$2 000 on 1 October 2011.

In the 1st month (Jan 2009), she has saved 10 dollars. In the 2nd (Feb 2009), she has saved $10 + (10 + 1 \times 3)$ dollars. So in the nth month, she has saved $10 + (10 + 1 \times 3) + (10 + 2 \times 3) + \dots [10 + (n-1) \times 3] = [20 + (n-1) \times 3] \times \dfrac{n}{2} = 8.5n + 1.5n^2$ dollars. Set $8.5n + 1.5n^2 = 2000$ and solve:

$$n = \frac{-8.5 \pm \sqrt{8.5^2 - 4(1.5)(-2000)}}{3} = \frac{-8.5 \pm 109.874}{3}$$

We can ignore the negative root. So $n = \dfrac{-8.5 + 109.874}{3} \approx 33.791$. So it is only in the 34th month that she has saved over \$2000. That's 1 October 2011.

(ii)(a) At the end of 2 years, her original \$10 has earned $10 \times 1.02^{24} - 10 \approx 6.084$ dollars in compound interest.

(b) Just after interest has been paid on the last day of the nth month, the total in her account is

$$10 \cdot 1.02^n + 10 \cdot 1.02^{n-1} + \cdots + 10 \cdot 1.02^1 = 10 \cdot 1.02 \cdot \frac{1.02^n - 1}{1.02 - 1} = 510 \left(1.02^n - 1\right) \text{ dollars.}$$

Hence, at the end of 2 years, just after interest has been paid on 31 December 2010, the total in her account is

$$510 \left(1.02^{24} - 1\right) \approx 310.303 \text{ dollars.}$$

(c) Let j be the smallest positive integer such that $510 \left(1.02^j - 1\right) > 2\,000$ or $510 \cdot 1.02^j > 2\,510$ or,

$$1.02^j > \frac{251}{51} \qquad \text{or} \qquad j \ln 1.02 > \ln \frac{251}{51} \qquad \text{or} \qquad j > \ln \frac{251}{51} \div \ln 1.02 \approx 80.476.$$

Thus, it is only after $j = 81$ complete months that her total savings first exceed \$2 000.

A596 (9233 N2008/II/2). Let a_n and g_n denote the nth terms of the arithmetic and geometric progressions. Let d and r be the corresponding common difference and ratio. We are given

$$a_2 + g_2 = \frac{1}{2} \qquad \text{or} \qquad \frac{1}{2} + d + \frac{r}{2} = \frac{1}{2} \qquad \text{or} \qquad d + \frac{r}{2} \overset{1}{=} 0.$$

And, $$a_3 + g_3 = \frac{1}{8} \qquad \text{or} \qquad \frac{1}{2} + 2d + \frac{r^2}{2} = \frac{1}{8} \qquad \text{or} \qquad 2d + \frac{r^2}{2} \overset{2}{=} -\frac{3}{8}.$$

$2\times \overset{1}{=}$ minus $\overset{2}{=}$ yields $r - r^2/2 = 3/8$ or $4r^2 - 8r + 3 = 0$ or,

$$r = \frac{8 \pm \sqrt{(-8)^2 - 4(4)(3)}}{2(4)} = 1 \pm \sqrt{1 - 3/4} = 1 \pm \frac{1}{2} = \frac{1}{2}, \frac{3}{2}.$$

Since the geometric progression converges, $|r| < 1$ and so $r = 1/2$. And thus, its sum to infinity is

$$\frac{g_1}{1-r} = \frac{1/2}{1-1/2} = 1.$$

A597 (9740 N2007/I/9)(i) Using your graphing calculator, $\alpha \approx 0.619$ and $\beta \approx 1.512$ (calculator).

(ii) Suppose $x_n \to L$. Then $x_{n+1} - x_n \to 0$. Or,

$$x_{n+1} - x_n = \frac{1}{3}e^{x_n} - x_n \to 0 \qquad \text{or} \qquad \frac{1}{3}e^L - L = 0.$$

Equivalently, L equals a solution to $\frac{1}{3}e^x - x = 0$. So, L equals α or β.

Remark 218. The answer to **(ii)** takes for granted certain results that students were not taught (even under the old 9740 syllabus). This question should never have been asked.

(iii) If $x_1 = 0$, then $x_2 = \frac{1}{3}$, $x_3 \approx 0.465$, $x_4 \approx 0.531$, $x_5 \approx 0.567$, $x_6 \approx 0.588$, ..., $x_{15} \approx 0.619$. So the sequence converges to $\alpha \approx 0.619$.

If $x_1 = 1$, then $x_2 \approx 0.906$, $x_3 \approx 0.825$, $x_4 \approx 0.761$, $x_5 \approx 0.713$, $x_6 \approx 0.680$, ..., $x_{17} \approx 0.619$. So the sequence converges to $\alpha \approx 0.619$.

If $x_1 = 2$, then $x_2 \approx 2.463$, $x_3 \approx 3.913$, $x_4 \approx 16.690$, $x_5 \approx 5\,903\,230.335$. "Clearly", the sequence diverges.

(iv) From the graph of $y = e^x - 3x$, we observe that if $\alpha < x_n < \beta$, then $e^x - 3x < 0$ or $\frac{1}{3}e^x < x$ or $x_{n+1} < x_n$.

Similarly, we observe that if $x < \alpha$ or $x > \beta$, then $e^x - 3x > 0$ or $\frac{1}{3}e^x > x$ or $x_{n+1} > x_n$.

(v) If $x_n > \beta \approx 1.512$, then **(iv)** tells us that $x_{n+1} > x_n$ and therefore that the sequence diverges. We saw this with $x_1 = 2$ in **(iii)**.

If $x_n \in (\alpha, \beta) \approx (0.619, 1.512)$, then **(iv)** tells us that $x_{n+1} < x_n$. We saw this with $x_1 = 1$ in **(iii)**.

If $x_n < \alpha \approx 0.619$, then **(iv)** tells us that $x_{n+1} > x_n$. We saw this with $x_1 = 0$ in **(iii)**.

A598 (9740 N2007/I/10)(i) We are given that the first term of the geometric progression equals a.

We are also given: $\qquad ra \overset{1}{=} a + 3d, \qquad r^2 a \overset{2}{=} a + 5d.$

Rearranging $\overset{1}{=}$ and $\overset{2}{=}$ so that d is on one side, we get

$$d = \frac{a(r-1)}{3} = \frac{a(r^2-1)}{5} \qquad \text{or} \qquad 5r - 5 = 3r^2 - 3 \qquad \text{or} \qquad 3r^2 - 5r + 2 = 0.$$

(ii) $3r^2 - 5r + 2 = (3r-2)(r-1) = 0$ and so $r = 2/3$ or $r = 1$. But if $r = 1$, then by $\overset{4}{=}$, $d = 0$, contradicting our assumption that $d \neq 0$. Hence, $r = 2/3$. Since $|r| < 1$, the geometric series converges to:

$$\frac{a}{1-r} = 3a.$$

(iii) From $\stackrel{1}{=}$, $d = \dfrac{ra-a}{3} = \dfrac{-a/3}{3} = -\dfrac{a}{9}$. And now,

$$S = [a + a + (n-1)d]\frac{n}{2} = an + \frac{n-1}{2}nd = an\left(1 - \frac{n-1}{18}\right) = an\frac{19-n}{18}.$$

$S > 4a \iff an\dfrac{19-n}{18} > 4a \iff n(19-n) > 72 \iff n^2 - 19n + 72 \stackrel{1}{<} 0$. By the quadratic formula, $x^2 - 19x + 72 = 0$ has solutions:

$$x = \frac{19 \pm \sqrt{(-19)^2 - 4(1)(72)}}{2(1)} = \frac{19 \pm \sqrt{73}}{2} \approx 5.228, 13.772.$$

So, $\stackrel{1}{<}$ holds if and only if $5.228 \lesssim n \lesssim 13.772$. Of course, n must be an integer. And so, the set of possible values of n for which $\stackrel{1}{<}$ or $S > 4a$ holds is $\{6, 7, 8, 9, 10, 11, 12, 13\}$.

A599 (9740 N2007/II/2)(i) Let $\mathbf{P}(k)$ be the following proposition:

$$u_k = \frac{1}{k^2}.$$

We show that $\mathbf{P}(1)$ is true:

$$u_1 = 1 = \frac{1}{1^2}. \qquad \checkmark$$

We next show that for all $j \in \mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$\begin{aligned} u_{j+1} &= u_j - \frac{2j+1}{j^2(j+1)^2} \stackrel{\mathbf{P}(j)}{=} \frac{1}{j^2} - \frac{2j+1}{j^2(j+1)^2} = \frac{(j+1)^2 - (2j+1)}{j^2(j+1)^2} \\ &= \frac{j^2}{j^2(j+1)^2} = \frac{1}{(j+1)^2}. \qquad \checkmark \end{aligned}$$

(ii)

$$\begin{aligned} \sum_{n=1}^{N} \frac{2n+1}{n^2(n+1)^2} &= \sum_{n=1}^{N}(u_n - u_{n+1}) \\ &= u_1 - u_2 + u_2 - u_3 + \cdots + u_N - u_{N+1} \\ &= u_1 - u_{N+1} = 1 - \frac{1}{(N+1)^2}. \end{aligned}$$

(iii) In the formula just found in **(ii)**, as $N \to \infty$, the second term vanishes, so that the series converges to 1.

(iv) First observe that $\dfrac{2n+1}{n^2(n+1)^2} = \dfrac{2(n+1)-1}{(n+1)^2(n+1-1)^2}$. Thus,

$$\sum_{n=2}^{N} \frac{2n-1}{n^2(n-1)^2} = \sum_{n=1}^{N-1} \frac{2n+1}{n^2(n+1)^2} = 1 - \frac{1}{N^2}.$$

A600 (9233 N2007/I/14). Let $\mathbf{P}(k)$ be the following proposition:

$$\sum_{r=1}^{k}\sin rx=\frac{\cos\frac{1}{2}x-\cos\left(k+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}.$$

We show that $\mathbf{P}(1)$ is true:

$$\sum_{r=1}^{1}\sin rx=\sin x=\frac{2\sin x\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\stackrel{1}{=}\frac{\cos\frac{1}{2}x-\cos\left(1+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}.\qquad\checkmark$$

($\stackrel{1}{=}$ used the last trigonometric identity printed on List MF26, p. 3.)

We next show that for all $j\in\mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$\begin{aligned}\sum_{r=1}^{j+1}\sin rx &= \sum_{r=1}^{j}\sin rx+\sin(j+1)x\\ &\stackrel{\mathbf{P}(j)}{=}\frac{\cos\frac{1}{2}x-\cos\left(j+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}+\sin(j+1)x\\ &=\frac{\cos\frac{1}{2}x-\cos\left(j+\frac{1}{2}\right)x+2\sin\frac{1}{2}x\sin(j+1)x}{2\sin\frac{1}{2}x}\\ &\stackrel{2}{=}\frac{\cos\frac{1}{2}x\cancel{-\cos\left(j+\frac{1}{2}\right)x}\cancel{+\cos\left(j+\frac{1}{2}\right)x}-\cos\left(j+\frac{3}{2}\right)x}{2\sin\frac{1}{2}x}\\ &=\frac{\cos\frac{1}{2}x-\cos\left(j+1+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}.\qquad\checkmark\end{aligned}$$

(Again, $\stackrel{2}{=}$ used the same trigonometric identity as before.)

A601 (9233 N2007/II/1). $\displaystyle\sum_{r=1}^{2n}3^{r+2}=9\sum_{r=1}^{2n}3^{r}=9\cdot3\frac{3^{2n}-1}{3-1}=\frac{27}{2}\left(3^{2n}-1\right).$

A602 (9233 N2006/I/1). The first term is $S_1=6-\dfrac{2}{3^{1-1}}=4.$

The common ratio is $(S_2-S_1)\div S_1=S_2\div S_1-1=\left(6-\dfrac{2}{3^{2-1}}\right)\div4-1=\dfrac{4}{3}-1=\dfrac{1}{3}.$

A603 (9233 N2006/I/11)(i) Let $\mathbf{P}(k)$ be the following proposition:

$$\sum_{r=1}^{k}r^3=\frac{1}{4}k^2(k+1)^2.$$

We show that $\mathbf{P}(1)$ is true:

$$\sum_{r=1}^{1}r^3=1^3=1=\frac{1}{4}\cdot1\cdot4=\frac{1}{4}1^2(1+1)^2.\qquad\checkmark$$

We next show that for all $j\in\mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$\sum_{r=1}^{j+1} r^3 = \sum_{r=1}^{j} r^3 + (j+1)^3 \overset{\mathbf{P}(j)}{=} \frac{1}{4} j^2 (j+1)^2 + (j+1)^3$$

$$= \frac{(j+1)^2}{4} j^2 + \frac{(j+1)^2}{4}(4j+4) = \frac{(j+1)^2}{4}\left(j^2+4j+4\right) = \frac{(j+1)^2}{4}(j+2)^2. \quad ✓$$

(ii) $2^3 + 4^3 + \cdots + (2n)^3 = \sum_{r=1}^{n} (2r)^3 = 8 \sum_{r=1}^{n} r^3 = 2n^2 (n+1)^2.$

(iii)

$$\sum_{r=1}^{n} (2r-1)^3 = 1^3 + 3^3 + \ldots (2n-1)^3 = 1^3 + 2^3 + \cdots + (2n)^2 - \left[2^3 + 4^3 + \cdots + (2n)^2\right]$$

$$= \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n} (2r)^3 = \frac{1}{4}(2n)^2 (2n+1)^2 - 2n^2 (n+1)^2$$

$$= n^2 \left[4n^2 + 4n + 1 - \left(2n^2 + 4n + 2\right)\right] = n^2 \left(2n^2 - 1\right).$$

155.3. Ch. 135 Answers (Vectors)

A604 (9758 N2019/I/12)(i) We're given $Q = P + \lambda(-2,-3,-6) = (2,2,4) + \lambda(-2,-3,-6)$ for some $\lambda \in \mathbb{R}$. By my Assumption 1, Q is on the plane $x+y+z = 1$. So, $2-2\lambda+2-3\lambda+4-6\lambda = 1$ or $8 - 11\lambda = 1$ or $\lambda = 7/11$. Hence, $Q = (2,2,4) + {}^{7}/_{11}(-2,-3,-6) = {}^{1}/_{11}(8,1,2)$.

Similarly, we're given $R = S + \mu(-2,-3,-6) = (-5,-6,-7) + \mu(-2,-3,-6)$ for some $\mu \in \mathbb{R}$. By my Assumption 2, R is on the plane $x + y + z = -9$. So, $-5 - 2\mu - 6 - 3\mu - 7 - 6\mu = -9$ or $-18 - 11\mu = -9$ or $\mu = -9/11$. Hence, $R = (-5,-6,-7) - {}^{9}/_{11}(-2,-3,-6) = {}^{1}/_{11}(-37,-39,-23)$.

(ii) $\cos\theta = \dfrac{|(-2,-3,-6)\cdot(1,1,1)|}{|(-2,-3,-6)||(1,1,1)|} = \dfrac{11}{7\sqrt{3}}$. $\qquad \overrightarrow{QR} = \dfrac{1}{11}(-45,-40,-25) = \dfrac{-5}{11}(9,8,5)$.

$$\cos\beta = \frac{|(9,8,5)\cdot(1,1,1)|}{|(9,8,5)||(1,1,1)|} = \frac{22}{\sqrt{510}}.$$

(iii) Consider the point $A = (1,0,0)$ on the plane $x + y + z = 1$. Let B be the point on the plane $x + y + z = -9$ that is closest to A. Then $B = A + \lambda(1,1,1) = (1+\lambda, \lambda, \lambda)$ and so $1 + \lambda + \lambda + \lambda = -9$ or $\lambda = -10/3$. Hence, the distance between the two planes is $\left|\overrightarrow{AB}\right| = |\lambda||(1,1,1)| = 10\sqrt{3}/3$.

(iv) $$k = \frac{\sin\theta}{\sin\beta} = \frac{\sqrt{1-\cos^2\theta}}{\sqrt{1-\cos^2\beta}} = \frac{\sqrt{1-121/147}}{\sqrt{1-484/510}} = \frac{\sqrt{26/147}}{\sqrt{26/510}} = \sqrt{\frac{170}{49}} \approx 1.86.$$

(v) Since $\theta, \beta \in [0, \pi/2]$, $\beta > \theta \implies k = \dfrac{\sin\theta}{\sin\beta} < 1$.

A605 (9758 N2019/II/5)(i) For some $\lambda, \mu \in \mathbb{R}$,

$$\overrightarrow{OX} = \overrightarrow{OA} + \lambda\overrightarrow{AC} = \overrightarrow{OB} + \mu\overrightarrow{BD} = \mathbf{a} + \lambda(\mathbf{a} + 4\mathbf{b}) = \mathbf{b} + \mu(5\mathbf{a}).$$

Comparing coefficients, $\lambda = 1/4$. Hence, $\overrightarrow{OX} = 1.25\mathbf{a} + \mathbf{b}$.

(ii) For some $\alpha, \beta \in [0,1]$ with $\alpha + \beta \overset{1}{=} 1$, $\overrightarrow{OY} = \alpha\overrightarrow{OC} + \beta\overrightarrow{OD} = (2\alpha + 5\beta)\mathbf{a} + (4\alpha + \beta)\mathbf{b}$.

For some $\gamma \in \mathbb{R}$, $\overrightarrow{OY} = \gamma\overrightarrow{OX} = 1.25\gamma\mathbf{a} + \gamma\mathbf{b}$. Comparing coefficients,

$$2\alpha + 5\beta \overset{2}{=} 1.25\gamma \qquad \text{and} \qquad 4\alpha + \beta \overset{3}{=} \gamma.$$

Now, $2 \times \overset{2}{=} - \overset{3}{=}$ yields $9\beta = 1.5\gamma$ or $\beta = \gamma/6 = 4\gamma/24$ and hence $\alpha = 5\gamma/24$.

Now, $\overset{1}{=}$ implies $\alpha = 5/9$ and $\beta = 4/9$. Hence,

$$\overrightarrow{OY} = (2\alpha + 5\beta)\mathbf{a} + (4\alpha + \beta)\mathbf{b} = \frac{2}{3}(5\mathbf{a} + 4\mathbf{b}) = \frac{8}{3}\left(\frac{5}{4}\mathbf{a} + \mathbf{b}\right) = \frac{8}{3}\overrightarrow{OX}.$$

And $\overrightarrow{OX} : \overrightarrow{OY} = 3/8$.

A606 (9758 N2018/I/6)(a) By the distributivity of the vector product (Fact 145),

$$\mathbf{a} \times 3\mathbf{b} = 2\mathbf{a} \times \mathbf{c} \iff \mathbf{a} \times 3\mathbf{b} - 2\mathbf{a} \times \mathbf{c} = \mathbf{0} \iff \mathbf{a} \times (3\mathbf{b} - 2\mathbf{c}) = \mathbf{0}.$$

By Corollary 25, $3\mathbf{b} - 2\mathbf{c} \parallel \mathbf{a}$ or equivalently, $3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$ for some λ.

(b) We're given $\mathbf{a} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{c} = 1$, $\mathbf{b} \cdot \mathbf{b} = 16$, and $\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}| |\mathbf{c}| \cos 60° = (4)(1)(0.5) = 2$.
Also, $\mathbf{b} \nparallel \mathbf{c}$, so $\lambda \neq 0$.
Now consider the scalar product of $3\mathbf{b} - 2\mathbf{c}$ with each of $\mathbf{c}$, $\mathbf{b}$, and $\mathbf{a}$:

- $(3\mathbf{b} - 2\mathbf{c}) \cdot \mathbf{c} = \lambda\mathbf{a} \cdot \mathbf{c} = 3\mathbf{b} \cdot \mathbf{c} - 2\mathbf{c} \cdot \mathbf{c} = 6 - 2 = 4 \implies \mathbf{a} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 4/\lambda$.
- $(3\mathbf{b} - 2\mathbf{c}) \cdot \mathbf{b} = \lambda\mathbf{a} \cdot \mathbf{b} = 3\mathbf{b} \cdot \mathbf{b} - 2\mathbf{c} \cdot \mathbf{b} = 48 - 4 = 44 \implies \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = 44/\lambda$.
- $(3\mathbf{b} - 2\mathbf{c}) \cdot \mathbf{a} = \lambda\mathbf{a} \cdot \mathbf{a} = \lambda = 3\mathbf{b} \cdot \mathbf{a} - 2\mathbf{c} \cdot \mathbf{a} = 132/\lambda - 8/\lambda = 124/\lambda$.

Rearranging, $\lambda^2 = 124$ or $\lambda = \pm\sqrt{124}$.

A607 (9758 N2018/II/3)(i) $\overrightarrow{AB} = \overrightarrow{DC} = B - A = (0, 8, -1)$.
So, $D = C - \overrightarrow{DC} = (-5, 4, 2) - (0, 8, -1) = (-5, -4, 3)$.

(ii) $\overrightarrow{EB} \times \overrightarrow{EC} = (5, 4, -10) \times (-5, 4, -8) = (8, 90, 40)$.
$(0, 0, 10) \cdot (8, 90, 40) = 400$.
So, the cartesian equation of the *plane* BCE is $\mathbf{r} \cdot (8, 90, 40) \stackrel{1}{=} 400$.

(iii) Plane $ABCD$ has normal vector $\overrightarrow{AB} \times \overrightarrow{BC} = (0, 8, -1) \times (-10, 0, 2) = (16, 10, 80)$.
The requested angle is the angle between the planes $ABCD$ and BCE, which is (Fact 172)

$$\cos^{-1} \frac{|(8, 90, 40) \cdot (16, 10, 80)|}{|(8, 90, 40)| |(16, 10, 80)|} = \cos^{-1} \frac{|128 + 900 + 3200|}{\sqrt{64 + 8100 + 1600}\sqrt{256 + 100 + 6400}}$$
$$= \cos^{-1} \frac{4228}{\sqrt{9764}\sqrt{6756}} \approx 1.02.$$

(iv) The midpoint of edge AD is $(0, -4, 2)$.
Let $(0, -4, 2) + k(8, 90, 40)$ be the point on the *plane* BCE that is closest to $(0, -4, 2)$.
This point must satisfy $\stackrel{1}{=}$:

$$[(0, -4, 2) + k(8, 90, 40)] \cdot (8, 90, 40) = 400 \iff 9764k - 360 + 80 = 400 \iff k = 680/9764.$$

Hence, the distance between the midpoint of edge AD and the *plane* BCE is

$$|k(8, 90, 40)| = \frac{680}{9764}\sqrt{9764} = \frac{680}{\sqrt{9764}} \approx 6.88.$$

A608 (9758 N2017/I/6)(i) Assuming t takes on all values in $\mathbb{R}$, the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ describes the line with direction vector $\mathbf{b}$ and which passes through the point with position vector $\mathbf{a}$.

(ii) The vector equation $\mathbf{r} \cdot \mathbf{n} = d$ describes the plane that has normal vector $\mathbf{n}$ and is of distance $|d|$ away from the origin. (See Corollary 35.)

(iii) Since $\mathbf{b} \cdot \mathbf{n} \neq 0$, the line is not parallel to the plane. Hence (Fact 171), the line and plane intersect at exactly one point. This point is given by the solution of the following system of (two vector) equations:

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \qquad \text{and} \qquad \mathbf{r} \cdot \mathbf{n} = d.$$

To solve these equations, plug the first into the second:

$$\left(\mathbf{a} + \hat{t}\mathbf{b}\right) \cdot \mathbf{n} = d \qquad \Longleftrightarrow \qquad \mathbf{a} \cdot \mathbf{n} + \hat{t}\mathbf{b} \cdot \mathbf{n} = d \qquad \Longleftrightarrow \qquad \hat{t} = \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}.$$

That is, $\hat{t}$ solves the above system of equations.

And this solution corresponds to the point at which the line and plane intersect. This point has position vector:

$$\mathbf{a} + \hat{t}\mathbf{b} = \mathbf{a} + \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}.$$

A609 (9758 N2017/I/10)(i) The existing cable corresponds to the line described by

$$\mathbf{r} = (0,0,0) + \lambda\,(3,1,-2) \qquad (\lambda \in \mathbb{R})$$

The new cable corresponds to the line with direction vector:

$$\overrightarrow{PQ} = Q - P = (5,7,a) - (1,2,-1) = (4,5,a+1)\,.$$

And so, this line may be described by

$$\mathbf{r} = (1,2,-1) + \mu\,(4,5,a+1) \qquad (\mu \in \mathbb{R}).$$

If the two lines intersect, then there exist real numbers $\hat{\lambda}$ and $\hat{\mu}$ such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \hat{\lambda}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \hat{\mu}\begin{pmatrix} 4 \\ 5 \\ a+1 \end{pmatrix} \quad \text{or} \quad \begin{aligned} 3\hat{\lambda} &\overset{1}{=} 1 + 4\hat{\mu}, \\ \hat{\lambda} &\overset{2}{=} 2 + 5\hat{\mu}, \\ -2\hat{\lambda} &\overset{3}{=} -1 + (a+1)\,\hat{\mu} \end{aligned} \qquad (\lambda \in \mathbb{R}).$$

$3\times \overset{2}{=}$ minus $\overset{1}{=}$ yields $0 = 5 + 11\hat{\mu}$ or $\hat{\mu} = -5/11$. And now from $\overset{2}{=}$, $\hat{\lambda} = -3/11$. If these values of $\hat{\lambda}$ and $\hat{\mu}$ satisfy $\overset{3}{=}$, then

$$-2\left(\frac{-3}{11}\right) \overset{3}{=} -1 + (a+1)\left(\frac{-5}{11}\right) \qquad \text{or} \qquad a \overset{3}{=} -\frac{22}{5}.$$

(i) Let $R = (0,0,0) + \tilde{\lambda}(3,1,-2) = \tilde{\lambda}(3,1,-2)$.

If $\angle PRQ$ is right, then $\overrightarrow{PR} \perp \overrightarrow{QR}$ or $\overrightarrow{PR} \cdot \overrightarrow{QR} = 0$. But,

$$\overrightarrow{PR} = R - P = \tilde{\lambda}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3\tilde{\lambda}-1 \\ \tilde{\lambda}-2 \\ -2\tilde{\lambda}+1 \end{pmatrix} \qquad \text{and}$$

$$\overrightarrow{QR} = R - Q = \tilde{\lambda}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 3\tilde{\lambda}-5 \\ \tilde{\lambda}-7 \\ -2\tilde{\lambda}+3 \end{pmatrix}.$$

So,

$$\begin{aligned} \overrightarrow{PR} \cdot \overrightarrow{QR} \quad &= \left(3\tilde{\lambda}-1\right)\left(3\tilde{\lambda}-5\right) + \left(\tilde{\lambda}-2\right)\left(\tilde{\lambda}-7\right) + \left(-2\tilde{\lambda}+1\right)\left(-2\tilde{\lambda}+3\right) \\ &= 14\tilde{\lambda}^2 - 35\tilde{\lambda} + 22. \end{aligned}$$

This is a quadratic expression in $\tilde{\lambda}$ with determinant $(-35)^2 - 4(14)(22) < 0$. Hence, $\overrightarrow{PR} \cdot \overrightarrow{QR} \neq 0$ or $\overrightarrow{PR} \not\perp \overrightarrow{QR}$ and thus $\angle PRQ$ cannot be right,

(iii) Let $R = (0,0,0) + \bar{\lambda}(3,1,-2) = \bar{\lambda}(3,1,-2)$.

$$\left|\overrightarrow{PR}\right| = \sqrt{\left(3\bar{\lambda}-1\right)^2 + \left(\bar{\lambda}-2\right)^2 + \left(-2\bar{\lambda}+1\right)^2} = \sqrt{14\bar{\lambda}^2 - 14\bar{\lambda} + 6}.$$

$\left|\overrightarrow{PR}\right|$ is minimised at "$-b/2a$" (Fact 34):

$$\bar{\lambda} = -\frac{-14}{2(14)} = 0.5.$$

Hence, $R = \bar{\lambda}(3,1,-2) = (1.5, 0.5, -1)$ and $\left|\overrightarrow{PR}\right| = \sqrt{14\bar{\lambda}^2 - 14\bar{\lambda} + 6} = \sqrt{3.5 - 7 + 6} = \sqrt{2.5}$.

A610 (9740 N2016/I/5)(i) Method 1. $\mathbf{u}+\mathbf{v} = (2+a, -1, 2+b)$, $\mathbf{u}-\mathbf{v} = (2-a, -1, 2-b)$, and so,

$$(\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v}) = \begin{pmatrix} 2+a \\ -1 \\ 2+b \end{pmatrix} \times \begin{pmatrix} 2-a \\ -1 \\ 2-b \end{pmatrix} = \begin{pmatrix} b-2+2+b \\ (2+b)(2-a) - (2+a)(2-b) \\ -2-a+2-a \end{pmatrix} = \begin{pmatrix} 2b \\ 4b-4a \\ -2a \end{pmatrix}.$$

Method 2. Recall that the vector product has the following three properties (Fact 145): the vector product is **distributive** and **anti-commutative**; moreover, **the vector product of a vector with itself is the zero vector**. Hence,

$$\begin{aligned} (\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v}) \quad &= \mathbf{u} \times (\mathbf{u}-\mathbf{v}) + \mathbf{v} \times (\mathbf{u}-\mathbf{v}) && \text{(Distributivity)} \\ &= \mathbf{u}\times\mathbf{u} - \mathbf{u}\times\mathbf{v} + \mathbf{v}\times\mathbf{u} - \mathbf{v}\times\mathbf{v} && \text{(``)} \\ &= \mathbf{u}\times\mathbf{u} + \mathbf{v}\times\mathbf{u} + \mathbf{v}\times\mathbf{u} - \mathbf{v}\times\mathbf{v} && \text{(Anti-commutativity)} \\ &= \mathbf{0} + \mathbf{v}\times\mathbf{u} + \mathbf{v}\times\mathbf{u} - \mathbf{0} && \text{(Self vector product is zero)} \\ &= 2\mathbf{v}\times\mathbf{u}. \end{aligned}$$

But, $$\mathbf{v}\times\mathbf{u}=\begin{pmatrix}a\\0\\b\end{pmatrix}\times\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}b\\2b-2a\\-a\end{pmatrix}.$$

So, $$(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=\begin{pmatrix}2b\\4b-4a\\-2a\end{pmatrix}.$$

(ii) We are given that $2b=-2a \iff b=-a$.

So, $$(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=\begin{pmatrix}2b\\4b-4a\\-2a\end{pmatrix}=\begin{pmatrix}-2a\\-8a\\-2a\end{pmatrix}.$$

And, $$|(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})|=\sqrt{(-2a)^2+(-8a)^2+(-2a)^2}=\sqrt{72a^2}=\sqrt{72}\,|a|.$$

If $\sqrt{72}\,|a|=1$, then $a=\pm1/\sqrt{72}$.

(c) Recall that the scalar product is distributive and commutative. Hence,

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=\mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{u}\cancel{-\mathbf{u}\cdot\mathbf{v}}\cancel{+\mathbf{v}\cdot\mathbf{u}}-\mathbf{v}\cdot\mathbf{v}=0.$$

Or, $$\mathbf{v}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{u}\text{ or }|\mathbf{v}|^2=|\mathbf{u}|^2\text{ or }|\mathbf{v}|=|\mathbf{u}|=\sqrt{2^2+(-1)^2+2^2}=\sqrt{9}=3.$$

A611 (9740 N2016/I/11)(i)(a) Let $\mathbf{w}=(-2,1,2)$ be l's direction vector. We show that $\mathbf{w}$ is perpendicular to the two non-parallel vectors $\mathbf{u}=(1,2,0)$ and $\mathbf{v}=(a,4,-2)=(0,4,-2)$ on p:

$$\begin{array}{lclclcl}
\mathbf{w}\cdot\mathbf{u} & = & (-2,1,2)\cdot(1,2,0) & = & -2+2+0 & =0, & \checkmark\\
\mathbf{w}\cdot\mathbf{v} & = & (-2,1,2)\cdot(0,4,-2) & = & 0+4+-4 & =0. & \checkmark
\end{array}$$

Since $\mathbf{w}\perp\mathbf{u},\mathbf{v}$, we have $\mathbf{w}\perp p$ and thus also $l\perp p$.

To find the point at which l and p intersect, write

$$\begin{pmatrix}1\\-3\\2\end{pmatrix}+\hat{\lambda}\begin{pmatrix}1\\2\\0\end{pmatrix}+\hat{\mu}\begin{pmatrix}0\\4\\-2\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}+\hat{t}\begin{pmatrix}-2\\1\\2\end{pmatrix}\quad\text{or}\quad\begin{aligned}1+\hat{\lambda}&\overset{1}{=}-1-2\hat{t},\\-3+2\hat{\lambda}+4\hat{\mu}&\overset{2}{=}\hat{t},\\2-2\hat{\mu}&\overset{3}{=}1+2\hat{t}.\end{aligned}$$

From $\overset{1}{=}$, $\hat{\lambda}\overset{4}{=}-2\left(1+\hat{t}\right)$. Plug $\overset{4}{=}$ into $\overset{2}{=}$ to get

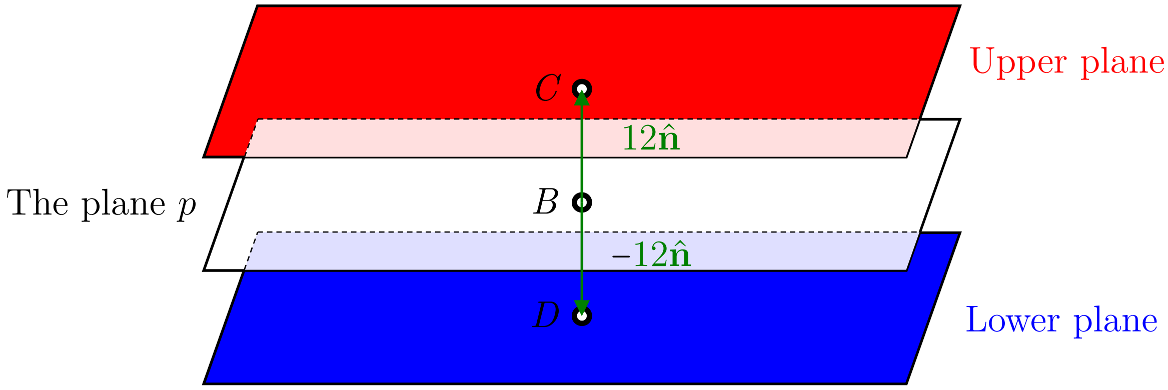
$$-3+2\left[-2\left(1+\hat{t}\right)\right]+4\hat{\mu}=\hat{t}\quad\text{or}\quad\hat{\mu}\overset{5}{=}\left(5\hat{t}+7\right)/4.$$

Plug $\overset{5}{=}$ into $\overset{3}{=}$ to get

$$2-2\left(5\hat{t}+7\right)/4=1+2\hat{t}\quad\text{or}\quad\hat{t}\stackrel{6}{=}-5/9.$$

And now plugging $\stackrel{6}{=}$ back into $\stackrel{4}{=}$ and $\stackrel{5}{=}$, we have $\hat{\lambda}=-8/9$ and $\hat{\mu}=19/18$.

(i)(b) "Obviously", the two planes must be parallel to p, with one "above" it and the other "below" it.



The plane p has normal vector $\mathbf{n}=(1,2,0)\times(0,4,-2)=(-4,2,4)$. We have $\hat{\mathbf{n}}=\mathbf{n}/|\mathbf{n}|=(-4,2,4)/\sqrt{36}=\dfrac{1}{3}(-2,1,2)$.

The plane p contains the point $B=(1,-3,2)$.

The upper plane contains the point $C=B+12\hat{\mathbf{n}}=B+4(-2,1,2)=(-7,1,10)$.

The lower plane contains the point $D=B-12\hat{\mathbf{n}}=B-4(-2,1,2)=(9,-7,-6)$.

Both planes have normal vector $(-2,1,2)$. We have

$$\overrightarrow{OC}\cdot(-2,1,2)=14+1+20=35\quad\text{and}\quad\overrightarrow{OD}\cdot(-2,1,2)=-18-7-12=-37.$$

And so, the two planes have cartesian equations:

$$-2x+y+z=35\quad\text{and}\quad-2x+y+2z=-37.$$

(ii) If a line and a plane intersect at zero or more than one points, then they are parallel (Fact 171). And so, $\mathbf{w}\perp\mathbf{n}$ or $\mathbf{w}\cdot\mathbf{n}=0$.

But $\mathbf{n}=(1,2,0)\times(a,4,-2)=(-4,2,4-2a)$.

So $\mathbf{w}\cdot\mathbf{n}=(-2,1,2)\cdot(-4,2,4-2a)=18-4a=0$ or $a=18/4=9/2$.

A612 (9740 N2015/I/7)(i) $\overrightarrow{OC}=0.6\mathbf{a}$ and $\overrightarrow{OD}=5/11\mathbf{b}$.

(ii) $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=0.6\mathbf{a}-\mathbf{b}$ and so the line BC can be written as $r=\mathbf{b}+\lambda(0.6\mathbf{a}-\mathbf{b})=0.6\lambda\mathbf{a}+(1-\lambda)\mathbf{b}$, for $\lambda\in\mathbb{R}$, as desired.

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=5/11\mathbf{b}-\mathbf{a}$ and so the line AD can be written as $r=\mathbf{a}+\mu(5/11\mathbf{b}-\mathbf{a})=(1-\mu)\mathbf{a}+5/11\mu\mathbf{b}$, for $\lambda\in\mathbb{R}$, as desired.

(iii) Where the lines meet, we have $0.6\lambda\mathbf{a}+(1-\lambda)\mathbf{b}=(1-\mu)\mathbf{a}+5/11\mu\mathbf{b}$. Equating the coefficients, we have $0.6\lambda\stackrel{1}{=}1-\mu$ and $5/11\mu\stackrel{2}{=}1-\lambda$. From $\stackrel{1}{=}$, we have $\mu=1-0.6\lambda$. Plugging this into $\stackrel{2}{=}$, we have $5/11(1-0.6\lambda)=1-\lambda\iff 1-0.6\lambda=11/5-11/5\lambda\iff 8/5\lambda=6/5\iff\lambda=3/4$. And $\mu=0.55$. Altogether then, the position vector of E is $0.45\mathbf{a}+0.25\mathbf{b}$.

$\overrightarrow{AE} = 0.55\mathbf{a} - 0.25\mathbf{b}$ and $\overrightarrow{ED} = -0.45\mathbf{a} + {}^9\!/\!_{44}\mathbf{b}$. We observe that $\overrightarrow{AE} = -{}^9\!/\!_{11}\overrightarrow{ED}$ and so the desired ratio is ${}^9\!/\!_{11}$.

A613 (9740 N2015/II/2)(i) The angle is

$$\cos^{-1}\left(\frac{(2,3,-6)\cdot(1,0,0)}{|(2,3,-6)||(1,0,0)|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{49}\sqrt{1}}\right) \approx 1.281.$$

(ii) The vector from P to a generic point on L is $(2,5,-6) - \mathbf{r} = (2,5,-6) - (1,-2,-4) - (2\lambda,3\lambda,-6\lambda) = (1-2\lambda, 7-3\lambda, -2+6\lambda)$. The length of this vector is

$$\sqrt{(1-2\lambda)^2 + (7-3\lambda)^2 + (-2+6\lambda)^2} = \sqrt{49\lambda^2 - 70\lambda + 54}.$$

$49\lambda^2 - 70\lambda + 54 = 33 \iff 49\lambda^2 - 70\lambda + 21 = 0 \iff 7\lambda^2 - 10\lambda + 3 = 0 \iff (7\lambda-3)(\lambda-1) = 0 \iff \lambda = {}^3\!/\!_7, 1$.

Hence, the two points are $(1,-2,-4)+{}^3\!/\!_7(2,3,-6) = {}^1\!/\!_7(13,-5,-46)$ and $(1,-2,-4)+(2,3,-6) = (3,1,-10)$.

$49\lambda^2 - 70\lambda + 54$ is a $\cup$-shaped quadratic with minimum point given by $98\lambda - 70 = 0$ or $\lambda = 5/7$. Hence, the closest point is $(1,-2,-4) + {}^5\!/\!_7(2,3,-6) = {}^1\!/\!_7(17,1,-58)$.

(iii) The plane is parallel to the vectors $(2,3,-6)$ and $(2,5,-6) - (1,-2,-4) = (1,7,-2)$. It thus has normal vector $(2,3,-6) \times (1,7,-2) = (36,-2,11)$. Moreover, we know that $(1,-2,-4)$ is on the plane. Hence, a cartesian equation is $36x - 2y + 11z = 36 \times 1 - 2 \times (-2) + 11 \times (-4) = -4$.

A614 (9740 N2014/I/3)(i) One possibility is that one or both are zero vectors. And if neither is a zero vector, then they point either in the same direction or in the exact opposite directions—or equivalently, either $\hat{\mathbf{a}} = \hat{\mathbf{b}}$ or $\hat{\mathbf{a}} = -\hat{\mathbf{b}}$.

(ii) $\widehat{(1,2,-2)} = \dfrac{(1,2,-2)}{3}$.

(iii) It is $\dfrac{|(1,2,-2)\cdot(0,0,1)|}{3\times 1} = \dfrac{2}{3}$.

A615 (9740 N2014/I/9)(i) The plane q is parallel to the vectors $(1,2,-3)$ and $(2,-1,4)$. It thus has normal vector $(1,2,-3) \times (2,-1,4) = (5,-10,-5)$ and hence also normal vector $(-1,2,1)$. It contains the point $(1,-1,3)$. Altogether then, it has cartesian equation $-x + 2y + z = 0$.

(ii) Line m has direction vector $(-1,2,1) \times (1,2,-3) = (-8,-2,-4)$ and hence also direction vector $(4,1,2)$.

To find a point that is on both planes, try plugging in $x = 0$. Then from the equation of q, we have $z = -2y$. Now plug this also into the equation of p to get $2y - 3(-2y) = 12$ or $y = 1.5$. Hence, an intersection point is $(0,1.5,-3)$.

Altogether then, the line m has vector equation $\mathbf{r} = (0,1.5,-3) + \lambda(4,1,2)$, for $\lambda \in \mathbb{R}$.

(iii) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (4\lambda, 1.5+\lambda, -3+2\lambda) - (1,-1,3) = (4\lambda-1, 2.5+\lambda, -6+2\lambda)$. So $\left|\overrightarrow{AB}\right|^2 = (4\lambda-1)^2 + (2.5+\lambda)^2 + (-6+2\lambda)^2 = 21\lambda^2 - 27\lambda + 43.25$. This lattermost expression is a $\cup$-shaped quadratic, with minimum point given by $42\lambda - 27 = 0$ or $\lambda = 9/14$. So $B = (4\lambda, 1.5+\lambda, -3+2\lambda) = (\frac{18}{7}, \frac{15}{7}, -\frac{12}{7})$.

A616 (9740 N2013/I/1)(i) From the equation for p, we have $z \overset{1}{=} 0.5x - 2$.

Plug $\overset{1}{=}$ into the equation for q to get $2x - 2y + 0.5x - 2 = 6 \iff y \overset{2}{=} 1.25x - 4$.

Now plug $\overset{1}{=}$ and $\overset{2}{=}$ into the equation for r to get $5x - 4(1.25x - 4) + \mu(0.5x - 2) = -9 \iff 0.5\mu x + 25 - 2\mu \overset{3}{=} 0 \iff x \overset{4}{=} \dfrac{4\mu - 50}{\mu}$.

So if $\mu = 3$, from $\overset{1}{=}$, $\overset{2}{=}$, and $\overset{4}{=}$, we have

$$x = -\frac{38}{3}, \qquad y = -\frac{119}{6}, \qquad z = -\frac{25}{3}.$$

(ii) From $\overset{1}{=}$, if $\mu = 0$, then we have 250, a contradiction. So the three planes do not intersect.

A617 (9740 N2013/I/6)(i) Every vector in the plane can be expressed as the linear combination of any two vectors with distinct directions (see Fact 125).

(ii) By the Ratio Theorem, $\overrightarrow{ON} = \dfrac{4\mathbf{a} + 3\mathbf{c}}{7}$.

(iii) The area of triangle ONC is

$$\begin{aligned}
0.5\left|\overrightarrow{ON} \times \overrightarrow{OC}\right| &= 0.5\left|\frac{4\mathbf{a}+3\mathbf{c}}{7} \times \mathbf{c}\right| \\
&= {}^1\!/_{14}\left|(4\mathbf{a}+3\mathbf{c}) \times \mathbf{c}\right| \\
&= {}^1\!/_{14}\left|4\mathbf{a} \times \mathbf{c} + 3\mathbf{c} \times \mathbf{c}\right| && \text{(distributivity of vector product)} \\
&= {}^1\!/_{14}\left|4\mathbf{a} \times \mathbf{c}\right| && (\mathbf{v} \times \mathbf{v} = \mathbf{0}) \\
&= {}^1\!/_{14}\left|4\mathbf{a} \times (\lambda\mathbf{a} + \mu\mathbf{b})\right| \\
&= {}^1\!/_{14}\left|4\mathbf{a} \times \lambda\mathbf{a} + 4\mathbf{a} \times \mu\mathbf{b}\right| && \text{(distributivity of vector product)} \\
&= {}^1\!/_{14}\left|4\mathbf{a} \times \mu\mathbf{b}\right| = \frac{2\mu}{7}.
\end{aligned}$$

Similarly, the area of triangle OMC is

$$\begin{aligned}
0.5\left|\overrightarrow{OM} \times \overrightarrow{OC}\right| &= 0.5\left|0.5\mathbf{b} \times \mathbf{c}\right| \\
&= {}^1\!/_4\left|\mathbf{b} \times \mathbf{c}\right| \\
&= {}^1\!/_4\left|\mathbf{b} \times (\lambda\mathbf{a} + \mu\mathbf{b})\right| \\
&= {}^1\!/_4\left|\mathbf{b} \times \lambda\mathbf{a} + \mathbf{b} \times \mu\mathbf{b}\right| && \text{(distributivity of vector product)} \\
&= {}^1\!/_4\left|\mathbf{b} \times \lambda\mathbf{a}\right| && (\mathbf{v} \times \mathbf{v} = \mathbf{0}) \\
&= {}^1\!/_4\lambda\left|\mathbf{b} \times \mathbf{a}\right|.
\end{aligned}$$

Altogether then, $2\mu/7 = \lambda/4$ or $\lambda = 8\mu/7$.

A618 (9740 N2013/II/4)(i) The angle θ between two planes is given by the scalar product of their normal vectors:

$$\cos\theta = \frac{(2,-2,1)\cdot(-6,3,2)}{|(2,-2,1)||(-6,3,2)|} = \frac{16}{\sqrt{9}\sqrt{49}} = \frac{16}{3\times 7} = \frac{16}{21}.$$

So $\theta = 0.705$.

(ii) The intersection line of two planes has direction vector given by the vector product of

their normal vectors: $(2,-2,1)\times(-6,3,2)=(-7,-10,-6)$.

A point that is on both planes satisfies both equations $2x-2y+z=1$ and $-6x+3y+2z=-1$. Plugging in $x=0$, the first equation yields $z=1+2y$, which when plugged into the second equation yields $y=-3/7$. So a point that is on both planes is $(0,-{}^3\!/\!_7,{}^1\!/\!_7)$.

(iii) The distance between a point a and a plane is given by $|d-\mathbf{a}\cdot\hat{\mathbf{n}}|$, where $\mathbf{n}$ is its normal vector and $d=\mathbf{r}\cdot\hat{\mathbf{n}}$.

For p_1, $d=1/3$ and for $p_2=d=-1/7$. Hence, the distance between $A(4,3,c)$ and the plane p_1 is

$$\left|\frac{1}{3}-\frac{(4,3,c)\cdot(2,-2,1)}{3}\right|=\left|\frac{1}{3}-\frac{2+c}{3}\right|=\left|-\frac{1+c}{3}\right|,$$

and the distance between $A(4,3,c)$ and the plane p_2 is

$$\left|-\frac{1}{7}-\frac{(4,3,c)\cdot(-6,3,2)}{7}\right|=\left|-\frac{1}{7}+\frac{15-2c}{7}\right|=\left|\frac{14-2c}{7}\right|.$$

Equating these two distances, we have

$$\begin{aligned} -\frac{1+c}{3}&=\frac{14-2c}{7}\iff -7-7c=42-6c\iff c=-49,\\ \text{OR}\quad \frac{1+c}{3}&=\frac{14-2c}{7}\iff \quad 7+7c=42-6c\iff c=35/13.\end{aligned}$$

A619 (9740 N2012/I/5)(i) The area of triangle OAC is

$$\begin{aligned} 0.5\left|\overrightarrow{OA}\times\overrightarrow{OC}\right| &= 0.5\,|\mathbf{a}\times(\lambda\mathbf{a}+\mu\mathbf{b})| \\ &= 0.5\,|\mathbf{a}\times\lambda\mathbf{a}+\mathbf{a}\times\mu\mathbf{b}| && \text{(distributivity of vector product)}\\ &= 0.5\,|\mathbf{a}\times\mu\mathbf{b}| && (\mathbf{v}\times\mathbf{v}=\mathbf{0})\\ &= 0.5\mu\,|(1,-1,1)\times(1,2,0)| \\ &= 0.5\mu\,|(-2,1,3)| = 0.5\sqrt{14}\mu. \end{aligned}$$

So $0.5\sqrt{14}\mu=\sqrt{126}$ or $\mu=2\times\sqrt{126/14}=2\times\sqrt{9}=2\times3=6$.

(ii) $\mathbf{c}=\lambda\mathbf{a}+\mu\mathbf{b}=\lambda\mathbf{a}+4\mathbf{b}=\lambda(1,-1,1)+4(1,2,0)=(4+\lambda,8-\lambda,\lambda)$. We are given that $|\mathbf{c}|=5\sqrt{3}$. So $(4+\lambda)^2+(8-\lambda)^2+\lambda^2=3\lambda^2-8\lambda+80=\left(5\sqrt{3}\right)^2=75\iff 3\lambda^2-8\lambda+5=0=(3\lambda-5)(\lambda-1)$, so $\lambda=5/3$ or 1. And $\mathbf{c}=(5{}^2\!/\!_3,6{}^1\!/\!_3,{}^5\!/\!_3)$ or $(5,7,1)$.

A620 (9740 N2012/I/9)(i) $\mathbf{r}=(7,8,9)+\lambda(8,16,8)$.

(ii) The position vector of N is given by $p+\left(\overrightarrow{pa}\cdot\hat{\mathbf{v}}\right)\hat{\mathbf{v}}$, where $\mathbf{v}$ is the direction vector of the line, p is a point on the line, and a is the given point $(1,8,3)$. Compute $\widehat{(8,16,8)}=\dfrac{(1,2,1)}{\sqrt{6}}$ and now,

$$(7,8,9)+\frac{(-6,0,-6)\cdot(1,2,1)}{\sqrt{6}}\frac{(1,2,1)}{\sqrt{6}}=(7,8,9)-\frac{12}{6}(1,2,1)=(5,4,7).$$

By the Ratio Theorem, the ratio $AN:NB=\alpha:1$ satisfies

$$\begin{aligned}(5,4,7) &= \frac{(7,8,9)+\alpha(-1,-8,1)}{\alpha+1}\\ &= \frac{1}{\alpha+1}(7-\alpha,8-8\alpha,9+\alpha).\end{aligned}$$

Solving $5=\dfrac{7-\alpha}{\alpha+1}$, we have $5\alpha+5=7-\alpha$ or $\alpha=1/3$. So the ratio $AN:NB=\alpha:1={}^1\!/_3:1=1:3$.

(iii) *To be written.*

A621 (9740 N2011/I/7)(i) $\mathbf{m}=0.5(\mathbf{p}+\mathbf{q})={}^1\!/_6\mathbf{a}+{}^3\!/_{10}\mathbf{b}$. The area of triangle OMP is

$$\begin{aligned}0.5|\mathbf{m}\times\mathbf{p}| &= 0.5|0.5(\mathbf{p}+\mathbf{q})\times\mathbf{p}|\\ &= 0.25|\mathbf{p}\times\mathbf{p}+\mathbf{q}\times\mathbf{p}| &&\text{(distributivity of vector product)}\\ &= 0.25|\mathbf{q}\times\mathbf{p}| &&(\mathbf{v}\times\mathbf{v}=\mathbf{0})\\ &= 0.25|{}^3\!/_5\mathbf{b}\times{}^1\!/_3\mathbf{a}|\\ &= 0.05|\mathbf{b}\times\mathbf{a}|=0.05|\mathbf{a}\times\mathbf{b}|.\end{aligned}$$

(ii) (a) Since $\mathbf{a}$ is a unit vector, $(2p)^2+(6p)^2+(3p)^2=1$ or $49p^2=1$ or $p=1/7$.

(b) $|\mathbf{a}\cdot\mathbf{b}|$ is the length of the projection vector of $\mathbf{b}$ on $\mathbf{a}$.

(c) $\mathbf{a}\times\mathbf{b}={}^1\!/_7(2,-6,3)\times(1,1,-2)={}^1\!/_7(9,7,8)$.

A622 (9740 N2011/I/11)(i) A normal vector to the plane is

$$\begin{aligned}&(4-(-2),-1-(-5),-3-2)\times(4-4,-1-(-3),-3-(-2))\\ =\ &(6,4,-5)\times(0,2,-1)=(6,6,12).\end{aligned}$$

Another normal vector to the plane is a scalar multiple of the above, namely $(1,1,2)$. We have $(4,-1,-3)\cdot(1,1,2)=-3$. Hence, a cartesian equation of p is $x+y+2z=-3$.

(ii) From the equations for l_1, we have $\dfrac{x-1}{2}=z+3 \iff x \overset{1}{=} 2(z+3)+1=2z+7$ and $\dfrac{y-2}{-4}=z+3 \iff y\overset{2}{=}-4z-10$.

Plug in $\overset{1}{=}$ and $\overset{2}{=}$ into the equations for l_2 to get

$$\frac{2z+7+2}{1}\overset{3}{=}\frac{-4z-10-1}{5}\overset{4}{=}\frac{z-3}{k}.$$

From $\overset{3}{=}$, we have $10z+45=-4z-11 \iff z=-56/14=-4$. Now from $\overset{4}{=}$, we have $k=5\dfrac{z-3}{-4z-11}=5\dfrac{-7}{5}=-7$.

(iii) The direction vector of l_1 is perpendicular to the normal vector of the plane p, as we can verify—$(2,-4,1)\cdot(1,1,2)=0$. Moreover, a point on l_1 is on p, as we can verify—$(1,2,-3)\cdot(1,1,2)=-3$. Altogether then, l_1 is on p.

From the equations for l_2, we have $y=5x+11$ and $z=-7x-11$. Plug these into the equation for the plane p to get $x+(5x+11)+2(-7x-11)=-3 \iff -8x-11=-3 \iff x=-1$. So $y=6$ and $z=-4$. The intersection point is $(-1,6,-4)$.

(iv) The angle θ between l_2 and the normal vector to p is given by

$$\theta = \cos^{-1}\left(\frac{(1,5,-7)\cdot(1,1,2)}{|(1,5,-7)|\,|(1,1,2)|}\right) = \cos^{-1}\left(\frac{-8}{\sqrt{75}\sqrt{4}}\right) = \cos^{-1}\left(\frac{-4}{5\sqrt{2}}\right) \approx 1.957.$$

So the acute angle between l_2 and p is $2.172 - \pi/2 \approx 0.387$.

A623 (9740 N2010/I/1)(i) $|\mathbf{b}|^2 = 1 + 2^2 + 2^2 = 9 = |\mathbf{a}|^2 = \left(2^2 + 3^2 + 6^2\right)p^2 = 49p^2$. So $p = 3/7$.

$$\textbf{(ii)} \quad (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \left(\frac{6}{7}+1, \frac{9}{7}+2, \frac{18}{7}+2\right)\cdot\left(\frac{6}{7}-1, \frac{9}{7}-2, \frac{18}{7}-2\right)$$
$$= -\frac{13}{49} - \frac{115}{49} + \frac{128}{49} = 0.$$

(*Optional.* Actually, more generally, since $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$, if $|\mathbf{a}| = |\mathbf{b}|$, then $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = 0$.)

A624 (9740 N2010/I/10)(i) The line has direction vector $(-3,6,9)$, which is a scalar multiple of the plane's normal vector $(1,-2,-3)$. So the line is perpendicular to the plane.

(ii) From the equations of the line, we have $y = -2x + 19$ and $z = -3x + 27$. Plug these in to the equation of the plane to get $x - 2(-2x+19) - 3(-3x+27) = 0 \iff 14x - 119 = 0 \iff x = 119/14 = 8.5$. And so $y = 2$ and $z = 1.5$. So the point of intersection is $(8.5, 2, 1.5)$.

(iii) We can easily verify that the given point satisfies the equations for the line: $\frac{-2-10}{-3} = 4 = \frac{23+1}{6} = \frac{33+3}{9}$. The point is therefore on the line.

The point of intersection we found in (ii) (call it X) is equidistant to both A and B. Moreover, these three points are collinear. Thus, $B = (19,-19,-30)$.

(iv) The area of triangle OAB is

$$0.5\,|\mathbf{a}\times\mathbf{b}| = 0.5\,|(-2,23,33)\times(19,-19,-30)|$$
$$= 0.5\,|(-63,567,-399)| \approx 348.$$

A625 (9740 N2009/I/10)(i) The angle θ between the two planes is given by

$$\theta = \cos^{-1}\left(\frac{(2,1,3)\cdot(-1,2,1)}{|(2,1,3)|\,|(-1,2,1)|}\right) = \cos^{-1}\left(\frac{3}{\sqrt{14}\sqrt{4}}\right) \approx 1.237.$$

(ii) The line l has direction vector $(2,1,3)\times(-1,2,1) = (-5,-5,5)$ and thus also direction vector $(1,1,-1)$.

A point (x,y,z) that lies on both planes satisfies $2x+y+3z \overset{1}{=} 1$ and $-x+2y+z \overset{2}{=} 2$. Plugging in $x = 0$, $\overset{1}{=}$ yields $y = 1 - 3z$ and now $\overset{2}{=}$ yields $z = 0$. So $(x,y,z) = (0,1,0)$.

Altogether then, the line l has vector equation $\mathbf{r} = (0,1,0) + \lambda(1,1,-1)$, for $\lambda \in \mathbb{R}$.

(iii) The line l is parallel to the plane p_3, as we now verify: $(1,1,-1)\cdot(2-k,1+2k,3+k) = 2-k+1+2k-3-k = 0$. Moreover, the point $(0,1,0)$, which is on the line l, is also on the plane p_3, as we now verify: $2\times0+1+3\times0-1+k(-0+2\times0+0-2) = 0$. Altogether then, the line l lies in p_3 for any constant k.

We want to find k such that $(2,3,4)$ satisfies $2x+y+3z-1+k(-x+2y+z-2)=0$. That is, $2\times2+3+3\times4-1+k(-2+2\times3+4-2)=18+6k$. So $k=-3$. So the plane is $2x+y+3z-1-3(-x+2y+z-2)=0$ or $5x-5y+5=0$ or $x-y+1=0$.

A626 (9740 N2009/II/2)(i) Let $\mathbf{p}=\overrightarrow{OP}$. By the Ratio Theorem, $\mathbf{p}=\dfrac{2(11,-13,2)+(14,14,14}{3}$ $(12,-4,6)$. So the point P is $(12,-4,6)$.

(ii) $\overrightarrow{AB}\cdot\mathbf{p}=(-3,-27,-12)\cdot(12,-4,6)=0$.

(iii) $\mathbf{c}=\widehat{(12,-4,6)}=\widehat{(6,-2,3)}=\dfrac{(6,-2,3)}{\sqrt{6^2+(-2)+3^2}}=\dfrac{(6,-2,3)}{7}$. $|\mathbf{a}\cdot\mathbf{c}|$ is the length of the projection vector of $\mathbf{a}$ on $\mathbf{p}$.

(iv) $\mathbf{a}\times\mathbf{p}=(140,84,-224)$. $|\mathbf{a}\times\mathbf{p}|$ is the area of the parallelogram formed with $\mathbf{a}$ and $\mathbf{p}$ as its sides, where the heads of $\mathbf{a}$ and $\mathbf{p}$ are the same point. The area of the triangle OAP is $|\mathbf{a}\times\mathbf{p}|=\sqrt{140^2+84^2+(-224)^2}\approx139$.

A627 (9740 N2008/I/3)(i) $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{OB}=(6,3,-3)$.

(ii) The angle AOB is equal to the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$:

$$\cos^{-1}\left(\frac{(1,4,-3)\cdot(5,-1,0)}{|(1,4,-3)||(5,-1,0)|}\right)=\cos^{-1}\left(\frac{1}{\sqrt{26}\sqrt{26}}\right)\approx1.532.$$

(iii) It is $\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|\approx25.981$.

A628 (9740 N2008/I/11). You can either find the intersection point using a graphing calculator or painfully by hand, as I do now:

From p_1, $z\overset{1}{=}1-\dfrac{2}{3}x+\dfrac{5}{3}y$. Plug $\overset{1}{=}$ into the equation for p_2 to get $3x+2y-5z=3x+2y-5\left(1-\dfrac{2}{3}x+\dfrac{5}{3}y\right)=-5$ or $\dfrac{19}{3}x-\dfrac{19}{3}y=0$ or $x\overset{2}{=}y$. And so from $\overset{1}{=}$, we have $z\overset{3}{=}1+x$. Now plug in $\overset{3}{=}$ and $\overset{2}{=}$ into the equation for p_3 to get $5x+\lambda x+17(1+x)=\mu$ or $(22+\lambda)x=\mu-17$ or $x=\dfrac{\mu-17}{22+\lambda}=-\dfrac{0.4}{1.1}=-\dfrac{4}{11}$. So the point of intersection is $\left(-\dfrac{4}{11},-\dfrac{4}{11},\dfrac{7}{11}\right)$.

(i) The line has direction vector $(2,-5,3)\times(3,2,-5)=(19,19,19)$ and thus also direction vector $(1,1,1)$. From our work above, $x=y$ at the intersection of the two planes. Plug in $x=0$ to find that the two planes intersect at $(0,0,1)$. Altogether then, the line has vector equation $\mathbf{r}=(0,0,1)+\alpha(1,1,1)$, for $\alpha\in\mathbb{R}$.

(ii) Two points on the line are $(0,0,1)$ and $(-1,-1,0)$. Plug these into the equation for plane p_3 to get $17=\mu$ and $-5-\lambda=\mu$, so that $\mu=-22$.

(iii) The line l must be parallel to the plane p_3, so that $(1,1,1)\cdot(5,\lambda,17)=0$ or $\lambda=-22$. Moreover, the point $(0,0,1)$ on the line is not on the plane, so that $\mu\neq17$.

(iv) Another vector that is parallel to the plane to be found is $(1,-1,3)-(0,0,1)=(1,-1,2)$. The plane thus has normal vector $(1,1,1)\times(1,-1,2)=(3,-1,-2)$. Compute also $d=(0,0,1)\cdot(3,-1,-2)=-2$. Altogether then, the plane has cartesian equation $3x-y-2z=-2$.

A629 (9233 N2008/I/11)(i) From the equations of the first line, we have $y = 2x - 2$ and $z = 5 - x$. Plugging these into the equations of the second line, we have

$$\frac{x-1}{-1} \overset{1}{=} \frac{2x-2+3}{-3} \overset{2}{=} \frac{5-x-4}{1}.$$

Both $\overset{1}{=}$ and $\overset{2}{=}$ imply that $x = 4$ and so indeed the two lines intersect. (If they didn't intersect, then $\overset{1}{=}$ would contradict $\overset{2}{=}$.) So the point of intersection is $(4, 6, 1)$.

(ii) The angle between the lines is given by

$$\cos^{-1}\frac{(1,2,-1)\cdot(-1,-3,1)}{|(1,2,-1)||(-1,-3,1)|} = \cos^{-1}\frac{-8}{\sqrt{4}\sqrt{11}} \approx 2.967.$$

This is obtuse. So the acute angle is $\pi - 2.967 \approx 0.175$.

A630 (9740 N2007/I/6)(i) $(1,-1,2)\cdot(2,4,1) = 0$.

(ii) By the Ratio Theorem, $\overrightarrow{OM} = {}^1\!/_3\left[2(1,-1,2) + (2,4,1)\right] = {}^1\!/_3(4,2,5)$.

(iii) The area of triangle OAC is

$$\begin{aligned} 0.5\left|\overrightarrow{OA} \times \overrightarrow{OC}\right| &= 0.5\,|(1,-1,2) \times (-4,2,2)| \\ &= 0.5\,|(-6,-10,-2)| \\ &= 0.5\sqrt{140} \approx 5.916. \end{aligned}$$

A631 (9740 N2007/I/8)(i) The line l has vector equation $\mathbf{r} = (1,2,4) + \lambda(-3,1,-3)$, for $\lambda \in \mathbb{R}$. Plugging this into the equation for the plane, we have $3(1-3\lambda)-(2+\lambda)+2(4-3\lambda) = 17 \iff 9 - 16\lambda = 17 \iff \lambda = -0.5$. So the point of intersection is $(1,2,4) - 0.5(-3,1,-3) = (2.5, 1.5, 5.5)$.

(ii) The angle between l and the normal vector to p is

$$\cos^{-1}\frac{(-3,1,-3)\cdot(3,-1,2)}{|(-3,1,-3)||(3,-1,2)|} = \cos^{-1}\frac{-16}{\sqrt{19}\sqrt{14}} \approx 2.946$$

So the angle between the line and the plane is $2.946 - \pi/2 \approx 1.376$.

(iii) $|d - \mathbf{a}\cdot\hat{\mathbf{n}}| = \dfrac{|17-(1,2,4)\cdot(3,-1,2)|}{\sqrt{14}} = \dfrac{|17-9|}{\sqrt{14}} = \dfrac{8}{\sqrt{14}} = \dfrac{4\sqrt{14}}{7} \approx 2.138.$

A632 (9233 N2007/I/7). The foot of the perpendicular a point A to a line is $Q + \left(\overrightarrow{QA}\cdot\hat{\mathbf{v}}\right)\hat{\mathbf{v}}$, where Q is any point on the line and $\mathbf{v}$ is the line's direction vector. Hence,

$$\begin{aligned} P &= (-3,8,3) + \frac{(4,-5,-5)\cdot(2,2,3)}{\sqrt{17}}\frac{(2,2,3)}{\sqrt{17}} \\ &= (-3,8,3) - \frac{17(2,2,3)}{17} = (-5,6,0). \end{aligned}$$

$\left|\overrightarrow{AP}\right| = |(-6,3,2)| = \sqrt{49} = 7.$

A633 (9233 N2007/II/2)(i) $\overrightarrow{OD} = 0.75(1,-3,4)$. So the line AD has direction vector $(3.25, 3.25, 0)$ and hence also direction vector $(1,1,0)$. So the line AD has equation $\mathbf{r} = (4,1,3) + \lambda(1,1,0)$, for $\lambda \in \mathbb{R}$.

(ii) $\overrightarrow{OC} = 0.25(4,1,3)$. So the line BC has direction vector $(0,3.25,-3.25)$ and hence also direction vector $(0,1,-1)$. So the line BC has equation $\mathbf{r} = (1,-3,4) + \mu(0,1,-1)$, for $\mu \in \mathbb{R}$.

Setting the equations of the two lines equal to each other, we have $4+\lambda = 1$, $1+\lambda = -3+\mu$, and $3 = 4-\mu$, so that $\lambda = -3$ and $\mu = 1$. And the point of intersection is $(1,-2,3)$.

A634 (9233 N2006/I/14). By the Ratio Theorem, $\overrightarrow{OP} = (1-\lambda)\overrightarrow{OA} + \lambda\overrightarrow{OB} = (1-\lambda)(1,-2,5) + \lambda(1,3,0) = (1,-2+5\lambda,5-5\lambda)$. And $\overrightarrow{OQ} = (1-\mu)\overrightarrow{OC} + \mu\overrightarrow{OD} = (1-\mu)(10,1,2) + \mu(-2,4,5) = (10-12\mu, 1+3\mu, 2+3\mu)$.

(i) PQ has direction vector $\overrightarrow{AB} \times \overrightarrow{CD} = (0,5,-5) \times (-12,3,3) = (30,60,60)$ and hence also direction vector $(1,2,2)$.

Moreover, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (9-12\mu, 3+3\mu-5\lambda, -3+3\mu+5\lambda)$, which must be a scalar multiple of $(1,2,2)$. And so $3+3\mu-5\lambda \stackrel{1}{=} 2(9-12\mu)$ and $-3+3\mu+5\lambda \stackrel{2}{=} 2(9-12\mu)$. Taking $\stackrel{2}{=}$ minus $\stackrel{1}{=}$, we have $-6+10\lambda = 0$ or $\lambda = 0.6$. Taking $\stackrel{2}{=}$ plus $\stackrel{1}{=}$, we have $6\mu = 4(9-12\mu)$ or $\mu = 2/3$. Altogether then, $\overrightarrow{PQ} = (1,2,2)$, as desired.

(ii) First observe that $\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = (10-12\mu, 1+3\mu, 2+3\mu) - (1,-2,5) = (2,3,4) - (1,-2,5) = (1,5,-1)$.

Now compute that the area of triangle ABQ is

$$\begin{aligned} 0.5\left|\overrightarrow{AB} \times \overrightarrow{AQ}\right| &= 0.5\left|(0,5,-5) \times (1,5,-1)\right| \\ &= 0.55\left|(20,-5,-5)\right| \approx 10.607. \end{aligned}$$

A635 (9758 N2019/I/1). (Note that $a \neq 0$.)[674]

Since all of the coefficients of this polynomial equation are real, by the Complex Conjugate Root Theorem, $2 - \mathrm{i}$ is also a root of this equation. Hence,

$$\begin{aligned} 0 = f(z) = az^3 + bz^2 + cz + d &= z^3 + (b/a) z^2 + (c/a) z + d/a \\ &= [z - (2+\mathrm{i})][z - (2-\mathrm{i})][z - (-3)] = \left[(z-2)^2 + 1\right](z+3) \\ &= \left(z^2 - 4x + 5\right)(z+3) = z^3 - z^2 - 7z + 15. \end{aligned}$$

Comparing coefficients, $b = -a$, $c = -7a$, and $d = 15a$.

A636 (9758 N2019/I/9)(i)(a) $w + \dfrac{1}{w} = \cos\theta + \mathrm{i}\sin\theta + \dfrac{1}{\cos\theta + \mathrm{i}\sin\theta}$

$= \cos\theta + \mathrm{i}\sin\theta + \dfrac{1}{\cos\theta + \mathrm{i}\sin\theta}\dfrac{\cos\theta - \mathrm{i}\sin\theta}{\cos\theta - \mathrm{i}\sin\theta} = \cos\theta + \mathrm{i}\sin\theta + \dfrac{\cos\theta - \mathrm{i}\sin\theta}{\cos^2\theta + \sin^2\theta}$

$= \cos\theta + \mathrm{i}\sin\theta + \cos\theta - \mathrm{i}\sin\theta = 2\cos\theta \in \mathbb{R}$.

(i)(b) $\dfrac{w-1}{w+1} = \dfrac{\cos\theta + \mathrm{i}\sin\theta - 1}{\cos\theta + \mathrm{i}\sin\theta + 1} = \dfrac{\cos\theta + \mathrm{i}\sin\theta - 1}{\cos\theta + \mathrm{i}\sin\theta + 1}\dfrac{\cos\theta - \mathrm{i}\sin\theta + 1}{\cos\theta - \mathrm{i}\sin\theta + 1}$

$= \dfrac{(\cos\theta) + (\mathrm{i}\sin\theta - 1)}{(\cos\theta + 1) + (\mathrm{i}\sin\theta)}\dfrac{(\cos\theta) - (\mathrm{i}\sin\theta - 1)}{(\cos\theta + 1) - (\mathrm{i}\sin\theta)} = \dfrac{\cos^2\theta - \left(-\sin^2\theta - 2\mathrm{i}\sin\theta + 1\right)}{\cos^2\theta + 2\cos\theta + 1 + \sin^2\theta}$

$= \dfrac{2\mathrm{i}\sin\theta}{2 + 2\cos\theta} = \dfrac{\mathrm{i}\sin\theta}{1 + \cos\theta} = \dfrac{\mathrm{i}\sin\theta}{2\cos^2\frac{\theta}{2}} = \dfrac{\mathrm{i}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \mathrm{i}\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \mathrm{i}\tan\dfrac{\theta}{2}$. So, $k = \mathrm{i}$.

Remark 219. With the following two formulae (not in the syllabus), we can solve **(i)(a)** and **(i)(b)** much more quickly:

$$\mathrm{i}\sin\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2}, \qquad \cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}.$$

(i)(a) $w + \dfrac{1}{w} = \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} = 2\cos\theta \in \mathbb{R}$.

(i)(b) $\dfrac{w-1}{w+1} = \dfrac{\mathrm{e}^{\mathrm{i}\theta} - 1}{\mathrm{e}^{\mathrm{i}\theta} + 1} = \dfrac{\mathrm{e}^{\mathrm{i}\theta/2}\left(\mathrm{e}^{\mathrm{i}\theta/2} - \mathrm{e}^{-\mathrm{i}\theta/2}\right)}{\mathrm{e}^{\mathrm{i}\theta/2}\left(\mathrm{e}^{\mathrm{i}\theta/2} + \mathrm{e}^{-\mathrm{i}\theta/2}\right)} = \dfrac{\mathrm{e}^{\mathrm{i}\theta/2} - \mathrm{e}^{-\mathrm{i}\theta/2}}{\mathrm{e}^{\mathrm{i}\theta/2} + \mathrm{e}^{-\mathrm{i}\theta/2}} = \dfrac{2\mathrm{i}\sin(\theta/2)}{2\cos(\theta/2)} = \mathrm{i}\tan\dfrac{\theta}{2}$.

(ii) Let $z = a + \mathrm{i}b$. We are given that $\sqrt{a^2 + b^2} = 1$ or $a^2 + b^2 = 1$.

$$\left|\frac{z - 3\mathrm{i}}{1 + 3\mathrm{i}z}\right| = \frac{|z - 3\mathrm{i}|}{|1 + 3\mathrm{i}z|} = \frac{\sqrt{a^2 + (b-3)^2}}{\sqrt{(1-3b)^2 + (3a)^2}} = \frac{\sqrt{a^2 + b^2 - 6b + 9}}{\sqrt{9b^2 - 6b + 1 + 9a^2}} = \frac{\sqrt{1 - 6b + 9}}{\sqrt{9 - 6b + 1}} = 1.$$

[674] If $a = 0$, then $f(z) = 0$ is a quadratic equation whose coefficients are real and hence cannot have roots $2 + \mathrm{i}$ and -3. So, $a \neq 0$.

A637 (9758 N2018/II/2)(i) Since all of the coefficients of this polynomial equation are real, by the Complex Conjugate Root Theorem, $2+3\mathrm{i}$ is also a root of this equation. Now,

$$[x-(2-3\mathrm{i})][x-(2+3\mathrm{i})]=(x-2)^2-(3\mathrm{i})^2=x^2-4x+4+9=x^2-4x+13.$$

Write $$\left(x^2-4x+13\right)\left(ax^2+bx+c\right)=4x^4-20x^3+sx^2-56x+t.$$

Or, $ax^4+(b-4a)x^3+(c-4b+13a)x^2+(-4c+13b)x+13c=4x^4-20x^3+sx^2-56x+t$

Comparing coefficients, $a=4$; $b-4a=-20$. So, $b=-4$, $-4c+13b=-56$, and $c=1$.

The other roots of the equation are $x=\left(-b\pm\sqrt{b^2-4ac}\right)/(2a)=\left(4\pm\sqrt{16-16}\right)/8=1/2$.

Hence the other three roots of the given equation are $2+3\mathrm{i}$ and $1/2$ (repeated).

Also, $s=c-4b+13a=69$ and $t=13c=13$.

(ii)(a) Let $a,b\in\mathbb{R}$ and $(a+\mathrm{i}b)^3=a^3+3\mathrm{i}a^2b-3ab^2-\mathrm{i}b^3=a^3-3ab^2+\mathrm{i}\left(3a^2b-b^3\right)=27$.

So, $a^3-3ab^2\overset{1}{=}27$, $3a^2b-b^3\overset{2}{=}0$. By $\overset{2}{=}$, $b^2\overset{3}{=}3a^2$.

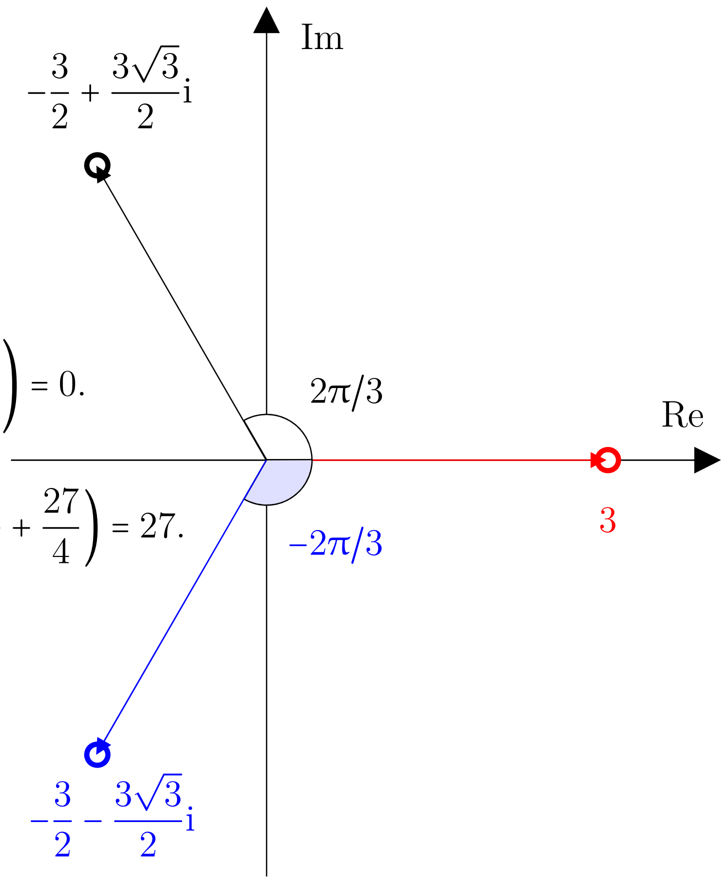
Plug $\overset{3}{=}$ into $\overset{1}{=}$ to get $a^3-9a^3=27$ or $a=(-27/8)^{1/3}=-3/2$ and hence $b=\pm3\sqrt{3}/2$.

Thus, the other two possible values of w are $-3/2\pm3\sqrt{3}\mathrm{i}/2$.

(ii)(b) $\left|-3/2\pm3\sqrt{3}\mathrm{i}/2\right|=\sqrt{9/4+27/4}=3$.

$\arg\left(-3/2\pm3\sqrt{3}\mathrm{i}/2\right)=\cos^{-1}[(\mp3/2)/3]=\cos^{-1}(\mp1/2)=\pm2\pi/3$.

Hence, $-3/2\pm3\sqrt{3}\mathrm{i}/2=3[\cos(2\pi/3)\pm\mathrm{i}\sin(2\pi/3)]$.



(ii)(c) Sum $=3+\left(-\frac{3}{2}+\frac{3\sqrt{3}}{2}\mathrm{i}\right)+\left(-\frac{3}{2}-\frac{3\sqrt{3}}{2}\mathrm{i}\right)=0$.

Product $=3\left(-\frac{3}{2}+\frac{3\sqrt{3}}{2}\mathrm{i}\right)\left(-\frac{3}{2}-\frac{3\sqrt{3}}{2}\mathrm{i}\right)=3\left(\frac{9}{4}+\frac{27}{4}\right)=27$.

A638 (9758 N2017/I/8)(a) By the quadratic formula:

$$z=\frac{\cancel{2}\pm\sqrt{(\cancel{-2})^{\cancel{2}\,1}-\cancel{4}(1-\mathrm{i})(5+5\mathrm{i})}}{\cancel{2}(1-\mathrm{i})}=\frac{1\pm\sqrt{1-(5+5\mathrm{i}-5\mathrm{i}+5)}}{1-\mathrm{i}}=\frac{1\pm\sqrt{-9}}{1-\mathrm{i}}=\frac{1\pm3\mathrm{i}}{1-\mathrm{i}}.$$

Multiply by $\dfrac{1+\mathrm{i}}{1+\mathrm{i}}$: $\quad\dfrac{1\pm3\mathrm{i}}{1-\mathrm{i}}=\dfrac{1\pm3\mathrm{i}}{1-\mathrm{i}}\dfrac{1+\mathrm{i}}{1+\mathrm{i}}=\dfrac{1+\mathrm{i}\pm3\mathrm{i}\mp3}{1^2+1^2}=-1+2\mathrm{i}$ or $2-\mathrm{i}$.

(b)(i) $\omega^2=(1-\mathrm{i})^2=1-1-2\mathrm{i}=-2\mathrm{i}$, $\omega^3=(1-\mathrm{i})(-2\mathrm{i})=-2-2\mathrm{i}$, $\omega^4=(1-\mathrm{i})(-2-2\mathrm{i})=-2-2\mathrm{i}+2\mathrm{i}-2=-4$.

We are given

$$\omega^4+p\omega^3+39\omega^2+q\omega+58=-4+p(-2-2\mathrm{i})+39(-2\mathrm{i})+q(1-\mathrm{i})+58=q-2p+54+\mathrm{i}(-q-2p-78)=0.$$

Hence, $q-2p+54\overset{1}{=}0$ and $-q-2p-78\overset{2}{=}0$. Taking $\overset{1}{=}$ plus $\overset{2}{=}$, we have $-4p-24=0$ or $p=-6$ and $q=-66$.

(c)(i) Since the coefficients of the given quartic equation are all real, by the Complex Conjugate Root Theorem (Theorem 21), $\omega^*=1+\mathrm{i}$ also solves the given equation. Thus, a quadratic factor of the given quadratic polynomial is $(\omega-1+\mathrm{i})(\omega-1-\mathrm{i})=\omega^2+1-2\omega+1=\omega^2-2\omega+2$.

Now write $$\omega^4-6\omega^3+39\omega^2-66\omega+58=\left(\omega^2-2\omega+2\right)\left(a\omega^2+b\omega+c\right)=a\omega^4+(b-2a)\omega^3+?\omega^2+?\omega+2c,$$

where ?'s are coefficients we didn't bother to compute. Comparing coefficients, we have $a=1$, $b=-4$, and $c=29$.

Thus, $$\omega^4-6\omega^3+39\omega^2-66\omega+58=\left(\omega^2-2\omega+2\right)\left(\omega^2-4\omega+29\right).$$

A639 (9740 N2016/I/7)(a) Simply plug in $-1+5\mathrm{i}$ and verify that the equation holds:

$$(-1+5\mathrm{i})^2+(-1-8\mathrm{i})(-1+5\mathrm{i})+(-17+7\mathrm{i})=1-25-10\mathrm{i}+1-5\mathrm{i}+8\mathrm{i}+40-17+7\mathrm{i}=0.\quad\checkmark$$

Note that the Complex Conjugate Roots Theorem (Theorem ???) does not apply here because the coefficients of the given quadratic equation are not all real.

Let $a+\mathrm{i}b$ be the other root. Then,

$$w^2+(-1-8\mathrm{i})w+(-17+7\mathrm{i})=(w+1-5\mathrm{i})(w-a-\mathrm{i}b)=w^2+(1-a-5\mathrm{i}-\mathrm{i}b)w+?,$$

where we didn't bother computing ? because it isn't necessary.

Comparing coefficients, we have $1-a=-1$ or $a=2$ and $-5\mathrm{i}-\mathrm{i}b=-8\mathrm{i}$ or $b=3$. Hence, the other root is $2+3\mathrm{i}$.

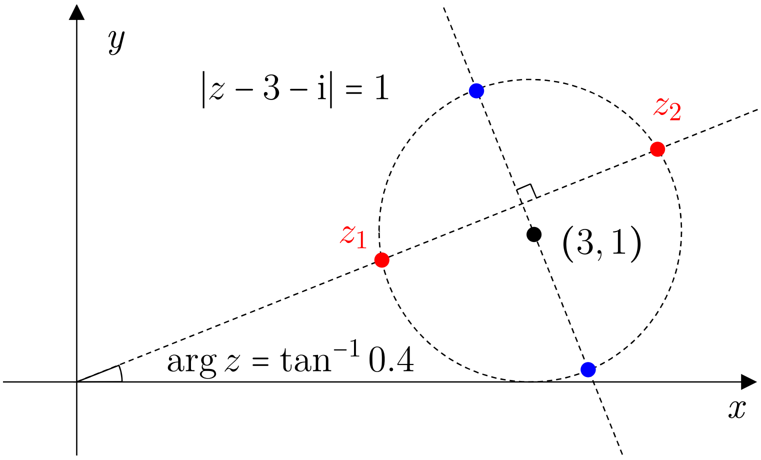
(b) Plug in $1+a\mathrm{i}$ into the given equation:

$$\begin{aligned}(1+a\mathrm{i})^3-5(1+a\mathrm{i})^2+16(1+a\mathrm{i})+k&=1+3a\mathrm{i}-3a^2-a^3\mathrm{i}-5\left(1+2a\mathrm{i}-a^2\right)+16+16a\mathrm{i}+k\\&=1-3a^2-5+5a^2+16+k+\mathrm{i}\left(3a-a^3-10a+16a\right)\\&=12+2a^2+k+\mathrm{i}\left(9a-a^3\right)=0.\end{aligned}$$

Comparing coefficients, we have $12 + 2a^2 + k \overset{1}{=} 0$ and $9a - a^3 \overset{2}{=} 0$. From $\overset{2}{=}$, we have $a = \pm 3, 0$. We are given that $a > 0$ and so $a \overset{3}{=} 3$. Plugging $\overset{3}{=}$ into $\overset{1}{=}$, we have $k = -30$.

A640 (9740 N2016/II/4)(a)(i) $|z - 3 - \mathrm{i}| \overset{1}{=} 1$ traces out a unit circle centred on $3 + \mathrm{i} = (3, 1)$.

$\arg z \overset{2}{=} \tan^{-1} 0.4$ traces out a ray from the origin, with gradient 0.4.



(a)(ii) We are asked to find the two points that are on the circle and equidistant from z_1 and z_2. These are the two blue points depicted in the figure above.

The line connecting these two blue points is perpendicular to the line with gradient 0.4—it thus has gradient $-1/0.4 = -2.5$ and direction vector $(2, -5)$, whose unit vector is $\frac{1}{\sqrt{29}}(2, -5)$. Moreover, it passes through the point $(3, 1)$. Thus, it may be described by the vector equation:

$$R = (3, 1) + \frac{\lambda}{\sqrt{29}}(2, -5) \qquad \lambda \in \mathbb{R}.$$

The circle's radius is 1. And so, the two blue points are given by

$$(3, 1) + \frac{1}{\sqrt{29}}(2, -5) = \left(3 + \frac{2}{\sqrt{29}}, 1 - \frac{5}{\sqrt{29}}\right) \qquad \text{or}$$
$$(3, 1) - \frac{1}{\sqrt{29}}(2, -5) = \left(3 - \frac{2}{\sqrt{29}}, 1 + \frac{5}{\sqrt{29}}\right).$$

(b)(i) $|w| = |2 - 2\mathrm{i}| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$ and $\arg w = \arg(2 - 2\mathrm{i}) = -\cos^{-1}\frac{2}{2\sqrt{2}} = -\cos^{-1}\frac{1}{\sqrt{2}} = -\frac{\pi}{4}$.

Thus, $w = 2\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}$ and the cube roots of w are $\left(2\sqrt{2}\right)^{1/3}\mathrm{e}^{-\mathrm{i}\pi/12} = \sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/12}$, $\sqrt{2}\mathrm{e}^{(-\pi/12+2\pi/3)\mathrm{i}} = \sqrt{2}\mathrm{e}^{7\mathrm{i}\pi/12}$, and $\sqrt{2}\mathrm{e}^{(-\pi/12-2\pi/3)\mathrm{i}} = \sqrt{2}\mathrm{e}^{-3\mathrm{i}\pi/4}$.

(b)(ii) $\arg(w^* w^n) = \arg w^* + n \arg w + 2k\pi = \frac{\pi}{4} - \frac{\pi n}{4} + 2k\pi = \frac{\pi}{2}\left(\frac{1-n}{2} + 4k\right)$, where $k = -1, 0, 1$.

So, $\frac{1-n}{2} + 4k = 1$ and $n > 0$. So we must have $k = 1$ and thus $\frac{1-n}{2} = -3$ or $n = 7$.

A641 (9740 N2015/I/9)(a)

$$\frac{w^2}{w^*} = \frac{(a+ib)^2}{a-ib} = \frac{a^2-b^2+2abi}{a-ib} = \frac{a^2-b^2+2abi}{a-ib} \times \frac{a+ib}{a+ib}$$

$$= \frac{a^3-ab^2+2a^2bi+a^2ib-ib^3-2ab^2}{a^2+b^2}$$

$$= \frac{1}{a^2+b^2}\left[\left(a^3-3ab^2\right)+i\left(3a^2b-b^3\right)\right]$$

is purely imaginary if and only if $a^3-3ab^2=0$. But $a^3-3ab^2 = a(a^2-3b^2) = a\left(a-\sqrt{3}b\right)\left(a+\sqrt{3}b\right)$. So either $b = \pm a/\sqrt{3}$ or $a = 0$ (but the latter is explicitly ruled out in the question).

Altogether, the possible values of $w = a+ib$ are given by $b = \pm a/\sqrt{3}$ and a is any non-zero real number.

(b)(i) $z^5 = 2^5 e^{i\pi(-0.5)} = 2^5 e^{i\pi(-0.5+2k)}$ for $k \in \mathbb{Z}$. So $z = 2e^{i\pi(-0.5+2k)/5}$ for $k = 0, \pm1, \pm2$. So $|z| = 2$ and $\arg z = -0.9\pi, -0.5\pi, -0.1\pi, 0.3\pi, 0.7\pi$.

(ii)
$$\begin{aligned} z_1 - z_2 &= 2e^{i\pi(0.7)} - 2e^{i\pi(-0.9)} \\ &= 2e^{i\pi(-0.1)}\left(e^{i\pi(0.8)} + e^{i\pi(-0.8)}\right) \\ &= 2e^{i\pi(-0.1)} 2i \sin 0.8\pi \\ &= (4\sin 0.8\pi)\, ie^{i\pi(-0.1)}. \end{aligned}$$

$$\begin{aligned} \text{So}\quad \arg(z_1 - z_2) &= \arg\left[(4\sin 0.8\pi)\, ie^{i\pi(-0.1)}\right] \\ &= \arg(4\sin 0.8\pi) + \arg i + \arg\left(e^{i\pi(-0.1)}\right) + 2k\pi \\ &= 0 + \pi/2 - 0.1\pi + 2k\pi = 0.4\pi \quad (k=0), \end{aligned}$$

$$\begin{aligned} \text{and}\qquad |z_1 - z_2| &= \left|(4\sin 0.8\pi)\, ie^{i\pi(-0.1)}\right| \\ &= |4\sin 0.8\pi| \underbrace{|i|}_{1} \underbrace{\left|e^{i\pi(-0.1)}\right|}_{1} = 4\sin 0.8\pi = 4\sin 0.2\pi, \end{aligned}$$

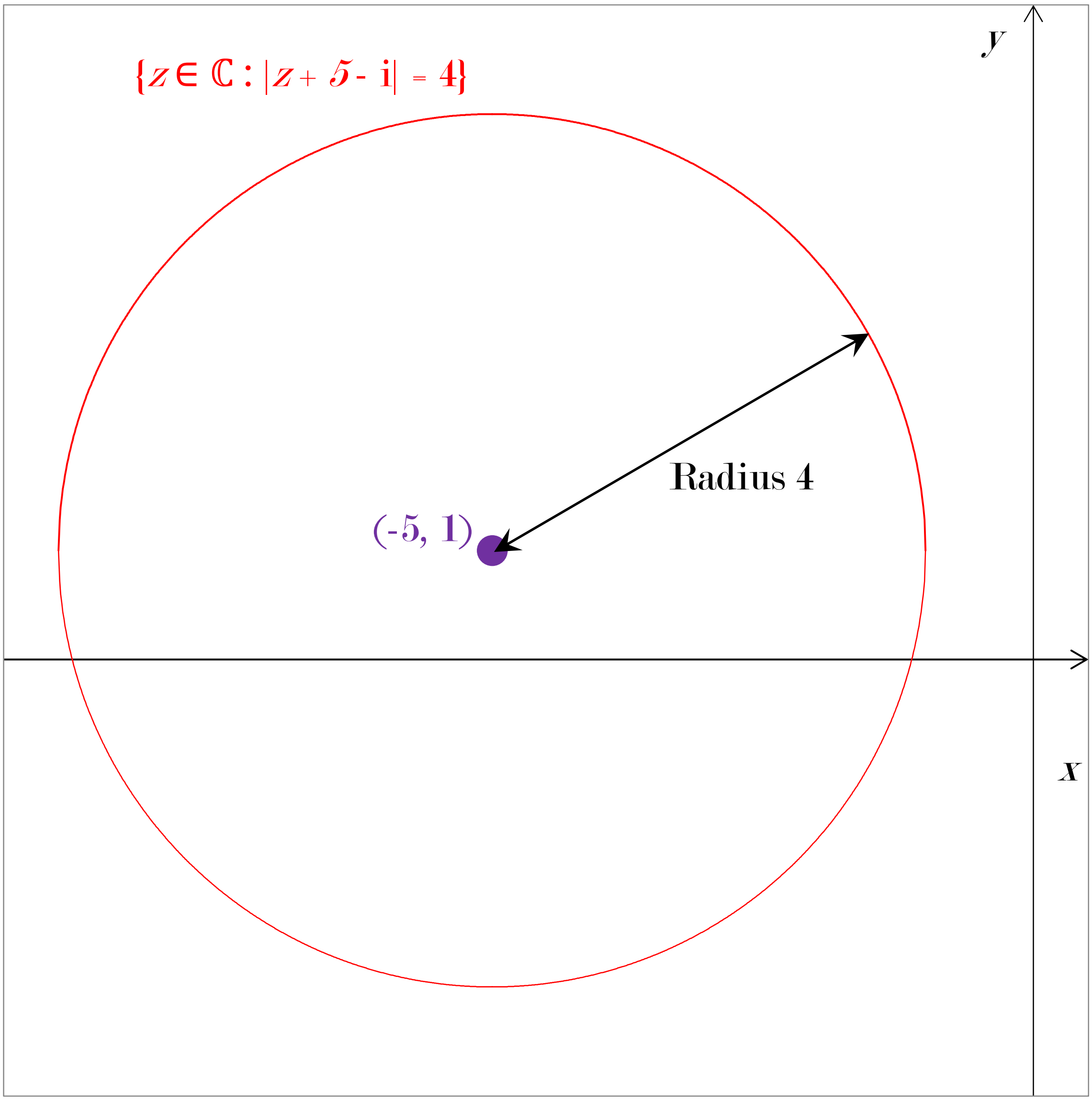
where the last line uses the fact that $\sin(\pi - x) = \sin x$.

A642 (9740 N2014/I/5)(i) $z^2 = (1+2i)^2 = 1^2 + (2i)^2 + 2(1)(2i) = 1 - 4 + 4i = -3 + 4i$. $z^3 = (-3+4i)(1+2i) = -3 - 6i + 4i + (4i)(2i) = -3 - 2i - 8 = -11 - 2i$. So,

$$\frac{1}{z^3} = \frac{1}{-11-2i} = \frac{1}{-11-2i} \times \frac{-11+2i}{-11+2i} = \frac{-11+2i}{11^2-(2i)^2} = \frac{-11+2i}{121+4} = \frac{-11+2i}{125}.$$

(ii) Since $pz^2 + \dfrac{q}{z^3} = p(-3+4i) + q\dfrac{-11+2i}{125} = \left(-3p - \dfrac{11}{125}q\right) + i\left(4p + \dfrac{2}{125}q\right)$ is real, we have $4p + 2\dfrac{q}{125} = 0$ or $q = -250p$. And $pz^2 + \dfrac{q}{z^3} = 19p$.

A643 (9740 N2014/II/4)(a)(i) This is simply a circle with radius 4 centred on the point $-5+i$. It has cartesian equation $(x+5)^2+(y-1)^2 \overset{1}{=} 4^2$.



(ii) The complex equation $|z-6i| = |z+10+4i|$ is equivalent to the cartesian equation $(x-0)^2+(y-6)^2 = (x+10)^2+(y+4)^2$ or $-12y+36 = 20x+100+8y+16$ or $y \overset{2}{=} -x-4$.

So to find the intersection points of the line and the circle, plug $\overset{2}{=}$ into $\overset{1}{=}$ to get $(x+5)^2+(-x-4-1)^2 = 4^2$ or $2(x+5)^2 = 4^2$ or $(x+5)^2 = 8$ or $x+5 = \pm\sqrt{8}$ or $x = -5\pm\sqrt{8}$. So the possible values of z are $-5\pm\sqrt{8}+\left(5\mp\sqrt{8}-4\right)i = -5\pm\sqrt{8}+\left(1\mp\sqrt{8}\right)i$.

(b)(i) $w=\sqrt{3}-i$, so $|w| = \sqrt{\left(\sqrt{3}\right)^2+(-1)} = 2$ and $\arg w = \tan^{-1}\dfrac{-1}{\sqrt{3}} = -\dfrac{\pi}{6}$. So $w = 2e^{i(-\pi/6)}$. And so $w^6 = 2^6e^{i(-\pi+2k\pi)} = 2^6e^{i\pi}$.

(ii) $\arg\left(\dfrac{w^n}{w^*}\right) = \arg w^n - \arg w^* + 2k\pi = n\arg w + \arg w + 2k\pi = (n+1)\arg w + 2k\pi = (n+1)\times(-\pi/6)+2k\pi$. A complex number z is real if and only if $\arg z = 0$ or $\arg z = \pi$. So by observation, the three smallest positive whole number values of n for which $\dfrac{w^n}{w^*}$ is real are 5, 11, and 17.

A644 (9740 N2013/I/4)(i) $(1+2i)^3 = 1+3\times 2i+3\times(2i)^2+(2i)^3 = 1+6i-12-8i = -11-2i$.

(ii) Since $w = 1+2i$ is a root for $az^3+5z^2+17z+b=0$, we have

$$\begin{aligned}0 &= a\left(1+2i\right)^3+5\left(1+2i\right)^2+17\left(1+2i\right)+b\\&= a(-11-2i)+5(-3+4i)+17+34i+b\\&= (-11a-15+17+b)+i(-2a+20+34)\\&= \underbrace{(2-11a+b)}_{0}+i\underbrace{(54-2a)}_{0}.\end{aligned}$$

$54-2a=0 \implies a=27$. And $2-11a+b=0 \implies b=11(27)-2=295$.

(iii) By the complex conjugate roots theorem, $1-2i$ is also a root for the equation. Write

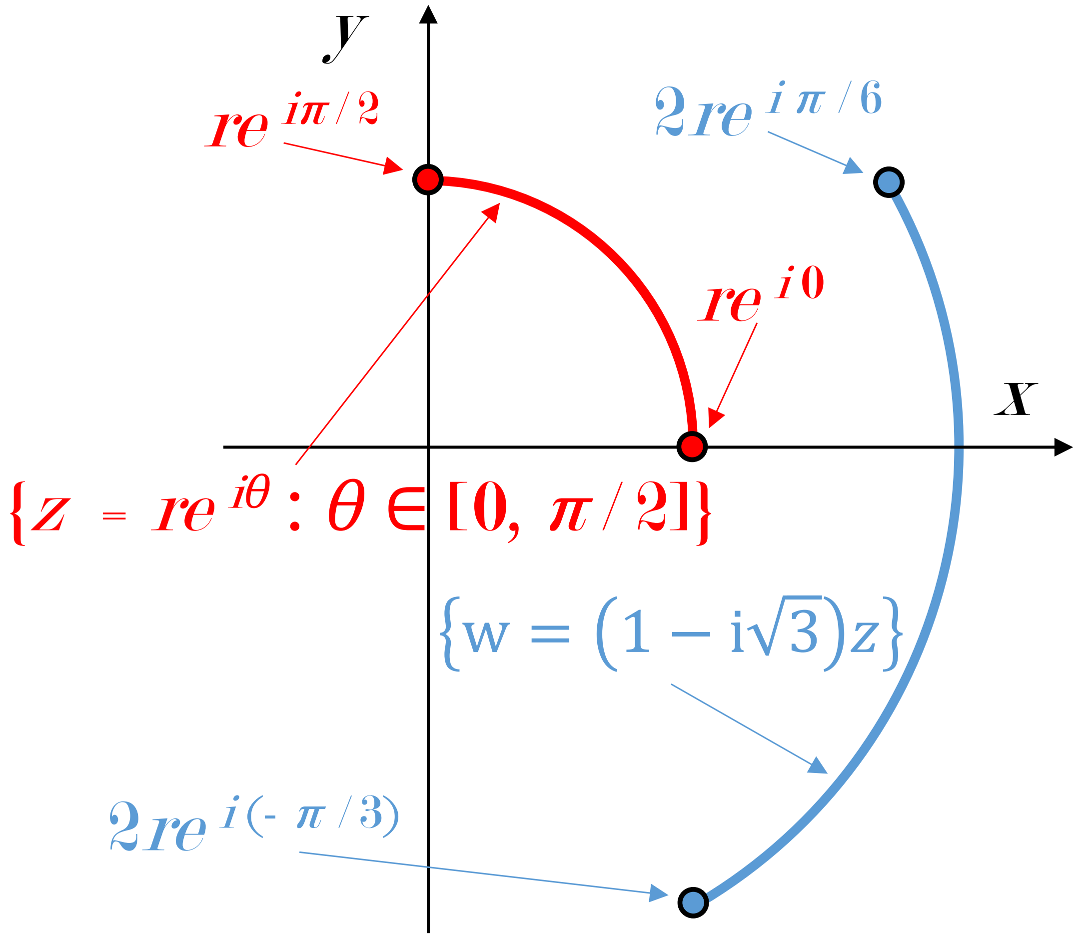
$$\begin{aligned}27z^3+5z^2+17z+295 &= 27\left[z-(1+2i)\right]\left[z-(1-2i)\right](z-k)\\&= 27\left[(z-1)^2-(2i)^2\right](z-k)\\&= 27\left(z^2-2z+5\right)(z-k)\\&= 27\left[z^3-(k+2)z^2-(2k+5)z-5k\right].\end{aligned}$$

So $295=27\times(-5k)$ or $k=-59/27$. The roots are $1\pm 2i$ and $-59/27$.

A645 (9740 N2013/I/8)(i) $|w| = \left|\left(1-i\sqrt{3}\right)z\right| = \left|1-i\sqrt{3}\right||z| = \sqrt{1^2+\left(\sqrt{3}\right)^2}\,|z| = 2|z|$. $\arg w = \arg\left[\left(1-i\sqrt{3}\right)z\right] = \arg\left(1-i\sqrt{3}\right)+\arg z+2k\pi = \tan^{-1}\left(-\sqrt{3}\right)+\theta+2k\pi = -\pi/3+\theta+2k\pi \in [-\pi/3+2k\pi, \pi/6+2k\pi]$. So we should choose $k=0$ and $\arg w = -\pi/3+\theta$.

(ii) z is the top-right quarter of the circumference of the circle of radius r, centred on the origin.

Take the position vector of z, rotate it clockwise by $\pi/3$ radians about the origin, double its length—this is the position vector of w.



(iii) $\arg\left(\dfrac{z^{10}}{w^2}\right) = \arg z^{10} - \arg w^2 + 2k\pi = 10\arg z - 2\arg w + 2k\pi = 10\theta - 2\left(-\dfrac{\pi}{3} + \theta\right) + 2k\pi = 8\theta + 2\dfrac{\pi}{3} + 2k\pi = \pi$, so $\theta = \dfrac{\pi}{24}$ (with $k = 0$).

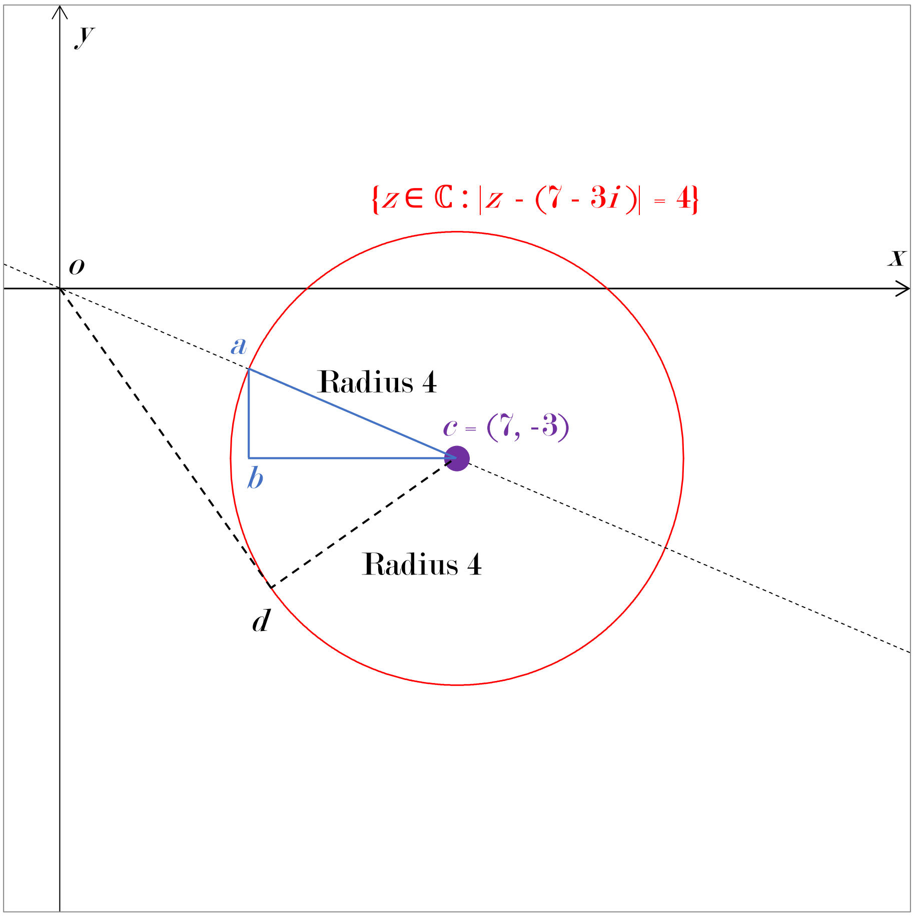
A646 (9740 N2012/I/6)(i) $z^3 = (1 + ic)^3 = 1^3 + 3(ic) + 3(ic)^2 + (ic)^3 = 1 + 3ic - 3c^2 - ic^3 = (1 - 3c^2) + i(3c - c^3)$.

(ii) z^3 is real if and only if $3c - c^3 = 0$ or $c = 0, \pm\sqrt{3}$. The question already ruled out $c = 0$. So $c = \pm\sqrt{3}$ and $z = 1 \pm i\sqrt{3}$.

(iii) $z = 1 - i\sqrt{3} = |z|\,\mathrm{e}^{i\arg z} = 2\mathrm{e}^{i(-\pi/3)}$. $|z^n| = 2^n > 1000$ if and only if $n > 9$. (The reason is that $2^9 = 512$ and $2^{10} = 1024$.) So the smallest positive integer n is 10.

$\left|z^{10}\right| = 2^{10}$ and $\arg z^{10} = 10(-\pi/3) + 2k\pi = 2\pi/3$ $(k = 2)$.

A647 (9740 N2012/II/2)(i) $|z - (7 - 3i)| = 4$ describes a circle with centre $7 - 3i$ and radius 4.



(ii)(a) a is the point on the circle's circumference that is closest to the origin a. The line l through the origin and the centre of the circle passes through a (see Fact **??**).

But the distance of the centre of the circle from the origin is $\sqrt{7^2+3^2} = \sqrt{58}$. The distance of the centre of the circle to the point a is 4 (this is simply the length of the radius). Hence, the distance of the origin to the point a is $\sqrt{58} - 4$.

(b) $\triangle abc$ is right. So $ab^2 + bc^2 = ca^2 = 4^2 = 16$.

But the line l has gradient $-\dfrac{3}{7}$ (because it runs through the origin and the point $7-3i$) and so $ab = \dfrac{3}{7}bc$. Hence, $\left(\dfrac{3}{7}\right)^2 \times bc^2 + bc^2 = 16$. Or $bc^2 = 16 \times \dfrac{49}{58}$. Or $bc = 4 \times \dfrac{7}{\sqrt{58}} = \dfrac{28}{\sqrt{58}}$. And $ab = \dfrac{12}{\sqrt{58}}$. Hence, $a = \left(7 - \dfrac{28}{\sqrt{58}}, -3 + \dfrac{12}{\sqrt{58}}\right)$.

(iii) By observation, d is the point where $|\arg z|$ is as large as possible. $\arg z = \arg(7-3i) + \angle cod$.

But $\triangle cod$ is right. So $\angle cod = \sin^{-1} \dfrac{4}{\sqrt{58}}$. Moreover, $\arg(7-3i) = \tan^{-1} \dfrac{-3}{7}$.

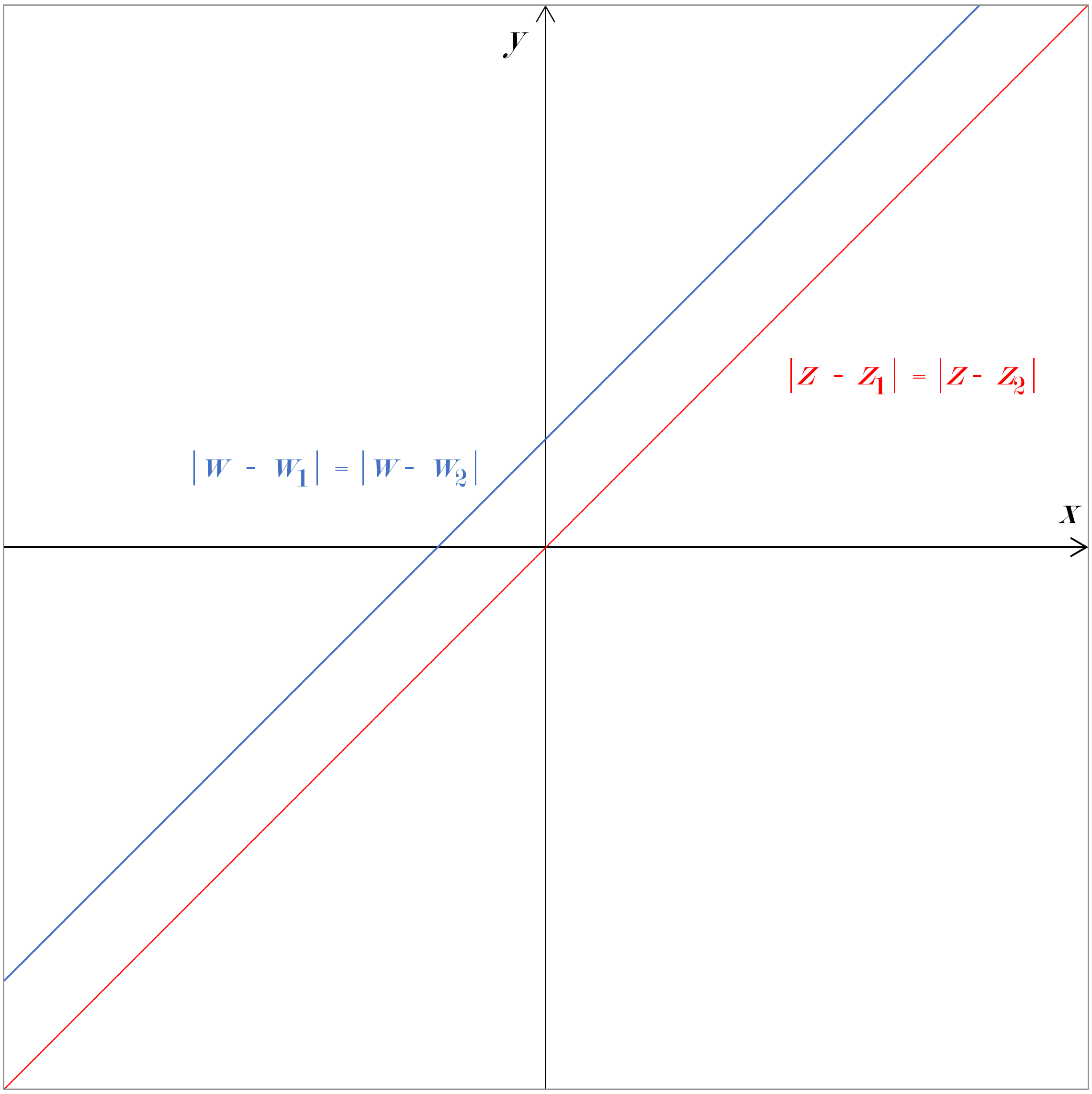
Altogether then, $\arg z = \tan^{-1} \dfrac{-3}{7} + \sin^{-1} \dfrac{4}{\sqrt{58}} = -0.9579$.

A648 (9740 N2011/I/10)(i) Let $(x+iy)^2 = x^2 - y^2 + i(2xy) = -8i$. So $x^2 - y^2 \stackrel{1}{=} 0$ and $2xy \stackrel{2}{=} -8$. From $\stackrel{2}{=}$, we observe that x and y must have opposite signs. From $\stackrel{1}{=}$, $x = \pm y$ and by our observation of the previous sentence, we must have $x = -y$. And now from $\stackrel{2}{=}$, we have $2(-y) \times y = -8$ or $-2y^2 = -8$ or $y = \pm 2$. Altogether then, $z_1 = -2 + 2i$ and $z_2 = 2 - 2i$.

(ii) Using the quadratic formula and part (i),

$$w=\frac{-4\pm\sqrt{4^2-4(1)(4+2i)}}{2}=-2\pm\sqrt{4-(4+2i)}=-2\pm\sqrt{-2i}=-2\pm(1+i)=-3-i,-1+i.$$

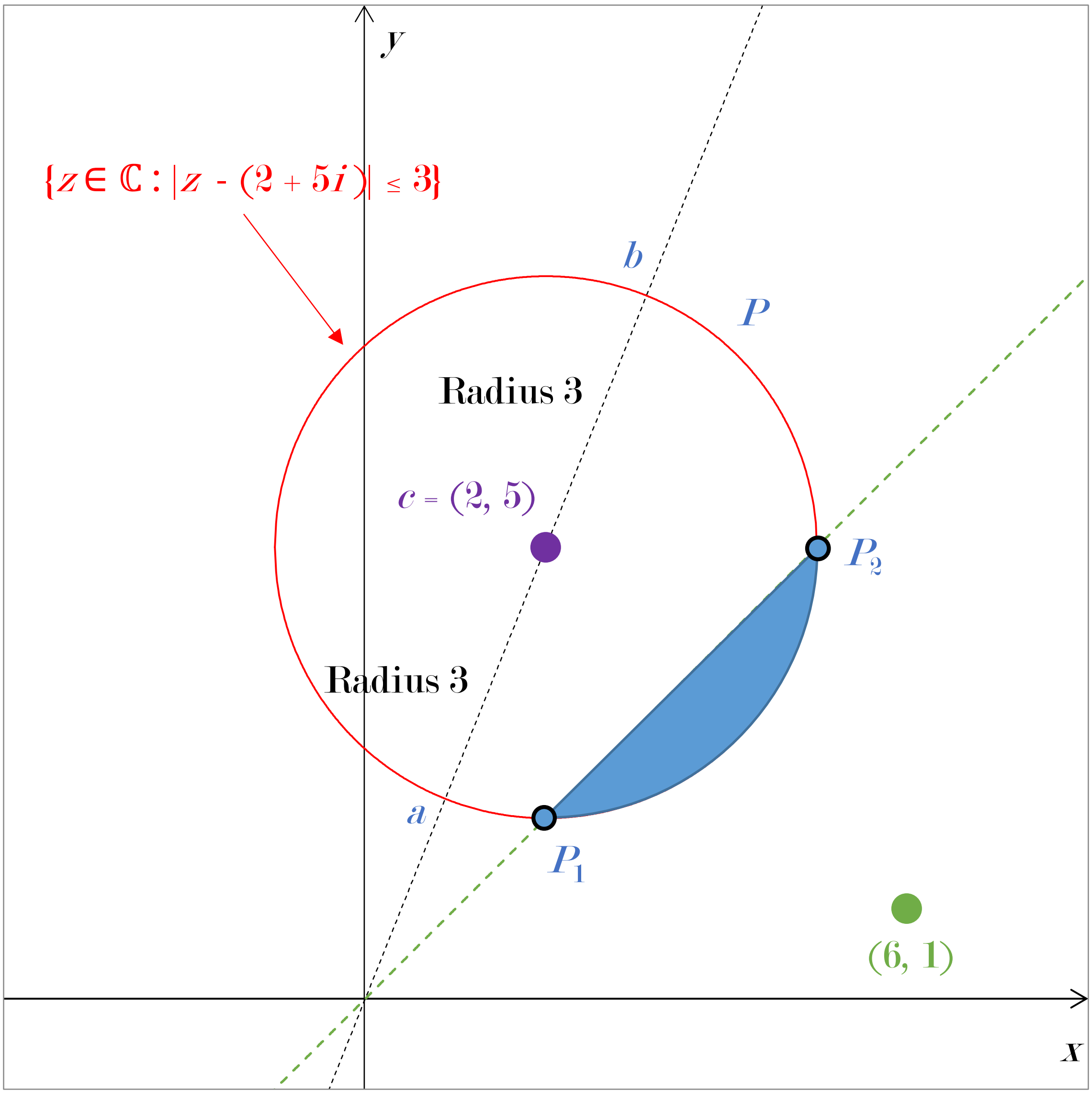
(iii)(a) This is simply the line that is equidistant to $z_1=(-2,2)$ and $z_2=(2,-2)$. By observation, it has cartesian equation $y=x$.



(b) This simply the line that is equidistant to $w_1=(-3,-1)$ and $w_2=(-1,1)$. By observation, it has cartesian equation $y=x+2$.

(iv) The two lines are parallel and do not intersect.

A649 (9740 N2011/II/1)(i) This is simply the circle with radius 3 and centre $2+5i$, including all the points within the circle.



(ii) The points on the circle's circumference that are closest to and furthest from the origin o are a and b. The line l through the origin and the centre of the circle passes through both a and b (see Fact **??**).

$oc = \sqrt{2^2+5^2} = \sqrt{29}$ and $ac = 3$. Hence, $oc = \sqrt{29}-3$. Symmetrically, $ob = \sqrt{29}+3$. The maximum and minimum possible values of $|z|$ are thus $\sqrt{29}\pm 3$.

(iii) The locus of points that satisfy both $|z-2-5i| \le 3$ and $0 \le \arg z \le \pi/4$ is the blue closed segment.

By observation, $|z-6-i|$ is maximised either at P_1 or P_2. These points are given by

$$(p-2)^2 + (p-5)^2 = 3 \iff 2p^2 - 14p + 20 = 0$$
$$\iff p^2 - 7p + 10 = 0 \iff (p-5)(p-2) = 0.$$

So $P_1 = (2,2)$ and $P_2 = (5,5)$. The distances of these points to the point $(6,1)$ are $\sqrt{4^2+1^2} = \sqrt{17}$ and $\sqrt{1^2+4^2} = \sqrt{17}$. So both are, equally, the furthest point from $(6,1)$.

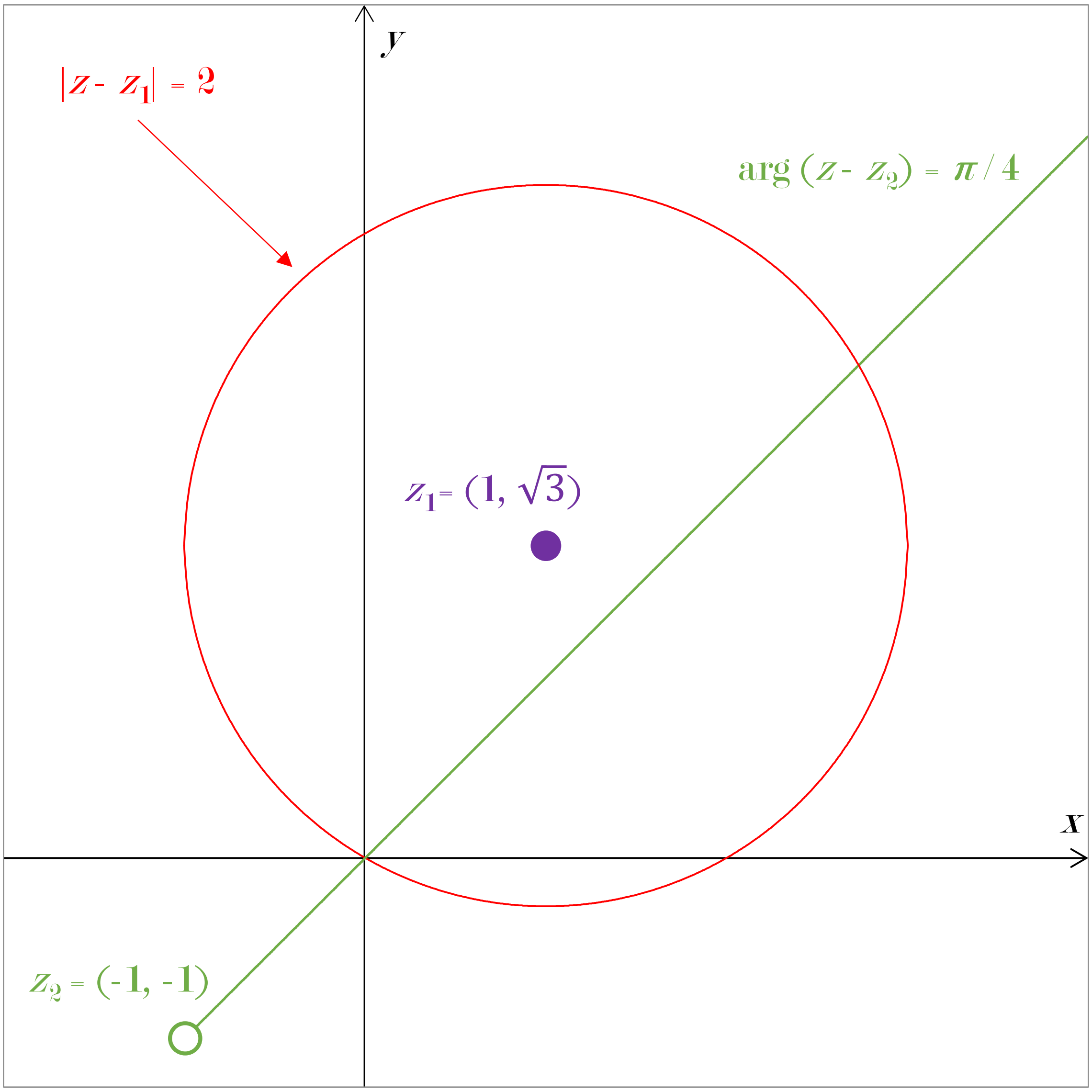
A650 (9740 N2010/I/8)(i) For z_1, $r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ and $\theta = \tan^{-1}\left(\sqrt{3}/1\right) = \pi/3$. For z_2, $r = \sqrt{(-1)+(-1)^2} = \sqrt{2}$ and $\theta = \tan^{-1}\dfrac{-1}{-1} = \dfrac{-3\pi}{4}$.

Altogether then, $z_1 = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$ and $z_2 = \sqrt{2}\left[\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right]$.

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{2}{\sqrt{2}} = \sqrt{2}$. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 = \pi/3 - \frac{-3\pi}{4} = \frac{13\pi}{12} = \frac{-11\pi}{12}$.

Hence, $\left(\frac{z_1}{z_2}\right)^* = \sqrt{2}\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right]$.

(iii)(a) This is simply the circle with centre z_1 and radius 2.



(b) This is simply the ray from the point z_2 (but excluding the point z_2) that makes an angle $\pi/4$ with the horizontal.

(iv) We want to find $x > 0$ such that $\left|(x,0) - \left(1,\sqrt{3}\right)\right| = 2$ or $(x-1)^2 + 3 = 4$ or $(x-1)^2 = 1$ or $x = 0, 2$. So $(2,0)$ is where the locus $|z - z_1| = 2$ meets the positive real axis.

A651 (9740 N2010/II/1)(i)

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(34)}}{2} = 3 \pm \frac{\sqrt{-100}}{2} = 3 \pm 5i.$$

(ii) Since $-2 + i$ is a root of $x^4 + 4x^3 + x^2 + ax + b = 0$, we have

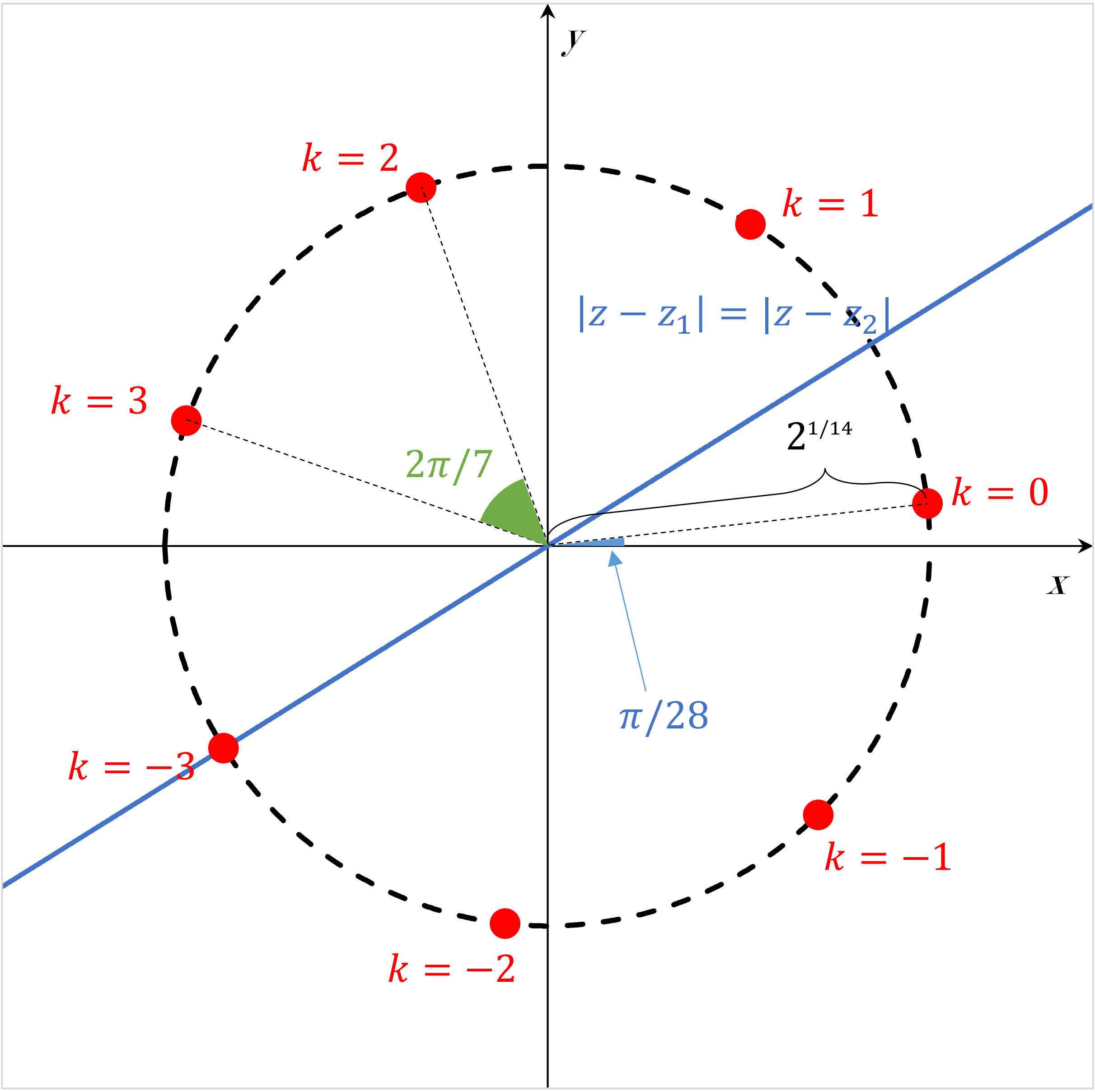
$$(-2+i)^4+4(-2+i)^3+(-2+i)^2+a(-2+i)+b=0$$
$$\vdots \quad \text{(tedious algebra)}$$
$$\underbrace{-12-2a+b}_{=0}+\underbrace{(16+a)}_{=0}i=0.$$

$16+a=0 \implies a=-16$. Moreover, $-12-2a+b=0 \implies -12-2(-16)+b=0 \implies b=-20$. By the complex conjugate roots theorem, $-2-i$ is also a root. So

$$\begin{aligned}
x^4+4x^3+x^2-16x-20 &= [x-(-2+i)][x-(-2-i)]\left(x^2+cx+d\right) \\
&= (x+2-i)(x+2+i)\left(x^2+cx+d\right) \\
&= \left[(x+2)^2-i^2\right]\left(x^2+cx+d\right) \\
&= \left(x^2+4x+5\right)\left(x^2+cx+d\right) \\
&= x^4+(4+c)x^3+(5c+4d)x+5d.
\end{aligned}$$

Comparing coefficients, we have $c=0$ and $d=-4$. So $x^2+cx+d=x^2-4=(x-2)(x+2)$. So the other two roots are ± 2.

A652 (9740 N2009/I/9)(i) $z^7=1+i=2^{1/2}e^{i\pi/4}=2^{1/2}e^{i\pi(1/4+2k)}$. By de Moivre's Theorem, $z=2^{1/14}e^{i\pi(1/28+2k/7)}$, for $k=0,\pm1,\pm2,\pm3$.



(ii)

(iii) $|z-z_1|=|z-z_2|$ is the line (blue) that is equidistant to the points $z_1=2^{1/14}e^{i\pi(1/28)}$ and $z_2=2^{1/14}e^{i\pi(1/28+2/7)}$

Explanation #1: 0 satisfies the equation $|z-z_1|=|z-z_2|$ as we can easily verify—$|0-z_1|=|0-z_2|=2^{1/14}$. So 0 is in the locus $|z-z_1|=|z-z_2|$.

Explanation #2: The perpendicular bisector of a chord runs through the centre of the circle. So in this case, the perpendicular bisector of the chord z_1z_2 runs through the origin (which is the centre of the circle).

A653 (9740 N2008/I/8)(i)

$$\left(1+\sqrt{3}i\right)^2\left(1+\sqrt{3}i\right)=\left(-2+2\sqrt{3}i\right)\left(1+\sqrt{3}i\right)=-2-6+\left(2\sqrt{3}-2\sqrt{3}\right)i=-8.$$

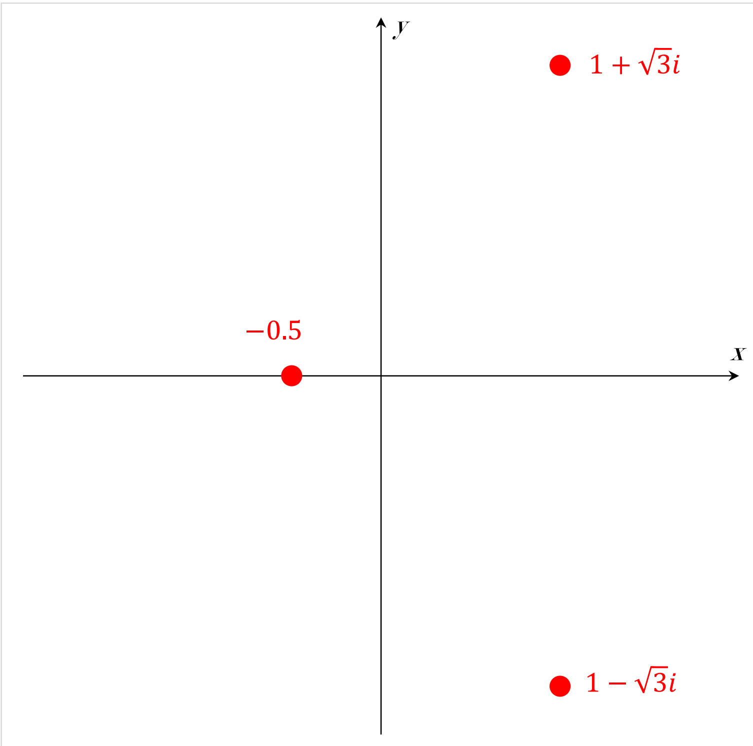
(ii)
$$\begin{aligned}0&=2z^3+az^2+bz+4\\&=2(-8)+a\left(-2+2\sqrt{3}i\right)+b\left(1+\sqrt{3}i\right)+4\\&=\underbrace{-12-2a+b}_{\overset{1}{=}0}+i\sqrt{3}\underbrace{(2a+b)}_{\overset{2}{=}0}\end{aligned}$$

Adding $\stackrel{1}{=}$ and $\stackrel{2}{=}$ together, we have $-12+2b=0$ or $b=6$. And now from $\stackrel{2}{=}$, $a=-3$.

(iii) By the complex conjugate roots theorem, another root is $1-\sqrt{3}i$. So

$$\begin{aligned}2z^3-3z^2+6z+4 &= 2\left[z-\left(1+\sqrt{3}i\right)\right]\left[z-\left(1-\sqrt{3}i\right)\right](z-c)\\ &= 2\left(z-1-\sqrt{3}i\right)\left(z-1+\sqrt{3}i\right)(z-c)\\ &= 2\left[(z-1)^2-\left(\sqrt{3}i\right)^2\right](z-c)\\ &= 2\left(z^2-2z+4\right)(z-c)\\ &= 2\left[z^3+(-c-2)z^2+(4+2c)z-4c\right].\end{aligned}$$

Comparing coefficients, we have $c=-0.5$, which is also the third root for the equation.



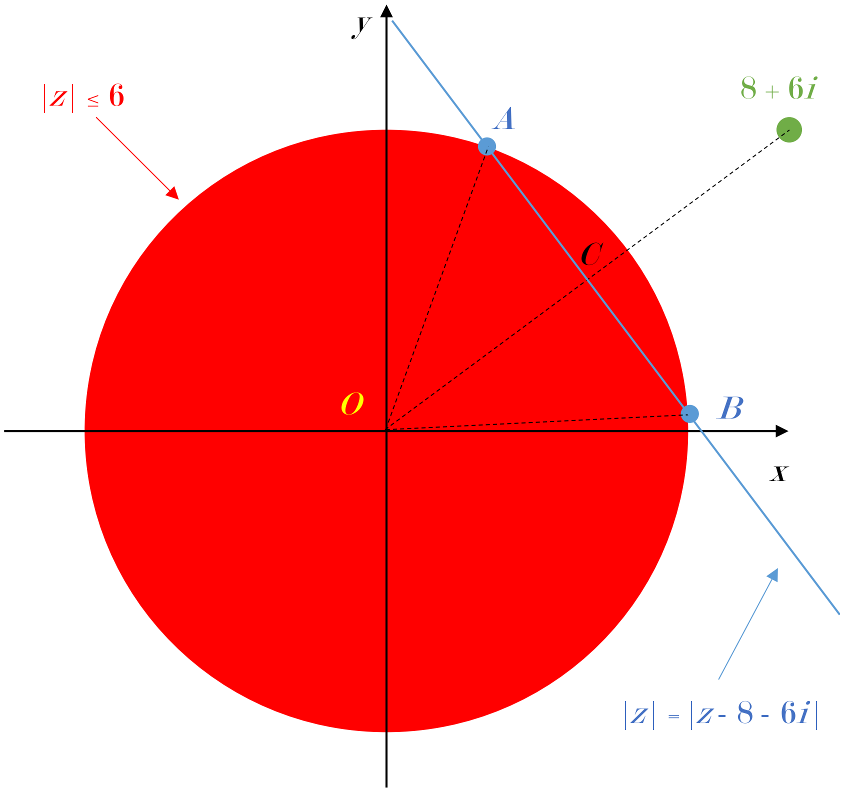
A654 (9740 N2008/II/3)(a) $|p| = \left|\frac{w}{w^*}\right| = \frac{|w|}{|w^*|} = 1$ and $\arg p = \arg \frac{w}{w^*} = \arg w - \arg w^* = \theta - (-\theta) = 2\theta$.

$\arg p^5 = 10\theta + 2k\pi$. The argument of a positive real number is $2m\pi$ for some integer m. Hence, $\theta = n\pi/5$ for integers n. Given also the restriction that $\theta \in (0, \pi/2)$, we have $\theta = \pi/5$ or $2\pi/5$.

(b) $|z| \le 6$ is a circle of radius 6 centred on the origin, including the interior of the circle.

$|z| = |z-8-6i|$ is a line that is equidistant to the origin and the point $(8,6)$.

So the locus of z is the line segment AB.



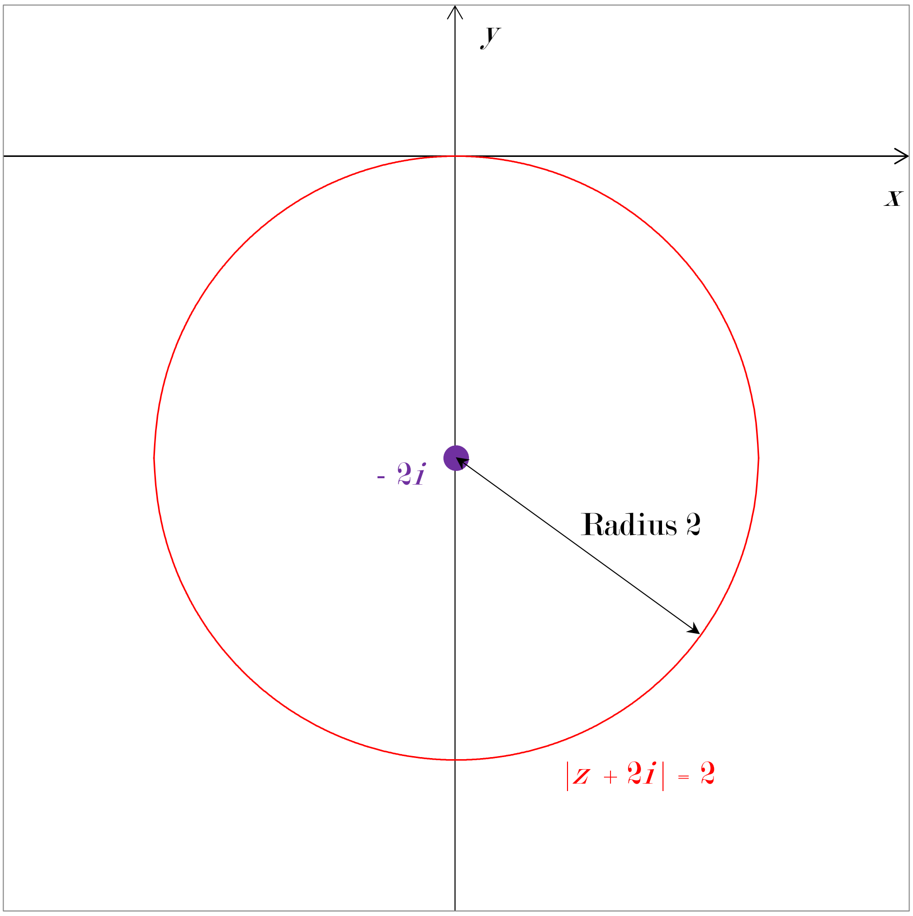
(i)

(ii) Observe that $\arg z$ is maximised and minimised at A and B. $\arg A = \angle COX + \angle AOC$, $\arg B = \angle COX - \angle BOC$. Moreover, $\angle COX = \arg(8+6i) = \tan^{-1}\frac{6}{8} = \tan^{-1}\frac{3}{4}$.

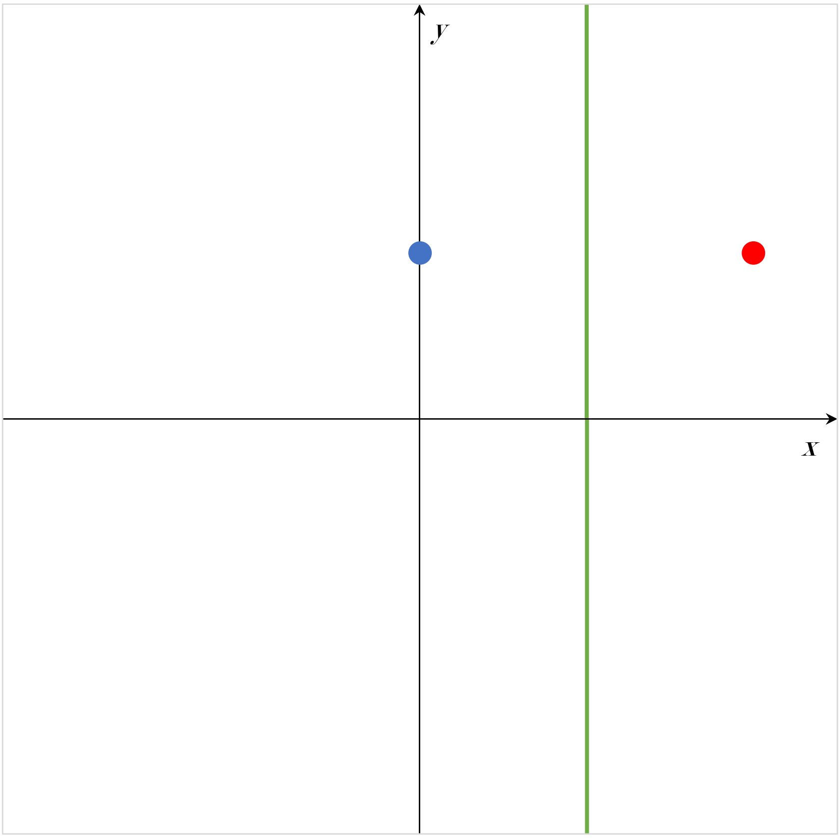
Note that $\triangle AOC$ is right and the length of OC is half of $|8+6i| = \sqrt{8^2+6^2} = 10$. So $OC = 10$. Thus, $\angle AOC = \angle BOC = \cos^{-1}\frac{OC}{OA} = \cos^{-1}\frac{OC}{OB} = \cos^{-1}\frac{5}{4}$.

Altogether then, $\arg A = \angle COX + \angle AOC = \tan^{-1}\frac{3}{4} + \cos^{-1}\frac{5}{4} \approx 1.229$ and $\arg B = \angle COX - \angle BOC = \tan^{-1}\frac{3}{4} - \cos^{-1}\frac{5}{4} \approx 0.058$.

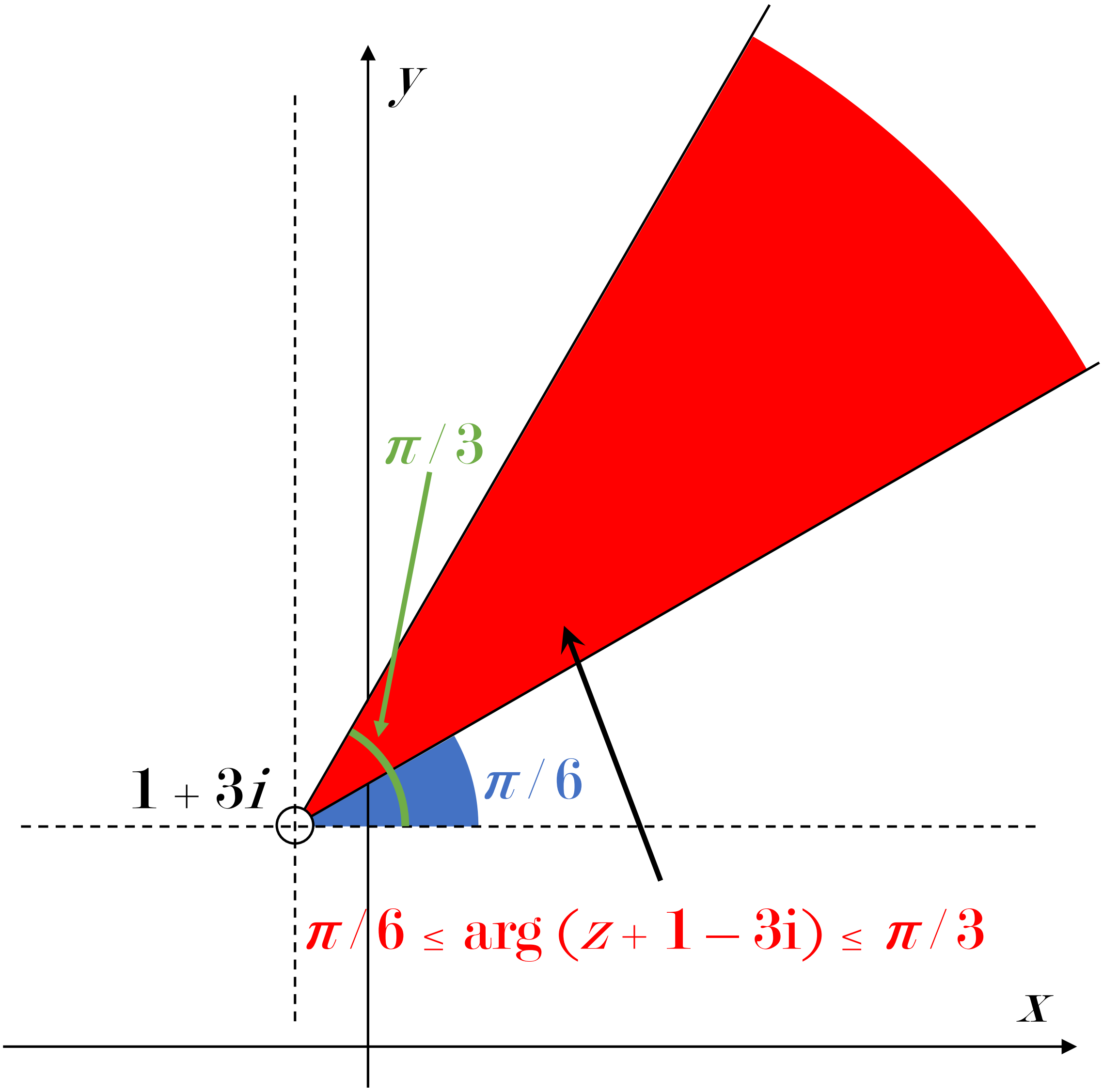
A655 (9233 N2008/I/9)(i) This is the circle centred on $-2i$ with radius 2.



(ii) This is the line that is equidistant to the points $2+i$ and i.



A655 (9233 N2008/I/9)(iii) This is the region bounded by and including the rays $\arg(z+1-3i)=\pi/6$ and $\arg(z+1-3i)=\pi/3$.



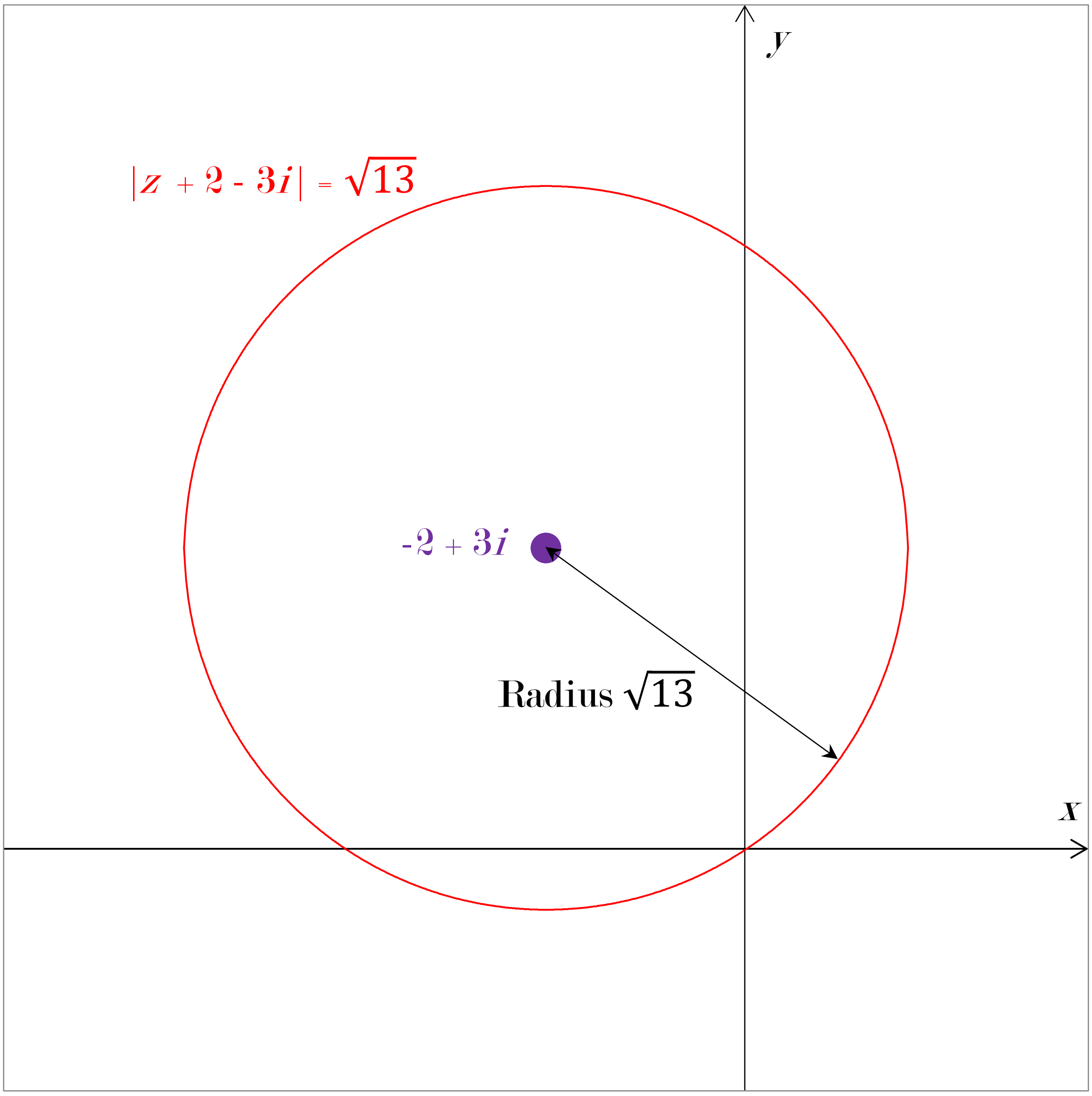
A656 (9233 N2008/II/3)(i) $(1-i)^2 = 1-1-2i = -2i$. ✓

The other root is $-1+i$, because $(-1+i)^2 = 1-1-2i = -2i$.

Remark 220. Do not make the mistake of concluding that by the complex conjugate roots theorem, $1+i$ is the other root of the equation $w^2 = -2i$. The theorem applies only for polynomials whose coefficients are all real. It does not apply here because there is an imaginary coefficient.

(ii) $$z = \frac{3+5i \pm \sqrt{(3+5i)^2 - 4(1)(-4)(1-2i)}}{2} = \frac{3+5i \pm \sqrt{9-25+30i+16(1-2i)}}{2}$$
$$= \frac{3+5i \pm \sqrt{-2i}}{2} = \frac{3+5i \pm (1-i)}{2} = 2+2i, 1+3i.$$

A657 (9740 N2007/I/3)(a) This is the circle with radius $\sqrt{13}$ centred on the point $-2+3i$.



(b) $(a+ib)(a-ib)+2(a+ib)=3+4i$ or $a^2+b^2+2a+2bi=3+4i$. Two complex numbers are equal if and only if their real and imaginary parts are equal. So $a^2+b^2+2a \stackrel{1}{=} 3$ and $2b \stackrel{2}{=} 4$. From $\stackrel{2}{=}$, $b=2$. Plug this into $\stackrel{1}{=}$ to find that $a^2+2a+1=0$ or $a=-1$. So $w=-1+2i$.

A658 **(9740 N2007/I/7)(i)** By the complex conjugate roots theorem, another root is $r\mathrm{e}^{-\mathrm{i}\theta}$. And so a quadratic factor of $P(z)$ is

$$\begin{aligned}\left(z+r\mathrm{e}^{i\theta}\right)\left(z-r\mathrm{e}^{-i\theta}\right) &= z^2+rz\mathrm{e}^{i\theta}-rz\mathrm{e}^{-i\theta}-\left(r\mathrm{e}^{i\theta}\right)\left(r\mathrm{e}^{-i\theta}\right)\\ &= z^2+rz\left(\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}\right)-r^2=z^2+rz\cos\theta-r^2.\end{aligned}$$

(ii) $z^6=-64=64\mathrm{e}^{\mathrm{i}\pi}=2^6\mathrm{e}^{\mathrm{i}\pi(1+2k)}$ for $k\in\mathbb{Z}$. So $z=2\mathrm{e}^{\mathrm{i}\pi(1+2k)/6}$ for $k=0,\pm1,\pm2,-3$.

(iii) We first use **(ii)**, then use **(i)**:

$$z^6 + 64$$

$$= \overbrace{\left(z - 2e^{i\pi/6}\right)\left(z - 2e^{-i\pi/6}\right)}^{z^2-2(2)\cos(\pi/6)+2^2} \underbrace{\left(z - 2e^{3i\pi/6}\right)\left(z - 2e^{-3i\pi/6}\right)}_{z^2-2(2)\cos(3\pi/6)+2^2} \overbrace{\left(z - 2e^{5i\pi/6}\right)\left(z - 2e^{-5i\pi/6}\right)}^{z^2-2(2)\cos(5\pi/6)+2^2}$$

$$= \left(z^2 - 2\sqrt{3} + 4\right)\left(z^2 + 4\right)\left(z^2 + 2\sqrt{3} + 4\right).$$

A659 (9233 N2007/I/9)(i) By the complex conjugate roots theorem, another root is $-ki$. Altogether then,

$$\begin{aligned} az^4 + bz^3 + cz^2 + dz + e &= a(z - ki)(z + ki)(z^2 + fz + g) \\ &= a\left(z^2 + k^2\right)(z^2 + fz + g) \\ &= a\left[z^4 + fz^3 + (k^2 + g)z^2 + k^2 fz + gk^2\right]. \end{aligned}$$

By comparing coefficients, we have $b \overset{1}{=} af$, $c = a(k^2 + g)$, $d \overset{2}{=} ak^2 f$, and $e = agk^2$. Now verify that indeed:

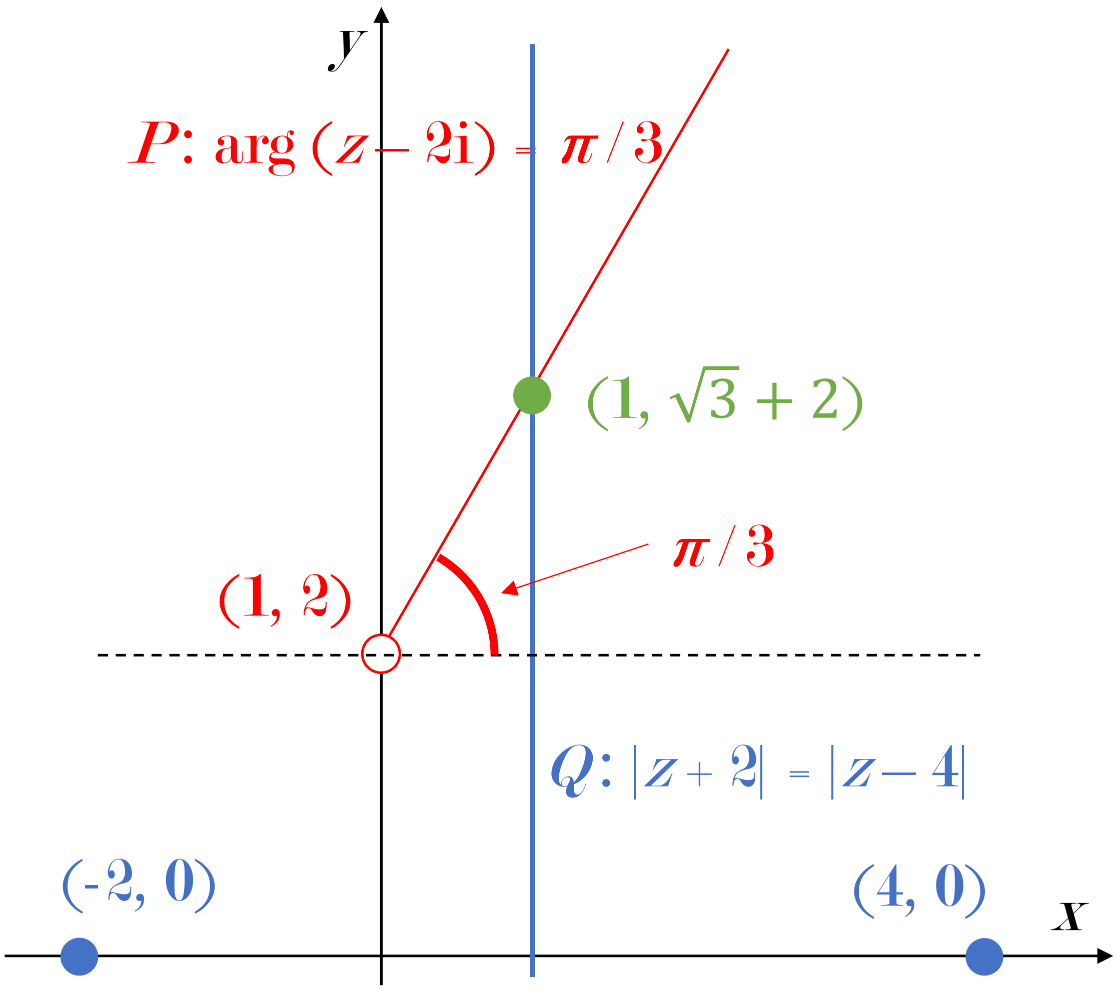
$$\begin{aligned} ad^2 + b^2 e &= a^3 k^4 f^2 + a^3 f^2 gk^2 \\ &= (af) \times \left[a(k^2 + g)\right] \times \left(ak^2 f\right) \\ &= b \times c \times d. \quad ✓ \end{aligned}$$

(ii) $a = 1, b = 3, c = 13, d = 27, e = 36$. So indeed $ad^2 + b^2 e = 1 \times 27^2 + 3^2 \times 36 = 1053 = 3 \times 13 \times 27 = bcd$. ✓

From $\overset{1}{=}$ above, $f = \dfrac{b}{a} = 3$. So from $\overset{2}{=}$, $k = \pm\sqrt{\dfrac{d}{af}} = \pm\sqrt{\dfrac{27}{1 \times 3}} = \pm\sqrt{9} = \pm 3$. So the two desired roots are $\pm 3i$.

A660 (9233 N2007/II/5). The locus of P is the ray from (but excluding) the point $2i$ that makes an angle $\pi/3$ with the horizontal. This is one half of the line with cartesian equation $y = x \tan \dfrac{\pi}{3} + 2 = \sqrt{3}x + 2$.

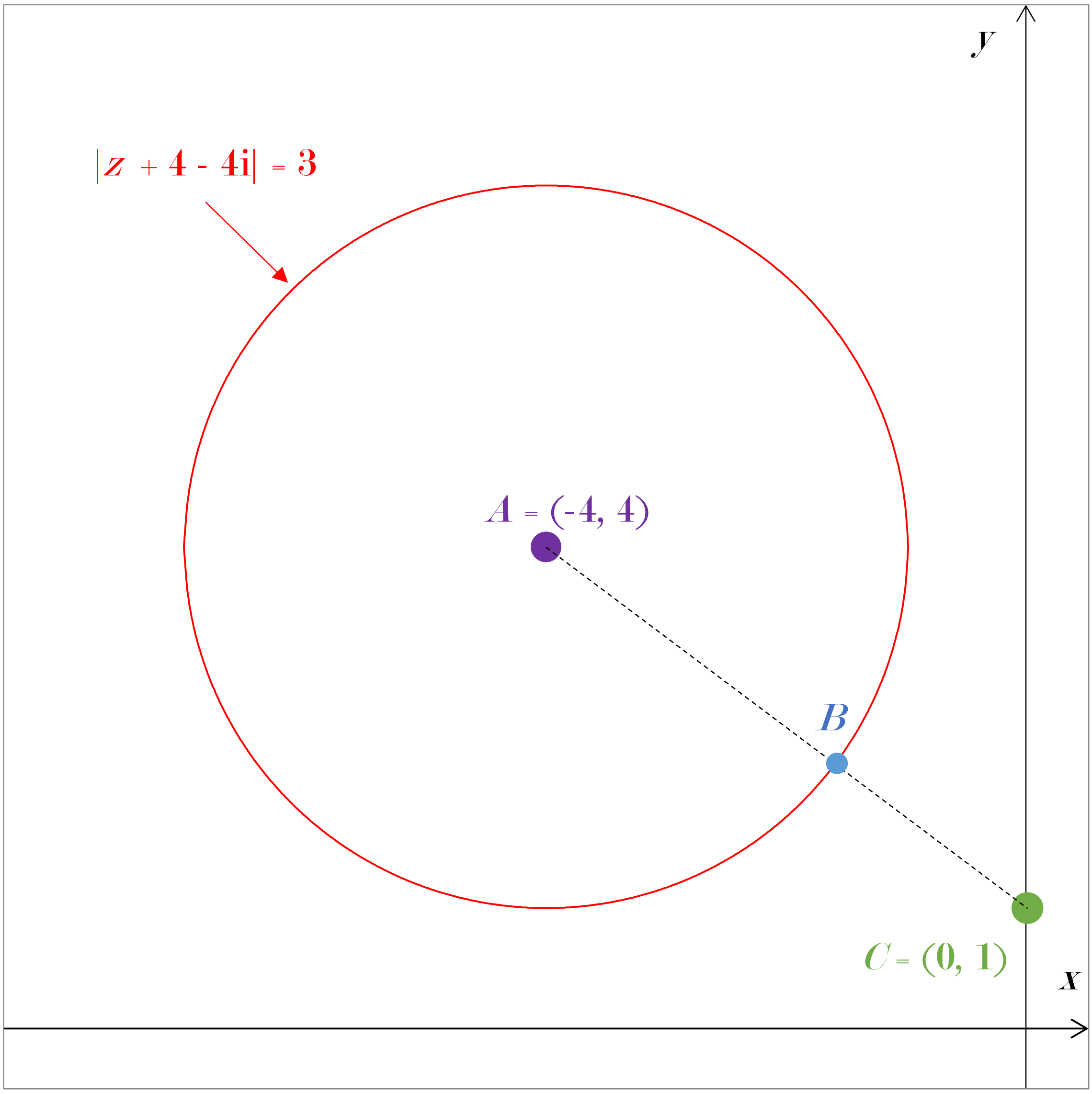
The locus of Q is the line that is equidistant to the points 4 and -2. It has cartesian equation $x = 1$.



The intersection of the two lines is $\left(1, \sqrt{3}+2\right)$ or $1+i\left(\sqrt{3}+2\right)$.

$\left[1+i\left(\sqrt{3}+2\right)\right]\left[1-i\left(\sqrt{3}+2\right)\right] = 1+\left(\sqrt{3}+2\right)^2 = 1+3+4+4\sqrt{3} = 8+4\sqrt{3}$. ✓

A661 (9233 N2006/I/5)(i) This is the circle with radius 3 centred on $-4+4i$.



(ii) Given a point (C here), the line connecting it to the centre of a circle (A here) also passes through the point on the circumference (B here) that is closest to the given point (see Fact **??**).

The distance between A and C is $\sqrt{(-4-0)^2+(4-1)^2} = \sqrt{4^2+3^2} = 5$. So the distance between B and C is $5-3=2$.

A662 (9233 N2006/I/6)(i)

$$\begin{aligned}0 &= (ki)^4 - 2(ki)^3 + 6(ki)^2 - 8(ki) + 8\\ &= k^4 + 2k^3 i - 6k^2 - 8ki + 8\\ &= \underbrace{k^4 - 6k^2 + 8}_{=0} + \underbrace{2k(k^2-4)}_{0} i.\end{aligned}$$

From $2k(k^2-4)=0$, we have $k=0,\pm2$. Only $k=\pm2$ also satisfies $k^4-6k^2+8=0$. So the equation has roots $\pm 2i$.

$$\textbf{(ii)}\quad \begin{aligned}z^4 - 2z^3 + 6z^2 - 8z + 8 &= (z-2i)(z+2i)(z^2+az+b)\\ &= \left(z^2+4\right)(z^2+az+b)\\ &= z^4 + az^3 + (4+b)z^2 + 4az + 4b.\end{aligned}$$

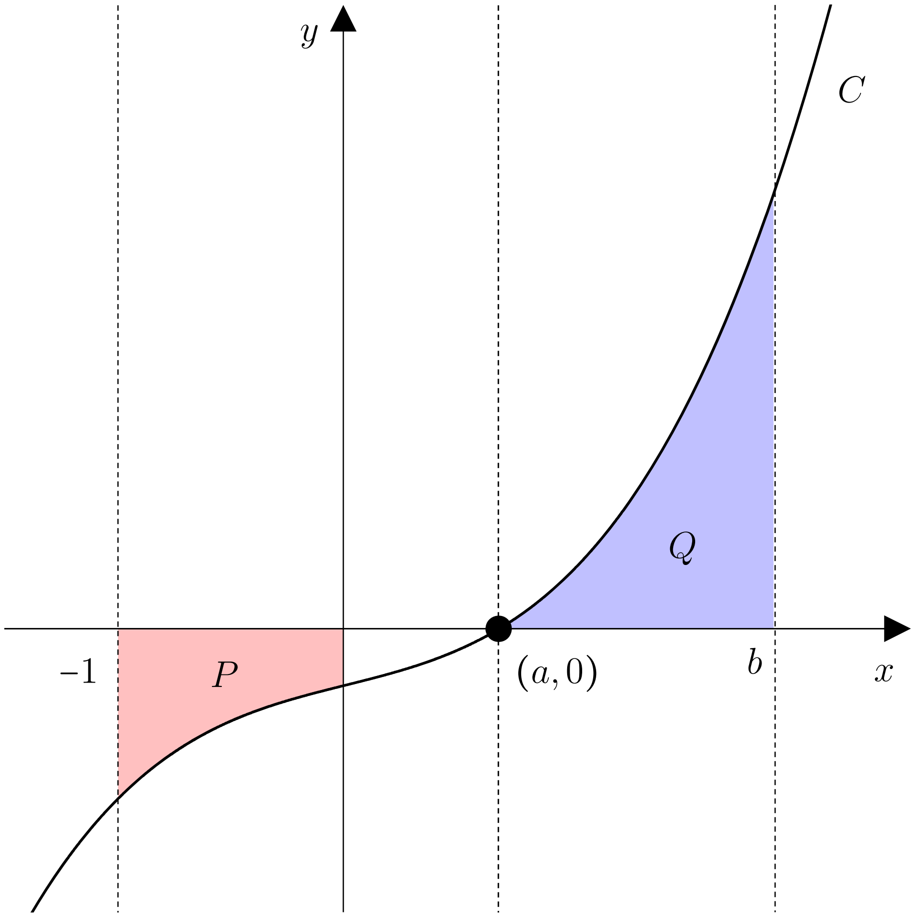
Comparing coefficients, $a = -2$ and $b = 2$. So $z^2 + az + b = z^2 - 2z + 2$, whose zeros are

$$z = \frac{2 \pm \sqrt{(-2) - 4(1)(2)}}{2} = 1 \pm \sqrt{1-2} = 1 \pm i.$$

Altogether then, the equation has roots $\pm 2i, 1 \pm i$.

A663 (9758 N2019/I/2)(i) Solve $a^3 + a - 1 = 0$ using your calculator: $a \approx 0.682$.

(ii) As usual, a quick sketch is helpful:



$$P = \int_{-1}^{0} y \, dx = \int_{-1}^{0} x^3 + x - 1 \, dx = \left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_{-1}^{0} = 0 - \left(\frac{1}{4} + \frac{1}{2} + 1\right) = -\frac{7}{4}.$$

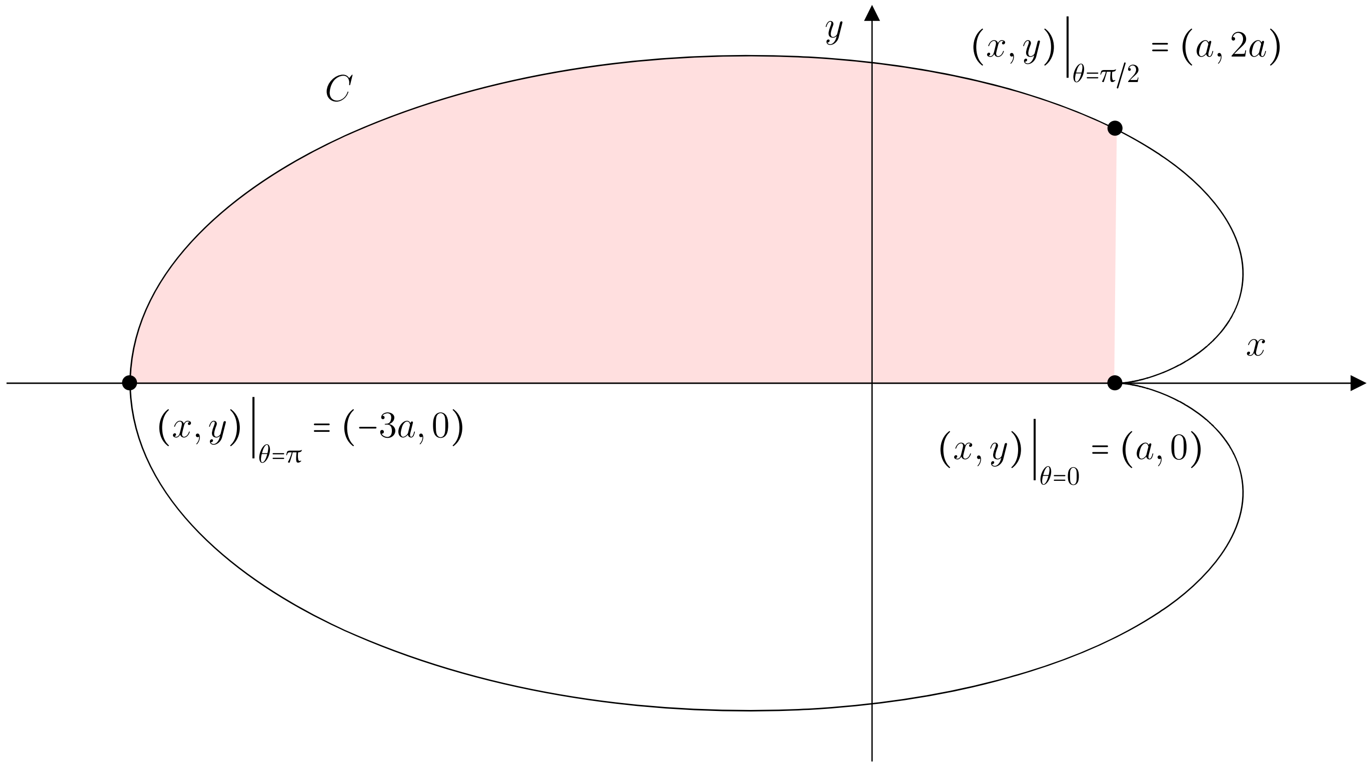
$$Q = \int_{a}^{b} y \, dx = \left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_{a}^{b} \approx \left(\frac{b^4}{4} + \frac{b^2}{2} - b\right) - \left(\frac{0.682^4}{4} + \frac{0.682^2}{2} - 0.682\right)$$

$$\approx \frac{b^4}{4} + \frac{b^2}{2} - b + 0.395.$$

We are given that $|Q| = 2|P|$ or $\dfrac{b^4}{4} + \dfrac{b^2}{2} - b + 0.395 = \dfrac{7}{2}$.

Solving this last equation on your calculator, you'll find four roots, of which three are negative and only one $(1.892\ldots)$ is positive. Since $b > 0$, conclude $b \approx 1.892$.

A664 **(9758 N2019/I/10)(i)** As usual, mindlessly use your calculator and copy:



By observation, a line of symmetry is $y = 0$.

(ii) If $y = 0$, then $0 = a\left(2\sin\theta - \sin 2\theta\right) = 2\sin\theta - \sin 2\theta = 2\sin\theta - 2\sin\theta\cos\theta = \sin\theta\left(1-\cos\theta\right)$ $\iff$ $\sin\theta = 0$ or $\cos\theta = 1$ $\iff$ $\theta = 0, \pi, 2\pi$.

(iii) As mentioned in Remark 191, to properly compute this area requires knowledge of multivariate (or vector) calculus and, in particular, a result called Green's Theorem.

But in the absence of such knowledge, I can only imagine that the exam-taker was supposed to have blindly applied the usual Substitution Rule without having any idea of what she was doing. In this case, it turns out that fortuitously, this produces the "correct answer", which is apparently all that matters in the Singapore education system:

"Answer". We observe that

$$(x,y)\Big|_{\theta=0} = (a,0), \qquad (x,y)\Big|_{\theta=\pi/2} = (a,2a), \quad \text{and} \quad (x,y)\Big|_{\theta=\pi} = (-3a,0).$$

So, it "seems" that the top half of the curve C corresponds to $\theta = [0,\pi]$.

Proceeding blindly, we guess that the requested area is

$$\int_{-3a}^{a} y \, \mathrm{d}x.$$

But why this should be so is hardly clear. Surely $\int_{-3a}^{a} y \, \mathrm{d}x$ should correspond only to the red shaded area in the above figure?

Aiyah who cares? We have no idea what we are doing anyway. So let's blindly apply the Substitution Rule as usual, cross our fingers, and hope we get the "correct answer":

(*Answer continues below ...*)

(*... Answer continued from above.*)

$$\begin{aligned}
\int_{-3a}^{a} y\,\mathrm{d}x &= \int_{x=-3a}^{x=a} y\frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x \overset{\text{s}}{=} \int_{x=-3a}^{x=a} y\frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} \overset{1}{=} \int_{\theta=\pi}^{\theta=0} y\frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta \\
&= \int_{\pi}^{0} a\left(2\sin\theta-\sin 2\theta\right)\frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta = \int_{\pi}^{0} a\left(2\sin\theta-\sin 2\theta\right)a\left(-2\sin\theta+2\sin 2\theta\right)\,\mathrm{d}\theta \\
&= \int_{\pi}^{0} a^2\left(-4\sin^2\theta+4\sin\theta\sin 2\theta+2\sin\theta\sin 2\theta-2\sin^2 2\theta\right)\,\mathrm{d}\theta \\
&= \int_{\pi}^{0} a^2\left(-4\sin^2\theta+6\sin\theta\sin 2\theta-2\sin^2 2\theta\right)\,\mathrm{d}\theta \\
&\overset{2}{=} \int_{0}^{\pi} a^2\left(4\sin^2\theta-6\sin\theta\sin 2\theta+2\sin^2 2\theta\right)\,\mathrm{d}\theta.
\end{aligned}$$

At $\overset{1}{=}$, we substitute "$x = a$" with "$\theta = \pi$". But since we also have $x\big|_{\theta=\pi/2} = a$, why couldn't we have substituted "$x = a$" with "$\theta = \pi/2$"? We don't really know why and we have in our minds only some vague and ill-formed argument along these lines: "Well, I know that $\theta = \pi$ is where the curve 'ends', so that's what I should probably use."

At $\overset{2}{=}$, just to get to the "correct" sign (as in the expression given in the question), we arbitrarily reverse the limits without knowing why on earth we're doing this. So, as requested, we submit the "correct answers" $\theta_1 = 0$ and $\theta_2 = \pi$.

This has all been a bit bizarre and we don't really know what on earth we just did. But who cares? Hooray! We've arrived at some expression they've told us to get and that's all that matters!

It turns out that fortuitously, the given definite integral *does* correspond correctly to the requested area. But the above "answer" arrived at this "correct answer" only by sheer luck. (For a proper solution to **(iii)**, see e.g. Brilliant.org.)

(iv) First, observe that $2\sin^2\theta = 1-\cos 2\theta$, $\sin\theta\sin 2\theta = 2\sin^2\theta\cos\theta$, and $\sin^2 2\theta = \dfrac{1-\cos 4\theta}{2}$.

Moreover, $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\sin^3\theta = 3\sin^2\theta\cos\theta$. Now,

$$\begin{aligned}
&\int_{0}^{\pi} a^2\left(4\sin^2\theta-6\sin\theta\sin 2\theta+2\sin^2 2\theta\right)\,\mathrm{d}\theta \\
&= 2a^2\int_{0}^{\pi} 2\sin^2\theta-3\sin\theta\sin 2\theta+\sin^2 2\theta\,\mathrm{d}\theta \\
&= 2a^2\int_{0}^{\pi} 1-\cos 2\theta-6\sin^2\theta\cos\theta+\frac{1-\cos 4\theta}{2}\,\mathrm{d}\theta \\
&= 2a^2\left[\theta-\frac{1}{2}\sin 2\theta-2\sin^3\theta+\frac{\theta}{2}-\frac{1}{8}\sin 4\theta\right]_0^{\pi} = 3\pi a^2.
\end{aligned}$$

The area bounded by C is double what we just found, or $6\pi a^2$.

A665 (9758 N2019/I/11)(i)(a) Write $\mathrm{d}\theta/\mathrm{d}t = -k(\theta - 16)$. Now,

$$t = \int \frac{\mathrm{d}t}{\mathrm{d}\theta}\,\mathrm{d}\theta = -\frac{1}{k}\int \frac{1}{\theta - 16}\,\mathrm{d}\theta = -\frac{1}{k}\ln|\theta - 16| + C = -\frac{1}{k}\ln(\theta - 16) + C.$$

(At the last step, we can remove the absolute value operator because $\theta > 16$.)

Plug in the initial conditions $(t, \theta) = (0, 80), (30, 32)$ to get $0 \stackrel{1}{=} -(\ln 64)/k + C$ and $30 \stackrel{2}{=} -(\ln 16)/k + C$. From $\stackrel{2}{=} - \stackrel{1}{=}$, $(\ln 64 - \ln 16)/k = 30$ or $k = \ln 4/30 = \ln 2/15$. Now from $\stackrel{1}{=}$, we also have $C = 15\ln 64/\ln 2 = 15 \times 6 = 90$. Hence, $t = -(15/\ln 2)\ln(\theta - 16) + 90$ or

$$\theta = \exp\frac{90 - t}{15/\ln 2} + 16 = \exp\frac{(90 - t)\ln 2}{15} + 16 = 2^{(90-t)/15} + 16 = \frac{64}{2^{t/15}} + 16.$$

(i)(b) $\theta\Big|_{t=45} = \dfrac{64}{2^{45/15}} + 16 = 8 + 16 = 24.$

(ii) Write $\mathrm{d}T/\mathrm{d}t = k/T$. Now, $t = \displaystyle\int \frac{\mathrm{d}t}{\mathrm{d}T}\,\mathrm{d}T = \int \frac{T}{k}\,\mathrm{d}T = \frac{1}{2k}T^2 + C.$

Plug in the initial conditions $(t, T) = (0, 0), (60, 1)$ to get

$$0 = C \qquad \text{and} \qquad 60 = 1/(2k) + C = 1/(2k) \text{ or } k = 1/120.$$

Hence, $t = 60T^2$. It's first safe to skate when $T = 3$ or at $t = 60 \cdot 3^2 = 540$.

A666 (9758 N2019/II/1)(i) Choose $u = x$:

$$\int \overbrace{x}^{u}\overbrace{(1-x)^{\frac{1}{2}}}^{v'}\,\mathrm{d}x = \overbrace{-\frac{2}{3}(1-x)^{\frac{3}{2}}}^{v}\overbrace{x}^{u} - \int \overbrace{1}^{u'}\overbrace{\left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]}^{v}\mathrm{d}x = -\frac{2}{3}x(1-x)^{\frac{3}{2}} - \frac{2}{3}\frac{2}{5}(1-x)^{\frac{5}{2}} + C$$

$$= -\frac{2}{3}(1-x)^{\frac{3}{2}}\left[x + \frac{2}{5}(1-x)\right] + C = -\frac{2}{15}(1-x)^{\frac{3}{2}}(2+3x) + C.$$

(ii) Apply $\dfrac{\mathrm{d}}{\mathrm{d}x}$ to $u \stackrel{1}{=} 1 - x^2$ to get $2u\dfrac{\mathrm{d}u}{\mathrm{d}x} \stackrel{2}{=} -1$.

Note that from $\stackrel{1}{=}$, we actually have $u = \pm\sqrt{1-x}$. So, let us use $u \stackrel{3}{=} \sqrt{1-x}$.[675] Now,

$$\int x(1-x)^{\frac{1}{2}}\,\mathrm{d}x \stackrel{1,3}{=} \int (1-u^2)\,u\,\mathrm{d}x \stackrel{2}{=} \int (1-u^2)\,u\left(-2u\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\right)\cancel{\mathrm{d}x}$$

$$= \int 2u^4 - 2u^2\,\mathrm{d}u = \frac{2}{5}u^5 - \frac{2}{3}u^3 + \bar{C} = \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + \bar{C}$$

$$= \frac{2}{15}(1-x)^{\frac{3}{2}}[3(1-x) - 5] + \bar{C} = -\frac{2}{15}(1-x)^{\frac{3}{2}}(2+3x) + \bar{C}.$$

(iii) Our answers in **(ii)** and **(iii)** differ by $C - \bar{C}$, which is a constant.

[675]We'd also arrive at the same answer if we instead used $u = -\sqrt{1-x}$.

A667 (9758 N2019/II/3). Let S and V be the surface area and the volume of the cylinder, respectively. Then $S = 2\pi r(r+h) = 900$ or $h \stackrel{1}{=} 450/(r\pi) - r$.

And $V = \pi r^2 h \stackrel{1}{=} \pi r^2 [450/(r\pi) - r] = 450r - \pi r^3$.

Compute $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 450 - 3\pi r^2$. The stationary points of V are given by

$$\left.\frac{\mathrm{d}V}{\mathrm{d}r}\right|_{r=\bar{r}} = 0 \qquad \Longleftrightarrow \qquad 450 - 3\pi\bar{r}^2 = 0 \qquad \Longleftrightarrow \qquad \bar{r} = \sqrt{\frac{150}{\pi}}.$$

Since $\dfrac{\mathrm{d}V}{\mathrm{d}r} > 0$ for all $r \in [0, \bar{r})$ and $\dfrac{\mathrm{d}V}{\mathrm{d}r} < 0$ for all $r \in (\bar{r}, \infty)$, $\bar{r}$ is a strict global maximum and the corresponding volume is

$$V(\bar{r}) = 450\sqrt{\frac{150}{\pi}} - \pi\sqrt{\frac{150}{\pi}}^3 = 450\sqrt{\frac{150}{\pi}} - \pi\frac{150}{\pi}\sqrt{\frac{150}{\pi}} = 300\sqrt{\frac{150}{\pi}} = 1\,500\sqrt{\frac{6}{\pi}}.$$

So, $$\frac{\bar{r}}{h(\bar{r})} = \frac{\sqrt{150/\pi}}{\frac{450}{\pi\sqrt{150/\pi}} - \sqrt{150/\pi}} = \frac{\frac{150}{\pi}}{\frac{450}{\pi} - \frac{150}{\pi}} = \frac{150}{450 - 150} = \frac{1}{2}.$$

A668 (9758 N2019/II/4)(i) $f'(x) = 2\sec 2x \tan 2x$.

$f''(x) = 2\left[f'(x)\tan 2x + 2\sec 2x \sec^2 2x\right] = 2f'(x)\tan 2x + 4\sec^3 2x$.

$f(0) = 1$, $f'(0) = 0$, $f''(0) = 4$. So, $\sec 2x \stackrel{1}{=} 1 + 2x^2 + \ldots$ (for $x \in (-\pi/2, \pi/2)$).

(ii) $$\int_0^{0.02} \sec 2x \,\mathrm{d}x \stackrel{1}{=} \int_0^{0.02} 1 + 2x^2 + \ldots \,\mathrm{d}x = \left[x + \frac{2}{3}x^3 + \ldots\right]_0^{0.02}$$
$$= 0.02 + (2/3) \times 0.02^3 + \cdots = 0.020005\overline{3} + \cdots \approx 0.02001$$

Remark 221. As discussed in Ch. 104.7, the step taken at $\stackrel{1}{=}$ requires justification.

(iii) $0.02000533546765866\cdots \approx 0.02001$

(iv) Very close. Good.

(v) Undefined at 0.

A669 (9758 N2018/I/1)(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\ln x}{x} = \dfrac{1}{x} \cdot \dfrac{1}{x} + \ln x\left(-\dfrac{1}{x^2}\right) = \dfrac{1}{x^2} - \dfrac{\ln x}{x^2}$.

(ii) $$\int \frac{\ln x}{x^2}\,\mathrm{d}x = -\int \frac{1}{x^2} - \frac{\ln x}{x^2}\,\mathrm{d}x + \int \frac{1}{x^2}\,\mathrm{d}x \stackrel{\text{(i)}}{=} -\frac{\ln x}{x} + \int \frac{1}{x^2}\,\mathrm{d}x = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

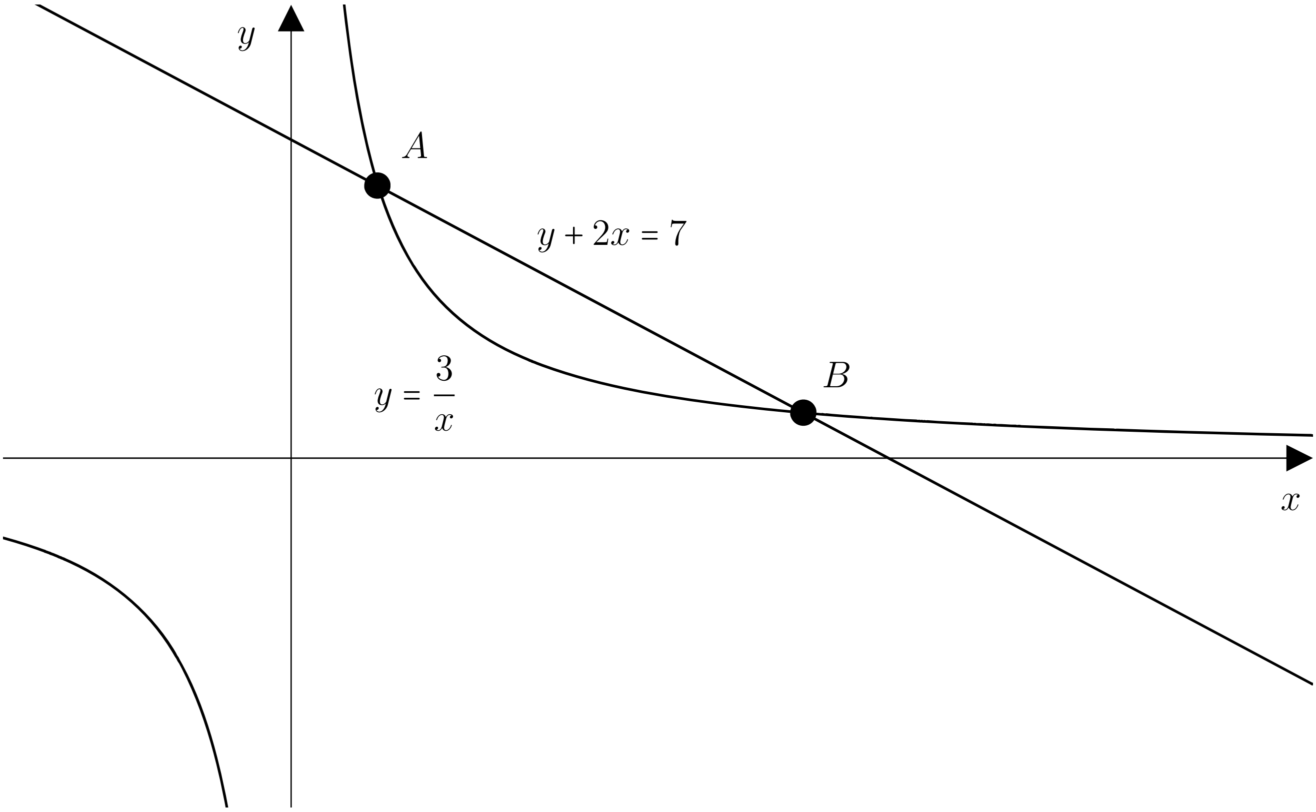
Thus, $$\int_1^{\mathrm{e}} \frac{\ln x}{x^2}\,\mathrm{d}x = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^{\mathrm{e}} = \left[-\frac{\ln \mathrm{e}}{\mathrm{e}} - \frac{1}{\mathrm{e}}\right] - \left[-\frac{\ln 1}{1} - \frac{1}{1}\right] = -\frac{2}{\mathrm{e}} + 1.$$

A670 (9758 N2018/I/2)(i) For $x \neq 0$, $\dfrac{3}{x} = 7 - 2x \iff 3 = 7x - 2x^2 \iff 2x^2 - 7x + 3 = 0$

$$\iff \qquad x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(3)}}{2(2)} = \frac{7 \pm \sqrt{25}}{4} = \frac{7 \pm 5}{4} = \frac{1}{2}, 3.$$

The x-coordinates of A and B are $1/2$ and 3.

(ii) As usual, a quick sketch can be helpful:



The requested volume is

$$\int_{1/2}^{3} \pi \left[(7-2x)^2 - \left(\frac{3}{x}\right)^2 \right] \mathrm{d}x = \pi \left[\frac{(7-2x)^3}{-2 \cdot 3} + \frac{9}{x} \right]_{1/2}^{3} = \pi \left[\left(\frac{1^3}{-6} + 3 \right) - \left(\frac{6^3}{-6} + 18 \right) \right] = 20\frac{5}{6}\pi.$$

A671 (9758 N2018/I/3)(i) $y \overset{1}{=} ux^2 \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} \overset{2}{=} \dfrac{\mathrm{d}u}{\mathrm{d}x}x^2 + 2ux.$

Plug $\overset{2}{=}$ and $\overset{1}{=}$ into the given equation $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y - 6$ to get $\dfrac{\mathrm{d}u}{\mathrm{d}x}x^3 + 2ux^2 = 2ux^2 - 6.$

Rearranging, $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\dfrac{6}{x^3}$ (for $x \neq 0$).

(ii) $u = \displaystyle\int -\frac{6}{x^3}\,\mathrm{d}x = \frac{3}{x^2} + C.$ So, $y = ux^2 = 3 + Cx^2.$

Plug the initial condition $(x, y) = (1, 2)$ into $\overset{3}{=}$ to get $2 = 3 + C \cdot 1^2$. So, $C = -1$ and $y = 3 - x^2$.

A672 (9758 N2018/I/7)(i) $\dfrac{x^2-4y^2}{x^2+xy^2}=\dfrac{1}{2} \implies x^2-4y^2=\dfrac{1}{2}\left(x^2+xy^2\right)$

$$\implies \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-4y^2\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(x^2+xy^2\right)\right] \iff 2x-8y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(2x+y^2+2xy\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$\iff \frac{\mathrm{d}y}{\mathrm{d}x}(8y+xy)=x-\frac{1}{2}y^2 \iff \frac{\mathrm{d}y}{\mathrm{d}x}\overset{1}{=}\frac{2x-y^2}{2xy+16y} \text{ (for } 2xy+16y\neq 0\text{).}$$

(ii) Plug 1 into the equation for C:

$$\frac{1^2-4y^2}{1^2+1y^2}=\frac{1}{2} \iff \frac{1-4y^2}{1+y^2}=\frac{1}{2} \iff 1-4y^2=\frac{1}{2}\left(1+y^2\right) \iff 1=9y^2 \iff y=\pm\frac{1}{3}.$$

Now plug in $(x,y)=(1,1/3)$ and $(x,y)=(1,-1/3)$ into $\overset{1}{=}$ to get

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1/3)}=\frac{2-1/9}{2/3+16/3}=\frac{17}{54} \quad\text{and}\quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,-1/3)}=-\frac{17}{54}.$$

So, the two tangent lines are $y-1/3\overset{2}{=}17(x-1)/54$ and $y+1/3\overset{3}{=}-17(x-1)/54$.

Taking $\overset{2}{=}+\overset{3}{=}$ yields $2y=0$ or $y=0$. And now from $\overset{2}{=}$, $x=-1/17$. Thus, the coordinates of N are $(x,y)=(-1/17,0)$.

A673 (9758 N2018/I/9)(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta}=\dfrac{4\sin\theta\cos\theta}{2-2\cos 2\theta}=\dfrac{2\sin\theta\cos\theta}{1-\cos 2\theta}=\dfrac{2\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)}$

$$=\frac{2\sin\theta\cos\theta}{2\sin^2\theta}=\frac{\cos\theta}{\sin\theta}=\cot\theta \text{ (for } \theta\neq 0,\pi\text{).}$$

(ii) The point on C at which $\theta=\alpha$ is $(x,y)=\left(2\alpha-\sin 2\alpha, 2\sin^2\alpha\right)$. The normal at this point has gradient $-1/\cot\alpha=-\tan\alpha$ and equation $y-2\sin^2\alpha\overset{1}{=}-\tan\alpha\left[x-(2\alpha-\sin 2\alpha)\right]$.

If $y=0$, then $x=\dfrac{0-2\sin^2\alpha}{-\tan\alpha}+(2\alpha-\sin 2\alpha)=2\sin\alpha\cos\alpha+2\alpha-2\sin\alpha\cos\alpha=2\alpha$ $(k=2)$.

(iii) First, note that since $\theta\in[0,\pi]$, $\sin\theta\overset{2}{\geq}0$. Now,

$$\int_\beta^\gamma\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta=\int_\beta^\gamma\sqrt{(2-2\cos 2\theta)^2+(4\sin\theta\cos\theta)^2}\,\mathrm{d}\theta$$

$$=\int_\beta^\gamma\sqrt{(2-2\cos 2\theta)^2+(2\sin 2\theta)^2}\,\mathrm{d}\theta=\int_\beta^\gamma\sqrt{4+4\cos^2 2\theta-8\cos 2\theta+4\sin^2 2\theta}\,\mathrm{d}\theta$$

$$=\int_\beta^\gamma\sqrt{4+4-8\cos 2\theta}\,\mathrm{d}\theta=\int_\beta^\gamma\sqrt{8(1-\cos 2\theta)}\,\mathrm{d}\theta=\int_\beta^\gamma\sqrt{8\left(2\sin^2\theta\right)}\,\mathrm{d}\theta$$

$$=4\int_\beta^\gamma|\sin\theta|\,\mathrm{d}\theta\overset{2}{=}4\int_\beta^\gamma\sin\theta\,\mathrm{d}\theta=4\left[-\cos\theta\right]_\beta^\gamma=4(\cos\beta-\cos\gamma).$$

A674 (9758 N2018/I/10)(i) Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ to $L\frac{\mathrm{d}I}{\mathrm{d}t}+RI+\frac{q}{C}=V$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}L\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{\mathrm{d}}{\mathrm{d}t}RI+\frac{\mathrm{d}}{\mathrm{d}t}\frac{q}{C}=\frac{\mathrm{d}}{\mathrm{d}t}V \qquad \text{or} \qquad L\frac{\mathrm{d}^2I}{\mathrm{d}t^2}+R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{I}{C}\overset{1}{=}\frac{\mathrm{d}V}{\mathrm{d}t},$$

If V is constant, then $\frac{\mathrm{d}V}{\mathrm{d}t}=0$ and $\overset{1}{=}$ becomes $L\frac{\mathrm{d}^2I}{\mathrm{d}t^2}+R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{I}{C}\overset{2}{=}0$.

(ii) Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ to $I\overset{3}{=}Ate^{-\frac{Rt}{2L}}$ twice:

$$\frac{\mathrm{d}I}{\mathrm{d}t}=A\left(\mathrm{e}^{-\frac{Rt}{2L}}-\frac{R}{2L}t\mathrm{e}^{-\frac{Rt}{2L}}\right)\overset{4}{=}A\mathrm{e}^{-\frac{Rt}{2L}}\left(1-\frac{R}{2L}t\right),$$

$$\frac{\mathrm{d}^2I}{\mathrm{d}t^2}=A\left[-\frac{R}{2L}\mathrm{e}^{-\frac{Rt}{2L}}\left(1-\frac{R}{2L}t\right)+\mathrm{e}^{-\frac{Rt}{2L}}\left(-\frac{R}{2L}\right)\right]=A\frac{R}{2L}\mathrm{e}^{-\frac{Rt}{2L}}\left(\frac{R}{2L}t-1-1\right)\overset{5}{=}A\frac{R}{2L}\mathrm{e}^{-\frac{Rt}{2L}}\left(\frac{R}{2L}t-2\right).$$

Now plug $\overset{3}{=}$, $\overset{4}{=}$, and $\overset{5}{=}$ into $\overset{2}{=}$:

$$\begin{aligned}0&=L\cancel{A}\frac{R}{2L}\cancel{\mathrm{e}^{-\frac{Rt}{2L}}}\left(\frac{R}{2L}t-2\right)+R\cancel{A\mathrm{e}^{-\frac{Rt}{2L}}}\left(1-\frac{R}{2L}t\right)+\frac{\cancel{A}t\cancel{\mathrm{e}^{-\frac{Rt}{2L}}}}{C}\\&=\frac{R}{2}\left(\frac{R}{2L}t-2\right)+R\left(1-\frac{R}{2L}t\right)+\frac{t}{C}=\frac{R}{2}\left(\frac{R}{2L}t\right)+R\left(-\frac{R}{2L}t\right)+\frac{t}{C}=-\frac{R^2t}{4L}+\frac{t}{C}=-\frac{R^2}{4L}+\frac{1}{C}.\end{aligned}$$

Rearranging, $C=\frac{4L}{R^2}$.

(iii) The stationary points of I (as a function of t) are given by

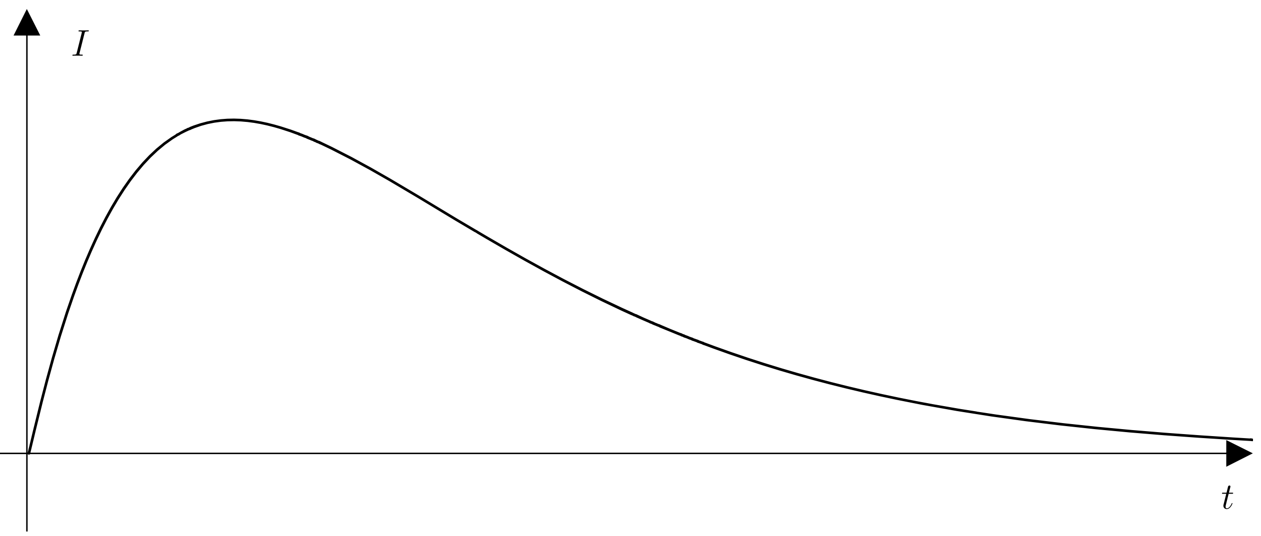
$$\left.\frac{\mathrm{d}I}{\mathrm{d}t}\right|_{t=\bar{t}}=0 \qquad \overset{4}{\iff} \qquad A\mathrm{e}^{-\frac{R\bar{t}}{2L}}\left(1-\frac{R}{2L}\bar{t}\right)=0 \qquad \text{or} \qquad \bar{t}=\frac{2L}{R}.$$

Note that A and $\mathrm{e}^{-\frac{Rt}{2L}}$ are always positive.

Since $\mathrm{d}I/\mathrm{d}t>0$ for all $t\in\left[0,\bar{t}\right)$ and $\mathrm{d}I/\mathrm{d}t<0$ for all $t\in\left(\bar{t},\infty\right)$, $\bar{t}$ is a strict global maximum of I and the corresponding maximum value of I is

$$I\left(\bar{t}\right)=A\bar{t}\mathrm{e}^{-\frac{R\bar{t}}{2L}}=A\frac{2L}{R}\mathrm{e}^{-1}=\frac{2AL}{\mathrm{e}R}=\frac{2A\cdot 3}{\mathrm{e}\cdot 4}=\frac{3A}{2\mathrm{e}}.$$

(iv)



A675 (9758 N2018/II/1)(i) $x = \int \frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y = \int \left(\frac{y}{3} - 15\right)^{-\frac{1}{3}} \mathrm{d}y = \frac{9}{2}\left(\frac{y}{3} - 15\right)^{\frac{2}{3}} + C.$

Plug in the initial condition $(x, y) = (0, 69)$ to get

$0 = \frac{9}{2}(23 - 15)^{\frac{2}{3}} + C = \frac{9}{2} \cdot 8^{\frac{2}{3}} + C = \frac{9}{2} \cdot 4 + C = 18 + C$ or $C = -18$. So, $x \overset{1}{=} \frac{9}{2}\left(\frac{1}{3}y - 15\right)^{\frac{2}{3}} - 18.$

Rearranging, $y = 3\left\{\left[\frac{2}{9}(x + 18)\right]^{\frac{3}{2}} + 15\right\} = \frac{1}{9}\left[2(x + 18)\right]^{\frac{3}{2}} + 45.$

(ii) $$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{y=\bar{y}} = 4 \qquad \Longleftrightarrow \qquad (\bar{y}/3 - 15)^{\frac{1}{3}} \overset{2}{=} 4 \qquad \Longleftrightarrow \qquad \bar{y} = 3\left(4^3 + 15\right) = 237.$$

Plug $\overset{2}{=}$ into $\overset{1}{=}$ to get the corresponding $\bar{x} = (9/2) \cdot 4^2 - 18 = 72 - 18 = 54$.

So the requested point are $(\bar{x}, \bar{y}) = (54, 237)$.

A676 (9758 N2018/II/4)(i) First, for all $x \in \mathbb{R}$,

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4x^6}{45} + \ldots.$$

Next, for $\cos 2x - 1 \in (-1, 1]$ or $\cos 2x \neq 0$ or $x \in [0, \pi/4)$, we have

$$\ln(\cos 2x) = \ln\left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4x^6}{45} + \ldots\right)$$

$$= \left(-2x^2 + \frac{2}{3}x^4 - \frac{4x^6}{45} + \ldots\right) - \frac{\left(-2x^2 + \frac{2}{3}x^4 - \ldots\right)^2}{2} + \frac{\left(-2x^2 + \ldots\right)^3}{3} + \ldots$$

$$= \left(-2x^2 + \frac{2}{3}x^4 - \frac{4x^6}{45} + \ldots\right) - \frac{\left(4x^4 - \frac{8}{3}x^6 + \ldots\right)}{2} + \frac{\left(-8x^6 + \ldots\right)}{3} + \cdots = -2x^2 - \frac{4}{3}x^4 - \frac{64}{45}x^6 + \ldots.$$

As already stated above, the expansion is not valid for $x = \pi/4$.

(ii) $$\int \frac{\ln(\cos 2x)}{x^2}\,\mathrm{d}x \overset{1}{=} \int \frac{-2x^2 - \frac{4}{3}x^4 - \frac{64}{45}x^6 + \ldots}{x^2}\,\mathrm{d}x = \int -2 - \frac{4}{3}x^2 - \frac{64}{45}x^4 + \ldots\,\mathrm{d}x$$

$$= -2x - \frac{4}{9}x^3 - \frac{64}{225}x^5 + \ldots$$

$$\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,\mathrm{d}x = \left[-2x - \frac{4}{9}x^3 - \frac{64}{225}x^5 + \ldots\right]_0^{0.5} = -1 - \frac{1}{18} - \frac{2}{225} + \cdots \approx -1.0644.$$

Remark 222. As discussed in Ch. 104.7, the step taken at $\overset{1}{=}$ requires justification.

(iii) $\int_0^{0.5} \frac{\ln(\cos 2x)}{x^2}\,\mathrm{d}x \approx -1.0670.$

A677 (9758 N2017/I/1). For $ax \in (-1, 1]$,

$$e^{2x} \ln(1+ax) = \left[1 + 2x + \frac{(2x)^2}{2!} + \dots\right]\left[ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \dots\right]$$

$$= ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} + 2ax^2 - a^2x^3 + 2ax^3 + \dots = ax + \left(2a - \frac{a^2}{2}\right)x^2 + \left(2a - a^2 + \frac{a^3}{3}\right)x^3 + \dots.$$

If $0 = 2a - a^2/2 = a(4-a)/2$, then $a = 0$ (discard) or $a = 4$.

A678 (9758 N2017/I/3)(i) Apply $\frac{d}{dx}$ to $y^2 - 2xy + 5x^2 - 10 \stackrel{1}{=} 0$ to get

$$2y\frac{dy}{dx} - 2\left(y + x\frac{dy}{dx}\right) + 10x \stackrel{2}{=} 0.$$

The stationary points are given by $\frac{dy}{dx} \stackrel{3}{=} 0$ or $-2y + 10x = 0$ or $y \stackrel{4}{=} 5x$.

Plug $\stackrel{4}{=}$ into $\stackrel{1}{=}$ to get

$$0 = 25x^2 - 10x^2 + 5x^2 - 10 = 20x^2 - 10 \quad \text{or} \quad x^2 = 1/2 \quad \text{or} \quad x = \pm 1/\sqrt{2} = \pm\sqrt{2}/2.$$

Thus, the x-coordinates of the two stationary points of C are $\pm\sqrt{2}/2$.

(ii) Let the stationary point with $x > 0$ be $P = \left(\sqrt{2}/2, 5\sqrt{2}/2\right)$.

Apply $\frac{d}{dx}$ to $\stackrel{2}{=}$ to get $2\left(\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right) - 2\left(\frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2}\right) + 10 \stackrel{5}{=} 0.$

Plug $\frac{dy}{dx} \stackrel{3}{=} 0$ into $\stackrel{5}{=}$ to find that at P,

$$2y\frac{d^2y}{dx^2} - 2x\frac{d^2y}{dx^2} + 10 \stackrel{4}{=} 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{5}{x-y} = \frac{1}{\sqrt{2}/2 - 5\sqrt{2}/2} = \frac{-\sqrt{2}}{4} < 0.$$

Since the second derivative is negative at P,[676] P is a maximum.

[676]See Proposition 11 (Second Derivative Test)

A679 (9758 N2017/I/7)(i) List MF26 (p. 3) contains this Sum-to-Product formula:

$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$

Equivalently, $\cos(A+B) - \cos(A-B) = -2\sin A\sin B$.

Rearranging, $\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$.

$$\text{So, } \int \sin 2mx\sin 2nx\,\mathrm{d}x = \frac{1}{2}\int \cos(2mx-2nx) - \cos(2mx+2nx)\,\mathrm{d}x$$

$$= \frac{1}{2}\int \cos 2(m-n)x - \cos 2(m+n)x\,\mathrm{d}x = \frac{1}{2}\left[\frac{\sin 2(m-n)x}{2(m-n)} - \frac{\sin 2(m+n)x}{2(m+n)}\right] + C.$$

(ii)
$$\int_0^\pi (f(x))^2\,\mathrm{d}x = \int_0^\pi (\sin 2mx + \sin 2nx)^2\,\mathrm{d}x$$

$$= \int_0^\pi \sin^2 2mx + \sin^2 2nx + 2\sin 2mx\sin 2nx\,\mathrm{d}x$$

$$= \int_0^\pi \frac{1-\cos 4mx}{2} + \frac{1-\cos 4nx}{2} + 2\sin 2mx\sin 2nx\,\mathrm{d}x$$

$$= \left[x - \frac{1}{8m}\sin 4mx - \frac{1}{8n}\sin 4nx + \frac{\sin 2(m-n)x}{2(m-n)} - \frac{\sin 2(m+n)x}{2(m+n)}\right]_0^\pi.$$

Since that $\sin(k\pi) = 0$ for any integer k, this last expression simply equals $[x]_0^\pi = \pi$.

A680 (9758 N2017/I/11)(i)(a) $\frac{\mathrm{d}v}{\mathrm{d}t} = c$. **(b)** $4 + 2.5c = 29$. So $c = 10$ and $v = 4 + 10t$.

(ii) $\frac{\mathrm{d}v}{\mathrm{d}t} = c - kv = 10 - kv$ (with $k > 0$). If $10 - kv \neq 0$, then $\frac{\mathrm{d}t}{\mathrm{d}v} \stackrel{1}{=} \frac{1}{10-kv}$. So,

$$\int \frac{\mathrm{d}t}{\mathrm{d}v}\,\mathrm{d}v = \int \frac{1}{10-kv}\,\mathrm{d}v \qquad \text{or} \qquad t \stackrel{2}{=} -\frac{1}{k}\ln|10-kv| + C_1,$$

or, $\quad k(C_1 - t) = \ln|10-kv| \quad$ or $\quad |10-kv| = \mathrm{e}^{k(C_1-t)} \quad$ or $\quad 10 - kv = \pm\mathrm{e}^{kC_1}\mathrm{e}^{-kt} = C_2\mathrm{e}^{-kt}$.
Plug in the initial value $(t, v) = (0, 0)$ to get $10 = C_2$. Thus, $10 - kv = 10\mathrm{e}^{-kt}$ or

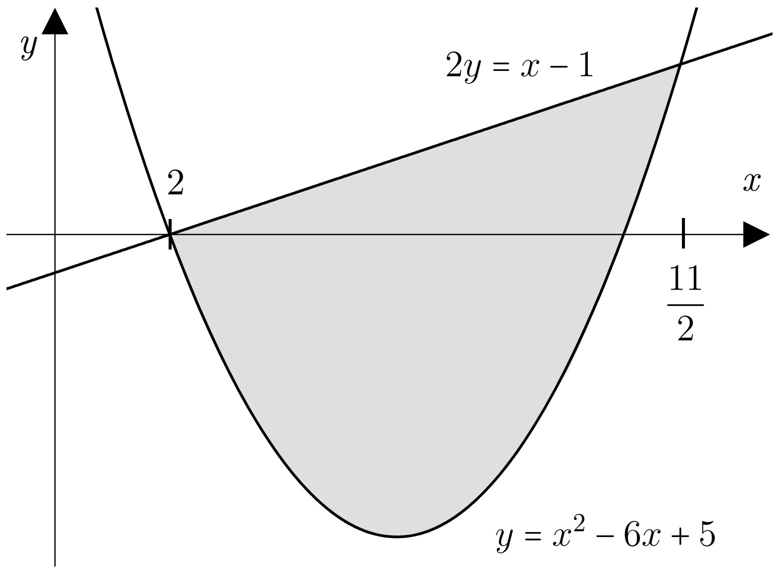
$$v = \frac{10}{k}\left(1 - \mathrm{e}^{-kt}\right).$$

(iii) This object's terminal velocity is $\lim_{t\to\infty} v = \lim_{t\to\infty} 10\left(1-\mathrm{e}^{-kt}\right)/k = 10/k = 40$.

So, for this object, $k = 1/4$. At 90% of terminal velocity, we have

$$1 - \mathrm{e}^{-kt} = 0.9 \qquad \text{or} \qquad 0.1 = \mathrm{e}^{-kt} \qquad \text{or} \qquad t = -\ln 0.1/k = 4\ln 10 \approx 9.21 \text{ seconds}.$$

A681 (9758 N2017/II/4)(a) As usual, a quick sketch can be helpful:



Find their intersection points by plugging the line's equation into the curve's:

$$(x-1)/2 = x^2 - 6x + 5 \qquad \iff \qquad 2x^2 - 13x + 11 = 0$$

$$\iff \qquad x = \frac{13 \pm \sqrt{(-13)^2 - 4(2)(11)}}{2(2)} = \frac{13 \pm 9}{4} = 1, \frac{11}{2}.$$

So, the area of the plate is

$$\int_1^{11/2} \frac{x-1}{2} - \left(x^2 - 6x + 5\right) \mathrm{d}x = \left[-\frac{x^3}{3} + \frac{13}{4}x^2 - \frac{11}{2}x\right]_1^{11/2} = \frac{243}{16} = 15.1875 \approx 15.2.$$

(b)(i) $$\int_0^1 \pi\left(\frac{\sqrt{y}}{a-y^2}\right)^2 \mathrm{d}y = \pi \int_0^1 \frac{y}{(a-y^2)^2} \mathrm{d}y = \frac{\pi}{2}\left[\frac{1}{a-y^2}\right]_0^1 = \frac{\pi}{2}\left(\frac{1}{a-1} - \frac{1}{a}\right) = \frac{\pi}{2(a^2-a)}.$$

(ii) The volume of the second container is four times that of the first:

$$\frac{\pi}{2(b^2-b)} = 4\frac{\pi}{2(a^2-a)} \qquad \iff \qquad a^2 - a = 4\left(b^2 - b\right) \qquad \iff \qquad 0 = 4b^2 - 4b + a - a^2.$$

By the quadratic formula, $b = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(a-a^2)}}{2(4)} = \dfrac{1 \pm \sqrt{1-a+a^2}}{2}$.

Observe that since $a > 1$, $-a + a^2 > 0$. So,

$$\frac{1 - \sqrt{1-a+a^2}}{2} < \frac{1-\sqrt{0}}{2} = \frac{1}{2}, \qquad \text{while } \frac{1+\sqrt{1-a+a^2}}{2} > \frac{1+\sqrt{1}}{2} = \frac{1}{2}.$$

Since $b > 1$, we may discard the smaller value and conclude:

$$b = \frac{1+\sqrt{1-a+a^2}}{2}.$$

A682 (9740 N2016/I/2). We'll do this without a calculator:

First, observe that $2^{\cos x} = e^{\ln 2^{\cos x}} = e^{(\cos x)(\ln 2)}$. So,

$$y'(x) = \frac{d}{dx}e^{(\cos x)(\ln 2)} = e^{(\cos x)(\ln 2)}(-\sin x)(\ln 2) = -2^{\cos x}\sin x(\ln 2).$$

Hence, $$y'(0) = 0 \text{ and } y'\left(\frac{\pi}{2}\right) = -\ln 2.$$

(ii) The tangent line at $x = 0$ is $y - y(0) = y'(0)(x-0)$ or $y - 2 = 0$ or $y \stackrel{1}{=} 2$.

The tangent line at $x = \pi/2$ is

$$y - y\left(\frac{\pi}{2}\right) = y'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) \qquad \text{or} \qquad y - 1 \stackrel{2}{=} -\ln 2\left(x - \frac{\pi}{2}\right).$$

Plug $\stackrel{1}{=}$ into $\stackrel{2}{=}$ to get $$1 = -\ln 2\left(x - \frac{\pi}{2}\right) \qquad \text{or} \qquad x = \frac{\pi}{2} - \frac{1}{\ln 2}.$$

So, the tangents meet at $\left(\frac{\pi}{2} - \frac{1}{\ln 2}, 2\right)$.

A683 (9740 N2016/I/8)(i) Compute

$f'(x) = a\sec^2(ax+b) = a\left[1 + \tan^2(ax+b)\right] = a + a\tan^2(ax+b) = a + ay^2 = a\left(1+y^2\right)$.

$f''(x) = 2ayf'(x) = 2a^2y\left(1+y^2\right)$.

$f'''(x) = 2a^2\left[f'(x)\left(1+y^2\right) + y\cdot 2yf'(x)\right] = 2a^2f'(x)\left(1+3y^2\right) = 2a^3\left(1+y^2\right)\left(1+3y^2\right)$.

(ii) $f(0) = \tan\left(a\cdot 0 + \frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1. \qquad f'(0) = a\left\{1 + [f(0)]^2\right\} = 2a.$

$f''(0) = 2a^2f(0)\left(1 + [f(0)]^2\right) = 2a^2\cdot 1\cdot 2 = 4a^2$.

$f'''(0) = 2a^3\left(1 + [f(0)]^2\right)\left(1 + 3[f(0)]^2\right) = 2a^3\cdot 2\cdot 4 = 16a^3$.

So, $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = 1 + 2ax + 2a^2x^2 + \frac{8}{3}a^3x^3 + \ldots$

(iii) $f(0) = \tan(2\cdot 0 + 0) = \tan 0 = 0. \qquad f'(0) = 2\left\{1 + [f(0)]^2\right\} = 2.$

$f''(0) = 2\cdot 2^2f(0)\left(1 + [f(0)]^2\right) = 0$.

$f'''(0) = 2\cdot 2^3\left(1 + [f(0)]^2\right)\left(1 + 3[f(0)]^2\right) = 2\cdot 2^3\cdot 1\cdot 1 = 16$. So,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = 2x + \frac{8}{3}x^3 + \ldots$$

A684 (9740 N2016/I/9)(i)(a) Plug $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ into the given equation to get

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 10 \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 10 - 2y.$$

(i)(b) For $10 - 2y \neq 0$, we have $\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{1}{10-2y}$. So,

$$t = \int \frac{\mathrm{d}t}{\mathrm{d}y}\,\mathrm{d}y = \int \frac{1}{10-2y}\,\mathrm{d}y = -\frac{1}{2}\int \frac{1}{y-5}\,\mathrm{d}y = -\frac{1}{2}\ln|y-5| + C_1,$$

or, $\quad 2(C_1 - t) = |y-5| \quad$ or $\quad y - 5 = \pm\mathrm{e}^{2(C_1-t)} = \pm\mathrm{e}^{2C_1}\mathrm{e}^{-2t} = C_2\mathrm{e}^{-2t} \quad$ or $\quad y \stackrel{1}{=} 5 + C_2\mathrm{e}^{-2t}$.

Plug the initial values $\left(t, x, y = \frac{\mathrm{d}x}{\mathrm{d}t}\right) \stackrel{2}{=} (0,0,0)$ into $\stackrel{1}{=}$ to get $0 = 5 + C_2$ or $C_2 = -5$.

So, $y \stackrel{1}{=} 5\left(1 - \mathrm{e}^{-2t}\right)$.

Next, $x = \int \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int y\,\mathrm{d}t = \int 5\left(1 - \mathrm{e}^{-2t}\right)\mathrm{d}t \stackrel{3}{=} 5\left(t + 0.5\mathrm{e}^{-2t}\right) + C_3 = 5t + 2.5\mathrm{e}^{-2t} + C_3$.

Again plugging $\stackrel{2}{=}$ into $\stackrel{3}{=}$, we get $0 = 0 + 2.5 \cdot 1 + C_3$ or $C_3 = -2.5$.

So, $x = 5t + 2.5\mathrm{e}^{-2t} - 2.5$.

(ii) First, $\frac{\mathrm{d}x}{\mathrm{d}t} = \int \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\mathrm{d}t = \int 10 - 5\sin\frac{1}{2}t\,\mathrm{d}t = 10t + 10\cos\frac{1}{2}t + C_4$.

Plug in the initial values $\stackrel{2}{=}$ to get $0 = 0 + 10 + C_4$ or $C_4 = -10$.

Next $x = \int \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int 10t + 10\cos\frac{1}{2}t - 10\,\mathrm{d}t = 10\left(\frac{t^2}{2} + 2\sin\frac{t}{2} - t\right) + C_5$.

Again, plug in $\stackrel{2}{=}$ to get $0 = 10(0 + 0 - 0) + C_5$ or $C_5 = 0$. Hence,

$$x = 5t^2 + 20\sin\frac{t}{2} - 10t.$$

(iii) For Model **(i)**, $x = 5 \implies 5 = 5t + 2.5\mathrm{e}^{-2t} - 2.5 \iff t \approx 1.47$ (calculator).

For Model **(ii)**, $x = 5 \implies 5 = 5t^2 + 20\sin t/2 - 10t \iff t \approx 1.05$ (calculator).

A685 (9740 N2016/II/1). Let $V(t)$ and $h(t)$ be the volume (m^3) and height (m) of water in the cone at time t (min).

From $\tan\alpha = 0.5$, we have $r/h = 0.5$ or $r = 0.5h$.

Plug this into $V = \dfrac{\pi}{3}r^2h$ to get $V = \dfrac{\pi}{12}h^3$ or $h = \left(\dfrac{12}{\pi}V\right)^{1/3}$.

By the Chain Rule, $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \dfrac{\pi}{12}3h^2\dfrac{\mathrm{d}h}{\mathrm{d}t}$ or $\dfrac{\mathrm{d}h}{\mathrm{d}t} \stackrel{1}{=} \dfrac{4}{\pi h^2}\dfrac{\mathrm{d}V}{\mathrm{d}t}$.

When $V = 3$, $h \stackrel{2}{=} \left(\dfrac{36}{\pi}\right)^{\frac{1}{3}}$. Also, $\dfrac{\mathrm{d}V}{\mathrm{d}t} \stackrel{3}{=} 0.1$ (for all t). Plug $\stackrel{2}{=}$ and $\stackrel{3}{=}$ into $\stackrel{1}{=}$ to get

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{4}{\pi(36/\pi)^{2/3}}0.1 = 0.1\pi^{-1/3}\left(\frac{2}{9}\right)^{2/3} \approx 0.0251\,\mathrm{m\,s^{-1}}.$$

A686 (9740 N2016/II/2)(a)(i) Use Integration by Parts twice:

$$\int x^2\cos nx\,\mathrm{d}x = x^2\frac{\sin nx}{n} - 2\int x\frac{\sin nx}{n}\,\mathrm{d}x = x^2\frac{\sin nx}{n} - \frac{2}{n}\left[x\frac{-\cos nx}{n} - \int\frac{-\cos nx}{n}\,\mathrm{d}x\right]$$

$$= x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3} + C = \frac{1}{n^3}\left(n^2x^2\sin nx + 2nx\cos nx - 2\sin nx\right) + C.$$

(ii) We'll use the facts that the value of sin at any integer multiple of π is 0 and the value of cos at any even (or odd) integer multiple of π is 1 (or -1).

$$\int_{\pi}^{2\pi} x^2\cos nx\,\mathrm{d}x = \frac{1}{n^3}\left[n^2x^2\sin nx + 2nx\cos nx - 2\sin nx\right]_{\pi}^{2\pi} \stackrel{1}{=} \frac{1}{n^3}\left[2nx\cos nx\right]_{\pi}^{2\pi}$$

$$= \frac{2}{n^2}(2\pi\cdot 1 - \pi\cos n\pi) = \frac{2\pi}{n^2}(2 - \cos n\pi) = \begin{cases} \dfrac{2\pi}{n^2}(2-1) = 2\dfrac{\pi}{n^2}, & \text{for } n \text{ even,} \\ \dfrac{2\pi}{n^2}[2-(-1)] = 6\dfrac{\pi}{n^2}, & \text{for } n \text{ odd.} \end{cases}$$

So, $a = 2$ if n is even and $a = 6$ if n is odd.

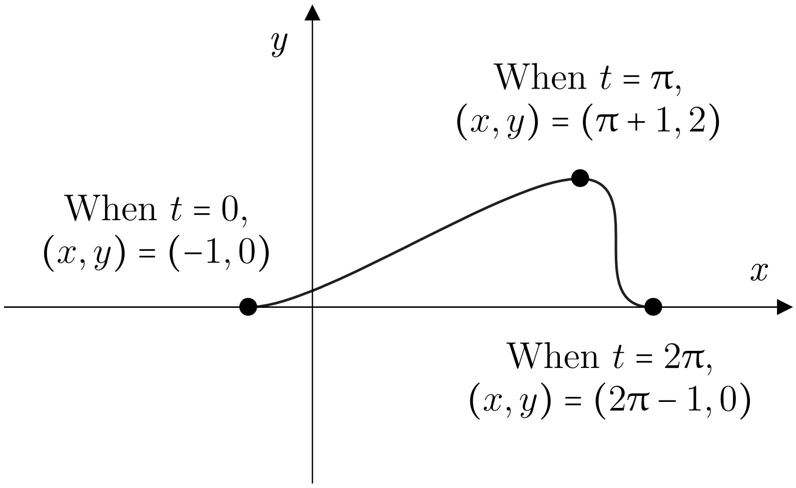
(b) From $u \stackrel{1}{=} 9 - x^2$, we have $\dfrac{\mathrm{d}u}{\mathrm{d}x} \stackrel{2}{=} -2x$. The volume of the solid is

$$\pi\int_0^2 y^2\,\mathrm{d}x = \pi\int_0^2\left(\frac{x\sqrt{x}}{9-x^2}\right)^2\mathrm{d}x = \pi\int_0^2\frac{x^3}{(9-x^2)^2}\,\mathrm{d}x \stackrel{1}{=} \pi\int_0^2\frac{9-u}{u^2}x^2\,\mathrm{d}x$$

$$\stackrel{2}{=} -\frac{\pi}{2}\int_{x=0}^{x=2}\frac{9-u}{u^2}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = -\frac{\pi}{2}\int_{u=9}^{u=5}\frac{9-u}{u^2}\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} \stackrel{1}{=} \frac{\pi}{2}\int_9^5\frac{u-9}{u^2}\,\mathrm{d}u = \frac{\pi}{2}\int_9^5\frac{1}{u} - \frac{9}{u^2}\,\mathrm{d}u$$

$$= \frac{\pi}{2}\left[\ln|u| + \frac{9}{u}\right]_9^5 = \frac{\pi}{2}\left(\ln 5 + \frac{9}{5} - \ln 9 - \frac{9}{9}\right) = \frac{\pi}{2}\left(\ln\frac{5}{9} + \frac{4}{5}\right) \approx 0.333.$$

A687 (9740 N2016/II/3)(i) If $y = 0$, then $\cos t = 1$, so $t = 0, 2\pi$ and $x = -1, 2\pi - 1$. So, the x-intercepts are $(-1, 0)$ and $(2\pi - 1, 0)$.



$y = 1 - \cos t$ is maximised at $\cos t = -1$ or $t = \pi$. So, the maximum point is $(\pi + 1, 2)$.

(ii) Compute $\dfrac{dx}{dt} \stackrel{1}{=} 1 + \sin t$. Now,

$$\int_{t=0}^{t=a} y\,dx = \int_{t=0}^{t=a} y \frac{dx}{dt}\frac{dt}{dx}\,dx \stackrel{1}{=} \int_{t=0}^{t=a} (1 - \cos t)(1 + \sin t)\frac{dt}{\cancel{dx}}\cancel{dt}$$

$$= \int_0^a 1 + \sin t - \cos t - \sin t \cos t\,dt = \int_0^a 1 + \sin t - \cos t - \frac{1}{2}\sin 2t\,dt$$

$$= \left[t - \cos t - \sin t + \frac{1}{4}\cos 2t\right]_0^a = a - \cos a - \sin a + \frac{1}{4}\cos 2a + \frac{3}{4}.$$

(iii) The given normal has gradient $-\left.\dfrac{dx}{dy}\right|_{t=\pi/2} = -\dfrac{dx}{dt} \div \left.\dfrac{dy}{dt}\right|_{t=\pi/2} = -\left.\dfrac{1 + \sin t}{\sin t}\right|_{t=\pi/2} = -2.$

So, its equation is

$$y - \left(1 - \cos\frac{\pi}{2}\right) = -2\left[x - \left(\frac{\pi}{2} - \cos\frac{\pi}{2}\right)\right] \qquad \text{or} \qquad y = -2x + \pi + 1.$$

Hence, $E = \left(\dfrac{\pi + 1}{2}, 0\right)$, $F = (0, \pi + 1)$, and the area of $\triangle OEF$ is

$$\frac{1}{2}\frac{\pi + 1}{2}(\pi + 1) = \frac{(\pi + 1)^2}{4}.$$

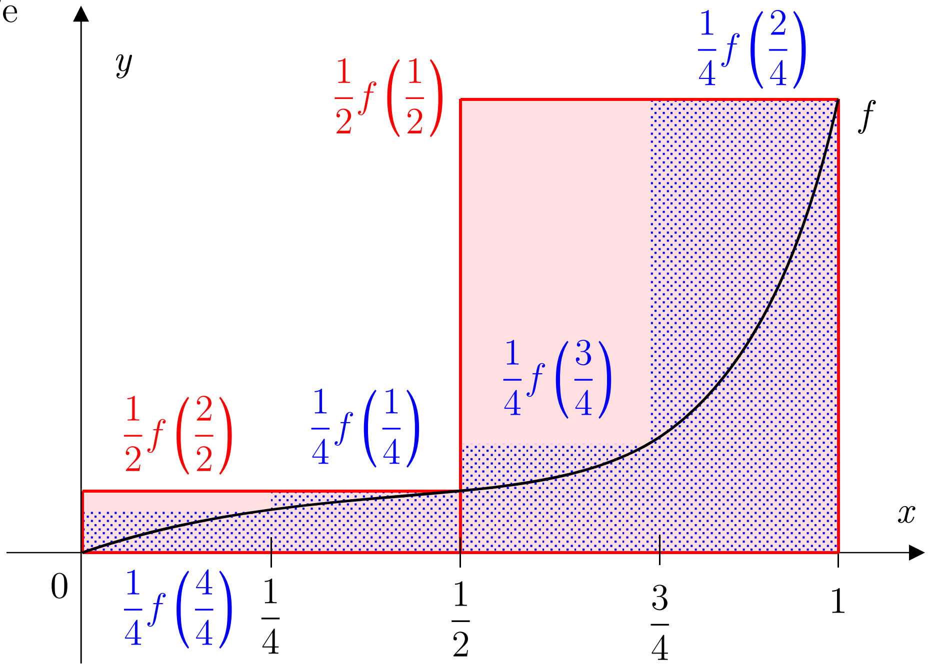
A688 (9740 N2015/I/3)(i) We have graphed some continuous function $f:[0,1]\to\mathbb{R}$.

The area bounded by f, the x-axis, and the vertical lines $x=0$ and $x=1$ is $\int_0^1 f(x)\,\mathrm{d}x$.

Consider the two large red rectangles. Their total area is

$$\frac{1}{2}\left[f\left(\frac{1}{2}\right)+f\left(\frac{2}{2}\right)\right],$$

which serves as a crude approximation of $\int_0^1 f(x)\,\mathrm{d}x$.



Next, consider the four smaller blue rectangles. Their total area is

$$\frac{1}{4}\left[f\left(\frac{1}{4}\right)+f\left(\frac{2}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{4}{4}\right)\right],$$

which serves as an improved approximation of A.

It is plausible then that as n grows (and the number of rectangles grow), the following expression serves as an ever-improved approximation of A:

$$\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right]$$

It is thus plausible that

$$\lim_{n\to\infty}\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right]\stackrel{1}{=}\int_0^1 f(x)\,\mathrm{d}x.$$

(Note: We haven't proven that $\stackrel{1}{=}$ is true. We have merely presented an informal argument for why it *might* be true. Which is all you need to know for A-Level Maths.)

(ii) Suppose $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\sqrt[3]{x}$. Then

$$\frac{1}{n}\left(\frac{\sqrt[3]{1}+\sqrt[3]{2}+\cdots+\sqrt[3]{n}}{\sqrt[3]{n}}\right)=\frac{1}{n}\left[\sqrt[3]{\frac{1}{n}}+\sqrt[3]{\frac{2}{n}}+\cdots+\sqrt[3]{\frac{n}{n}}\right]=\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right].$$

Hence, by **(i)**, $$\frac{1}{n}\left(\frac{\sqrt[3]{1}+\sqrt[3]{2}+\cdots+\sqrt[3]{n}}{\sqrt[3]{n}}\right)=\int_0^1 f(x)\,\mathrm{d}x=\int_0^1\sqrt[3]{x}\,\mathrm{d}x=\frac{3}{4}\left[x^{\frac{4}{3}}\right]_0^1=\frac{3}{4}.$$

A689 (9740 N2015/I/4). The rectangle's perimeter is $2(x+y)$. The semicircle's is $2x+\pi x$. Adding these up, we have

$$d = 2(x+y) + 2x + \pi x = (4+\pi)x + 2y \qquad \text{or} \qquad y \stackrel{1}{=} \frac{d}{2} - \left(2+\frac{\pi}{2}\right)x.$$

The rectangle's area is xy. The semicircle's is $\frac{\pi}{2}x^2$. So, the total area is

$$xy + \frac{\pi}{2}x^2 \stackrel{1}{=} x\left[\frac{d}{2} - \left(2+\frac{\pi}{2}\right)x\right] + \frac{\pi}{2}x^2 = -2x^2 + \frac{d}{2}x.$$

This is a ∩-shaped quadratic polynomial that is maximised at $x = -\frac{\text{`}b\text{'}}{2\text{`}a\text{'}} = -\frac{d/2}{2(-2)} = \frac{d}{8}$.

So, the maximised area is $-2\left(\frac{d}{8}\right)^2 + \frac{d}{2}\left(\frac{d}{8}\right) = -\frac{d^2}{32} + \frac{d^2}{16} = \frac{d^2}{32}$ (and $k = \frac{1}{32}$).

A690 (9740 N2015/I/6)(i) For $2x \in (-1,1]$ or $x \in (-0.5, 0.5]$,

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \cdots = 2x - 2x^2 + \frac{8}{3}x^3 - \ldots$$

(ii) For $|bx| < 1$,

$$\begin{aligned} ax(1+bx)^c &= ax\left[1 + c(bx) + \frac{c(c-1)(bx)^2}{2!} + \frac{c(c-1)(c-2)(bx)^3}{3!} + \ldots\right] \\ &= ax + abcx^2 + \frac{1}{2}ab^2c(c-1)x^3 + \frac{1}{6}ab^3c(c-1)(c-2)x^4 + \ldots \end{aligned}$$

Comparing coefficients, $a \stackrel{1}{=} 2$, $abc \stackrel{2}{=} -2$, $\frac{1}{2}ab^2c(c-1) \stackrel{3}{=} \frac{8}{3}$.

Plug $\stackrel{1}{=}$ into $\stackrel{2}{=}$ to get $b \stackrel{4}{=} -1/c$. Then plug $\stackrel{1}{=}$ and $\stackrel{4}{=}$ into $\stackrel{3}{=}$ to get

$$\frac{c-1}{c} = \frac{8}{3} \qquad \Longleftrightarrow \qquad 1 - \frac{1}{c} = \frac{8}{3} \qquad \Longleftrightarrow \qquad c = -\frac{3}{5}.$$

And now from $\stackrel{4}{=}$, we also have $b = \frac{5}{3}$. So, the coefficient of x^4 is

$$\frac{1}{6}ab^3c(c-1)(c-2) = \frac{1}{6}\cdot 2\left(\frac{\cancel{5}}{3}\right)^3\left(-\frac{3}{\cancel{5}}\right)\left(-\frac{8}{\cancel{5}}\right)\left(-\frac{13}{\cancel{5}}\right) = \frac{1}{\cancel{3}}\left(\frac{1}{3}\right)^3(-\cancel{3})(-8)(-13) = -\frac{104}{27}.$$

A691 (9740 N2015/I/10)(i) $A_1 + A_2 = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} \stackrel{1}{=} 1.$

By symmetry, the area under $y = \cos x$ from O to P is

$$A_2 + \frac{A_1}{2} = \int_0^{\pi/4} \cos x \, dx = [\sin x]_0^{\pi/4} \stackrel{2}{=} \frac{\sqrt{2}}{2}.$$

Taking $\stackrel{1}{=} - \stackrel{2}{=}$ yields

$$\frac{A_1}{2} = 1 - \frac{\sqrt{2}}{2} \qquad \text{or} \qquad \text{or } A_1 \stackrel{3}{=} 2 - \sqrt{2}.$$

Now plug $\stackrel{3}{=}$ into $\stackrel{1}{=}$ to get $A_2 = 1 - A_1 = \sqrt{2} - 1$. So,

$$\frac{A_1}{A_2} = \frac{2-\sqrt{2}}{\sqrt{2}-1} = \frac{2-\sqrt{2}}{\sqrt{2}-1}\frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{2\sqrt{2}-2+2-\sqrt{2}}{2-1} = \sqrt{2}.$$

(ii) The volume of the solid is $\pi \int_0^{\sqrt{2}/2} x^2 \, dy = \pi \int_0^{\sqrt{2}/2} \left(\sin^{-1} y\right)^2 dy.$

(iii) From $y = \sin u$, taking $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have $u = \sin^{-1} y$. Also, $\frac{dy}{du} \stackrel{2}{=} \cos u$. So,

$$\pi \int_0^{\sqrt{2}/2} \left(\sin^{-1} y\right)^2 dy \stackrel{1}{=} \pi \int_0^{\sqrt{2}/2} u^2 \, dy = \pi \int_{y=0}^{y=\sqrt{2}/2} u^2 \frac{dy}{du}\frac{du}{\cancel{dy}} \cancel{dy} \stackrel{2}{=} \pi \int_{u=0}^{u=\pi/4} u^2 \cos u \, du$$

$$= \pi \left[u^2 \sin u - \int 2u \sin u \, du\right]_0^{\pi/4} = \pi \left[u^2 \sin u + 2u \cos u - \int 2 \cos u \, du\right]_0^{\pi/4}$$

$$= \pi \left[u^2 \sin u + 2u \cos u - 2 \sin u\right]_0^{\pi/4} = \pi \left[\left(\frac{\pi^2}{16}\frac{\sqrt{2}}{2} + \frac{\pi}{2}\frac{\sqrt{2}}{2} - \sqrt{2}\right) - 0\right] = \pi \frac{\sqrt{2}}{2}\left(\frac{\pi^2}{16} + \frac{\pi}{2} - 2\right).$$

(So, $a = 0$ and $b = \frac{\pi}{4}$.)

A692 (9740 N2015/I/11)(i) For $\theta \neq 0, \pi/2$ or $\theta \in (0, \pi/2)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \div \frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{6\sin\theta\cos^2\theta - 3\sin^3\theta}{3\sin^2\theta\cos\theta} = \frac{2\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = 2\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = 2\cot\theta - \tan\theta.$$

(ii) Stationary point $\iff 0 = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\bar\theta} = 2\cot\bar\theta - \tan\bar\theta \iff 2 = \tan^2\bar\theta \iff \tan\bar\theta = \pm\sqrt{2}$.

Since $\bar\theta \in (0, \pi/2)$, $\tan\bar\theta > 0$. So, we can discard $\tan\bar\theta = -\sqrt{2}$ and conclude that the sole stationary point is at $\tan\bar\theta \stackrel{1}{=} \sqrt{2}$.

At this stationary point, $\sin\bar\theta = \sqrt{2/3}$ and $\cos\bar\theta = \sqrt{1/3}$. So,

$$(x, y) = \left(\sqrt{\frac{2}{3}}^3, 3 \cdot \frac{2}{3} \cdot \sqrt{\frac{1}{3}}\right) = \left(\left(\frac{2}{3}\right)^{3/2}, \frac{2}{\sqrt{3}}\right).$$

Next, compute $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}(2\cot\theta - \tan\theta) = \left(-2\operatorname{cosec}^2\theta - \sec^2\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{-2\operatorname{cosec}^2\theta - \sec^2\theta}{3\sin^2\theta\cos\theta}$.

For all $\theta \in (0, \pi/2)$, this second derivative is negative (because it has negative numerator and positive denominator). Hence, $\bar\theta$ in particular is a strict local maximum[677] and hence also a turning point $(k = 2)$.

(iii) Since $\theta \in [0, \pi/2]$, we have $y = 3\sin^2\theta\cos\theta \geq 0$. So, C is entirely above the x-axis. Moreover, $x = \sin^3\theta$ is strictly increasing in θ. Hence, the requested area is simply

$$\begin{aligned}\int_{\theta=0}^{\theta=\pi/2} y\,\mathrm{d}x &= \int_{\theta=0}^{\theta=\pi/2} 3\sin^2\theta\cos\theta\,\mathrm{d}x = \int_{\theta=0}^{\theta=\pi/2} 3\sin^2\theta\cos\theta\frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} \\ &= \int_{\theta=0}^{\theta=\pi/2} 3\sin^2\theta\cos\theta\left(3\sin^2\theta\cos\theta\right)\mathrm{d}\theta = \int_0^{\pi/2} 9\sin^4\theta\cos^2\theta\,\mathrm{d}\theta \approx 0.884.\end{aligned}$$

(iv) Plug C's equations into the line's equation to get their intersection points:

$$y = ax \iff 3\sin^2\theta\cos\theta = a\sin^3\theta \iff \sin^2\theta\,(3\cos\theta - a\sin\theta) = 0.$$

So, they intersect at $\sin\theta = 0$ (the origin) and $3\cos\theta - a\sin\theta = 0$ or $3/a \stackrel{1}{=} \tan\theta$ (the point P).

If the line passes through the maximum point of C, then P must be the maximum point of C.

From **(ii)**, the maximum point of C is at $\tan\bar\theta = \sqrt{2}$—plug this into $\stackrel{1}{=}$ to get

$$\frac{3}{a} = \sqrt{2} \qquad \text{or} \qquad a = \frac{3}{\sqrt{2}}.$$

[677] Proposition 11 (Second Derivative Test).

A693 (9740 N2015/II/1)(i) The maximum height is attained when

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0 \qquad \text{or} \qquad \frac{1}{10}\sqrt{16 - \frac{1}{2}h} = 0 \qquad \text{or} \qquad h = 32.$$

(ii) For $h < 32$, $\dfrac{\mathrm{d}t}{\mathrm{d}h} \overset{1}{=} \dfrac{10}{\sqrt{16 - \frac{1}{2}h}}$. So,

$$t = \int \frac{\mathrm{d}t}{\mathrm{d}h}\,\mathrm{d}h = \int \frac{10}{\sqrt{16 - \frac{1}{2}h}}\,\mathrm{d}h = -40\sqrt{16 - \frac{1}{2}h} + C = -20\sqrt{64 - 2h} + C.$$

Plug in the initial condition $(h, t) = (0, 0)$ to get $0 = -20\sqrt{64} + C$ or $C = 160$.

Hence, $t = -20\sqrt{64 - 2h} + 160$ and

$$t\big|_{h=16} = -20\sqrt{64 - 2 \cdot 16} + 160 = -20\sqrt{32} + 160 = 160 - 80\sqrt{2}.$$

A694 (9740 N2014/I/2). Apply $\dfrac{\mathrm{d}}{\mathrm{d}x}$ to $x^2y + xy^2 + 54 \overset{1}{=} 0$ to get

$$2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + x \cdot 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Plug in $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -1$ to get

$$0 = 2xy - x^2 + y^2 - 2xy = y^2 - x^2 \qquad \text{or} \qquad y^2 = x^2 \qquad \text{or} \qquad y \overset{2}{=} \pm x.$$

Plug $\overset{2}{=}$ into $\overset{1}{=}$ to get $\pm x^3 + x^3 + 54 \overset{3}{=} 0$.

Since $54 \neq 0$, we can discard the negative value in $\overset{2}{=}$ and conclude that $y = x$.

Now from $\overset{3}{=}$, we have $2x^3 + 54 = 0$ or $x = -3$.

Hence, the unique point on C at which the gradient is 1 is $(-3, -3)$.

A695 (9740 N2014/I/7)(i) $\alpha \approx 1.885$ (calculator).

$$f(x) = -7 \quad \Longleftrightarrow \quad x^6 - 3x^4 - 7 = -7 \quad \Longleftrightarrow \quad x^4\left(x^2 - 3\right) = 0 \quad \Longleftrightarrow \quad x = 0, \pm\sqrt{3}.$$

Since $\beta > 0$, we must have $\beta = \sqrt{3}$.

(ii) $\displaystyle\int_\beta^\alpha f(x)\,\mathrm{d}x = \int_\beta^\alpha x^6 - 3x^4 - 7\,\mathrm{d}x = \left[\frac{x^7}{7} - \frac{3}{5}x^5 - 7x\right]_\alpha^\beta \approx -0.597.$

(iii) $\displaystyle\int_0^\beta f(x)\,\mathrm{d}x = \left[\frac{x^7}{7} - \frac{3}{5}x^5 - 7x\right]_0^{\sqrt{3}} = \frac{27}{7}\sqrt{3} - \frac{27}{5}\sqrt{3} - 7\sqrt{3}.$

So, the requested area is $\displaystyle -\left(\frac{27}{7}\sqrt{3} - \frac{27}{5}\sqrt{3} - 7\sqrt{3}\right) - 7\sqrt{3} = \frac{54}{35}\sqrt{3}.$

(iv) $f(-x) = (-x)^6 - 3(-x)^4 - 7 = x^6 - 3x^4 - 7 = f(x)$. ✓

If γ solves $f(x) = 0$, then so too does $-\gamma$.

So for example, since $\alpha \approx 1.885$ solves $f(x) = 0$, so too does $-\alpha \approx -1.885$.

Remark 223. The last question was strangely open-ended. I *think* the above answer should have sufficed for the full four marks. But of course, who can divine what was on the mind of those who wrote this question?

Here are two other things that could also have been "said":

1. Compute $f'(x) = 6x^5 - 12x^3 = 6x^3\left(x^2 - 2\right)$. So, for $x \geq 0$, $f'(x) > 0 \iff x > \sqrt{2}$. Hence, the only positive root is α and the only other real root is $-\alpha$, while the other four roots are complex.
2. These we can even find using only what we've learnt in H2 Maths.[678] Write

$$x^6 - 3x^4 - 7 = (x - \alpha)(x + \alpha)\left(x^4 + bx^2 + c\right) = \left(x^2 - \alpha^2\right)\left(x^4 + bx^2 + c\right).$$

Comparing coefficients on the x^4 and constant terms, we have $-3 = -\alpha^2 + b$ and $-7 = -\alpha^2 c$. And thus, $b = \alpha^2 - 3$ and $c = 7/\alpha^2$.

Now let $z = x^2$, so that $x^4 + bx^2 + c = z^2 + bz + c = 0$. By the quadratic formula,

$$z = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{3 - \alpha^2 \pm \sqrt{\alpha^4 - 6\alpha^2 + 9 - 28/\alpha^2}}{2}$$

So, the other four (complex) roots of $f(x) = 0$ are

$$x = \pm\sqrt{z} = \pm\sqrt{\frac{3 - \alpha^2 \pm \sqrt{\alpha^4 - 6\alpha^2 + 9 - 28/\alpha^2}}{2}}.$$

[678] In particular, without knowing how to solve cubic equations.

A696 (9740 N2014/I/8)(i) $\int f(x)\,\mathrm{d}x = \int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{3} + C$.[679]

(ii) $f(x) = \frac{1}{\sqrt{9-x^2}} = \frac{1}{3}\frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} = \frac{1}{3}\left[1-\left(\frac{x}{3}\right)^2\right]^{-1/2}$

$$= \frac{1}{3}\left[1 + \left(-\frac{1}{2}\right)\left[-\left(\frac{x}{3}\right)^2\right] + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left[-\left(\frac{x}{3}\right)^2\right]^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left[-\left(\frac{x}{3}\right)^2\right]^3}{3!} + \dots\right]$$

$$= \frac{1}{3} + \frac{1}{54}x^2 + \frac{1}{648}x^4 + \frac{5}{34\,992}x^6 + \dots$$

(iii) $\sin^{-1}\frac{x}{3} \stackrel{\text{(i)}}{=} \int f(x)\,\mathrm{d}x - C \stackrel{\text{(ii)}}{=} \int \frac{1}{3} + \frac{1}{54}x^2 + \frac{1}{648}x^4 + \frac{5}{34\,992}x^6 + \dots\,\mathrm{d}x - C$

$$= C_1 + \frac{1}{3}x + \frac{1}{162}x^3 + \frac{1}{3\,240}x^5 + \frac{5}{244\,944}x^7 + \dots.$$

Plug in $x = 0$ to find that $C_1 = 0$. So,

$$\sin^{-1}\frac{x}{3} = \frac{1}{3}x + \frac{1}{162}x^3 + \frac{1}{3\,240}x^5 + \frac{5}{244\,944}x^7 + \dots$$

Remark 224. As discussed in Ch. 104.7, the step taken at $\stackrel{\text{(ii)}}{=}$ requires justification.

[679] See List MF26, p. 4.

A697 (9740 N2014/I/10) $\dfrac{\mathrm{d}x}{\mathrm{d}t} \overset{1}{=} k\left(1+x-x^2\right)$, $\left(t, x, \dfrac{\mathrm{d}x}{\mathrm{d}t}\right) \overset{2}{=} \left(0, \dfrac{1}{2}, -\dfrac{1}{4}\right)$.

(i) Plug $\overset{2}{=}$ into $\overset{1}{=}$ to get $-\dfrac{1}{4} = k\left[1+\dfrac{1}{2}-\left(\dfrac{1}{2}\right)^2\right] = \dfrac{5}{4}k$ or $k = -\dfrac{1}{5}$.

(ii) $1+x-x^2 = \dfrac{5}{4}-\left(x-\dfrac{1}{2}\right)^2$. So,

$$t = \int \frac{1}{k\left(1+x-x^2\right)}\,\mathrm{d}x = \int \frac{1}{-\frac{1}{5}\left[\frac{5}{4}-\left(x-\frac{1}{2}\right)^2\right]}\,\mathrm{d}x = -5\int \frac{1}{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}\,\mathrm{d}x$$

$$= \frac{-5}{2\sqrt{5/4}}\ln\frac{\sqrt{5/4}+(x-1/2)}{\sqrt{5/4}-(x-1/2)}+C = \frac{\overset{\sqrt{5}}{\cancel{5}}}{\cancel{2}\sqrt{\cancel{5}/\cancel{4}}}\ln\frac{\sqrt{5/4}-(x-1/2)}{\sqrt{5/4}+(x-1/2)}+C = \sqrt{5}\ln\frac{\sqrt{5}-2x+1}{\sqrt{5}+2x-1}+C.$$

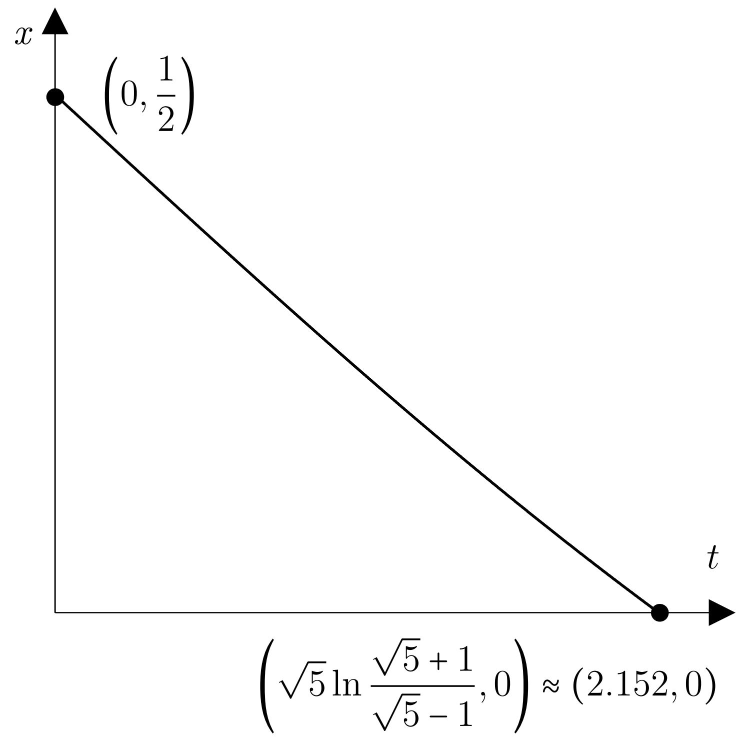
Plug in $\overset{2}{=}$ to get $0 = \sqrt{5}\ln 1 + C$ or $C = 0$. So, $t \overset{3}{=} \sqrt{5}\ln\dfrac{\sqrt{5}-2x+1}{\sqrt{5}+2x-1}$, for $x \in \left[0, \dfrac{1}{2}\right]$.

(iii)(a) $t\big|_{x=1/4} = \sqrt{5}\ln\dfrac{\sqrt{5}+1/2}{\sqrt{5}-1/2}$.[680]

(iii)(b) $t\big|_{x=0} = \sqrt{5}\ln\dfrac{\sqrt{5}+1}{\sqrt{5}-1} \approx 2.152$.

(iv) Take $\overset{3}{=}$ and rearrange: $\mathrm{e}^{t/\sqrt{5}} = \dfrac{\sqrt{5}-2x+1}{\sqrt{5}+2x-1} = -1+\dfrac{2\sqrt{5}}{\sqrt{5}+2x-1}$

$$\iff \frac{\sqrt{5}}{\mathrm{e}^{t/\sqrt{5}}+1} = \frac{\sqrt{5}+2x-1}{2} = x+\frac{\sqrt{5}-1}{2} \iff x = \frac{\sqrt{5}}{\mathrm{e}^{t/\sqrt{5}}+1}+\frac{1-\sqrt{5}}{2}, \text{ for } t \in \left[0, \sqrt{5}\ln\frac{\sqrt{5}+1}{\sqrt{5}-1}\right].$$



[680] We can further simplify: $\sqrt{5}\ln\dfrac{\sqrt{5}+1/2}{\sqrt{5}-1/2} = \sqrt{5}\ln\dfrac{2\sqrt{5}+1}{2\sqrt{5}-1} = \sqrt{5}\ln\left(\dfrac{2\sqrt{5}+1}{2\sqrt{5}-1}\dfrac{2\sqrt{5}+1}{2\sqrt{5}+1}\right) = \sqrt{5}\ln\dfrac{21+4\sqrt{5}}{19}$.

A698 (9740 N2014/I/11)(i) By Pythagoras' Theorem, $h = \sqrt{4^2 - r^2} = \sqrt{16 - r^2}$.

The hemisphere's volume is $2\pi r^3/3$. The cone's is $\pi r^2 h/3 = \pi r^2\sqrt{16-r^2}/3$. So, the toy's total volume is $V(r) = 2\pi r^3/3 + \pi r^2\sqrt{16-r^2}/3$, for $r \in [0,4]$. Next, compute

$$V'(r) = 2\pi r^2 + \frac{\pi}{3}\left(2r\sqrt{16-r^2} - \frac{r^3}{\sqrt{16-r^2}}\right) = \pi r\left[2r + \frac{1}{3}\left(2\sqrt{16-r^2} - \frac{r^2}{\sqrt{16-r^2}}\right)\right]$$

$$= \pi r\left[2r + \frac{1}{3}\frac{2(16-r^2)-r^2}{\sqrt{16-r^2}}\right] = \pi r\left(2r + \frac{32-3r^2}{3\sqrt{16-r^2}}\right) = \frac{\pi r}{3\sqrt{16-r^2}}\left(6r\sqrt{16-r^2} + 32 - 3r^2\right).$$

Any stationary points $\bar{r}$ are given by $V'(\bar{r}) = 0 \iff \bar{r}_a = 0$ OR $6\bar{r}_b\sqrt{16-\bar{r}_b^2} + 32 - 3\bar{r}_b^2 \overset{1}{=} 0$.

Since $V(\bar{r}_a) = V(0) = 0$, $\bar{r}_a$ does not maximise V. Now, from $\overset{1}{=}$,

$$6\bar{r}_b\sqrt{16-\bar{r}_b^2} = 3\bar{r}_b^2 - 32 \overset{\text{Square}}{\Longrightarrow} 36\bar{r}_b^2\left(16-\bar{r}_b^2\right) = 9\bar{r}_b^4 - 192\bar{r}_b^2 + 1\,024 \text{ or } 45\bar{r}_b^4 - 768\bar{r}_b^2 + 1\,024 \overset{2}{=} 0.$$

Making the dubious assumption that r_1 must be a stationary point of f, r_1 satisfies $\overset{2}{=}$.

(ii) Equation $\overset{2}{=}$ is a quadratic in $\bar{r}_b^2$. By the quadratic formula,

$$\bar{r}_b^2 = \frac{768 \pm \sqrt{768^2 - 4(45)(1\,024)}}{2(45)} = \frac{384 \pm \sqrt{101\,376}}{45} \approx 15.609, 1.458.$$

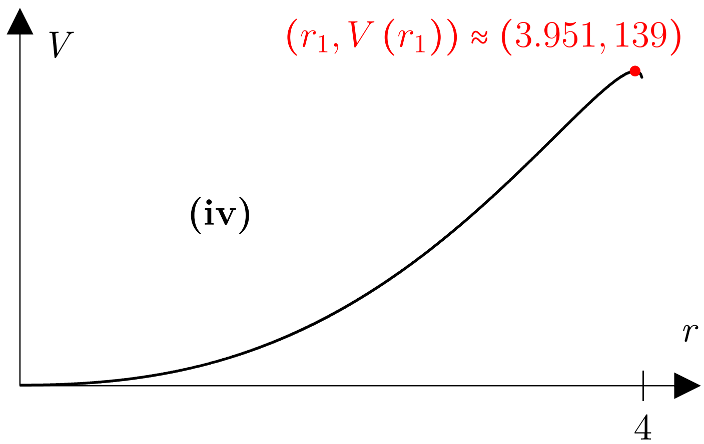
So, in $\overset{2}{=}$, the two positive solutions for $\bar{r}_b$ are $\sqrt{15.609} \approx 3.951$ and $\sqrt{1.458} \approx 1.207$.

(iii) We can verify that 3.951 satisfies $\overset{1}{=}$, while 1.207 does not.[681]

Hence, $r_1 \approx 3.951$ and

$h(r_1) = \sqrt{16 - r_1^2} \approx 0.625$.

(iv)



Remark 225. The above would probably have sufficed for full marks on the A-Level exam.

However, the above work is in fact incomplete (and therefore incorrect), because all we've shown is that r_1 is a stationary point of V. We have not shown that V attains a strict global maximum at $r_1 \approx 3.951$. Here's how we can show this:

Observe that $V(0) = 0$, $V(4) = 128\pi/3 \approx 134$, and $V(r_1) \approx V(3.951) \approx 139$. Altogether, r_1 is the unique stationary point of V in $(0,4)$ and $V(r_1)$ is greater than $V(0)$ and $V(4)$. Hence, by Proposition 10, V attains a strict global maximum at $r_1 \approx 3.951$.

[681] It was at $\overset{\text{Square}}{\Longrightarrow}$ that the *extraneous solution* 1.207 was introduced—see Ch. 42.1.

A699 (9740 N2014/II/2). (See List MF26, p. 2.)

$$\frac{9x^2+x-13}{(2x-5)(x^2+9)}=\frac{A}{2x-5}+\frac{Bx+C}{x^2+9}=\frac{(A+2B)x^2+(2C-5B)x+9A-5C}{(2x-5)(x^2+9)}.$$

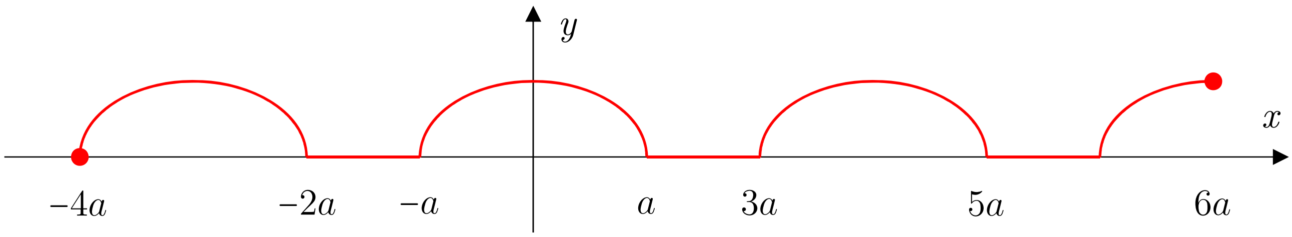
Comparing coefficients, $A+2B\overset{1}{=}9$, $2C-5B\overset{2}{=}1$, and $9A-5C\overset{3}{=}-13$.

Solving, $A=3$, $B=3$, $C=8$. Hence, $\displaystyle\int_0^2\frac{9x^2+x-13}{(2x-5)(x^2+9)}\,\mathrm{d}x$ equals

$$\int_0^2\frac{3}{2x-5}+\frac{3x}{x^2+9}+\frac{8}{x^2+9}\,\mathrm{d}x=\left[\frac{3}{2}\ln|2x-5|+\frac{3}{2}\ln\left(x^2+9\right)+\frac{8}{3}\tan^{-1}\frac{x}{3}\right]_0^2$$

$$=\frac{3}{2}(\ln 1-\ln 5)+\frac{3}{2}(\ln 13-\ln 9)+\frac{8}{3}\left(\tan^{-1}\frac{2}{3}-\tan^{-1}\frac{0}{3}\right)=\frac{3}{2}\ln\frac{13}{45}+\frac{8}{3}\tan^{-1}\frac{2}{3}.$$

A700 (9740 N2013/I/5)(i) On $[-a,a]$, we have $[f(x)]^2+(x/a)^2=1$, with $f(x)\geq 0$. So, this portion of the graph is a semi-ellipse with y-intercept 1 and x-intercepts $\pm a$.



We have an identical semi-ellipse on$[-4a,-2a]$ (which is $3a$ to the left of $[-a,a]$).

And again on $[2a,4a]$ (which is $3a$ to the right of $[-a,a]$).

And again on $[5a,7a]$ (which is $6a$ to the right of $[-a,a]$). But we're asked to graph only up to $x=6a$. So, we just have a quarter-ellipse on $[5a,6a]$.

For all other x, we have $y=0$.

(ii) From $x\overset{1}{=}a\sin\theta$, we have $\mathrm{d}x/\mathrm{d}\theta\overset{2}{=}a\cos\theta$. We specify also that $\theta\in[-\pi/2,\pi/2]$, so that $\sqrt{\cos^2\theta}\overset{3}{=}\cos\theta$. Note that both $a/2$ and $\sqrt{3}a/2$ (the lower and upper limits of integration) are in $[-a,a]$. Now,

$$\int_{a/2}^{\sqrt{3}a/2}f(x)\,\mathrm{d}x=\int_{a/2}^{\sqrt{3}a/2}\sqrt{1-\frac{x^2}{a^2}}\,\mathrm{d}x\overset{1}{=}\int_{x=a/2}^{x=\sqrt{3}a/2}\sqrt{1-\sin^2\theta}\frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x}$$

$$\overset{2}{=}\int_{\theta=\pi/6}^{\theta=\pi/3}\sqrt{1-\sin^2\theta}\,a\cos\theta\,\mathrm{d}\theta=\int_{\pi/6}^{\pi/3}\sqrt{\cos^2\theta}\,a\cos\theta\,\mathrm{d}\theta\overset{3}{=}a\int_{\pi/6}^{\pi/3}\cos^2\theta\,\mathrm{d}\theta$$

$$=a\int_{\pi/6}^{\pi/3}\frac{\cos 2\theta-1}{2}\,\mathrm{d}\theta=a\left[\frac{1}{4}\sin 2\theta-\frac{1}{2}\theta\right]_{\pi/6}^{\pi/3}=\frac{\pi}{12}a.$$

A701 (9740 N2013/I/10)(i) Since $z < 3/2$, $3 - 2z \overset{1}{>} 0$. So, $\dfrac{\mathrm{d}x}{\mathrm{d}z} = \dfrac{1}{3-2z}$ and

$$x = \int \frac{\mathrm{d}x}{\mathrm{d}z}\,\mathrm{d}z = \int \frac{1}{3-2z}\,\mathrm{d}z = -\frac{1}{2}\ln|3-2z| + C_1 \overset{1}{=} -\frac{1}{2}\ln(3-2z) + C_1,$$

or $\ln(3-2z) = 2(C_1 - x)$, or $3 - 2z = \mathrm{e}^{-2C_1}\mathrm{e}^{-2x} = C_2\mathrm{e}^{-2x}$, or $z = \dfrac{3}{2} - \dfrac{C_2}{2}\mathrm{e}^{-2x} = \dfrac{3}{2} + C_3\mathrm{e}^{-2x}$.

(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = z = \dfrac{3}{2} + C_3\mathrm{e}^{-2x}$. So,

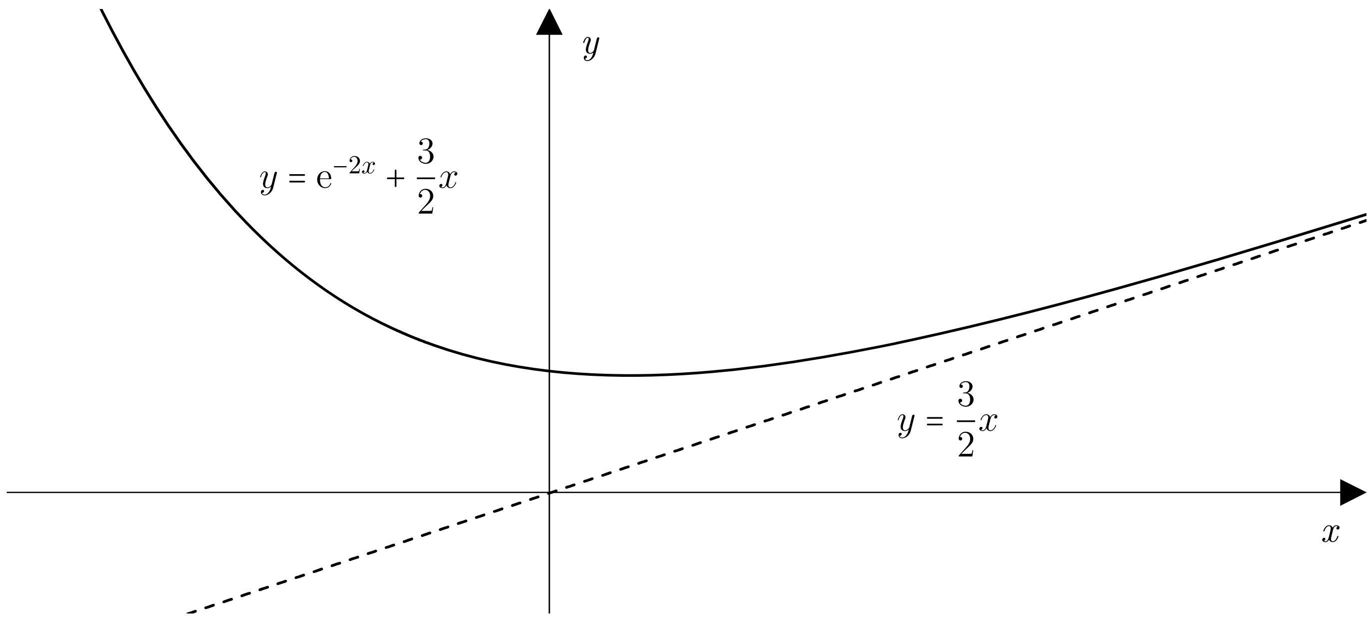
$$y = \int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{3}{2} + C_3\mathrm{e}^{-2x}\,\mathrm{d}x = \frac{3}{2}x + \frac{C_3}{-2}\mathrm{e}^{-2x} + C_4 \overset{2}{=} \frac{3}{2}x + C_5\mathrm{e}^{-2x} + C_4.$$

(iii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2C_3\mathrm{e}^{-2x} = -2\left(\dfrac{3}{2} + C_3\mathrm{e}^{-2x}\right) + 3 = -2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3$ $(a = -2,\ b = 3)$.

(iv) In $\overset{2}{=}$, set $C_5 = 0$, $C_4 = 0$ to get one line $y = \dfrac{3}{2}x$.

Set $C_5 = 0$, $C_4 = 1$ to get another line $y = \dfrac{3}{2}x + 1$.

Set $C_5 = 1$, $C_4 = 0$ to get $y = \mathrm{e}^{-2x} + \dfrac{3}{2}x$, a non-linear member of the family that has the line $y = \dfrac{3}{2}x$ as its asymptote.



A702 (9740 N2013/I/11)(i) Compute $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = 6t^2 \div (6t) = t$.

So the tangent has equation $y - 2t^3 = t\left(x - 3t^2\right)$ or $y = tx - t^3$.

(ii) The two tangent lines are $y \stackrel{1}{=} px - p^3$ and $y \stackrel{2}{=} qx - q^3$. Let $R = (x_R, y_R)$. Plug $\stackrel{1}{=}$ into $\stackrel{2}{=}$:

$$px_R - p^3 = qx_R - q^3 \iff (p-q)\,xx_R = p^3 - q^3 \stackrel{3}{\iff} x_R = \frac{p^3 - q^3}{p - q} = p^2 + pq + q^2.$$

(At $\stackrel{3}{\iff}$, we assume $p \neq q$ so that it's OK to divide by $p - q$.)
Next, $y_R = px_R - p^3 = p\left(p^2 + pq + q^2\right) - p^3 = p\left(pq + q^2\right)$.
Suppose $pq = -1$. Then R is on the curve $x = y^2 + 1$, as we now verify:

$$\begin{aligned} y_R^2 + 1 &= p^2\left(pq + q^2\right)^2 + 1 = p^2\left(p^2q^2 + 2pq^3 + q^4\right) + 1 = p^2\left(1 - 2q^2 + q^4\right) + 1 \\ &= p^2 - 2p^2q^2 + p^2q^4 + 1 = p^2 - 2 + q^2 + 1 = p^2 - 1 + q^2 = p^2 + pq + q^2 = x_R. \end{aligned}$$

(iii) Plug $x = 3t^2$ and $y = 2t^3$ into $x = y^2 + 1$ to get $3t^2 = 4t^6 + 1$ or $4t^6 - 3t^2 + 1 \stackrel{3}{=} 0$ at any point at which the curves C and L intersect and, in particular, at M.
Observe that $t^2 = -1$ solves $\stackrel{3}{=}$. So, write $4t^6 - 3t^2 + 1 = \left(t^2 + 1\right)\left(at^4 + bt^2 + c\right)$.
Comparing coefficients, $4 = a$, $-3 = a + b$ (so, $b = -4$), and $1 = c$. Hence,

$$4t^6 - 3t^2 + 1 = \left(t^2 + 1\right)\left(4t^4 - 4t^2 + 1\right) = \left(t^2 + 1\right)\left(2t^2 - 1\right)\left(2t^2 - 1\right).$$

The real solutions to $\stackrel{3}{=}$ are $2t^2 - 1 = 0$ or $t = \pm\sqrt{1/2}$. Since the y-coordinate of M is positive (and real) and $y = 2t^3$, we must have $t > 0$ at M. Thus,

$$M = \left(3\left(\sqrt{\frac{1}{2}}\right)^2, 2\left(\sqrt{\frac{1}{2}}\right)^3\right) = \left(\frac{3}{2}, \sqrt{\frac{1}{2}}\right).$$

(iv) In the given region, C and L may be described by $y = 2(x/3)^{1.5}$ and $y = \sqrt{x - 1}$.
The area under C, above the x-axis, and from 0 to M is

$$\int_0^{3/2} 2\left(\frac{x}{3}\right)^{1.5} \mathrm{d}x = \left[\frac{2}{3^{1.5} \cdot 2.5} x^{2.5}\right]_0^{3/2} = \frac{2}{3^{1.5} \cdot 2.5}\left(\frac{3}{2}\right)^{2.5} = \frac{3}{5\sqrt{2}} = \frac{3\sqrt{2}}{10}.$$

L intersects the x-axis at $x = 0$. The area under L, above the x-axis, and from 1 to M is

$$\int_1^{3/2} \sqrt{x - 1}\,\mathrm{d}x = \left[\frac{2}{3}(x - 1)^{1.5}\right]_1^{3/2} = \frac{2}{3}\frac{1}{2^{3/2}} = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}.$$

Hence, the requested area is $\frac{3\sqrt{2}}{10} - \frac{\sqrt{2}}{6} = \frac{2}{15}\sqrt{2}$.

A703 (9740 N2013/II/2). Figures reproduced, but with the points D, E, and F added:

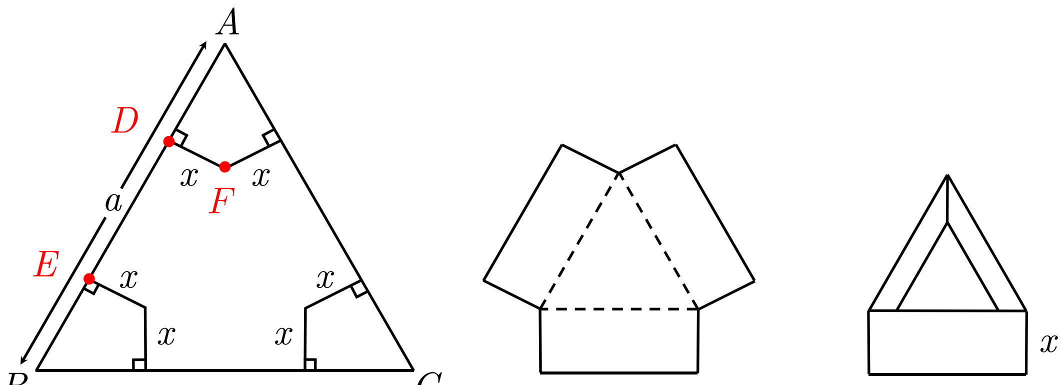


Fig. 1 **Fig. 2** **Fig. 3**

(i) Observe that $\angle AFD = \pi/3$. And $\tan \angle AFD = |AD| / x$. So, $|AD| = \sqrt{3}x$.

Hence, $|DE| = |AB| - (|AD| + |BE|) \overset{1}{=} a - 2\sqrt{3}x$.

The equilateral triangle in Fig. 2 has sides of length $|DE|$ each and hence area

$$\frac{1}{2}|DE|^2 \sin\frac{\pi}{3} = \frac{1}{2}\left(a - 2\sqrt{3}x\right)^2 \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}\left(a - 2\sqrt{3}x\right)^2.$$

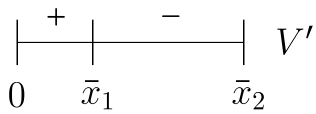
The prism has height x and thus volume $V(x) = \dfrac{\sqrt{3}}{4}\left(a - 2\sqrt{3}x\right)^2 x$.

(ii) Note that $|DE| \overset{1}{=} a - 2\sqrt{3}x \geq 0$, so $x \leq \dfrac{a}{2\sqrt{3}}$. Hence, $x \in \left[0, \dfrac{a}{2\sqrt{3}}\right]$.

Define the function $V : \left[0, \dfrac{a}{2\sqrt{3}}\right] \to \mathbb{R}$ by $V(x) = \dfrac{\sqrt{3}}{4}\left(a - 2\sqrt{3}x\right)^2 x$. Compute

$$V'(x) = \frac{\sqrt{3}}{4}\left[\left(a - 2\sqrt{3}x\right)^2 + 2x\left(a - 2\sqrt{3}x\right)\left(-2\sqrt{3}\right)\right] = \frac{\sqrt{3}}{4}\left(a - 2\sqrt{3}x\right)\left(a - 6\sqrt{3}x\right).$$

So, $V'(\bar{x}) = 0 \iff \bar{x}_1 = \dfrac{a}{2\sqrt{3}}$ OR $\bar{x}_2 = \dfrac{a}{6\sqrt{3}}$.



Since $V'(x) > 0$ for all $x \in [0, \bar{x}_2)$ and $V'(x) < 0$ for all $x \in \left(\bar{x}_2, \dfrac{a}{2\sqrt{3}}\right)$, we conclude that V attains its strict global maximum at $\bar{x}_2$, and

$$V(\bar{x}_2) = \frac{\sqrt{3}}{4}\left(a - 2\sqrt{3}\bar{x}_2\right)^2 \bar{x}_2 = \frac{\cancel{\sqrt{3}}}{4}\left(a - 2\cancel{\sqrt{3}}\frac{a}{6\cancel{\sqrt{3}}}\right)^2 \frac{a}{6\cancel{\sqrt{3}}} = \frac{a}{24}\left(a - \frac{a}{3}\right)^2 = \frac{a}{24}\left(\frac{2a}{3}\right)^2 = \frac{a^3}{54}.$$

A704 (9740 N2013/II/3)(i) $f(0) = \ln(1 + 2\sin 0) = \ln(1 + 0) = \ln 1 = 0.$

$$f'(x) = \frac{2\cos x}{1 + 2\sin x}, \qquad f''(x) = 2\left[\frac{-\sin x}{1 + 2\sin x} - \frac{2\cos^2 x}{(1 + 2\sin x)^2}\right]$$

$$f'''(x) = 2\left[\frac{-\cos x}{1 + 2\sin x} + \frac{2\sin x\cos x}{(1 + 2\sin x)^2} + \frac{4\cos x\sin x}{(1 + 2\sin x)^2} + 2\frac{4\cos^3 x}{(1 + 2\sin x)^3}\right]$$

So, $f'(0) = 2$, $f''(0) = 2(0 - 2) = -4$, and $f'''(0) = (-1 + 0 + 0 + 8) = 14$. Altogether,

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = 0 + 2x - 2x^2 + \frac{7}{3}x^3 + \ldots$$

(ii) $\mathrm{e}^{ax}\sin nx = \left[1 + ax + \frac{(ax)^2}{2!} + \ldots\right]\left[nx - \frac{(nx)^3}{3!} + \ldots\right] = nx + anx^2 + \left(\frac{a^2 n}{2} - \frac{n^3}{6}\right)x^3 + \ldots.$

Comparing coefficients, $n = 2$ and $an = -2$ (so, $a = -1$). Hence,

$$\left(\frac{a^2 n}{2} - \frac{n^3}{6}\right)x = \left[\frac{(-1)^2 \cdot 2}{2} - \frac{2^3}{6}\right]x = \left(1 - \frac{8}{6}\right)x^3 = -\frac{x^3}{3}$$

A705 (9740 N2012/I/2)(i) $\int \frac{x^3}{1 + x^4}\,\mathrm{d}x = \frac{1}{4}\ln\left(1 + x^4\right) + C.$[682]

(ii) From $u \overset{1}{=} x^2$, $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ or $x \overset{2}{=} \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x}$. Now,

$$\int \frac{x}{1 + x^4}\,\mathrm{d}x \overset{2}{=} \int \frac{1}{2}\frac{1}{1 + x^4}\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} \overset{1}{=} \frac{1}{2}\int \frac{1}{1 + u^2}\,\mathrm{d}u \overset{\star}{=} \frac{1}{2}\tan^{-1} u + C \overset{1}{=} \frac{1}{2}\tan^{-1} x^2 + C.$$

(Note that $\overset{\star}{=}$ is given on List MF26, p. 4.)

(iii) $\int_0^1 \left(\frac{x}{1 + x^4}\right)^2 \mathrm{d}x \approx 0.186$ (calculator).

[682]For $x \in \mathbb{R}$, $1 + x^4 > 0$, so we don't need the absolute value operator.

A706 (9740 N2012/I/4)(i) Observe that $\angle ACB = \pi/4 - \theta$. So,

$$\sin\angle ACB = \sin\left(\frac{\pi}{4}-\theta\right) = \sin\left(\frac{\pi}{4}\right)\cos\theta - \sin\theta\cos\left(\frac{\pi}{4}\right) \overset{1}{=} (\cos\theta - \sin\theta)\,\frac{\sqrt{2}}{2}.$$

By the Law of Sines, $\dfrac{|AC|}{\sin\angle ABC} = \dfrac{|AB|}{\sin\angle ACB}$. Rearranging, $|AC|$ equals

$$\frac{|AB|}{\sin\angle ACB}\sin\angle ABC = \frac{1}{\sin(\pi/4-\theta)}\sin\frac{3\pi}{4} \overset{1}{=} \sin\frac{1}{(\cos\theta-\sin\theta)\sqrt{2}/2.}\frac{\sqrt{2}}{2} = \frac{1}{\cos\theta-\sin\theta}.$$

(ii) By the small-angle approximations,[683]

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \qquad \text{and} \qquad \sin\theta \approx \theta.$$

So, by the Maclaurin series expansion for $(1+x)^n$,

$$\frac{1}{\cos\theta-\sin\theta} \approx \frac{1}{1-\frac{\theta^2}{2}-\theta} = 1 + (-1)\left(-\frac{\theta^2}{2}-\theta\right) + \frac{(-1)(-2)}{2}\left(-\frac{\theta^2}{2}-\theta\right)^2 + \cdots = 1 + \theta + \frac{3}{2}\theta^2 + \ldots$$

The above holds for $\left|-\theta^2/2-\theta\right| < 1$, which in turn holds since θ is "sufficiently small". And since θ is "sufficiently small", $\theta^k \approx 0$ for $k \geq 0$, so that

$$\frac{1}{\cos\theta-\sin\theta} \approx 1 + \theta + \frac{3}{2}\theta^2, \qquad \text{for } a = 1,\ b = \frac{3}{2}.$$

A707 (9740 N2012/I/8)(i) Apply $\dfrac{\mathrm{d}}{\mathrm{d}x}$ to $x - y = (x+y)^2$:

$$1 - \frac{\mathrm{d}y}{\mathrm{d}x} = 2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 2(x+y) + 2(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 2y + (2x+2y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Rearranging, $1 + \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 + \dfrac{1-2x-2y}{2x+2y+1} \overset{1}{=} \dfrac{2}{2x+2y+1}$ (for $2x+2y+1 \neq 0$).

(ii) Apply $\mathrm{d}/\mathrm{d}x$ to $\overset{1}{=}$ to get

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-2}{(2x+2y+1)^2}\left(2+2\frac{\mathrm{d}y}{\mathrm{d}x}\right) = -\frac{4}{(2x+2y+1)^2}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right) \overset{1}{=} -\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3.$$

(iii) By definition, the turning point is a stationary point and so occurs where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

But at any such point, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} \overset{\textbf{(ii)}}{=} -\left(1+\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^3 = -(1+0)^3 = -1 < 0$.

So, by the Second Derivative Test,[684] it is a maximum.

[683] In List MF26 (p. 2), read off the first few terms in the Maclaurin series expansions for sine and cosine.
[684] Proposition 11.

A708 (9740 N2012/I/10)(i) The model's volume is $k \overset{1}{=} \pi r^2 h + \dfrac{2}{3}\pi r^3$.

Rearranging, $h \overset{2}{=} \dfrac{k}{\pi r^2} - \dfrac{2}{3}r$. So, the model's external surface area is

$$A = \pi r^2 + 2\pi r h + 2\pi r^2 = \pi r\,(2h + 3r) \overset{2}{=} \pi r\left[2\left(\frac{k}{\pi r^2} - \frac{2}{3}r\right) + 3r\right] \overset{3}{=} \frac{2k}{r} + \frac{5}{3}\pi r^2.$$

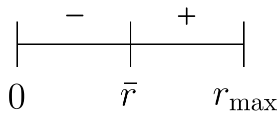
Compute $\dfrac{\mathrm{d}A}{\mathrm{d}r} = -\dfrac{2k}{r^2} + \dfrac{10}{3}\pi r = \dfrac{1}{r^2}\left(\dfrac{10}{3}\pi r^3 - 2k\right)$. Now look for any stationary points $\bar{r}$:

$$\left.\frac{\mathrm{d}A}{\mathrm{d}r}\right|_{r=\bar{r}} = -\frac{2k}{\bar{r}^2} + \frac{10}{3}\pi\bar{r} = 0 \qquad \text{or} \qquad \frac{10}{3}\pi\bar{r}^3 - 2k = 0 \qquad \text{or} \qquad \bar{r} = \left(\frac{3k}{5\pi}\right)^{1/3}.$$

So, this is the only stationary point.

The minimum value of r is, of course, 0. The maximum value $r_{\max}$ occurs where $h = 0$ (and can be expressed in terms of k using $\overset{1}{=}$).

Sign diagram for $\mathrm{d}A/\mathrm{d}r$ (assuming $k > 0$):



Since $\mathrm{d}A/\mathrm{d}r < 0$ for all $r \in [0, \bar{r})$ and $\mathrm{d}A/\mathrm{d}r > 0$ for all $r \in (\bar{r}, r_{\max}]$, A indeed attains its strict global minimum at $\bar{r}$. And the corresponding value of h is

$$\begin{aligned}\bar{h} &\overset{2}{=} \frac{k}{\pi\bar{r}^2} - \frac{2}{3}\bar{r} = \frac{k}{\pi}\left(\frac{3k}{5\pi}\right)^{-2/3} - \frac{2}{3}\left(\frac{3k}{5\pi}\right)^{1/3} \\ &= \left(\frac{3k}{5\pi}\right)^{1/3}\left[\frac{\cancel{k}}{\cancel{\pi}}\left(\frac{3\cancel{k}}{5\cancel{\pi}}\right)^{-1} - \frac{2}{3}\right] = \left(\frac{3k}{5\pi}\right)^{1/3}\left(\frac{5}{3} - \frac{2}{3}\right) = \left(\frac{3k}{5\pi}\right)^{1/3}.\end{aligned}$$

(ii) Plug $A = 180$ and $k = 200$ into $\overset{3}{=}$ to get

$$180 \overset{3}{=} \frac{2 \times 200}{r} + \frac{5}{3}\pi r^2 \qquad \text{or} \qquad \frac{5}{3}\pi r^3 - 180r + 400 \overset{4}{=} 0.$$

Solve $\overset{4}{=}$ using your calculator: $r \approx -6.76$ (reject), 3.04, or 3.72.

The two corresponding values of h are $h \overset{2}{=} \dfrac{200}{\pi r^2} - \dfrac{2}{3}r \approx 4.88, 2.12$.

Since $r < h$, we conclude $(r, h) \approx (3.04, 4.88)$.

A709 (9740 N2012/I/11)(i) For $\theta \neq 0, 2\pi$,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} \stackrel{1}{=} \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}.$$

The gradient of C at $\theta = \pi$ is $\left.\frac{dy}{dx}\right|_{\theta=\pi} = \cot\frac{\pi}{2} = 0$.

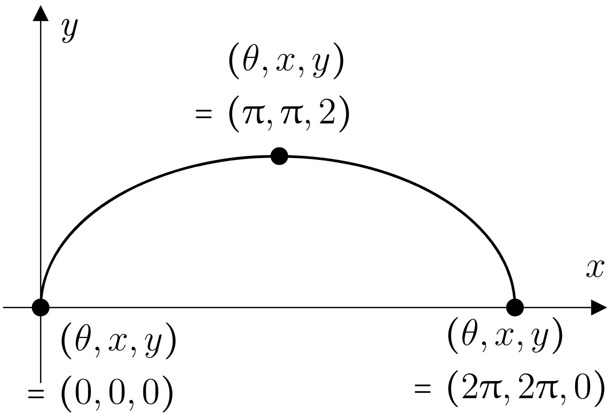
As $\theta \to 0$, $\frac{dy}{dx} \to \infty$; as $\theta \to 2\pi$, $\frac{dy}{dx} \to -\infty$. That is, the tangents approach verticality.

(ii) You're supposed to just copy from your calculator. But here as an exercise, we won't:

At the endpoints, $(\theta, x, y) = (0, 0, 0)$ and $(\theta, x, y) = (2\pi, 2\pi, 0)$, and the tangents are vertical. At the midpoint, $(\theta, x, y) = (\pi, \pi, 2)$ and the tangent is horizontal.

Between the endpoints, $dy/dx = \cot(\theta/2)$ which is positive for $\theta < \pi$ and negative for $\theta > \pi$.

Moreover, $d^2y/dx^2 = -\operatorname{cosec}^2(\theta/2)$ which is always negative (if defined). So, the gradient is decreasing.



(iii) We have $y \stackrel{2}{=} 1 - \cos\theta$ and $dx/d\theta \stackrel{3}{=} 1 - \cos\theta$. So, the area is

$$\int_0^{2\pi} y\,dx = \int_{x=0}^{x=2\pi} y\frac{dx}{d\theta}\frac{d\theta}{\cancel{dx}}\cancel{dx} = \int_{\theta=0}^{\theta=2\pi} y\frac{dx}{d\theta}\,d\theta$$

$$\stackrel{2,3}{=} \int_0^{2\pi} (1-\cos\theta)^2\,d\theta = \int_0^{2\pi} 1 + \cos^2\theta - 2\cos\theta\,d\theta$$

$$= \int_0^{2\pi} 1 + \frac{\cos 2\theta + 1}{2} - 2\cos\theta\,d\theta = \left[\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta - 2\sin\theta\right]_0^{2\pi} = 3\pi.$$

(iv) The normal has gradient $-1 \div \frac{dy}{dx} \stackrel{1}{=} \frac{\cos p - 1}{\sin p}$ and hence equation

$$y - (1 - \cos p) = \frac{\cos p - 1}{\sin p}\left[x - (p - \sin p)\right].$$

For the x-intercept, plug in $y = 0$ to get

$$\cancel{\cos p - 1} = \frac{\cancel{\cos p - 1}}{\sin p}\left[x - (p - \sin p)\right] \qquad \text{or} \qquad \sin p = x - p + \sin p \qquad \text{or} \qquad x = p.$$

A710 **(9740 N2012/II/1)(a)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\displaystyle\int\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\,\mathrm{d}x=\int 16-9x^2\,\mathrm{d}x=16x-3x^3+C_1.$

$$y=\int\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x=\int 16x-3x^3+C_1\,\mathrm{d}x=8x^2-\frac{3}{4}x^4+C_1x+C_2.$$

(b) For $16-9u^2\neq 0$ or $u\neq 4/3$, we have $\mathrm{d}t/\mathrm{d}u=1/\left(16-9u^2\right)$ and so

$$t=\int\frac{1}{16-9u^2}\,\mathrm{d}u=\frac{1}{9}\int\frac{1}{(4/3)^2-u^2}\,\mathrm{d}u=\frac{1}{9}\frac{1}{2\cdot(4/3)}\ln\left|\frac{4+3u}{4-3u}\right|+C=\frac{1}{24}\ln\left|\frac{4+3u}{4-3u}\right|+C.$$

Plug in the initial condition $(t,u)=(0,1)$ to find that $C=-(\ln 7)/24$. Hence,

$$t=\frac{1}{24}\left(\ln\left|\frac{4+3u}{4-3u}\right|-\ln 7\right)=\frac{1}{24}\ln\left|\frac{4+3u}{7(4-3u)}\right|,\qquad\text{for } u\neq\frac{4}{3}.$$

A711 **(9740 N2011/I/3)(i)** Compute $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}\div\dfrac{\mathrm{d}x}{\mathrm{d}t}=-\dfrac{2}{t^2}\div(2t)=-\dfrac{1}{t^3}.$

So the requested tangent is $y-\dfrac{2}{p}=-\dfrac{1}{p^3}\left(x-p^2\right)$ or $y=-\dfrac{x}{p^3}+\dfrac{3}{p}$.

(ii) If $y=0$, then $0=-x/p^3+3/p$ or $x=3p^2$, so $Q=\left(3p^2,0\right)$.

If $x=0$, then $y=3/p$. So, $R=(0,3/p)$.

(iii) The mid-point is $\left(1.5p^2,1.5/p\right)$ and has equation $xy^2=1.5p^2(1.5/p)^2=1.5^3$.

A712 **(9740 N2011/I/4)(i)** Plug in the Maclaurin expansion for cosine:

$$\cos^6 x=\left(1-\frac{x^2}{2}+\frac{x^4}{24}+\dots\right)^6=1+6\left(-\frac{x^2}{2}+\frac{x^4}{24}+\dots\right)+\frac{6\cdot 5}{2!}\left(-\frac{x^2}{2}+\frac{x^4}{24}+\dots\right)^2+\dots$$

$$=1-3x^2+\frac{1}{4}x^4+15\left(\frac{x^4}{4}\right)+\dots=1-3x^2+4x^4+\dots$$

(ii)(a) $\displaystyle\int_0^a\cos^6 x\,\mathrm{d}x\overset{1}{=}\int_0^a 1-3x^2+4x^4+\dots\,\mathrm{d}x=\left[x-x^3+\frac{4}{5}x^5+\dots\right]_0^a=a-a^3+\frac{4}{5}a^5+\dots.$

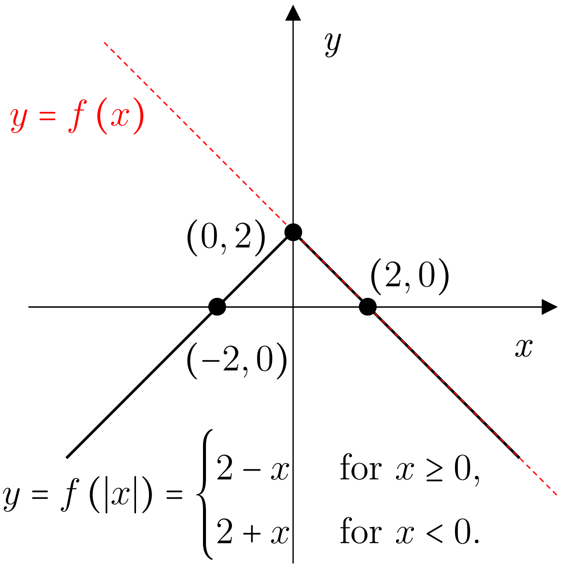
So, if $a=\dfrac{\pi}{4}$, then $\displaystyle\int_0^{\pi/4}\cos^6 x\,\mathrm{d}x\approx\frac{\pi}{4}-\left(\frac{\pi}{4}\right)^3+\frac{4}{5}\left(\frac{\pi}{4}\right)^5\approx 0.540.$

(b) By our calculator, $\displaystyle\int_0^{\pi/4}\cos^6 x\,\mathrm{d}x\approx 0.475.$

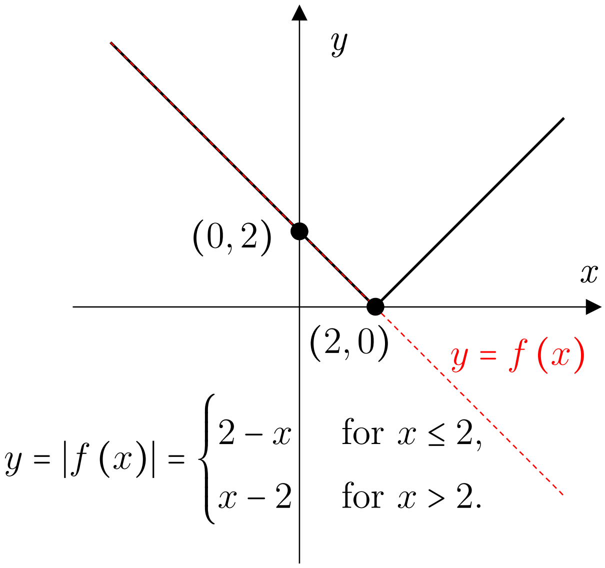
Using the first few terms of the Maclaurin series as an approximation would work well if $\pi/4$ were near 0. But it isn't and so this approximation doesn't work well.

Remark 226. As discussed in Ch. 104.7, the step taken at $\overset{1}{=}$ requires justification.

A713 **(9740 N2011/I/5)(i)** Where $x < 0$, $y = f(|x|)$ and f are reflections of each other in the y-axis. Where $x \geq 0$, they are identical.



$$y = f(|x|) = \begin{cases} 2 - x & \text{for } x \geq 0, \\ 2 + x & \text{for } x < 0. \end{cases}$$



$$y = |f(x)| = \begin{cases} 2 - x & \text{for } x \leq 2, \\ x - 2 & \text{for } x > 2. \end{cases}$$

Where $f(x) < 0$, $y = |f(x)|$ and f are reflections of each other in the x-axis. Where $f(x) \geq 0$, they are identical.

(ii) $x \in [0,2]$.

(iii) $\int_{-1}^{1} f(|x|)\,\mathrm{d}x = \int_{-1}^{0} 2 + x\,\mathrm{d}x + \int_{0}^{1} 2 - x\,\mathrm{d}x = \left[2x + \frac{x^2}{2}\right]_{-1}^{0} + \left[2x - \frac{x^2}{2}\right]_{0}^{1} \overset{1}{=} 3.$

For any $a \geq 2$, we have

$$\int_1^a |f(x)|\,\mathrm{d}x = \int_1^2 (2-x)\,\mathrm{d}x + \int_2^a (x-2)\,\mathrm{d}x = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^a$$

$$= \left(4 - 2 - 2 + \frac{1}{2}\right) + \left(\frac{a^2}{2} - 2a - 2 + 4\right) \overset{2}{=} \frac{a^2}{2} - 2a + \frac{5}{2}.$$

Set $\overset{1}{=}$ and $\overset{2}{=}$ to be equal: $\quad 3 = \frac{a^2}{2} - 2a + \frac{5}{2} \quad$ or $\quad a^2 - 4a - 1 = 0.$

By the quadratic formula, $\quad a = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)} = 2 \pm \sqrt{4+1} = 2 \pm \sqrt{5}.$

We can discard $a = 2 - \sqrt{5} < 2$. So, $a = 2 + \sqrt{5}$.

A714 (9740 N2011/I/8)(i) For $100 - v^2 \neq 0$ or $v \neq 10$,

$$\int \frac{1}{100 - v^2} \, dv = \frac{1}{20} \ln \left| \frac{10 + v}{10 - v} \right| + C_1.$$

(ii)(a) For $100 - v^2 \neq 0$ or $v \neq 10$, $\dfrac{dt}{dv} = \dfrac{1}{10 - 0.1v^2} = 10\dfrac{1}{100 - v^2}$.

So, $\quad t = \displaystyle\int \frac{dt}{dv} \, dv = \int 10 \frac{1}{100 - v^2} \, dv \overset{(i)}{=} \frac{1}{2} \ln \left| \frac{10 + v}{10 - v} \right| + C_2$ or $2(t - C_2) = \ln \left| \dfrac{10 + v}{10 - v} \right|$,

or, $\quad \left| \dfrac{10 + v}{10 - v} \right| = e^{2(t - C_2)} = e^{-2C_2} e^{2t} \quad$ or $\quad \dfrac{10 + v}{10 - v} = \pm e^{-2C_2} e^{2t} = C_3 e^{2t}$,

or, $\quad \dfrac{10 + v}{10 - v} = -1 + \dfrac{20}{10 - v} = C_3 e^{2t} \quad$ or $\quad v = 10 - \dfrac{20}{C_3 e^{2t} + 1}$.

Plug in the given initial condition $(t, v) = (0, 0)$ to get $0 = 10 - 20/(C_3 + 1)$ or $C_3 = 1$.

Hence, $v(t) = 10 - \dfrac{20}{e^{2t} + 1}$ $(t \geq 0)$ and

$$v = 5 \iff 10 - \frac{20}{e^{2t} + 1} = 5 \iff 5 = \frac{20}{e^{2t} + 1} \iff e^{2t} + 1 = 4 \iff e^{2t} = 3 \iff t = \frac{1}{2} \ln 3.$$

(ii)(b) $v(1) = 10 - \dfrac{20}{e^2 + 1}$. $\qquad$ **(ii)(c)** $\displaystyle\lim_{t \to \infty} v(t) = 10$.

A715 (9740 N2011/II/2)(i) The box has length $2(n - x)$, breadth $n - 2x$, height x, and hence volume $V = 2(n - x)(n - 2x)x = 2(n^2 - 3nx + 2x^2)x = 2n^2x - 6nx^2 + 4x^3$.

(ii) Compute $dV/dx = 2n^2 - 12nx + 12x^2$. Any stationary points $\bar{x}$ are given by

$$\left.\frac{dV}{dx}\right|_{x=\bar{x}} = 0 \qquad \text{or} \qquad 2n^2 - 12n\bar{x} + 12\bar{x}^2 = 0 \qquad \text{or} \qquad n^2 - 6n\bar{x} + 6\bar{x}^2 = 0.$$

By the quadratic formula,

$$\bar{x} = \frac{6n \pm \sqrt{(-6n)^2 - 4 \cdot 6n^2}}{2 \cdot 6} = \frac{3n \pm \sqrt{9n^2 - 6n^2}}{6} = \frac{1}{2}n \pm \frac{\sqrt{3}}{6}n = \frac{1}{2}\left(1 \pm \frac{\sqrt{3}}{3}\right)n.$$

We discard the larger value of $\bar{x}$ because it implies negative breadth:

$$n - 2 \cdot \frac{1}{2}\left(1 + \frac{\sqrt{3}}{3}\right)n = -\frac{\sqrt{3}}{3}n < 0.$$

Thus, the only stationary value of V is $\bar{x} = \left(1 - \dfrac{\sqrt{3}}{3}\right)\dfrac{n}{2}$.

A716 (9740 N2011/II/4)(a)(i) Use Integration by Parts twice:

$$\int_0^n x^2 \mathrm{e}^{-2x} \, \mathrm{d}x = \left[x^2 \left(\frac{-1}{2} \right) \mathrm{e}^{-2x} - \int 2x \left(\frac{-1}{2} \right) \mathrm{e}^{-2x} \, \mathrm{d}x \right]_0^n = \left[-\frac{1}{2} x^2 \mathrm{e}^{-2x} + \int x \mathrm{e}^{-2x} \, \mathrm{d}x \right]_0^n$$

$$= \left[-\frac{1}{2} x^2 \mathrm{e}^{-2x} + x \left(\frac{-1}{2} \right) \mathrm{e}^{-2x} - \int \left(\frac{-1}{2} \right) \mathrm{e}^{-2x} \, \mathrm{d}x \right]_0^n = \left[-\frac{1}{2} x^2 \mathrm{e}^{-2x} - \frac{1}{2} x \mathrm{e}^{-2x} - \frac{1}{4} \mathrm{e}^{-2x} \right]_0^n$$

$$= \left[-\frac{1}{4} \mathrm{e}^{-2x} \left(2x^2 + 2x + 1 \right) \right]_0^n = -\frac{1}{4} \mathrm{e}^{-2n} \left(2n^2 + 2n + 1 \right) + \frac{1}{4}.$$

(a)(ii) $\displaystyle\int_0^\infty x^2 \mathrm{e}^{-2x} \, \mathrm{d}x = \lim_{n \to \infty} \left[-\frac{1}{4} \mathrm{e}^{-2n} \left(2n^2 + 2n + 1 \right) + \frac{1}{4} \right] = \frac{1}{4}.$

(b) From $x \stackrel{1}{=} \tan \theta$, we have $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} \stackrel{2}{=} \sec^2 \theta$. The volume is

$$\pi \int_0^1 y^2 \, \mathrm{d}x = \pi \int_0^1 \left(\frac{4x}{x^2+1} \right)^2 \mathrm{d}x \stackrel{1}{=} \pi \int_0^1 \frac{16 \tan^2 \theta}{\left(\tan^2 \theta + 1 \right)^2} \mathrm{d}x = \pi \int_0^1 \frac{16 \tan^2 \theta}{\left(\sec^2 \theta \right)^2} \mathrm{d}x$$

$$\stackrel{2}{=} \pi \int_{x=0}^{x=1} \frac{16 \tan^2 \theta}{\sec^2 \theta} \frac{\mathrm{d}\theta}{\cancel{\mathrm{d}x}} \cancel{\mathrm{d}x} = \pi \int_{\theta=0}^{\theta=\pi/4} \frac{16 \tan^2 \theta}{\sec^2 \theta} \mathrm{d}\theta = 16\pi \int_0^{\pi/4} \sin^2 \theta \, \mathrm{d}\theta$$

$$= 8\pi \int_0^{\pi/4} 1 - \cos 2\theta \, \mathrm{d}\theta = 8\pi \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/4} = 2\pi \left(\pi - 2 \right).$$

A717 (9740 N2010/I/2)(i) $\mathrm{e}^x = 1 + x + \dfrac{1}{2} x^2 + \dots$ and $1 + \sin 2x = 1 + 2x + \dots$

So, $$\mathrm{e}^x \left(1 + \sin 2x \right) = \left(1 + x + \frac{1}{2} x^2 + \dots \right) \left(1 + 2x + \dots \right) = 1 + 3x + \frac{5}{2} x^2 + \dots$$

(ii) $$\left(1 + \frac{4x}{3} \right)^n = 1 + n \frac{4x}{3} + \frac{n(n-1)}{2} \left(\frac{4x}{3} \right)^2 + \dots = 1 + \frac{4n}{3} x + \frac{8n(n-1)}{9} x^2 + \dots$$

So, $3 = \dfrac{4n}{3}$ or $n = \dfrac{9}{4}$. And $\dfrac{8n(n-1)}{9} = \dfrac{8(9/4)(5/4)}{9} = \dfrac{5}{2}$. ✓

A718 (9740 N2010/I/4)(i) Apply $\dfrac{\mathrm{d}}{\mathrm{d}x}$ to $x^2 - y^2 + 2xy + 4 \stackrel{1}{=} 0$ to get[685]

$$2x - 2y \frac{\mathrm{d}y}{\mathrm{d}x} + 2y + 2x \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \iff 2(x-y) \frac{\mathrm{d}y}{\mathrm{d}x} = -2(x+y) \stackrel{2}{\iff} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{y-x}.$$

(ii) Tangent parallel to x-axis $\iff$ $\mathrm{d}y/\mathrm{d}x = 0$ $\iff$ $y \stackrel{3}{=} -x$.

Plug $\stackrel{3}{=}$ into $\stackrel{1}{=}$ to get $x^2 - x^2 - 2x^2 + 4 = 0$ or $x^2 = 2$ or $x = \pm\sqrt{2}$.

So, the two points are $\left(\mp\sqrt{2}, \pm\sqrt{2} \right)$.

[685] If $x = y$, then the given equation becomes $2x^2 + 4 = 0$ which has no real solutions. Hence, $x \neq y$ and at $\stackrel{2}{\iff}$, it's OK to divide by $y - x$.

A719 (9740 N2010/I/6)(i) $\beta \approx 0.347$, $\gamma \approx 1.532$ (calculator).

(ii) $\left|\int_\beta^\gamma x^3 - 3x + 1 \, dx\right| = \left[\frac{x^4}{4} - \frac{3}{2}x^2 + x\right]_\beta^\gamma \approx 0.781.$

(iii) The curve and line intersect at $x = -\sqrt{3}$. So, the shaded area is

$$\int_{-\sqrt{3}}^0 x^3 - 3x \, dx = \left[\frac{x^4}{4} - \frac{3}{2}x^2\right]_{-\sqrt{3}}^0 = -\frac{9}{4} + \frac{3}{2}\cdot 3 = \frac{9}{4}.$$

(iv) At the turning points ± 1, the corresponding values of y are

$$(-1)^3 - 3(-1) + 1 = 3 \qquad \text{and} \qquad 1^3 - 3\cdot 1 + 1 = -1.$$

The equation $x^3 - 3x + 1 = k$ has three distinct real roots if and only if k is strictly between the above two values. That is, $k \in (-1, 3)$.

A720 (9740 N2010/I/7)(i) The differential equation is $d\theta/dt \overset{1}{=} k(20-\theta)$.

Plug the initial condition $(t, \theta, \theta') \overset{2}{=} (0, 10, 1)$ into $\overset{1}{=}$ to get $1 = k(20-10)$ or $k = 0.1$.

So, $\frac{d\theta}{dt} = 0.1(20-\theta)$ If $\theta \neq 20$, then $\frac{dt}{d\theta} = \frac{10}{20-\theta}$. Hence,

$$t = \int \frac{dt}{d\theta} d\theta = \int \frac{10}{20-\theta} d\theta = -10\int \frac{1}{\theta - 20} d\theta = -10\ln|\theta - 20| + C_1,$$

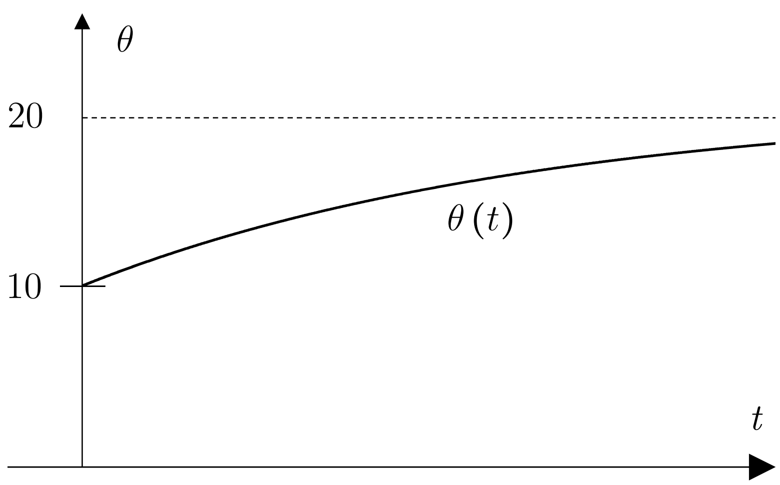
or, $\qquad 0.1(C_1 - t) = \ln|\theta - 20| \qquad$ or $\qquad |\theta - 20| = e^{0.1(C_1-t)} = e^{0.1C_1}e^{-0.1t}$,

or, $\qquad \theta - 20 = \pm e^{0.1C_1}e^{-0.1t} = C_2e^{-0.1t} \qquad$ or $\qquad \theta \overset{3}{=} 20 + C_2e^{-0.1t}$.

Plug $\overset{2}{=}$ into $\overset{3}{=}$ to get $10 = 20 + C_20$ or $C_2 = -10$. Hence, $\theta = 20 - 10e^{-0.1t}$.

(ii) $\theta = 15 \iff 15 = 20 - 10e^{-0.1t}$
$\iff e^{-0.1t} = 0.5$
$\iff -0.1t = \ln 0.5$
$\iff t = 10\ln 2.$

$$\lim_{t\to\infty}\theta(t) = \lim_{t\to\infty}\left(20 - 10e^{-0.1t}\right) = 20.$$



A721 (9740 N2010/I/9)(i) The box has volume $3x^2y = 300$. Rearranging,

$$xy \stackrel{1}{=} \frac{100}{x}.$$

The lid's external surface area is

$$2(kxy + 3xky) + 3x^2 = 8kxy + 3x^2 \stackrel{1}{=} k\frac{800}{x} + 3x^2.$$

The box's is $\qquad 2(xy + 3xy) + 3x^2 = 8xy + 3x^2 \stackrel{1}{=} \frac{800}{x} + 3x^2.$

So, the total external surface area is $\qquad A = \frac{800}{x}(1+k) + 6x^2.$

Compute $\frac{\mathrm{d}A}{\mathrm{d}x} = -\frac{800}{x^2}(1+k) + 12x = \frac{1}{x^2}\left[12x^3 - 800(1+k)\right].$

So, any stationary points $\bar{x}$ are given by

$$\left.\frac{\mathrm{d}A}{\mathrm{d}x}\right|_{x=\bar{x}} = 0 \qquad \text{or} \qquad 12\bar{x}^3 - 800(1+k) = 0 \qquad \text{or} \qquad \bar{x} = \left[\frac{200}{3}(1+k)\right]^{1/3}.$$

So, there is only one stationary point $\bar{x}$.

Since $\mathrm{d}A/\mathrm{d}x < 0$ for all $x \in (0, \bar{x})$ and $\mathrm{d}A/\mathrm{d}x > 0$ for all $x \in (\bar{x}, \infty)$, we conclude that A attains a strict global minimum at $\bar{x}$.

(ii) Let $\bar{y}$ be the value of y that corresponds to $\bar{x}$. Then $\frac{\bar{y}}{\bar{x}} \stackrel{1}{=} \frac{100}{\bar{x}^3} = \frac{3}{2(1+k)}$.

(iii) $k \in (0,1] \iff 1+k \in (1,2] \iff \frac{1}{(1+k)} \in \left[\frac{1}{2}, 1\right) \iff \frac{3}{2(1+k)} \in \left[\frac{3}{4}, \frac{3}{2}\right)$.

(iv) $x = y \iff \frac{3}{2(1+k)} = 1 \iff k = \frac{1}{2}$.

A722 (9740 N2010/I/11)(i) For $t \neq 0, 1$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1 + t^{-2}}{1 - t^{-2}} = \frac{t^2 + 1}{t^2 - 1}.$$

So, provided $p \neq 0, 1$, the tangent at point P has equation

$$y - \left(p - \frac{1}{p}\right) = \frac{p^2 + 1}{p^2 - 1}\left[x - \left(p + \frac{1}{p}\right)\right] \qquad \text{or} \qquad \left(p^2 - 1\right) y - \frac{\left(p^2 - 1\right)^2}{p} = \left(p^2 + 1\right)\left(x - \frac{p^2 + 1}{p}\right),$$

or,

$$\left(p^2 + 1\right) x - \left(p^2 - 1\right) y = \frac{\left(p^2 + 1\right)^2}{p} - \frac{\left(p^2 - 1\right)^2}{p} = \frac{4p^2}{p} \overset{1}{=} 4p.$$

(ii) Plug $y = x$ into $\overset{1}{=}$ to get $\left(p^2 + 1\right) x - \left(p^2 - 1\right) x = 4p$ or $x = 2p$. So, $A = (2p, 2p)$.

Plug $y = -x$ into $\overset{1}{=}$ to get $\left(p^2 + 1\right) x + \left(p^2 - 1\right) x = 4p$ or $x = \dfrac{2}{p}$. So, $B = \left(\dfrac{2}{p}, -\dfrac{2}{p}\right)$.

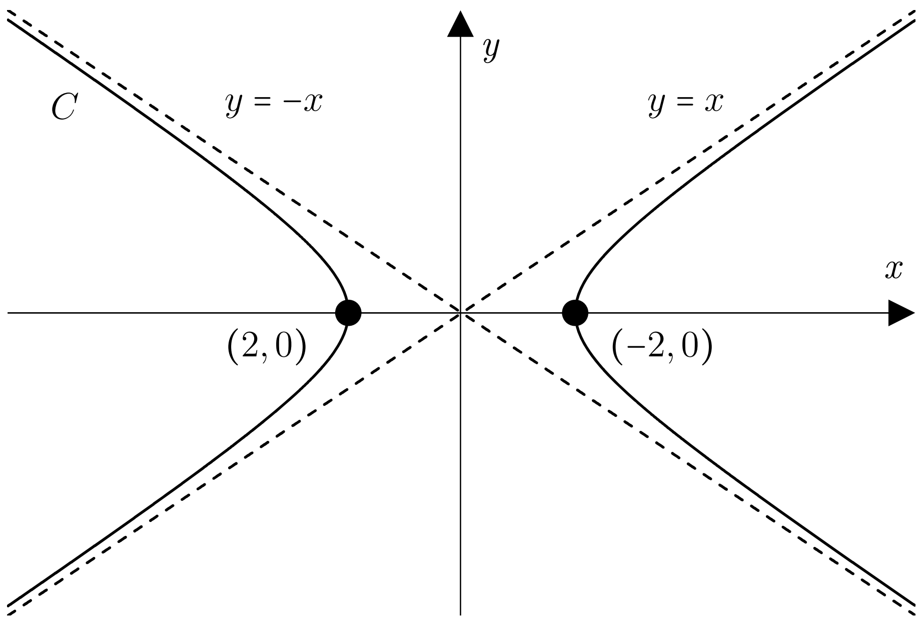
The lines $y = x$ and $y = -x$ are perpendicular and pass through the origin O. So, OA and OB are perpendicular. Hence, the area of $\triangle OAB$ is

$$\frac{1}{2}|OA||OB| = \frac{1}{2}\left|\sqrt{(2p)^2 + (2p)^2}\right|\left|\sqrt{\left(\frac{2}{p}\right)^2 + \left(-\frac{2}{p}\right)^2}\right| = \frac{1}{2}\sqrt{8p^2}\sqrt{\frac{8}{p^2}} = 4,$$

which is independent of p.

(iii) Observe that $x + y = 2t$ and $x - y = \dfrac{2}{t}$. So, $x^2 - y^2 = (x + y)(x - y) = 4$.

This is an east-west hyperbola with x-intercepts $(\pm 2, 0)$ and asymptotes $y = \pm x$.



A723 (9740 N2010/II/3)(i) $\dfrac{dy}{dx} = y'(x) = \sqrt{x+2} + \dfrac{x}{2\sqrt{x+2}} = \dfrac{2(x+2)+x}{2\sqrt{x+2}} = \dfrac{3x+4}{2\sqrt{x+2}}$.

Any stationary points $\bar{x}$ are given by $y'(\bar{x}) = 0$ or $3\bar{x}+4=0$ or $\bar{x} = -4/3$. So, this is the only stationary point (and hence also only possible turning point).

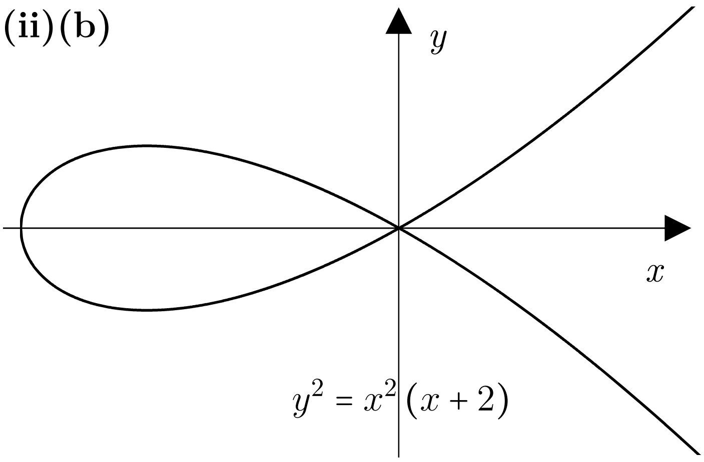
Since $y'(x) < 0$ for $x < \bar{x}$ and $y'(x) > 0$ for $x > \bar{x}$, $\bar{x}$ is a strict local minimum.[686]

Since $\bar{x}$ is both a stationary point and a strict extremum, it is a turning point.[687]

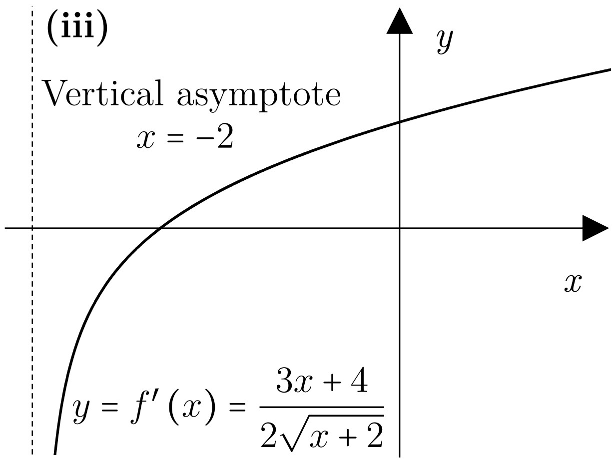
(ii)(a) $y^2 = x^2(x+2) \iff y \stackrel{1}{=} \pm x\sqrt{x+2}$.

From **(i)**, $\dfrac{dy}{dx} = y'(x) = \pm\dfrac{3x+4}{2\sqrt{x+2}}$. So, $y'(0) = \pm\dfrac{4}{2\sqrt{2}} = \pm\sqrt{2}$.

(ii)(b)



(iii)



A724 (9740 N2009/I/2). $\displaystyle\int_0^1 \frac{1}{4-x^2}\,dx = \left[\frac{1}{2\cdot 2}\ln\left|\frac{2+x}{2-x}\right|\right]_0^1 = \frac{1}{4}\left(\ln\frac{3}{1} - \ln\frac{2}{2}\right) = \frac{1}{4}\ln 3.$

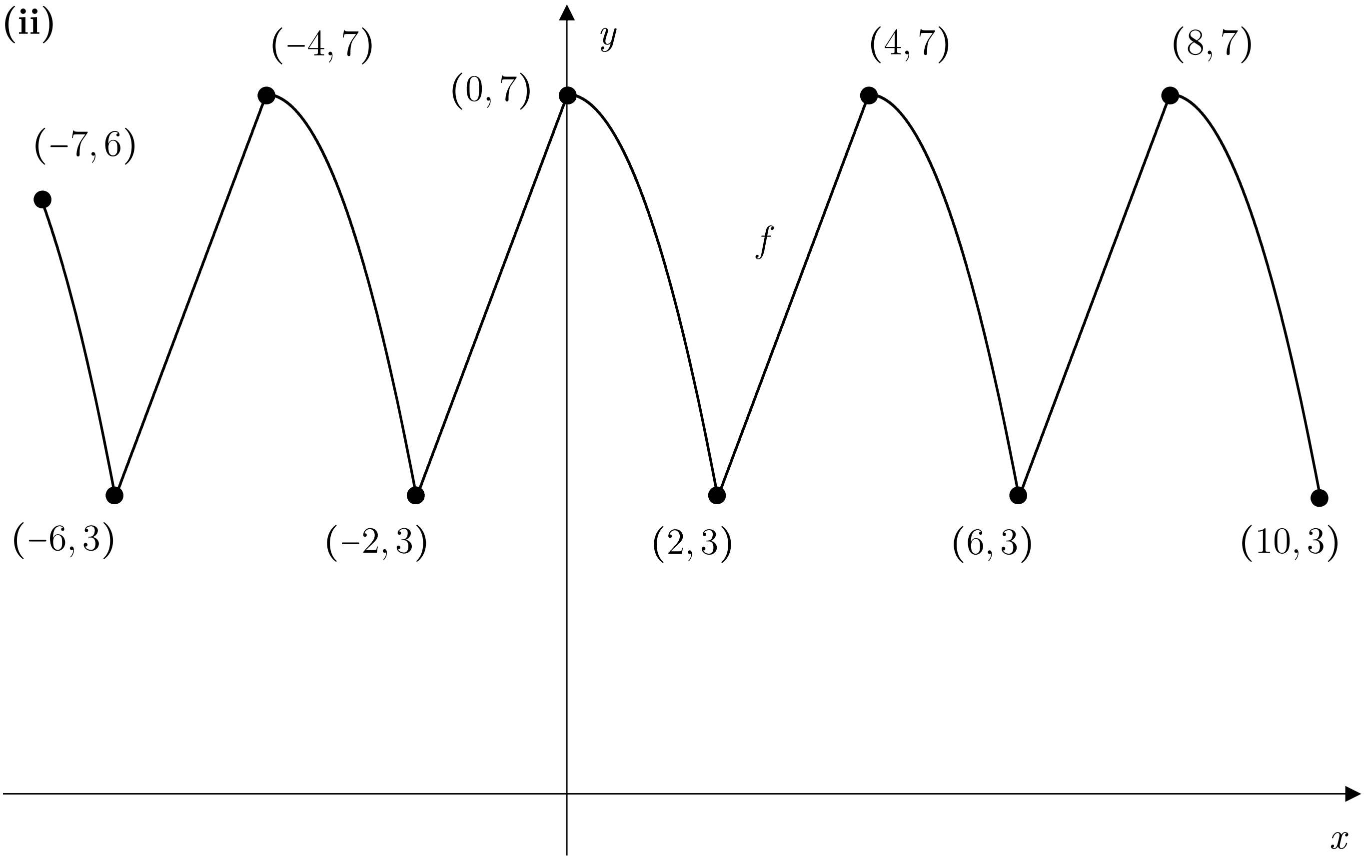
$$\int_0^{1/2p} \frac{1}{\sqrt{1-p^2x^2}}\,dx = \frac{1}{p}\int_0^{1/2p}\frac{1}{\sqrt{1/p^2-x^2}}\,dx = \frac{1}{p}\left[\sin^{-1}\frac{x}{1/p}\right]_0^{1/2p} = \frac{1}{p}\sin^{-1}\frac{1}{2} = \frac{\pi}{6p}.$$

So,
$$\frac{1}{4}\ln 3 = \frac{\pi}{6p} \iff p = \frac{4\pi}{6\ln 3} = \frac{2\pi}{3\ln 3}.$$

[686]See Proposition 9 (First Derivative Test for Extrema).
[687]See Definition ??.

A725 (9740 N2009/I/4)(i) $f(27)+f(45) = f(3)+f(1) = (2\cdot 3 - 1)+\left(7-1^2\right) = 5+6 = 11$.

(ii)



(iii) $\int_{-4}^{3} f(x)\,\mathrm{d}x = \int_{-4}^{-2} f(x)\,\mathrm{d}x + \int_{-2}^{0} f(x)\,\mathrm{d}x + \int_{0}^{2} f(x)\,\mathrm{d}x + \int_{2}^{3} f(x)\,\mathrm{d}x$

$$= \int_{0}^{2} f(x)\,\mathrm{d}x + \int_{2}^{4} f(x)\,\mathrm{d}x + \int_{0}^{2} f(x)\,\mathrm{d}x + \int_{2}^{3} f(x)\,\mathrm{d}x$$

$$= 2\int_{0}^{2} f(x)\,\mathrm{d}x + \int_{2}^{4} f(x)\,\mathrm{d}x + \int_{2}^{3} f(x)\,\mathrm{d}x$$

$$= 2\int_{0}^{2} 7 - x^2\,\mathrm{d}x + \int_{2}^{4} 2x-1\,\mathrm{d}x + \int_{2}^{3} 2x-1\,\mathrm{d}x$$

$$= 2\left[7x - \frac{1}{3}x^3\right]_0^2 + \left[x^2-x\right]_2^4 + \left[x^2-x\right]_2^3 = 22\frac{2}{3} + 12 - 2 + 6 - 2 = 36\frac{2}{3}.$$

A726 (9740 N2009/I/7)(i) $f'(x) = \mathrm{e}^{\cos x}(-\sin x)$, $f''(x) = \mathrm{e}^{\cos x}(-\sin x)^2 + \mathrm{e}^{\cos x}(-\cos x)$. So, $f(0) = \mathrm{e}^{\cos 0} = \mathrm{e}^1 = \mathrm{e}$, , $f'(0) = \mathrm{e}^{\cos 0}(-\sin 0) = 0$, and $f''(0) = 0 - \mathrm{e}^1\cdot 1 = -\mathrm{e}$. Hence,

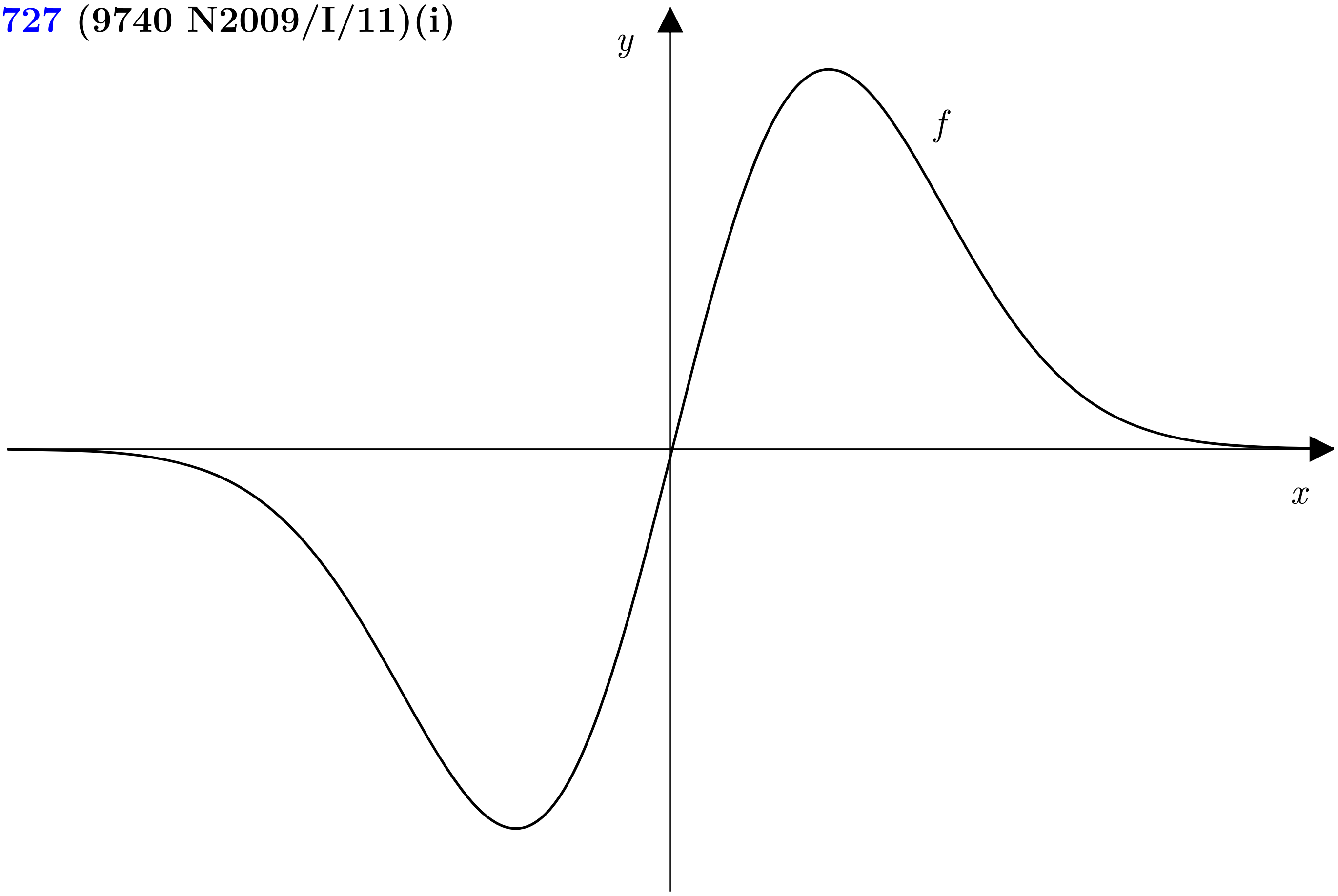
$$\mathrm{e}^{\cos x} = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots = \mathrm{e} + 0x + \frac{-\mathrm{e}}{2}x^2 + \cdots = \mathrm{e} - \frac{\mathrm{e}}{2}x^2 + \ldots$$

(ii) Let $g(x) = \dfrac{1}{a+bx^2}$. Then $g'(x) = -\dfrac{2bx}{\left(a+bx^2\right)^2}$ and $g''(x) = -\dfrac{2b}{\left(a+bx^2\right)^2} + \dfrac{8b^2x^2}{\left(a+bx^2\right)^3}$.

So, $g(0) = 1/a$, $g'(0) = 0$, and $g''(0) = -2b/a^2$.

Hence, $f(0) = g(0)$ or $a = 1/\mathrm{e}$; and $f''(0) = g''(0)$ or $-\mathrm{e} = -2b/a^2 = -2b\mathrm{e}^2$ or $b = 1/2\mathrm{e}$.

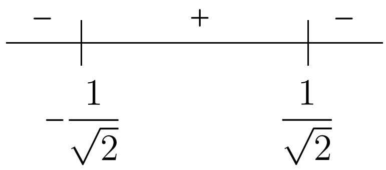
A727 (9740 N2009/I/11)(i)



(ii) $f'(x) = \mathrm{e}^{-x^2} + x\mathrm{e}^{-x^2}(-2x) = \mathrm{e}^{-x^2}\left(1 - 2x^2\right)$.

$f'(x) = 0 \iff x = \pm\dfrac{1}{\sqrt{2}}$. So, the two stationary points are $\left(\pm\dfrac{1}{\sqrt{2}}, \pm\dfrac{\mathrm{e}^{1/2}}{\sqrt{2}}\right)$.

Each is also a turning point because f' changes sign at each point:



(iii) From $u \overset{1}{=} x^2$, we have $\mathrm{d}u/\mathrm{d}x \overset{2}{=} 2x$. Now,

$$\int_0^n x\mathrm{e}^{-x^2}\,\mathrm{d}x \overset{1}{=} \int_0^n x\mathrm{e}^{-u}\,\mathrm{d}x \overset{2}{=} \frac{1}{2}\int_{x=0}^{x=n} \mathrm{e}^{-u}\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \frac{1}{2}\int_{u=0}^{u=n^2} \mathrm{e}^{-u}\,\mathrm{d}u = \frac{1}{2}\left[-\mathrm{e}^{-u}\right]_0^{n^2} = \frac{1}{2}\left(1 - \mathrm{e}^{-n^2}\right).$$

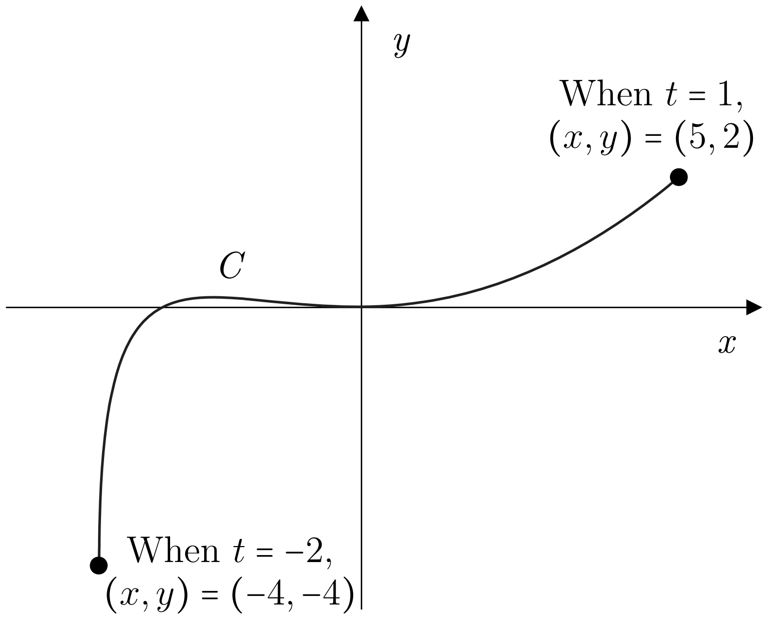
Hence, the requested area is $\lim_{n\to\infty}\int_0^n x\mathrm{e}^{-x^2}\,\mathrm{d}x = \lim_{n\to\infty}\frac{1}{2}\left(1 - \mathrm{e}^{-n^2}\right) = \frac{1}{2}$.

(iv) By symmetry, $\int_{-2}^{2}\left|x\mathrm{e}^{-x^2}\right|\mathrm{d}x = 2\int_0^2 x\mathrm{e}^{-x^2}\,\mathrm{d}x = 2\cdot\frac{1}{2}\left(1 - \mathrm{e}^{-n^2}\right)\Big|_{n=2} = 1 - \mathrm{e}^{-4}$.

(v) $\pi\int_0^1 y^2\,\mathrm{d}x = \pi\int_0^1\left(x\mathrm{e}^{-x^2}\right)^2\mathrm{d}x \approx 0.363$ (calculator).

A728 (9740 N2009/II/1)(i) Observations:

1. $x = t^2 + 4t = t(t+4)$. So, x is minimised at $t = -2$ and is strictly increasing for $t \in [-2, 1]$.
2. $y'(t) = 3t^2 + 2t = t(3t+2)$. So, y has turning points at $t = -2/3, 0$ (or $x = -20/9, 0$).



(ii) Compute $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{3t^2+2t}{2t+4}$. So, $\left.\dfrac{dy}{dx}\right|_{t=2} = \dfrac{3 \cdot 2^2 + 2 \cdot 2}{2 \cdot 2 + 4} = \dfrac{16}{8} = 2$. And l is

$$y - \left(2^3 + 2^2\right) = 2\left[x - \left(2^2 + 4 \cdot 2\right)\right] \qquad \text{or} \qquad y - 12 = 2x - 24 \qquad \text{or} \qquad y \overset{1}{=} 2x - 12.$$

(iii) Plug $x = t^2 + 4t$ and $y = t^3 + t^2$ into $\overset{1}{=}$ to get $t^3 + t^2 = 2\left(t^2 + 4t\right) - 12$ or $t^3 - t^2 - 8t + 12 \overset{2}{=} 0$. We already know that $t = 2$ solves $\overset{2}{=}$, because P is an intersection point. So, write

$$t^3 - t^2 - 8t + 12 = (t-2)\left(at^2 + bt + c\right).$$

Comparing coefficients, $a = 1$, $-2a + b = -1$ (so, $b = 1$), and $-2c = 12$ (so, $c = -6$). Hence,

$$t^3 - t^2 - 8t + 12 = (t-2)\left(t^2 + t - 6\right) = (t-2)(t-2)(t+3).$$

So, the only other intersection point is at $t = -3$ and

$$Q = \left((-3)^2 + 4(-3), (-3)^3 + (-3)^2\right) = (-3, -18).$$

A729 (9740 N2009/II/4)(i) $\dfrac{\mathrm{d}n}{\mathrm{d}t} = \displaystyle\int \dfrac{\mathrm{d}^2 n}{\mathrm{d}t^2}\,\mathrm{d}t = \int 10 - 6t\,\mathrm{d}t = 10t - 3t^2 + C$.

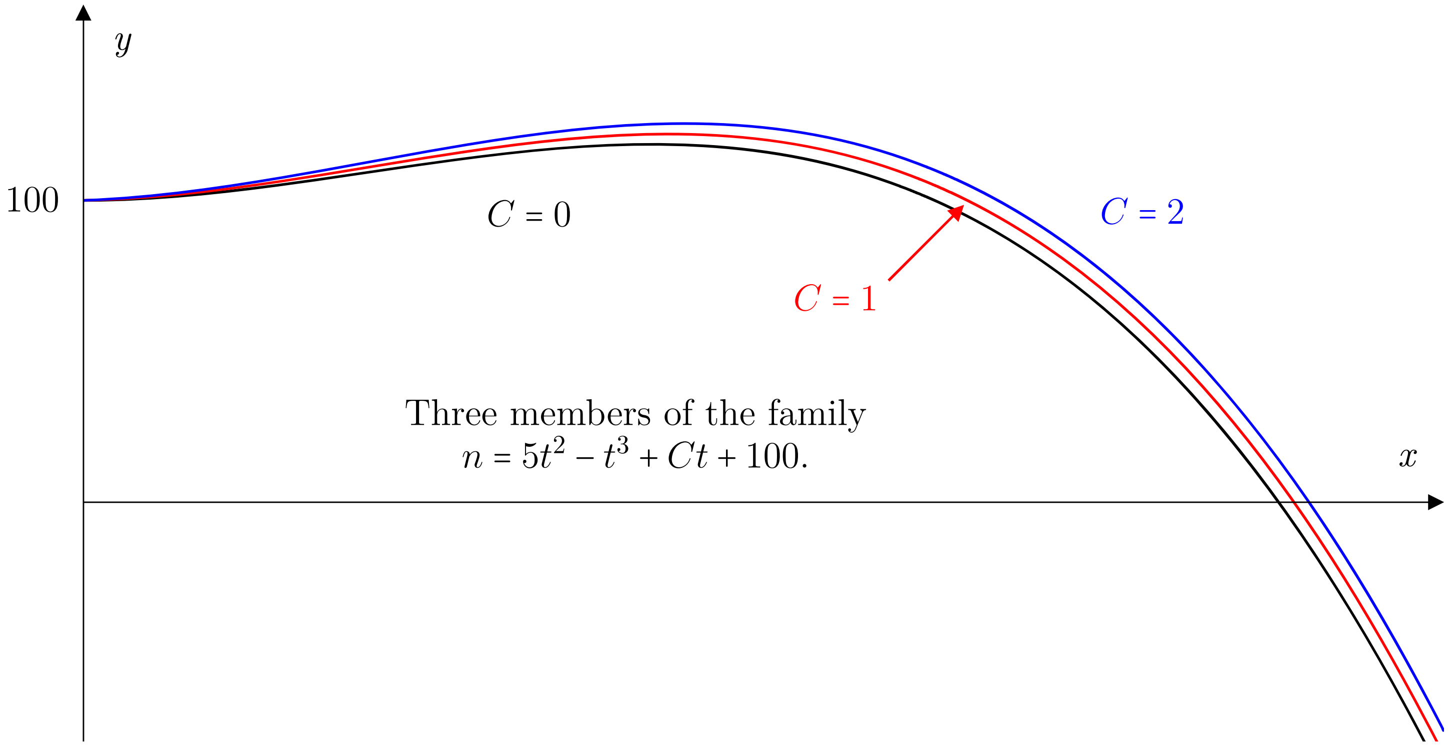
$n = \displaystyle\int \dfrac{\mathrm{d}n}{\mathrm{d}t}\,\mathrm{d}t = \int 10t - 3t^2 + C\,\mathrm{d}t \stackrel{1}{=} 5t^2 - t^3 + Ct + D$.

The remainder of the answer for **(i)** is no longer in the current 9758 syllabus.

Plug the initial condition $(t, n) = (0, 100)$ into $\stackrel{1}{=}$ to get $100 = D$.

So, the family of curves is $n = 5t^2 - t^3 + Ct + 100$.

Sketched below are three members of this family, corresponding to $C = 0$, $C = 1$, and $C = 2$:



(ii) For $3 - 0.02n \neq 0$ or $n \neq 150$, we have $\dfrac{\mathrm{d}t}{\mathrm{d}n} = \dfrac{1}{3 - 0.02n} = \dfrac{50}{150 - n}$ and

$$t = \int \frac{\mathrm{d}t}{\mathrm{d}n}\,\mathrm{d}n = \int \frac{50}{150 - n}\,\mathrm{d}n = -50\int \frac{1}{n - 150}\,\mathrm{d}n = -50\ln|n - 150| + E,$$

or, $\ln|n - 150| = (E - t)/50$ or $|n - 150| = \mathrm{e}^{(E-t)/50}$ or $n - 150 = \pm\mathrm{e}^{E/50}\mathrm{e}^{-t/50} = F\mathrm{e}^{-t/50}$.

Plug in the given initial condition $(t, n) = (0, 100)$ to get $-50 = F$. Hence, $n = 150 - 50\mathrm{e}^{-t/50}$.

As $t \to \infty$, $n \to 150$. So, the population will approach 150 thousand.

A730 (9740 N2008/I/1). The dotted area is $\displaystyle\int_a^4 \sqrt{y}\,\mathrm{d}y = \left[\frac{2}{3}y^{3/2}\right]_a^4 = \frac{2}{3}\left(8 - a^{3/2}\right)$.

The shaded area is $\displaystyle\int_1^2 x^2\,\mathrm{d}x = \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}(8 - 1) = \frac{7}{3}$. These two areas are equal:

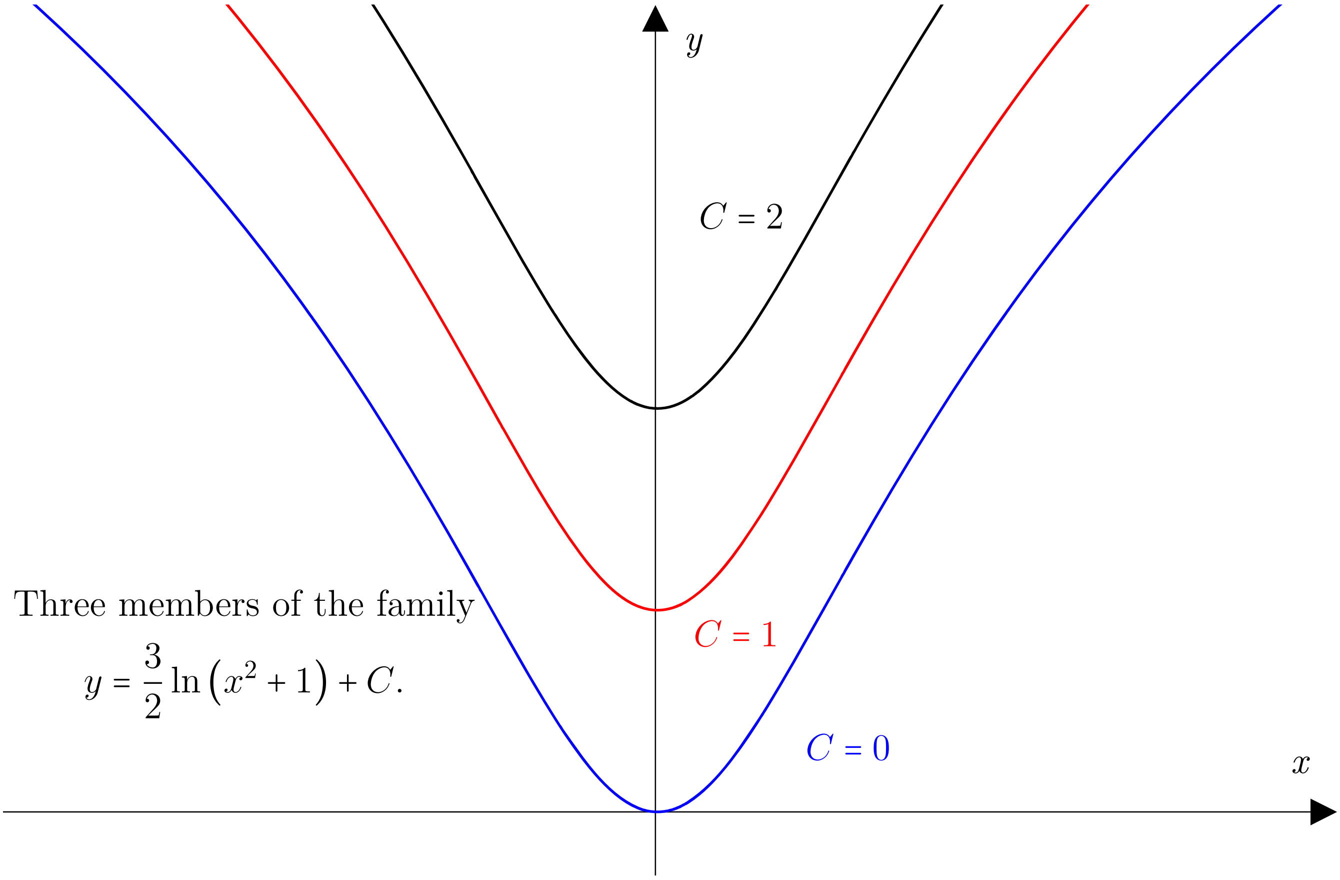
$$\frac{2}{3}\left(8 - a^{3/2}\right) = \frac{7}{3} \quad\iff\quad 8 - a^{3/2} = \frac{7}{2} \quad\iff\quad a = \left(\frac{9}{2}\right)^{2/3} \approx 2.73.$$

A731 (9740 N2008/I/4)(i) $y = \int \frac{dy}{dx}\,dx = \int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\ln\left(x^2+1\right) + C.$[688]

(ii) Plug in the initial condition $(x,y) = (0,2)$ to get $2 = C$. So, $y = 1.5\ln\left(x^2+1\right) + 2$.

(iii) As $x \to \pm\infty$, $dy/dx \to 0$.

(iv)



A732 (9740 N2008/I/5)(i)

$$\int_0^{1/\sqrt{3}} \frac{1}{1+9x^2}\,dx = \frac{1}{9}\int_0^{1/\sqrt{3}} \frac{1}{(1/3)^2+x^2}\,dx = \frac{1}{9}\left[3\tan^{-1}3x\right]_0^{1/\sqrt{3}} = \frac{1}{3}\tan^{-1}\sqrt{3} = \frac{\pi}{9}.$$

(ii) Use Integration by Parts:

$$\int_1^{e} x^n \ln x\,dx = \left[\frac{1}{n+1}x^{n+1}\ln x\right]_1^{e} - \int_1^{e} \frac{1}{n+1}x^{n+1}\frac{1}{x}\,dx = \frac{1}{n+1}e^{n+1} - \left[\frac{1}{(n+1)^2}x^{n+1}\right]_1^{e}$$

$$= \frac{1}{n+1}e^{n+1} - \frac{1}{(n+1)^2}\left(e^{n+1} - 1^{n+1}\right) = \frac{(n+1)e^{n+1} + 1 - e^{n+1}}{(n+1)^2} = \frac{ne^{n+1}+1}{(n+1)^2}.$$

[688] Since $x^2+1 > 0$, we don't need the absolute value operator.

A733 (9740 N2008/I/6)(a) By the Law of Cosines,[689]

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos\theta = 1 + 9 - 6\cos\theta = 10 - 6\cos\theta.$$

So, $AC = \sqrt{10 - 6\cos\theta}$.[690] Plug in the small angle approximation $\cos\theta \approx 1 - \theta^2/2$ to get

$$AC \approx \sqrt{10 - 6\left(1 - \frac{\theta^2}{2}\right)} = \sqrt{4 + 3\theta^2}.$$

And now, by the first "standard" Maclaurin expansion,

$$\sqrt{4 + 3\theta^2} = 2\left(1 + \frac{3}{4}\theta^2\right)^{1/2} = 2\left[1 + \frac{1}{2}\left(\frac{3}{4}\theta^2\right) + \dots\right] = \underbrace{2}_{a} + \underbrace{\frac{3}{4}}_{b}\theta^2 + \dots$$

(b) $f'(x) = 2\sec^2\left(2x + \frac{\pi}{4}\right)$ and $f''(x) = 8\sec^2\left(2x + \frac{\pi}{4}\right)\tan\left(2x + \frac{\pi}{4}\right)$.

$f(0) = \tan\frac{\pi}{4} = 1$, $f'(0) = 2\sec^2\frac{\pi}{4} = 2\cdot 2 = 4$, and $f''(0) = 8\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 8\cdot 2\cdot 1 = 16$.

Hence, $f(x) = 1 + 4x + \dfrac{16x^2}{2!} + \dots = 1 + 4x + 8x^2 + \dots$

A734 (9740 N2008/I/7). The straight parts have length $x + 2y$. The semicircular part has length $\pi x/2$. So, the total time to build the wall is

$$3(x + 2y) + 9\pi\frac{x}{2} = 180.$$

Rearranging, $$y \stackrel{1}{=} 30 - (2 + 3\pi)\frac{x}{4}.$$

Next, the total area is

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 \stackrel{1}{=} 30x - (2 + 3\pi)\frac{x^2}{4} + \pi\frac{x^2}{8} = 30x - (4 + 5\pi)\frac{x^2}{8}.$$

This is a quadratic polynomial with negative coefficient on x^2. Hence, A is maximised at

$$\bar{x} = -\frac{\text{`}b\text{'}}{2\text{`}a\text{'}} = -\frac{30}{-2(4 + 5\pi)x^2/8} \stackrel{2}{=} \frac{120}{4 + 5\pi} \approx 6.09.$$

From $\stackrel{1}{=}$, the corresponding value of y is

$$\bar{y} = 30 - \frac{2 + 3\pi}{4}\frac{120}{4 + 5\pi} = 30\left(1 - \frac{2 + 3\pi}{4 + 5\pi}\right) = 30\frac{2 + 2\pi}{4 + 5\pi} = 60\frac{1 + \pi}{4 + 5\pi} \approx 12.6.$$

[689] Proposition 7(c).

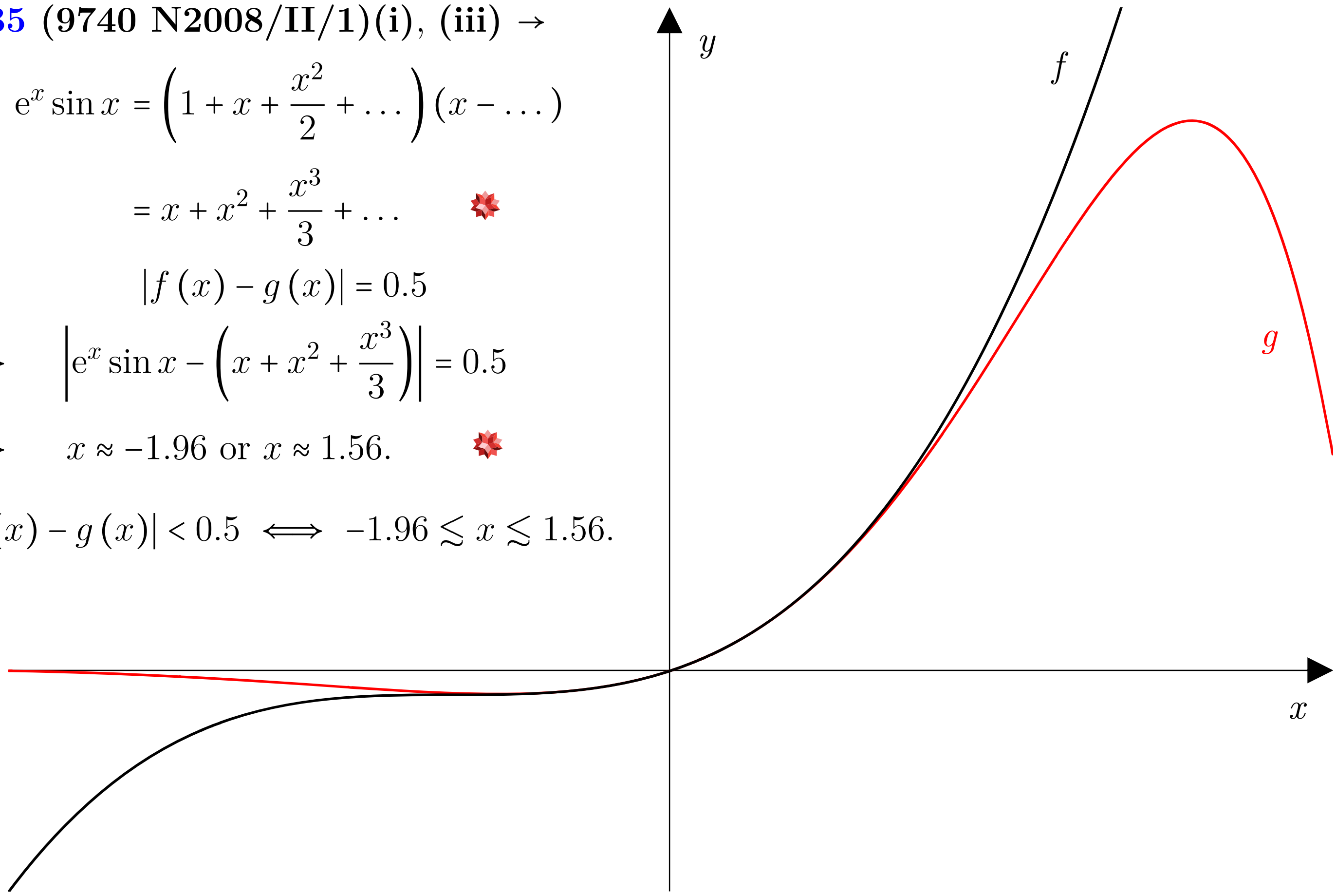
[690] We can discard the negative square root since length must be non-negative.

A735 (9740 N2008/II/1)(i), (iii) →

(ii) $$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \dots\right)(x - \dots)$$
$$= x + x^2 + \frac{x^3}{3} + \dots$$

(iv) $$|f(x) - g(x)| = 0.5$$
$$\iff \quad \left|e^x \sin x - \left(x + x^2 + \frac{x^3}{3}\right)\right| = 0.5$$
$$\iff \quad x \approx -1.96 \text{ or } x \approx 1.56.$$

$$|f(x) - g(x)| < 0.5 \iff -1.96 \lesssim x \lesssim 1.56.$$



A736 (9740 N2008/II/2)(i) The upper half of the curve C has equation $y \overset{1}{=} \sqrt{x\sqrt{1-x}}$. And so, by symmetry, $R = 2\int_0^1 \sqrt{x\sqrt{1-x}}\,dx \approx 0.999$.

(ii) From $u \overset{2}{=} 1 - x$, $du/dx \overset{3}{=} -1$. And the volume is

$$\pi\int_0^1 y^2\,dx = \pi\int_0^1 x\sqrt{1-x}\,dx \overset{2}{=} \pi\int_0^1 (1-u)\sqrt{u}\,dx \overset{3}{=} \pi\int_{x=0}^{x=1} (u-1)\sqrt{u}\frac{du}{\cancel{dx}}\cancel{dx}$$
$$= \pi\int_{u=1}^{u=0} u^{3/2} - u^{1/2}\,du = \pi\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^0 = \pi\left(\frac{2}{3} - \frac{2}{5}\right) = \frac{4}{15}\pi.$$

(iii) Apply d/dx to $\overset{1}{=}$ to get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x\sqrt{1-x}}}\left(\sqrt{1-x} - \frac{1}{2\sqrt{1-x}}\right) = \frac{2(1-x)-1}{4\sqrt{1-x}\sqrt{x\sqrt{1-x}}} = \frac{2-3x}{4\sqrt{1-x}\sqrt{x\sqrt{1-x}}}.$$

So, $dy/dx\big|_{x=\bar{x}} = 0 \iff \bar{x} = 2/3$. This is indeed the global maximum point because $dy/dx > 0$ for $x \in (0, \bar{x})$ and $dy/dx < 0$ for $x \in (\bar{x}, 1)$.

A737 (9233 N2008/I/2). For small x, $\cos 2x \approx 1 - (2x)^2/2! = 1 - 2x^2$ and

$$\frac{1}{\sqrt{1+x^2}} = \left(1 + x^2\right)^{-1/2} \approx 1 + \left(\frac{-1}{2}\right)x^2 = 1 - \frac{1}{2}x^2.$$

So, $$\frac{\cos 2x}{\sqrt{1+x^2}} \approx \left(1 - 2x^2\right)\left(1 - \frac{1}{2}x^2\right) \approx \underbrace{1}_{a}\underbrace{-\frac{5}{2}}_{b}x^2$$

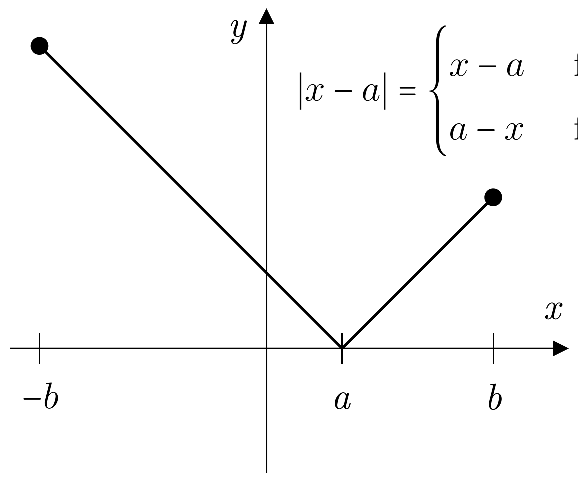
A738 (9233 N2008/I/3). Use Integration by Parts:

$$\int_0^1 x\mathrm{e}^{-2x}\,\mathrm{d}x = -\frac{1}{2}\left[x\mathrm{e}^{-2x}\right]_0^1 + \frac{1}{2}\int_0^1 \mathrm{e}^{-2x}\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-2} - \frac{1}{4}\left[\mathrm{e}^{-2x}\right]_0^1 = \frac{1}{4} - \frac{3}{4}\mathrm{e}^{-2}.$$

A739 (9233 N2008/I/4). From $t \overset{1}{=} \ln x$, we have $\mathrm{d}t/\mathrm{d}x \overset{2}{=} 1/x$. So,

$$\int_{\mathrm{e}}^{\mathrm{e}^3} \frac{1}{x(\ln x)^2}\,\mathrm{d}x \overset{1,2}{=} \int_{x=\mathrm{e}}^{x=\mathrm{e}^3} \frac{1}{t^2}\frac{\mathrm{d}t}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = \int_{t=1}^{t=3} \frac{1}{t^2}\,\mathrm{d}t = \left[-\frac{1}{t}\right]_1^3 = \frac{2}{3}.$$

A740 (9233 N2008/I/6)(i)



$$|x-a| = \begin{cases} x-a & \text{for } x \geq a, \\ a-x & \text{for } x < a. \end{cases}$$

(ii) This is simply the area of two triangles: $\dfrac{1}{2}\left[a-(-b)\right]^2 + \dfrac{1}{2}(b-a)^2 = a^2 + b^2$.

A741 (9233 N2008/I/8). $\displaystyle\int_a^\infty \frac{1}{4+x^2}\,\mathrm{d}x = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_a^\infty = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{a}{2}.$

$$\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \left[\sin^{-1}x\right]_{1/2}^{\sqrt{3}/2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

If the two expressions are equal, then

$$\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{a}{2} = \frac{\pi}{6} \quad\Longleftrightarrow\quad \frac{\pi}{6} = \tan^{-1}\frac{a}{2} \quad\Longleftrightarrow\quad a = 2\tan\frac{\pi}{6} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.$$

A742 (9233 N2008/I/10)(i) From $y \overset{1}{=} xz$, we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} \overset{2}{=} z + x\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

Plug $\overset{1}{=}$ and $\overset{2}{=}$ into $xy\dfrac{\mathrm{d}y}{\mathrm{d}x} \overset{3}{=} x^2 + y^2$ to get

$$x^2z\left(z + x\frac{\mathrm{d}z}{\mathrm{d}x}\right) = x^2 + x^2z^2 \qquad \text{or} \qquad 0 = x^2\left(z^2 + xz\frac{\mathrm{d}z}{\mathrm{d}x} - z^2 - 1\right) = x^2\left(xz\frac{\mathrm{d}z}{\mathrm{d}x} - 1\right).$$

So, if $x \neq 0$, then $xz\dfrac{\mathrm{d}z}{\mathrm{d}x} \overset{4}{=} 1$. (If $x = 0$, then $\overset{3}{=}$ becomes $y = 0$.)

(ii) Assume $x \neq 0$. Rearrange $\overset{4}{=}$ to get $z\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{1}{x}$. And now,

$$\int z\frac{\mathrm{d}z}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x \qquad \text{or} \qquad \int z\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x,$$

or,
$$z^2 = 2\ln|x| + C_1 \qquad \text{or} \qquad z \overset{5}{=} \pm\sqrt{2\ln|x| + C}.$$

Plug $\overset{5}{=}$ into $\overset{1}{=}$ to get $y = \pm x\sqrt{2\ln|x| + C}$.

Plug in the initial condition $(x, y) = (2, 6)$ to get $6 = \pm 2\sqrt{2\ln 2 + C}$. Since $6 > 0$, it must be that $y = x\sqrt{2\ln|x| + C}$. And now from $6 = 2\sqrt{2\ln 2 + C}$, we also have $C = 9 - 2\ln 2$.

Altogether,
$$y = \begin{cases} 0, & \text{for } x = 0, \\ x\sqrt{2\ln|x| + 9 - 2\ln 2}, & \text{for } x \neq 0. \end{cases}$$

A743 (9233 N2008/I/13)(i) $-\dfrac{\mathrm{d}x}{\mathrm{d}y} = -\dfrac{\mathrm{d}x}{\mathrm{d}t} \div \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{-3\cos^2 t\,(-\sin t)}{3\sin^2 t\cos t} = \dfrac{\cos t}{\sin t}$.

So, the normal is $y - \sin^3 t = \dfrac{\cos t}{\sin t}\left(x - \cos^3 t\right)$ or $x\cos t - y\sin t \overset{1}{=} \cos^4 t - \sin^4 t$.

(ii) $\cos^4 t - \sin^4 t = \left(\cos^2 t + \sin^2 t\right)\left(\cos^2 t - \sin^2 t\right) = (1)(\cos 2t) \overset{2}{=} \cos 2t$.

(iii) Plug $\overset{2}{=}$ into $\overset{1}{=}$ to get $x\cos t - y\sin t \overset{3}{=} \cos 2t$.

If $y = 0$, then $x\cos t = \cos 2t$ or $x = \cos 2t/\cos t$. So, $A = (\cos 2t/\cos t, 0)$

If $x = 0$, then $-y\sin t = \cos 2t$ or $y = -\cos 2t/\sin t$. So, $B = (0, -\cos 2t/\sin t)$.

$$|AB| = \sqrt{\left(\frac{\cos 2t}{\cos t}\right)^2 + \left(-\frac{\cos 2t}{\sin t}\right)^2} = \cos 2t\sqrt{\frac{1}{\cos^2 t} + \frac{1}{\sin^2 t}} = \cos 2t\sqrt{\frac{\sin^2 t + \cos^2 t}{\sin^2 t\cos^2 t}}$$
$$= \cos 2t\frac{1}{\sin t\cos t} = \frac{\cos 2t}{0.5\sin 2t} = 2\cot 2t. \qquad \text{(So, } k = 2.\text{)}$$

(Note that since $t \in (0, \pi/4)$, $\sin t, \cos t, \cos 2t \in (0, 1)$.)

A744 (9233 N2008/I/14)(i) Let $\mathbf{P}(k)$ be this proposition:

$$1+2x+3x^2+\cdots+kx^{k-1}=\frac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}$$

We show that $\mathbf{P}(1)$ is true: $\dfrac{1-(1+1)x^1+1\cdot x^{1+1}}{(1-x)^2}=\dfrac{1-2x+x^2}{(1-x)^2}=\dfrac{(x-1)^2}{(1-x)^2}=1.$ ✓

We next show that for all $j\in\mathbb{N}$, if $\mathbf{P}(j)$ is true, then $\mathbf{P}(j+1)$ is also true:

$$1+2x+3x^2+\cdots+jx^{j-1}+(j+1)x^j \overset{\mathbf{P}(j)}{=} \frac{1-(j+1)x^j+jx^{j+1}}{(1-x)^2}+(j+1)x^j$$

$$=\frac{1+(j+1)x^j\left[(1-x)^2-1\right]+jx^{j+1}}{(1-x)^2}=\frac{1+(j+1)x^j\left(x^2-2x\right)+jx^{j+1}}{(1-x)^2}$$

$$=\frac{1-(j+2)x^{j+1}+(j+1)x^{j+2}}{(1-x)^2}. \quad ✓$$

(ii) $1+2x+3x^2+\cdots+nx^{n-1}=\dfrac{d}{dx}\displaystyle\int 1+2x+3x^2+\cdots+nx^{n-1}\,dx$

$$=\frac{d}{dx}\left(x+x^2+x^3+\cdots+x^n+C\right)=\frac{d}{dx}\left(x\frac{1-x^n}{1-x}+C\right)=\frac{1-x^n}{1-x}+x\left[-nx^{n-1}\frac{1}{1-x}+\frac{1-x^n}{(1-x)^2}\right]$$

$$==\frac{(1-x^n)(1\cancel{-x})-nx^n(1-x)+\cancel{x(1-x^n)}}{(1-x)^2}=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}.$$

A745 (9233 N2008/II/1). $\cos 4x-\cos 6x=-2\sin\left(\dfrac{4+6}{2}x\right)\sin\left(\dfrac{4-6}{2}x\right)$

$$=-2\sin 5x\sin(-x)=2\sin 5x\sin x.$$

$$\int_0^{\pi/3}\sin 5x\sin x\,dx=\frac{1}{2}\int_0^{\pi/3}\cos 4x-\cos 6x\,dx=\frac{1}{2}\left[\frac{\sin 4x}{4}-\frac{\sin 6x}{6}\right]_0^{\pi/3}=-\frac{\sqrt{3}}{16}.$$

A746 (9233 N2008/II/5)(i) Define $f:(-1,\infty)\to\infty$ by $f(x)=\ln(1+x)-\dfrac{2x}{x+2}$.

So, $f'(x)=\dfrac{1}{1+x}-\dfrac{(x+2)\cdot 2-2x\cdot 1}{(x+2)^2}=\dfrac{(x+2)^2-4(1+x)}{(1+x)(x+2)^2}=\dfrac{x^2}{(1+x)(x+2)^2}$.

Indeed, $f'(x)\geq 0$ for all $x\in\text{Domain } f=(-1,\infty)$.[691]

(ii) Observe that $f(0)=0$. Since $f'(x)\geq 0$ for all $x\in\text{Domain } f=(-1,\infty)$, for all $x\geq 0$,

$$f(x)\geq f(0) \qquad \text{or} \qquad \ln(1+x)-\frac{2x}{x+2}\geq 0 \qquad \text{or} \qquad \ln(1+x)\geq\frac{2x}{x+2}.$$

[691] It turns out that more generally, the domain of the natural logarithm function ln is $\mathbb{C}\setminus\{0\}$, that is, the set of all complex numbers excluding 0. In which case, it is perfectly possible that $1+x<0$ and the conclusion to which this question leads is false.

A747 (9740 N2007/I/4)(i) For $2-3I \neq 0$ or $I \neq 2/3$, we have $\mathrm{d}t/\mathrm{d}I = 4/(2-3I)$. So,

$$t = \int \frac{\mathrm{d}t}{\mathrm{d}I}\,\mathrm{d}I = \int \frac{4}{2-3I}\,\mathrm{d}I = -4\int \frac{1}{3I-2}\,\mathrm{d}I = -\frac{4}{3}\ln|3I-2| + C_1,$$

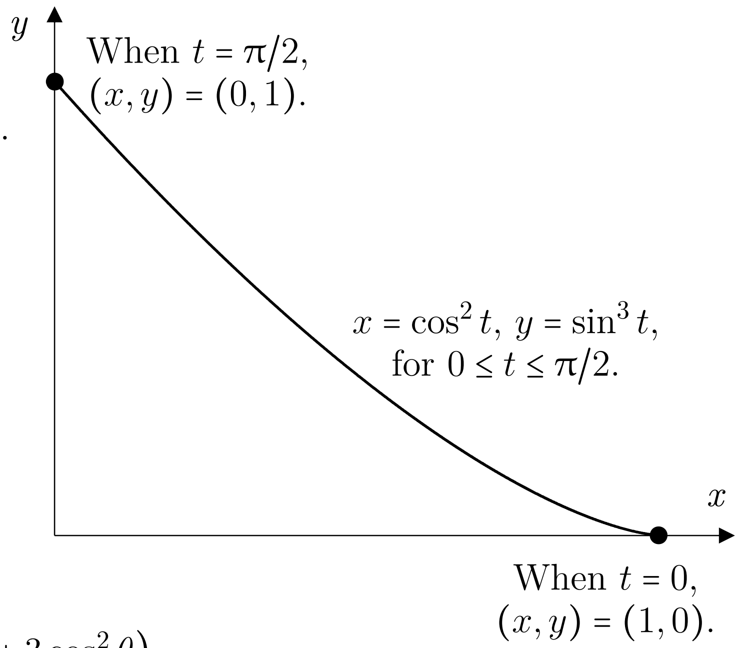
or, $\quad \ln|3I-2| = 3(C_1 - t)/4 \quad$ or $\quad |3I-2| = \mathrm{e}^{3(C_1-t)/4} = \mathrm{e}^{3C_1/4}\mathrm{e}^{-3t/4}$,

or, $\quad 3I-2 = \pm\mathrm{e}^{3C_1/4}\mathrm{e}^{-3t/4} = C_2\mathrm{e}^{-3t/4} \quad$ or $\quad I = C_3\mathrm{e}^{-3t/4} + 2/3$.

Plug in $(t, I) = (0, 2)$ to get $2 = C_3 + 2/3$ or $C_3 = 4/3$. So, $I = 2\left(2\mathrm{e}^{-3t/4} + 1\right)/3$.

(ii) $\lim\limits_{t\to\infty} I = 2/3$.

A748 (9740 N2007/I/11)(i) →



(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t} \div \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{3\sin^2 t\cos t}{2\cos t\,(-\sin t)} = -\dfrac{3}{2}\sin t.$

So, the tangent is

$$y - \sin^3\theta = -\frac{3}{2}\sin\theta\left(x - \cos^2\theta\right).$$

If $y = 0$, then $-\sin^3\theta \overset{1}{=} -\dfrac{3}{2}\sin\theta\left(x_Q - \cos^2\theta\right)$,

$x_Q = \dfrac{2}{3}\sin^2\theta + \cos^2\theta = \dfrac{1}{3}\left(2\sin^2\theta + 3\cos^2\theta\right)$.

If $x = 0$, then $y_R - \sin^3\theta \overset{1}{=} -\dfrac{3}{2}\sin\theta\left(-\cos^2\theta\right)$,

$y_R = \sin\theta\left(\sin^2\theta + \dfrac{3}{2}\cos^2\theta\right) = \dfrac{1}{2}\sin\theta\left(2\sin^2\theta + 3\cos^2\theta\right)$.

So, the area of $\triangle OQR$ is

$$\frac{1}{2}x_Q y_R = \frac{1}{2}\frac{1}{3}\left(2\sin^2\theta + 3\cos^2\theta\right)\frac{1}{2}\sin\theta\left(2\sin^2\theta + 3\cos^2\theta\right) = \frac{1}{12}\left(2\sin^2\theta + 3\cos^2\theta\right)^2.$$

(iii) The area is $\quad \displaystyle\int_0^1 y\,\mathrm{d}x = \int_{x=0}^{x=1} y\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int_{t=\pi/2}^{t=0} y\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$

$$= \int_{\pi/2}^{0} \sin^3 t \cdot 2\cos t\,(-\sin t)\,\mathrm{d}t = 2\int_0^{\pi/2}\cos t\sin^4 t\,\mathrm{d}t.$$

From $u \overset{1}{=} \sin t$, we have $\mathrm{d}u/\mathrm{d}t \overset{2}{=} \cos t$ and

$$2\int_0^{\pi/2}\cos t\sin^4 t\,\mathrm{d}t \overset{1,2}{=} 2\int_{t=0}^{t=\pi/2}\frac{\mathrm{d}u}{\cancel{\mathrm{d}t}}u^4\,\cancel{\mathrm{d}t} = 2\int_{u=0}^{u=1} u^4\,\mathrm{d}u = 2\left[\frac{1}{5}u^5\right]_0^1 = \frac{2}{5}.$$

A749 (9740 N2007/II/3)(i) Let $f(x) = (1+x)^n$.

$f'(x) = n(1+x)^{n-1}$, $f''(x) = n(n-1)(1+x)^{n-2}$, and $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$.
$f(0) = 1$, $f'(0) = n$, $f''(0) = n(n-1)$, and $f'''(0) = n(n-1)(n-2)$. So,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots.$$

(ii) $\left(1+2x^2\right)^{\frac{3}{2}} = 1 + 1.5\left(2x^2\right) + \cdots = 1 + 3x^2 + \ldots$

$$(4-x)^{\frac{3}{2}} = 8\left(1 - \frac{1}{4}x\right)^{\frac{3}{2}}$$

$$= 8\left[1 + \frac{3}{2}\left(-\frac{1}{4}x\right) + \frac{\frac{3}{2}\frac{1}{2}}{2!}\left(-\frac{1}{4}x\right)^2 + \frac{\frac{3}{2}\frac{1}{2}\frac{-1}{2}}{3!}\left(-\frac{1}{4}x\right)^3 + \ldots\right] = 8 - 3x + \frac{3}{16}x^2 + \frac{1}{128}x^3 + \ldots$$

So, $$(4-x)^{\frac{3}{2}}\left(1+2x^2\right)^{\frac{3}{2}} = \left(8 - 3x + \frac{3}{16}x^2 + \frac{1}{128}x^3 + \ldots\right)\left(1 + 3x^2 + \ldots\right)$$

$$= 8 - 3x + \frac{3}{16}x^2 + \frac{1}{128}x^3 + 24x^2 - 9x^3 + \cdots = 8 - 3x + 24\frac{3}{16}x^2 - 8\frac{127}{128}x^3 + \ldots$$

(iii) The expansions are valid for $|-x/4| < 1$ AND $\left|2x^2\right| < 1$ or $|x| < 1/\sqrt{2}$.

A750 (9740 N2007/II/4)(i)

$$\int_0^{5\pi/3} \sin^2 x \,\mathrm{d}x = \int_0^{5\pi/3} \frac{1-\cos 2x}{2}\,\mathrm{d}x = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{5\pi/3} = \frac{5\pi}{6} + \frac{\sqrt{3}}{8}.$$

$$\int_0^{5\pi/3} \cos^2 x \,\mathrm{d}x = \int_0^{5\pi/3} 1 - \sin^2 x \,\mathrm{d}x = [x]_0^{5\pi/3} - \left(\frac{5\pi}{6} + \frac{\sqrt{3}}{8}\right) = \frac{5\pi}{6} - \frac{\sqrt{3}}{8}.$$

(ii)(a) Use Integration by Parts twice:

$$R = \int_0^{\pi/2} x^2 \sin x \,\mathrm{d}x = \left[x^2(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} 2x(-\cos x)\,\mathrm{d}x = 0 + 2\int_0^{\pi/2} x\cos x \,\mathrm{d}x$$

$$= 2\left[x\sin x - \int \sin x \,\mathrm{d}x\right]_0^{\pi/2} = 2\left[x\sin x + \cos x\right]_0^{\pi/2} = 2\left(\frac{\pi}{2} + 0 - 0 - 1\right) = \pi - 2.$$

(ii)(b) $\pi \int_0^{\pi/2} \left(x^2 \sin x\right)^2 \mathrm{d}x \approx 5.391$ (calculator).

A751 (9233 N2007/I/2). $(4+3x)^{\frac{5}{2}} = 4^{\frac{5}{2}}(1+3x/4)^{\frac{5}{2}}$.

The first negative coefficient is the 4th Maclaurin coefficient:

$$4^{\frac{5}{2}}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{4!}\left(\frac{3}{4}\right)^4 = 2^5 \cdot \frac{-15/2^4}{4!}\left(\frac{3}{4}\right)^4 = \frac{-15}{12}\cdot\frac{81}{256} = \frac{-5}{4}\cdot\frac{81}{256} = -\frac{405}{1\,024}.$$

A752 (9233 N2007/I/3). $V = \pi \int_{1/2}^{\sqrt{3}/2} y^2 \, dx = \pi \int_{1/2}^{\sqrt{3}/2} \frac{1}{1+4x^2} \, dx$

$$= \frac{\pi}{4} \int_{1/2}^{\sqrt{3}/2} \frac{1}{(1/2)^2 + x^2} \, dx = \frac{\pi}{4} \left[2 \tan^{-1} 2x \right]_{1/2}^{\sqrt{3}/2}$$

$$= \frac{\pi}{2} \left(\tan^{-1} \sqrt{3} - \tan^{-1} 1 \right) = \frac{\pi}{2} \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{\pi^2}{24}. \text{ (So, } k = \frac{1}{24}.)$$

A753 (9233 N2007/I/8)(i) From $t \stackrel{1}{=} \sin u$, $u \stackrel{2}{=} \sin^{-1} t$ and $du/dt \stackrel{3}{=} 1/\cos u$. Now,

$$\int \frac{\left(\sin^{-1} t\right) \cos \left[\left(\sin^{-1} t \right)^2 \right]}{\sqrt{1-t^2}} \, dt \stackrel{1,2}{=} \int \frac{u \cos u^2}{\sqrt{1 - \sin^2 u}} \, dt = \int \frac{u \cos u^2}{|\cos u|} \, dt \stackrel{4}{=} \int \frac{u \cos u^2}{\cos u} \, dt$$

$$\stackrel{3}{=} \int u \cos u^2 \frac{du}{\cancel{dt}} \cancel{dt} = \int u \cos u^2 \, du = \frac{1}{2} \sin u^2 + C \stackrel{2}{=} \frac{1}{2} \sin \left(\sin^{-1} t \right)^2 + C.$$

(At $\stackrel{4}{=}$, we can remove the absolute value operator because $\cos u \in (0,1)$.)

(ii) $\int_0^1 \frac{\left(\sin^{-1} t\right) \cos \left[\left(\sin^{-1} t \right)^2 \right]}{\sqrt{1-t^2}} \, dt = \left[\frac{1}{2} \sin \left(\sin^{-1} t \right)^2 \right]_0^1 = \frac{1}{2} \left[\sin \left(\frac{\pi}{2} \right)^2 - \sin 0^2 \right] = \frac{1}{2} \sin \frac{\pi^2}{4}.$

A754 (9233 N2007/I/10)(i) For $x \in [0, 2\pi]$, $\cos x = \sin x \iff x = \pi/4, 5\pi/4$. From a sketch, we see that $\cos x > \sin x$ on the left of the first intersection point and the right of the second—i.e., $x \in [0, \pi/4) \cup (5\pi/4, 2\pi]$.

(ii) $\int_0^{2\pi} |\cos x - \sin x| \, dx = \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx + \int_{\frac{5\pi}{4}}^{2\pi} \cos x - \sin x \, dx$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{5\pi/4} + [\sin x + \cos x]_{5\pi/4}^{2\pi}$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0 + 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 4\sqrt{2}.$$

A755 (9233 N2007/I/11). $\frac{5x+4}{(x-5)(x^2+4)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+4}$

$$= \frac{Ax^2 + 4A + Bx^2 - 5Bx + Cx - 5C}{(x-5)(x^2+4)} = \frac{(A+B)x^2 + (C-5B)x + 4A - 5C}{(x-5)(x^2+4)}.$$

Comparing coefficients, $A + B \stackrel{1}{=} 0$, $C - 5B \stackrel{2}{=} 5$, $4A - 5C \stackrel{3}{=} 4$.

Solving, we have $C = 0$, $B = -1$, and $A = 1$. So,

$$\int_1^4 \frac{5x+4}{(x-5)(x^2+4)} \, dx = \int_1^4 \frac{1}{x-5} - \frac{x}{x^2+4} \, dx = \left[\ln|x-5| - \frac{1}{2} \ln \left(x^2 + 4 \right) \right]_1^4$$

$$= \ln 1 - \ln 4 - \frac{1}{2} \ln 20 + \frac{1}{2} \ln 5 = \ln \frac{\sqrt{5}}{4 \cdot \sqrt{20}} = \ln \frac{1}{8} = -\ln 8.$$

A756 (9233 N2007/I/13)(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sec x}\sec x \tan x = \tan x.$ $\qquad \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sec^2 x.$

$\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 2\sec^2 x \tan x = 2\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x}.$

(ii) $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} = 2\left(2\sec^2 x \tan^2 x + \sec^4 x\right).$ $\qquad$ So, $\left.\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4}\right|_{x=0} = 2(0+1) = 2.$

(iii) $\left.y\right|_{x=0} = 0$, $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 0$, $\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0} = 1$, and $\left.\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}\right|_{x=0} = 0$. So,

$$\ln(\sec x) = 0 + 0x + \frac{1}{2!}x^2 + 0 + \frac{2}{4!}x^4 + \cdots = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \ldots.$$

(iv) $\ln 2 = 2\ln\sqrt{2} = 2\ln\left(\sec\dfrac{\pi}{4}\right) \overset{\text{(iii)}}{\approx} 2\left[\dfrac{1}{2}\left(\dfrac{\pi}{4}\right)^2 + \dfrac{1}{12}\left(\dfrac{\pi}{4}\right)^4\right] = \dfrac{\pi^2}{16} + \dfrac{\pi^4}{1\,536}.$

A757 (9233 N2007/I/14)(i) Apply $\mathrm{d}/\mathrm{d}x$ to $x^2 - y^2 \stackrel{1}{=} Ax$ to get

$$2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = A \quad \text{or} \quad 2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} \stackrel{1}{=} \frac{x^2 - y^2}{x} \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - \frac{x^2-y^2}{x}}{2y} = \frac{x^2+y^2}{xy} \ (\text{for } x, y \neq 0).$$

(ii) From $y \stackrel{2}{=} vx$, $\mathrm{d}y/\mathrm{d}x \stackrel{3}{=} x\mathrm{d}v/\mathrm{d}x + v$. Now,

$$x\frac{\mathrm{d}v}{\mathrm{d}x} \stackrel{3}{=} \frac{\mathrm{d}y}{\mathrm{d}x} - v = -\frac{2xy}{x^2+y^2} - v \stackrel{2}{=} -\frac{2vx^2}{x^2+v^2x^2} - v = -\left(\frac{2v}{1+v^2} + v\right) = -\frac{3v+v^3}{1+v^2}.$$

(iii) Rearranging, $\dfrac{1}{x}\dfrac{\mathrm{d}x}{\mathrm{d}v} \stackrel{4}{=} -\dfrac{1+v^2}{3v+v^3}$. Now, $\displaystyle\int \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}v}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x \stackrel{5}{=} \ln x + C_1.$

Also, $\displaystyle\int \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}v}\,\mathrm{d}v \stackrel{4}{=} \int -\frac{1+v^2}{3v+v^3}\,\mathrm{d}v \stackrel{6}{=} -\frac{1}{3}\ln\left|3v+v^3\right| + C_2.$

Putting $\stackrel{5}{=}$ and $\stackrel{6}{=}$ together, $\ln x + C_3 = -\dfrac{1}{3}\ln\left|3v+v^3\right|$ or $\mathrm{e}^{-3(\ln x + C_3)} = \left|3v+v^3\right|$,

or, $$3v + v^3 = \pm\mathrm{e}^{-3(\ln x + C_3)} = \pm\mathrm{e}^{-3C_3}\mathrm{e}^{-3\ln x} = C\mathrm{e}^{-3\ln x},$$

or, $$C = \left(3v+v^3\right)\mathrm{e}^{3\ln x} = \left(3v+v^3\right)x^3 = 3vx^3 + v^3x^3 = 3x^2y + y^3.$$

A758 (9233 N2006/I/7). Let $V(t)$, $h(t)$, and $r(t)$ be the volume (cm^3), height (cm), and radius (cm) of the cone formed by the liquid t seconds after the start of the experiment.

Given the angle 45°, $h = r$. So, $V(t) = \dfrac{\pi}{3}[r(t)]^2 h(t) = \dfrac{\pi}{3}[h(t)]^3.$

At the 3-minute mark, $V(180) = 390 - 180 \times 2 = 30$ and $h(180) = \left[\dfrac{3}{\pi}V(180)\right]^{1/3} = \left(\dfrac{90}{\pi}\right)^{1/3}.$

Next, compute $V'(t) = \pi[h(t)]^2 h'(t)$. We are also given $V'(t) = -2$ for all t. So,

$$h'(180) = \frac{V'(180)}{\pi[h(180)]^2} = \frac{-2}{\pi(90/\pi)^{2/3}} = \frac{-2}{\pi^{1/3}90^{2/3}}\,\mathrm{cm\,s^{-1}}.$$

A759 (9233 N2006/I/8). Apply $\frac{d}{dx}$ to $3x^2 + xy + y^2 \stackrel{1}{=} 33$ to get

$$6x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{6x+y}{x+2y}.$$

So, $$\frac{dy}{dx} = 0 \qquad \Longleftrightarrow \qquad 6x + y = 0 \qquad \Longleftrightarrow \qquad y \stackrel{2}{=} -6x.$$

Plug $\stackrel{2}{=}$ into $\stackrel{1}{=}$ to get $3x^2 + x(-6x) + (-6x)^2 = 33$ or $33x^2 = 33$ or $x = \pm 1$.
So, the two points at which the tangent is parallel to the x-axis are $(\pm 1, \mp 6)$.

A760 (9233 N2006/I/9)(i) $\frac{d}{d\theta}\sec\theta = \frac{d}{d\theta}\frac{1}{\cos\theta} = -\frac{-\sin\theta}{\cos^2\theta} = \frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta} = \sec\theta\tan\theta.$

(ii) From $x \stackrel{1}{=} \sec\theta - 1$, $\frac{dx}{d\theta} \stackrel{2}{=} \sec\theta\tan\theta$ and $x^2 + 2x \stackrel{3}{=} (x+1)^2 - 1 = \sec^2\theta - 1$.

$$\int_{\sqrt{2}-1}^{1} \frac{1}{(x+1)\sqrt{x^2+2x}}\,dx \stackrel{1,3}{=} \int_{\sqrt{2}-1}^{1} \frac{1}{\sec\theta\sqrt{\sec^2\theta - 1}}\,dx = \int_{\sqrt{2}-1}^{1} \frac{1}{\sec\theta\sqrt{\tan^2\theta}}\,dx$$

$$= \int_{\sqrt{2}-1}^{1} \frac{1}{\sec\theta\,|\tan\theta|}\,dx \stackrel{4}{=} \int_{\sqrt{2}-1}^{1} \frac{1}{\sec\theta\tan\theta}\,dx \stackrel{2}{=} \int_{x=\sqrt{2}-1}^{x=1} \frac{d\theta}{\cancel{dx}}\cancel{dx} = \int_{\theta=\pi/4}^{\theta=\pi/3} d\theta = [\theta]_{\pi/4}^{\pi/3} = \frac{\pi}{12}.$$

(At $\stackrel{4}{=}$, we can remove the absolute value operator because $\tan\theta > 0$.)

A761 (9233 N2006/I/12)(i) $$\frac{1 + x - 2x^2}{(2-x)(1+x^2)} = \frac{A}{2-x} + \frac{Bx+C}{1+x^2}$$
$$= \frac{A + 2C + (2B - C)x + (A - B)x^2}{(2-x)(1+x^2)}.$$

Comparing coefficients, $A + 2C = 1$, $2B - C = 1$, and $A - B = -2$.

Solving, $A = -1$, $B = 1$, and $C = 1$. So, $\frac{1+x-2x^2}{(2-x)(1+x^2)} \stackrel{1}{=} -\frac{1}{2-x} + \frac{x+1}{1+x^2}$.

(ii) $$-\frac{1}{2-x} = -\frac{1}{2}\left(1 - \frac{x}{2}\right)^{-1} = -\frac{1}{2}\left[1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 + \dots\right]$$
$$\stackrel{2}{=} -\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 - \dots.$$

$$\frac{x+1}{1+x^2} = (x+1)\left[1 + (-1)x^2 + \dots\right] = (x+1)\left(1 - x^2 + \dots\right) \stackrel{3}{=} 1 + x - x^2 + \dots.$$

Plug $\stackrel{2}{=}$ and $\stackrel{3}{=}$ into $\stackrel{1}{=}$ to get

$$\frac{1+x-2x^2}{(2-x)(1+x^2)} = \left(-\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 - \dots\right) + \left(1 + x - x^2 + \dots\right) = \frac{1}{2} + \frac{3}{4}x - \frac{9}{8}x^2 + \dots$$

(iii) It is valid if $|-x/2| < 1$ AND $\left|x^2\right| < 1$ or simply $|x| < 1$.

A762 (9233 N2006/I/14)(i) $\dfrac{y_R - y_Q}{x_R - x_Q} = \dfrac{c/r - c/q}{cr - cq} = \dfrac{1/r - 1/q}{r - q} = \dfrac{(q-r)/(qr)}{r-q} = -\dfrac{1}{qr}$.

(ii) The described line has gradient qr and passes through $P(cp, c/p)$. So, its equation is

$$y - \frac{c}{p} \overset{1}{=} qr(x - cp).$$

This line passes through V. So, plug $(x, y) = (cv, c/v)$ into $\overset{1}{=}$ to get

$$\frac{c}{v} - \frac{c}{p} = qr(cv - cp) \quad \text{or} \quad \frac{1}{v} - \frac{1}{p} = qr(v - p) \quad \text{or} \quad \frac{p-v}{vp} = qr(v-p) \quad \text{or} \quad v = -\frac{1}{pqr}.$$

(iii) Observe that $xy = c^2$. Apply $\dfrac{\mathrm{d}}{\mathrm{d}y}$ to get $\dfrac{\mathrm{d}x}{\mathrm{d}y}y + x = 0$.

Rearrange to find that the requested gradient is $-\dfrac{\mathrm{d}x}{\mathrm{d}y}\Big|_{t=p} = \dfrac{x}{y}\Big|_{t=p} = \dfrac{cp}{c/p} = p^2$.

(iv) The normal at P is $y - c/p \overset{2}{=} p^2(x - cp)$.

This line passes through S. So, plug $(x, y) = (cs, c/s)$ into $\overset{2}{=}$ to get

$$\frac{c}{s} - \frac{c}{p} = p^2(cs - cp) \quad \text{or} \quad \frac{1}{s} - \frac{1}{p} = p^2(s-p) \quad \text{or} \quad \frac{p-s}{sp} = p^2(s-p) \quad \text{or} \quad s = -\frac{1}{p^3}.$$

(v) Since $QP \perp PR$, their gradients must be negative reciprocals of each other.

Using **(i)**, QP has gradient $-1/qp$ and PR has gradient $-1/pr$. So,

$$-\frac{1}{qp} = pr \qquad \text{or} \qquad -\frac{1}{qr} \overset{3}{=} p^2.$$

Again from **(i)**, $-1/qr$ is the gradient of QR. From **(iii)**, p^2 is the gradient of the normal at P. Hence, $\overset{3}{=}$ says that QR is parallel to the normal at P.

A763 (9233 N2006/II/2)(i) Use the Quotient Rule:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\sqrt{x^2+32} - x\left(\frac{1}{2}\right)(x^2+32)^{-1/2}(2x)}{x^2+32} = \frac{x^2+32-x^2}{(x^2+32)^{3/2}} = \frac{32}{(x^2+32)^{3/2}}.$$

(ii) $$\int_2^7 \frac{1}{(x^2+32)^{3/2}}\,\mathrm{d}x \overset{\text{(i)}}{=} \frac{1}{32}\left[\frac{x}{(x^2+32)^{\frac{1}{2}}}\right]_2^7 = \frac{1}{32}\left[\frac{7}{\sqrt{81}} - \frac{2}{\sqrt{36}}\right]_2^7 = \frac{1}{32}\left(\frac{7}{9} - \frac{2}{6}\right) = \frac{1}{72}.$$

A776 (9758 N2017/II/5). XXX

A777 (9758 N2017/II/6). XXX

A778 (9758 N2017/II/7). XXX

A779 (9758 N2017/II/8). XXX

A780 (9758 N2017/II/9). XXX

A781 (9758 N2017/II/10). XXX

A782 (9740 N2016/II/5). XXX

A783 (9740 N2016/II/6). XXX

A784 (9740 N2016/II/7). XXX

A785 (9740 N2016/II/8). XXX

A786 (9740 N2016/II/9). XXX

A787 (9740 N2016/II/10). XXX

A788 (9740 N2015/II/5)(i) The manager may not have all the required information to properly implement stratified sampling. For example, he may not know what proportion of the sampling population each age group composes.

(ii) Decide what the age groups are and how many he wishes to survey from each group. (That is, for each age group, set a quota of respondents to be surveyed.) Then simply go around surveying customers he sees in the supermarket, until he meets the quota for each age group.

(iii) The manager may unconsciously gravitate towards customers that look more friendly. He may thus not get a representative sample of his customers (many of whom look unfriendly).

A789 (9740 N2015/II/6)(i) Let X be the number of red sweets in the packet.

$$\begin{aligned}
P(X \geq 4) &= 1 - P(X < 4) = 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) \\
&= 1 - 0.75^{10} - \binom{10}{1} 0.75^9 0.25 - \binom{10}{2} 0.75^2 0.25^8 - \binom{10}{3} 0.75^3 0.25^7 \\
&= 1 - 0.75^{10} - \binom{10}{1} 0.75^9 0.25 - \binom{10}{2} 0.75^2 0.25^8 - \binom{10}{3} 0.75^3 0.25^7 \\
&\approx 0.247501
\end{aligned}$$

(ii) $X \sim B(100, 0.25)$. Since $np = 25 > 5$ and $n(1-p) > 5$, the normal approximation $Y \sim N(25, 18.75)$ is suitable. Hence, using also the continuity correction,

$$\begin{aligned}
P(X \geq 30) = 1 - P(X < 30) \approx 1 - P(Y < 29.5) = 1 - \Phi\left(\frac{29.5 - 25}{\sqrt{18.75}}\right) \\
\approx 1 - \Phi(1.039) \approx 1 - 0.8506 = 0.1494.
\end{aligned}$$

(iii) Let $p = P(X \geq 30) \approx 0.1494$ and $q = 1 - P(X \geq 30) \approx 0.8506$. Then the desired

probability is

$$\binom{15}{0}q^{15}+\binom{15}{1}pq^{14}+\binom{15}{2}p^2q^{13}+\binom{15}{3}p^3q^{12}\approx 0.8245.$$

A790 (9740 N2015/II/7)(i) The rate at which errors are made is independent of the number of errors that have already been made.

The rate at which errors are made is constant throughout the newspaper.

(ii) Let $E \sim \text{Po}(6\cdot 1.3)=\text{Po}(7.8)$. Then

$$\text{P}(E>10)=1-\text{P}(E\le 10)=1-e^{-7.8}\left(\frac{7.8^0}{0!}+\frac{7.8^1}{1!}+\cdots+\frac{7.8^{10}}{10!}\right)\approx 0.164770.$$

(iii) Let $F\sim\text{Po}(1.3n)$. We are given that $\text{P}(F<2)<0.05$. That is,

$$e^{-1.3n}\left(\frac{(1.3n)^0}{0!}+\frac{(1.3n)^1}{1!}\right)<0.05 \qquad \text{or} \qquad e^{-1.3n}(1+1.3n)<0.05.$$

Let $f(n)=\mathrm{e}^{-1.3n}(1+1.3n)$. From calculator, $f(1),f(2),f(3)>0.05$ and $f(4)<0.05$. Hence, the smallest possible integer value of n is 4.

A791 (9740 N2015/II/8)(i)

$$\bar{x}=\frac{0.80+1.000+0.82+0.85+0.93+0.96+0.81+0.89}{8}=\frac{\sum x_i}{n}=0.8825,$$

$$s^2=\frac{\sum(x_i-\bar{x})^2}{n-1}=\frac{(0.80-0.8825)^2+(1.000-0.8825)^2+\cdots+(1.000-0.8825)^2}{7}\approx 0.005592857.$$

(ii) The null hypothesis is $H_0:\mu_0=0.9$ and the alternative hypothesis is $H_A:\mu_0<0.9$.

$$t=\frac{\bar{x}-\mu_0}{s/\sqrt{n}}=\frac{0.8825-0.9}{0.005592857/\sqrt{9}}\approx -0.661860.$$

Since, $|t|<t_{7,0.1}=-1.415$, we are unable to reject the null hypothesis at the 10% significance level.

A792 (9740 N2015/II/9)(i) By indep., $\text{P}(B|A)=\text{P}(B)=0.4$.

$$\textbf{(ii)}\text{P}(A\cup B\cup C)=\text{P}(A)+\text{P}(B)+\text{P}(C)-\text{P}(A\cap B)-\text{P}(A\cap C)-\text{P}(B\cap C)+\text{P}(A\cap B\cap C)$$

$$=0.45+0.4+0.3-0.45\cdot 0.4-0.45\cdot 0.3-\text{P}(B\cap C)+0.1=0.935-\text{P}(B\cap C)$$

$$\Longrightarrow \text{P}(A'\cap B'\cap C')=1-\text{P}(A\cup B\cup C)=0.065+\text{P}(B\cap C).$$

The above is true even if B and C are not independent.

And if B and C are independent, $\text{P}(B\cap C)=0.4\cdot 0.3=0.12$ and $\text{P}(A'\cap B'\cap C')=0.185$.

(iii) We know that $\text{P}(A\cap B'\cap C)=\text{P}(A\cap C)-\text{P}(A\cap B\cap C)=0.135-0.1=0.035$.

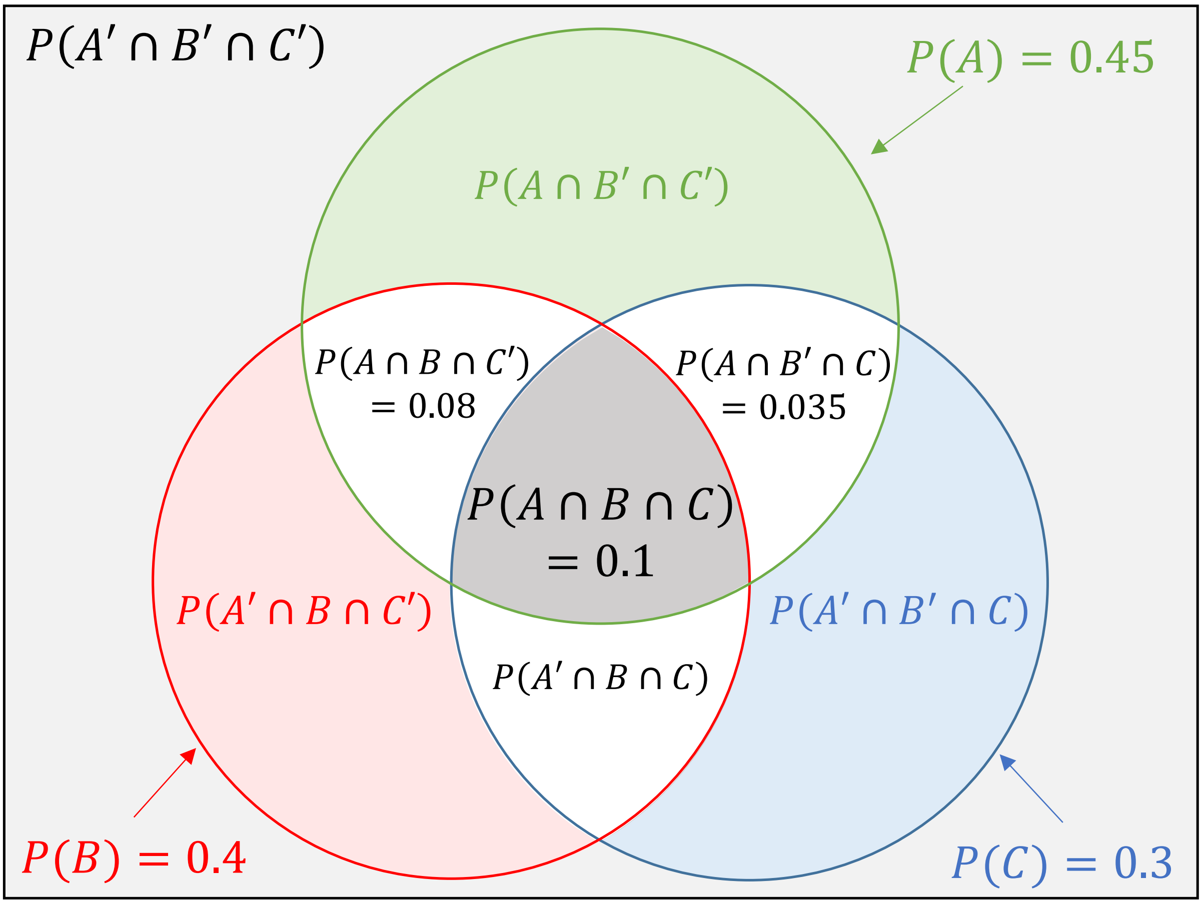
We want to find lower and upper bounds for $\text{P}(B\cap C)$. Refer to diagram below.

At one extreme, it could be that $P(A' \cap B \cap C) = 0$, in which case $P(B \cap C) = 0.1$.

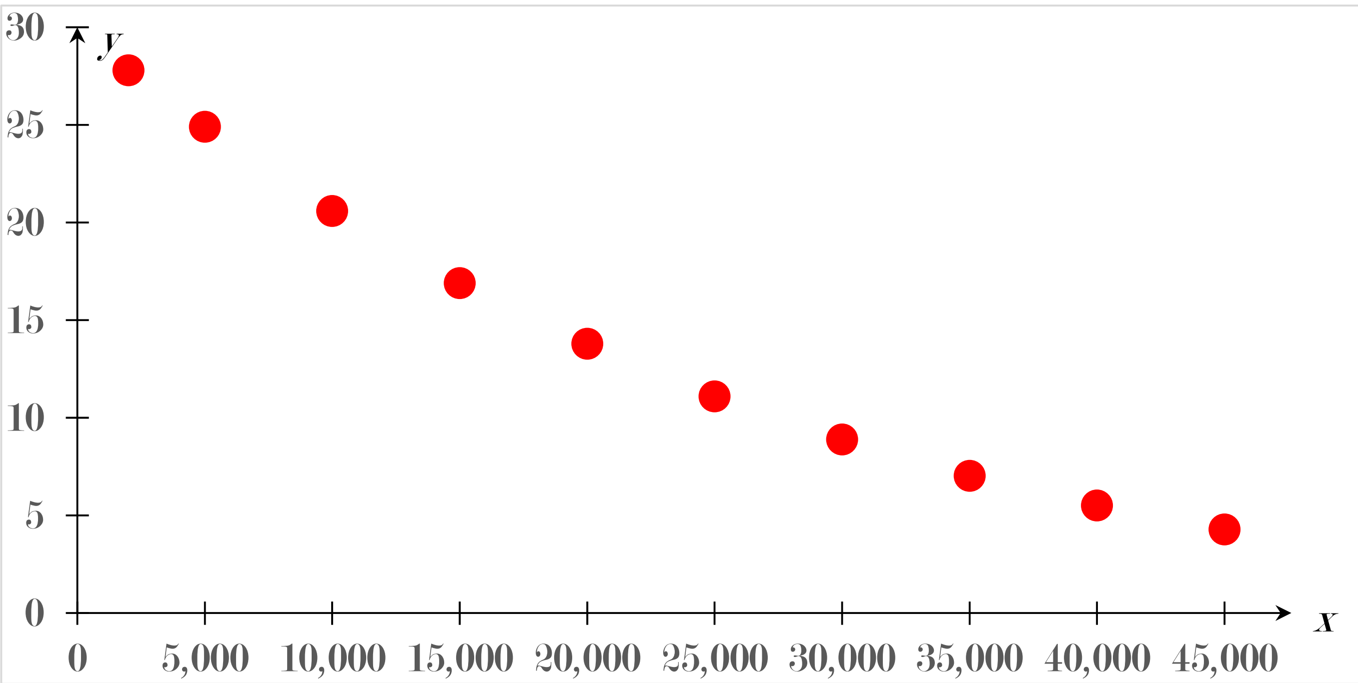
At the other extreme, it could be that $P(A' \cap B' \cap C) = 0$, in which case $P(B \cap C) = 0.265$.

Altogether, $P(A' \cap B' \cap C') = 0.065 + P(B \cap C) \in [0.165, 0.33]$

In $P(B \cap C) \geq P(A \cap B \cap C) = 0.1$, $P(B \cap C) \leq P(B) = 0.4$, $P(B \cap C) \leq P(C) = 0.3$.



A793 (9740 N2015/II/10)(i)



(ii) (a) PMCC ≈ -0.9807.

(ii) (b) PMCC ≈ -0.9748.

(ii) (c) PMCC ≈ -0.9986.

(iii) We are apparently supposed to presume that the greater the PMCC, the "better" or the "more appropriate". So we are supposed to use (c) from part (ii).

The estimated regression equation is $y - \bar{y} = b(x - \bar{x})$, where $b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2$. So in this case, the estimated regression equation is

$$P - 14.083 = -0.147\left(\sqrt{h} - 140.986\right).$$

(iv) Let x be the height given in metres. Then $3x = h$. Thus, the above equation may be rewritten as

$$P - 14.083 = -0.147\left(\sqrt{3x} - 140.986\right).$$

A794 (9740 N2015/II/11)(i) $8!/(2!2!) = 10080$.

(ii) There is only one arrangement where the letters are in alphabetical order, namely AABBCEGS. Hence, the number of these arrangements in which the letters are **not** in alphabetical order is $10080 - 1 = 10079$.

(iii) Treating the two A's as a single unit and the two B's as a single unit, we have 6 units altogether, so there are 6! arrangements.

(iv) Treating the two A's as a single unit, we have 7 units altogether, so there are $7!/2!$ arrangements.

Treating the two B's as a single unit, we have 7 units altogether, so there are $7!/2!$ arrangements.

Hence, the number of arrangements where there are at least two adjacent letters is $7!/2! + 7!/2! - 6! = 7! - 6!$, where the subtraction of 6! is to avoid double counting.

Hence, he number of different arrangements with no two adjacent letters the same is $8!/(2!2!) - (7! - 6!) = 5760$.

A795 (9740 N2015/II/12)(i) Let A_1, A_2, A_3, A_4, A_5 be independent random variables with the identical distribution $\mathrm{N}\left(300, 20^2\right)$. Then $F = A_1 + A_2 + A_3 + A_4 + A_5 \sim \mathrm{N}\left(5 \cdot 300, 5 \cdot 20^2\right)$ and

$$\begin{aligned}\mathrm{P}(F > 1600) &= 1 - \mathrm{P}(F \le 1600) = 1 - \Phi\left(\frac{1600 - 1500}{\sqrt{520}}\right) \\ &= 1 - \Phi\left(\sqrt{5}\right) \approx 1 - \Phi(2.236) \approx 1 - 0.9873 = 0.0127.\end{aligned}$$

(ii) Let $\mathrm{P} \sim \mathrm{N}\left(200, 15^2\right)$. Then $E = P_1 + P_2 + \cdots + P_8 \sim \mathrm{N}\left(8 \cdot 200, 8 \cdot 15^2\right)$. Then $F - E \sim \mathrm{N}\left(5 \cdot 300 - 8 \cdot 200, 5 \cdot 20^2 + 8 \cdot 15^2\right) = \mathrm{N}(-100, 3800)$ and

$$\begin{aligned}\mathrm{P}(F > E) &= \mathrm{P}(F - E > 0) = 1 - \mathrm{P}(F - E \le 0) = 1 - \Phi\left(\frac{0 - (-100)}{\sqrt{3810}}\right) \\ &= 1 - \Phi\left(\frac{10}{\sqrt{38}}\right) \approx 1 - \Phi(1.622) \approx 1 - 0.9476 = 0.0524.\end{aligned}$$

(iii) $0.85F + 0.9E \sim \mathrm{N}\left(0.85 \cdot 5 \cdot 300 + 0.9 \cdot 8 \cdot 200, 0.85^2 \cdot 5 \cdot 20^2 + 0.9^2 \cdot 8 \cdot 15^2\right) = \mathrm{N}(2715, 2903)$.

$$\mathrm{P}(0.85F + 0.9E < 2750) = \Phi\left(\frac{2750 - 2715}{\sqrt{2903}}\right) \approx \Phi(0.650) \approx 0.7422.$$

A796 (9740 N2014/II/5)(i) Arrange these 10000 customers by name, alphabetically. If two customers have the exact same same, then randomly pick one to precede the other.

From this list of alphabetically sorted customers, pick every 20th customer to survey.

(ii) Advantage: Each customer has equal probability of being surveyed.

Disadvantage: There is the small risk that there is some periodic pattern that could bias the sample. For example, it could be that the customers are all in some country (or concentration camp), where each person has a 9-digit number for a name (e.g. 001533123) and only the most-privileged persons have 7 as the last digit of their name. If so, our proposed method would omit all the most-privileged persons.

Such a pattern is obviously contrived and absurdly unlikely. In practice, it is unlikely that my proposed method of systematic sampling is any different from purely random sampling.

A797 (9740 N2014/II/6)(i) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = 31\,500.$

(ii) Ways to include only the midfielder brother $= \begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}.$

Ways to include only the attacker brother $= \begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix}.$

In total, 16 800 ways.

(iii) The club now has 3 goalkeepers, 8 defenders, 3 midfielders, 5 attackers, and one player (call him Apu) who can either be a midfielder or a defender.

Ways to form a team without Apu $= \begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = 3\,150.$

Ways to form a team with Apu as a midfielder $= \begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = 3\,150.$

Ways to form a team with Apu as a defender $= \begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 8 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = 2\,520.$

In total, 8 820 ways.

A798 (9740 N2014/II/7)(i) Let X be the number of sixes rolled. Then $X \sim \mathrm{B}\left(10, \frac{1}{6}\right)$.

And $\mathrm{P}(X = 3) = \begin{pmatrix} 10 \\ 3 \end{pmatrix}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^7 = \frac{10!5^7}{3!6^{10}} = 0.155045.$

(ii) Let Y be the number of sixes rolled. Then $Y \sim \mathrm{B}\left(60, \frac{1}{6}\right)$. We have $np > 5$ and $n(1-p) > 5$. So $Z = \mathrm{N}\left(10, \frac{50}{6}\right)$ is a suitable approximate distribution for Y. Using also the continuity correction, we have

$$P(5 \le Y \le 8) \approx P(4.5 < Z < 8.5) = \Phi\left(\frac{8.5-10}{\sqrt{50/6}}\right) - \Phi\left(\frac{4.5-10}{\sqrt{50/6}}\right) \approx \Phi(-0.520) - \Phi(-1.905)$$

$$= \Phi(1.905) - \Phi(0.520) \approx 0.9716 - 0.6985 = 0.2731.$$

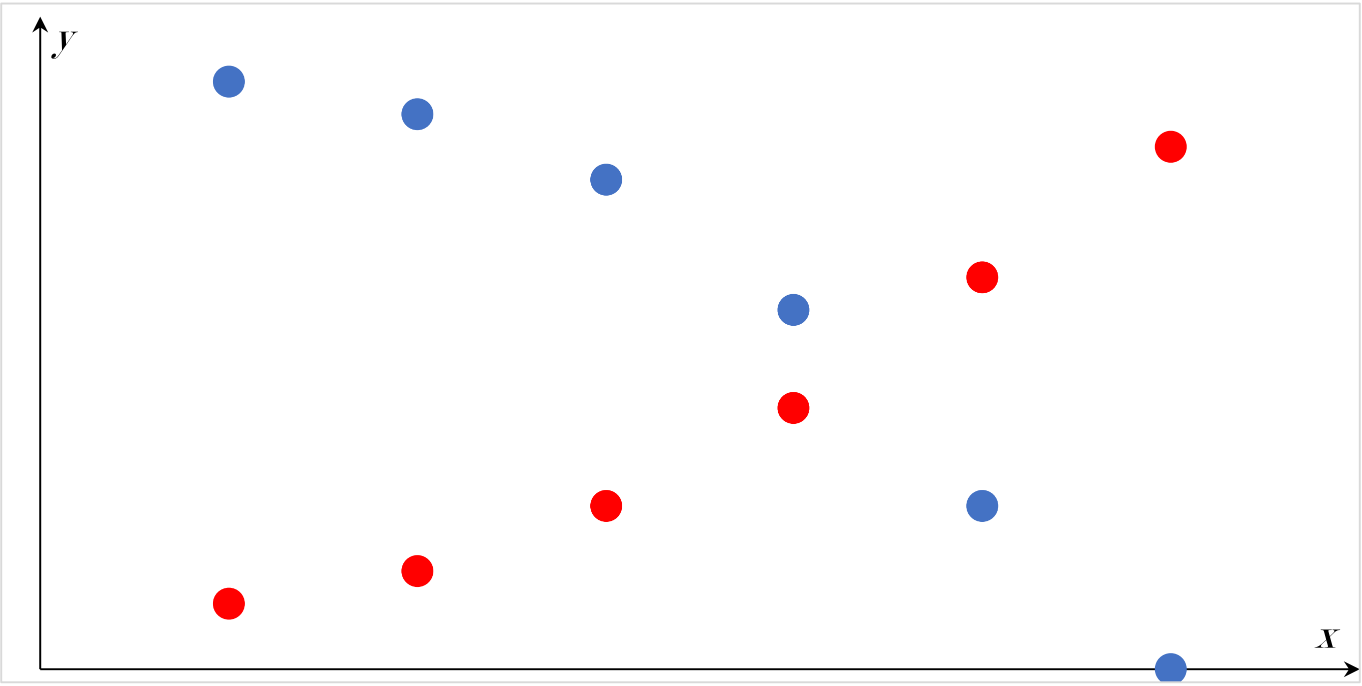
(Without using an approximation, $P(5 \le Y \le 8) \approx 0.291854$.)

(iii) Let A be the number of sixes rolled. Then $A \sim B\left(60, \frac{1}{15}\right)$. We have $n > 20$ and $np < 5$. So $B = \text{Po}(4)$ is a suitable approximate distribution for A.

$$P(5 \le A \le 8) \approx P(5 \le B \le 8) = e^{-4}\left(\frac{4^5}{5!} + \frac{4^6}{6!} + \frac{4^7}{7!} + \frac{4^8}{8!}\right) \approx 0.349800.$$

(Without using an approximation, $P(5 \le A \le 8) \approx 0.353659$.)

A799 (9740 N2014/II/8)(a) Case **(i)** is in red and case **(ii)** is in blue.



(b) (i) (A) PMCC ≈ -0.9470452.

(b) (i) (B) PMCC ≈ -0.974921.

(ii) It's not at all clear which is the better model. But apparently we are supposed to say that since the second model is better because the magnitude of its PMCC is greater.

In general, the estimated regression equation is $y - \bar{y} = b(x - \bar{x})$, where $b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2$. So in this case, the estimated regression equation is

$$P - 72590 \approx -33659.728(\ln m - 3.657)$$
$$\iff P \approx -33659.728 \ln m + 195693.560.$$

(iii) $P(50) \approx -33659.728 \ln 50 + 195693.560 \approx 64016$.

A800 (9740 N2014/II/9)(i) Let X be the number of minutes a bus is late after the new company has taken over. We'll assume $X \sim N(\mu, \sigma^2)$.

Our null hypothesis is $H_0 : \mu = \mu_0 = 4.3$ and our alternative hypothesis is $H_A : \mu < \mu_0 = 4.3$.

(ii) The null hypothesis is not rejected if $\bar{t} > \mu_0 - t_{9,0.1} \cdot k/\sqrt{n} = 4.3 - 1.383 \cdot \sqrt{3.2}/\sqrt{10} \approx 3.518$.

(iii) The null hypothesis is rejected if $\bar{t} < \mu_0 - t_{9,0.1} \cdot k/\sqrt{n}$ or $4.0 < 4.3 - 1.383 \cdot k/\sqrt{10}$ or $k > 0.3\sqrt{10}/1.383$ or $k^2 > 0.3^2 \cdot 10/1.383^2 \approx 0.471$.

A801 (9740 N2014/II/10)(i)(a) $0.1 \cdot 0.2 \cdot 0.1 = 0.002$.

(i)(b) The probability that no $\star$ is displayed is $0.9 \cdot 0.8 \cdot 0.9 = 0.648$. And so the probability that t least one $\star$ symbol is displayed is $1 - 0.648 = 0.352$.

(i)(c) $\mathrm{P}(\times \times +) = 0.3 \cdot 0.1 \cdot 0.2$, $\mathrm{P}(\times + \times) = 0.3 \cdot 0.3 \cdot 0.4$, $\mathrm{P}(+ \times \times) = 0.4 \cdot 0.1 \cdot 0.4$.

Thus, the desired probability $0.006 + 0.036 + 0.016 = 0.058$.

(ii) The probability that there is exactly one $\star$ is $\mathrm{P}(\star \not\star \not\star) + \mathrm{P}(\not\star \star \not\star) + \mathrm{P}(\not\star \not\star \star) = 0.1 \cdot 0.8 \cdot 0.9 + 0.9 \cdot 0.2 \cdot 0.9 + 0.9 \cdot 0.8 \cdot 0.1 = 0.306$.

The probability that the symbols are $\star$, $+$, $\bigcirc$ (in any order) is

$$\begin{aligned}
&\mathrm{P}(\star + \bigcirc) + \mathrm{P}(\star \bigcirc +) + \mathrm{P}(+ \star \bigcirc) + \mathrm{P}(\bigcirc \star +) + \mathrm{P}(+ \bigcirc \star) + \mathrm{P}(\bigcirc + \star) \\
&= 0.1(0.3 \cdot 0.3 + 0.4 \cdot 0.2) + 0.2(0.4 \cdot 0.3 + 0.2 \cdot 0.2) + 0.1(0.4 \cdot 0.4 + 0.2 \cdot 0.3) \\
&= 0.017 + 0.032 + 0.022 = 0.071
\end{aligned}$$

Hence, the desired probability is $0.071/0.306 = 71/306 \approx 0.232026$.

A802 (9740 N2014/II/11)(i)(a) Let $O \sim \mathrm{Po}(2)$ and $P \sim \mathrm{Po}(11)$. Then

$$\mathrm{P}(P > 8) = 1 - \mathrm{P}(P \le 8) = 1 - e^{-11}\left(\frac{11^0}{0!} + \frac{11^1}{1!} + \cdots + \frac{11^8}{8!}\right) \approx 0.768015.$$

(i)(b) $O + P \sim \mathrm{Po}(13)$. So

$$\mathrm{P}(O + P < 15) = e^{-13}\left(\frac{13^0}{0!} + \frac{13^1}{1!} + \cdots + \frac{13^{14}}{14!}\right) \approx 0.675132.$$

(ii) Let $Q \sim \mathrm{Po}(2n)$. We are given that $\mathrm{P}(Q < 3) < 0.01$. That is,

$$\mathrm{P}(Q < 3) = e^{-2n}\left(\frac{(2n)^0}{0!} + \frac{(2n)^1}{1!} + \frac{(2n)^2}{2!}\right) = e^{-2n}\left(1 + 2n + 2n^2\right) < 0.01.$$

Let $f(n) = \mathrm{e}^{-2n}\left(1 + 2n + 2n^2\right)$. From calculator, $f(1), f(2), f(3), f(4) > 0.01$ and $f(5) < 0.01$. Hence, the smallest possible integer value of n is 5.

(iii) Let $R \sim \mathrm{Po}(52 \cdot 11) = \mathrm{Po}(572)$. Given a large sample, we can use the normal distribution $S \sim \mathrm{N}(572, 572)$as an approximation. Hence, using also the continuity correction,

$$\mathrm{P}(R > 550) \approx \mathrm{P}(S > 550.5) = 1 - \mathrm{P}(S < 550.5) = 1 - \Phi\left(\frac{550.5 - 572}{\sqrt{572}}\right)$$

$$\approx 1 - \Phi(-0.898960) = \Phi(0.898960) \approx 0.8158.$$

(iv) Sales may be seasonal—e.g. it may be that art collectors make most of their purchases in the northern hemisphere's summer months.

The sales of originals and prints may not be independent of each other. E.g., an art collector who buys an original Picasso might wish to also buy a few copies thereof.

A803 (9740 N2013/II/5)(i) Use a computer program to randomly sort the 100000 employees into an ordered list. Pick the first 90 employees on the list.

The Chief Executive's idea of a representative sample might be to have each country's employees proportionally represented. For example, if 10% of employees are from India, then she may want 9 of the invited employees to be from India.

(ii) Stratified sampling is more appropriate. If say 10% of employees are from India, 30% from China, 20% from Thailand, and 40% from Singapore, then we could instead pick from the list the first 9 Indian employees, the first 27 Chinese employees, the first 18 Thai employees, and the first 36 Singaporean employees.

A804 (9740 N2013/II/6). $\mathrm{P}(Y < 2a) = \mathrm{P}\left(Z < \frac{2a-\mu}{\sigma}\right) = 0.95 \implies \frac{2a-\mu}{\sigma} \approx 1.645 \iff \frac{2a-\mu}{1.645} \overset{1}{\approx} \sigma.$

$\mathrm{P}(Y < a) = \mathrm{P}\left(Z < \frac{a-\mu}{\sigma}\right) = 0.25 \implies \frac{a-\mu}{\sigma} \approx -0.674 \iff \mu - a \approx 0.674\sigma \overset{1}{\approx} 0.674\frac{2a-\mu}{1.645}$
$\iff \mu\left(1+\frac{0.674}{1.645}\right) \approx \left(1+\frac{2\cdot 0.674}{1.645}\right)a \iff \mu \approx 1.29a$. That is, $k \approx 1.29$.

A805 (9740 N2013/II/7)(i) The probability that one packet contains a free gift is independent of why another packet contains a free gift.

There is no possibility that any one packet contains two or more free gifts.

(ii) Let $F \sim \mathrm{B}\left(20, \frac{1}{20}\right)$. Then $\mathrm{P}(F=1) = \binom{20}{1}\left(\frac{1}{20}\right)\left(\frac{19}{20}\right)^{19} \approx 0.377354$.

(iii) Let $F \sim \mathrm{B}\left(60, \frac{1}{20}\right)$. Since $n = 60$ is large and $np = 3$ is small, a suitable approximation for F is $G \sim \mathrm{Po}(3)$.

$$\mathrm{P}(F \geq 5) \approx \mathrm{P}(G \geq 5) = 1 - e^{-3}\left(\frac{3^0}{0!}+\frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\frac{3^5}{5!}\right) \approx 0.184737.$$

(By comparison, the actual probability is $\mathrm{P}(F \geq 5) \approx 0.180335$.)

A806 (9740 N2013/II/8)(i) $\mathrm{P}(B \cap A') = \mathrm{P}(B|A')\mathrm{P}(A') = 0.8 \times 0.3 = 0.24$.

(ii) $\mathrm{P}(A' \cap B') = 1 - [\mathrm{P}(A) + \mathrm{P}(B \cap A')] = 1 - 0.7 - 0.24 = 0.06$.

(iii) $\mathrm{P}(A'|B') = 1 - \mathrm{P}(A|B') = 0.18$.

$\mathrm{P}(B') = \frac{\mathrm{P}(A' \cap B')}{\mathrm{P}(A'|B')} = \frac{0.06}{0.12} = 0.5$

$$\begin{aligned}\mathrm{P}(A \cap B) &= 1 - [\mathrm{P}(A') + \mathrm{P}(A \cap B')] = 1 - 0.3 - \mathrm{P}(A \cap B') \\ &= 0.7 - \mathrm{P}(A|B')\mathrm{P}(B') = 0.7 - 0.88 \times 0.5 = 0.26.\end{aligned}$$

A807 (9740 N2013/II/9)(i)

$$\bar{x} = \frac{\sum x_i}{n} = \frac{14.0+12.5+11.0+11.0+12.5+12.6+15.6+13.2}{8} = 12.8,$$
$$s^2 = \frac{\sum(x_i-\bar{x})^2}{n-1} = \frac{(14.0-12.8)^2+(12.5-12.8)^2+\cdots+(13.2-12.8)^2}{7} \approx 2.305714.$$

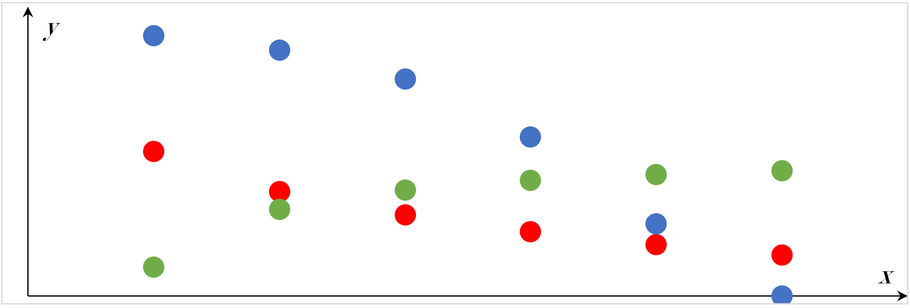
(ii) The necessary assumption is that the population is normally distributed.

The null hypothesis is $H_0 : \mu_0 = 13.8$ and the alternative hypothesis is $H_A : \mu_0 < 13.8$.

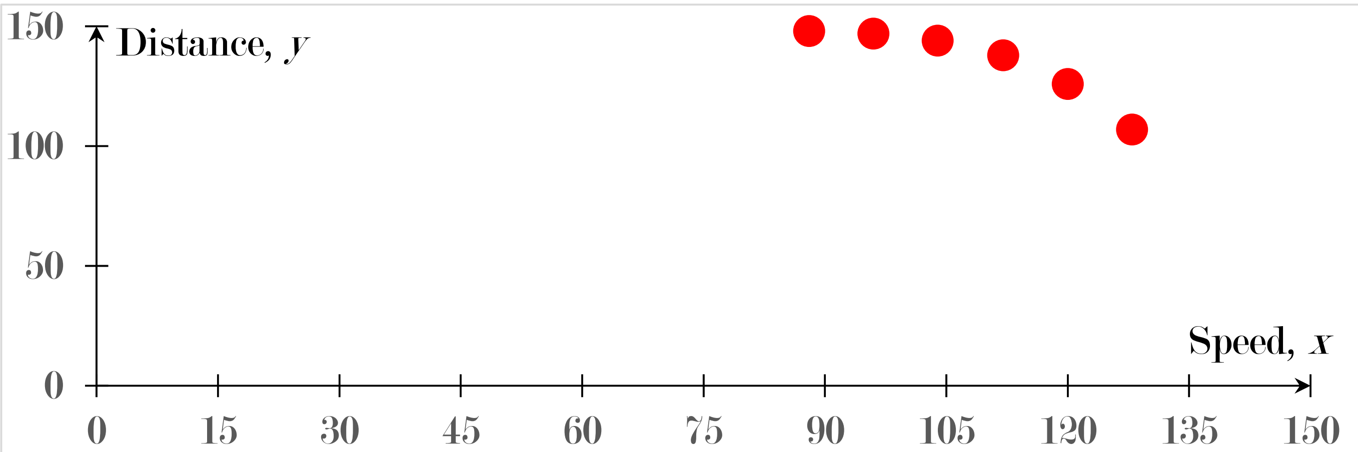
$$t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{12.8-13.8}{\sqrt{2.305714/8}} \approx -1.862697.$$

Since $|t| < t_{7,0.05} = 1.895$, we are unable to reject the null hypothesis at the 5% significance level.

A808 (9740 N2013/II/10)(i) In blue is case **(A)**, in red is case **(B)**, and in green is case **(C)**.



(ii)



(iii) As a function of speed, the distance travelled decreases at an increasing rate. So (A) is the most appropriate.

PMCC ≈ -0.939203.

(iv) In general, the estimated regression equation is $y-\bar{y} = b(x-\bar{x})$, where $b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2$. So in this case, the estimated regression equation is

$$y - 135 \approx -0.00461978\left(x^2 - 11850.66667\right)$$
$$\iff y \approx -0.00461978x^2 + 189.747528.$$

Thus, $y(110) \approx -0.00461978(110)^2 + 189.747528 \approx 134$.

A809 (9740 N2013/II/11)(i) The total number of ways to choose a code is $26^3 9^2$ (1423656). The number of ways to choose a code with three different letters and two different digits is 26·25·24·9·8 (1123200). Hence, the desired probability is $26{\cdot}25{\cdot}24{\cdot}9{\cdot}8/\left(26^3 \cdot 9^2\right) = \dfrac{400}{507} \approx 0.78895$.

(ii) The number of ways to choose the two digits so that the second digit is larger than the first is $1+2+\cdots+8 = 36$. Hence, the desired probability is $(1+2+\cdots+8)/9^2 = \dfrac{4}{9} = 0.\dot{4}$.

(iii) The number of ways to choose a code with exactly two letters the same, but not two digits the same is

$$\overbrace{26}^{\text{Repeated letter}} \times \overbrace{25}^{\text{Third letter}} \times \overbrace{\frac{3!}{2!}}^{\text{Arrange these three letters}} \times \overbrace{9 \cdot 8}^{\text{Two digits}} = 26{\cdot}25{\cdot}3{\cdot}9{\cdot}8 = 140400.$$

The number of ways to choose a code with exactly two digits the same, but not exactly two letters the same is

$$\overbrace{9}^{\text{Repeated digit}} \times \left(\overbrace{26 \cdot 25 \cdot 24}^{\substack{\text{All three}\\ \text{letters different}}} + \overbrace{26}^{\substack{\text{All three}\\ \text{letters same}}} \right) = 9 \cdot 26 \cdot 601 = 140634.$$

Hence the desired probability is $\dfrac{26 \cdot 25 \cdot 3 \cdot 9 \cdot 8 + 9 \cdot 26 \cdot 601}{26^3 \cdot 9^2} = \dfrac{25 \cdot 3 \cdot 8 + 601}{26^2 \cdot 9} = \dfrac{1201}{6084} \approx 0.197$.

(iv) There are 4 ways to choose the even digit, 5 to choose the odd digit then 2 ways to arrange these two digits. Hence, there are $4 \cdot 5 \cdot 2 = 40$ ways to choose the two digits.

There are 5 ways to choose the vowel. There are 21^2 ways to choose the two consonants. We can now slot in the vowel amidst the consonants in 3 different ways. Hence, there are $5 \cdot 21^2 \cdot 3$ ways to choose the three letters.

Altogether then, there are $5 \cdot 21^2 \cdot 3 \cdot 4 \cdot 5 \cdot 2$ ways to choose a code with exactly one vowel and exactly one even digit.

Hence the desired probability is $\dfrac{5 \cdot 21^2 \cdot 3 \cdot 4 \cdot 5 \cdot 2}{26^3 9^2} = \dfrac{5 \cdot 7^2 \cdot 5}{13^3 \cdot 3} = \dfrac{1225}{6591} \approx 0.18586$.

A810 (9740 N2013/II/12)(i) #1. The number of people sick on a particular day is independent of how many were sick the previous day. #2. The average number of illnesses in any span of 30 days is the same, throughout the course of the year.

Condition #1 may not be met if the illness is contagious. If so, we'd expect the number of people sick on a particular day to depend (positively) on how many were sick the previous day.

Condition #2 may not be met if the illnesses are seasonal. For example, due to influenza, illnesses may be more common during the winter than during the summer.

Let $A \sim \text{Po}(1.2)$ and $M \sim \text{Po}(2.7)$.

(ii) Let $B \sim \text{Po}(1.2n)$. Then $\text{P}(B = 0) < 0.01 \iff \text{e}^{-1.2n} < 0.01 \iff n > (\ln 0.01)/(-1.2) \approx 3.8$. Hence, the smallest number of days is 4.

(iii) Let C be the total number of days of absence across both departments, over a 5-day period. Then $C \sim \text{Po}(19.5)$ and

$$\text{P}(C > 20) = 1 - \text{P}(C \leq 20) = 1 - e^{-19.5} \sum_{i=0}^{20} \frac{19.5^i}{i!} \approx 0.396583.$$

(iv) Let D be the total number of days of absence across both departments, over a 60-day period. Then $D \sim \text{Po}(234)$. Since $\lambda_D = 234$ is large, the normal distribution is a suitable approximation. Let $E \sim \text{N}(234, 234)$. Then,

$$\text{P}(200 \leq D \leq 250) \approx \text{P}(199.5 \leq E \leq 250.5) = \Phi\left(\frac{250.5 - 234}{\sqrt{234}}\right) - \Phi\left(\frac{199.5 - 234}{\sqrt{234}}\right)$$

$$\approx \Phi(1.0786) - \Phi(-2.2553) \approx 0.8597 - 0.0120 = 0.8477.$$

A811 (9740 N2012/II/5)(i)(a) Let $+$, $-$, D, and N denote the events "positive result", "negative result", "has disease", and "no disease". Then,

$$\text{P}(+) = \text{P}(+|D)\text{P}(D) + \text{P}(+|N)\text{P}(N) = p \cdot 0.001 + (1-p) \cdot 0.999 = 0.999 - 0.998p = 0.00599.$$

(i)(b) $\text{P}(D|+) = \text{P}(D \cap +) \div \text{P}(+) = \text{P}(D)\text{P}(+|D) \div \text{P}(+) = 0.001p \div 0.00599 \approx 0.166110.$

(ii) as$\text{P}(D|+) = 0.75$. But

$$\text{P}(D|+) = \frac{0.001p}{0.999 - 0.998p}.$$

So $3(0.999 - 0.998p) = 4(0.001p)$ or $2.997 = 2.998p$ or $p \approx 0.999666$.

A812 (9740 N2012/II/6)(i) $H_0 : \mu_0 = 14.0$, $H_A : \mu_0 \neq 14.0$.

(ii) $H_0 : \bar{x} \sim \text{N}\left(14.0, 3.8^2\right)$. Since $Z_{0.025} = 1.96$, the values of $\bar{x}$ for which the null hypothesis would not be rejected are

$$\bar{x} \in \left(\mu - Z_{0.025}\frac{\sigma}{\sqrt{n}}, \mu + Z_{0.025}\frac{\sigma}{\sqrt{n}}\right) = \left(14.0 - 1.96\frac{3.8}{\sqrt{20}}, 14.0 + 1.96\frac{3.8}{\sqrt{20}}\right) \approx (12.335, 15.665).$$

(iii) The null hypothesis is rejected.

A813 (9740 N2012/II/7)(i) There are 15! ways to arrange the 15 individuals.

There are 2 ways to arrange the 2 sisters as a single unit. Counting the 2 sisters as a single unit, we have 14 units total, and there are 14! ways to arrange these 14 units. So, there are in total $2 \cdot 14!$ ways to arrange the 15 individuals so that the two sisters are together.

Hence, the probability that the sisters are next to each other is $2 \cdot 14!/15! = 2/15 = 0.1\dot{3}$.

(ii) There are 3! ways to arrange the 3 brothers as a single unit. Counting the 3 brothers as a single unit, we have 13 units total, and there are 13! ways to arrange these 13 units. So, there are in total $3! \cdot 13!$ ways to arrange the 15 individuals so that the three brothers

are together. We do not want the three brothers to be together.

Hence, the desired probability is $1-3!\cdot 13!/15! = 1-6/(14\cdot 15) = 1-1/35 = 34/35 \approx 0.97142857$.

(iii) There are 2 ways to arrange the 2 sisters as a single unit and 3! ways to arrange the 3 brothers as a single unit. Counting the 2 sisters as a single unit and also the 3 brothers as a single unit, we have 12 units in total, and there are 12! ways to arranges these 12 units. So, there are in total $2\cdot 3!\cdot 12!$ ways to arrange the 15 individuals so that the 2 sisters are together and the 2 brothers are together.

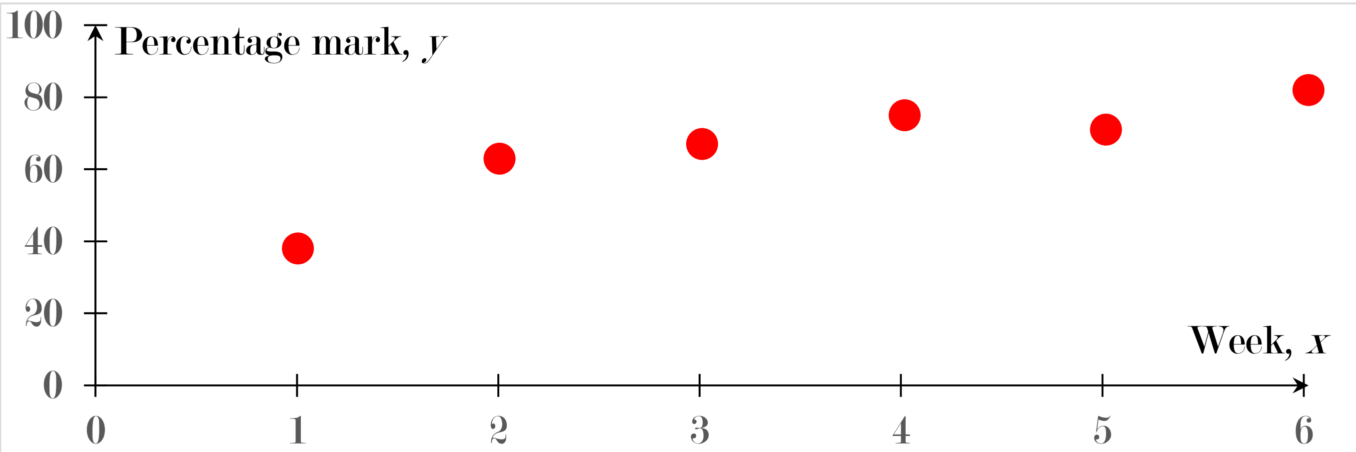
Hence, the desired probability is $2\cdot 3!\cdot 12!/15! = 12/(13\cdot 14\cdot 15) = 2/(13\cdot 7\cdot 5) = 2/455 \approx 0.0043956$.

(iv) Let A and B denote the events that "the sisters are next to each other" and "the brothers are next to each other". Our desired probably is $\mathrm{P}(A\cup B)$.

$$\mathrm{P}(A\cup B) = \mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B) = \frac{2}{15}+\frac{1}{35}-\frac{2}{455} = \frac{91\cdot 2}{3\cdot 455}+\frac{13}{455}-\frac{2}{455}$$

$$= \frac{91\cdot 2}{3\cdot 455}+\frac{33}{3\cdot 455} = \frac{43}{3\cdot 91} = \frac{43}{243} \approx 0.17695.$$

A814 (9740 N2012/II/8)(i)



(ii) The trend is one of steady improvement. After a terrible performance in Week 1, Amy resolves to work hard. Her work pays off, with her mark improving week after week.

The only deviation from trend occurs on Week 5, because Amy happened to be experimenting with drugs that week.

(iii) A linear model would suggest that she eventually breaks the 100% barrier, which is quite impossible.

A quadratic model would suggest that her mark eventually starts falling and moreover at an increasing rate, which is quite improbable, unless of course she gets hooked on drugs.

(iv) PMCC ≈ -0.929744.

(v) We are supposed to say that the most appropriate choice is wherever the magnitude of the PMCC is the largest. Hence, $L = 92$ is the most appropriate.

(vi) In general, the estimated regression equation is $y-\bar{y} = b(x-\bar{x})$, where $b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2$. So in this case, the estimated regression equation is

$$\ln(92-y) - 3.125912 \approx -0.279599(x-3.5)$$
$$\iff \ln(92-y) \approx -0.279599x + 4.104510.$$

$y \geq 90 \iff -0.28x + 4.10 \leq \ln 2 \iff x \gtrsim 12.2$. So she'll get at least 90% in Week 13.

(vii) As $x \to \infty$, $y \to L$. An interpretation is thus that L is the best mark she can ever hope to get, no matter how long she spends studying.

A815 (9740 N2012/II/9)(i) The choice must be binary—a voter must be said to either support the Alliance Party or not support it.

The probability that any one polled voter supports the Party is independent of whether another polled voter supports the party.

(ii) $\mathrm{P}(A=3) + \mathrm{P}(A=4) = \begin{pmatrix} 30 \\ 3 \end{pmatrix} p^3(1-p)^{27} + \begin{pmatrix} 30 \\ 4 \end{pmatrix} p^4(1-p)^{26} \approx 0.373068.$

(iii)(a) $np = 16.5 > 5$ and $n(1-p) = 13.5 > 5$ are both large and so yes, the normal distribution $\mathrm{N}(16.5, 16.5 \cdot 0.45)$would be a suitable approximation for A.

(iii)(b) p is large. And so, while it is certainly *possible* to use the Poisson distribution as an approximation, it would fare poorly.

(iv) $\mathrm{P}(A=15) = \begin{pmatrix} 30 \\ 15 \end{pmatrix} p^{15}(1-p)^{15} \approx 0.06864.$

Thus, $p(1-p) = p - p^2 \approx \left[0.06864 \Big/ \begin{pmatrix} 30 \\ 15 \end{pmatrix}\right]^{1/15} \approx 0.237900.$

Rearranging, $p^2 - p + 0.237900 = 0$. By the quadratic formula, $p \approx 0.39, 0.61$. Given that $p < 0.5$, we have $p \approx 0.39$.

A816 (9740 N2012/II/10)(i) The number of gold coins in a randomly chosen square metre is independent of how many gold coins there are in the square metre to its left.

No two coins are stacked exactly on top of each other.

(ii) Let $G \sim \mathrm{Po}(0.8)$. Then,

$$\mathrm{P}(G \geq 3) = 1 - e^{-0.8}\left(\frac{0.8^0}{0!} + \frac{0.8^1}{1!} + \frac{0.8^2}{2!}\right) \approx 0.0474226.$$

(iii) Let $H \sim \mathrm{Po}(0.8x)$. Then $\mathrm{P}(H=1) = \mathrm{e}^{-0.8x}(0.8x) = 0.2$.

By calculator, $x \approx 0.323964, 3.1783$. So $x \approx 0.323964$.

(iv) Let $I \sim \mathrm{Po}(80)$. Since λ is large, the normal distribution $J \sim \mathrm{N}(80, 80)$ is a suitable approximation. Using also the continuity correction:

$$\mathrm{P}(I \geq 90) \approx \mathrm{P}(J \geq 89.5) = 1 - \Phi\left(\frac{89.5 - 80}{\sqrt{80}}\right) \approx 1 - \Phi(1.062) \approx 1 - 0.8559 = 0.1441.$$

(v) Let $P \sim \mathrm{Po}(3)$. Let Z be the number of gold coins and pottery shards found in $50\,\mathrm{m}^2$. Then $Z \sim \mathrm{Po}(190)$. Since λ is large, the normal distribution $Q \sim \mathrm{N}(190, 190)$ is a suitable approximation for Z. Using also the continuity correction,

$$\mathrm{P}\left(Z \geq 200\right) \approx \mathrm{P}(Q \geq 199.5) = 1 - \Phi\left(\frac{199.5 - 190}{\sqrt{190}}\right) \approx 1 - \Phi(0.6892) \approx 1 - 0.7546 = 0.2454.$$

(vi) Let X and Y be, respectively, the numbers of gold coins and pottery shards found in $50\,\mathrm{m}^2$. Then $X \sim \mathrm{Po}(40)$ and $Y \sim \mathrm{Po}(150)$. Our goal is to find $\mathrm{P}(Y \geq 3X) = \mathrm{P}(Y - 3X \geq 0)$.

Since $\lambda_X = 40$ and $\lambda_Y = 150$ are both large, the normal distributions $A \sim \mathrm{N}(40, 40)$ and $B \sim \mathrm{N}(150, 150)$ are suitable approximations for X and Y, respectively. And in turn, $B - 3A \sim \mathrm{N}(150 - 3 \cdot 40, 150 + 3^2 \cdot 40) = \mathrm{N}(30, 510)$ is a good approximation for $Y - 3X$. Hence, using also the continuity correction,

$$\mathrm{P}(Y - 3X \geq 0) \approx \mathrm{P}(B - 3A \geq -0.5) = 1 - \Phi\left(\frac{-0.5 - 30}{\sqrt{510}}\right) = \Phi\left(\frac{30.5}{\sqrt{510}}\right) \approx \Phi(1.3506) \approx 0.9116.$$

A817 (9740 N2011/II/5)(i) $\mathrm{P}(X < 40.0) = \mathrm{P}\left(Z < \dfrac{40.0 - \mu}{\sigma}\right) = 0.05 \iff \dfrac{40.0 - \mu}{\sigma} \approx -1.645 \iff \mu \overset{1}{\approx} 1.645\sigma + 40.0$

$\mathrm{P}(X < 70.0) = \mathrm{P}\left(Z < \dfrac{70.0 - \mu}{\sigma}\right) = 0.975 \iff \dfrac{70.0 - \mu}{\sigma} \approx 1.96 \iff \mu \overset{2}{\approx} -1.96\sigma + 70.0.$

Comparing $\overset{1}{\approx}$ and $\overset{2}{\approx}$, we have $1.645\sigma + 40.0 \approx -1.96\sigma + 70.0 \iff 3.605\sigma \approx 30.0 \iff \sigma \approx 8.3$ and $\mu \approx 53.7$.

A818 (9740 N2011/II/6)(i) Decide what the age groups will be. Decide how many from each age group are to be interviewed (these are our *quotas*). Then pick, at random, residents on the street to be interviewed, until the quota for every age group is fulfilled.

(ii) Residents who are on the street may not be a representative sample of the population.

(iii) Random sampling. Acquire a complete list of the city suburb's population. Use a computer program to randomly pick a sample. Interview this sample.

No it is not realistic. First, one may be able to acquire a complete list of the city suburb's population. Second, one may not be able to contact every member of one's sample.

A819 (9740 N2011/II/7)(i) #1. I do indeed make an actual attempt to contact n different friends.

#2. The probability that one friend is contactable is independent of whether another friend is contactable.

(ii) Assumption #1 may not hold because if say $n = 100$, I may run out of time before I attempt to contact all 100 different friends.

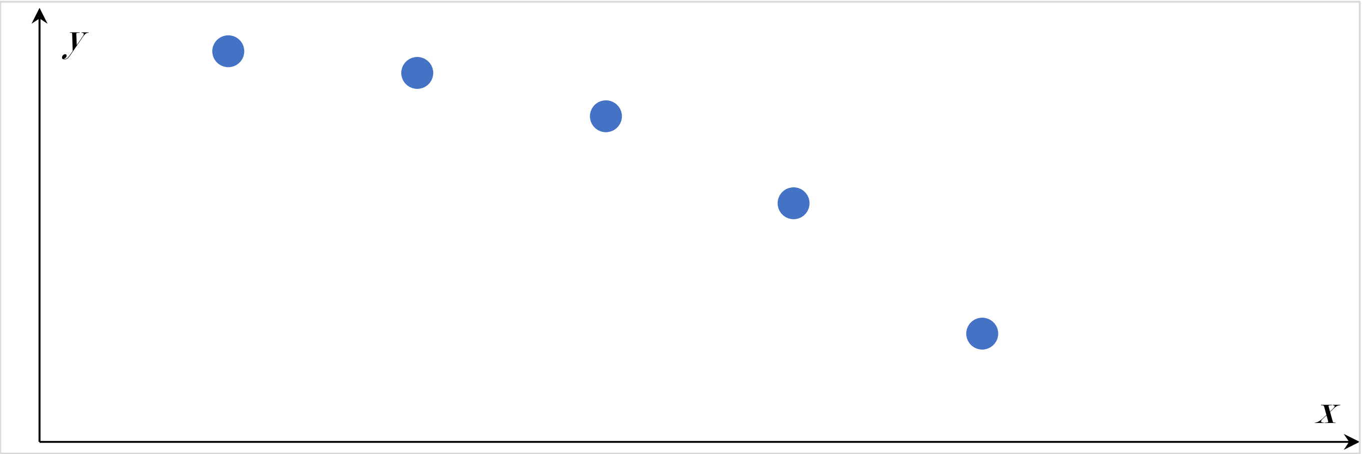
Assumption #2 may not hold because my friends probably know each other and so they might be watching a movie together and their handphones are switched off. This would mean that the probability that one friend is contactable is dependent on whether another friend is contactable.

(iii) $\mathrm{P}(R \geq 6) = 1 - \displaystyle\sum_{i=0}^{5} \mathrm{P}(R = i) = 1 - \sum_{i=0}^{5} \binom{5}{i} 0.7^i 0.3^{5-i} \approx 0.551774.$

(iv) Since $np = 28 > 5$ and $n(1-p) = 12 > 5$ are both large, a suitable approximation to R is the normal distribution $S \sim \mathrm{N}\left(28, 8.4\right)$. Using also the continuity correction, we have

$$P(R < 25) \approx P(S < 24.5) = \Phi\left(\frac{24.5 - 28}{\sqrt{8.4}}\right) \approx \Phi(-1.2076)$$
$$= 1 - \Phi(1.2076) \approx 1 - 0.8863 = 0.1137.$$

A820 (9740 N2011/II/8)(i)



(ii) The PMCC is ≈ -0.992317 which is very large in magnitude. But this merely means that the correlation between x and y is very strong. It does not also imply that their true relationship is definitely linear. Indeed in this case, it appears that the relationship is not linear.

(iii) We are supposed to say that the larger the magnitude of the PMCC, the better the model. In this case, the PMCC of y and x^2 is -0.999984. And so we're supposed to conclude that $y = a + bx^2$ is the better model.

(iv) In general, the estimated regression equation is $y - \bar{y} = b(x - \bar{x})$, where $b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2$. So in this case, the estimated regression equation is

$$y - 10.885714 \approx -0.856210\left(x^2 - 13.25\right)$$
$$\iff y \approx -0.856210x^2 + 22.230492.$$

$y(3.2) = -0.856210 \cdot (3.2)^2 + 22.230492 \approx 13.5.$

A821 (9740 N2011/II/9)(i)(a) $0.6 \cdot 0.05 + 0.4 \cdot 0.07 = 0.03 + 0.028 = 0.058.$

(i)(b) $0.03/0.058 = 15/29 \approx 0.517241.$

(ii)(a) P(Exactly one faulty) = P(First faulty, second not) + P(Second faulty, first not) = $0.058(1 - 0.058) + (1 - 0.058)0.058 = 2 \cdot 0.058 \cdot 0.942 = 0.109272.$

$$\textbf{(ii)}\ (b)\mathrm{P}(\overbrace{\text{Both made by } A}^{E}|\overbrace{\text{Exactly one faulty}}^{F}) = \frac{\mathrm{P}(E \cap F)}{\mathrm{P}(F)}.$$

But $P(E \cap F) = P(E)P(F|E) = 0.6^2(0.05 \cdot 0.95 + 0.95 \cdot 0.05) = 0.0342$. Hence $P(E|F) = 0.0342/0.109272 \approx 0.312980.$

A822 (9740 N2011/II/10)(i) We are given that $T \sim N(5.0, 38.0)$.

Let X be the time taken to install the component after background music is introduced. Assume that X remains normally distributed with standard deviation 5.0 (these are ques-

tionable assumptions, but without these we cannot proceed). That is, $X \sim \mathrm{N}\left(\mu_0, 5.0^2\right)$. The null hypothesis is $H_0 : \mu_0 = 38.0$ and the alternative hypothesis is $H_A : \mu_0 < 38.0$.

(ii) $Z_{0.05} \approx 1.645$. So to reject the null hypothesis, we must have $\bar{t} < \mu_0 - Z_{0.05}\sigma/\sqrt{n} = 38.0 - 1.645 \cdot 5.0/\sqrt{50} \approx 36.8$.

(iii) Since the null is not rejected with $\bar{t} = 37.1$, we must have $\bar{t} = 37.1 > \mu_0 - Z_{0.05}\sigma/\sqrt{n} = 38.0 - 1.645 \cdot 5.0/\sqrt{n}$. Rearranging, $n < (1.645 \cdot 5.0/0.9)^2 \approx 83.5$. Thus, $n \in \{1, 2, \ldots, 83\}$.

A823 (9740 N2011/II/11)(i) There are in total $C(30, 10)$ ways to choose the committee. There are $C(18, 4) \times C(12, 6)$ ways to choose a committee with exactly 4 women. Hence, the desired probability is

$$\binom{18}{4}\binom{12}{6}\Big/\binom{30}{10} = \frac{[(18 \cdot 17 \cdot 16 \cdot 15)/4!]\,[(12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7)/6!]}{(30 \cdot 29 \cdot \cdots \cdot 21)/10!}$$

$$= \frac{17 \cdot 48}{29 \cdot 13 \cdot 23} \approx 0.9410679.$$

(ii) The number of ways to choose a committee with exactly r women is

$$\binom{18}{r}\binom{12}{10-r}.$$

And the number of ways to choose a committee with exactly $r+1$ women is

$$\binom{18}{r+1}\binom{12}{9-r}.$$

We are told that the first number is greater than the second, i.e.

$$\binom{18}{r}\binom{12}{10-r} > \binom{18}{r+1}\binom{12}{9-r}$$

$$\iff \qquad \frac{18!}{(18-r)!r!}\frac{12!}{(2+r)!(10-r)!} > \frac{18!}{(17-r)!(r+1)!}\frac{12!}{(3+r)!(9-r)!}$$

$$\iff (17-r)!(r+1)!(3+r)!(9-r)! > (18-r)!r!(2+r)!(10-r)! \qquad \text{(as desired)}.$$

Continuing with the algebra, we have $(r+1)(3+r) > (18-r)(10-r) \iff r^2 + 4r + 3 > r^2 - 28r + 180 \iff 32r > 177 \iff r > 5 + 17/32$.

We have just proven that $\mathrm{P}(R = r) > \mathrm{P}(R = r + 1)$ if and only if $r = 6, 7, 8, 9$. That is, we have just shown that $\mathrm{P}(R = 6) > \mathrm{P}(R = 7) > \mathrm{P}(R = 8) > \mathrm{P}(R = 9) > \mathrm{P}(R = 10)$, but $\mathrm{P}(R = 0) \le \mathrm{P}(R = 1) \le \mathrm{P}(R = 2) \le \mathrm{P}(R = 3) \le \mathrm{P}(R = 4) \le \mathrm{P}(R = 5) \le \mathrm{P}(R = 6)$.

We have thus shown that 6 is a most-probable-number-of-women and that 7, 8, 9, 10 are not. We must rule out that 5 (or any smaller number) is a most-probable-number-of-women. But clearly, $6!4! \ne 5!5!$, so that

$$\binom{18}{6}\binom{12}{4} \neq \binom{18}{5}\binom{12}{5}.$$

Hence, it is indeed the case that $\mathrm{P}(R=5) < \mathrm{P}(R=6)$. Thus, 6 is indeed the unique most-probable-number-of-women.

A824 (9740 N2011/II/12)(i) Let X be the number of people who join the queue in a period of 4 minutes. Then $X \sim \mathrm{Po}(4.8)$ and

$$\mathrm{P}(X \geq 8) = 1 - \mathrm{P}(X \leq 7) = 1 - e^{-4.8} \sum_{i=0}^{7} \frac{4.8^i}{i!} \approx 0.113334.$$

(ii) Let Y be the number of people who join the queue in a period of t minutes. Then $Y \sim \mathrm{Po}(1.2t/60) = \mathrm{Po}(0.02t)$. We are told that $\mathrm{P}(Y \leq 1) = 0.7$. That is,

$$\mathrm{P}(Y \leq 1) = e^{-0.02t}(1 + 0.02t) = 0.7.$$

By calculator, $t \approx 54.8675$.

(iii) Let Z be the number of people who leave the queue over 15 minutes. Then $Z \sim \mathrm{Po}(27)$.

Let B be the number of people who join the queue over 15 minutes. Then $B \sim \mathrm{Po}(18)$.

We wish to find $\mathrm{P}(35 + B - Z \geq 24) = \mathrm{P}(Z - B \leq 11)$.

Since $\lambda_Z = 27$ is large, a suitable approximation for Z is the normal distribution is $A \sim \mathrm{N}(27, 27)$. Since $\lambda_B = 18$ is large, a suitable approximation for B is the normal distribution is $C \sim \mathrm{N}(18, 18)$. In turn, a suitable approximation for $Z - B$ is $A - C \sim \mathrm{N}(9, 45)$. Hence, using also the continuity correction,

$$\mathrm{P}(Z - B \leq 11) \approx \mathrm{P}(A - C \leq 11.5) = \Phi\left(\frac{11.5 - 9}{\sqrt{45}}\right) \approx \Phi(0.3727) \approx 0.6453.$$

(iv) There might be certain periods of time when more planes arrive and other periods when fewer arrive. So the rate at which people join the queue will probably not be constant.

A825 (9740 N2010/II/5)(i) Say we wish to stratify the spectators by age group. One problem is that we may not know what proportion of the spectators belongs to each age group. As such, it would may be difficult to get a representative sample.

(ii) Order the spectators by their names, alphabetically. Choose every 100th spectator on the list to survey.

A826 (9740 N2010/II/6)(i)

$$\bar{t} = \frac{\sum t}{n} = \frac{454.3}{11} = 41.3,$$

$$s^2 = \frac{\sum t^2 - \left(\sum t\right)^2/11}{n-1} = \frac{18779.43 - 454.3^2/11}{10} = 1.684.$$

(ii) The null hypothesis is $H_0 : \mu_0 = 42.0$ and the alternative hypothesis is $H_A : \mu_0 \neq 42.0$.

$$T=\frac{\bar{t}-\mu_0}{s/\sqrt{n}}=\frac{41.3-42.0}{\sqrt{1.684/11}}\approx -1.789.$$

Since $|T|<t_{10,0.05}=1.812$, we are unable to reject the null hypothesis.

A827 (9740 N2010/II/7)(i) $\mathrm{P}(A\cap B')=\mathrm{P}(A|B')\mathrm{P}(B')=0.8\cdot 0.4=0.32$.

(ii) $\mathrm{P}(A\cup B)=\mathrm{P}(B)+\mathrm{P}(A\cap B')=0.92$.

(iii) $\mathrm{P}(B'|A)=\mathrm{P}(B'\cap A)\div\mathrm{P}(A)=0.32\div 0.7=16/35\approx 0.457142857$.

(iv) $\mathrm{P}(A'\cap C)=\mathrm{P}(A')\mathrm{P}(C)=0.3\cdot 0.5=0.15$.

(v) $\mathrm{P}(A'\cap B\cap C)\le 0.15$.

A828 (9740 N2010/II/8)(i) The probability that the number is greater than 30000 is the probability that the first digit is 3, 4, or 5. Answer: $3/5=0.6$.

(ii) The first three digits are odd and there are 3! ways to arrange them. The last two are even and there are 2! ways to arrange them. The total number of ways to arrange the five digits is 5!. Answer: $3!2!/5!=1/10=0.1$.

(iii) If the first digit is 3, the last digit must be 1or 5, and in each case, there are 3! ways to arrange the middle 3 digits.

Similarly, if the first digit is 5, the last digit must be 1 or 3, and in each case, there are 3! ways to arrange the middle 3 digits.

If the first digit is 4, the last digit can be 1, 3, or 5, and in each case, there are 3! ways to arrange the middle 3 digits.

Altogether then, there are $7\cdot 3!$ ways to get such a number and the desired probability is $7\cdot 3!/5!=7/20=0.35$.

A829 (9740 N2010/II/9)(i) Our desired probability is $\mathrm{P}(Y>2X)=\mathrm{P}(Y-2X>0)$. Now, $Y-2X\sim\mathrm{N}\left(400-2\cdot 180,60^2+2^2 30^2\right)=\mathrm{N}\left(40,7200\right)$. So

$$\mathrm{P}(Y-2X>0)=1-\Phi\left(\frac{0-40}{\sqrt{7200}}\right)\approx\Phi(0.4714)\approx 0.6813.$$

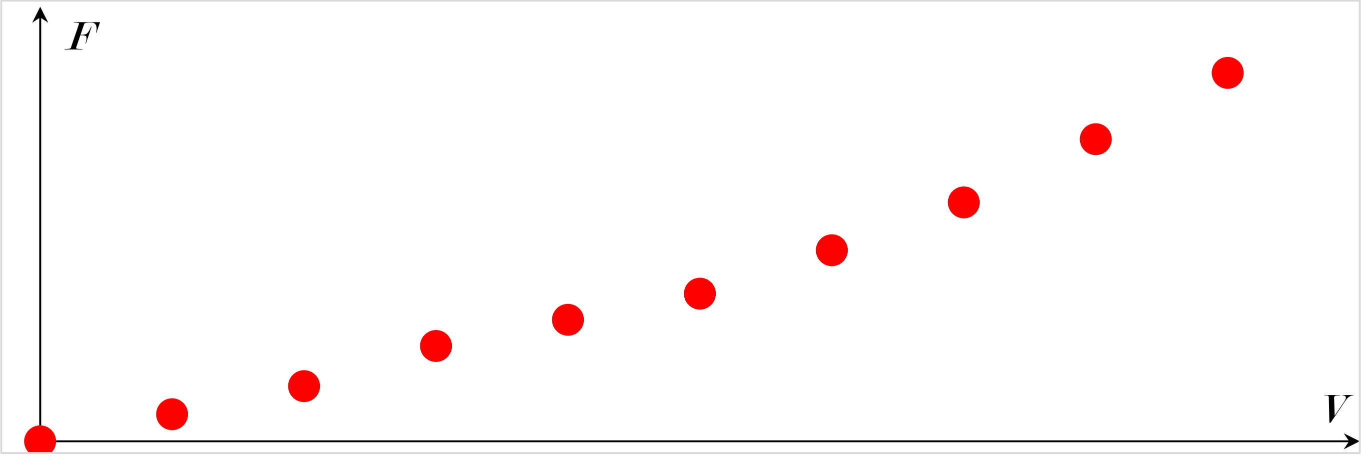
(ii) Our desired probability is $\mathrm{P}(0.12X+0.05Y>45)$. Now,

$$0.12X+0.05Y\sim\mathrm{N}\left(0.12\cdot 180+0.05\cdot 400,0.12^2\cdot 30^2+0.05^2\cdot 60^2\right)=\mathrm{N}\left(41.6,21.96\right)$$
$$\Longrightarrow\ \mathrm{P}(0.12X+0.05Y>45)=1-\Phi\left(\frac{45-41.6}{\sqrt{21.96}}\right)\approx 1-\Phi(0.7255)\approx 1-0.7658=0.2342.$$

(iii) Our desired probability is $\mathrm{P}\left(0.12X_1+0.12X_2>45\right)$. Now,

$$0.12X_1+0.12X_2\sim\mathrm{N}\left(0.12\cdot 180+0.12\cdot 180,0.12^2\cdot 30^2+0.12^2\cdot 30^2\right)=\mathrm{N}(43.2,25.92)$$
$$\mathrm{P}\left(0.12X_1+0.12X_2>45\right)=1-\Phi\left(\frac{45-43.2}{\sqrt{25.92}}\right)\approx 1-\Phi(0.3536)\approx 1-0.6381=0.3619.$$

A830 (9740 N2010/II/10)(i)



(ii)(a) PMCC ≈ 0.986024.

(ii)(b) PMCC ≈ 0.990681.

(iii) We are, as usual, supposed to say that the larger the magnitude of the PMCC, the better the model. So $F = c + dv^2$ is the better model.

(iv) In general, the estimated regression equation is $y - \bar{y} = b(x - \bar{x})$, where $b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2$. So in this case, the estimated regression equation is

$$F - 14.25 \approx 0.0242420\left(x^2 - 456\right)$$
$$\iff F \approx 0.0242420x^2 + 3.195652.$$

And $F = 26.0 \iff x \approx \sqrt{(26.0 - 3.195652)/0.0242420} \approx 30.7$.

To predict a value of v given a value of F, it would be more appropriate to use a regression where v (or a function of v) is the independent variable and F (or a function of F) is the dependent variable.

A831 (9740 N2010/II/11)(i) Let X be the number of calls received in a randomly chosen period of 4 minutes. Then $X \sim \text{Po}(12)$ and

$$\text{P}(X = 8) = e^{-12}\frac{12^8}{8!} \approx 0.0655233.$$

(ii) Let Y be the number of calls received in a randomly chosen period of t seconds. Then $Y \sim \text{Po}(3t/60) = \text{Po}(0.05t)$ and $\text{P}(Y = 0) = \text{e}^{-0.05t} = 0.2$. So $t = (\ln 0.2)/(-0.05) \approx 32$.

$$\text{P}(Y = 0) = e^{-12}\frac{12^8}{8!} \approx 0.0655233.$$

(iii) Let Z be the number of calls received in a randomly chosen period of 12 hours. Then $Z \sim \text{Po}(2160)$ and a suitable approximation therefor is the normal distribution $A \sim \text{N}(2160, 2160)$. Hence, using also the continuity correction,

$$\text{P}(Z > 2200) \approx \text{P}(A \geq 2200.5) = 1 - \Phi\left(\frac{2200.5 - 2160}{\sqrt{2160}}\right) \approx 1 - \Phi(0.8714) \approx 1 - 0.8082 = 0.1918.$$

(iv) $\displaystyle\binom{6}{2} 0.1918^2 0.8082^4 \approx 0.2354$.

(v) Let B be the number of busy days out of 30. Since $np \approx 5.754 > 5$ and $n(1-p) > 5$, a suitable approximation to B is the normal distribution $C \sim \mathrm{N}(5.754, 4.650)$. So using also the continuity correction,

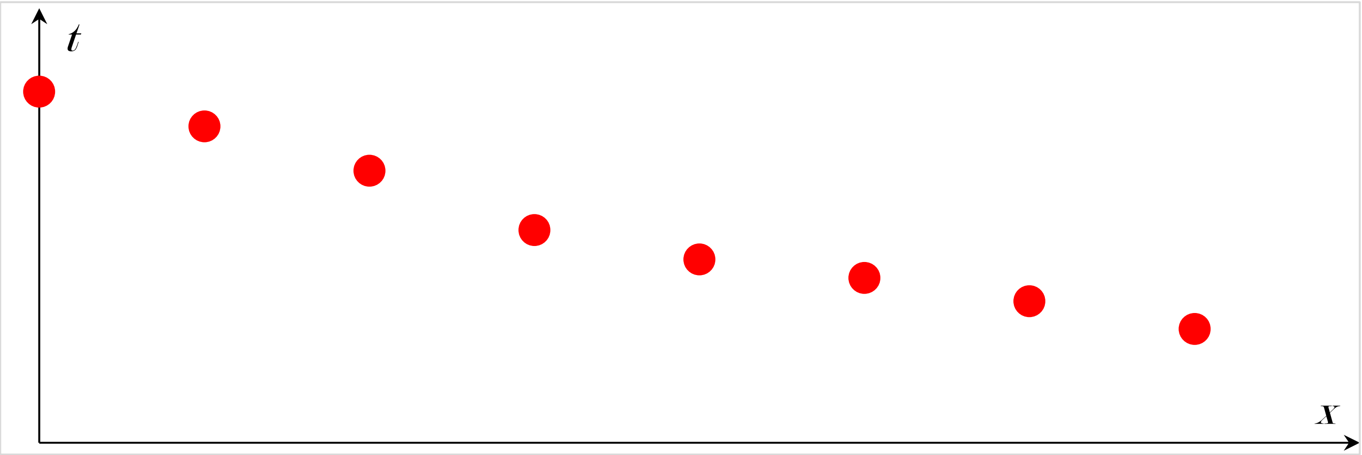
$$\mathrm{P}(B \le 10) \approx \mathrm{P}(C \le 10.5) = \Phi\left(\frac{10.5 - 5.754}{\sqrt{4.650}}\right) \approx \Phi(2.201) \approx 0.9861.$$

(Without using any approximation, $\mathrm{P}(B \le 10) \approx 0.980906$.)

A832 (9740 N2009/II/5). Simply survey people standing outside the theatre waiting for the movie to start. Stop once the quota of 100 persons is met.

A disadvantage is that this may not be a representative sample. For example, there will be no late-comers in our sample of 100.

A833 (9740 N2009/II/6)(i)



(ii) No. A linear model would imply that several centuries hence, the time taken to run a mile would be negative, which is clearly impossible.

The scatter diagram similarly suggests that the rate of improvement is tapering off, rather than linear.

(iii) A quadratic model would imply that the world record time taken to run a mile eventually bottoms out, then starts increasing. But by definition, it is impossible that the world record time increases.

(iv) In general, the estimated regression equation is $y-\bar{y} = b(x-\bar{x})$, where $b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2$. So in this case, the estimated regression equation is

$$\ln t - 3.161647 \approx -0.0161280(x - 1965)$$
$$\iff \ln t \approx -0.0161280x + 34.853071.$$

$t(2010) \approx \mathrm{e}^{-0.0161280(2010)+34.853071} \approx 11.4$. So the predicted world record time on 1st January 2010 is 3 m 41.4 s.

Our range of data is 1930-2000. We are extrapolating our data, which might not always work out reliably.

A834 (9740 N2009/II/7)(i) Let E and F be the events that "a randomly chosen component that is faulty" and "a randomly chosen component was supplied by A". Then,

$$\mathrm{P}(E) = 0.01p \cdot 0.05 + 0.01(1-p)0.03 = 0.03 + 0.02 \cdot 0.01p = 0.035$$

(ii) $$f(p) = \mathrm{P}(F|E) = \frac{\mathrm{P}(F \cap E)}{F(E)} = \frac{0.01p \cdot 0.05}{0.03 + 0.02 \cdot 0.01p} = \frac{0.05p}{3 + 0.02p} = 2.5 - \frac{7.5}{3 + 0.02p}.$$

$f'(p) = 7.5(3+0.02p)^{-2}(0.02) > 0$. This shows that the probability that a randomly chosen component that is faulty was supplied by A is increasing in the percentage of electronic components bought from A. Which is not very surprising.

A835 (9740 N2009/II/8)(i) We have 8 letters total, 3 of which are repeated. Hence, there are $8!/3! = 6720$ possible permutations.

(ii) Let TD or DT be a single letter. Then we have 7 "letters" total, 3 of which are repeated, so there are $2! \times 7!/3!$ possible permutations that we do not want. So there are $6720 - 2! \times 7!/3! = 5040$ possible permutations that we do want.

(iii) The 4 consonants by themselves have 4! possible permutations. The 4 vowels by themselves have $4! \div 3! = 4$ possible permutations. The first letter can either be a consonant or a vowel. Hence, there are in total $2 \times 4! \times 4 = 192$ possible permutations.

(iv) There are only four broad possibilities: E _ _ E _ _ E _ , E _ _ E _ _ _ E, E _ _ _E _ _ E, and _ E _ _E _ _ E. Each of which have 5! possible permutations. Hence, there are in total $4 \times 5! = 480$ possible permutations.

A836 (9740 N2009/II/9)(i) $\bar{M} \sim \mathrm{N}\left(2.5, \frac{0.1^2}{n}\right)$. So $\mathrm{P}\left(\bar{M} > 2.53\right) = 1 - \Phi\left(\frac{2.53 - 2.5}{\sqrt{0.1^2/n}}\right) = 1 - \Phi\left(0.3\sqrt{n}\right) = 0.0668 \iff \Phi\left(0.3\sqrt{n}\right) = 0.9332 \iff 0.3\sqrt{n} = 1.5 \iff n = 25$.

(ii) Assuming the thicknesses of the textbooks are independently distributed,

$$X = M_1 + \cdots + M_{21} + S_1 + \ldots S_{24} \sim \mathrm{N}\left(21 \cdot 2.5 + 24 \cdot 2.0, 21 \cdot 0.1^2 + 24 \cdot 0.08^2\right) = \mathrm{N}\left(100.5, 0.3636\right).$$

Now, $\mathrm{P}(X \le 100) = \Phi\left(\frac{100 - 100.5}{\sqrt{0.3636}}\right) \approx 1 - \Phi\left(0.8292\right) \approx 1 - 0.7964 = 0.2036$.

(iii) Again assuming the thicknesses of the textbooks are independently distributed, our desired probability is $\mathrm{P}\left(S_1 + S_2 + S_3 + S_4 < 3M\right) = \mathrm{P}\left(S_1 + S_2 + S_3 + S_4 - 3M < 0\right)$. Now, $S_1 + S_2 + S_3 + S_4 - 3M \sim \mathrm{N}\left(4 \cdot 2.0 - 3 \cdot 2.5, 4 \cdot 0.08^2 + 3^2 \cdot 0.1^2\right) = \mathrm{N}\left(0.5, 0.1156\right)$. Hence,

$$\mathrm{P}\left(S_1 + S_2 + S_3 + S_4 - 3M < 0\right) = \Phi\left(\frac{0 - 0.5}{\sqrt{0.1156}}\right) \approx 1 - \Phi\left(1.4706\right) \approx 1 - 0.9293 = 0.0707.$$

(iv) The thicknesses of the textbooks are independently distributed.

A837 (9740 N2009/II/10)(i)

$$\bar{x} = \frac{\sum x}{n} = \frac{86.4}{9} = 9.6,$$

$$s^2 = \frac{\sum x^2 - \left(\sum x\right)^2/n}{n-1} = \frac{835.92. - 86.4^2/9}{8} \approx 0.81.$$

(ii) A necessary assumption is that X is normally distributed. The null hypothesis is $H_0 : \mu_0 = 10$ and the alternative hypothesis is $H_A : \mu_0 \neq 10$.

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{9.6 - 10}{\sqrt{0.81/9}} = -\frac{4}{3}.$$

Since $|t| < t_{8,0.025} = 2.306$, we are unable to reject the null hypothesis.

The sample size is small. And so we are unable to appeal to the CLT and claim that a normal distribution is a suitable approximate distribution for $\bar{x}$.

(Author's remark: It actually makes no sense to say that "the CLT does not apply in this context". The CLT certainly applies. It is merely that the normal distribution is a poor approximation for the sample mean.)

(iii) We'd use the Z-test instead.

A838 (9740 N2009/II/11)(i) The probability that any observed car is red is independent of whether any other observed car is red.

Each car is either strictly red or strictly not red.

(ii) $\mathrm{P}(4 \leq R < 8)$

$$= \binom{20}{4} 0.15^4 0.85^{16} + \binom{20}{5} 0.15^5 0.85^{15} + \binom{20}{6} 0.15^6 0.85^{14} + \binom{20}{7} 0.15^7 0.85^{13}$$

$$\approx 0.346354.$$

(iii) Since np and $n(1-p)$ are large, a suitable approximation to R is the normal distribution $X \sim \mathrm{N}(72, 50.4)$. Hence, using also the continuity correction,

$$\mathrm{P}(R < 60) \approx \mathrm{P}(X < 59.5) = \Phi\left(\frac{59.5 - 72}{\sqrt{50.4}}\right) \approx 1 - \Phi(1.761) \approx 1 - 0.9609 = 0.0391.$$

(iv) Since n is large and p is small, a suitable approximation to R is the normal distribution $Y \sim \mathrm{Po}(4.8)$. Hence,

$$\mathrm{P}(R = 3) = e^{-4.8}\frac{4.8^3}{3!} \approx 0.152.$$

$$\textbf{(v)}\mathrm{P}(R = 0) + \mathrm{P}(R = 1) = \binom{20}{0} p^0 (1-p)^{20} + \binom{20}{1} p^1 (1-p)^{19}$$
$$= (1-p)^{19}(1 - p + 20p) = 0.2.$$

By calculator, $p \approx 0.142432$.

A839 (9740 N2008/II/5)(i) Take any ordered list of the 950 pupils. From the list, pick every 19th student.

(ii) We might want each level to be equally well-represented. For example, we might like approximately one-sixth of the sample to be from Primary 1, another sixth from Primary

2, etc.

In which case we'd probably prefer to do a stratified sample. The method might be something like this: Pick from the aforementioned ordered list the first 108 Primary 1 students, the first 108 Primary 2 students, etc.

A840 (9740 N2008/II/6). Let the mass of calcium in a bottle (after the extreme weather) be $X \sim \mathrm{N}\left(\mu_0, \sigma^2\right)$. (We have made the necessary assumption that X is normally distributed.)

The null hypothesis is $H_0 : \mu_0 = 78$ and the alternative hypothesis is $H_0 : \mu_0 \neq 78$. Now,

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{\sum x/n - 78}{\sqrt{\left[\sum x^2 - \left(\sum x\right)^2/n\right]/(n-1)}/\sqrt{n}} \approx -1.207.$$

Since $|t| < t_{14,0.025} \approx 2.145$, we are unable to reject the null hypothesis.

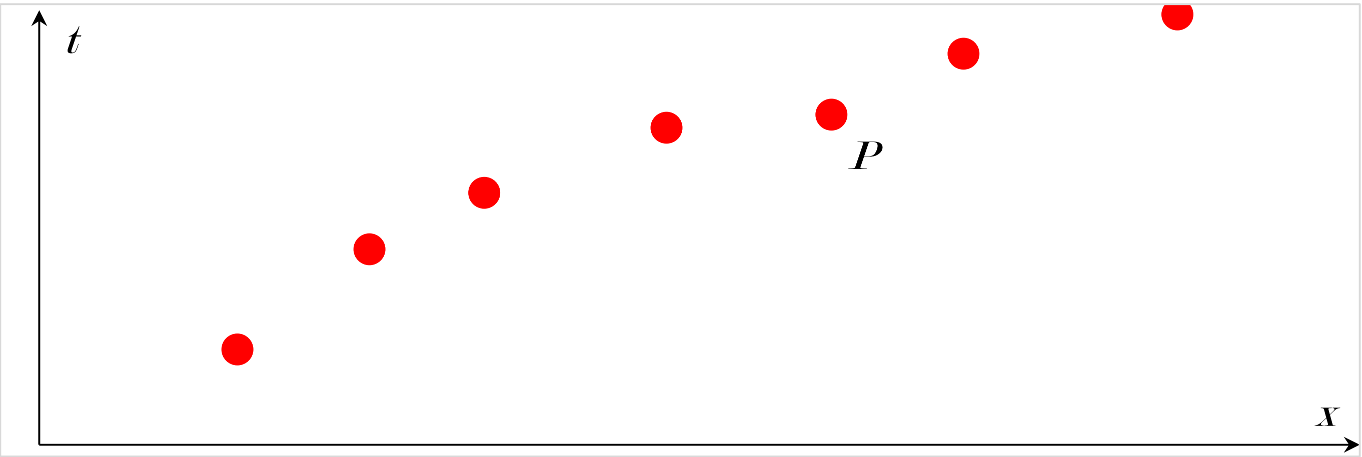
A841 (9740 N2008/II/7)(i) Let A_1 denote the event that A wins the first set. Similarly define A_2, A_3, B_1, B_2, and B_3. $\mathrm{P}\left(A_2\right) = \mathrm{P}\left(A_1 \cap A_2\right) + \mathrm{P}\left(B_1 \cap A_2\right) = 0.6 \cdot 0.7 + 0.4 \cdot 0.2 = 0.5$.

(ii) $\mathrm{P}\left(A \text{ wins}\right) = \mathrm{P}\left(A_1 \cap A_2\right) + \mathrm{P}\left(A_1 \cap B_2 \cap A_3\right) + \mathrm{P}\left(B_1 \cap A_2 \cap A_3\right) = 0.42 + 0.6 \cdot 0.3 \cdot 0.2 + 0.4 \cdot 0.2 \cdot 0.7 = 0.42 + 0.036 + 0.056 = 0.512$.

(iii) $\mathrm{P}\left(B_1 \cap A_2 \cap A_3\right)/\mathrm{P}\left(A \text{ wins}\right) = 0.056/0.512 = 0.109375$.

A842 (9740 N2008/II/8)(i) PMCC ≈ 0.9695281468. This large PMCC merely suggests that there is a strong (positive) linear relationship between x and t. However, the true relationship between x and t could be something other than linear.

(ii)



(iii) Without P, it appears that t is increasing, but at a decreasing rate. So a log model might be appropriate.

(iv) In general, the estimated regression equation is $y - \bar{y} = b(x - \bar{x})$, where

$$b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2.$$

So in this case, the estimated regression equation is

$$t - 6.45 \approx 4.396563 \left(\ln x - 1.143002\right)$$
$$\iff t \approx 4.396563 \ln x + 1.424722.$$

So for the model $t = a + b \ln x$, the least square estimates are $a \approx 1.4$ and $b \approx 4.4$.

(v) $t(x = 4.8) \approx 4.4 \ln(4.8) + 1.4 \approx 8.3$.

(vi) This would be an extrapolation of the data, which may or may not be wise.

A843 (9740 N2008/II/9)(i) $X \sim \text{Po}(1.8)$.

$$\text{P}(X \geq 4) = 1 - \text{P}(X \leq 3) = 1 - e^{-1.8}\left(\frac{1.8^0}{0!} + \frac{1.8^1}{1!} + \frac{1.8^2}{2!}\right) \approx 0.108708.$$

(ii) Let Y be the total number of pianos sold in a given week. Then $Y \sim \text{Po}(4.4)$. $\text{P}(Y = 4) = \text{e}^{-4.4} 4.4^4/4! \approx 0.191736$.

(iii) Let Z be the number of grand pianos sold in 50 weeks. Then $Z \sim \text{Po}(90)$. Since λ_Z is large, a suitable approximation is the normal distribution $A \sim \text{N}(90, 90)$. Hence, using also the continuity correction,

$$\text{P}(Z < 80) \approx \text{P}(A < 79.5) = \Phi\left(\frac{79.5 - 90}{\sqrt{90}}\right) \approx 1 - \Phi(1.1068) \approx 1 - 0.8657 = 0.1343.$$

(iv) An organisation might buy a relatively large number of grand pianos on any given day. So it is not likely that the rate at which grand pianos are sold is constant throughout the year.

A844 (9740 N2008/II/10)(i) $\binom{3}{2}\binom{4}{3}\binom{5}{3} = 3 \cdot 4 \cdot 10 = 120$.

(ii) $\binom{9}{8} = 9$.

(iii) $\binom{5}{4}\binom{7}{4} + \binom{5}{5}\binom{7}{3} = 5 \cdot 35 + 1 \times 35 = 210$.

(iv) The number of ways to have

- No diplomats from K (i.e. only diplomats from L and M) is $\binom{9}{8}$;
- No diplomats from L is $\binom{8}{8}$;
- No diplomats from M is 0.

The total number of ways to choose the diplomats is $\binom{12}{8}$. Hence the number of ways to have at least 1 diplomat from each island is $\binom{12}{8} - \left[\binom{9}{8} + \binom{8}{8}\right] = 495 - (9 + 1) = 485$.

A845 (9740 N2008/II/11)(i) $X_1 + X_2 \sim \mathrm{N}\left(100, 2 \cdot 8^2\right)$. So,

$$\mathrm{P}(X_1 + X_2 > 120) = 1 - \Phi\left(\frac{120 - 100}{\sqrt{2 \cdot 8^2}}\right) \approx 1 - \Phi(1.768) \approx 1 - 0.9615 = 0.0385.$$

(ii) $X_1 - X_2 \sim \mathrm{N}\left(0, 2 \cdot 8^2\right)$. So,

$$\mathrm{P}(X_1 > X_2 + 15) = \mathrm{P}(X_1 - X_2 > 15) = 1 - \Phi\left(\frac{15 - 0}{\sqrt{2 \cdot 8^2}}\right) \approx 1 - \Phi(1.3258) \approx 1 - 0.9075 = 0.0925.$$

(iii) $\mathrm{P}(Y < 74) = \Phi\left(\dfrac{74 - \mu}{\sigma}\right) = 0.0668 \iff \dfrac{74 - \mu}{\sigma} = -1.5.$

$\mathrm{P}(Y > 146) = 1 - \Phi\left(\dfrac{146 - \mu}{\sigma}\right) = 0.0668 \iff \Phi\left(\dfrac{146 - \mu}{\sigma}\right) = 0.9332 \iff \dfrac{146 - \mu}{\sigma} = 1.5.$

$\dfrac{146 - \mu}{\sigma} - \dfrac{74 - \mu}{\sigma} = 1.5 - (-1.5) = \dfrac{72}{\sigma} = 3 \iff \sigma = 24$ and $\mu = 110$.

Since $\sigma = 8a$ and $\mu = 50a + b$, $a = 3$ and $b = -40$.

A846 (9233 N2008/I/1). 3 ways to arrange the 3 groups of books. And within each group of books, we can permute them as usual. So there are $3!6!5!4! = 12\,441\,600$ ways.

A847 (9233 N2008/II/23). By independence, $p_{A \cap B} = p_A p_B$. Also $p_{A \cup B} = p_A + p_B - p_{A \cap B} = p_A + p_B - p_A p_B$. Plugging in the given numbers, we have $0.4 = 0.2 + p_B - 0.2 p_B$, so $p_B = 0.25$. $p_B p_C = 0.25 \cdot 0.4 = 0.1 = p_{B \cap C}$, so that by definition, B and C are indeed independent.

A848 (9233 N2008/II/26)(i) Let $X \sim \mathrm{Po}(3)$. $\mathrm{P}(X > 2) = 1 - \mathrm{P}(X \leq 0) = 1 - \mathrm{e}^{-3}(1 + 3 + 9/2) = 1 - 8.5\mathrm{e}^{-3} \approx 1 - 0.423 = 0.577$.

(ii) Let Y be the number of times the machine will break down in a period of four weeks. Then $Y \sim \mathrm{Po}(12)$.

$$\mathrm{P}(Y \leq 3) = e^{-12}\left(1 + 12 + 12^2/2 + 12^3/6\right) \approx 0.00229.$$

(iii) Let Z be the number of times the machine will break down in a period of 16 weeks. Then $Z \sim \mathrm{Po}(48)$. Since λ_Z is large, a suitable approximation for Z is the normal distribution $A \sim \mathrm{N}(48, 48)$. Hence, using also the continuity correction,

$$\mathrm{P}(Z > 50) \approx \mathrm{P}(A > 50.5) = 1 - \Phi\left(\frac{50.5 - 48}{\sqrt{48}}\right) \approx 1 - \Phi(0.3608) \approx 1 - 0.6409 = 0.3591.$$

A849 (9233 N2008/II/27)(i) Let the mass after the adjustment be $X \sim \mathrm{N}\left(\mu_0, \sigma^2\right)$. It is necessary to assume that these masses remain normally distributed. The null hypothesis is $H_0 : \mu_0 = 32.40$ and the alternative hypothesis is $H_A : \mu_0 \neq 32.40$. Now,

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{32.00 - 32.40}{\sqrt{2.892/80}} \approx -2.104.$$

Since $|t| > t_{79,0.025} \approx 1.99$, we can reject the null hypothesis.

(ii) This means that if H_0were true and we tested infinitely many size-80 samples (as done above), we'd reject H_0 in 5% of the samples.

(iii) The one-tailed p-value is ≈ 0.0193. So the least level of significance is 1.93%.

A850 (9233 N2008/II/29)(i) Let $X \sim \mathrm{N}\left(50, 4^2\right)$. The probability that Mr Sim is late on any given day is

$$\mathrm{P}(X > 55) = 1 - \Phi\left(\frac{55-50}{4}\right) = 1 - \Phi(1.25) \approx 1 - 0.8944 = 0.1056.$$

Assuming that the probability that he's late each day is independent of whether he was late on any other day, the probability that he will be late no more than once in 5 days is

$$\binom{5}{0} 0.1056^0 0.8944^5 + \binom{5}{1} 0.1056^1 0.8944^4 \approx 0.910.$$

(ii) Let $Y \sim \mathrm{N}\left(40, 5^2\right)$. Our desired probability is $\mathrm{P}(X - Y - 5 < 0)$. Assuming the journey times of Messrs Sim and Lee are independent, $X - Y - 5 \sim \mathrm{N}\left(5, 4^2 + 5^2\right)$. Thus,

$$\mathrm{P}(X - Y - 5 < 0) = \Phi\left(\frac{0-5}{\sqrt{4^2+5^2}}\right) \approx 1 - \Phi(0.7809) \approx 1 - 0.7826 = 0.2174.$$

(iii) Assume that the journey times of Messrs Sim and Lee each day are independent. Then the desired probability is

$$\binom{5}{3} 0.2174^3 0.7826^2 + \binom{5}{4} 0.2174^4 0.7826^1 + \binom{5}{5} 0.2174^5 0.7826 \approx 0.0722.$$

A851 (9233 N2008/II/30)(i) Let $M \sim \mathrm{N}\left(\mu, \sigma^2\right)$. $\mathrm{P}(M < 86.50) = \Phi\left(\frac{86.50-\mu}{\sigma}\right) = 0.12$ $\iff \frac{86.50-\mu}{\sigma} \overset{1}{=} -1.175$.

$\mathrm{P}(M > 92.25) = 1 - \Phi\left(\frac{92.25-\mu}{\sigma}\right) = 0.2 \iff \Phi\left(\frac{92.25-\mu}{\sigma}\right) = 0.8 \iff \frac{92.25-\mu}{\sigma} \overset{2}{=} 0.842$.

$\overset{2}{=}$minus $\overset{1}{=}$ yields $\frac{5.75}{\sigma} = 2.017 \iff \sigma \approx 2.85$. And now $\mu \approx 89.85$.

(ii) Let $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$. $\mathrm{P}\left(\mu - 2 \le X \le \mu + 2\right) = 0.8 \implies \mathrm{P}\left(X \le \mu + 2\right) = 0.9 \iff \Phi\left(\frac{2}{\sigma}\right) = 0.9 \iff \frac{2}{\sigma} \approx 1.281 \iff \sigma \approx 1.56$.

(iii) Let $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$. Then $\mathrm{P}(\bar{X} \ge \mu + 0.50) \le 0.1 \iff 1 - \Phi\left(\frac{0.50}{\sigma/\sqrt{n}}\right) \le 0.1 \iff 0.9 \le \Phi\left(\frac{0.50\sqrt{n}}{\sigma}\right) \iff \frac{0.50\sqrt{n}}{\sigma} \gtrsim 1.281 \iff \frac{0.50\sqrt{n}}{\sigma} \ge \frac{2}{\sigma} \iff 0.50\sqrt{n} \ge 2 \iff n \ge 16$.

A852 (9740 N2007/II/5)(i) Consider a survey of whether students like a particular teacher. A quota of 10 students is to be chosen. Take a list of the teacher's students, sort their names alphabetically, and pick the first 10 students on the list.

One disadvantage is that this sample of 10 students might not be representative. For example, they might all be siblings from the same family of Angs.

(ii) Yes. If say the teacher teaches 10 different classes, we could stratify our sample by class and pick 1 student from each class.

A853 (9740 N2007/II/6).

$$\binom{10}{0}0.24^0 0.76^{10}+\binom{10}{1}0.24^1 0.76^9+\cdots++\binom{10}{4}0.24^4 0.76^6\approx 0.933.$$

(i) Let $X\sim \mathrm{B}(1000,0.24)$ be the number of people in a sample of 1000 that have gene A. Since $np=240>5$ and $n(1-p)=760>5$ are both large, a suitable approximation for X is the normal distribution $Y\sim \mathrm{N}(240,182.4)$. Hence, using also the continuity correction,

$$\begin{aligned}\mathrm{P}(230\le X\le 260)\approx \mathrm{P}(229.5\le Y\le 260.5)&=\Phi\left(\frac{260.5-240}{\sqrt{182.4}}\right)-\Phi\left(\frac{229.5-240}{\sqrt{182.4}}\right)\\&\approx\Phi(1.5179)-\Phi(-0.7775)\approx 0.9355-0.2180\approx 0.7175.\end{aligned}$$

(ii) Let $Z\sim \mathrm{B}(1000,0.003)$ be the number of people in a sample of 1000 that have gene B. Since n is large and p is small, a suitable approximation for Y is the Poisson distribution $A\sim \mathrm{Po}(3)$. Hence,

$$\begin{aligned}\mathrm{P}(2\le Z<5)\approx \mathrm{P}(2\le Y<5)&=\mathrm{P}(Y=2)+\mathrm{P}(Y=3)+\mathrm{P}(Y=4)\\&=e^{-3}\left(3^2/2+3^3/6+3^4/24\right)\approx 0.616.\end{aligned}$$

A854 (9740 N2007/II/7)(i)

$$\bar{x}=\frac{\sum x}{n}=\frac{4626}{150}=30.84\qquad\text{and}\qquad s^2=\frac{\sum x^2-\left(\sum x\right)^2/n}{n-1}\approx 33.7259.$$

(ii) Let $H_0:\mu_0=30$ and $H_A:\mu_0>30$ be the null and alternative hypotheses. Now,

$$Z=\frac{\bar{x}-\mu_0}{s/\sqrt{n}}\approx\frac{30.84-30}{\sqrt{33.7259/150}}\approx 1.772.$$

Since $Z>Z_{0.05}=1.645$, we can reject the null hypothesis.

(iii) We used the Z-test. The sample size is large, so the normal distribution is a good approximation provided the underlying distribution is "nice enough".

A855 (9740 N2007/II/8)(i) Let C be the weight of a randomly chosen chicken. Then $C\sim \mathrm{N}\left(2.2,0.5^2\right)$. Then $3C\sim \mathrm{N}\left(3\cdot 2.2,3^2\cdot 0.5^2\right)=\mathrm{N}\left(6.6,1.5^2\right)$ and

$$\mathrm{P}(3C>7)=1-\Phi\left(\frac{7-6.6}{1.5}\right)=1-\Phi\left(\frac{4}{15}\right)\approx 0.3949.$$

(ii) Let T be the weight of a randomly chosen turkey. Then $T\sim \mathrm{N}\left(10.5,2.1^2\right)$. Then $5T\sim \mathrm{N}\left(5\cdot 10.5,5^2\cdot 2.1^2\right)=\mathrm{N}\left(52.5,10.5^2\right)$ and

$$\mathrm{P}(5T>55)=1-\Phi\left(\frac{55-52.5}{10.5}\right)=1-\Phi\left(\frac{5}{21}\right)\approx 0.405904.$$

Thus, $\mathrm{P}(3C>7)\cdot \mathrm{P}(5T>55)\approx 0.160$.

(iii) $3C + 5T \sim \mathrm{N}\left(6.6 + 52.5, 1.5^2 + 10.5^2\right) = \mathrm{N}(59.1, 112.5)$. So,

$$\mathrm{P}(3C + 5T > 62) = 1 - \Phi\left(\frac{62 - 59.1}{\sqrt{112.5}}\right) = 1 - \Phi\left(\frac{5}{21}\right) \approx 0.392.$$

(iv) The event "both chicken costs more than \$7 and turkey costs more than \$55" is a proper subset of the event "chicken and turkey together cost \$62". By the monotonicity of probability, the probability of the latter is greater than the latter.

A856 (9740 N2007/II/9)(i)(a) $12!$ **(b)** $6! \cdot 2^6$.

(ii)(a) $11!$

(ii)(b) Fix any man. Then we must have to his right: Woman, man, woman, man, etc. So $6!5!$

(ii)(c) Fix any man A. Then we must have

- To his right: "Wife A, some other man, that some other man's wife, etc."; OR
- To his left: "Wife A, some other man, that some other man's wife, etc.".

In the first scenario, we have $5!$ possible arrangements. Likewise in the second. Altogether $2 \cdot 5!$ possible arrangements.

A857 (9740 N2007/II/10).

Figure to be
inserted here.

(i) $\mathrm{P}(1,1,1) = \frac{1}{8} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{64}$.

(ii) $\mathrm{P}(1,1) + \mathrm{P}(1,0,1) + \mathrm{P}(0,1,1) = \frac{1}{8} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{8} \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{8 + 2 \cdot 3 + 7}{256} = \frac{21}{256}$.

(iii) Let E and F be the events that "the third throw is successful" and "exactly two of the three throws are successful".

$\mathrm{P}(E \cap F) = \mathrm{P}(1,0,1) + \mathrm{P}(0,1,1) = \frac{1}{8} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{8} \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{13}{256}$.

$\mathrm{P}(F) = \mathrm{P}(E \cap F) + \mathrm{P}(E' \cap F) = \frac{13}{256} + \mathrm{P}(1,1,0) = \frac{17}{256}$.

Thus, $\mathrm{P}(E|F) = \mathrm{P}(E \cap F) \div \mathrm{P}(F) = 13/17$.

A858 (9740 N2007/II/11)(i) In general, the estimated regression equation is $y - \bar{y} = b(x - \bar{x})$, where $b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2$. So in this case, the estimated regression equation is

$$x - 131.667 \approx -0.260\,(t - 32) \qquad \text{or} \qquad x \approx -0.260t + 66.194.$$

(ii) $x(t = 300) \approx -0.259701 \cdot (300) + 66.194030 \approx -11.7$.

From the scatter diagram, the linear model does not appear to be suitable. Moreover, the linear model predicts that at $t = 300$, $x < 0$, which is impossible.

(iii) PMCC ≈ -0.993839. Its magnitude is larger than -0.912 and very close to -1. It would appear that the regression of $\ln x$ on t is a more appropriate model.

(iv) In general, the estimated regression equation is $y-\bar{y} = b(x-\bar{x})$, where $b = \sum \hat{x}_i \sum \hat{y}_i \Big/ \sum \hat{x}_i^2$. So in this case, the estimated regression equation is

$$\ln x - 2.995391 \approx -0.0123434\,(t - 131.666667)$$
$$\iff \ln x \approx -0.0123434t + 4.620609$$

(v) $x = 15 \implies t \approx (4.620609 - \ln 15)/0.0123434 \approx 155$.

A859 (9233 N2007/I/4)(i) It cannot be that all three vertices are collinear. Thus, one vertex must be chosen from the upper line segment and the other must be chosen from the lower line segment. Hence, there are $3 \times 6 = 18$ possible triangles.

(ii) Consider triangles that do not have A as a vertex. Two vertices must be chosen from one line segment and the third must be chosen from the other. So there are $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot 7 + 4 \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 21 + 60 = 81$ possible triangles. Now, including also triangles with A as a vertex, we have 99 possible triangles.

A860 (9233 N2007/II/23)(i) $X \sim (30, 5^2) \implies \bar{X}_{100} \sim (30, 5^2/100) = (30, 0.25)$. Since the sample size is sufficiently large, by the Central Limit Theorem, a suitable approximation for $\bar{X}_{100}$ is the normal distribution $Y \sim \mathrm{N}\,(30, 0.25)$. So

$$\mathrm{P}\left(29.2 \le \bar{X}_{100} \le 30.8\right) \approx \mathrm{P}\,(29.2 \le Y \le 30.8) \approx 0.945201 - 0.054799 \approx 0.890.$$

(ii) The distribution is "sufficiently nice" that with a sample size of 100, it is appropriate to use the CLT.

A861 (9233 N2007/II/25)(i) $\mathrm{P}(W|B) = 20/52 = 5/13 \approx 0.384615$.

(ii) $\mathrm{P}(B|W) = 20/40 = 0.5$.

(iii) $\mathrm{P}(B \cup W) = (40 + 32)/90 = 72/100 = 0.72$.

(iv) $\mathrm{P}(W)\mathrm{P}(B) = 0.4 \cdot 0.52 \ne \mathrm{P}(B \cup W)$ and so W and B are not independent.

There are men who take chemistry (equivalently, $\mathrm{P}(M \cap C) \ne 0$), so M and C are not mutually exclusive.

A862 (9233 N2007/II/26)(i) Let X be the number of genuine call-outs in a randomly chosen two-week period. Then $X \sim \mathrm{Po}\,(4)$ and

$$\mathrm{P}(X < 6) = e^{-4}\left(1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right) \approx 0.785130.$$

(ii) Let Y be the total number of call-outs in a randomly chosen six-week period. Then $Y \sim \mathrm{Po}(15)$ and since λ_Y is large, a suitable approximation for Y is the normal distribution $Z \sim \mathrm{N}\,(15, 15)$. Hence, using also the continuity correction,

$$\mathrm{P}(Y > 19) \approx \mathrm{P}(Z > 19.5) \approx 1 - \Phi\left(\frac{19.5 - 15}{\sqrt{15}}\right) \approx 0.123.$$

A863 (N2007/II/27-9233)(i) $L + H \sim \mathrm{N}\left(5 + 3, 0.1^2 + 0.05^2\right) = \mathrm{N}(8, 0.0125)$. So

$$\mathrm{P}(7.9 \le L + H \le 8.2) = \Phi\left(\frac{8.2 - 8.0}{\sqrt{0.0125}}\right) - \Phi\left(\frac{7.9 - 8.0}{\sqrt{0.0125}}\right) \approx 0.963 - 0.185 \approx 0.778.$$

(ii) $0.74L + 0.86H \sim \mathrm{N}\left(0.74 \cdot 5 + 0.86 \cdot 3, 0.74^2 \cdot 0.1^2 + 0.86^2 \cdot 0.05^2\right) = \mathrm{N}(6.28, 0.00728225)$.

$$\text{So,} \quad \mathrm{P}(6.1 \le 0.74L + 0.86H \le 6.2) = \Phi\left(\frac{6.2 - 6.28}{\sqrt{0.00728225}}\right) - \Phi\left(\frac{6.1 - 6.28}{\sqrt{0.00728225}}\right)$$

$$\approx 0.183 - 0.021 \approx 0.162.$$

A867 (9233 N2006/II/26)(i) Let X be the number of severe floods in a randomly chosen 100-year period. Then $X \sim \mathrm{Po}(2)$. So

$$[\mathrm{P}(X = 1)]^2 = \left(e^{-2} \cdot 2\right)^2 = 4e^{-4} \approx 0.0733.$$

(ii) Let Y be the number of severe floods in a randomly chosen 1000-year period. Then $Y \sim \mathrm{Po}(20)$. Since λ_Y is large, a suitable approximation for Y is the normal distribution $Z \sim \mathrm{N}(20, 20)$. Hence, using also the continuity correction,

$$\mathrm{P}(Y > 25) \approx \mathrm{P}(Z > 25.5) = 1 - \Phi\left(\frac{25.5 - 20}{\sqrt{20}}\right) \approx 0.109.$$

A864 (9233 N2006/I/4). We could have

- All three identical—1 possibility.
- Two identical—5 possibilities.
- One identical—$\binom{5}{2} = 10$ possibilities.
- None identical—$\binom{5}{3} = 10$ possibilities.

So total 26 possibilities.

A865 (9233 N2006/II/23)(i) $\mathrm{P}(A) = 1/3$.

The sum of two scores is 9 if the dice are $(3,6)$, $(4,5)$, $(5,4)$, or $(6,3)$. So $\mathrm{P}(B) = 4/36 = 1/9$.

$\mathrm{P}(A \cap B) = 2/36 = 1/18 \ne \mathrm{P}(A)\mathrm{P}(B)$, so A and B are not independent.

(ii) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 1/3 + 1/9 - 1/18 = 7/18$.

A866 (9233 N2006/II/25)(i) The null hypothesis is $H_0 : \mu = \mu_0 = 10000$ and the alternative hypothesis is $H_A : \mu < 10000$

$$Z=\frac{\bar{x}-\mu_0}{s/\sqrt{n}}=\frac{\sum(x-10000)/n+10000-\mu_0}{\sqrt{\left\{\sum(x-10000)^2-\left[\sum(x-10000)\right]^2/n\right\}/(n-1)}/\sqrt{n}}$$

$$=\frac{-2510/80+10000-10000}{\sqrt{\left\{2010203-(-2510)^2/80\right\}/79}/\sqrt{80}}\approx-1.795.$$

Since $|Z|>Z_{0.05}=1.645$, we can reject the null hypothesis.

(ii) If H_0 is true and we conduct the above test on infinitely many size-80 samples, we'd (falsely) reject H_0 for 5% of the samples.

A868 (9233 N2006/II/28)(i) Let the speed of any car (in $\mathrm{km\,h^{-1}}$) be $X\sim\mathrm{N}\left(\mu,\sigma^2\right)$. We are given that $\mathrm{P}(X>125)=1/80$ and $\mathrm{P}(X<40)=1/10$.

$$\mathrm{P}(X>125)=\frac{1}{80}\iff 1-\Phi\left(\frac{125-\mu}{\sigma}\right)=\frac{1}{80}\iff\Phi\left(\frac{125-\mu}{\sigma}\right)=\frac{79}{80}\iff\frac{125-\mu}{\sigma}\overset{1}{\approx}2.240.$$

$$\mathrm{P}(X<40)=\frac{1}{10}\iff\Phi\left(\frac{40-\mu}{\sigma}\right)=\frac{1}{10}\iff\frac{40-\mu}{\sigma}\overset{2}{\approx}-1.282.$$

$\overset{1}{\approx}$ minus $\overset{2}{\approx}$ yields $\dfrac{85}{\sigma}\approx3.522\iff\sigma\approx24.1$ and $\mu\approx70.9$.

(ii) $$\binom{10}{0}0.1^0 0.9^{10}+\binom{10}{1}0.1^1 0.9^9+\binom{10}{2}0.1^2 0.9^8+\binom{10}{3}0.1^3 0.9^7\approx0.987.$$

(iii) Let Y be the number of cars out of a random sample of 100 that are travelling at speed less than 40 km h^{-1}. Then $Y\sim\mathrm{B}(100,0.1)$. Since $np=10>5$ and $n(1-p)=90>5$ are both large, a suitable approximation to Y is the normal distribution $Z\sim\mathrm{N}(10,9)$. Hence, using also the continuity correction:

$$\mathrm{P}(Y\le8)\approx\mathrm{P}(Z\le8.5)=\Phi\left(\frac{8.5-10}{\sqrt{9}}\right)=1-\Phi(0.5)\approx1-0.6915=0.3085.$$

Abbreviations Used in This Textbook

We give the number of the page on which each abbreviation is first used.

2D	two-dimensional	
3D	three-dimensional	
9233	subject code of 2002–08 H2-Maths-equivalent syllabus	
9740	subject code of 2007–17 H2 Maths syllabus	
9758	subject code of current H2 Maths syllabus (2017–?)	
A Level	Singapore-Cambridge Advanced Level	
'A' Maths	Additional Mathematics (an O-Level subject)	
ASEAN	Association of Southeast Asian Nations	70
BBC	British Broadcasting Corporation	xl
CLT	Central Limit Theorem	??
COI	constant of integration	1032
CPFE	Correct Procedure for Finding Extrema	??
DOS	disk operating system	xxxvii

ESGS	Every School a Good School	xli
FDTE	First Derivative Test for Extrema	935, 1688
FDTI	First Derivative Test for Inflexion Points	??, ??
FLC	four-letter campaign	xli
FSSPFE	the Flawed Secondary School Procedure for Finding Extrema	??
FTCs	Fundamental Theorems of Calculus	
GCT	Goh Chok Tong	
GDP	gross domestic product	xl
GUI	graphical user interface	

H1, H2, H3	Higher 1, Higher 2, Higher 3 (see A Level)	
HCI	Hwa Chong Institution	
IET	Interior Extremum Theorem	??
IMHO	in my humble opinion	xxxvii
IMO	International Mathematical Olympiad	
J1, J2	the first and second years of junior college	
JC	junior college	xxxv
JSYK	just so you know	
LHL	Lee Hsien Loong	
List MF26	List of Formulae and Statistical Tables	xxxv
LKY	Lee Kuan Yew	
MOE	Ministry of Education (Singapore)	xxxvii

O Level	Singapore-Cambridge Ordinary Level	
PAP	People's Action Party	
PDF	portable document format	
PDF	probability density function	
PISA	Programme for International Student Assessment	
PM	Prime Minister	69
PSC	Public Service Commission (Singapore)	xlii
PSLE	Primary School Leaving Examination	4
Q&A	question and answer	xxxvi
SDN	Social Development Network	
SDTE	Second Derivative Test for Extrema	944
SDTI	Second Derivative Test for Inflexion Points	??
SDU	Social Development Unit	
SE	Stack Exchange	
SEAB	Singapore Examinations and Assessment Board	xxxvii
TI84	TI-84 PLUS Silver Edition calculator	xxxvii
TLA	three-letter abbreviation[692]	xxix
TLLM	Teach Less, Learn More	xli
TLT	Tangent Line Test	965
TPL	Tin Pei Ling	??
TSLN	Thinking Schools, Learning Nation	xli
TYS	Ten Year Series	xl

Singlish Used in This Textbook

We give the number of the page on which each Singlish expression is first used.

Note that there is often no single standardised spelling of a Singlish word or phrase (e.g. *angmoh* and *sibei* might be spelt *angmo* and *seepeh*).

angmoh	white person (Hokkien, literally "red-haired")	366
Familee	the family of Lee Kuan Yew	70
Gahmen	government	xli
kiasu	literally, "afraid to lose" (Hokkien)	xxxv
mug	to study hard, especially in rote fashion	xliii
promos	the J1 end-of-year *promotional* examinations	xliv
sibei	very (literally, "dead father")	1034

Notation Used in the Main Text

We list only notation not already already listed on pp. 14–18 of your H2 Maths syllabus. This is a fairly short list because we've generally tried to stick closely to the notation used in your syllabus and exams.

We give the number of the page on which each piece of notation is first used and/or defined.

$\because$	because	3
$\therefore$	therefore	3
"	ditto	3
$\mathbb{Z}_0^+$	the set of non-negative integers, $\{0, 1, 2, 3, \ldots\}$	53
$\mathbb{Z}_0^-$	the set of non-positive integers, $\{\ldots, -3, -2, -1, 0\}$	53
$\mathbb{Q}_0^-$	the set of non-positive rational numbers, $\{x \in \mathbb{Q} : x \leq 0\}$	53
$\mathbb{R}_0^-$	the set of non-positive real numbers, $\{x \in \mathbb{R} : x \leq 0\}$	53
$\subsetneq$	proper subset of (used by other writers, not used in this textbook)	58
$\setminus$	set minus (or set difference)	63
Domain f	the domain of the function f	
Codomain f	the codomain of the function f	
Range f	the range of the function f	
$\lim\limits_{x \to a^-} f(x)$	the left-hand limit of f as x tends to a	**??**, 1654
$\lim\limits_{x \to a^-} f(x)$	the right-hand limit of f as x tends to a	**??**, 1654
$\gtreqless$	any inequality, i.e. any of $\geq$, $<$, or $\leq$.	

Notation Used (Appendices)

We list only notation not already listed on the previous page or on pp. 14–18 of your H2 Maths syllabus.

We give the number of the page on which each piece of notation is first used and/or defined.

$A \times B$	the cartesian product of the sets A and B	1556
$\mathrm{N}_\varepsilon(a)$	the ε-neighbourhood of a—in $\mathbb{R}$, the set $(a-\varepsilon, a+\varepsilon)$	1652
$\mathrm{N}_\varepsilon^-(a)$	the left ε-neighbourhood of a—in $\mathbb{R}$, the set $(a-\varepsilon, a)$	1652
$\mathrm{N}_\varepsilon^+(a)$	the right ε-neighbourhood of a—in $\mathbb{R}$, the set $(a, a+\varepsilon)$	1652
$\cancel{\mathrm{N}}_\varepsilon(a)$	the deleted (or punctured) ε-neighbourhood of a—in $\mathbb{R}$, the set $(a-\varepsilon, a) \cup (a, a+\varepsilon)$	1652

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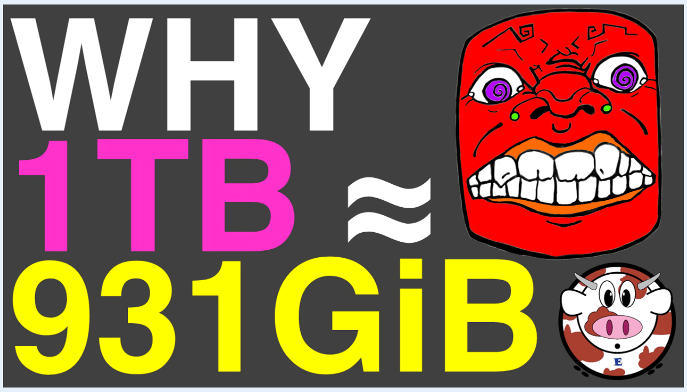

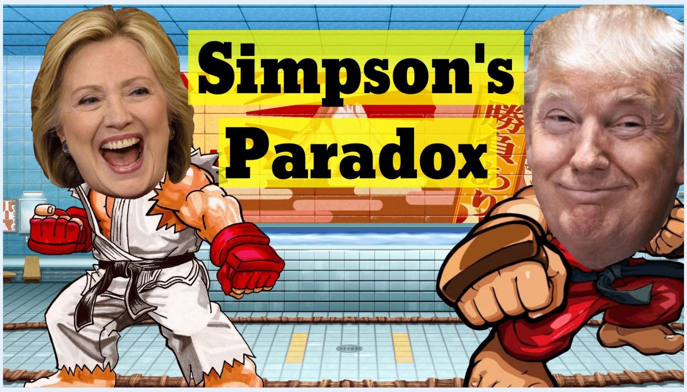

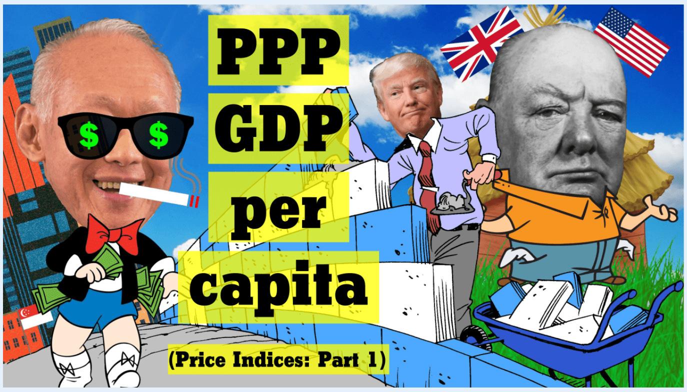

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